**DIRECT BOUNDARY INTEGRAL EQUATION FOR POTENTIAL EQUATION WITH DIRICHLET BOUNDARY CONDITIONS IN 2D PIECEWISE SMOOTH DOMAIN**

**By**

**Sarbessa Adugna**

**A Thesis Submitted to the Program of Applied Mathematics**

**School of Applied Natural Science**

**Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics**

**Office of Graduate Studies**

**Adama Science and Technology University**

**Adama, Ethiopia**

**September, 2017**

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# Abstract

*In this thesis, analysis of direct boundary integral equations(DBIEs) for potential equation subject to Dirichlet boundary condition in 2D were analyzed. Using fundamental solution in green's identity, the considered Dirichlet boundary value problems is reduced to a system of boundary integral equations. Unique solvability of boundary integral equation system were proved. The equivalence of the obtained boundary integral equation systems to the original boundary value problems and invertiblity of the boundary integral equation operators were analyzed.*

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# Notations

✓ **The notations used in this thesis are introduced as follows;**

$\nabla$ - Nabla Operator.

$\Delta a$ - Laplacian of a , $\Delta a = \nabla.\nabla a$.

$\overline{\Omega}$ - closure of the $\Omega$.

$\Gamma := \partial\Omega$ - the boundary of the domain $\Omega$.

$\nabla a$ - Gradient of a.

$\nabla.a$ - divergence of a.

$\delta(x)$ - Dirac delta distribution.

$\Omega$ - the domain.

$C(\Omega)$ - The space of continuous function in $\Omega$.

$C^1(\Omega)$ - A set of all continuous functions and differentiable on $\Omega$.

$C^2(\Omega)$ - The set of all continuous functions and twice differentiable on $\Omega$.

$W_p^k$ - Sobolev spaces order $k$

# Chapter 1

# Introduction

## 1.1 Background of the study

A differential equation is an equation involving derivatives of a function $u(x)$. So it should be clear then that an integral equation is an equation involving integrals of $u(x)$, possibly (usually) with some weighting function involved. Boundary value problems are a problems which given by boundary conditions (Chkadua.O.. etal (2013) ).

Boundary integral equations(BIEs) are reformulations of boundary value problems for partial differential equations. They have been present in the mathematics literature for almost two centuries, and engineers have made much use of them for solving a wide variety of real-world problems. The system of boundary integral equations can be evaluated on a computer(E.Penney(2004)).

The Laplace equation is one of the most widely used partial differential equations for modeling science and engineering problems. It typically comes from the physical consequence of combining a phenomenological gradient law (such as the Fourier law in heat conduction and the Darcy law in groundwater flow) with a conservation law (such as the heat energy conservation and the mass conservation of an in compressible material). For example, Fourier law was presented by Jean Baptiste Joseph Fourier in 1822 .It states that the heat flux in a thermal conducting medium is proportional to the spatial

$$q = k\nabla T$$

where $q$ is the heat flux vector, $k$ is the thermal conductivity, and $T$ is the temperature. The steady state heat energy conservation requires that at any point in space the divergence of the flux equals to zero:

$$\nabla . q = 0$$

combining the above two equations and assuming that $k$ is a constant, we obtain the Laplace equation

$$\nabla^2 T = 0$$

The Laplace equation, however, had been used earlier in the context of hydrodynamics by Leonhard Euler in 1755 , and by Lagrange in 1760 . But Laplace was credited for making it a standard part of mathematical physics. Hence it is a fundamental solution of the Laplace equation (K.Atkinson(1982)).

There fore the Laplace equation $\Delta u = 0$ is one of the basic classical equations of mathematical physics. Its solution is represented as the integral of the product of some function (potential density) and the fundamental solution of the Laplace equation. An integral of this kind is said to be a potential integral. If we look for a solution of a boundary value problem in the form of a potential, then for potential density we obtain the Fredholm integral equation, where the integration is performed over the boundary of a given domain. The potential theory can be naturally extended to more complicated elliptic equations and other equations of mathematical physics(M.Costable(1988)).

## 1.2 Statement of the problem

Let $\Omega$ be an open domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. We shall find and investigate solution of the potential equation by the method of Direct Boundary Integral

Equation for the following Dirichlet BVP: Find a function $u \in H^1(\Omega)$ satisfying

$$\Delta u(x) = \sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^2} = f(x) \quad in \quad \Omega \tag{1.2.1}$$

$$\gamma^+ u(x) = g(x) \quad on \quad \partial\Omega \tag{1.2.2}$$

where $f \in L_2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions.

This study is intended to answer the following basic questions:

• what is the necessary condition for the solvablity of this BVP ?

• Is the BVP uniquely solvable ?

• How to transform this Dirichlet BVP to BIE ?

• Is the BIE equivalent to BVP ?

• Is the BIE Invertible ?

## 1.3 Objective of the study

### 1.3.1 General Objective

• The general objective of this study is to Solve the Potential equation using Direct Boundary Integral equation method subject to the Dirichlet Boundary conditions in 2D Piecewise smooth Domain.

### 1.3.2 Specific Objectives

The specific objectives of this study are :

• to give a suitable condition for the existence and uniqueness of BIE.

• to justify whether BVP is uniquely solvable or not.

• to explain the invertiblity of the BIE.

• to investigate the equivalence of the corresponding BIE systems to the original Dirichlet BVPs.

# Chapter 2

# Literature Review

The Potential equation was first introduced at the end of the 18th century by P.Laplace , J.Lagrange and L. Euler for problems of hydrodynamics. They say the notion of a potential being considered as a function whose gradient is a vector field which due to Gauss. The properties of the simple layer potential were first studied by Coulomb and Poisson. The great contribution to the development of the potential theory was made by Green. Now a days the potential theory is an actively developed tool for studying and solving problems in different fields(C.Miranda(1970)).

The theory behind integral equations was developed by WT Ang (2007), while applications followed later in potential theory developed by W. Dijkstra, Eindhoven (2008) and elastostatics (Muskhelishvili (1953); Kupradze (1965)). A comprehensive treatise on the theory of singular integral equations is by Mikhlin (1965), while the numerical treatment of integral equations was elaborated later in such those by Baker (1977), Delves and Mohamed (1992). The earliest numerical methods for the solution of engineering problems based on integral equations are attributed to Trefftz(1926) and to Prager (1928). It was only after the introduction of the electronic computer as a computational tool in the mid-1950s that a new impetus was given to the subject of computational methods. Specifically, we have the emergence of boundary integral methods in potential theory (Smith and Pierce, 1958; Hess, 1962; Jawson, 1963; Symm, 1963), in acoustics (Friedman and Shaw, 1962; Banaugh and Goldsmith, 1963), and in elastostatics (Massonnet, 1965; Oliveira, 1968). At about the same time, boundary integral methods of the direct type, which were extremely well suited for the solution of boundary value problems (BVP), were introduced in two-dimensional (2D) and three-dimensional (3D) elastostatics by Rizzo (1967) and Cruse (1969), respectively, and in transient elastodynamics by Cruse and Rizzo (1968). Also, the relation between integral methods and the general weighed residual statement was identified by Zienkiewicz et al. (1977) and by Brebbia (1980). Numerous articles have since appeared on the use of numerical methods based on boundary integral equation formulations in all fields of engineering science such as potential theory, potential fluid flow, acoustics, torsion, electric and magnetic field theories, elastostatics, elastodynamics, plates and shells, heat conduction, viscoelasticity, thermoelasticity, fracture mechanics, plasticity, water waves, viscous flow, ground water flow, corrosion, contact problems, coupled-field problems, etc., as can be surmised by looking through a number of textbooks that have appeared in the last twenty years or so (Jawson and Symm, 1977; Liggett and Liu, 1983; Brebbia et al., 1984; Beskos, 1987; Manolis and Beskos, 1988; Brebbia and Dominguez, 1989; Beskos, 1991; Antes and Panayiotopoulos, 1992; Pozrikidis, 1992; Dominguez, 1993; Kane, 1994; Banerjee, 1994; Melnikov, 1995; Canas and Paris, 1997; Sladek and Sladek, 1998; Bonnet, 1999).

From the above literature survey it is very relevant to point out that no attempt has been made to study the Boundary Integral Equation method on the domain of Piecewise smooth.This thesis study, analysis of direct Boundary Integral Equations of Potential equation where the domain is piecewise smooth.

# Chapter 3

# Preliminaries

## 3.1 Distributions

Distributions are generalizations of the classical concept of functions. One of the most useful aspects of the theory of distribution is that, in application, discontinuous functions can be handled as easily as continuous functions. Distributions also allow us to express in mathematical form idealized concepts such as, the density of a material point,the strength of a point charge, the density of a simple or double layer, the strength of an instantaneous point source, the intensity of a force applied at a point, and so on.

**Example 3.1.1.** Density created by a material point mass 1

**Definition 3.1.1.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Then , the set

$$\mathcal{D}(\Omega) = \{\varphi \in C^\infty(\Omega) \colon \ \text{supp}(\varphi) \text{ is compact in } \Omega\}$$

is called the set of test functions.

**Example 3.1.2.** The function

$$h(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if } |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

is a test function.

**Definition 3.1.2.** (Convergence in $\mathcal{D}$)
A sequence $\{\varphi_j\}_{j=1}^\infty \subset \mathcal{D}(\Omega)$ is said to be convergent in $\mathcal{D}(\Omega)$ to $\varphi \in \mathcal{D}(\Omega)$, we write $\varphi_j \to \varphi$ as $j \to \infty$, if there is a compact $K \subset \Omega$ with

$$\text{supp } \varphi_j \subset K, \quad j \in \mathbb{N}$$

and

$$D^\alpha \varphi_j \Longrightarrow D^\alpha \varphi \quad \forall \alpha \in \mathbb{N}_0^n$$

That means *uniform convergence* for all derivatives, that is,

$$||\varphi_j - \varphi||_{C^m(\Omega)} \to 0 \text{ if } j \to \infty \quad \forall m \in \mathbb{N}_0$$

**Definition 3.1.3.** Distribution/Generalized function
Let $\Omega$ be a domain in $\mathbb{R}^n$ and let $\mathcal{D}(\Omega)$ be the space of test function.
$\mathcal{D}'(\Omega)$ is the collection of all complex-valued linear continuous functions $T$ over $\mathcal{D}(\Omega)$, i.e

$$T : \mathcal{D}(\Omega) \to \mathbb{C}.$$
$$T : \varphi \mapsto T(\varphi) := T\varphi := \langle T, \varphi \rangle$$
$$T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T\varphi_1 + \lambda_2 T\varphi_2, \quad \lambda_1, \lambda_2 \in \mathbb{C},\ \varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$$

and

$$T\varphi_j \to T\varphi \text{ for } j \to \infty \text{ in } \mathbb{C}. \text{ whenever } \varphi_j \to \varphi \text{ in } \in \mathcal{D}(\Omega)$$

Convergence in $\mathcal{D}'(\Omega)$

$$T_j \to T \text{ in } \mathcal{D}'(\Omega)$$

means

$$T_j\varphi \to T\varphi \text{ in } \mathbb{C} \text{ if } j \to \infty \text{ for any } \varphi \in \mathcal{D}(\Omega)$$

The simplest example of generalized function is the functional generated by a function $f(x) \in L_1^{loc}(\Omega)$

$$\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

is a distribution.

Generalized functions which are definable in terms of functions locally integrable are said to be *regular generalized functions.* The remaining generalized functions are called *singular generalized functions.*

### 3.1.1 The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$

One of the most effective means of solving problems in mathematical physics is by means of the transform method. In this section we shall see the basic theory of Fourier transform theory for so-called generalized functions of slow growth (tempered distributions). The remarkable feature of the class of slow growth is that the Fourier transform does not cause us to go beyond the limits of this class.

**Definition 3.1.4.** For $n \in \mathbb{N}$, $\quad \mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : ||\varphi||_{k,l} < \infty$ for all $k \in \mathbb{N}_0, l \in \mathbb{N}_0\}$, where $||\varphi||_{k,l} = \sup(1+|x|^2)^{\frac{k}{2}} \sum_{|\alpha|\le l} |D^\alpha \varphi(x)|$

A sequence $\varphi_j$ in $\mathcal{S}(\mathbb{R}^n)$ is said to converge to $\varphi$ in $\mathcal{S}(\mathbb{R}^n)$ if $||\varphi_j - \varphi||_{k,l} \to 0$ for $j \to \infty$ and all $k, l \in \mathbb{N}_0$.

*Remark* 3.1.1. Let $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ and $l = 0$

$$\begin{aligned}
\varphi(x) \in \mathcal{S}(\mathbb{R}^n) \quad &\Rightarrow \quad ||\varphi||_{k,0} = sup(1+|x|^2)^{\frac{k}{2}} |\varphi(x)| \le c_k \\
&\Rightarrow \quad (1+|x|^2)^{\frac{k}{2}} |\varphi(x)| \le c_k \\
&\Rightarrow \quad |\varphi(x)| \le \frac{c_k}{(1+|x|^2)^{\frac{k}{2}}} \le \frac{c_k}{1+|x|^k} \\
&\Rightarrow \quad |\varphi(x)| \le c_k(1+|x|^k)^{-1}
\end{aligned}$$

Similarly for all derivatives $D^\alpha \varphi(x), \alpha \in \mathbb{N}_0^n$. This explains why $\mathcal{S}(\mathbb{R}^n)$ is usually called the Schwartz space of all rapidly decreasing infinitely differentiable functions in $\mathbb{R}^n$.
$\mathcal{S}(\mathbb{R}^n)$ all function of the class $C^\infty(\mathbb{R}^n)$ which decreases as $|x| \to \infty$, together with all their derivatives, faster than any power of $|x|^{-1}$.

**Definition 3.1.5.** The space $\mathcal{S}'(\mathbb{R}^n)$ is the collection of all complex-valued linear functionals $T$ over $\mathcal{S}(\mathbb{R}^n)$. i.e.,

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}.$$

$$T : \varphi \mapsto T(\varphi) := T\varphi := \langle T, \varphi \rangle$$

$$T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T\varphi_1 + \lambda_2 T\varphi_2, \quad \lambda_1, \lambda_2 \in \mathbb{C},\ \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^n)$$

**Fourier transforms**

(**Font**)

**Definition 3.1.6.** Let $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$. Then

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) dx \quad , \xi \in \mathbb{R}^n,$$

is called the Fourier transform of $\varphi$, and

$$(\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \varphi(x) dx \quad , \xi \in \mathbb{R}^n$$

is the inverse Fourier transform of $\varphi$.

*Remark* 3.1.2.

- Since $|\varphi(x)| \le c|x|^{-n-1}$ for $|x| \ge 1$, definition 3.1.6 make sense

- It is known that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ are continuous linear operators.

- It is known that the Fourier transform $\mathcal{F}$ can be extended by continuity to $L_2(\mathbb{R}^n)$, and $\mathcal{F}$ is unitary operator in $L_2(\mathbb{R}^n)$.

$$\mathcal{F} : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$$

also

$$||f||_{L_2(\mathbb{R}^n)} = ||\mathcal{F}f||_{L_2(\mathbb{R}^n)}$$

**Definition 3.1.7.** The Fourier Transform in $\mathcal{S}'(\mathbb{R}^n)$
Let $T \in \mathcal{S}'(\mathbb{R}^n)$ then the Fourier transform $\mathcal{F}T$ and the inverse Fourier transform $\mathcal{F}^{-1}T$ are given by

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle$$

and

$$\langle \mathcal{F}^{-1}T, \varphi \rangle = \langle T, \mathcal{F}^{-1}\varphi \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

### 3.1.2 Dirac-delta function

The $\delta$-function is defined by the following properties:

1. $$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

2. $\int \int \delta(x, y) dA = 1$

3. $\int \int f(x, y)\delta(x - a, y - b) dA = f(a, b)$

## 3.2 Green's Formula

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $\partial\Omega = \Gamma$ be boundary of $\Omega$ is $C^1$

**Theorem 3.2.1.** *The Gauss-Green Theorem*

*i. Suppose $u \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_{x_i} ds \tag{3.2.1}$$

**Theorem 3.2.2.** ***(first Green's identity)***
*Let $u \in C^1(\bar{\Omega})$ and $w \in C^2(\bar{\Omega})$.*
*Then* $\int_\Omega (u \triangle w)dx = -\int_\Omega \nabla u \nabla w dx + \int_{\partial\Omega} u \frac{\partial w}{\partial v} d\sigma$

*Proof.* : note that:- $u\nabla w \in C^1(\bar{\Omega}; \Re^n)$, then by divergence theorem we have

$\int_\Omega div(u\nabla w)dx = \int_{\partial\Omega} u\nabla w v d\sigma$

$= \int_{\partial\Omega} u \frac{\partial w}{\partial v} d\sigma$

consider the vector field $u\nabla w$ $(uwx_1, uwx_2, uwx_3, ...uwx_n)$.

$div(uDw) = \sum_1^n \frac{\partial}{\partial x_i}(uwx_i)$

$= \sum_1^n (ux_i wx_i + uwx_i x_i)$

$= \nabla u \nabla w + u \triangle w$

$\int (u\nabla w + \nabla u \nabla w)dx = \int_\Omega div(u\nabla w)dx$

$= \int_{\partial\Omega} u \frac{\partial w}{\partial v} d\sigma$

Then by rearranging we have,

$\int_\Omega (u\nabla w)dx = -\int_\Omega \nabla u \nabla w dx + \int_{\partial\Omega} u \frac{\partial w}{\partial v} d\sigma.$ □

**Theorem 3.2.3.** ***( The second Green's identities)***
*Let $u, w \in C^2(\bar{\Omega})$. Then* $\int_\Omega (u \triangle w - w \triangle u)dx$
$= \int_{\partial\Omega} (u\frac{\partial w}{\partial v} - w\frac{\partial u}{\partial v})d\sigma$
*That is* $\int_\Omega (u \triangle w - w \triangle u)dx$
$= \int_{\partial\Omega} (u\frac{\partial w}{\partial v} - w\frac{\partial u}{\partial v})d\sigma$

*Proof.* In the first Greens identity interchange the given of $w$ and $u$.
Then we have:

i, $\int_\Omega (w\Delta u)dx = -\int_\Omega \nabla u \nabla w dx + \int_{\partial\Omega} (w\frac{\partial u}{\partial v})d\sigma.$

ii. $\int_\Omega (u\Delta w)dx - \int_\Omega \nabla u \nabla w dx + \int_{\partial\Omega} (u\frac{\partial w}{\partial v})d\sigma.$

Then by substructing $(i)$ from $(ii)$ we obtain:-
$\int_\Omega (u \triangle w - w \triangle u)dx = \int_{\partial\Omega} (u\frac{\partial w}{\partial v} - w\frac{\partial u}{\partial v})d\sigma.$ □

## 3.3 Fundamental solution

**Definition 3.3.1.** (C.Miranda(1970)) Let $L$ be an operator with constant coefficients $a_\alpha$:

$$L(D) = \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha \tag{3.3.1}$$

The generalized function $F \in \mathcal{D}'$ which satisfies equation,

$$L(D)F = \delta(x) \quad \text{in } \mathbb{R}^n \tag{3.3.2}$$

is said to be the fundamental solution(the function of influence) of the differential operator $L(D)$

By means of fundamental solution $F(x)$ of the operator $L(D)$ it is possible to construct a solution of the equation

$$L(D)u = f(x) \quad \text{with an arbitrary function } f. \tag{3.3.3}$$

**Theorem 3.3.1.** *(D.Dorothe(2008)) Let $f \in \mathcal{D}'$ be such that the convolution $F * f$ exists in $\mathcal{D}'$. Then the solution of equation* (3.3.3) *exist in $\mathcal{D}'$ and is given by the formula*

$$u = F * f$$

*This solution is unique in the class of distributions belonging to $\mathcal{D}'$ for which a convolution with $F$ exists.*

*Proof:*

$$\begin{aligned} L(D)(F * f) &= \sum_{|\alpha|=0}^{m} a_\alpha D^\alpha (F * f) \\ &= (\sum_{|\alpha|=0}^{m} a_\alpha D^\alpha F) * f \\ &= L(D)F * f \\ &= \delta * f = f \end{aligned}$$

Therefore, the formula $u = F * f$ is the solution of 3.3.3. To see the uniqueness consider the corresponding homogeneous equation

$$L(D)u = 0$$

Then $u = u * \delta = u * L(D)F = L(D)u * F = 0$

hence $u = 0$ is the only solution for the corresponding homogeneous equation.

Let $u_1, u_2$ be solution of the equation 3.3.3. i.e

$$L(D)u_1 = f(x) \text{ and } L(D)u_2 = f(x)$$

This implies $L(D)(u_1 - u_2) = 0 \Leftrightarrow u_1 - u_2 = 0 \Leftrightarrow u_1 = u_2$

*Remark* 3.3.1. **On physical sense of the solution $u = F * f$:** Let us represent the source $f(x)$ in the form of a "sum" of the point sources $f(\xi)\delta(x - \xi)$,

$$f(x) = \delta * f(x) = \int f(\xi)\delta(x - \xi)d\xi$$

Since $L(D)F = \delta$, each point source $f(\xi)\delta(x - \xi)$ defines the influence $f(\xi)F(x - \xi)$ Therefore the solution

$$u(x) = F * f(x) = \int f(\xi)F(x - \xi)d\xi \tag{3.3.4}$$

is the superposition of these influences.

### 3.3.1 Derivation of fundamental solution for Laplace operator

The operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is Laplace operator.

A function $u$ which satisfies the Laplace's equation $\Delta u = 0$ is called Harmonic function.

We will find a radial solution.

i.e function of $r = r(x) = |x| = \sqrt{\sum_{j=1}^{n} x_j^2}$, $x \in \mathbb{R}^n$,$n \geq 2$

Let $u(x) = v(r)$ be radially symmetric solution of the Laplace equation

$$\Delta u(x) = 0 \text{ in } \mathbb{R}^n \backslash \{0\}$$

Now

$$\frac{\partial u}{\partial x_j} = \frac{dv}{dr}\frac{\partial r}{\partial x_j} = \frac{dv}{dr}\frac{x_j}{r}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_j^2} &= \frac{\partial}{\partial x_j}(\frac{dv}{dr}\frac{x_j}{r}) \\
&= \frac{\partial}{\partial x_j}(\frac{dv}{dr})\frac{x_j}{r} + \frac{dv}{dr}\frac{\partial}{\partial x_j}(\frac{x_j}{r}) \\
&= \frac{d^2v}{dr^2}\frac{\partial r}{\partial x_j}\frac{x_j}{r} + \frac{dv}{dr}(\frac{1}{r} - \frac{x_j^2}{r^3}) \\
&= \frac{d^2v}{dr^2}\frac{x_j^2}{r^2} + \frac{dv}{dr}(\frac{1}{r} - \frac{x_j^2}{r^3})
\end{aligned}$$

so

$$\begin{aligned}
0 &= \Delta u(x) \\
&= \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} \\
&= \sum_{j=1}^{n}(\frac{d^2v}{dr^2}\frac{x_j^2}{r^2} + \frac{dv}{dr}(\frac{1}{r} - \frac{x_j^2}{r^3})) \\
&= \frac{1}{r^2}\frac{d^2v}{dr^2}\sum_{j=1}^{n} x_j^2 + \frac{n}{r}\frac{dv}{dr} - \frac{1}{r^3}\frac{dv}{dr}\sum_{j=1}^{n} x_j^2 \\
&= \frac{d^2v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr} \qquad , r > 0
\end{aligned}$$

This is a second order homogeneous ordinary differential equation. We can reduce to a first order ordinary differential equation by substituting $w = w(r)$ for $\frac{dv}{dr}$ and

$$\frac{dw}{dr} + \frac{n-1}{r}w = 0$$

**Case 1.** If $n = 2$, then

$$\begin{aligned}
\frac{w'}{w} &= \frac{1}{r} \\
\ln w &= -\ln r + c_0 \\
rw &= c_1 \\
w &= \frac{c_1}{r} \\
so \quad \frac{dv}{dr} &= \frac{c_1}{r} \\
which\ implies \qquad v(r) &= c_1 \ln r + c_2
\end{aligned}$$

**Case 2.** If $n \geq 3$, then

$$\begin{aligned}
\frac{w'}{w} &= \frac{1-n}{r} \\
\log w &= 1 - n \log r + c_0 \\
\log w + (n-1)\log r &= +c_0 \\
r^{n-1} w &= c_1 \\
w &= \frac{c_1}{r^{n-1}} \\
so \quad \frac{dv}{dr} &= \frac{c_1}{r^{n-1}} \\
\text{which implies} \quad v(r) &= c_1 \frac{r^{2-n}}{2-n} + c_2
\end{aligned}$$

Therefore ,for any constant $c_1$ and $c_2$ the function $u$ given by,

$$u(x) = \begin{cases} c_1 ln|x| + c_2, & \text{If } n = 2 \\ \frac{c_1}{(2-n)|x|^{n-2}} + c_2, & \text{If } n \geq 3 \end{cases} \tag{3.3.5}$$

is a solution of Laplace's equation in $\mathbb{R}^n \backslash \{0\}$. Note that the function $u$ satisfies $\Delta u = 0$ for $x \neq 0$, We claim that, we can choose constants $c_1$ and $c_2$ appropriately so that $\Delta_x u = \delta(x)$ in the sense of distribution.

Define

$$F(x) = \begin{cases} \frac{1}{2\pi} ln|x|, & \text{If } n = 2 \\ \frac{-1}{n(n-2)\alpha(n)|x|^{n-2}}, & \text{If } n \geq 3 \end{cases} \tag{3.3.6}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. and $\Delta F(x) = \delta(x)$. This proofs that F is the fundamental solution of the Laplace's equation.

*Remark* 3.3.2. If we shift the origin to a new point $y$, the PDE $\Delta u = 0$ is unchanged. If $u(x)$ is harmonic then $u(x-y)$ is also harmonic for $x \neq y$
Therefore,

$$F(x,y) = \begin{cases} \frac{1}{2\pi} ln|x-y|, & \text{If } n = 2 \\ \frac{-1}{n(n-2)\alpha(n)|x-y|^{n-2}}, & \text{If } n \geq 3 \end{cases} \tag{3.3.7}$$

Is a fundamental solution for Laplace equation. i.e

$$\Delta F(x,y) = \delta(x-y)$$

**Definition 3.3.2.** (**Sobolev space**)(**W.Mclean(200)**): Let $\Omega$ be an open set in $\mathbb{R}^n$. Let $k \in \mathbb{N} \cup \{0\}$. and let $1 \leq p \leq +\infty$. The sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions $u$ defined on $\Omega$ such that for every multi-index $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$ with $|\alpha| \leq k$ where $|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n$ , the mixed partial derivative

$$\mathbb{D}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

is both locally integrable and belongs to $L^p(\Omega)$ i.e $||\mathbb{D}^\alpha u||_{L^p}$.
In short, the sobolev space on $\Omega$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} \in L^p(\Omega), \forall |\alpha| \leq k\}$$

$k \in \mathbb{N}$ is called order of sobolev space. If $p = 2$ one often uses the notation $H^k(\Omega)$ or $L_2(\Omega)$. If $p = 2$ and $k = 0$, we get $H^0(\Omega) = L^2(\Omega)$. In particular if $p = 2$ and $k = 2$

$$H^k(\Omega) = \{u \in L^2(\Omega) : \tfrac{\partial u}{\partial x_i} \in L_2 \forall\, |\alpha| \leq k\}$$

**Theorem 3.3.2.** ***(Trace theorem)(D.Dorothee(2008))****: Let $\Omega$ be a $C^{k-1,1}$ domain. For $\frac{1}{2} < s \leq k$ the interior trace operator*

$$\gamma_0 : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma),$$

*where $\gamma_0 V := V/\Gamma$, is bounded. There exist $C_T > 0$ such that*

$$||\gamma_0 V||_{H^{s-\frac{1}{2}}}(\Gamma) \leq C_T ||V||_{H^s}(\Omega)$$

*for all $V \in H^s(\Omega)$ (Lawrence C.Evans)*

# Chapter 4

# Boundary Integral Equation for Dirichlet BVP in a Domain with a Corner on the boundary

## 4.1 Boundary Value problem

Let $\Omega$ be an open domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. The Dirichlet boundary value problem is defined as follows: Find a function $u \in H^1(\Omega)$ satisfying the condition

$$\Delta u(x) = f(x) \quad \text{in } \Omega, \tag{4.1.1}$$

$$\gamma^+ u(x) = \varphi_0(x) \quad \text{on } \partial\Omega. \tag{4.1.2}$$

where $f \in L_2(\Omega)$ and $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ are given functions.

Here equation (4.1.1) is under stood in distributional sense and equation (4.1.2) is in trace sense.

**Theorem 4.1.1.** *(**Steinbach.O(1970)**) Let $\Omega$ be a bounded open domain in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Then the Dirichlet Boundary Value Problem (4.1.1)-(4.1.2), has a unique solution $u \in H^1(\Omega)$, satisfying*

$$\|u\|_{H^1(\Omega)} \leqslant C(\|f\|_{L_2(\Omega)}) + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

*for some $C > 0$ depending only on $\Omega$.*

## 4.2 Green's Identities and Integral Representation

For the Laplace operator $\Delta$, and functions $u, v \in C^2(\overline{\Omega})$, the following identity is true(W.Mclean(2000)).

$$v\Delta u = \nabla \cdot (v\nabla u) - \nabla u \cdot \nabla v \tag{4.2.1}$$

Integrating both side over $\Omega$ and applying the divergence theorem and put $E(u,v) := \nabla u . \nabla v$ we obtain, the *first Green's identity*

$$\int_\Omega (v\Delta u + E(u,v))d\Omega = \int_\Gamma vTu(x)dx \tag{4.2.2}$$

where $Tu := \frac{\partial u}{\partial n} = \nabla u \cdot n$ is the conormal derivative. Interchanging the roles of $u$ and $v$ in (4.2.2)

$$\int_\Omega (u\Delta v + E(u,v))dx = \int_\Gamma uTv(x)dx \tag{4.2.3}$$

Subtracting (4.2.3) from (4.2.2), we arrive at the *second Greens identity*

$$\int_\Omega [v\Delta u - u\Delta v]dx = \int_\Gamma [vTu - uTv]dx \tag{4.2.4}$$

Taking $u(x)$ as a solution of (4.1.1) and $v(x)$ as the fundamental solution $F(x,y)$ in (4.2.4),

$$\int_\Omega [F\Delta u - u\Delta F]dx = \int_{\partial\Omega} [FTu - uTF]d\Gamma(x)$$

$$\int_\Omega F(x,y)f(x)dx - \int_\Omega u(x)\delta(x-y)dx = \int_{\partial\Omega} [F(x,y)Tu(x) - u(x)T_xF(x,y)]dx \tag{4.2.5}$$

We know that $u(x) * \delta(x-y) = \int_\Omega u(x)\delta(x-y)dx = u(y)$ ,

Hence equation (4.2.5) becomes,

$$u(y) - \int_{\partial\Omega} [u(x)T_xF(x,y) - F(x,y)Tu(x)]dx = \int_\Omega F(x,y)f(x)dx \tag{4.2.6}$$

The functions $F(x,y)$ and $\dfrac{\partial F}{\partial \vec{n}} = T_xF$ in equation (4.2.6) are both known quantities. These are the fundamental solution of the Laplace equation and its normal derivative at point $x$ of the boundary which are given by,

$$F(x,y) = \frac{1}{2\pi}\ln r,\ r = |x-y|$$

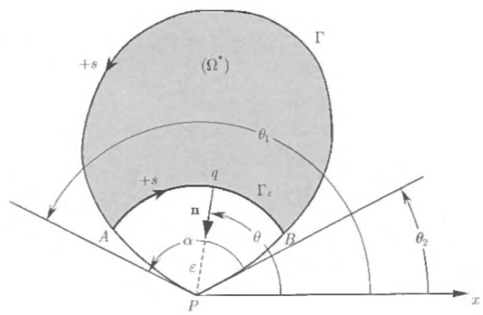


Figure 4.1: A Domain with no-smooth boundary point.

$$\frac{\partial F}{\partial n} = \frac{1}{2\pi}\frac{cos\phi}{r}$$

where $\phi$ is the angle between $\vec{r}$ and $\vec{n}$.

The expression (4.2.6) solved for $u$ is the solution of the differential equation (4.1.1) at any point $y \in \Omega$ (not on the boundary). So, what is the integral representation of $u$ for points on the boundary $\partial\Omega = \Gamma$?

Since we have consider the case where the boundary is piecewise smooth, in general let $y$ be a corner point on the boundary which is located at the point indicated on Figure 4.1.

Let $\Omega^*$ be obtained from $\Omega$ by subtracting(cutting out) a small circular section with center $P$, radius $\varepsilon$ and confined by the arcs $PA$ and $PB$. And $\Gamma_\varepsilon$ denotes the arc $AB$, $l$ the sum of arcs $AP$ and $PB$, $\vec{n}$ the outward normal to $\Gamma_\varepsilon$ coincides with the radius $\varepsilon$ and is directed towards the corner $P$ and $\alpha$ the angle between the tangents of the boundary at the point $P$.

We can easily see that

$$\lim_{\varepsilon\to 0}(\theta_1 - \theta_2) = \alpha$$

$$\lim_{\varepsilon\to 0}\Gamma_\varepsilon = 0$$

$$\lim_{\varepsilon\to 0}(\Gamma - l) = \Gamma$$

If we consider the expression (4.2.5) on the domain $\Omega^*$ ,since $y$ lies out side the domain

$\Omega^*$, where it is $\delta(x-y)=0$. This implies,

$$\int_{\Omega^*} u(x)\delta(x-y)dx = 0$$

And hence (4.2.6) becomes

$$\int_{\partial\Omega^*}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx = \int_{\Omega^*} F(x,y)f(x)dx$$

Or

$$\int_{\Gamma-l}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx+\int_{\Gamma_\varepsilon}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx = \int_{\Omega^*} F(x,y)f(x)dx \tag{4.2.7}$$

Now,

$$\lim_{\varepsilon\to 0}\int_{\Gamma-l}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx = \int_{\Gamma}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx$$

And ,

$$\lim_{\varepsilon\to 0}\int_{\Omega^*} F(x,y)f(x)dx = \int_{\Omega} F(x,y)f(x)dx$$

And

$$\int_{\Gamma_\varepsilon}[F(x,y)Tu(x)-u(x)T_xF(x,y)]dx = \int_{\Gamma_\varepsilon}\frac{1}{2\pi}\ln r\frac{\partial u}{\partial\vec{n}}dx - \int_{\Gamma_\varepsilon}u(x)\frac{1}{2\pi}\frac{cos\phi}{r}dx$$

For the circular arc $\Gamma_\varepsilon$, it is $r=\varepsilon$, and $\phi=\pi$. Also, $ds=\varepsilon(-d\theta)$,because the angle $\theta$ is positive in counter-clockwise sense, which is opposite to that of increasing $s$.

Thus,

$$\int_{\Gamma_\varepsilon}\frac{1}{2\pi}\ln r\frac{\partial u}{\partial\vec{n}}dx = \int_{\theta_1}^{\theta_2}\frac{1}{2\pi}\varepsilon\ln\varepsilon\frac{\partial u}{\partial\vec{n}}d(-\theta)$$

By mean value theorem for integral, the last integral will be,

$$=\frac{1}{2\pi}\left[\frac{\partial u}{\partial n}\right]_o\varepsilon\ln\varepsilon(\theta_1-\theta_2)$$

when $\varepsilon\to 0$, the point $O$ of the arc approaches to $y$. In this case the derivative $\left[\frac{\partial u}{\partial n}\right]_y$, though not defined, it is bounded. Therefore ,

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\frac{1}{2\pi}\ln r\frac{\partial u}{\partial\vec{n}}dx = \frac{1}{2\pi}\left[\frac{\partial u}{\partial n}\right]_y(\theta_1-\theta_2)\lim_{\varepsilon\to 0}\varepsilon\ln\varepsilon = 0$$

Similarly, (Please check the role of "-" sign in the following relations!!!)

$$\lim_{\varepsilon\to 0} -\int_{\Gamma_\varepsilon} u(x)\frac{1}{2\pi}\frac{cos\phi}{r}dx = \lim_{\varepsilon\to 0} -\int_{\theta_1}^{\theta_2} \frac{1}{2\pi}u(\frac{-1}{\varepsilon})\varepsilon d(-\theta)$$

$$= \lim_{\varepsilon\to 0} -\frac{1}{2\pi}u_O(\theta_2-\theta_1) = \lim_{\varepsilon\to 0}\frac{\theta_1-\theta_2}{2\pi}u_O = \frac{\alpha}{2\pi}u(y)$$

Hence

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}[F(x,y)Tu(x) - u(x)T_xF(x,y)]dx = \frac{\alpha}{2\pi}u(y)$$

Therefore, as $\varepsilon\to 0$,(4.2.7) will be,

$$\frac{\alpha}{2\pi}u(y) + \int_\Gamma[F(x,y)Tu(x) - u(x)T_xF(x,y)]dx = \int_\Omega F(x,y)f(x)dx \tag{4.2.8}$$

The expression (4.2.8) is the integral representation of the solution for the Poisson equation at point $y\in\Gamma$, where the boundary is not smooth. For points $y$, where the boundary is smooth, $\alpha=\pi$. Hence (4.2.8) will have the form:

$$\frac{1}{2\pi}u(y) + \int_{\partial\Omega}[F(x,y)Tu(x) - u(x)T_xF(x,y)]dx = \int_\Omega F(x,y)f(x)dx$$

*Remark* 4.2.1. A comparison between (4.2.6) and (4.2.8) shows that the function $u$ is discontinuous when the point $y\in\Omega$ approaches to a point to boundary. So we will have a jump condition equal to $(1-\frac{\alpha}{2\pi})$ for a corner points. And a jump condition equal to $\frac{1}{2}$ for smooth points.

If the point $y$ is located outside the domain $\Omega$, then $\delta(x-y)=0$ and hence

$$\int_\Omega u(x)\delta(x-y)dx = 0$$

So from equation (4.2.5),

$$\int_{\partial\Omega}[F(x,y)Tu(x) - u(x)T_xF(x,y)]dx = \int_\Omega F(x,y)f(x)dx \tag{4.2.9}$$

Equations (4.2.6),(4.2.8) and (4.2.9) can be combined in a single general equation as:

$$c(y)u(y) - \int_\Gamma[u(x)T_xF(x,y) - F(x,y)Tu(x)]dx = \int_\Omega F(x,y)f(x)dx \qquad y\in\Gamma\cup\Omega \tag{4.2.10}$$

where

$$c(y) = c(y;\Omega) = \begin{cases} 1, & \text{if } y \in \Omega \\ 0, & \text{if } y \notin \Omega \\ \frac{\alpha}{2\pi}, & \text{if } y \in \Gamma. \end{cases}$$

Here $\alpha$ is an interior space angle at a corner point $P$ of the boundary $\Gamma$.

Different combination of representation of (4.2.10) with boundary condition (4.1.2) leads to different integral or integro-differential systems, which, in turn, can lead to different numerical realizations.

*Remark* 4.2.2. For the formulation of the boundary integral equation, let $u$ be a solution of (4.1.1). Then, substituting the Dirichlet boundary condition (4.1.2), $\gamma^+ u(x) = \varphi_0(x)$, in the integral representation (4.2.10), we have for $y \in \Gamma \cup \Omega$.

$$c(y)u(y) - \int_\Gamma [\varphi_0(x) T_x F(x,y) - F(x,y) T u(x)] dx = \int_\Omega F(x,y) f(x) dx \tag{4.2.11}$$

Or

$$c(y)u(y) + \int_{\partial\Omega} F(x,y) T u(x) dx = F_0(y) \tag{4.2.12}$$

where

$$F_0(y) = \int_{\partial\Omega} \varphi_0(x) T_x F(x,y) dx + \int_\Omega F(x,y) f(x) dx$$

## 4.3 Integral Operators and Properties

The logarithmic potential operator appeared in (4.2.10)(**Steinbach.O(1970)**), is defined by

$$\mathcal{P}g(y) := \int_\Omega F(x,y) g(x) dx$$

where $g(x)$ is some density function.

The single and double layer potential operators corresponding to the fundamental solution $F(x,y)$ appeared in (4.2.10)are given by;

$$Vg(y) := -\int_\Gamma F(x,y) g(x) dx \qquad y \notin \Gamma \tag{4.3.1}$$

$$Wg(y) := -\int_\Gamma T_x F(x,y) g(x) dx \qquad y \notin \Gamma \tag{4.3.2}$$

Where $g$ is some scalar density function.

Also the following boundary integral operators are defined by,

$$\mathcal{V}g(y) \;:=\; -\gamma^{+}\int_{\Gamma} F(x,y)g(x)ds_x, \tag{4.3.3}$$

$$\mathcal{W}g(y) \;:=\; -\gamma^{+}\int_{\Gamma} T_xF(x,y)g(x)ds_x \tag{4.3.4}$$

**Theorem 4.3.1.** *The following operators are continuous*

$$V : H^{-\frac{1}{2}}(\Gamma) \to H^{1}(\Omega) \tag{4.3.5}$$

$$W : H^{\frac{1}{2}}(\Gamma) \to H^{1}(\Omega) \tag{4.3.6}$$

$$\mathcal{V} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \tag{4.3.7}$$

$$\mathcal{W} : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \tag{4.3.8}$$

**Theorem 4.3.2.** *Let $s \in \mathbb{R}$. The following operators are compact.*

$$\mathcal{V} \;:\; H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \tag{4.3.9}$$

$$\mathcal{W} \;:\; H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \tag{4.3.10}$$

**Theorem 4.3.3.** *Let $\Omega$ be a bounded open region $\mathbb{R}^2$ with closed, infinitely smooth boundary $\Gamma$. The following operators are continuous.*

$$\mathcal{P} \;:\; \tilde{H}^{s}(\Omega) \to H^{s+2}(\Omega) \qquad s \in \mathbb{R} \tag{4.3.11}$$

$$\;:\; H^{s}(\Omega) \to H^{s+2}(\Omega), \qquad s \in (\frac{-1}{2}, \infty); \tag{4.3.12}$$

$$\gamma^{+}\mathcal{P} \;:\; \tilde{H}^{s}(\Omega) \to H^{s+\frac{3}{2}}(\Gamma), \qquad s \in (\frac{-3}{2}, \infty); \tag{4.3.13}$$

$$\;:\; H^{s}(\Omega) \to H^{s+\frac{3}{2}}(\Gamma), \qquad s \in (\frac{-1}{2}, \infty); \tag{4.3.14}$$

$$T^{+}\mathcal{P} \;:\; \tilde{H}^{s}(\Omega) \to H^{s+\frac{1}{2}}(\Gamma), \qquad s \in (\frac{-1}{2}, \infty); \tag{4.3.15}$$

$$\;:\; H^{s}(\Omega) \to H^{s+\frac{1}{2}}(\Gamma), \qquad s \in (\frac{-1}{2}, \infty); \tag{4.3.16}$$

**Lemma 4.3.4.** ***(i)**: Let either $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)$ and $diam(\Omega) < 1$ ,or $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)_*$. If $V\psi^* = 0$ in $\Omega$ ,then $\psi^* = 0$ on $\partial\Omega$.*
***(ii)** Let $\phi^* \in H^{\frac{1}{2}}$. If $W\phi^* = 0$ ,then $\phi^* = 0$ on $\partial\Omega$.*

*Proof.* **(i)**: Taking the Trace of equation in Lemma 4.4.1 on $\partial\Omega$ , by the jump relation ,we have $\mathcal{V}\psi^*(y) = 0$ on $(\partial\Omega)$. If $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)$ and $diam(\Omega) < 1$ , the the result follows from the *invertibility* of the single layer potential given by above theorem.
On the other hand if $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)_*$ , then the result follows. □

*Proof.* **(ii)**: Let us take the trace of equation in lemma 4.4.1 on $\partial\Omega$ and use the jump relation to obtain,
$-\dfrac{1}{2}\phi^* + \mathcal{W}\phi^* = 0 \qquad \text{on} \qquad \partial\Omega.$
Multiplying this equation by $a(y)$, denoting $\hat{\phi}^* = a\phi^*$ and using the second relation in 4.3.1, we obtain equation
$\dfrac{-1}{2}\hat{\phi}^* + \mathcal{W} \vartriangle \hat{\phi}^* = 0 \qquad \text{on} \qquad \partial\Omega.$
It is well known that this equation has only the trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_{\vartriangle}$, since $a(y)$ not equal to zero, then the result follows. □

## 4.4 Boundary Integral Equation

Let us write the representation formula (the Green's third identity) using the potential operator notations,

$$u - VT^+u + W\gamma^+u = \mathcal{P}\Delta u \quad in \quad \Omega \tag{4.4.1}$$

Applying the trace operator to the equation, we have

$$\frac{\alpha}{2\pi}\gamma^+u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}\Delta u \quad on \quad \partial\Omega \tag{4.4.2}$$

For some functions $f, \psi, \phi$ let us consider a more general indirect integral relation associated with equation 4.4.1.

$$u - V\psi + W\phi = \mathcal{P}f \quad in \quad \Omega. \tag{4.4.3}$$

**Lemma 4.4.1.** *Let $u \in H^1(\Omega), f \in L_2(\Omega), \psi \in H^{\frac{-1}{2}}(\partial\Omega)$ and $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy equation* 4.4.3. *Then $u$ belongs to $H^{1,0}(\Omega;\Delta)$ and is a solution of partial differential equation $\Delta u = f$ in $\Omega$ and*

$$V(\psi - T^+u)(y) - W(\phi - \gamma^+u)(y) = 0, \qquad y \in \Omega.$$

By substituting the Dirichlet condition into (4.4.1) and into its trace (4.4.2) on $\partial\Omega$, we can reduce the BVP (4.1.1)-(4.1.2) to a system of Boundary Integral Equation. Thus we will find the unknown function $u \in H^{1,0}(\Omega;\Delta)$ and $\psi := T^+u \in H_*^{-\frac{1}{2}}(\Gamma)$(alternatively we can assume $diam(\Omega) < 1$ and in this case $\psi := T^+u \in H^{-\frac{1}{2}}(\Gamma)$ ). The BIE system obtained using equation (4.4.1) in $\Omega$, and equation (4.4.2) on $\Gamma$ is;

$$\begin{aligned} u - V\psi &= \mathcal{P}f - W\varphi_0 \qquad \text{in } \Omega \\ -\mathcal{V}\psi &= \mathcal{P}f - \frac{\alpha}{2\pi}\varphi_0 - \mathcal{W}\varphi_0 \qquad \text{on } \Gamma \end{aligned}$$

We can denote the system in matrix form as;

$$\mathcal{A}^1\mathcal{U} = \mathcal{F}^1$$

Where, the matrix operator is given by;

$$\mathcal{A}^1 := \begin{bmatrix} I & -V \\ 0 & -\mathcal{V} \end{bmatrix}, \ \mathcal{F}^1 := \begin{bmatrix} \mathcal{P}f - W\varphi_0 \\ \mathcal{P}f - \frac{\alpha}{2\pi}\varphi_0 - \mathcal{W}\varphi_0 \end{bmatrix}$$

And

$$\mathcal{U} := [u, \psi]^t \in H^{1,0}(\Omega; \Delta) \times H_*^{-\frac{1}{2}}(\Gamma)$$

From the mapping properties of $W$ in Theorem 4.3.1 and $\mathcal{P}$ by 4.3.12, we get the inclusion $\mathcal{F}^1 \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$.

Due to the mapping properties of operators involving in the matrix operator $\mathcal{A}^1$, the operator

$$\mathcal{A}^1 : H^{1,0}(\Omega; A) \times H_*^{-\frac{1}{2}}(\Gamma) \to H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$$

is bounded which implies that it is continuous.

*Remark* 4.4.1. $\mathcal{F}^1 = 0$ if and only if $(f, \varphi_0) = 0$.

Indeed, the latter equality evidently implies the former.

Inversely, if $\mathcal{F}^1 = 0$ then, $\mathcal{P}f - W\varphi_0$ and $\mathcal{P}f - \frac{\alpha}{2\pi}\varphi_0 - \mathcal{W}\varphi_0$. Which implies $\varphi_0 = 0$ and then $\mathcal{P}f = 0$ in $\Omega$. Multiplying this by $a$ and applying Laplace operator, we get $f = 0$ .

## 4.5 Equivalence and Invertability Theorems

In the following theorem we shall see the equivalence of the original Direchlet boundary value problem with the boundary integral equation system.

**Theorem 4.5.1.** *Let* $\varphi_0 \in H^{\frac{1}{2}}(\Gamma)$ *and* $f \in L_2(\Omega)$ *be given.*

*(i) If some* $u \in H^1(\Omega)$ *solves the BVP(4.1.1)-(4.1.2), then the pair* $(u, \psi)$*, where*

$$\psi = T^+ u \in H_*^{-\frac{1}{2}}(\Gamma) \tag{4.5.1}$$

*solves the BIE system.*

*(ii) If a pair* $(u, \psi) \in H^1(\Omega) \times H_*^{-\frac{1}{2}}(\Gamma)$ *solves the BDIE system, then* $u$ *solves the BVP(4.1.1)-(4.1.2) and* $\psi$ *satisfies (4.5.1).*

*Proof.* (i) Let $u \in H^1(\Omega)$ be solution of the BVP(4.1.1)-(4.1.2). Then following the same procedure in constructing the BIE system, we can show that $(u, \psi)$ solves the systems.

(ii) Let now a pair $(u, \psi) \in H^1(\Omega) \times H_*^{-\frac{1}{2}}(\Gamma)$ solves the BIE system. Then the conditions in Lemma (4.4.1) are satisfied, hence the result follows.

□

**Theorem 4.5.2.** *The fallowing operator is invertible.*

$$\mathcal{A}^1 \quad : \quad H^{1,0}(\Omega; A) \times H_*^{-\frac{1}{2}}(\Gamma) \to H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$$

*Proof.* The operator $\mathcal{A}^1$ is injective, i.e., $\ker \mathcal{A}^1 = \{0\}$. To see this,

$$\begin{aligned} \mathcal{A}^1 \mathcal{U} = 0 \quad &\Rightarrow \quad \mathcal{F}^1 = 0 \\ &\Leftrightarrow \quad (f, \varphi_0) = 0 \qquad \text{by remark (4.4.1)} \end{aligned}$$

This means $Au = 0, \varphi_0 = 0$, hence $u = 0, \psi = 0$. Therefore, $\mathcal{U} = 0$.
The equation

$$\mathcal{A}^1 \mathcal{U} = \mathcal{F}^1 \tag{4.5.2}$$

Or component wise,

$$u - V\psi = \tilde{F}_1 \qquad \text{in } \Omega \tag{4.5.3}$$

$$-\mathcal{V}\psi = \tilde{F}_2 \qquad \text{on } \Gamma \tag{4.5.4}$$

With an unknown vector $\mathcal{U} = (u, \psi)^t \in H^{1,0}(\Omega; A) \times H_*^{-\frac{1}{2}}(\Gamma)$ and a given vector $\tilde{\mathcal{F}} := (\tilde{F}_1, \tilde{F}_2) \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$. Due to Theorem (4.5.1), equation (4.5.4) is uniquely solvable. i.e, for arbitrary $\tilde{F}_2 \in H^{\frac{1}{2}}(\Gamma)$ there exist a unique $\psi \in H_*^{-\frac{1}{2}}(\Gamma)$ satisfying (4.5.4). Hence substituting the result into (4.5.3) we can uniquely solve of $u$,

$$u = V\psi + \tilde{F}_1 \qquad \text{in } \Omega \tag{4.5.5}$$

That is, the function $u \in H^1(\Omega)$ is uniquely defined. We conclude that $\mathcal{A}^1$ is invertible.

□

# Chapter 5

# Conclusion

In this thesis,we have considered the Dirichlet Problem for Potential equations in two-dimensional domain,where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$. The BVP was reduced to a system of BIEs using the method of fundamental solution and Green's identity. The equivalence of the the boundary integral equation to the original boundary value problem was investigated and invertiblity of the boundary integral operators were shown.

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