ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

School of Applied Natural Science

Program of Applied Mathematics

Numerical solution of 1-D and 2-D wave equations

using finite difference methods

A thesis submitted in partial fulfillment of the requirements for the degree of master's in numerical analysis.

By Dessu Argaw ID-Gsu/0424/06

Advisor Dr.Tekle Gemechu (PhD)

January 2/ 2017

Adama Ethiopia

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Declaration

Dessu Argaw declare that this is my original work and has not been presented for a degree in any other University and all sources of material used for this thesis have been duly acknowledged.

Name	signature	Date

This thesis has been submitted for examination with my approval as thesis advisor.

Name	signature	Date

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In writing this thesis, I owe a great debt to the joint contribution of several

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Dessu Argaw

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List of Acronyms

L Length of string ties at both ends

T Maximum time (one time variable)

C The speed of wave propagation

N The total number of spatial or temporal nodes

ρ The mass of the undeflected string per unit length

i Unit vector in x -direction

j Unit vector in y- direction

Δx Space step size in the x -direction

Δy Space step size in the y- direction

Δt Time step sizes.

$u_{i,j}$ The approximate numerical solution at time level j.

$o(\Delta x)$ First order spatial truncation error

$o(\Delta x)^2$ Second order spatial truncation error

$o(\Delta t)$ First order temporal error.

$o(\Delta t)^2$ Second order temporal error.

List of Abbreviations

ADI	Alternating Direction Implicit
CFL	Courant-Friedrichs-Lewy
FDM	Finite Difference Method
FEM	Finite Element Method
FVM	Finite Volume Method
GFDM	General Finite Difference Method
PDE	Partial Differential Equation
IVP	Initial Value Problem
BVP	Boundary Value Problem
ICs	Initial Conditions
BCs	Boundary Conditions
1-D	One Dimensional
2-D	Two Dimensional

Abstract

This study focuses on applications of finite difference methods for solving 1D and 2D wave equations. Explicit and implicit finite different schemes based on domain discretization are analysed for 1D wave equation. We also considered explicit scheme for 2D wave equation. The partial differential equations are replaced by difference equations at the interior nodal points (meshes) and systems of algebraic equation are solved numerically. We also investigate how the space and time step sizes Δx *,* Δy *and* Δt *affect the stability, convergence and consistency of explicit FDM in one dimensional and two dimensional wave equations. We finally present test* examples *and show the results using math lab codes.*

CHAPTER ONE

1. Introduction

The growth in computer technology has made it possible to consider the application of partial deferential equations in science and engineering on a larger scale than ever. When experimental work is cost prohibitive, well-formed theory with numerical methods may be used to obtain very valuable information. In engineering, experimental and numerical solutions are viewed as complimentary to one another in solving problems. It is common to use the experimental work to verify the numerical method and then extend the numerical method to solve new design and system.(Lyengar.S.R.K and M.K. Jain ,2010) The fast growing computational capacity also makes it practical to use numerical methods to solve problems even for non-technical people. It is a common encounter that finite difference (FD) or finite element (FE) or finite volume (FV) numerical methods-based applications are used to solve or simulate complex scientific and engineering problems. Furthermore, advances in mathematical models, methods, and computational capacity have made it possible to solve problems not only in science and engineering but also in social science, medicine, and economics (J.W. Thomas, 1995). Finite elements and finite difference methods are the most frequently applied numerical approximations, although several numerical methods are available. Finite element method (FEM) utilizes discrete elements to obtain the approximate solution of the governing differential equation. The final FEM system equation is constructed from the discrete element equations. However, the finite difference method (FDM) uses direct discrete points system interpretation to define the equation and uses the combination of all the points to produce the system equation. Both systems generate large linear and/or nonlinear system equations that can be solved by the computer. Finite element and finite difference methods are widely used in numerical procedures to solve differential equations in science and engineering. They are also the basis for countless engineering computing and computational software. As the boundaries of numerical method applications expand to non-traditional fields, there is a greater need for basic understanding of numerical simulation (Ya Yan Lu,2012) . This thesis intended to give basic insight into finite difference method (FDM) by demonstrating simple examples and working through the solution process. Simple one- and two-dimensional wave equations are used to illustrate particularly finite difference numerical method.

1.1 **Background of the study**

Partial differential equation (PDES) arises in many fields of engineering and science. Most real physical processes such as propagation of heat pressure, wave in fluid, fluid dynamics, and quantum mechanics are governed by differential equations and involve more than one dependent variable and the corresponding partial differential equations (Dr.B.S Grewal, 2009). Only a few of these equations can be solved by analytic methods. On the other hand some of these equations are complicated and can be solved by using advanced mathematical technique. This is known as numerical method. In most case, it is easier to develop approximate solutions by numerical methods. There are different methods like FDM, FVM, FEM and spectral method which are applicable to find numerical solution. Of all the numerical methods available for the solution of PDEs, FDM is the commonly used method to solve initial boundary value problems (IBVP) such as one and two-dimensional second ordered partial differential wave equations (S.S.Sastery, 2006). The finite difference method consists of replacing each derivative by a difference quotient in the classic formulation. It is simple to code and economic to compute. In a sense, a finite difference formulation offers a more direct approach to the numerical solution ofone and two - dimensional second ordered partial differential wave equation by using explicit and implicit Schemes and using implicit scheme for two dimensional wave partial differential equations than does a method based on other formulation (Mitchell.A.R, 2007).The finite difference approach is one of the premier mathematical tools employed to solve partial differential equations. It is a means of obtaining numerical solutions to partial differential equations. The most common finite difference methods for the solution of partial differential equation are:

- Explicit difference method
- Implicit difference method

There are three components to be considered:

• The partial differential equation.

• The region of space-time on which the partial differential equation is required to be satisfied.

• The initial and boundary conditions.

Classification of Linear Second-Order PDEs

An equation which involves partial derivatives of unknown function of two or more independent variables is called a partial differential equation. These equations arise in connection with numerous physical and geometric problems. A partial differential equation can be classified according to various properties. Some of the characteristic properties are as follows. The order is the order of the highest derivatives that appears in the equation, the dimension is the number of independent variables in the equation, sometimes for initial value problems, dimension refers to the number of "Space "variables while "time" is not counted. An equation is said to be linear if it is linear in the unknown variable and its derivatives. A nonlinear differential equation which is linear in the derivatives of the unknown function. (Sandip Mazumder, 2016)

The general second – order linear PDE with two independent variables in its domain is written of the form:

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = W(x,y,u,u_x,u_y) \tag{1.1}$$

Depending on the numerical relationship between A, B, and C, equation (1.1) is classified as either being hyperbolic, parabolic, or elliptic. The classification is as follows:

$$\mathrm{B}^2 - 4\mathrm{AC} > 0 \implies \text{PDE is hayperbolic} \tag{1.1a}$$

$$\mathrm{B}^2 - 4\mathrm{AC} = 0 \implies \text{PDE is parabolic} \tag{1.1b}$$

$$\mathrm{B}^2 - 4\mathrm{AC} < 0 \implies \text{PDE is elliptic} \tag{1.1c}$$

The particular cases of equation (1.1) namely

$$u_{xx} + u_{yy} = 0 \ \ (The\ laplace\ equation) \tag{1.2}$$

$$u_{tt} = c^2\, u_{xx} \quad (The\ wave\ equation) \tag{1.3}$$

$$u_t = c^2\, u_{xx} \quad (The\ heat\ conducion\ equation) \tag{1.4}$$

Where (x, y) space coordinates and t are is the time coordinate. It is easy to see that the Laplace equation is of elliptic type, that the wave equation is of hyperbolic type and that the heat conduction equation is of parabolic type.

The boundary conditions are important in the study of partial differential equations. The boundary conditions are of different types. Dirichlet condition consists in prescribing only the values of the function on a hyper surface. For Neumann condition, only the values of derivative of the function along the normal to a hyper surface are specified Cauchy condition define both function value as well as the normal derivative along a hyper surface. These boundary condition supplies to second – order differential equations. For equations of higher order, boundary condition may involve derivatives of higher order.The general form of hyperbolic type linear PDE having one, two and three space variables are as follows:

$$u_{tt} = c^2 u_{xx} \ , \ u_{tt} = c^2 \left(u_{xx} + u_{yy}\right), \ u_{tt} = c^2 \left(u_{xx} + u_{yy} + u_{zz}\right)$$

with $u(x.t)$, $u(x.y,t)$, $u(x,y,z,t)$ respectively 1-D ,2-D and 3-D .Here an attempt is made to solve 1-D and 2-D wave equations using FDM's by considering different space and time step sizes Δx,Δy and Δt to investigate the effects of mesh size on the numerical solution of 1-D and 2-D wave equations.

1.2 Statement of the problem

Hyperbolic equations arise typically in problems involving the propagation of disturbances, for example, in wave mechanics, in vibrating string, water wave and sound wave. A solution of wave equation is sought satisfying the initial and boundary conditions. For most partial differential equations we are not able to find their exact solutions, and sometimes we do not even know whether a unique solution exists. For these reasons, in most cases the only way to solve partial differential equations arising in engineering and scientific problems is to approximate their solutions numerically (K. W. Morton.D. F.Mayers, 2005). Numerical methods which are used to solve partial differential equations play a great roll in the field of modern engineering and science. There are several different numerical methods such as FDM, FEM and FVM available for solving elliptic, parabolic and hyperbolic equations.

An evolution of the method of finite differences has been the development of generalized finite difference (GFDM) that can be applied to irregular grids or clouds of points. Lizska and Orkisz proposed a generalized finite difference method (GFDM) on irregular grids (T. Liszka,., 1984). However, these GFD formulations were later essentially improved and extended by many other authors and the most advanced version was given by J. Orkisz, including: mesh generation, local approximation, generation of finite difference (FD) formulae and FD equations resulting from local(collocation) or global formulations (J. Orkisz,1998) . Benito, Ureña and Gavete have made interesting contributions to the development of this method (Aklilu T.G.Giorgis, 2011).

The finite difference method is an interpretation of the differential equation into discrete domain so that it can be solved using a numerical method. It is a representation of the governing equation (S.S.Sastery, 2006). Using the discontinuous but connected regions, the governing equation is defined within the interval. In addition to direct interpretation, the deferential equation, the basic finite difference form, also can be derived from the Taylor-series expansion. Here we are motivated to focus on finite difference schemes for solving 1-D and 2-D wave equations and to give some generalizations on the investigations:

- Effects of reducing the time step sizes Δt on stability, convergence and consistency in 1D wave equation with fixed space step sizeΔx .

- Effects of reducing the space step sizes Δx and Δy on stability, convergence and consistency in 2D wave equations with fixed time step size Δt to answer the following questions.

1. What are the effects of reducing mesh size Δt with fixed space step size Δx on 1-D wave equation?

2. What are the effects of reducing mesh sizes Δx and Δy with fixed time step size Δt on 2D wave equations?

3. Which FDM (explicit or implicit) is more consistent, stable and convergent?

4. How to make error analysis?

1.3 Objective of the study

1.3.1 General objective of the study

To investigate the effect of mesh sizes Δx , Δy and Δt for solving 1-D and 2-D wave equations using FDMs.

1.3.2 Specific objectives of the study

To this end the study is intended to explore the following specific objectives:.

- To analyse convergence, consistency and stability of explicit and implicit schemes.
- To analyse errors of the explicit and implicit schemes.
- To compare performances of explicit and implicit schemes

1.4 Significance of the study

- This particular study may serve as reference for any individual who conduct a thesis on the study of numerical solution methods for 1-D and 2-D wave equations.
- It may help for individuals to solve problems involving PDEs arises in physical propagation of disturbance associated with wave equations.

1.5 Delimitation (scope) of the study

The research study is mainly delimilted on numerical solutions of hyperbolic one and two dimensional second ordered wave equation for finite difference scheme (FDM).

By doing so the main focus was on comparison of the solution obtained with finite difference method and analytic method of one dimensional second ordered wave equation problem using the explicit and implicit method of finite difference and also the researcher consider numerical solution of two dimensional wave equations using explicit method.

1.6 Hypothesis

The expected results are:

- Which scheme is effective explicit or implicit to get a numerical solution for problems of wave equations?
- Which scheme is better convergent explicit or implicit?
- Which mesh size affects more the stability and convergence?

CHAPTER TWO

2. Review of related literature

Many physical processes in nature, whose correct understanding, prediction, and control are important to people, are described by equations that involve physical quantities together with their spatial and temporal rates of change (partial derivatives). Among such processes are the flow of liquids, deformation of solid bodies, heat transfer, propagation of wave in different media, and many others. Equations involving partial derivatives are called partial differential equations (Wakefield.M.A, 2003).These equations involve two or more independent variables that determine the behavior of the dependent variable, usually of second or higher order.

As it was mentioned above the solution of PDEs involve arbitrary functions. That is, in order to solve the system in question completely, additional conditions are needed. These conditions can be given in the form of initial and boundary conditions. Initial conditions define the values of the dependent variables at the initial stage (e.g. at $t = 0$), whereas the boundary conditions give the information about the value of the dependent variable or its derivative on the boundary of the area of interest. Typically, one distinguishes:

Dirichlet conditions: -specify the values of the dependent variables of the boundary points.
Neumann conditions: -specify the values of the normal gradients of the boundary.
Robin conditions: -defines a linear combination of the Dirichlet and Neumann conditions.

Moreover, it is useful to classify the PDE in terms of initial value problem (IVP) and boundary value problem (BVP).

Initial value problem: PDE in question describes time evolution, i.e., the initial space-distribution is given; the goal is to find how the dependent variable propagates in time.

Boundary value problem: A static solution of the problem should be found in some region-and the dependent variable is specified on its boundary.

Different types of partial differential equation require different combination of initial and boundary conditions, and this in turn means a different numerical procedure (Mitchell.A.R, 2007).In this thesis we start with a brief discussion of finite difference formulae used to numerically approximate second order derivatives of PDEs. Practical overview of numerical solution to one and two dimensional wave equations using finite difference schemes is presented. And also some of the most basic finite difference schemes for one dimensional wave equation will be established and analyzed.

The explicit and implicit schemes are discussed and applied to a simple problem involving the one and two dimensional wave equations. There are different numerical methods for solving partial differential equations such as finite difference method, the finite element method, finite volume method, spectral method, mesh free method, domain decomposition method, and multi-grid method. The finite difference method is a numerical solution of PDEs introduced by Euler in the 18th century. The starting point is the conservation equation in differential form .The solution domain is covered by grid. At each grid point ,the differential is approximated by replacing the partial derivatives by approximation in terms of the nodal values of the functions .The result is one algebraic equation per grid node ,in which the variable at that and a certain number of neighbor nodes appear as unknowns.

In principle, the FD method can be applied to any grid type .However in all applications of the FD method known; it has been applied to structured grids. Taylor series Expansion or polynomial fitting is used to obtain approximations to the first and second derivatives of the variables with respect to the coordinates. When necessary, these methods are also used to obtain values at locations other than grid (nodes) interpolation.

Finite difference method is to replace continuous derivative with so-called difference formulas that involve only the discrete values associated with positions on the mesh. These methods are widely dominant in the numerical solution of differential equations and their application. It is based on the Taylor series expansion or polynomial approximation and proceeds difference approximation to obtain the numerical solution of various types of partial differential equations such as elliptic, parabolic and hyperbolic; it has been extensively used in several engineering fields, such as fluid mechanics, heat and so on (Wakefield.M.A, 2003). It is widely believed that the seminal work of Courant, Friedrichs, and Lewy in 1928 is the first published example of five-point difference approximations to derivatives for solving the elliptic Laplace equation. Using the same method, the same researcher also proceeded to solve the hyperbolic wave equation in the same paper and established the so-called CFL criterion for the stability of hyperbolic partial differential equations (sandip, 2016).

In connection to this Shuonan Dong did a research on finite difference method for wave equations. He compared the general performance of each FD schemes for 1-D and 2-D wave equations. ON his study he concludes that the overall performance of the explicit difference method is quite good for stable values of r and small Δx , Δy and Δt with second order accuracy. The stability could be improved by going to an implicit difference method, but often implicit schemes tend to be too complicated to use in a simple manner. But this thesis specifically concentrates on 1-D and 2-D second ordered partial differential wave equations using FDMs.
We investigate the effect of reducing space sizes Δx and Δy in both 1-D and 2-D with fixed time step Δt and to make conclusions on the performance of each FD schemes.

CHAPTER THREE

3. Methodology of the study

In this chapter the researcher tries to consider the study design, study sight, sampling, instrumentation, data collection and data analysis.

3.1 Study design

Documentary and experimental design. .

3.2 procedures of the study

To obtain numerical solution of PDEs using finite difference method basically we did the following procedures:

1 .Dividing the domain into mesh points (Discretization).

2. Replacing partial differential equation by finite difference approximations that relates the solutions to grid points to create linear systems

3 Solving system of linear equations subject to the prescribed boundary conditions and/or initial conditions using mat lab codes.

4. Presenting the results with graphs and tables.

3.3 Significance of the study

- This particular study may serve as reference for any individual who conduct a thesis on the study of numerical solution methods for 1-D and 2-D wave equations.
- It may help for individuals to solve complicated problems involving PDES arises in physical propagation of disturbance associated with wave equations.

CHAPTER FOUR

4.1 Derivation of 1-D wave equation from forces of vibrating string

To obtain the derivation of the PDE modeling small transverse vibrations of an elastic string, we have to consider the following physical assumptions: (Erwin Kreyszig, 2006).

1. The mass of the string per unit length is constant and does not offer any resistance to bending

2. Gravitational force on the string (trying to pull the string down a little) can be neglected.

3. The string performs small transverse up-down motion (i.e. only in y-direction).

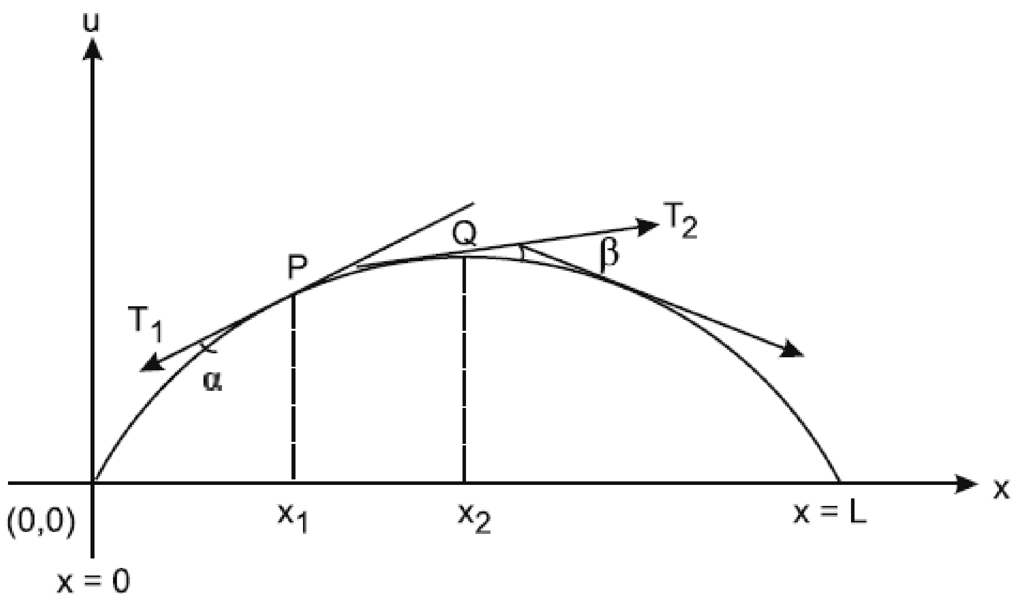


Fig (1) a homogeneous string of length L, tied at both ends.

Consider the forces acting on a small portion of the string as shown above on the Fig (1).Since the string offers no resistance to bending, the tension is tangential to the curve of the string at each point. Let T_1 and T_2 be the tension at the endpoints *P* and *Q* of that portion. Since the points of the string move vertically, there is no motion in the horizontal direction.

Horizontally: $T_2cos\beta - T_1\ cos\alpha = 0\ \Rightarrow T_2cos\beta = T_1\ cos\alpha = T(constant)$ (4.1.1)

In the vertical direction we have two forces, namely$-T_1\ sin\alpha\ and\ T_2sin\beta$. By Newton's second law the resultant of these two forces is equal to the *m*ass($\rho\Delta x$) of the portion times the acceleration $\left(\frac{\partial^2 u}{\partial t^2}\right)$ evaluated at some point between x and $x + \Delta x$; here ρ is the mass of the un deflected string per unit length, and Δx is the length of the portion of the un deflected string.

Vertically: $T_2 sin\beta - T_1\ sin\alpha = (\rho\Delta x)\left(\frac{\partial^2 u}{\partial t^2}\right)$ (4.1.2)

Dividing equation (4.1.2) by equation (4.1.1) we obtain:

$$\frac{T_2 sin\beta}{T_2 sin\beta} - \frac{T_1\ sin\alpha}{T_1\ sin\alpha} = tan\beta - tan\alpha = \left(\frac{\rho\Delta x}{T}\right)\left(\frac{\partial^2 u}{\partial t^2}\right) \tag{4.1.3}$$

Now $tan\alpha$ and $tan\beta$ are the slops of the string at x and $x + \Delta x$;

$$tan\alpha = \left(\frac{\partial u}{\partial x}\right)\Big|_x \ and \quad tan\beta = \left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} \tag{4.1.4}$$

Using the slops in equation (4.1.4) and dividing equation (4.1.3) by Δx we thus obtain an equation:

$$\frac{1}{\Delta x}\left[\left(\frac{\partial u}{\partial x}\right)\Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)\Big|_x\right] = \left(\frac{\rho}{T}\right)\left(\frac{\partial^2 u}{\partial t^2}\right) \tag{4.1.5}$$

If we let Δx approach to zero, we obtain the linear PDE of the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{, where the physical constant } \frac{T}{\rho} = c^2 \tag{4.1.6}$$

Hence this PDE is called one-dimensional wave equation.

4.2 Derivation of 2-D wave equation from forces of vibrating membrane

The derivation of the PDE for 2-D wave equation is similar to that of 1-D vibration problem of the PDE derived in above section. Her physical assumptions are:

1. The mass of the membrane per unit area is constant; the membrane is perfectly flexible and offers no resistance to bending.

2. The membrane is stretched and then fixed along its entire boundary in the xy plane.

3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Now consider the Vibrating membrane that is stretched and then fixed along its edge on xy-plane given on the figure below.

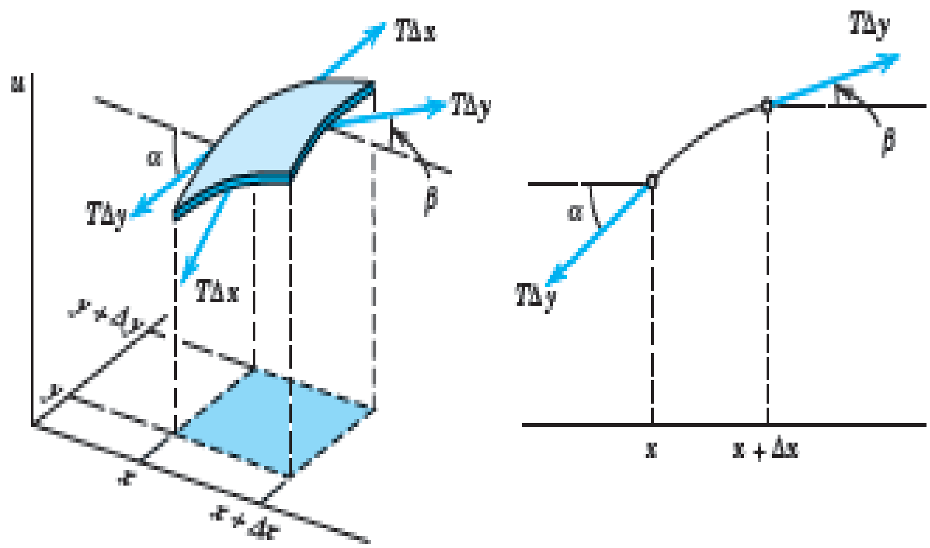


Fig.(2) Vibrating membrane

The horizontal components of the forces at opposite sides are approximately equal and negligibly small. Therefore, the motion of the particles of the membrane as transversal; that is, each particle moves vertically and vertical components of the forces along the right side and the left side are $T\Delta y \sin\beta$ and $-T\Delta y \sin\alpha$, respectively.

Since the angles are small, we may replace their shines by their tangents. Hence the resultant of those two vertical components is:

$$T\Delta y(\sin\beta - \sin\alpha) \approx T\Delta y(\tan\beta - \tan\alpha)$$

$$= T\Delta y[u_x(x+\Delta x, y_1) - u_x(x, y_2)] \tag{4.2.1}$$

Where subscripts x denote partial derivatives y_1 and y_2 are values between y and $y + \Delta y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the portion is:

$$T\Delta x\left[u_y(x_1, y+\Delta y) - u_y(x_2, y)\right] \tag{4.2.2}$$

Where x_1 and x_2 are values between x and $x + \Delta x$.

By Newton's second law the sum of the forces given by (4.2.1) and (4.2.2) is equal to the mass of that small portion times the acceleration here is the mass $(\rho\Delta A)$ of that small portion times the acceleration$\left(\frac{\partial^2 u}{\partial t^2}\right)$, here ρ is the mass of the undeflected membrane per unit area and, $\Delta A = \Delta x\Delta y$ is the area of that portion when there is no deflection (Erwin Kreyszig, 2006).

$$Thus\ \ \rho\Delta x\Delta y\frac{\partial^2 u}{\partial\, t^2} = T\Delta y[u_x\ (x+\Delta x, y_1) - u_x\ (\ x\ , y_2\ )] + T\Delta x\big[u_y(\ x_1, y+\Delta y) - u_y\ (x_2, y\ )\big]$$

Where the derivative on the left is evaluated at some suitable point $(\tilde{x}, \tilde{y}\ )$ corresponding to that portion. Dividing the above equation by $\rho\Delta x\Delta y$ and we obtain:

$$\frac{\partial^2 u}{\partial\, t^2} = \frac{T}{\rho}\left[\frac{u_x\ (x+\Delta x, y_1) - u_x\ (\ x\ , y_2\ )}{\Delta x} - \frac{u_y(\ x_1, y+\Delta y) - u_y\ (x_2, y\ )}{\Delta y}\right] \tag{4.2.3}$$

If we let Δx and Δy approach zero in equation (4.2.3), then we obtain the PDE model:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \text{Where } \frac{T}{\rho} = c^2 \tag{4.2.4}$$

Hence this PDE is called the two-dimensional wave equation.

4.3 Finite difference method

The finite difference method (FDM) is one of the ways for solving partial differential equations. The PDE's are converted by transforming the continuous domain of the state variables to a network of discrete points into a set of finite difference equations that can be solved subject to the appropriate boundary conditions. Applying the finite difference method to a differential equation involves replacing all derivatives with difference formulas. In the wave equation there are derivatives with respect to time (t) and derivatives with respect to space (x)

4.4 Solution of 1-D wave equation

The wave equation is a second order linear hyperbolic PDE that describes the propagation of a variety of waves, such as oscillating strings, sound waves, water waves, light waves, wave in a pond and so on. It arises in different fields such as acoustics, electromagnetic, or fluid dynamics. Suppose that the function $u(x,t)$ gives the height of the wave at position x and time t as illustrated in fig (3) below. Then u satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\ \frac{\partial^2 u}{\partial x^2}\quad ,$$ where c is the speed that the wave propagates.

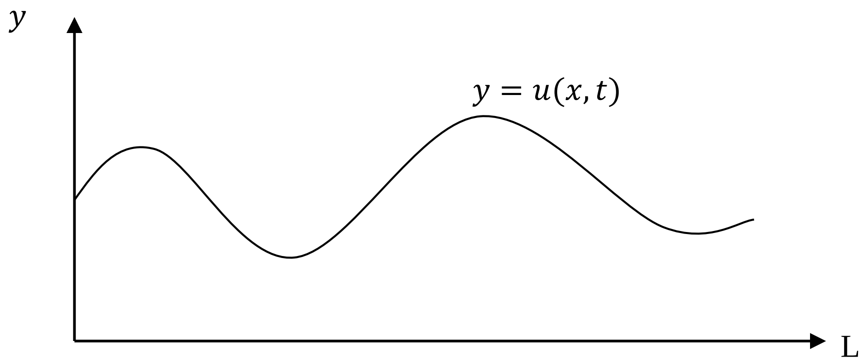


Fig. (3) Wave propagation of string

We begin our study of wave equations by simulating one-dimensional waves on an oscillating string; say on a guitar or violin string. Let the string in the deformed state coincide with the interval $[o, L]$ on the x axis, and let $u(x,t)$ be the displacement at time t in the y direction of a point initially at x. The displacement function u is governed by the following mathematical model:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \epsilon\ (0, L],\ \ t\ \epsilon\ (0, T\ ] and\ \ c > 0 \tag{4.4.1}$$

$$u(x,0) = f(x),\ \ u_t(x,0) = g(x),\ \ \ x \in [0, L] \tag{4.4.2}$$

$$u(0,t) = 0,\ \ u(1,t) = 0\ \ ,\ \ t\ \epsilon\ (0\ ,T] \tag{4.4.3}$$

The constant c and the function $f(x) = u(x,t)$ must be prescribed. Equation (4.4.1) is known as the one-dimensional wave equation. Since this PDE contains a second-order derivative in time, we need two initial conditions, here (4.4.2) specifying the initial shape of the string $f(x)$ and reflecting that the initial velocity of the string is zero. In addition, PDEs need boundary conditions, here (4.4.3), specifying that the string is fixed at the ends, i.e., that the displacement u is zero. The solution $u(x,t)$ varies in space and time and describes waves that are moving with velocity c to the left and right. Sometimes the wave equation can be written compactly as: $u_{tt} = c^2\ u_{xx}\ \ \ x\epsilon\ (0, L],\ \ t\ \epsilon\ (0, T\ ] and\ \ c > 0$, with initial and boundary problems.

4.5 Discretizing the domain

The purpose of discretization is to obtain a problem that can be solved by a finite difference procedure. The temporal domain [0, T] is represented by a finite number of mesh points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T \tag{4.5.1}$$

Similarly, the spatial domain [0, L] is replaced by a set of mesh points

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = L \tag{4.5.2}$$

One may view the mesh as two-dimensional in the x, t plane, consisting of points

$(x_i\ ,\ t_j)$, with $i = 0, \dots . N_x$, and $j = 0, \dots , N_t$

For uniformly distributed mesh points we can introduce the constant mesh spacing Δt and Δx. We have that:

$$x_i = i\Delta x\ , i = 0, \dots , N_x \qquad t_j = j\Delta t\ \ , j = 0, \dots , N_t \tag{4.5.3}$$

We also have that $\Delta x = x_i - x_{i-1}\ , i = 1, \dots , N_x$ and $\Delta t = t_j - t_{j-1}\ , j = 1, \dots , N_t$

Figure (4) displays a mesh in the x, t plane with $N_t = 5\ , N_x = 5$ and constant mesh spacing.

4.6 The discrete solution

The solution $u(x, t)$ is sought at the mesh points. We introduce the mesh function $u_{i,j}$, which approximates the exact solution at the mesh point (x_i, t_j), for $i = 0, \dots \dots , N_x$ and $j = 0, \dots \dots , N_t$. using the finite difference method, we shall develop algebraic equations for computing the mesh function. The black squares in figure (4) illustrate neighboring mesh points where values of $u_{i,j}$ are connected through an algebraic equation. In this particular case, $u_{i,j-i}$, $u_{i-1,j}\ ,\ u_{i,j}\ ,\ u_{i+1,j}\ and\ \ u_{i,j+1}$ are connected in an algebraic equation associated with the center point $u_{i,j}$. The term stencil is often used about the algebraic equation at a mesh point, and the geometry of a typical stencil is illustrated in figure (4) given below. One also often refers to the algebraic equations as discrete equations or a finite difference scheme.

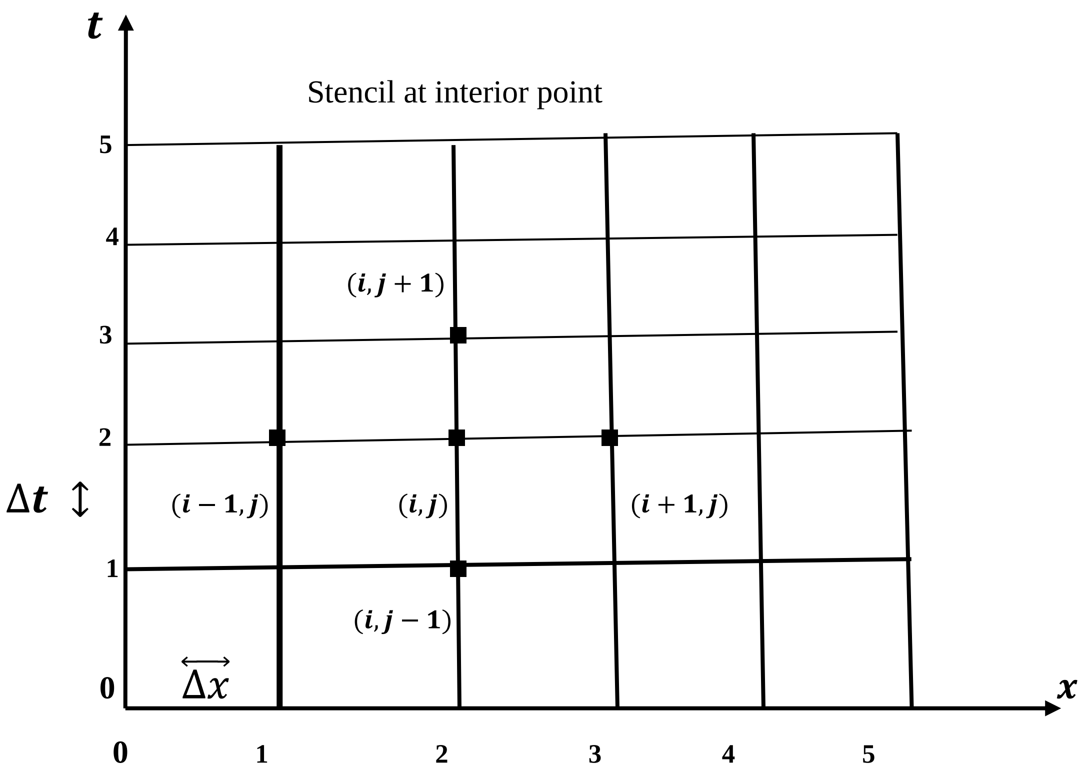


Fig. (4) Mesh in space and time for a 1-D wave equation

4.7 Wave equation at the mesh points

For a numerical solution by the finite difference method, we relax the condition that (4.4.1) at all points in the space-time domain $(0, L) \times (0, T]$ to the requirement that the PDE is fulfilled at the interior mesh points.

$$\frac{\partial^2}{\partial t^2} u(x_i, t_j) = c^2 \frac{\partial^2}{\partial x^2} u(x_i, t_j) \tag{4.7.1}$$

For $i = 1, \ldots\ldots, N_{x-1}$ and $j = 1, \ldots\ldots, N_{t-1}$ for $j = 0$ we have the initial conditions $u = f(x)$ and $u_t = 0$ and at the boundaries $i = 0, N_x$ we have the boundary condition $u = 0$.

4.8 Finite difference approximations for 1-D and 2-D wave equations

Here our interest is to approximate solutions to differential equations, i.e., to find a function (or some discrete approximation to the function) which satisfies a given relationship between various of its derivatives on some given region of space and/or time, along with some boundary conditions along the edges of this domain. In general this is a difficult problem and only rarely can an analytic formula be found for the solution. A finite difference method proceeds by replacing the derivatives in the deferential equations by finite difference approximations. This gives a large algebraic system of equations to be solved in place of the deferential equation, something that is easily solved on a computer.

4.9 Taylor series expansions for 1-D and 2-D wave equations

4.9.1 First order forwarded, back ward and central difference approximations

Consider Taylor series expansion of analytical functions of $u(x)$.

$$u(x+\Delta x) = u(x) + \sum_{n=1}^{\infty} \frac{\Delta x^n}{n!}\frac{\partial^n u}{\partial x^n} = u(x) + (\Delta x)\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots \quad (4.9.1.1)$$

$$u(x-\Delta x) = u(x) \mp \sum_{n=1}^{\infty} \frac{\Delta x^n}{n!}\frac{\partial^n u}{\partial x^n} = u(x) - (\Delta x)\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots \quad (4.9.1.2)$$

Then solving the first derivative $\left(\frac{\partial u}{\partial x}\right)$ from equation (4.9.1.1) one obtains:

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x)-u(x)}{\Delta x} - \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} - \cdots \quad (4.9.1.3)$$

If we neglect the right hand side of (4.9.1.3) after the first term, as truncation error then the last equation becomes:

$$\frac{\partial u}{\partial x} = \frac{\mathrm{u(x+\Delta x)-u(x)}}{\Delta \mathrm{x}} + O(\Delta x) \implies \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) \quad (4.9.1.4)$$

Equation (4.9.1.4) is called **first order forward difference**.

Similarly solving the first derivative $\left(\frac{\partial u}{\partial x}\right)$ from equation (4.9.1.2) one obtains:

$$\frac{\partial u}{\partial x} = \frac{u(x)-u(x-\Delta x)}{\Delta x} + \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots \quad (4.9.1.5)$$

If we neglect the right hand side of (4.7.1.5) after the first term, as truncation error then the last equation becomes:

$$\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = \frac{\mathrm{u(x)-u(x-\Delta x)}}{\Delta \mathrm{x}} + \mathrm{O}(\Delta \mathrm{x}) \implies \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \quad (4.9.1.6)$$

This equation (4.9.1.6) is called **first order backward difference**. One can see that both forward and backward differences are of the order $O(\Delta x)$. We can combine these two first order forwarded and backward difference approaches to derive a central difference approach, which yields a more accurate approximation. If we subtract equation (4.9.1.2) from equation (4.9.1.1) we obtain:

$$u(x+\Delta x)-\ u(x-\Delta x)=(2\Delta x)\frac{\partial u}{\partial x}+\frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (4.9.1.7)$$

If we neglect the right hand side of (4.9.1.7) and solving for $\left(\frac{\partial u}{\partial x}\right)$ then the last equation becomes:

$$\frac{\partial u}{\partial x}=\frac{\boldsymbol{u}(\boldsymbol{x}+\Delta\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x}-\Delta\boldsymbol{x})}{2\Delta x}+O(\Delta x)^2 \implies \left(\frac{\partial u}{\partial x}\right)_{i,j}=\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+O(\Delta x)^2 \qquad (4.9.1.8)$$

This equation (4.9.1.8) is called **first order central difference** is of the order of $O(\Delta x^2)$.

4.9.2 Second order forward, backward and central difference approximations

Finite difference approximations to higher order derivatives can be obtained with the additional manipulations of the Taylor series expansion about $u(x)$. The second order derivatives can be found in the same way using the linear combination of different Taylor series expansions.

For instance, consider the following two expansions

$$u(x+2\Delta x)=u(x)+(2\Delta x)\frac{\partial u}{\partial x}+\frac{(2\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}+\frac{(2\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (4.9.2.1)$$

$$u(x-2\Delta x)=u(x)-(2\Delta x)\frac{\partial u}{\partial x}+\frac{(2\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}-\frac{(2\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (4.9.2.2)$$

Subtracting two times of eq. (4.9.1.1) from eq. (4.9.2.1), one gets the following equation

$$u(x+2\Delta x)-2u(x+\Delta x)=-u(x)+(\Delta x)^2\frac{\partial^2 u}{\partial x^2}+(\Delta x)^3\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (4.9.2.3)$$

If we neglect the right hand side of (4.9.2.3) and solving for $\left(\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}\right)$, then one can approximate the second derivative in terms of **forward difference** as follows:

$$\frac{\partial^2 u}{\partial x^2}=\frac{u(x+2\Delta x)-2u(x+\Delta x)+u(x)}{(\Delta x)^2} \qquad (4.9.2.4)$$

Similarly subtracting 2 times of eq. (4.9.1.2) from eq. (4.9.2.2), one can obtain the equation

$$\mathrm{u}(x-2\Delta x)-2u(x-\Delta x)=-u(x)+(\Delta x)^2\frac{\partial^2 u}{\partial x^2}-(\Delta x)^3\frac{\partial^3 u}{\partial x^3}+\cdots \qquad (4.9.2.5)$$

If we neglect the right hand side of (4.9.2.5) and solving for $\left(\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}\right)$, then one can approximate the second derivative in terms of **backward difference** as follows:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-2\Delta x)-2u(x-\Delta x)+u(x)}{(\Delta x)^2} + o(\Delta x) \qquad (4.9.2.6)$$

Finally, if we add eqn. (4.9.1.1) and (4.9.1.2) we obtain an expression for the **central difference for second derivative** as follows:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} = \frac{\boldsymbol{u}(\boldsymbol{x}+\Delta\boldsymbol{x})-\mathbf{2}\boldsymbol{u}(\boldsymbol{x})+\boldsymbol{u}(\boldsymbol{x}-\Delta\boldsymbol{x})}{(\Delta x)^2} + o(x^2) \Rightarrow \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} - u_{i-1,j}}{(\Delta x)^2} + O(x)^2 \qquad (4.9.2.7)$$

One can see that approximation (4.9.2.7) provides more accurate approximation as (4.9.2.5) and (4.9.2.6) in an analogous way one can obtain finite difference approximations to higher order derivatives and differential operators.

Recall that the second derivative of a function can be approximated by the central difference:

$$\mathrm{f}''(\mathrm{x}) \approx \frac{\mathrm{f}(\mathrm{x}+\Delta\mathrm{x})-2\mathrm{f}(\mathrm{x})+\mathrm{f}(\mathrm{x}-\Delta\mathrm{x})}{(\Delta\mathrm{x})^2} \qquad (4.9.2.8)$$

The same technique can be used to approximate the second derivatives of a function $u(x,t)$ as follows:

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} \approx \frac{\mathrm{u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)}}{(\Delta \mathrm{t})^2} \qquad (4.9.2.9)$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\mathrm{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}}{(\Delta \mathrm{x})^2} \qquad (4.9.2.10)$$

Using these approximations, an approximation for 1-D wave equation can be found replacing the second derivatives in to equation (4.2.1) by this approximation.

$$\frac{\mathrm{u(x,t+\Delta t)-2u(x,t)+u(x,t-\Delta t)}}{(\Delta \mathrm{t})^2} = c^2 \frac{\mathrm{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}}{(\Delta \mathrm{x})^2} \qquad (4.9.2.11)$$

Note that one term is at time $t + \Delta t$,which we will refer to a future time. The rest involve time t (current time) or $t - \Delta t$ (past time). If we solve for this future time term we get;

$$\mathrm{u(x, t + \Delta t)} = \alpha^2(\mathrm{u(x + \Delta x, t) + u(x - \Delta x\,, t)}) + 2(1 - \alpha^2)u(x,t) - u(x, t - \Delta t) \qquad (4.9.2.12)$$

With, $\alpha = \frac{c\Delta t}{\Delta x}$. It turns out that as long as Δt is chosen small enough so that $\alpha < \frac{1}{2}$ the simulation will be stable. Otherwise, the waves will continue to grow larger and larger.

4.10 Explicit difference method for 1-D wave equation

Consider the wave equation in one-dimensional space given below

$u_{tt} \;\; = c^2\, u_{xx} \;\; , x \in (0\,, L], \;\; t\epsilon(0, T]$ (4.10.1)

Where $c\ is\ a\ \ positive\ constant\ and\ u(x,t)\ subject\ to\ the\ intial\ conditions$:

$u(x, 0) = f(x),\;\; u_t(x, 0) = g(x), \quad x \in [0, L]$ (4.10.2)

and the boundary conditions:

$u(0, t) = 0,\;\; u(L, t) = 0\;\; ,\;\; \mathrm{t} \in (0\,, \mathrm{T}]$ (4.10.3)

The analytical solution of equation (4.10.1) as given by d'Alembert formula as found in [Hildebrand - 1968] is

$u(x, t) = \frac{1}{2}\, (f(x - ct) + f(x + ct) + \;\frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$ (4.10.4)

For the purposes of demonstrating the effectiveness of the finite difference methods, I will assume

$u_t(x, 0) = g(x) = 0$ (4.10.5)

Now consider a rectangular mesh in the plane with spacing h along x- direction with $h = \Delta x$ and k along time t- direction with $k = \Delta t$ in the fig (5) given below**:**

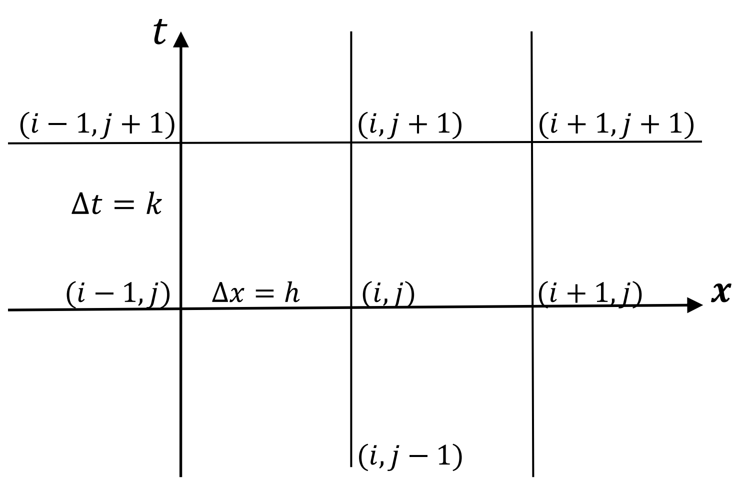


Fig. (5) Explicit FDM for 1D wave equation

Denoting a mesh point $u(x,t) = (ih, jk)$ $where\ x = i\text{h}$, $i = 1,2,3 \ldots\ldots$ $and\ t = jk$, $j = 1,2,3 \ldots.$ and substituting the PDES of equation (4.10.1) by central finite difference approximation, we have

$$u_{tt} = \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta t)^2}\right) = \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right) \text{ And}$$

$$u_{xx} = \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}\right) = \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\text{h}^2}\right)$$ so equation (4.10.1) becomes:

$$\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}\right) = c^2 \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\text{h}^2}\right) \tag{4.10.6}$$

When we solve for $u_{i,j+1}$ at a time step $j+1$, then equation (4.10.6) yields:

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \frac{k^2 c^2}{\text{h}^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \tag{4.10.7}$$

Using $u_{tt} = c^2 u_{xx}$ and $\alpha = \frac{k^2 c^2}{\text{h}^2}$ we solve for $u_{i,j+1}$ at a time step $j+1$ and rearranging the terms, we obtain an equation :

$$u_{i,j+1} = 2(1-\alpha)u_{i,j} + \alpha\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \tag{4.10.8}$$

Again if the value of $\alpha = \frac{k^2 c^2}{\text{h}^2} = 1$, then equation (4.10.8) can be simplified as the forom :

$$u_{i,j+1} = \left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \tag{4.10.9}$$

$$u_{i,j+1} = \underbrace{-u_{i,j-1}}_{solution\ at\ t_{j-1}} + \underbrace{2(1-\alpha)u_{i,j} + \alpha\left(u_{i-1,j} + u_{i+1,j}\right)}_{solution\ at\ t_j} \tag{4.10.10}$$

4.11 Implicit difference method for 1-D wave equation

Consider the wave equation in one-dimensional space given below

$$u_{tt} = c^2 u_{xx}\ , x \in (0, L],\ \ t \epsilon (0,T] \tag{4.11.1}$$

Where $c\ is\ a\ positive\ constant\ and\ u(x,t)\ subject\ to\ the\ intial\ conditions$:

$$u(x,0) = f(x),\ \ u_t(x,0) = g(x),\ \ \ x \in [0,L] \tag{4.11.2}$$

and the boundary conditions:

$u(0,t) = 0,\ \ u(L,t) = 0\ ,\ t \in (0\,,\mathrm{T}]$ (4.11.3)

Substituting the PDEs of equation (4.11.1) by central finite difference approximation, we obtain the equation

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2k^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \tag{4.11.4}$$

one can try to overcome the problems with conditional stability by introducing an implicit scheme. The simplest way to do it is just to replace all terms on the right hand side of (4.11.4) by an average from the values to the time steps j+1 and $j-1$, we obtain an equation :

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2k^2}{2h^2}\left((u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}) + (u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1})\right) \tag{4.11.5}$$

This equation (4.11.5) is called implicit difference method for one dimensional wave equation.

Assume that boundary condition $u(0,t) = 0, u(L,t) = 0$ and the intial codition $\mathrm{u_t}(x,0) = 0$

Is replacing by central finite difference approximation, that is

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0 \ \Rightarrow u_{i,j+1} = u_{i,j-1} \tag{4.11.6a}$$

$$\Rightarrow u_{i+1,j+1} = u_{i+1,j-1} \tag{4.11.6b}$$

$$\Rightarrow u_{i-1,j+1} = u_{i-1,j-1} \tag{4.11.6c}$$

When solving $u_{i,j}$ from equation (4.11.5) by putting $\alpha = \frac{k^2c^2}{\mathrm{h}^2}$ and using the relations equation (4.11.6a)−(4.11.6c) after rearranging the terms, we obtain an equation of the form:

$$-0.5\ \alpha u_{i-1,j+1} + (\alpha+1)u_{i,j+1} - 0.5\ \alpha u_{i+1,j+1} = u_{i,j} \tag{4.11.7}$$

Furthermore we can rewrite equation (4.11.7) in matrix vector form as:

$$\begin{bmatrix} (\alpha+1) & -0.5\alpha & 0 & 0 \\ -0.5\alpha & (\alpha+1) & -0.5\alpha & 0 \\ 0 & -0.5\alpha & (\alpha+1) & -0.5\alpha \\ 0 & 0 & -0.5\alpha & (\alpha+1) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix} \tag{4.11.8}$$

$for\ j = 0,1,2,\ldots..N\ \ and\ i = 1,2,\ldots N, for\ each\ entry.$

4.12 Explicit difference method for 2-D wave equation

The method employed for the solution of one dimensional wave equation (4.10.1) can be readily extended to the wave equation in a two dimensional form . Thus we consider the second order wave equation in 2-D given below:

$$u_{tt} = c^2(u_{xx} + u_{yy}), c > 0, t > 0 \; and \; (x, y) \in [0, a] \times [0, b] \tag{4.12.1}$$

$u(x, y, t)$ *is subjected to the intial conditions*:

$$u(x, y, 0) = f(x, y), \; u_t(x, y, 0) = g(x, y) \tag{4.12.2}$$

and the boundary conditions:

$$u(0, y, t) = 0, \; u(a, y, t) = 0, \; u(x, 0, t) = 0, and \; u(x, b, t) = 0 \tag{4.12.3}$$

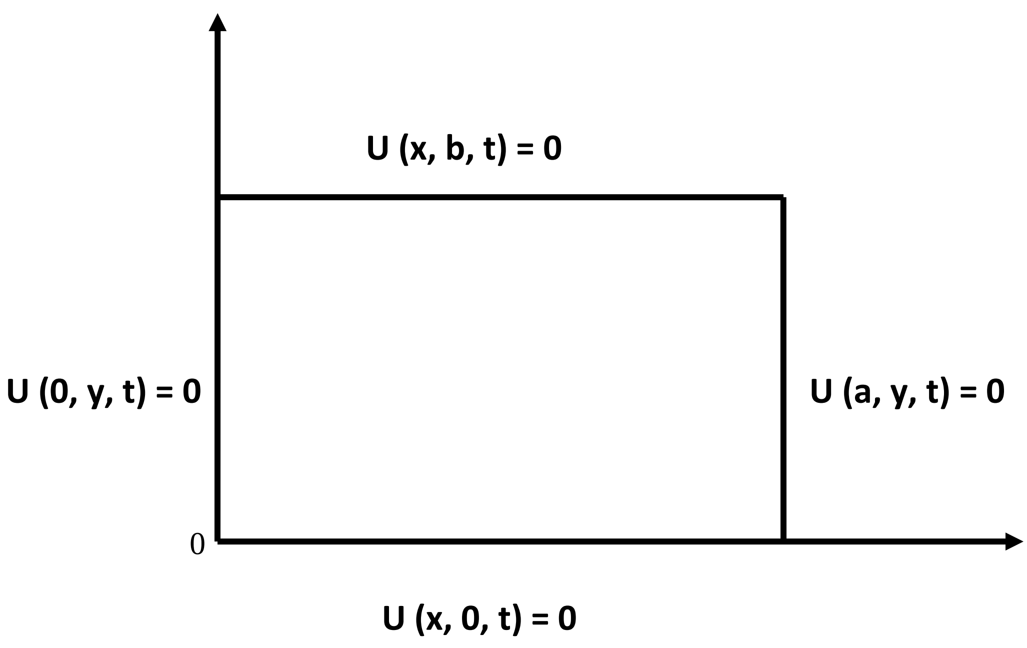


Fig. (6) Initial and boundary condition of 2-D wave equation.

Consider a square region $0 \leq x \leq a, 0 \leq y \leq b$ and assume that U is known at all points within and on the boundary of this square. If $\Delta x = h$, $\Delta y = k$ and $\Delta t = l$ be the step-sizes then a mesh point $(x, y, t) = (ih, jk, nl)$ may be denoted as simply (i, j, n) as shown in the fig. (7) below.

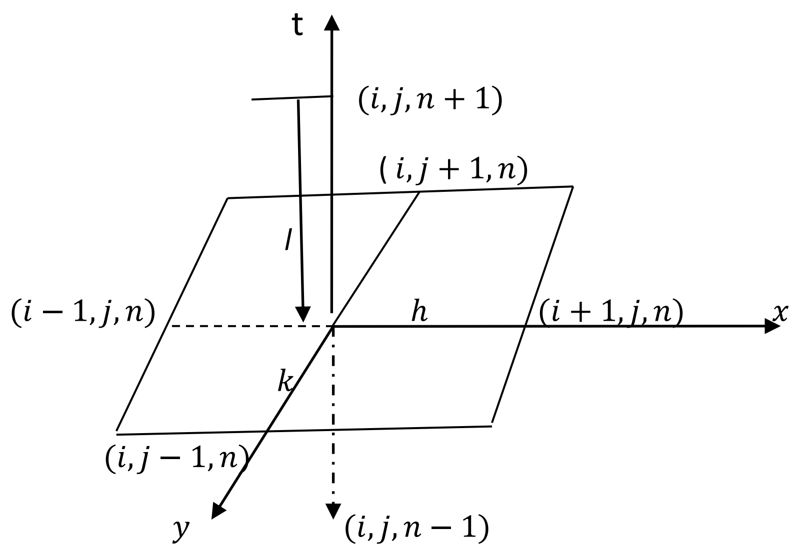


Fig.(7) explicit FDM for 2-D wave equation

The central difference approximations for the second derivatives with respect to t, x and y given by the equations:

$$u_{tt} = \left(\frac{u_{i,j,n-1} - 2u_{i,j,n} + u_{i,j,n+1}}{(\Delta t)^2}\right) = \left(\frac{u_{i,j,n-1} - 2u_{i,j,n} + u_{i,j,n+1}}{l^2}\right) \quad (4.12.4)$$

$$u_{xx} = \left(\frac{u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}}{(\Delta x)^2}\right) = \left(\frac{u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}}{h^2}\right) \quad (4.12.5)$$

$$u_{yy} = \left(\frac{u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}}{(\Delta y)^2}\right) = \left(\frac{u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}}{k^2}\right) \quad (4.12.6)$$

Replacing equation (4.12.4), (4.12.5) and (4.12.6) in to equation (4.12.1) and solving for $u_{i,j,n+1}$ we obtain the equation of the form:

$$u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} + \frac{c^2 l^2}{h^2}\left(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}\right) + \frac{c^2 l^2}{k^2}\left(u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}\right) \quad (4.12.7)$$

Putting $\beta = \frac{c^2 l^2}{h^2}, \gamma = \frac{c^2 l^2}{k^2}$ in to equation (4.12.7) and rearranging the terms yields an equation:

$$u_{i,j,n+1} = 2(1-\beta-\gamma)\, u_{i,j,n} - u_{i,j,n-1} + \beta\left(u_{i-1,j,n} + u_{i+1,j,n}\right) + \gamma\left(u_{i,j-1,n} + u_{i,j+1,n}\right) \qquad (4.12.8)$$

Equation (4.12.8) is called explicit finite difference method for 2-D wave equation and needs five points available, i.e. $[(i-1,j,n),(i+1,j,n),(i,j+1,n),(i,j-1,n), and (i,j,n)]$ on the three dimensional plane as indicated on fig. 4. The computational process consists of point by point evaluation in the (n+1) th plane using the points on the nth plane and this gives the system of equations of the form:

$Au = B$ and we solve it by using Thomas algorithm. To compute $\vec{U}^{n+1}$ we need to use the solution at $\vec{U}^{n}$ and $\vec{U}^{n-1}$, for the first time step $u_{i,j,1}$ needs also $u_{i,j,0}$ and $u_{i,j,-1}$. Again we have to use the initial conditions to find the ghost point at $u_{i,j,-1}$. Consider the initial condition of the right side of equation (4.12.2) with its central difference approximation that is:

$$u_t\,(x,y,0) = \frac{u_{i,j,1} - u_{i,j,-1}}{2\Delta t} = g(x,y) \Rightarrow u_{i,j,-1} = u_{i,j,1} - 2\Delta t g(x,y).$$

So the solution at the first time step n = 0 is given by the equation:

$$u_{i,j,1} = (1-\beta-\gamma)\, u_{i,j,0} + \Delta t g(x,y) + \frac{\beta}{2}\left(u_{i-1,j,0} + u_{i+1,j,0}\right) + \frac{\gamma}{2}\left(u_{i,j-1,0} + u_{i,j+1,0}\right) \quad (4.12.9)$$

4.13 Implicit difference method for 2-D wave equation

Consider the two dimensional wave equations given below:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (4.13.1)$$

By applying Crank-Nicolson scheme for 2-D wave equation, equation (4.13.1) can be written as form:

$$\left(\frac{u_{i,j,n+1} - 2u_{i,j,n} + u_{i,j,n-1}}{(\Delta t)^2}\right) = \frac{\boldsymbol{c^2}}{\boldsymbol{2}}\left({\delta_x}^2 + {\delta_y}^2\right)\left(u_{i,j,n+1} + u_{i,j,n}\right) \qquad (4.13.2)$$

$$\Rightarrow u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} + \frac{\boldsymbol{c^2(\Delta t)^2}}{\boldsymbol{2}}\left({\delta_x}^2 + {\delta_y}^2\right)\left(u_{i,j,n+1} + u_{i,j,n}\right) \qquad (4.13.3)$$

Where $\delta_x{}^2 = \left(\frac{u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}}{\Delta x^2}\right)$ and $\delta_y{}^2 = \left(\frac{u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n}}{\Delta y^2}\right)$

Equation (4.13.3) is said to be **implicit finite difference scheme for 2-D wave equation** and clearly this equation has 5 unknowns $\left(u_{i,j,n+1}\,, u_{i+1,j,n+1}\,, u_{i-1,j,n+1}\,, u_{i,j+1,n+1}\,, u_{i,j-1,n+1}\right)$ and the system of linear equations obtained from these unknowns is not tri-diagonal**.** Therefore even when we simplify this to a system of linear equations it is very complicated and time-consuming to solve. We thus, use Alternating Direction Implicit (ADI) Method to remove these difficulties.

4.14 Stability analysis, consistency, convergence of FDMs.

4.14.1 Stability analysis of explicit difference method for wave equation.

Consider the explicit central difference approximation of 1-D wave equation given below:

$$u_{n,k+1} = r^2\, u_{n+1,k} + 2(1-r^2)u_{n,k} + r^2 u_{n-1,k} - u_{n,k-1}\,,\quad r = \frac{c\Delta t}{\Delta x} \tag{4.14.1.1}$$

We assume a trial solution of the form: $u_{n,k} = \phi_k\, e^{in\Delta x\theta}$ (4.14.1.2)

Substituting in to equation (4.14.1.1) ,we obtain

$$e^{in\Delta x\theta}\phi_{k+1} = \left(r^2\, e^{i\Delta x\theta} + 2(1-r^2) + r^2\, e^{-i\Delta x\theta}\right) e^{in\Delta x\theta}\phi_k - \ e^{in\Delta x\theta}\phi_{k-1} \tag{4.14.1.3}$$

Canceling terms and using the double angle formulae i.e. $\cos(\,\Delta x\theta) - 1)) = -2sin^2\left(\frac{\theta\Delta x}{2}\right)$

$$\phi_{k+1} = 2(1 + r^2(\cos(\theta\,\Delta x) - 1))\phi_k - \phi_{k-1}$$

$$= 2\left(1 - 2r^2 sin^2\left(\frac{\theta\Delta x}{2}\right)\right)\phi_k - \phi_{k-1} \tag{4.14.1.4}$$

If now assume that ϕ_k has the exponential form $g^k = \phi_k$ then equation (4.14.1.4) reduced in the following quadratic equation:

$$g^2 - 2\beta g + 1 = 0, where\ \beta = \left(1 - 2r^2 sin^2\left(\frac{\theta\Delta x}{2}\right)\right) \tag{4.14.1.5}$$

The solution of this quadratic equation is given by:

$$g_{1,2} = \beta \pm \sqrt{(\beta^2 - 1)} \tag{4.14.1.6}$$

Now since g_1 and g_2 are the roots of this quadratic equation we may conclude that

$$(g - g_1)\,(\,g - g_2\,)\ = g^2 - (g_1 + g_2\,)\,g + g_1 g_2 = 0 = g^2 - 2\beta g + 1 \tag{4.14.1.7}$$

$$\Longrightarrow g_1 g_2 = 1 \tag{4.14.1.8}$$

However, for stability of solutions for the form $g^k = \phi_k$ we require that $|g_1| \leq 1$ and $|g_2| \leq 1$. Given the constraint (4.14.1.8), the only possibility, if the solutions are to be stable, is that $|g_1| = |g_2| = 1$. Thus g must fall on the unit disk, which implies that $|\beta| \leq 1$.

Thus $\left|1 - 2r^2 sin^2\left(\frac{\theta\Delta x}{2}\right)\right| \leq 1 \ or \ -1 \leq 1 - 2r^2 sin^2\left(\frac{\theta\Delta x}{2}\right) \leq 1$ (4.14.1.9)

So that $-2 \leq -2r^2 sin^2\left(\frac{\theta\Delta x}{2}\right) \leq 0$ (4.14.1.10)

The second inequality in (4.14.1.10) is satisfied automatically, while the first leads to the condition:

$$r^2 sin^2\left(\frac{\theta\Delta x}{2}\right) \leq 1 \qquad (4.14.1.11)$$

Since the maximum value that $sin^2\left(\frac{\Delta x\theta}{2}\right)$ can achieve is 1, we conclude that the condition for stability is given by:

$$r = \frac{c\Delta t}{\Delta x} \leq 1 \ \ or \ \ \Delta t = \frac{\Delta x}{c} \qquad (4.14.1.12)$$

The condition (4.14.1.12), which imposes an upper bound on the time step that can be used, is known as the Courant-Friedrichs-Lewy or CFL condition.

The stability of the explicit finite difference method for 2-D wave equation is dictated by the CFL condition $\beta, \gamma \leq \frac{1}{\sqrt{s}} \leq \frac{1}{\sqrt{2}} \cong 0.7071$, where $\beta = \gamma = \frac{c^2 l^2}{h^2}$ and $\Delta x = \Delta y$ where s is the space dimension of the wave equation.

4.14.2 Stability analysis of implicit difference method for wave equation.

In order to investigate the stability of the implicit difference method we have to consider equation (4.11.5) and the assumed trial solution of wave equation using the method applied in

(4.14.1). after rearranging the terms these equations leads to solution of the equation for

$$\beta g^2 - 2g + \beta = 0 \qquad \text{With } \beta = 1 + \alpha^2 \, sin^2\left(\frac{k\Delta x}{2}\right) \qquad (4.14.2.1)$$

one can see that $\beta \geq 1$ for all k .Hence the solutions $g_{1,2}$ take the form:

$$g_{1,2} = \frac{1 + i\sqrt{(1-\beta^2)}}{\beta} \ \text{ And } \ |g|^2 = \frac{1-(1-\beta^2)}{\beta^2} = 1 \qquad (4.14.2.2)$$

This shows that the implicit scheme (4.11.5) is absolute stable.

4.14.3 Consistency of finite difference method

Consistency depends on differencing Scheme**.** A FDE of a PDE is said to be consistent if we can show that the difference between the PDE and its FDE vanishes as the mesh (approaches zero)

$$\lim_{mesh\to 0}(PDE - FDE) = \lim_{mesh\to 0}(TE) = 0 \qquad (4.14.3.1)$$

For the schemes where truncation error (TE) like O(Δx) or O(Δt) (or its higher orders) is as mesh tends to zero, then TE would tend to zero. But schemes with TE like $O\left(\frac{\Delta t}{\Delta x}\right)$, we have searching further. In such cases FDE may be consistent only if as the mesh tends to zero, also $\left(\frac{\Delta t}{\Delta x}\right)$ tend to 0. However if Δt , Δx individually tend to without $\left(\frac{\Delta t}{\Delta x}\right)$ tending to zero then FDE becomes inconsistent.

4.14.4 Convergence of finite difference method

Lax Convergence theorem (J.W,THOMAS)

For a well-posed initial and boundary value problem, If a finite difference scheme is consistent with the partial differential equation, then the stability is the necessary and sufficient condition for convergence that is **:**

Consistency + stability ⟹ convergence. (4.14.4.1)

A solution of the algebraic equations that approximate a partial differential equation (PDE) is convergent if the approximate solution approaches the exact solution of the PDE for each value of the independent variable as the grid spacing tends to zero. The requirement is

$$U_{i,j} = U(x_i,t_j) \quad as\ \Delta x\ ,\Delta t \ \longrightarrow 0 \qquad (4.14.4.2)$$

Where LHS of equation (4.14.4.2) is the numerical solution from FDE and RHS is the exact solution of the equation of the PDE obtained analytically.

4.15 Numerical experiment for 1-D wave equation

Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ (4.15.1)

Subject to the boundary and intial conditions:

$u(0, t) = 0$,$(1, t) = 0$, for $t \geq 0$

$\frac{\partial u}{\partial t}(x, 0) = 0$ and $u(x, 0) = sin^3(\pi x)$ $, for$ $0 \leq \mathrm{x} \leq 1$,

Up to four time steps using explicit and implicit methods based on data given below:

Table 1 trial data for 1-D wave equation

Experiment	Space and time step sizes		Speed of wave (C)	$r = \frac{c\,\Delta t}{\Delta x}$	$\alpha = r^2$
	$h = \Delta x$	$k = \Delta t$			
Trial 1	0.20	0.1	1	0.5	2.50×10^{-1}
Trial 2	0.20	0.05	1	0.25	6.25×10^{-2}
Trial 3	0.20	0.025	1	0.125	1.56×10^{-2}
Trial 4	0.20	0.0125	1	0.0025	3.90×10^{-3}

4.15.1 Numerical solution of 1-D wave equation using explicit method

This problem can be solved exactly using d'Alambert method given by equation

$$u(x, t) = \frac{1}{2}\,(f(x - ct) + f(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \quad (4.15.1.1)$$

In the case of zero initial velocity this problem admits an exact solution which is given by:

$$u\,(x, t) = \frac{3}{4}\sin(\pi x)\cos(\pi t) - \frac{1}{4}\sin(3\pi x)\cos(3\pi t) \quad (4.15.1.2)$$

Using explicit scheme of equation (4.10.8) that is:

$$u_{i,j+1} = 2(1 - \alpha)u_{i,j} + \alpha\big(u_{i-1,j} + u_{i+1,j}\big) - u_{i,j-1} \quad (4.15.1.3)$$

Where $\alpha = \frac{k^2 c^2}{\mathrm{h}^2}$ and let $\Delta x = h = 0.2$ and $\Delta t = k = 0.1$ for $c = 1. Hence\ \alpha = 0.25$

Let $u_{i,j} = u\ (ih, jk)$ so that the boundary and initial conditions becomes:

$$u_{0,j} = 0 = u_{5,j} \text{ And } u_{i,0} = \sin^3(ih)\ , i = 1,2,3,4,5 \tag{4.15.1.4}$$

The values of the initial points using RHS of equation (4.15 .5) are:

$u_{0,0} = sin^3(\pi \times 0) = 0 \quad u_{1,0} = sin^3(\pi \times 0.2) = 0.2030 \quad u_{2,0} = sin^3(\pi \times 0.4) = 0.8602$

$u_{3,0} = sin^3(\pi \times 0.6) = 0.8602 \quad u_{4,0} = sin^3(\pi \times 0.8) = 0.2030 \quad u_{5,0} = sin^3(\pi \times 1) = 0$

Using central difference approximation of $\frac{\partial u}{\partial t}(x, 0) = 0 \Rightarrow \frac{u_{i,1} - u_{i,-1}}{2k} = 0$

$$\Longrightarrow u_{i,1} = u_{i,-1}\ ,\ u_{i,j+1} = u_{i,j-1}\ and\ u_{i+1,j+1} = u_{i+1,j-1} \tag{4.15.1.5}$$

Using the relation $u_{i,j+1} = u_{i,j-1}$ and putting in to equation (4.15.1.3), we obtain an equation of the form:

$$(1 - \alpha)u_{i,j} + 0.5\alpha\left(u_{i-1,j} + u_{i+1,j}\right) = u_{i,j+1} \tag{4.15.1.6}$$

Furthermore we can rewrite equation (4.15.1.6) in matrix and vector form to get an approximate solution at each grid point for the 1st entry.

$$\begin{bmatrix} (1-\alpha) & 0.5\alpha & 0 & 0 \\ 0.5\alpha & (1-\alpha) & 0.5\alpha & 0 \\ 0 & 0.5\alpha & (1-\alpha) & 0.5\alpha \\ 0 & 0 & 0.5\alpha & (1-\alpha) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \end{bmatrix} \tag{4.15.1.7}$$

By putting $\mathrm{j} = 0$, $\mathrm{i} = 1,2,3,4$ and $\alpha = 0.25$ in to equation (4.15.1.6)we obtain a system

of algebraic equations that can be solved by tri-diagonal matrix as follows:

$$\begin{cases} 0.125u_{0,0} + 0.125\, u_{2,0} + 0.75\, u_{1,0} = u_{1,1} \\ 0.125\, u_{1,0} + 0.125\, u_{3,0} + 0.75\, u_{2,0} = u_{2,1} \\ 0.125\, u_{2,0} + 0.125\, u_{4,0} + 0.75\, u_{3,0} = u_{3,1} \\ 0.125\, u_{3,0} + 0.125u_{5,0} + 0.75\, u_{4,0} = u_{4,1} \end{cases} \tag{4.15.1.8}$$

Hence the approximate solution for the 1st entry evaluated as follows:

$$\begin{bmatrix} 0.75 & 0.125 & 0 & 0 \\ 0.125 & 0.75 & 0.125 & 0 \\ 0 & 0.125 & 0.75 & 0.125 \\ 0 & 0 & 0.125 & 0.75 \end{bmatrix} \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ u_{4,0} \end{bmatrix} = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} \tag{4.15.1.9}$$

$$\Rightarrow u_{1,1} = 0.2598\ ,\ u_{2,1} = 0.7781\ ,\ u_{3,1} = 0.7781\ \ u_{4,1} = 0.2598$$

Form the 2nd time step to the nth –time step the explicit equation (4.15.1.3) that is can be rewrite in matrix and vector form to get an approximate solution at each grid point for the nth entry.

$$\begin{bmatrix} 2(1-\alpha) & \alpha & 0 & 0 \\ \alpha & 2(1-\alpha) & \alpha & 0 \\ 0 & \alpha & 2(1-\alpha) & \alpha \\ 0 & 0 & \alpha & 2(1-\alpha) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ u_{4,j-1} \end{bmatrix} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \end{bmatrix} \quad (4.15.1.10)$$

By putting $j = 1$, $i = 1,2,3,4$ and $\alpha = 0.25$ in to equation (4.15.1.3) we obtain a system

of algebraic equations that can be solved by tri-diagonal matrix as follows:

$$\begin{cases} 0.25u_{0,1} + 0.25\,u_{2,1} + 1.5\,u_{1,1} - u_{1,0} = u_{1,2} \\ 0.25\,u_{1,1} + 0.25\,u_{3,1} + 1.5\,u_{2,1} - u_{2,0} = \mathrm{u}_{2,2} \\ 0.25\,u_{2,1} + 0.25\,u_{4,1} + 1.5\,u_{3,1} - u_{3,0} = u_{3,2} \\ 0.25\,u_{3,1} + 0.25u_{5,1} + 1.5\,u_{4,1} - u_{4,0} = u_{4,2} \end{cases} \quad (4.15.1.11)$$

Hence the approximate solution for the 2nd entry evaluated as follows:

$$\begin{bmatrix} 1.5 & 0.25 & 0 & 0 \\ 0.25 & 1.5 & 0.25 & 0 \\ 0 & 0.25 & 1.5 & 0.25 \\ 0 & 0 & 0.25 & 1.5 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} - \begin{bmatrix} u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ u_{4,0} \end{bmatrix} = \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} \quad (4.15.1.12)$$

$$\Rightarrow u_{1,2} = 0.3812 \;,\; u_{2,2} = 0.5664 \;,\; u_{3,2} = 0.5664 \;,\; u_{4,2} = 0.3812$$

Similarly by putting $j = 2$, $i = 1,2,3,4$ and $\alpha = 0.25$ in to equation (4.15.1.3) we obtain

a system of algebraic equations that can be solved by tri-diagonal matrix as follows:

$$\begin{cases} 0.25u_{0,2} + 0.25\,u_{2,2} + 1.5\,u_{1,2} - u_{1,1} = u_{1,3} \\ 0.25\,u_{1,2} + 0.25\,u_{3,2} + 1.5\,u_{2,2} - u_{2,1} = \mathrm{u}_{2,3} \\ 0.25\,u_{2,2} + 0.25\,u_{4,2} + 1.5\,u_{3,2} - u_{3,1} = u_{3,3} \\ 0.25\,u_{3,2} + 0.25u_{5,2} + 1.5\,u_{4,2} - u_{4,1} = u_{4,3} \end{cases} \quad (4.15.1.13)$$

Hence the approximate solution for the 3rd entry evaluated as follows:

$$\begin{bmatrix} 1.5 & 0.25 & 0 & 0 \\ 0.25 & 1.5 & 0.25 & 0 \\ 0 & 0.25 & 1.5 & 0.25 \\ 0 & 0 & 0.25 & 1.5 \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} - \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} = \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3} \end{bmatrix} \qquad (4.15.1.14)$$

$$\Rightarrow u_{1,3} = 0.4536 \ , \ u_{2,3} = 0.3084 \ , \ u_{3,3} = 0.3084 \ , u_{4,3} = 0.4536$$

Finaly by putting $j = 3$, $i = 1,2,3,4$ and $\alpha = 0.25$ in to equation (4.15.1.3)we obtain

a system of algebraic equations that can be solved by tri-diagonal matrix as follows:

$$\begin{cases} 0.25u_{0,3} + 0.25\, u_{2,3} + 1.5\, u_{1,3} - u_{1,2} = u_{1,4} \\ 0.25\, u_{1,3} + 0.25\, u_{3,3} + 1.5\, u_{2,3} - u_{2,2} = \mathrm{u}_{2,4} \\ 0.25\, u_{2,3} + 0.25\, u_{4,3} + 1.5\, u_{3,3} - u_{3,2} = u_{3,4} \\ 0.25\, u_{3,3} + 0.25u_{5,3} + 1.5\, u_{4,3} - u_{4,2} = u_{4,4} \end{cases} \qquad (4.15.1.15)$$

Hence the approximate solution for the 4^{rd} entry evaluated as follows:

$$\begin{bmatrix} 1.5 & 0.25 & 0 & 0 \\ 0.25 & 1.5 & 0.25 & 0 \\ 0 & 0.25 & 1.5 & 0.25 \\ 0 & 0 & 0.25 & 1.5 \end{bmatrix} \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3} \end{bmatrix} - \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \\ u_{4,4} \end{bmatrix} \qquad (4.15.1.16)$$

$$\Rightarrow u_{1,4} = 0.3763 \ , \ u_{2,4} = 0.0867 \ , \ u_{3,4} = 0.0867 \ , u_{4,4} = 0.3763$$

4.15.2 Exact solution of 1-D wave equation for $\Delta x = 0.2$ and $\Delta t = 0.1$

The problem given on the equation (4.15.1) admits an exact solution given by:

$$u\,(x,t) = \frac{3}{4}\sin(\pi x)\cos(\pi t) - \frac{1}{4}\sin(3\pi x)\cos(3\pi t) \qquad (4.15.2.1)$$

$$\Rightarrow u\left(x_i\,, t_j\right) = \frac{3}{4}\sin(\pi x)\cos(\pi t) - \frac{1}{4}\sin(3\pi x)\cos(3\pi t) \qquad (4.15.2.2)$$

So the exact solutions at the 1^{st} entry using equation (4.15.2.2) when $\Delta x = 0.2$ and $t = 0.1$ are:

$U_{1,1} = u(0.2,0.1) = 0.2795$ $\qquad U_{4,1} = u(0.8,0.1) = 0.2795$

$U_{2,1} = u(0.4,0.1) = 0.7648$ $\qquad U_{5,1} = u(1,0.1) = 0.0000$

$U_{3,1} = u(0.6,0.1) = 0.7648$

Similarly when $\Delta x = 0.2$ and $t = 0.2$ the exact solutions at the 2^{nd} entry are:

$U_{1,2} = u(0.2,0.2) = 0.4301$ $\qquad U_{4,2} = u(0.8,0.2) = 0.4301$

$U_{2,2} = u(0.4,0.2) = 0.5317$ $\qquad U_{5,2} = u(1,0.2) = 0.0000$

$U_{3,2} = u(0.6,0.2) = 0.5317$

Again when $\Delta x = 0.2$ and $t = 0.3$ the exact solutions at the 3^{rd} entry are:

$U_{1,3} = u(0.2,0.3) = 0.4852$ $\qquad U_{4,3} = u(0.8,0.3) = 0.4852$

$U_{2,3} = u(0.4,0.3) = 0.2795$ $\qquad U_{5,3} = u(1,0.3) = 0.0000$

$U_{3,3} = u(0.6,0.3) = 0.2795$

Finally when $\Delta x = 0.2$ and $t = 0.4$ the exact solutions at the 4^{th} entry are:

$U_{1,4} = u(0.2,0.4) = 0.3286$ $\qquad U_{4,4} = u(0.8,0.4) = 0.3286$

$U_{2,4} = u(0.4,0.4) = 0.1015$ $\qquad U_{5,4} = u(1,0.4) = 0.0000$

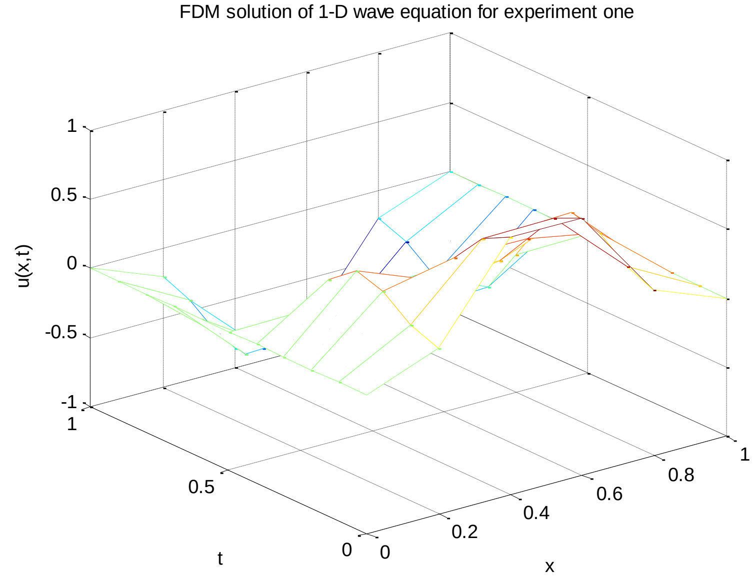
$U_{3,4} = u(0.6,0.4) = 0.1015$

Table 2 Summarized Exact and Explicit solutions for 1-Dtrial data given on table one.

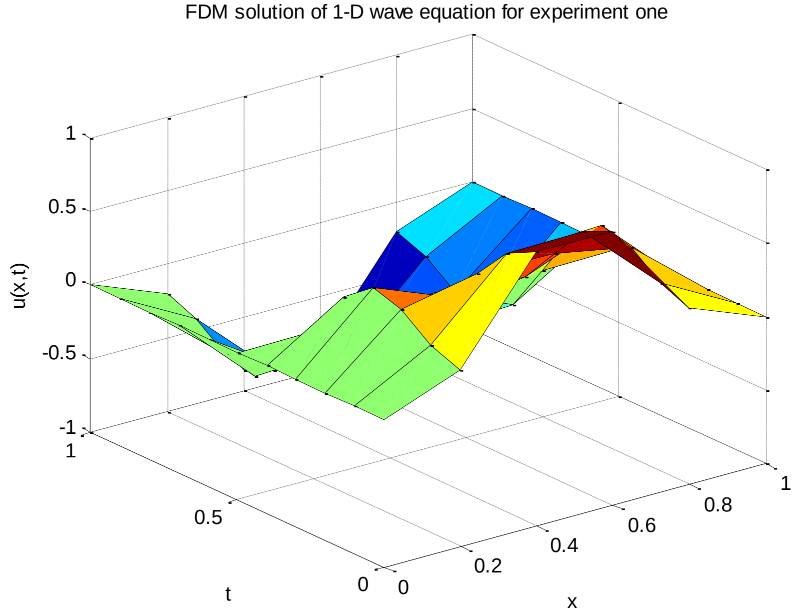
Entry	Node	**Trial one** $\Delta \boldsymbol{x} = \mathbf{0.2}$ and $\Delta t = 0.1$			**Trial two** $\Delta \boldsymbol{x} = \mathbf{0.2}$ and $\Delta t = 0.05$			**Trial three** $\Delta \boldsymbol{x} = \mathbf{0.2}$ and $\Delta t = 0.025$			**Trial four** $\Delta \boldsymbol{x} = \mathbf{0}.$ and $\Delta t = 0.0125$		
		Exact	Explicit	Error	Exact	Explicit	Error	Exact	Explicit	Error	Exact	Explicit	Error
1st	$u_{1,1}$	0.2795	0.2598	0.0197	0.2236	0.2172	0.0064	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005
	$u_{2,1}$	0.7648	0.7781	0.0133	0.8354	0.8397	0.0043	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002
	$u_{3,1}$	0.7648	0.7781	0.0133	0.8354	0.8397	0.0043	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002
	$u_{4,1}$	0.2795	0.2598	0.0197	0.2236	0.2172	0.0064	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005
2nd	$u_{1,2}$	0.4301	0.3812	0.0489	0.2795	0.2567	0.0228	0.2236	0.2170	0.0066	0.2083	0.2065	0.0018
	$u_{2,2}$	0.5317	0.5663	0.0346	0.7648	0.7802	0.0154	0.8354	0.8398	0.0044	0.8540	0.8551	0.0011
	$u_{3,2}$	0.5317	0.5663	0.0346	0.7648	0.7802	0.0154	0.8354	0.8398	0.0044	0.8540	0.8551	0.0011
	$u_{4,2}$	0.4301	0.3812	0.0489	0.2795	0.2567	0.0228	0.2236	0.2170	0.0066	0.2083	0.2065	0.0018
3rd	$u_{1,3}$	0.4852	0.4536	0.0316	0.3556	0.3129	0.0427	0.2479	0.2338	0.0141	0.2147	0.2109	0.0038
	$u_{2,3}$	0.2795	0.3083	0.0288	0.6585	0.6881	0.0296	0.8053	0.8148	0.0095	0.8462	0.8487	0.0025
	$u_{3,3}$	0.2795	0.3083	0.0288	0.6585	0.6881	0.0296	0.8053	0.8148	0.0095	0.8462	0.8487	0.0025
	$u_{4,3}$	0.4852	0.4536	0.0316	0.3556	0.3129	0.0427	0.2479	0.2338	0.0141	0.2147	0.2109	0.0038
4th	$u_{1,4}$	0.3286	0.3763	0.0477	0.4301	0.3730	0.0571	0.2795	0.2560	0.0235	0.2236	0.2170	0.0066
	$u_{2,4}$	0.1015	0.0866	0.0149	0.5317	0.5725	0.0408	0.7648	0.7807	0.0159	0.8354	0.8398	0.0044
	$u_{3,4}$	0.1015	0.0866	0.0149	0.5317	0.5725	0.0408	0.7648	0.7807	0.0159	0.8354	0.8398	0.0044
	$u_{4,4}$	0.3286	0.3763	0.0477	0.4301	0.3730	0.0571	0.2795	0.2560	0.0235	0.2236	0.2170	0.0066

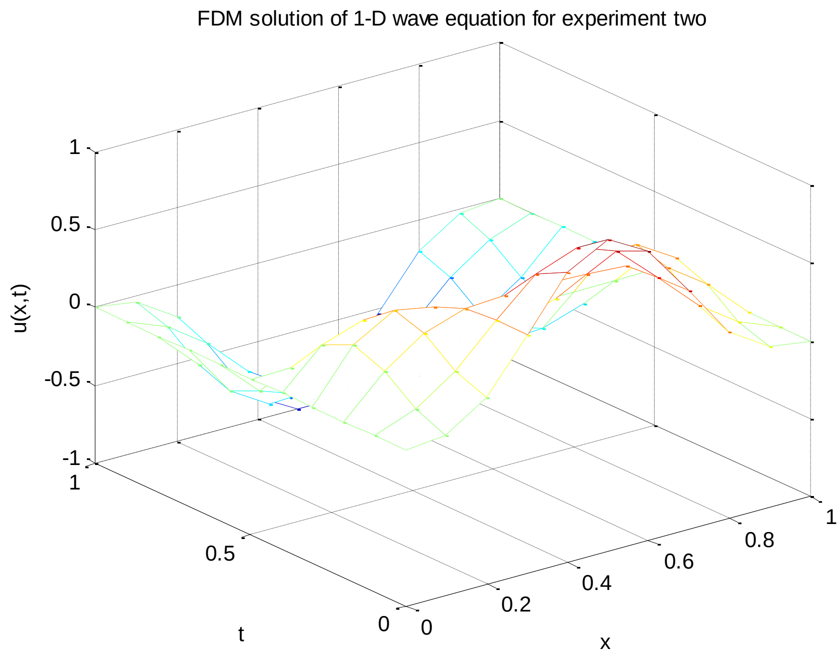
From the table (2) we can see that when decreasing time discretization step ($\Delta t \rightarrow 0$) with fixed space discretization step (Δx) the numerical solutions of 1-D wave equations which are obtained by the explicit method is more smoothly approaches to the exact solution at each entry and also the truncation error in this method approaches to zero when decreasing time step size ($\Delta t \rightarrow 0$). So from this we can conclude that stable numerical values obtained by this method which are converge more smoothly to the exact solution.

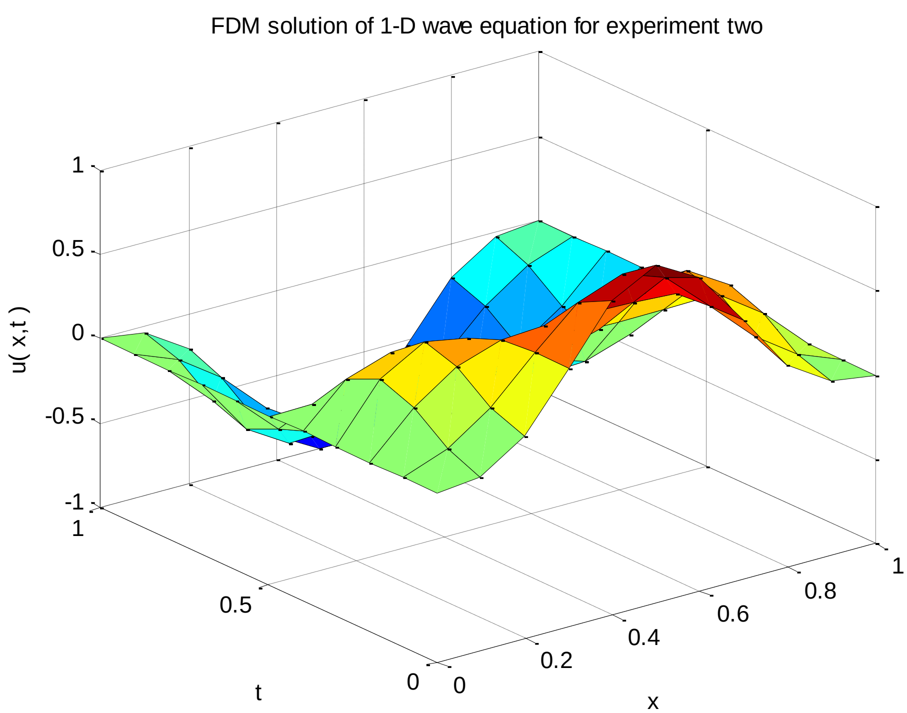
FDM solution of 1-D wave equation for experiment one

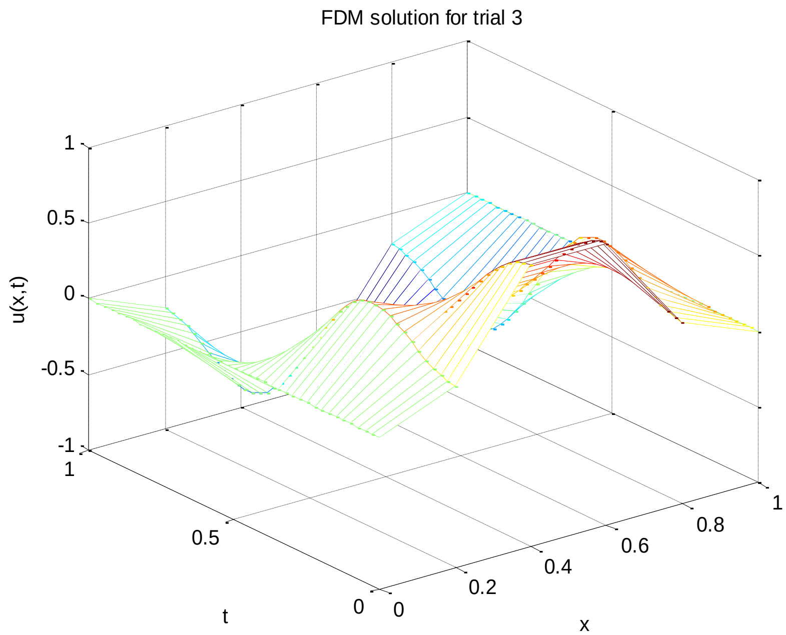


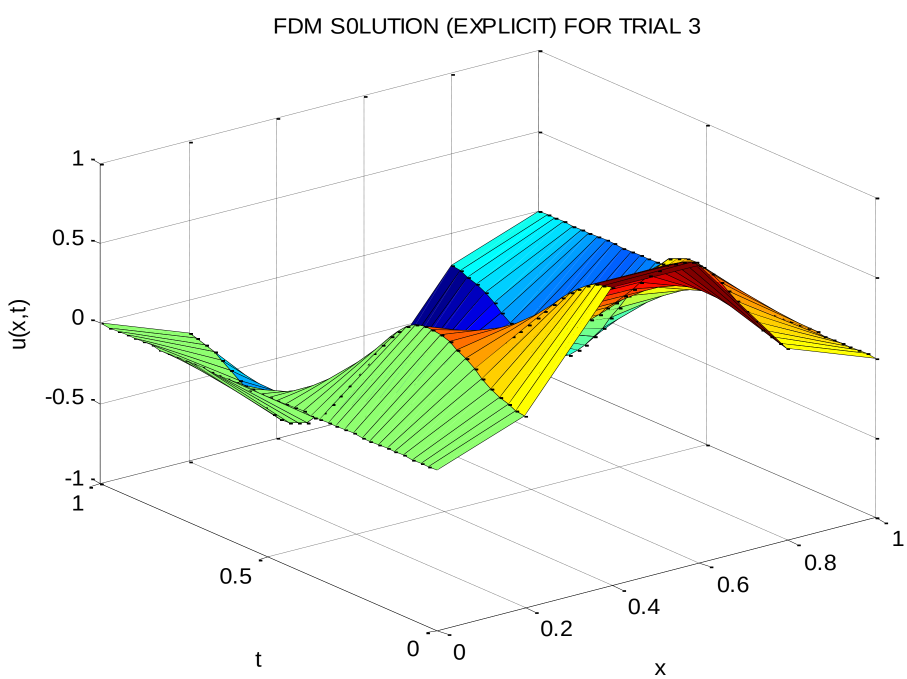
FDM solution of 1-D wave equation for experiment one

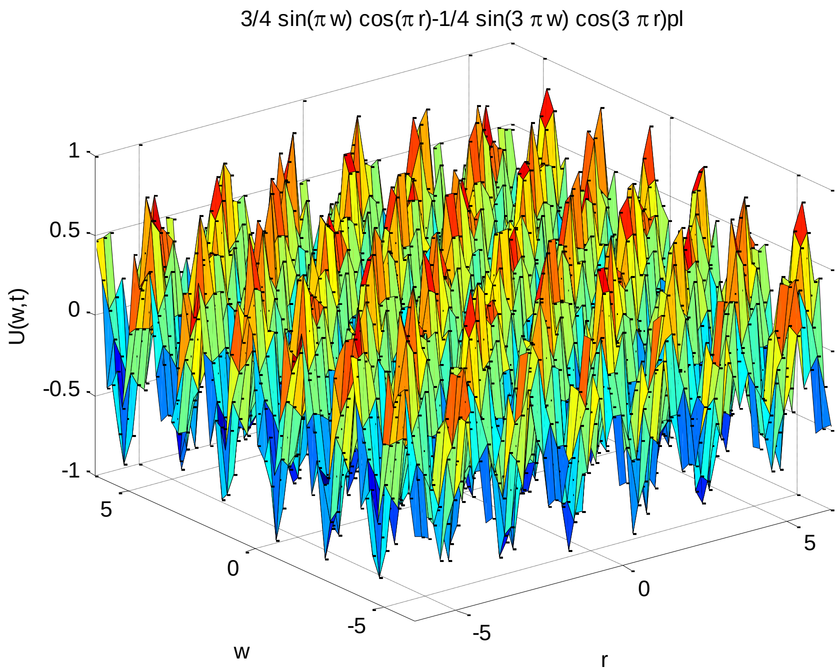



FDM solution of 1-D wave equation for experiment two
u(x,t)
1
0.5
0
-0.5
-1
t
x
0
0.2
0.4
0.6
0.8
1


FDM solution of 1-D wave equation for experiment two
u(x,t)
1
0.5
0
-0.5
-1
t
x
0
0.2
0.4
0.6
0.8
1


FDM solution for trial 3
u(x,t)
1
0.5
0
-0.5
-1
1
0.5
0
t
0
0.2
0.4
0.6
0.8
1
x


FDM S0LUTION (EXPLICIT) FOR TRIAL 3
u(x,t)
1
0.5
0
-0.5
-1
1
0.5
0
t
0
0.2
0.4
0.6
0.8
1
x



ezsurf plot for the exact solution with unbounded domain

4.15.3 Numerical solution of 1-D wave equation using implicit method

The problem given by equation (4.15.1) that is

$$\frac{\partial^2 u}{\partial\, t^2} = \frac{\partial^2 u}{\partial\, x^2}$$

Subject to the boundary conditions:

$u(0,t) = 0$, $(1,t) = 0$, for $t \geq 0$ and the intial conditions:

$\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = sin^3(\pi x)$ $, for$ $0 \leq \mathrm{x} \leq 1,$ Can be solved by using implicit scheme given by equation (4.5) i.e.

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{c^2\, k^2}{2h^2}\Big((u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\Big)$$

$$+(u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1})) \qquad (4.15.3.1)$$

Let $\Delta x = h = 0.2$ and $\Delta t = k = 0.1$ for $c = 1$ and putting $\alpha = \frac{k^2 c^2}{\mathrm{h}^2}$ equation (4.15.3.1) can be written as simplified form:

$$-0.5\, \alpha\, u_{i-1,j+1} + (\alpha + 1)u_{i,j+1} - 0.5\alpha\, u_{i+1,j+1} = u_{i,j} \qquad (4.15.3.2)$$

To approximate the numerical solutions for each entry, for $i = 1,2$, ... and $j = 0,1,2$, ...
we obtain a system of linear equations from equation (4.15.3.2) that can be solved by using tri-diagonal matrix as follows:

$$\begin{bmatrix} (\alpha+1) & -0.5\alpha & 0 & 0 \\ -0.5\alpha & (\alpha+1) & -0.5\alpha & 0 \\ 0 & -0.5\alpha & (\alpha+1) & -0.5\alpha \\ 0 & 0 & -0.5\alpha & (\alpha+1) \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ u_{4,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ u_{4,j} \end{bmatrix} \qquad (4.15.3.3)$$

.

Hence the approximate solutions at each grid points for the 1^{st} entry when $j = 0\; and\; i = 1,2,3,4$ are solutions of system of linear equations obtained from (4.15.3.2) that can be solved by using tri-diagonal matrix.

$$\begin{cases} -0.125\; u_{0,1} + 1.25\; u_{1,1} - 0.125\; u_{2,1} = u_{1,0} \\ -0.125\; u_{1,1} + 1.25\; u_{2,1} - 0.125\; u_{3,1} = u_{2,0} \\ -0.125\; u_{2,1} + 1.25\; u_{3,1} - 0.125\; u_{4,1} = u_{3,0} \\ -0.125\; u_{3,1} + 1.25\; u_{4,1} - 0.125\; u_{5,1} = u_{4,0} \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1.25 & -0.125 & 0 & 0 \\ -0.125 & 1.25 & -0.125 & 0 \\ 0 & -0.125 & 1.25 & -0.125 \\ 0 & 0 & -0.125 & 1.25 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} = \begin{bmatrix} 0.2030 \\ 0.8602 \\ 0.8602 \\ 0.2030 \end{bmatrix}$$

$\Rightarrow u_{1,1} = 0.2415$, $u_{2,1} = 0.7915$, $u_{3,1} = 0.7915$ and $u_{4,1} = 0.2415$

Similarly the approximate solution for 2^{nd} entry when $j = 1$ and $i = 1,2,3,4$

$$\begin{cases} -0.125\, u_{0,2} + 1.25\, u_{1,2} - 0.125\, u_{2,2} = u_{1,1} \\ -0.125\, u_{1,2} + 1.25\, u_{2,2} - 0.125\, u_{3,2} = u_{2,1} \\ -0.125\, u_{2,2} + 1.25\, u_{3,2} - 0.125\, u_{4,2} = u_{3,1} \\ -0.125\, u_{3,2} + 1.25\, u_{4,2} - 0.125\, u_{5,2} = u_{4,1} \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1.25 & -0.125 & 0 & 0 \\ -0.125 & 1.25 & -0.125 & 0 \\ 0 & -0.125 & 1.25 & -0.125 \\ 0 & 0 & -0.125 & 1.25 \end{bmatrix} \begin{bmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{bmatrix} = \begin{bmatrix} 0.2415 \\ 0.7915 \\ 0.7915 \\ 0.2415 \end{bmatrix}$$

$\Rightarrow u_{1,2} = 0.2665$, $u_{2,2} = 0.7332$, $u_{3,2} = 0.7332$ $u_{3,2} = 0.2665$

Again the approximate solution for the 3^{rd} entry when $j = 2$ and $i = 1,2,3,4$

$$\begin{cases} -0.125\, u_{0,3} + 1.25\, u_{1,3} - 0.125\, u_{2,3} = u_{1,2} \\ -0.125\, u_{1,3} + 1.25\, u_{2,3} - 0.125\, u_{3,3} = u_{2,2} \\ -0.125\, u_{2,3} + 1.25\, u_{3,3} - 0.125\, u_{4,3} = u_{3,2} \\ -0.125\, u_{3,3} + 1.25\, u_{4,3} - 0.125\, u_{5,3} = u_{4,2} \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1.25 & -0.125 & 0 & 0 \\ -0.125 & 1.25 & -0.125 & 0 \\ 0 & -0.125 & 1.25 & -0.125 \\ 0 & 0 & -0.125 & 1.25 \end{bmatrix} \begin{bmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3} \end{bmatrix} = \begin{bmatrix} 0.2665 \\ 0.7332 \\ 0.7332 \\ 0.2665 \end{bmatrix}$$

$\Rightarrow u_{1,3} = 0.2815$, $u_{2,3} = 0.6830$, $u_{3,3} = 0.6830$ $u_{4,3} = 0.2815$

Finally the approximate solution for the 3^{rd} entry when $j = 3$ and $i = 1,2,3,4$

$$\begin{cases} -0.125\, u_{0,4} + 1.25\, u_{1,4} - 0.125\, u_{2,4} = u_{1,3} \\ -0.125\, u_{1,4} + 1.25\, u_{2,4} - 0.125\, u_{3,4} = u_{2,3} \\ -0.125\, u_{2,4} + 1.25\, u_{3,4} - 0.125\, u_{4,4} = u_{3,3} \\ -0.125\, u_{3,4} + 1.25\, u_{4,4} - 0.125\, u_{5,4} = u_{4,3} \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1.25 & -0.125 & 0 & 0 \\ -0.125 & 1.25 & -0.125 & 0 \\ 0 & -0.125 & 1.25 & -0.125 \\ 0 & 0 & -0.125 & 1.25 \end{bmatrix} \begin{bmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \\ u_{4,4} \end{bmatrix} = \begin{bmatrix} 0.2815 \\ 0.6830 \\ 0.6830 \\ 0.2815 \end{bmatrix}$$

$\Rightarrow u_{1,4} = 0.2891$, $u_{2,4} = 0.6392$, $u_{3,4} = 0.6392$ $u_{4,4} = 0.2891$

Table 3 Summarized of Exact and Implicit solutions for 1-D wave equations obtained from trial data given on table one

Entry	Node	Trial one $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.1$			Trial two $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.05$			Trial three $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.025$			Trial four $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.0125$		
		Exact	Implicit	Error	Exact	Implicit	Error	Exact	Implicit	Error	Exact	Implicit	Error
1st	$u_{1,1}$	0.2795	0.2415	0.0560	0.2236	0.2158	0.0078	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005
	$u_{2,1}$	0.7647	0.7915	0.0268	0.8354	0.8407	0.0053	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002
	$u_{3,1}$	0.7648	0.7945	0.0268	0.8354	0.8407	0.0053	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002
	$u_{4,1}$	0.2795	0.2415	0.0560	0.2236	0.2158	0.0078	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005
2nd	$u_{1,2}$	0.4301	0.2666	0.1635	0.2795	0.2273	0.0522	0.2236	0.2098	0.0138	0.2083	0.2048	0.0035
	$u_{2,2}$	0.5317	0.7331	0.2015	0.7648	0.8221	0.0573	0.8354	0.8501	0.0147	0.8540	0.8576	0.0036
	$u_{3,2}$	0.5317	0.7331	0.2015	0.7648	0.8221	0.0573	0.8354	0.8501	0.0147	0.8540	0.8576	0.0036
	$u_{4,2}$	0.4301	0.2666	0.1635	0.2795	0.2273	0.0522	0.2236	0.2098	0.0138	0.2083	0.2048	0.0035
3rd	$u_{1,3}$	0.4852	0.2815	0.2037	0.3556	0.2376	0.1180	0.2479	0.2131	0.0348	0.2147	0.2056	0.0091
	$u_{2,3}$	0.2795	0.6830	0.4035	0.6585	0.8044	0.1459	0.8053	0.8452	0.0399	0.8462	0.8564	0.0102
	$u_{3,3}$	0.2795	0.6830	0.4035	0.6585	0.8044	0.1459	0.8053	0.8452	0.0399	0.8462	0.8564	0.0102
	$u_{4,3}$	0.4852	0.2815	0.2037	0.3556	0.2376	0.1180	0.2479	0.2131	0.0348	0.2147	0.2056	0.0091
4th	$u_{1,4}$	0.3286	0.2891	0.0394	0.4301	0.2467	0.1834	0.2795	0.2163	0.0632	0.2236	0.2065	0.0171
	$u_{2,4}$	0.1015	0.6392	0.5377	0.5317	0.7875	0.2498	0.7648	0.8403	0.0755	0.8354	0.8551	0.0197
	$u_{3,4}$	0.1015	0.6392	0.5377	0.5317	0.7875	0.2498	0.7648	0.8403	0.0755	0.8354	0.8551	0.0197
	$u_{4,4}$	0.3286	0.2891	0.0394	0.4301	0.2467	0.1834	0.2795	0.2163	0.0632	0.2236	0.2065	0.0171

We can clearly observe from the table (3) that when time discretization steps, $\Delta t \to 0$ with fixed space discretization step Δx the numerical solutions of 1-D wave equations which are obtained by the implicit method are less smoothly approaches to the exact solution from that of the counterpart of the explicit shames at each entry. The truncation error in this method will be decrease as $\Delta t \to 0$ approaches to zero so from this we can conclude that the numerical values obtained by this method less smoothly converges to the exact solution.

4.15.4 Table 4 Comparison of Exact, Explicit and Implicit solutions for1-D wave equation on trials 3&4.

Entry	Node	Trial three Explicit $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.025$			Trial three Implicit $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.025$			Trial four Explicit $\Delta x = \mathbf{0.2}$ and $\Delta t = 0.025$			Trial four Implicit $\Delta x = \mathbf{0}.$ and $\Delta t = 0.025$		
		Exact	Explicit	Error	Exact	Implicit	Error	Exact	Explicit	Error	Exact	Implicit	Error
1st	$u_{1,1}$	0.2083	0.2065	0.0018	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005	0.2044	0.2039	0.0005
	$u_{2,1}$	0.8540	0.8551	0.0011	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002	0.8587	0.8589	0.0002
	$u_{3,1}$	0.8540	0.8551	0.0011	0.8540	0.8551	0.0011	0.8587	0.8589	0.0002	0.8587	0.8589	0.0002
	$u_{4,1}$	0.2083	0.2065	0.0018	0.2083	0.2065	0.0018	0.2044	0.2039	0.0005	0.2044	0.2039	0.0005
2nd	$u_{1,2}$	0.2236	0.2170	0.0066	0.2236	0.2098	0.0138	0.2083	0.2065	0.0018	0.2083	0.2048	0.0035
	$u_{2,2}$	0.8354	0.8398	0.0044	0.8354	0.8501	0.0147	0.8540	0.8551	0.0011	0.8540	0.8576	0.0036
	$u_{3,2}$	0.8354	0.8398	0.0044	0.8354	0.8501	0.0147	0.8540	0.8551	0.0011	0.8540	0.8576	0.0036
	$u_{4,2}$	0.2236	0.2170	0.0066	0.2236	0.2098	0.0138	0.2083	0.2065	0.0018	0.2083	0.2048	0.0035
3rd	$u_{1,3}$	0.2479	0.2338	0.0141	0.2479	0.2131	0.0348	0.2147	0.2109	0.0038	0.2147	0.2056	0.0091
	$u_{2,3}$	0.8053	0.8148	0.0095	0.8053	0.8452	0.0399	0.8462	0.8487	0.0025	0.8462	0.8564	0.0102
	$u_{3,3}$	0.8053	0.8148	0.0095	0.8053	0.8452	0.0399	0.8462	0.8487	0.0025	0.8462	0.8564	0.0102
	$u_{4,3}$	0.2479	0.2338	0.0141	0.2479	0.2131	0.0348	0.2147	0.2109	0.0038	0.2147	0.2056	0.0091
4th	$u_{1,4}$	0.2795	0.2560	0.0235	0.2795	0.2163	0.0632	0.2236	0.2170	0.0066	0.2236	0.2065	0.0171
	$u_{2,4}$	0.7648	0.7807	0.0159	0.7648	0.8403	0.0755	0.8354	0.8398	0.0044	0.8354	0.8551	0.0197
	$u_{3,4}$	0.7648	0.7807	0.0159	0.7648	0.8403	0.0755	0.8354	0.8398	0.0044	0.8354	0.8551	0.0197
	$u_{4,4}$	0.2795	0.2560	0.0235	0.2795	0.2163	0.0632	0.2236	0.2170	0.0066	0.2236	0.2065	0.0171

From the above comparison table (4), we can clearly observe that when time discretization steps, $\Delta t \to 0$ with fixed space discretization step Δx the numerical solutions of 1-D wave equations which are obtained by the explicit method are more smoothly approaches to the exact solution from that of the counterpart of the implicit shames at each entry. On the other hand the truncation error will be decrease and approaches to zero so from this we can conclude that the numerical values obtained by the two methods smoothly converges to the exact solution

4.16 Numerical experiment for 2-D wave equation

Solve $\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad 0 \le x \le 1\,,\ 0 \le y \le 1 \text{ and } t \ge 0$ (4.16.1)

Subject to the initial and Dirichlet boundary conditions:

$U(x, y, 0) = f(x, y) = \sin(\pi x)\sin(\pi y) + x^2 - y^2 \text{ and } U_t\,(x, y, 0) = g(x, y) = 0$ (4.16.2)

$U(0, y, t) = 0$,$(1, y, t) = 0$, $U(x, 0, t) = 0$ and $U(x, 1, t) = 0$ (4.16.3)

Using explicit method based on trial data given below:

Table 5 trial data for 2-D wave equation

Experiment	Space and time step sizes			Speed of wave (C)	$r = \frac{c\,\Delta t}{\Delta x}$	$\alpha = r^2$
	$h = \Delta x$	$k = \Delta y$	$l = \Delta t$			
Trial 1	0.25	0.25	0.01	1	0.04	0.0016
Trial 2	0.20	0.20	0.01	1	0.05	0.0025
Trial 3	0.10	0.10	0.01	1	0.1	0.01
Trial 4	0.01	0.01	0.01	1	1	1

By applying the method of separation of variables the given IVBVP has an exact solution:

$U(x, y, t) = \cos(\sqrt{2}\pi t)\sin(\pi x)\sin(\pi y) + x^2 - y^2$ (4.16.4)

4.16.1 Numerical solution of 2-D wave equation using explicit method

Suppose we solve for $n = 3$ time steps since we have Dirichlet boundary conditions so we need to find the interior values of the region, $u_{i,j,n+1}$ for $1 \le i \le n$ and $1 \le j \le n$ we let $\Delta x = \Delta y = h = 0.25$, $\Delta t = k = 0.01$ and $\beta = \frac{c^2\,\Delta t^2}{\Delta x^2} = \frac{c^2\,\Delta t^2}{\Delta y^2} = 0.0016$, then to solve for $u_{i,j,n+1}$ using explicit scheme that is:

$$u_{i,j,n+1} = 2(1-2\beta)\, u_{i,j,n} - u_{i,j,n-1} + \beta\left(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n}\right) \quad (4.16.1.1)$$

$$\Rightarrow u_{i,j,n+1} = (1-2\beta)\, u_{i,j,n} + 0.5\beta\left(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n}\right) \quad (4.16.1.2)$$

Using the computational model the mesh points at zero level for all $i, j = 0,1,2,3$ equation (4.16.2) and (4.16.3) are becomes: $U(i,j,0) = \sin(\pi i)\sin(\pi j) + i^2 - j^2$ (4.16.1.3)

$U(0,j,n) = 0$,$(1,j,n) = 0$, $U(i,0,n) = 0$ and $U(i,1,n) = 0$ (4.16.1.4)

Now putting $\beta = 0.0016$ in to (4.16.1.2) then the equation become:

$$\Rightarrow u_{i,j,1} = 0.9968\, u_{i,j,0} + 0.0008\left(u_{i-1,j,0} + u_{i+1,j,0} + u_{i,j-1,0} + u_{i,j+1,0}\right) \quad (4.16.1.5)$$

The mesh values at the 1^{st} time level when $n = 0$ are:

$u_{1,1,1} = 0.9968(u_{1,1,0}) + 0.0008\left(u_{0,1,0} + u_{2,1,0} + u_{1,0,0} + u_{1,2,0}\right)$, for $i = j = 1$

$= 0.9968(0.5) + 0.0008(0 + 0.8946 + 0 + 0.5196) = 0.4995$

$u_{2,1,1} = 0.9968(u_{2,1,0}) + 0.0008\left(u_{1,1,0} + u_{3,1,0} + u_{2,0,0} + u_{2,2,0}\right)$, for $i = 2,\ j = 1$

$= 0.9968(0.8946) + 0.0008(0.5 + 1 + 0 + 1) = 0.8937$

$u_{1,2,1} = 0.9968(u_{1,2,0}) + 0.0008\left(u_{0,2,0} + u_{2,2,0} + u_{1,1,0} + u_{1,3,0}\right)$, for $i = 1, j = 2$

$= 0.9968(0.5196) + 0.0008(0 + 1 + 0.5 + 0) = 0.5191$

$u_{2,2,1} = 0.9968(u_{2,2,0}) + 0.0008\left(u_{1,2,0} + u_{3,2,0} + u_{2,1,0} + u_{2,3,0}\right)$, for $i = j = 2$

$= 0.9968(1) + 0.0008(0.5196 + 1.0196 + 0.8946 + 0.3946) = 0.9991$

Similarly the mesh values at the 2^{nd} time level when $n = 1$ are:

$$\Rightarrow u_{i,j,2} = 0.9968\, u_{i,j,1} + 0.0008\left(u_{i-1,j,1} + u_{i+1,j,1} + u_{i,j-1,1} + u_{i,j+1,1}\right) \quad (4.16.1.6)$$

$u_{1,1,2} = 0.9968\, u_{1,1,1} + 0.0008\left(u_{0,1,1} + u_{2,1,1} + u_{1,0,1} + u_{1,2,1}\right)$, for $i = j = 1$

$= 0.9968(0.4995) + 0.0008(0 + 0.8937 + 0 + 0.5191) = 0.4990$

$u_{1,2,2} = 0.9968\, u_{1,2,1} + 0.0008\big(u_{0,2,1} + u_{2,2,1} + u_{1,1,1} + u_{1,3,1}\big)$, for $i = 1,\ j = 2$

$= 0.9968(0.5191) + 0.0008(0 + 0.9990 + 0.4995 + 0) = 0.5186$

$u_{2,1,2} = 0.9968\, u_{2,1,1} + 0.0008\big(u_{1,1,1} + u_{3,1,1} + u_{2,0,1} + u_{2,2,1}\big)$, for $i = 2,\ j = 1$

$= 0.9968(0.8937) + 0.0008(0.4995 + 1 + 0 + 0.9990) = 0.8928$

$u_{2,2,2} = 0.9968\, u_{2,2,1} + 0.0008\big(u_{1,2,1} + u_{3,2,1} + u_{2,1,1} + u_{2,3,1}\big)$, for $i = 2,\ j = 2$

$= 0.9968(0.9990) + 0.0008(0.5191 + 1.0196 + 0.8937 + 0.3946) = 0.9981$

Finally the mesh values at the 3^{rd} time level when $n = 2$ are:

$$\Longrightarrow u_{i,j,3} = 0.9968\, u_{i,j,2} + 0.0008\big(u_{i-1,j,2} + u_{i+1,j,2} + u_{i,j-1,12} + u_{i,j+1,2}\big) \qquad (4.16.1.7)$$

$u_{1,1,3} = 0.9968\, u_{1,1,2} + 0.0008\big(u_{0,1,2} + u_{2,1,2} + u_{1,0,2} + u_{1,2,2}\big)$, for $i = j = 1$

$= 0.9968(0.4990) + 0.0008(0 + 0.8928 + 0 + 0.5186) = 0.4985$

$u_{1,2,3} = 0.9968\, u_{1,2,2} + 0.0008\big(u_{0,2,2} + u_{2,2,2} + u_{1,1,2} + u_{1,3,2}\big)$, for $i = 1,\ j = 2$

$= 0.9968(0.5186) + 0.0008(0 + 0.9981 + 0.4990 + 0) = 0.5181$

$u_{2,1,3} = 0.9968\, u_{2,1,2} + 0.0008\big(u_{1,1,2} + u_{3,1,2} + u_{2,0,2} + u_{2,2,2}\big)$, for $i = 2,\ j = 1$

$= 0.9968(0.8928) + 0.0008(0.4990 + 1 + 0 + 0.9981) = 0.8919$

$u_{2,2,3} = 0.9968\, u_{2,2,2} + 0.0008\big(u_{1,2,2} + u_{3,2,2} + u_{2,1,2} + u_{2,3,2}\big)$, for $i = 2,\ j = 2$

$= 0.9968(0.9981) + 0.0008(0.5186 + 1.0196 + 0.8928 + 0.3946) = 0.9972$

Table (6) Numerical solutions of 2-D wave equation obtained by explicit method with

$\Delta x = \Delta y = 0.25$ And $\Delta t = 0.01$

1st entry	$U_{1,1,1}$	$U_{2,1,1}$	$U_{1,2,1}$	$U_{1,2,1}$
Node value	0.4995	0.8937	0.5191	0.9991
2nd entry	$U_{1,1,2}$	$U_{1,2,2}$	$U_{2,1,2}$	$U_{2,2,2}$
Node value	0.4990	0.5186	0.8928	0.9981
3rd entry	$U_{1,1,3}$	$U_{1,2,3}$	$U_{2,1,3}$	$U_{2,2,3}$
Node value	0.4985	0.5181	0.8919	0.9972

4.16.2 Exact solution of 2-D wave equation

The problem given on the equation (4.16.1) admits an exact solution of the form:

$$(x, y, t) = \cos(\sqrt{2}\pi t)\sin(\pi x)\sin(\pi y) + x^2 - y^2$$

So the exact solutions at the 1^{st} time level, when $\Delta x = \Delta y = 0.25$ and $t = 0.01$ are:

$U_{1,1,1} = U(0.25,0.25,0.01) = 0.4995$ $\quad$ $U_{1,2,1} = U(0.25,0.5,0.01) = 0.5189$

$U_{2,1,1} = U(0.5,0.25,0.01) = 0.8939$ $\quad$ $U_{1,2,1} = U(0.5,0.5,0.01) = 0.9990$

Similarly when $\Delta x = \Delta y = 0.25$ and $t = 0.02$ the exact solutions at the 2^{nd} time level are:

$U_{1,1,2} = U(0.25,0.25,0.02) = 0.4980$ $\quad$ $U_{1,2,2} = U(0.25,0.5,0.02) = 0.5187$

$U_{2,1,2} = U(0.5,0.25,0.02) = 0.8917$ $\quad$ $U_{2,2,2} = U(0.5,0.5,0.02) = 0.9960$

Finally when $\Delta x = \Delta y = 0.25$ and $t = 0.03$ the exact solutions at the 3^{rd} time level are

$U_{1,1,3} = U(0.25,0.25,0.03) = 0.4956$ $\quad$ $U_{1,2,3} = U(0.25,0.5,0.03) = 0.5133$

$U_{2,1,3} = U(0.5,0.25,0.03) = 0.8833$ $U_{2,2,3} = U(0.5,0.5,0.03) = 0.9911$

Table (7) exact solutions of 2-D wave equation with $\Delta x = \Delta y = h = 0.25$ and time discretization steps $\Delta t = k = 0.01$

1st entry	$U_{1,1,1}$	$U_{2,1,1}$	$U_{1,2,1}$	$U_{2,2,1}$
Node value	0.4995	0.8939	0.5189	0.9990
2nd entry	$U_{1,1,2}$	$U_{1,2,2}$	$U_{2,1,2}$	$U_{2,2,2}$
Node value	0.4980	0.5183	0.8917	0.9960
3rd entry	$U_{1,1,3}$	$U_{1,2,3}$	$U_{2,1,3}$	$U_{2,2,3}$
Node value	0.4956	0.5133	0.8883	0.9911

4.16.3 Table 8 Comparison of exact and numerical solutions for 2-D wave equations obtained from trial 1 to 4

Entry	Nodes	Trial one $\boldsymbol{\Delta x = \Delta y = 0.25}$ and $\Delta t = 0.01$			Trial two $\boldsymbol{\Delta x = \Delta y = 0.2}$ and $\Delta t = 0.01$			Trial three $\boldsymbol{\Delta x = \Delta y = 0.1}$ and $\Delta t = 0.01$			Trial four $\boldsymbol{\Delta x = \Delta y = 0.01}$ and $\Delta t = 0.01$		
		Exact	Explicit	Error	Exact	Explicit	Error	Exact	Explicit	Error	Exact	Explicit	Error
1st	$u_{1,1,1}$	0.4995	0.4995	0.0000	0.3452	0.3452	0.0000	0.0954	0.0954	0.0000	0.0010	0.0010	0.0000
	$u_{2,1,1}$	0.8939	0.8937	0.0002	0.6785	0.6784	0.0001	0.2115	0.2112	0.0003	0.0023	0.0021	0.0002
	$u_{1,2,1}$	0.5189	0.5191	0.0002	0.4385	0.4387	0.0002	0.1515	0.1516	0.0001	0.0017	0.0019	0.0002
	$u_{2,2,1}$	0.9990	0.9991	0.0001	0.9036	0.9036	0.0000	0.3452	0.3452	0.0000	0.0039	0.0041	0.0002
2nd	$u_{1,1,2}$	0.4980	0.4990	0.0010	0.3441	0.3419	0.0022	0.0951	0.0953	0.0002	0.0010	0.0010	0.0000
	$u_{1,2,2}$	0.5183	0.5186	0.0003	0.4368	0.4354	0.0014	0.1509	0.1516	0.0007	0.0017	0.0020	0.0003
	$u_{2,1,2}$	0.8917	0.8928	0.0011	0.6768	0.6725	0.0043	0.2109	0.2108	0.0001	0.0023	0.0024	0.0001
	$u_{2,2,2}$	0.9960	0.9981	0.0008	0.9009	0.8951	0.0058	0.3441	0.3449	0.0008	0.0039	0.0039	0.0000
3rd	$u_{1,1,3}$	0.4956	0.4985	0.0029	0.3424	0.3387	0.0037	0.0946	0.0952	0.0006	0.0010	0.0012	0.0002
	$u_{1,2,3}$	0.5133	0.5181	0.0048	0.4341	0.4321	0.0020	0.1500	0.1516	0.0016	0.0017	0.0016	0.0001
	$u_{2,3,1}$	0.8883	0.8919	0.0036	0.6741	0.6667	0.0074	0.2100	0.2107	0.0007	0.0023	0.0020	0.0003
	$u_{2,2,3}$	0.9911	0.9972	0.0061	0.8965	0.8868	0.0097	0.3424	0.3446	0.0022	00039	0.0043	0.0004

From table (8), we can clearly observe that when decreasing the space discretization steps $(\Delta x, \Delta y \to 0)$ with fixed time discretization step (Δt), the numerical solutions of 2-D wave equations which are obtained by the explicit method are more smoothly approaches to the exact solutions as the result truncation errors in each entry decreases and approaches to zero. So from this we can conclude that decreasing the space step sizes with fixed time step size gives stable and consistent numerical values which are smoothly converges to the exact solution.

4.17 DISCUSSION ON THE RESULTS

This thesis devoted to numerical solution methods and their applications for some wave propagation namely one and two dimensional wave equations which are subjected to the initial and boundary conditions.

In the first part of chapter 4, derivation of 1-D and 2-D wave equations were considered. Tayler series is also considered to obtain 1st and 2nd order forward, backward and central difference approximations. Explicit and Implicit difference methods were applied to compute the numerical solutions at each grid points for 1-D and 2-D wave equations. In 4.15 part of chapter 4, the focus was carrying out numerical experiment of one dimensional wave equation based on four different trial data by using Explicit and Implicit difference methods. The result obtained from the experiment as indicated on tables (2) and (3), when reducing the time discretization ($\Delta t \rightarrow 0$) with fixed space time discretization (Δx) the explicit numerical solution was more smoothly converges to the exact solution for that of the counterpart of implicit solution.

Similarly in 4.16 part of chapter 4, numerical experiment was done on 2-D wave equation with four different trial data using explicit difference method and the result obtained from the experiment as indicated on table (8) reducing the space discretization step ($\Delta x, \Delta y \longrightarrow 0$) with fixed time discretization step (Δt) the numerical solution obtained from explicit method smoothly converges to the exact solution.

CHAPTER FIVE: CONCLSION AND RECOMMENDATION

5.1 CONCLUSION

It is evident that many physical phenomena can be modeled using partial differential equations and can be solved by analytic methods. But in other cases, some partial differential equations are complicated to solve analytically and can be solved by using advanced numerical methods such as explicit and implicit finite difference methods to get valuable information and deep understanding about physical phenomena which are associated to partial differential equations. In this study we observed that the applications of FDMS are limited to the cases of one and two dimensional wave equations where the PDE'S under consideration are well behaved. To get the approximate solution we apply the two schemes by discretizing the domain between the boundary conditions as to obtain the nodal values.

In this particular study, explicit and implicit schemes were applied to one and two dimensional wave equations. We made numerical computation and compared the results from analytical point of view that the two methods worked well accordingly. Each scheme helped us to produce reasonable approximate result of each $u_{i,j}$ to form systems difference equations which are solved by tri-diagonal matrix .The finite difference method has its own advantages and disadvantages from tables and figures, it is clear that among the numerical methods used to solve partial differential equations for one and two dimensional wave equations, explicit method provides better accuracy, and also the error generated from explicit method is smaller than that of its counter parts. The overall performance of the explicit difference method is quite good for stable values for $r \leq 1$, fixed Δx and small Δt for 1-D wave equation and $\beta, \gamma \leq \frac{1}{\sqrt{2}} \cong 0.7071$ for 2-D wave equations for small $\Delta x, \Delta y$ and fixed Δt , with second order accuracy.

The stability problem could be improved by going to an implicit difference method, but often implicit schemes tend to be too complicated to obtain numerical solution of 2-D wave equation. Explicit method is very easy to set up, to calculate numerically and is conditionally stable. Implicit method are unconditionally stable, computer time required at each step is higher.

Depending on the results obtained from test examples this study shows that:

1. Reducing the time step size (Δt) with fixed space step size(Δx) , the explicit method gives stable and consistent numerical solution which is more smoothly convergent to the analytical solution than the implicit method for 1-D wave equation.

2. Reducing the space step sizes (Δx and Δy) with fixed space step size (Δt) explicit method gives stable and consistent numerical solution which is smoothly convergent to the analytical solutions for 2-D wave equation.

5.2 RECOMMENDATION

This study can be extended to three dimensional wave equations in the future works.

CHAPTER SIX

6. References

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CHAPTER SEVEN

7. Appendix

A complete script file (program) that creates grid and then makes a meshe or surface plot of the FDM solution for 1-D wave equation is given as follows:

For figure (1)

```
≫ x=0:0.2:1; t=0:0.1:1; [X,T] = meshgrid(x,t);
≫ U = 0.75 .* sin (pi * X).*cos (pi * T) – 0.25.* sin(3*pi*X).*cos(3*pi*T) ;
≫ xlable(‘x’) ;  ylable(‘t’) ;  zlable(‘u’) ;
≫ mesh(X,T,U)
≫ surf (X,T,U)
```

For figure (2)

```
≫ x=0:0.2:1;t=0:0.05:1;[X,T] = meshgrid(x,t);
≫ U = 0.75 .* sin (pi * X).*cos (pi * T) – 0.25.* sin(3*pi*X).*cos(3*pi*T) ;
≫ xlable(‘x’) ;  ylable(‘t’) ;  zlable(‘u’) ;
≫ mesh(X,T,U)
≫ surf (X,T,U)
```

For figure (3)

```
≪syms  x  X  t  T  U  w  r
≪ x = 0: 0.2: 1; t= 0: 0.1: 1; [X,T] = meshgrid(x,t) ;
≪ U = 0.75.*sin (pi*X).*cos (pi*T)−0.25.*sin (3*pi*X).*cos (3*pi*T); U
≪ U = 0.75.*sin (pi*w).*cos (pi*r)−0.25.*sin(3*pi*w).*cos(3*pi*r);
≪ ezsur(U)
```

For figure (4)

```
≪syms  x  X  t  T  U  w  r
≪ x = 0: 0.2: 1; t= 0: 0.1: 1; [X,T] = meshgrid(x,t) ;
≪ U = 0.75.*sin (pi*X).*cos (pi*T)−0.25.*sin (3*pi*X).*cos (3*pi*T); U
≪ U = 0.75.*sin (pi*w).*cos (pi*r)−0.25.*sin(3*pi*w).*cos(3*pi*r);
≪ ezsur(U)
```