ENHANCED STATIONARY ENTANGLEMENT INDUCED BY OPTICAL PARAMETRIC AMPLIFIER IN OPTOMECHANICAL SYSTEM

Firomsa Feyissa Hordofa

A Thesis Submitted to Department of Applied Physics
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Quantum Optics

Office of Graduate Studies

Adama Science and Technology University

September 1, 2021
Adama, Ethiopia

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Declaration

I hereby declare that this Master Thesis entitled “Enhanced Stationary Entanglement Induced by Optical Parametric Amplifier in Optomechanical System”is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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Name of the Student	Signature	Date

Recommendation

I, the advisor of this thesis, hereby certify that I have read the revised version of the thesis entitled “Enhanced Stationary Entanglement Induced by Optical Parametric Amplifier in Optomechanical System”prepared under my guidance by Firomsa Feyissa submitted in Partial Fulfillment of the Requirement for the Degree of Master's Quantum Optics. Therefore, I recommend the submission of revised version of the thesis to the department following the applicable procedures.

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Name of Principal Advisor	Signature	Date

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Finally, I must express my very profound gratitude to my family for providing me with unfailing support and continuous encouragement through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

Thank you.

List of Abbreviations

CM Correlation Matrix.

CV Continuous Variable.

OPA Optical Parametric Amplifier.

OPO Optical Parametric Oscillator.

PPT Positive Partial Transpose.

QLE Quantum Langevin Equation.

List of Figures

Abstract

In this thesis, we theoretically studied the enhanced stationary entanglement induced by optical parametric amplifiers in optomechanical system. To this aim, we constructed the model and Hamiltonian of system. While, the system consists of one fixed mirror and one movable mirror and a degenerate parametric amplifier placed between them. Thus, the cavity mode is driven by a laser field and degenerate parametric amplifier is pumped by another laser at a frequency of ω_L*. Accordingly, the dynamics of the system can be obtained by using the nonlinear quantum Langevin equations and linearization approximation. Under the linearization approximation, the bipartite entanglement is quantified through logarithmic negativity. Accordingly, our results show that the introduction of optical parametric amplifier makes the entanglement more robust. Interestingly, we show that the effects gain, mass and temperature have great contribution for the enhancement of the stationary continuous-variable entanglement. Furthermore, under the resolved sideband regime, moderate optimal nonlinear gain and cryogenic temperatures the entanglement enhances as compared with system without optical parametric amplifier. Thus, when an optical parametric amplifier is added inside a cavity, the optomechanical coupling and the bipartite entanglement is improved. Such entanglements results may have spectacular contribution for constructing long-distance quantum communication network and technologies.*

Contents

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INTRODUCTION

1.1 Background of the Study

For the last two decade, there has been outstanding growth in the field of quantum optics, which deals with the interactions between matter and the radiation field where quantum effects are very important (Kassahun, 2008). Much of the fundamental interest in quantum optics is connected with its implications for the conceptual foundations of quantum mechanics. However, the major quantum optics problem is whether we have to quantize the electromagnetic field to obtain the correct picture of the interaction between matter and the field (Ficek and Wahiddin, 2014).

Recently, one of the young growing research area of quantum optics is quantum optomechanics, that promised new technology and advanced human civilization. Optomechanics as its name indicates contains two words “opto” and “mechanics”, opto refers to light (in the more general electromagnetic field) and mechanics deals with the motion of a massive object (Aspelmeyer et al., 2010). Thus, it deals with coupling electromagnetic field with mechanical motion through radiation pressure force, which is a scattering force that arises due to the reflection of light, which of course, has some momentum associated with it. An optomechanical system is a system that includes coupling between an optical and mechanical element. Therefore, the optomechanical system becomes an ideal candidate for studying the quantum behavior of macroscopic objects (Tesfahannes, 2021). A simple example some-times referred to as cavity-optomechanics of Fabry-Pérot cavity consists of one mirror which is fixed while the other one is free to move in a harmonic potential. The cavity modes of frequency are driven with a laser and the light inside the cavity will push the mirror due to radiation pressure (Aspelmeyer et al., 2014). Putting the mechanical oscillators into the quantum regime could allow us to explore quantum phenomena/properties in entirely new ways such away creation of entanglement between cavity field and mechanical motion (Tesfahannes, 2020), and control and manipulation of the quantum states for the optomechanical system (Altintas and Eryigit, 2013). Quantum entanglement is one of the most striking features of quantum mechanics and allows for correlations between two or more objects to be much stronger than is allowed classically. Though this behavior troubled the founders of quantum mechanics (Einstein et al., 1935), entanglement is now very well established and is viewed as a a powerful resource for quantum technologies, such as quantum metrology (Giovannetti et al., 2011), quantum decoding (Kachigar and Tillich, 2017), quantum error correction (Terhal, 2015), quantum cryptography (Yin et al., 2020), quantum communication (Gisin et al., 2002) and for tests of fundamental physics (Bell, 1964).

Recently, quantum entanglement is being actively studied both theoretically (Zou, 2021) and experimentally (Wagenknecht et al., 2010). Notably, the first demonstration of entanglement between spatially distinct mechanical oscillators have been achieved using trapped ions (Jost et al., 2009), which reported the observation of non-classical correlations between the motion of two separated pairs of ions. One route to study motional entanglement for more massive systems is via cavity quantum optomechanics, which utilizes radiation-pressure and other optical forces, such as electrostriction, to generate and study non-classical motional states of mechanical oscillators from the zeptogram to kilogram scale (Bougouffa and Al-Hmoud, 2020).

The most important quantum state of light which enhances entanglement quantization of an optomechanical system is squeezed states of light. Squeezed light can be generated from light in a coherent state by means of certain nonlinear optical interactions. One of the quantum optical processes which can generate squeezed light is a degenerate parametric oscillator (Wang et al., 2021).

A degenerate parametric oscillator has been considered as a typical source of squeezed light. It is one of the most interesting and well-characterized optical devices in quantum optics. In this device, a pump photon interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons. If these photons have the same frequency the device is called a degenerate parametric oscillator, otherwise it is called a non-degenerate parametric oscillator. However, in this thesis, we consider only a degenerate parametric oscillator (Schnabel, 2017).

1.2 Statement of the Problem

Both the theoretical and experimental studies of entanglement have been achieved for the past decade. One route to study entanglement for more massive systems is via cavity quantum optomechanics, which utilizes radiation-pressure and other optical forces. The simple optomechanics consists of two mirrors, where one is fixed and the other is movable under harmonic oscillator. The entanglement for such system have been achieved. Moreover, the enhancement of entanglement was achieved by using optical parametric amplifier. Currently, significant progress has been reported on the study of the bipartite entanglement between two optical modes and mechanical mode by placing optical parametric amplifier between them. Placing optical parametric amplifier between cavity mode and mechanical mode enhances the formation of stationary entanglement. The nonlinear crystal placed between cavity mode and mechanical mode induced by optical parametric amplifier generates squeezed light to enhance the entanglement.

Therefore, in this thesis we study the bipartite entanglement dynamics in an optomechanical system with OPA to generate the stationary continuous-variable entanglement. Particularly, we study on the enhancement of stationary entanglement induced by optical parametric amplifier in optomechanical system and quantify their entanglement by using logarithmic negativity.

1.3 Objectives of the Study

1.3.1 General Objective of the Study

The general objective of this thesis is to enhance the stationary bipartite entanglement induced by optical parametric amplifier in an optomechanical system.

1.3.2 Specific Objectives of the Study

The specific objectives of this thesis are:

- to construct the model for the system
- to determine the quantum Langevin equations of the system
- to extract a drift matrix
- to quantify the entanglement by using a method of logarithmic negativity.

1.4 Significance of the Study

This study has great importance to achieve quantum control of mechanical resonators in quantum information processing and for the construction of long-distance quantum communication network and technologies. This can be achieved if we could transfer state from one location to another by means of quantum entanglement. So, the results of this research will contribute for these applications that are used for technological developments. In addition to that, the results of this research will be used as input for other researchers or it will be served as a platform for other scientific researches.

1.5 Outline of the Thesis

In general, this thesis consists of five chapters. Each chapter will discuss on different issues related to the thesis. The first chapter describes an overview of the thesis background, statement of the problem, objective of the thesis and the significance of the study. Chapter two focuses on the review from previous studies that was made helpful to gain knowledge and to understand the thesis better and discusses about the optomechanics, a degenerate parametric oscillator, quantum Langevin equations and logarithmic negativity. Chapter three gives detail description about the model Hamiltonian and dynamics of the system. Chapter four presents result and discussion. Finally, chapter five provides conclusion and recommendations of the thesis.

2

LITERATURE REVIEW

2.1 Overview of the Study

In this chapter, we describe a cavity optomechanical system, a degenerate parametric oscillator which is used to generate squeezed light, Hamiltonian formulation of optomechanical system, the Heisenberg quantum Langevin equation to determine the system dynamics and logarithmic negativity which is used as a tool to quantify the generated bipartite entanglement of an optomechanical system with OPA.

2.2 Cavity Optomechanical System

Whether in classical or in quantum physics, the harmonic oscillator is the simplest known example of a study. It's mechanical performance, namely, mechanical oscillator, is probably the most concrete and intuitive implementation subject. In the past few decades, the mechanical oscillator has been investigated with different science and technologies (Aspelmeyer et al., 2014). Its applications range from atomic force microscope cantilever for micro size to nano-mechanical oscillator mirror interferometer for the detection of gravitational waves, displacement etc. (Meystre, 2013; Kippenberg and Vahala, 2008). The quantum control of the mechanical motion is state of the art, which spans the size range from hundreds of nanometers in the case of nano-electro-mechanical or nano-opto-mechanical systems (NEMS/NOMS) to tens of centimeters in the case of gravitational wave antennae (Gebremariam et al., 2019).

The pioneering theoretical work of cavity optomechanical system was performed by Braginsky (Caves, 1979), Caves and others. The field has undergone rapid development over the last decade with the separate development of new methods for fabricating small bulk mechanical resonators of various forms; nano-scale beams coupled to microwave cavities (Milburn, 2011), photonic-phononic crystals (Merklein et al., 2017), toroidal optical microresonators (Heylman et al., 2016), doubly clamped beams with integrated mirrors (Pruessner et al., 2008) and drumhead capacitors in superconducting microwave resonators (Milburn, 2011).

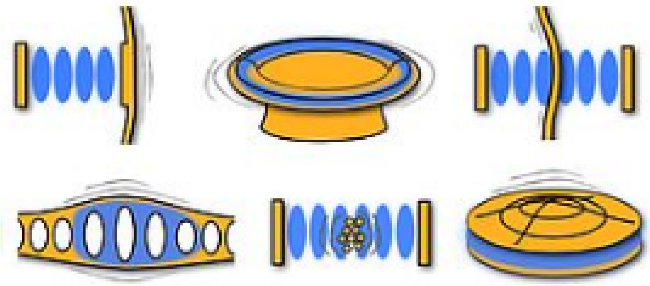

Fig. 2.1: Optomechanical Setups (schematic), from top left: Fabry-Perot cavity with a moveable end-mirror, microtoroid, membrane or nanotube inside a cavity, photonic crystal, atoms trapped inside a cavity, vibrating capacitor plates in a microwave LC circuit (Marquardt and Girvin, 2009).

In optomechanical systems, an optical field is used for the measurement and control of the mechanical resonator. The optical field is usually confined within a cavity, providing resonant enhancement of the field strength and sensitivity to mechanical displacements. The study of mechanical systems near the quantum limit encompasses systems having a wide range of sizes and frequencies. In order to study these mechanical resonators, some auxiliary system is required to measure and control them (Gröblacher, 2012). This auxiliary system may take the form of an electrical circuit (quantum electromechanics), an optical cavity (quantum optomechanics), or even an atomic system.

As mentioned above optomechanics is one of the emerging fields exploring the interaction between light and mechanical motion. Such interaction originates from the mechanical effect of light, i.e., optical force. Radiation pressure force (or scattering force) and optical gradient force (or dipole force) are two typical categories of optical forces. The radiation pressure force which originates from the fact that light carries momentum plays an astonishing role in optomechanics (Ulbricht, 2021).

2.3 Radiation Pressure in Quantum Optomechanical System

In 1619 Johannes Kepler postulates the existence of a force that pushes away from the sun the tail of comets. It states that the direct rays of the sun strike upon the comet, penetrate its substance, draw away with them a portion of this matter, and issue thence to form the track of light we call the tails. In the two tails formed, the dust part is called radiation pressure and the ions part is known as solar wind (Nichols and Hull, 1901).

Moreover, radiation pressure is the mechanical pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field. These incudes the momentum of light or electromagnetic radiation of any wavelength which is absorbed, reflected or otherwise emitted by matter on any scale. The radiation pressure force originates from the fact that light carries momentum. The momentum transfer from light to a mechanical object exerts a pressure force on the object.

According to the quantum theory of light, every photon with wavenumber, k carries a momentum $p = \hbar k$, where $\hbar$ is the Planck constant. This means that a photon reflected off a mirror surface transfers a momentum $\Delta p = 2\hbar k$ onto the mirror due to the conservation of momentum. This effect is extremely small and cannot be observed on most everyday objects; it becomes more significant when the mass of the mirror is very small and /or the number of photons is very large (i.e. high intensity of the light) (Metzger and Karrai, 2004).

Since the momentum of photons is extremely small and not enough to change the position of a suspended mirror significantly, the interaction needs to be enhanced.

One possible way to do this is by using optical cavities. If a photon is enclosed between two mirrors, where one is the oscillator and the other is a heavy fixed one, it will bounce off the mirrors many times and transfer its momentum every time it hits the mirrors.

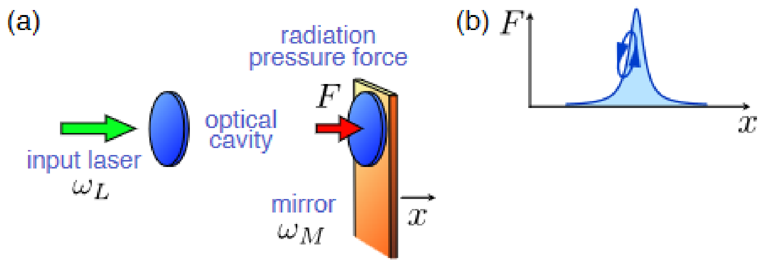

Fig. 2.2: (a) Schematic of basic optomechanical setup (b) The plot of radiation pressure force versus the position (Marquardt and Girvin, 2009).

In the number of times of oscillation, a photon can transfer its momentum is directly related to the finesse of the cavity, which can be improved with highly reflective mirror surfaces. The radiation pressure of the photons do not simply shift the suspended mirror further and further away as the effect on the cavity light field must be taken into account: if the mirror is displaced, the cavity's length changes, which also alters the cavity resonance frequency. Therefore, the detuning, which determines the light amplitude inside the cavity, between the changed cavity and the unchanged laser driving frequency is modified. It determines the light amplitude inside the cavity at smaller levels of detuning more light actually enters the cavity because it is closer to the cavity resonance frequency (Aspelmeyer et al., 2014).

Since the light amplitude, i.e. the number of photons inside the cavity, causes the radiation pressure force and consequently the displacement of the mirror, the loop is closed: the radiation pressure force effectively depends on the mirror position. Another advantage of optical cavities is that the modulation of the cavity length through an oscillating mirror can directly be seen in the spectrum of the cavity (Metzger et al., 2007).

2.4 Degenerate Parametric Oscillator

A degenerate parametric oscillator has been considered as a typical source of squeezed light. It is one of the most interesting and well-characterized optical devices in quantum optics. In this device, a pump photon interacts with a nonlinear crystal inside a

Fig. 2.3: A degenerate parametric amplifier with nonlinear crystal pumped by coherent light (Getahun, 2019).

cavity and is down-converted into two highly correlated photons (Kassahun, 2014). If these photons have the same frequency the device is called a degenerate parametric oscillator, otherwise, it is called a non-degenerate parametric oscillator (Kassahun, 2008).

Physically, a single-photon state in the squeezed cavity mode corresponds to an exponentially growing number of photons in the original cavity mode as a function of increasing squeezing strength. Consequently, the optomechanical interaction in units of the squeezed-cavity-mode photons can be enhanced, e.g., into the single-photon strong-coupling regime by tuning the intensity (or frequency) of the driving field that induced the squeezing (Lü et al., 2015).

In a degenerate parametric oscillator system, a pump photon of frequency 2ω is down-converted into a pair of signal photons each of frequency ω. In Fig. (2.3) we can see that how to pump the OPA to produce a signal mode that generates squeezed light. This generated squeezed light is useful to entangle the state of the system.

Squeezed light was first produced in 1985 by Slusher et al. using four-wave- mixing in sodium atoms in an optical cavity (Slusher et al., 1985). Shortly after, squeezed light also was generated by four-wave-mixing in optical fibre and by degenerate parametric down-conversion (PDC) in a 2^{nd}-order nonlinear crystal placed in an optical cavity (Wu et al., 1986). The pumped cavity was operated below its oscillation threshold, i.e. the parametric gain did not fully compensate the round trip losses, which is also called 'cavity-enhanced optical-parametric amplification (OPA)'.

Placing the nonlinear crystal inside a cavity can greatly enhance the down-conversion efficiency, but not only that. A cavity introduces a threshold for the pump power above which the parametric gain is infinite, just limited by the finite pump power. In this case, the vacuum uncertainty of the input field at frequency ω is amplified to a bright laser field at frequency ω. The device is then called an optical-parametric oscillator (OPO) (Dunn and Ebrahim-Zadeh, 2000). For the generation of squeezed states, however, the pump power is usually kept (slightly) below threshold. Due to nonzero optical loss, there exists a pump power smaller than the threshold above which the tiny improvement of squeezing is not noticeable anymore. The optical cavity that is built around the nonlinear crystal is vital for squeezed-light generation. Generally, the mode propagating away from a cavity is the result of interference at the cavity coupling mirror.

2.5 Basic Tools and Techniques of Optomechanics

2.5.1 Hamiltonian Formulation of Harmonic Oscillator

The Hamiltonian of the quantum system plays a decisive role in the property-ascription rule that selects the definite-valued observables whose values become actual (Kong et al., 2017). It is the sum of the kinetic energies of all particles, plus the potential energy of the particles associated with the system.

The Hamiltonian takes different forms and can be simplified in some cases by taking into account the concrete characteristics of the system under analysis, such as single or several particles in the system, the interaction between particles, time-varying potential or time-independent one.

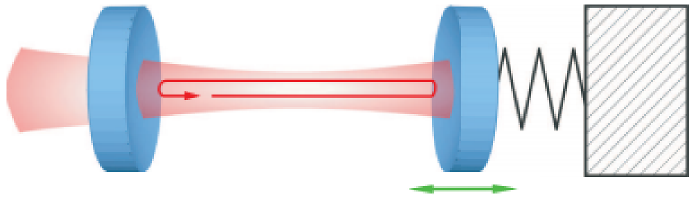

Fig. 2.4: Typical shape of cavity optomechanical system consisting of a Fabry Perot cavity with a harmonically bound end mirror (Kippenberg and Vahala, 2007).

The starting point of all our subsequent discussions will be the Hamiltonian describing the coupled system of a radiation mode interacting with a vibrational mode Fig. (2.4). The system with one movable mirror provides an additional mechanical degree of freedom. This system can be mathematically described by a single optical cavity mode coupled to a single mechanical mode. The coupling originates from the radiation pressure of the light field that eventually moves the mirror, which changes the cavity length and resonance frequency. The optical mode is driven by an external laser.

Starting from the derivation of quantum harmonic oscillator (Deniz, 2019) we can get standard Hamiltonian of optomechanical system as starting point for our system. Then, the free Hamiltonian (the sum of the kinetic energies of all particles, plus the potential energy of the particles associated with the system) is

$$\hat{H} = \hat{P}_E + \hat{K}_E \tag{2.1}$$

where $\hat{P}_E = \frac{1}{2}kx^2$ is potential energy and $\hat{K}_E = \frac{1}{2}mv^2$ is kinetic energy.
Here $\omega = \sqrt{\frac{k}{m}}$ then, $k = \omega^2 m$ and from $\hat{p} = m\hat{v}$, $\hat{v} = \frac{\hat{p}}{m}$, where k is force constant, v is

speed of a particle, $\hat{x}$ is the position (displacement) operator of particle and $\hat{p}$ is the momentum operator of the particle. Then, the Hamiltonian of Eq. (2.1) becomes

$$\hat{H} = \frac{1}{2m}\big[m^2\omega^2\hat{x}^2 + \hat{p}^2\big]. \tag{2.2}$$

Now, let by trick

$$\begin{aligned}(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) &= m^2\omega^2\hat{x}^2 + im\omega\hat{x}\hat{p} - im\omega\hat{p}\hat{x} + \hat{p}^2\\ &= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw(\hat{x}\hat{p} - \hat{p}\hat{x})\\ &= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw[\hat{x}, \hat{p}]\\ &= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw(i\hbar)\\ &= m^2\omega^2\hat{x}^2 + \hat{p}^2 - mw\hbar\end{aligned}$$

$$\Rightarrow m^2\omega^2\hat{x}^2 + \hat{p}^2 = (mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) + m\omega\hbar. \tag{2.3}$$

Inserting Eq. (2.3) into Eq. (2.2) we get

$$\begin{aligned}\hat{H} &= \frac{1}{2m}\Bigg[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) + m\omega\hbar\Bigg]\\ &= \frac{1}{2m}\Big[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\Bigg] + \frac{\omega\hbar}{2}\\ &= \frac{\omega\hbar}{\omega\hbar}\frac{1}{2m}\Bigg[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\Bigg] + \frac{\omega\hbar}{2}\\ &= \frac{\omega\hbar}{2m\omega\hbar}\Bigg[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\Bigg] + \frac{\omega\hbar}{2}\\ &= \omega\hbar\Bigg[(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}})(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}})\Bigg] + \frac{\omega\hbar}{2}\end{aligned}$$

Therefore,

$$\hat{H} = \omega\hbar\Bigg[(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}})(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}}) + \frac{1}{2}\Bigg]. \tag{2.4}$$

From Eq. (2.4) the terms

$$(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}}) = \hat{a}^\dagger,$$

which is called creation operator and

$$(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}}) = \hat{a}$$

is known as annihilation operator. Then, the harmonic Hamiltonian is written as

$$\hat{H} = \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}\right]\omega\hbar. \tag{2.5}$$

The corresponding commutation relations are $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$. The offset term, in Eq. (2.5) corresponding to the zero-point energy, makes no contribution to the Hamiltonian dynamics, and hence is often omitted. We have neglected the $\frac{1}{2}\omega\hbar$, since this will not affect our results. Thus, the Hamiltonian of optical field reads as (Law, 1995)

$$\hat{H}_{om} = \hbar\omega_c(x)\hat{a}^\dagger\hat{a} \tag{2.6}$$

where ω_c is the frequency of the optical mode and x is the position of the mechanical resonator.

In the same manner we can determine the Hamiltonian of mechanical resonator as

$$\hat{H}_{mech} = \hbar\omega_m\hat{b}^\dagger\hat{b}, \tag{2.7}$$

where $\hat{b}$ and $\hat{b}^\dagger$ are annihilation and creation operator of mechanical resonators respectively and ω_m is mechanical mode frequency. They are defined as

$$\hat{b} = \left(\frac{m\omega_m\hat{x} + i\hat{p}}{\sqrt{2m\omega_m\hbar}}\right) \tag{2.8}$$

and

$$\hat{b}^\dagger = \left(\frac{m\omega_m\hat{x} - i\hat{p}}{\sqrt{2m\omega_m\hbar}}\right) \tag{2.9}$$

and the corresponding commutation relations are $[\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0$.

Meanwhile, we need to see how the two systems communicate to one another. Because the optical cavity and mechanical oscillator are coupled by radiation pressure and by the mechanical oscillator physically changing the length of the optical cavity. We note that $\omega_c = \frac{2\pi nc}{L}$ where $n = 1, 2, 3, ...$ and when the mirror moves by a distance x, the length of the optical cavity increases by $L + x$. Then,

$$\omega_c(x) = \frac{2\pi nc}{L + x}. \tag{2.10}$$

By Taylor expansion Eq. (2.10) becomes

$$\begin{aligned}\omega_c(x) &= 2\pi nc\left[\frac{1}{L(1+\frac{x}{L})}\right]\\ &= \frac{2\pi nc}{L}\left[1-\frac{x}{L}+\frac{x^2}{L^2}-\frac{x^3}{L^3}+...\right].\end{aligned}$$

Since $x \ll L$, by taking the first order and rejecting the other terms we get

$$\omega_c(x) = \frac{2\pi nc}{L}\left[1-\frac{x}{L}\right] \tag{2.11}$$

$$\omega_c(x) = \omega_c - \frac{\omega_c x}{L}. \tag{2.12}$$

After solving Eq. (2.8) and Eq. (2.9) simultaneously we get

$$\hat{x} = \frac{\hat{b}+\hat{b}^\dagger}{\sqrt{2\hbar/m\omega_m}}. \tag{2.13}$$

Then, by replacing x by operator $\hat{x}$ and inserting Eq. (2.13) into Eq. (2.12) we get

$$\omega_c(x) = \omega_c - \frac{\omega_c}{\sqrt{2}L}x_{zpt}(\hat{b}+\hat{b}^\dagger) \tag{2.14}$$

where $x_{zpt} = \sqrt{\frac{\hbar}{m\omega_m}}$ is the size of mechanical zero point fluctuation or width of the mechanical ground state wave function.

From Eq. (2.14) let's define the term $g = \frac{\omega_c}{\sqrt{2}L}x_{zpt}$ which is called single photon optomechanical coupling interaction (strength). It determines the amount of cavity resonance frequency shift if the mechanical oscillator is displaced by zero point uncertainty. Then, Eq. (2.14) becomes

$$\omega_c(x) = \omega_c - g(\hat{b}+\hat{b}^\dagger). \tag{2.15}$$

Finally, inserting Eq. (2.15) into Eq. (2.6) we can extract the optomechanical interaction Hamiltonian as follows,

$$\hat{H}_{opt} = \hbar\left[\omega_c - g(\hat{b}+\hat{b}^\dagger)\right]\hat{a}^\dagger\hat{a} \tag{2.16}$$

$$\hat{H}_{opt} = \hbar\omega_c \hat{a}^\dagger \hat{a} - \hbar g(\hat{b} + \hat{b}^\dagger)\hat{a}^\dagger \hat{a}. \tag{2.17}$$

Then, from Eq. (2.7) and Eq. (2.17) we get

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}^\dagger \hat{b} - \hbar g(\hat{b} + \hat{b}^\dagger)\hat{a}^\dagger \hat{a}. \tag{2.18}$$

Equation (2.18) serve as standard optomechanical Hamiltonian of the optical and mechanical mode with the optomechanical interaction between the optical mode and the mechanical mode with out driving field for any optomechanical systems.

2.5.2 Quantum Langevin Equations

A Langevin equation is a stochastic differential equation describing the time evolution of a subset of degrees of freedom. These degrees of freedom typically are collective (macroscopic) variables changing only in comparison to the other (microscopic) variables of the system. The fast (microscopic) variables are responsible for the stochastic nature of the Langevin equation (Ford and Kac, 1987).

The quantum Langevin equations are used to describe the full dynamics of the system including the fluctuation and the dissipation term affecting the optomechanical system interact with the environment. It interacts in the external environment have corresponding decay or damping rate β and input noise $\hat{z}_{in}$ respectively. The general form of quantum Langevin equation in the optomechanics for any operator $\hat{z}$ can be written as

$$\frac{d\hat{z}}{dt} = \frac{-i}{\hbar}[\hat{z}, \hat{H}] - \beta\hat{z} + \sqrt{2\beta}\hat{z}_{in} \tag{2.19}$$

where $\hat{H}$ is the total Hamiltonian of the system.

2.5.3 The Linearization Approximation Approach

The quantum dynamics of the full non-linear system are difficult to analyze. So we linearize the dynamics of the quantum fluctuations around the semi-classical fixed points where the cavity is intensely driven under a strong field. We expand each Heisenberg operators as sums of its semi-classical steady-state value plus quantum fluctuation for any relevant operators (Tesfahannes and Alemu, 2021).

The general form of linearization approach for any operator $\hat{o}$ given by (Aspelmeyer et al., 2014)

$$\hat{o} = \hat{O}_s + \delta\hat{o} \tag{2.20}$$

where $\hat{o}$ is an operator $\hat{O}_s$ is a real numbers denoting the optical and mechanical amplitudes and $\delta\hat{o}$ is the quantum fluctuation operator.

2.5.4 Logarithmic Negativity

The logarithmic negativity is an entanglement measure which is easily computable and an upper bound to the distillable entanglement (Plenio, 2005).
It is defined as

$$E_N = max[0, -ln2\bar{\eta}], \tag{2.21}$$

where $\bar{\eta}$ is the minimum symplectic eigenvalue of partial transposed cavity mode corresponding to the bipartite system of interest which is given by

$$\bar{\eta} = \sqrt{\frac{[\sum(V) - \sqrt{\sum(V)^2 - 4detV}]}{2}} \tag{2.22}$$

where F can be expressed in a form of block matrix as

$$V = \begin{pmatrix} F_A & F_C \\ F_C^T & F_B \end{pmatrix} \tag{2.23}$$

and $\sum(V) = detF_A + detF_B - 2detF_C$ where F_A, F_C and F_B are 2×2 are matrices extracted from the correlation matrix. E_N is entangled if $detF_C < 0$ and $\bar{\eta} \leq 1$. Based on those frame work we can quantify the bipartite entanglement dynamics behavior of an optomechanical system with OPA.

3

METHODOLOGY

3.1 Overview of the Study

In this chapter, we investigate theoretically bipartite entanglement of an optomechanical system with OPA. The system is driven by a laser field and the degenerate OPA is pumped by another one laser at a frequency of ω_L. The dynamics of the system in terms of quantum Langevin equation for relevant operators are studied and using logarithmic negativity to quantify the generated and enhanced entanglement of an optomechanical system with OPA.

3.2 Theoretical Model and Hamiltonian Formulation

In this thesis, we quantify the bipartite entanglement of an optomechanical system with OPA. Naturally mirrors are not perfect so that some tunneling between the cavity mode and the continuum of modes outside the cavity is always present. As the cavity mirrors are not perfect the intra-cavity field can permeate through the dielectric material and tunnel to the outside. At the same time, the outside field can also tunnel inside the cavity. If this were not the case then the cavity would be useless as no driving would be possible. By this principle, our system consists of a partially reflected mirror (fixed mirror) and a fully reflected mirror (movable mirror), which is in harmonic oscillator and a degenerate parametric amplifier placed between them and the cavity mode is driven by a laser field and the degenerate parametric amplifier is pumped by another laser at a frequency of ω_L.

Therefore, in this system we have cavity mode, mechanical mode, the interaction between cavity mode and mechanical mode, the interaction between the driving field and the cavity mode, and finally parametric interaction (optomechanical system with OPA).

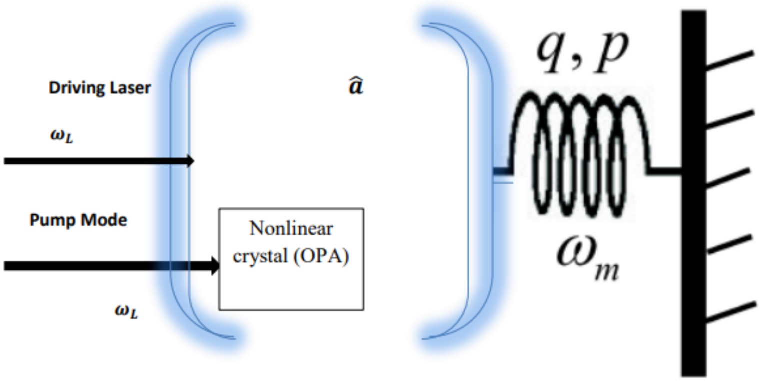

Fig. 3.1: Schematic model of an optomechanical system that consist of two mirrors and OPA inside the cavity to squeeze the cavity field by optical parametric process.

The total Hamiltonian of the whole system is written as (Huang and Agarwal, 2009)

$$\begin{aligned}\hat{H} =& \hbar\omega_c \hat{a}^\dagger \hat{a} + \frac{1}{2}\left[\frac{p^2}{m} + m\omega_m^2 q^2\right] - \hbar g a^\dagger \hat{a} q + i\hbar E\left[\hat{a}^\dagger e^{-i\omega_L t} - \hat{a} e^{i\omega_L t}\right] \\ &+ i\hbar G\left[e^{i\theta}\hat{a}^{\dagger 2} e^{-i\omega_L t} - e^{-i\theta}\hat{a}^2 e^{i\omega_L t}\right].\end{aligned} \tag{3.1}$$

Where ω_c resonance frequency, ω_L is driving and pump mode frequency, ω_m is mechanical frequency, q and p are dimensionless position and momentum of a mechanical resonator respectively, $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operator of cavity mode respectively, g is the single optomechanical coupling strength between the cavity field and mechanical mirror, E is amplitude of the field driving laser and the cavity field (input field intensity related to the input power $E = \sqrt{\frac{2\kappa P}{\hbar\omega_L}}$, where $\kappa = \frac{\pi c}{2FL}$ is optical decay rate with F is the cavity finesse and L is cavity length), G is a nonlinear gain of the OPA, which is proportional to the power of the pump mode and θ is the phase of the optical field driving the OPA.

The first term of Eq. (3.1) is cavity mode energy. The second term is the energy of movable mirror (cantilever), the third term is the interaction energy between the cavity field and mechanical mirror, the fourth term is the coupling energy between the driving field and the cavity mode, and the last term is parametric interaction energy (an optomechanical system with OPA).

3.3 Dynamics of the System

The dynamics of the optomechanics with OPA system can be described by a set of nonlinear quantum Langevin equations in which the dissipation and fluctuation terms are added to the Heisenberg equations of motion as shown in Eq. (2.19). To obtain the non-linear QLE for our system operators $\hat{q}$, $\hat{p}$ and $\hat{a}$, let's start from Eq. (2.8) and Eq.(2.9). In order to have phonon operators in the form of dimensionless position and momentum operators let's apply Eq. (2.8) and Eq. (2.9),

$$\hat{b}^{\dagger}\hat{b} = (\frac{m\omega_m\hat{x} - i\hat{p}}{\sqrt{2m\omega_m\hbar}})(\frac{m\omega_m\hat{x} + i\hat{p}}{\sqrt{2m\omega_m\hbar}}). \tag{3.2}$$

It then follows that

$$\hat{b}^{\dagger}\hat{b} = \frac{(m\omega_m\hat{x})^2 + im\omega_m(\hat{x}\hat{p} - \hat{p}\hat{x}) + \hat{p}^2}{2m\omega_m\hbar}. \tag{3.3}$$

From position and momentum commutation relation we do have

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{3.4}$$

After inserting Eq. (3.4) into Eq. (3.3) one can get

$$\hat{b}^{\dagger}\hat{b} = \frac{m\omega_m\hat{x}^2}{2\hbar} + \frac{\hat{p}^2}{2\hbar m\omega_m} - \frac{1}{2}. \tag{3.5}$$

Now, lets define $\hat{x}$ and $\hat{p}$ as

$$\hat{x} = q\sqrt{\frac{\hbar}{m\omega_m}} \tag{3.6}$$

and

$$\hat{p} = p\sqrt{\hbar m\omega_m}. \tag{3.7}$$

By inserting Eq. (3.6) and Eq. (3.7) into Eq. (3.5) we get

$$\hat{b}^\dagger\hat{b} = \frac{q^2}{2} + \frac{p^2}{2} - \frac{1}{2}. \tag{3.8}$$

Again the term $\hat{b} + \hat{b}^\dagger$ of Eq. (2.18) by applying Eq. (2.8)and Eq.(2.9) is

$$\hat{b} + \hat{b}^\dagger = (\frac{m\omega_m\hat{x} - i\hat{p}}{\sqrt{2m\omega_m\hbar}}) + (\frac{m\omega_m\hat{x} + i\hat{p}}{\sqrt{2m\omega_m\hbar}}) = \frac{2m\omega_m\hat{x}}{\sqrt{2\hbar m\omega_m}}. \tag{3.9}$$

Now, afrer inserting Eq.(3.7) into Eq.(3.9) with $\frac{2}{\sqrt{2}} \approx 1$ we get

$$\hat{b} + \hat{b}^\dagger = q. \tag{3.10}$$

Finally, by inserting Eq.(3.6) and Eq.(3.10)into Eq.(2.18) we get the Hamiltonian for the cavity and mechanical mode with their interaction

$$\hat{H}_{cm} = \hbar\omega_c\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \frac{1}{2}\hbar\omega_m - \hbar g\hat{a}^\dagger\hat{a}\hat{q}. \tag{3.11}$$

The offset term, in Eq. (3.11) corresponding to the zero-point energy, makes no contribution to the Hamiltonian dynamics, and hence is often omitted. We have neglected the $\frac{1}{2}\omega\hbar$, since this will not affect our results (Milburn, 2011). Hamiltonian for the cavity and mechanical mode $(\hat{H}_{cm})$ with their interaction read as

$$\hat{H}_{cm} = \hbar\omega_c\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \hbar g\hat{a}^\dagger\hat{a}\hat{q}. \tag{3.12}$$

From our system we have driving Hamiltonian $(\hat{H}_d)$ and parametric Hamiltonian $(\hat{H}_p)$ as

$$\hat{H}_d = i\hbar E(\hat{a}^\dagger e^{-i\omega_L t} - \hat{a}e^{i\omega_L t}) \tag{3.13}$$

and

$$\hat{H}_p = i\hbar G(e^{i\theta}\hat{a}^{\dagger 2}e^{-i\omega_L t} - e^{-i\theta}\hat{a}^2 e^{i\omega_L t}). \tag{3.14}$$

To obtain standard optomechanical Hamiltonian of this system we have to apply unitary transformation (Axline, 2018) to avoid the time dependent terms of Eq. (3.13), which obeys the Hamiltonian to recover the original wave function,

$$\tilde{H}_d = U\hat{H}_d U^\dagger + i\hbar\frac{dU}{dt}U^\dagger. \tag{3.15}$$

The transformation terms are given below

$$\begin{aligned}\hat{a} &= \hat{a}e^{-i\omega_L t}\\ \hat{a}^\dagger &= \hat{a}^\dagger e^{i\omega_L t}\\ \hat{U} &= e^{i\omega_L \hat{a}^\dagger \hat{a} t}\\ \hat{U}^\dagger &= e^{-i\omega_L \hat{a}^\dagger \hat{a} t}\\ \hat{U}\hat{U}^\dagger &= 1.\end{aligned} \tag{3.16}$$

These unitary transformation have applications in spatial translations and time evolution of operators.By using terms of equation (3.16) we obtain

$$\tilde{H}_d = U\left[i\hbar E(\hat{a}^\dagger e^{i\omega_L t}e^{-i\omega_L t} - \hat{a}e^{-i\omega_L t}e^{i\omega_L t})\right]U^\dagger + i\hbar\frac{d}{dt}(e^{i\omega_L \hat{a}^\dagger \hat{a} t})e^{-i\omega_L \hat{a}^\dagger \hat{a} t} \tag{3.17}$$

$$\tilde{H}_d = \hat{U}\left[i\hbar E(\hat{a}^\dagger - \hat{a})\right]U^\dagger + i\hbar(i\omega_L \hat{a}^\dagger \hat{a}). \tag{3.18}$$

Thus, the driving Hamiltonian $\hat{H}_d$ is

$$\tilde{H}_d = i\hbar E(\hat{a}^\dagger - \hat{a}) - \hbar\omega_L \hat{a}^\dagger \hat{a}. \tag{3.19}$$

Similarly, to obtain the standard optomechanical Hamiltonian of this system we have to apply a unitary transformation to avoid the time-dependent terms of Eq. (3.14), which obeys the Hamiltonian to recover the original wave function as

$$\tilde{H}_p = U\hat{H}_p U^\dagger + i\hbar\frac{dU}{dt}U^\dagger. \tag{3.20}$$

Likewise, the transformation terms are given below

$$
\begin{aligned}
\hat{a}^2 =& \hat{a}^2 e^{-i\omega_L t} \\
\hat{a}^{\dagger 2} =& \hat{a}^{\dagger 2} e^{i\omega_L t} \\
\hat{U} =& exp(i\omega_L (\hat{a}^\dagger \hat{a})^2 t) \\
\hat{U}^\dagger =& exp(-i\omega_L (\hat{a}^\dagger \hat{a})^2 t) \\
\hat{U}\hat{U}^\dagger =& 1.
\end{aligned} \tag{3.21}
$$

These unitary transformation have applications in spatial translations and time evolution of operators. By using terms of equation (3.21), we get

$$
\begin{aligned}
\tilde{H}_p =& \hat{U}\left[i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} e^{i\omega_L t} e^{-i\omega_L t} - e^{-i\theta}\hat{a}^2 e^{-i\omega_L t} e^{i\omega_L t})\right] U^\dagger + \\
& i\hbar \frac{d}{dt}(exp(i\omega_L (\hat{a}^\dagger \hat{a})^2 t)) exp(-i\omega_L (\hat{a}^\dagger \hat{a})^2 t).
\end{aligned} \tag{3.22}
$$

This could be rewritten as

$$
\tilde{H}_p = i\hbar G\left[e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2 \right] - \hbar\omega_L (\hat{a}^\dagger \hat{a})^2. \tag{3.23}
$$

The last term of Eq. (3.23) vanishes, it does not affect our result and then the equation becomes

$$
\tilde{H}_p = i\hbar G\left[e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2 \right]. \tag{3.24}
$$

Finally, from Eq. (3.12), Eq. (3.19) and Eq. (3.24) the total optomechanical Hamiltonian of our system is given by

$$
\hat{H} = \hbar(\omega_c - \omega_L)\hat{a}^\dagger \hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \hbar g \hat{a}^\dagger \hat{a}\hat{q} + i\hbar E(\hat{a}^\dagger - \hat{a}) + i\hbar G\left[e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2 \right]. \tag{3.25}
$$

The nonlinear QLE for this system can be calculated for $\dot{\hat{q}}$, $\dot{\hat{p}}$ and $\dot{\hat{a}}$ by inserting Eq. (3.25) into Eq. (2.19).

For $\dot{\hat{q}}$,

$$
\frac{d\hat{q}}{dt} = \dot{\hat{q}} = \frac{-i}{\hbar}[\hat{q}, \hat{H}] - \beta\hat{q} + \sqrt{2\beta}\hat{z}_{in}. \tag{3.26}
$$

Then, the commutation relation for Eq. (3.25) becomes

$$\begin{aligned}[\hat{q},\hat{H}] =&[\hat{q},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}+\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)-\hbar g\hat{a}^\dagger\hat{a}\hat{q}+i\hbar E(\hat{a}^\dagger-\hat{a})\\&+i\hbar G(e^{i\theta}\hat{a}^{\dagger 2}-e^{-i\theta}\hat{a}^2)]\end{aligned} \tag{3.27}$$

$$\begin{aligned}[\hat{q},\hat{H}] =&[\hat{q},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]+[\hat{q},\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)]-[\hat{q},\hbar g\hat{a}^\dagger\hat{a}\hat{q}]+[\hat{q},i\hbar E(\hat{a}^\dagger-\hat{a})]\\&+[\hat{q},i\hbar G(e^{i\theta}\hat{a}^{\dagger 2}-e^{-i\theta}\hat{a}^2)].\end{aligned} \tag{3.28}$$

Now, lets solve Eq.(3.28) term by term,

$$[\hat{q},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]=\hbar(\omega_c-\omega_L)[\hat{q},\hat{a}^\dagger\hat{a}]. \tag{3.29}$$

From the commutation relation we have

$$[\hat{A},\hat{B}\hat{C}]=\hat{B}[\hat{A},\hat{C}]+[\hat{A},\hat{B}]\hat{C} \tag{3.30}$$

and

$$[\hat{A}\hat{B},\hat{C}]=\hat{A}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{B}. \tag{3.31}$$

By using Eq. (3.30), the very right hand side of Eq. (3.29) becomes

$$[\hat{q},\hat{a}^\dagger\hat{a}]=\hat{a}^\dagger[\hat{q},\hat{a}]+[\hat{q},\hat{a}^\dagger]\hat{a}. \tag{3.32}$$

But

$$[\hat{q},\hat{a}]=[\hat{q},\hat{a}^\dagger]=0. \tag{3.33}$$

Now, by inserting Eq. (3.33) into Eq. (3.32) we get

$$[\hat{q},\hat{a}^\dagger\hat{a}]=0. \tag{3.34}$$

Thus, by inserting Eq. (3.34) into Eq. (3.29) one can get

$$[\hat{q},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]=0 \tag{3.35}$$

and

$$[\hat{q},\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)]=\frac{\hbar\omega_m}{2}[\hat{q},(\hat{p}^2+\hat{q}^2)], \tag{3.36}$$

$$[\hat{q},(\hat{p}^2+\hat{q}^2)]=[\hat{q},(\hat{p}^2+\hat{q}^2)]=[\hat{q},\hat{p}^2]+[\hat{q},\hat{q}^2)]. \tag{3.37}$$

But

$$[\hat{q}, \hat{p}^2] = \hat{p}[\hat{q}, \hat{p}] + [\hat{q}, \hat{p}]\hat{p}, \tag{3.38}$$

and

$$[\hat{q}, \hat{q}^2] = \hat{q}[\hat{q}, \hat{q}] + [\hat{q}, \hat{q}]\hat{q} = 0. \tag{3.39}$$

Thus, by using Eq. (3.4), Eq. (3.37), Eq. (3.38) and Eq. (3.39), Eq. (3.36) becomes

$$[\hat{q}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = i\hbar^2\omega_m\hat{p}. \tag{3.40}$$

For

$$[\hat{q}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = \hbar g[\hat{q}, \hat{a}^\dagger\hat{a}\hat{q}] = \hbar g\bigg(\hat{q}[\hat{q}, \hat{a}^\dagger\hat{a}] + [\hat{q}, \hat{q}\hat{a}^\dagger\hat{a}]\bigg). \tag{3.41}$$

By using Eq. (3.34) and Eq. (3.39), Eq. (3.41) becomes

$$[\hat{q}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = 0. \tag{3.42}$$

For

$$[\hat{q}, i\hbar E(\hat{a}^\dagger - \hat{a})] = i\hbar E[\hat{q}, \hat{a}^\dagger - \hat{a}] = i\hbar E([\hat{q}, \hat{a}^\dagger]) - [\hat{q}, \hat{a}]). \tag{3.43}$$

By using Eq. (3.33), Eq. (3.35) becomes

$$[\hat{q}, i\hbar E(\hat{a}^\dagger - \hat{a})] = 0. \tag{3.44}$$

For

$$[\hat{q}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = i\hbar G\bigg(e^{i\theta}[\hat{q}, \hat{a}^{\dagger 2}] - e^{-i\theta}[\hat{q}, \hat{a}^2]\bigg), \tag{3.45}$$

again, by using Eq. (3.41), Eq. (3.45) becomes

$$[\hat{q}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = 0. \tag{3.46}$$

Finally, by inserting Eq. (3.35), Eq. (3.40), Eq. (3.42), Eq. (3.44) and Eq. (3.45) into Eq. (3.27) we get

$$[\hat{q}, \hat{H}] = i\hbar^2\omega_m\hat{p}. \tag{3.47}$$

Thus, by using Eq. (3.47), Eq. (3.26) becomes

$$\frac{d\hat{q}}{dt} = \dot{\hat{q}} = \hbar\omega_m\hat{p}. \tag{3.48}$$

For $\hat{p}$,

$$\frac{d\hat{p}}{dt} = \dot{\hat{p}} = \frac{-i}{\hbar}[\hat{p}, \hat{H}] - \beta\hat{p} + \sqrt{2\beta}\hat{z}_{in}. \tag{3.49}$$

Then,

$$\begin{aligned}[\hat{p}, \hat{H}] =& [\hat{p}, \hbar(\omega_c - \omega_L)\hat{a}^\dagger \hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \hbar g \hat{a}^\dagger \hat{a}\hat{q} + i\hbar E(\hat{a}^\dagger - \hat{a}) \\ & + i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)],\end{aligned} \tag{3.50}$$

$$\begin{aligned}[\hat{p}, \hat{H}] =& [\hat{p}, \hbar(\omega_c - \omega_L)\hat{a}^\dagger \hat{a}] + [\hat{p}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] - [\hat{p}, \hbar g \hat{a}^\dagger \hat{a}\hat{q}] + [\hat{p}, i\hbar E(\hat{a}^\dagger - \hat{a})] \\ & + [\hat{p}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)].\end{aligned} \tag{3.51}$$

Again, let's solve Eq.(3.51) term by term,

$$[\hat{p}, \hbar(\omega_c - \omega_L)\hat{a}^\dagger \hat{a}] = \hbar(\omega_c - \omega_L)[\hat{p}, \hat{a}^\dagger \hat{a}]. \tag{3.52}$$

In the same manner by using Eq. (3.30), the very right hand side of Eq. (3.52) becomes

$$[\hat{p}, \hat{a}^\dagger \hat{a}] = \hat{a}^\dagger[\hat{p}, \hat{a}] + [\hat{p}, \hat{a}^\dagger]\hat{a}. \tag{3.53}$$

But

$$[\hat{p}, \hat{a}] = [\hat{p}, \hat{a}^\dagger] = 0. \tag{3.54}$$

Now, by inserting Eq. (3.54) into Eq. (3.53) we get

$$[\hat{p}, \hat{a}^\dagger \hat{a}] = 0. \tag{3.55}$$

Thus, by inserting Eq. (3.55) into Eq. (3.52) we get

$$[\hat{p}, \hbar(\omega_c - \omega_L)\hat{a}^\dagger \hat{a}] = 0. \tag{3.56}$$

$$[\hat{p}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = \frac{\hbar\omega_m}{2}[\hat{q}, (\hat{p}^2 + \hat{q}^2)] \tag{3.57}$$

$$[\hat{p}, (\hat{p}^2 + \hat{q}^2)] = [\hat{p}, (\hat{p}^2 + \hat{q}^2)] = [\hat{p}, \hat{p}^2] + [\hat{p}, \hat{q}^2)] \tag{3.58}$$

$$[\hat{p}, \hat{p}^2] = \hat{p}[\hat{p}, \hat{p}] + [\hat{p}, \hat{p}]\hat{p} = 0 \tag{3.59}$$

and

$$[\hat{p}, \hat{q}^2)] = \hat{q}[\hat{p}, \hat{q}] + [\hat{p}, \hat{q}]\hat{q} = -2i\hbar\hat{q}. \tag{3.60}$$

Now, by using Eq. (3.4), Eq. (3.59) and Eq. (3.60, Eq. (3.57 becomes

$$[\hat{p}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = -i\hbar^2\omega_m\hat{q}. \tag{3.61}$$

For

$$[\hat{p}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = \hbar g[\hat{p}, \hat{a}^\dagger\hat{a}\hat{q}] = \hbar g\Bigg(\hat{q}[\hat{p}, \hat{a}^\dagger\hat{a}] + [\hat{p}, \hat{q}\hat{a}^\dagger\hat{a}]\Bigg), \tag{3.62}$$

here

$$[\hat{p}, \hat{a}^\dagger\hat{a}] = 0. \tag{3.63}$$

By using Eq. (3.4) and Eq. (3.63) and Eq. (3.62 becomes

$$[\hat{p}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = -i\hbar^2 g\hat{a}^\dagger\hat{a}. \tag{3.64}$$

For

$$[\hat{p}, i\hbar E(\hat{a}^\dagger - \hat{a})] = i\hbar E[\hat{p}, \hat{a}^\dagger - \hat{a}] = i\hbar E([\hat{p}, \hat{a}^\dagger]) - [\hat{p}, \hat{a}]), \tag{3.65}$$

hence

$$[\hat{p}, \hat{a}^\dagger] = [\hat{p}, \hat{a}] = 0. \tag{3.66}$$

By inserting Eq. (3.66) into Eq. (3.65) we get

$$[\hat{p}, i\hbar E(\hat{a}^\dagger - \hat{a})] = 0. \tag{3.67}$$

For

$$[\hat{p}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = i\hbar G\Bigg(e^{i\theta}[\hat{p}, \hat{a}^{\dagger 2}] - e^{-i\theta}[\hat{p}, \hat{a}^2]\Bigg), \tag{3.68}$$

again, by using Eq. (3.66), Eq. (3.68) becomes

$$[\hat{q}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = 0 \tag{3.69}$$

Finally, by inserting Eq. (3.56), Eq. (3.61), Eq. (3.64), Eq. (3.67) and Eq. (3.69) into Eq. (3.51) we get

$$[\hat{p}, \hat{H}] = -i\hbar^2\omega_m\hat{q} + i\hbar^2 g\hat{a}^\dagger\hat{a}. \tag{3.70}$$

Thus, by using Eq. (3.70), Eq. (3.50) becomes

$$\frac{d\hat{p}}{dt} = \dot{\hat{p}} = -\hbar\omega_m\hat{q} + \hbar g\hat{a}^\dagger\hat{a} - \gamma_m\hat{p} + \xi. \tag{3.71}$$

For $\hat{a}$,

$$\frac{d\hat{a}}{dt} = \dot{\hat{a}} = \frac{-i}{\hbar}[\hat{a}, \hat{H}] - \kappa\hat{a} + \sqrt{2\kappa}\hat{a}_{in} \tag{3.72}$$

$$[\hat{a},\hat{H}] =[\hat{a},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}+\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)-\hbar g\hat{a}^\dagger\hat{a}\hat{q}+i\hbar E(\hat{a}^\dagger-\hat{a}) + i\hbar G(e^{i\theta}\hat{a}^{\dagger 2}-e^{-i\theta}\hat{a}^2)] \tag{3.73}$$

$$[\hat{a},\hat{H}] =[\hat{a},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]+[\hat{a},\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)]-[\hat{a},\hbar g\hat{a}^\dagger\hat{a}\hat{q}]+[\hat{a},i\hbar E(\hat{a}^\dagger-\hat{a})] + [\hat{a},i\hbar G(e^{i\theta}\hat{a}^{\dagger 2}-e^{-i\theta}\hat{a}^2)], \tag{3.74}$$

Again, let's solve Eq.(3.77) one by one,

$$[\hat{a},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]=\hbar(\omega_c-\omega_L)[\hat{a},\hat{a}^\dagger\hat{a}]. \tag{3.75}$$

In the same manner by using Eq. (3.30), the very right hand side of Eq. (3.75) becomes

$$[\hat{a},\hat{a}^\dagger\hat{a}]=\hat{a}^\dagger[\hat{a},\hat{a}]+[\hat{a},\hat{a}^\dagger]\hat{a}. \tag{3.76}$$

But,

$$[\hat{a},\hat{a}]=0 \tag{3.77}$$

and

$$[\hat{a},\hat{a}^\dagger]=1. \tag{3.78}$$

Now, by inserting Eq. (3.77) and Eq. (3.78) into Eq. (3.76) we get

$$[\hat{a},\hat{a}^\dagger\hat{a}]=\hat{a}. \tag{3.79}$$

Finally, after inserting Eq. (3.79) into Eq. (3.75) we get

$$[\hat{a},\hbar(\omega_c-\omega_L)\hat{a}^\dagger\hat{a}]=\hbar(\omega_c-\omega_L)\hat{a}. \tag{3.80}$$

$$[\hat{a},\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)]=\frac{\hbar\omega_m}{2}[\hat{a},(\hat{p}^2+\hat{q}^2)] \tag{3.81}$$

$$[\hat{a},(\hat{p}^2+\hat{q}^2)]=[\hat{a},(\hat{p}^2+\hat{q}^2)]=[\hat{a},\hat{p}^2]+[\hat{a},\hat{q}^2)]. \tag{3.82}$$

Here

$$[\hat{a},\hat{p}^2]=[\hat{a},\hat{q}^2)]=0. \tag{3.83}$$

Now, by inserting Eq. (3.83) into Eq. (3.81) we get

$$[\hat{a}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = 0. \tag{3.84}$$

For

$$[\hat{a}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = \hbar g[\hat{a}, \hat{a}^\dagger\hat{a}\hat{q}] = \hbar g\bigg(\hat{q}[\hat{a}, \hat{a}^\dagger\hat{a}] + [\hat{a}, \hat{q}\hat{a}^\dagger\hat{a}]\bigg), \tag{3.85}$$

from Eq. (3.79) we have

$$[\hat{a}, \hat{a}^\dagger\hat{a}] = \hat{a} \tag{3.86}$$

and

$$[\hat{a}, \hat{q}] = 0. \tag{3.87}$$

By using Eq. (3.86) and Eq. (3.87), Eq. (3.85 becomes

$$[\hat{a}, \hbar g\hat{a}^\dagger\hat{a}\hat{q}] = \hbar g\hat{a}\hat{q}. \tag{3.88}$$

For

$$[\hat{a}, i\hbar E(\hat{a}^\dagger - \hat{a})] = i\hbar E[\hat{a}, \hat{a}^\dagger - \hat{a}] = i\hbar E([\hat{a}, \hat{a}^\dagger]) - [\hat{a}, \hat{a}]), \tag{3.89}$$

By using Eq. (3.76) and Eq. (3.77), Eq. (3.89) becomes

$$[\hat{a}, i\hbar E(\hat{a}^\dagger - \hat{a})] = i\hbar E. \tag{3.90}$$

For

$$[\hat{a}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = i\hbar G\bigg(e^{i\theta}[\hat{a}, \hat{a}^{\dagger 2}] - e^{-i\theta}[\hat{a}, \hat{a}^2]\bigg), \tag{3.91}$$

from commutation relation we have

$$[\hat{a}, \hat{a}^2] = 0. \tag{3.92}$$

With the help of commutation relation of Eq. (3.30) one can have

$$[\hat{a}, \hat{a}^{\dagger 2}] = [\hat{a}, \hat{a}^\dagger]\hat{a}^\dagger + \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] = 2\hat{a}^\dagger. \tag{3.93}$$

By using Eq. (3.92) and Eq. (3.93), Eq. (3.91) becomes

$$[\hat{a}, i\hbar G(e^{i\theta}\hat{a}^{\dagger 2} - e^{-i\theta}\hat{a}^2)] = 2i\hbar G\hat{a}^\dagger e^{i\theta}. \tag{3.94}$$

Finally, by inserting Eq. (3.80), Eq. (3.84), Eq. (3.88), Eq. (3.90) and Eq. (3.94) into Eq. (3.73) we get

$$[\hat{a}, \hat{H}] = \hbar(\omega_c - \omega_L)\hat{a} - \hbar g\hat{a}\hat{q} + i\hbar E + 2i\hbar G\hat{a}^\dagger e^{i\theta}. \tag{3.95}$$

Thus, by using Eq. (3.95), Eq. (3.72) becomes

$$\frac{d\hat{a}}{dt} = \dot{\hat{a}} = -i(\omega_c - \omega_L)\hat{a} + ig\hat{a}\hat{q} + E + 2G\hat{a}^{\dagger}e^{i\theta} - \kappa\hat{a} + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.96}$$

Therefore, the following are nonlinear QLEs of our model

$$\dot{\hat{q}} = \hbar\omega_m\hat{p} \tag{3.97}$$

$$\dot{\hat{p}} = -\hbar\omega_m\hat{q} + \hbar g\hat{a}^{\dagger}\hat{a} - \gamma_m\hat{p} + \xi \tag{3.98}$$

and

$$\dot{\hat{a}} = -i(\omega_c - \omega_L)\hat{a} + ig\hat{a}\hat{q} + E + 2G\hat{a}^{\dagger}e^{i\theta} - \kappa\hat{a} + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.99}$$

The optical cavity amplitude decay at a rate κ is affected by the radiation input noise operator $\hat{a}_{in}$ with zero mean value; its correlation function (Gardiner and Zoeller, 1991) are as follows; where $\hat{a}_{in}$ is the input vacuum noise operator

$$\langle\delta\hat{a}_{in}(t)\delta\hat{a}_{in}^{\dagger}(t')\rangle = \delta(t-t') \tag{3.100}$$

and

$$\langle\delta\hat{a}_{in}(t)\delta\hat{a}_{in}(t')\rangle = \langle\delta\hat{a}_{in}^{\dagger}(t)\delta\hat{a}_{in}(t')\rangle = 0. \tag{3.101}$$

The mechanical resonator affected by the quantum Brownian noise $\xi(t)$ leading to the damping of its oscillation with a rate γ_m possessing the correlation function

$$\langle\xi(t)\xi(t')\rangle \approx \gamma_m(2\bar{n}+1)\delta(t-t') \tag{3.102}$$

where $\bar{n} = [exp(\hbar\omega_m/k_BT) - 1]^{-1}$ the mean thermal phonon number, which is assumed to stay at the same environmental temperature T. This means that the evolution of the mechanical resonator is a Markovian process (Pontin et al., 2014).

3.4 Steady State Solutions of the Equations of Motion

The quantum dynamics of the full nonlinear system is difficult to analyze. So we can proceed with linearization of the system operators around the steady state under strong intense laser driving intra-cavity field amplitude (Genes et al., 2008). So we write each operator of the system as the sum of its steady state mean value and a small fluctuation with zero mean value as follows

$$\hat{q} = q_s + \delta\hat{q}, \tag{3.103}$$

$$\hat{p} = P_s + \delta\hat{p}, \tag{3.104}$$

$$\hat{a} = \alpha_s + \delta\hat{a} \tag{3.105}$$

and

$$\hat{a}^\dagger = \alpha_s^* + \delta\hat{a}^\dagger \tag{3.106}$$

where P_s, q_s and α_s represent the amplitudes of the system, whereas $\delta\hat{q}$, $\delta\hat{p}$, and $\delta\hat{a}$ are the fluctuations with zero-mean values. The corresponding average values are obtained by setting the fluctuations with zero-mean values and applying derivatives to zero.
Eq. (3.97) becomes and it's mean value of momentum is zero.

$$\delta\dot{\hat{q}} = \hbar\omega_m(P_s + \delta\hat{p}) = 0 \tag{3.107}$$

The momentum P_s in the steady-state is equal to zero and the mechanical damping does not affect the steady-state of the system. Then,

$$P_s = 0. \tag{3.108}$$

To find q_s, lets insert Eq. (3.103) , Eq. (3.104), Eq. (3.105) and Eq. (3.106) into Eq. (3.98)

$$\delta\dot{\hat{p}} = -\hbar\omega_m(q_s + \delta\hat{q}) + \hbar g((\alpha_s^* + \delta\hat{a}^\dagger)(\alpha_s + \delta\hat{a})) - \gamma_m(P_s + \delta\hat{p}) + \xi = 0. \tag{3.109}$$

Then,

$$-\hbar\omega_m q_s - \hbar\omega_m\delta\hat{q} + \hbar g(\alpha_s^*\alpha_s + \alpha_s^*\delta\hat{a} + \alpha_s\delta\hat{a}^\dagger + \delta\hat{a}\hat{a}^\dagger) - \gamma_m P_s - \gamma_m\delta\hat{p} + \xi = 0. \tag{3.110}$$

Since the mean values of fluctuation terms $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$ and $\delta\hat{a}^\dagger$ are zero and due to the momentum in the steady-state, P_s , being equal to zero, the mechanical damping, ξ, does not affect the steady-state of the system and neglecting the small terms of $\delta\hat{a}\delta\hat{a}^\dagger$. Thus, q_s can be written as

$$q_s = \frac{g|\alpha_s|^2}{\omega_m}. \tag{3.111}$$

Since α_s, α_s^* are hermitian operators

$$\alpha_s = \alpha_s^*, \tag{3.112}$$

$\delta\hat{a}\hat{a}^\dagger$ and P_s are approximately zero for linearization.
Then, Eq. (3.110) becomes

$$\delta\dot{\hat{p}} = -\hbar\omega_m q_s - \hbar\omega_m\delta\hat{q} + \hbar g(|\alpha_s|^2 + \alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger)) - \gamma_m\delta\hat{p} + \xi. \tag{3.113}$$

Now after inserting Eq. (3.111) into Eq. (3.113) we obtain

$$\delta\dot{\hat{p}} = -\hbar\omega_m \frac{g|\alpha_s|^2}{\omega_m} - \hbar\omega_m\delta\hat{q} + \hbar g(|\alpha_s|^2 + \alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger)) - \gamma_m\delta\hat{p} + \xi. \tag{3.114}$$

Equation (3.113) can be simplified as

$$\delta\dot{\hat{p}} = -\hbar g|\alpha_s|^2 + \hbar g|\alpha_s|^2 - \hbar\omega_m\delta\hat{q} + \hbar g\alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger) - \gamma_m\delta\hat{p} + \xi. \tag{3.115}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} + \hbar g\alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger) - \gamma_m\delta\hat{p} + \xi. \tag{3.116}$$

Again after inserting Eq. (3.103), Eq. (3.105) and Eq. (3.106) into Eq. (3.99) we get

$$\begin{aligned} \delta\dot{\hat{a}} =& - i(\omega_c - \omega_L)(\alpha_s + \delta\hat{a}) + ig(\alpha_s + \delta\hat{a})(q_s + \delta\hat{q}) + E + 2G(\alpha_s^* + \delta\hat{a}^\dagger)e^{i\theta} \\ & - \kappa(\alpha_s + \delta\hat{a}) + \sqrt{2\kappa}\hat{a}_{in} = 0. \end{aligned} \tag{3.117}$$

Equation (3.117) can be written as

$$\begin{aligned} & - i(\omega_c - \omega_L)\alpha_s - i(\omega_c - \omega_L)\delta\hat{a} + ig(\alpha_s q_s + \alpha_s\delta\hat{q} + \delta\hat{a}q_s + \delta\hat{a}\delta\hat{q}) + E \\ & + 2G\alpha_s^* e^{i\theta} + 2G\delta\hat{a}^\dagger e^{i\theta} - \kappa\alpha_s - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in} = 0. \end{aligned} \tag{3.118}$$

Similarly, the mean values of fluctuation terms $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$ and $\delta\hat{a}^\dagger$ are zero and cavity input noise $\hat{a}_{in}$, being equal to zero,it does not affect the steady-state of the system and neglecting the small terms of $\delta\hat{a}\delta\hat{q}$ and also by using Eq. (3.112) we obtain

$$-i(\omega_c - \omega_L)\alpha_s + ig\alpha_s q_s + E + 2G\alpha_s e^{i\theta} - \kappa\alpha_s = 0. \tag{3.119}$$

By collecting similar terms of Eq. (3.119) one can obtain

$$(-i(\omega_c - \omega_L) + igq_s + 2Ge^{i\theta} - \kappa)\alpha_s + E = 0. \tag{3.120}$$

Then,

$$(-i(\omega_c - \omega_L) + igq_s + 2Ge^{i\theta} - \kappa)\alpha_s = -E. \tag{3.121}$$

Then, Eq. (3.121) can be written as

$$\alpha_s = \frac{E}{i(\omega_c - \omega_L - gq_s) - 2Ge^{i\theta} + \kappa}. \tag{3.122}$$

But, from Euler equations we have

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{3.123}$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{3.124}$$

and defining Δ, which is called the effective cavity detuning, including the radiation pressure effects as

$$\Delta = \omega_c - \omega_L - gq_s. \tag{3.125}$$

Then, again after using Eq. (3.123) and Eq. (3.125) into Eq. (3.122) we get

$$\alpha_s = \frac{E}{(\kappa - 2G\cos\theta) + i(\Delta - 2G\sin\theta)}. \tag{3.126}$$

In the same manner Eq. (3.117) can be written as

$$\begin{aligned}\delta\dot{\hat{a}} =& -i(\omega_c - \omega_L)\alpha_s - i(\omega_c - \omega_L)\delta\hat{a} + ig(\alpha_s q_s + \alpha_s\delta\hat{q} + \delta\hat{a}q_s + \delta\hat{a}\delta\hat{q}) + E \\ &+ 2G\alpha_s^* e^{i\theta} + 2G\delta\hat{a}^\dagger e^{i\theta} - \kappa\alpha_s - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in}.\end{aligned} \tag{3.127}$$

BY neglecting the small terms of $\delta\hat{a}\delta\hat{q}$ and also using Eq. (3.112), Eq. (3.127)becomes

$$\begin{aligned}\delta\dot{\hat{a}} =& (-i(\omega_c - \omega_L) + igq_s + 2Ge^{i\theta} - \kappa)\alpha_s + E - i(\omega_c - \omega_L + gq_s)\delta\hat{a} + ig\alpha_s\delta\hat{q} \\ &+ 2Ge^{i\theta}\delta\hat{a}^\dagger - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in}.\end{aligned} \tag{3.128}$$

After inserting Eq. (3.121) and Eq. (3.125) with their corresponding terms we obtain

$$\delta\dot{\hat{a}} = -i\Delta\delta\hat{a} + ig\alpha_s\delta\hat{q} + 2Ge^{i\theta}\delta\hat{a}^\dagger - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.129}$$

Therefore, from Eq. (3.97),Eq. (3.116) and Eq. (3.129)the linearized quantum Langevin equation for the fluctuation operators can be written as

$$\delta\dot{\hat{q}} = \hbar\omega_m \delta\hat{p}, \tag{3.130}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m \delta\hat{q} + \hbar g\alpha_s(\delta\hat{a}^\dagger + \delta\hat{a}) - \gamma_m \delta\hat{p} + \xi, \tag{3.131}$$

$$\delta\dot{\hat{a}} = -i\Delta\delta\hat{a} + ig\alpha_s\delta\hat{q} + 2Ge^{i\theta}\delta\hat{a}^\dagger - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in} \tag{3.132}$$

and

$$\delta\dot{\hat{a}}^\dagger = i\Delta\delta\hat{a}^\dagger - ig\alpha_s\delta\hat{q} + 2Ge^{-i\theta}\delta\hat{a} - \kappa\delta\hat{a}^\dagger + \sqrt{2\kappa}\hat{a}_{in}^\dagger. \tag{3.133}$$

3.5 Quantum Fluctuation

The squeezed lighted generated by OPA and optical cavity with the mechanical resonator is able to enhance bipartite robust CV entanglement in system (Vitali et al., 2007b) that is the quantum correlations between appropriate quadratures of the two intra-cavity fields and the position and momentum of the resonator.
Introducing the cavity field quadratures

$$\delta\hat{x} = (\frac{\delta\hat{a}^\dagger + \delta\hat{a}}{\sqrt{2}}) \tag{3.134}$$

and

$$\delta\hat{y} = i(\frac{\delta\hat{a}^\dagger - \delta\hat{a}}{\sqrt{2}}). \tag{3.135}$$

and the input noise quadratures

$$\hat{x}_{in} = (\frac{\hat{a}_{in}^\dagger + \hat{a}_{in}}{\sqrt{2}}) \tag{3.136}$$

and

$$\hat{y}_{in} = i(\frac{\hat{a}_{in}^\dagger - \hat{a}_{in}}{\sqrt{2}}). \tag{3.137}$$

After inserting Eq. (3.135) into Eq. (3.131) we get

$$\delta\dot{\hat{p}} = -\hbar\omega_m \delta\hat{q} + \hbar g\alpha_s(\sqrt{2}\delta\hat{x}) - \gamma_m \delta\hat{p} + \xi. \tag{3.138}$$

But, let's define the term $G_0 = \sqrt{2}g\alpha_s$ which is the effective coupling of the optical cavity fluctuation with mechanical resonator. Then, Eq. (3.138) becomes

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} + \hbar G_0\delta\hat{x} - \gamma_m\delta\hat{p} + \xi. \tag{3.139}$$

Now, let's derivate Eq. (3.134) on both sides with respect to time

$$\delta\dot{\hat{x}} = (\frac{\delta\dot{\hat{a}}^\dagger + \delta\dot{\hat{a}}}{\sqrt{2}}). \tag{3.140}$$

After inserting Eq. (3.132) and Eq. (3.133) into Eq. (3.140) we get

$$\begin{aligned}\delta\dot{\hat{x}} =&(\frac{i\Delta\delta\hat{a}^\dagger - ig\alpha_s\delta\hat{q} + 2Ge^{-i\theta}\delta\hat{a} - \kappa\delta\hat{a}^\dagger + \sqrt{2\kappa}\hat{a}_{in}^\dagger - i\Delta\delta\hat{a} + ig\alpha_s\delta\hat{q}}{\sqrt{2}})\\ &+ (\frac{2Ge^{i\theta}\delta\hat{a}^\dagger - \kappa\delta\hat{a} + \sqrt{2\kappa}\hat{a}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.141}$$

Now after inserting Eq. (3.123) and Eq. (3.124) into Eq. (3.141) and rearranging we get

$$\begin{aligned}\delta\dot{\hat{x}} =&i\Delta(\frac{\delta\hat{a}^\dagger - \delta\hat{a}}{\sqrt{2}}) + 2G\cos\theta(\frac{\delta\hat{a}^\dagger + \delta\hat{a}}{\sqrt{2}}) + 2iG\sin\theta(\frac{\delta\hat{a}^\dagger - \delta\hat{a}}{\sqrt{2}}) - \kappa(\frac{\delta\hat{a}^\dagger + \delta\hat{a}}{\sqrt{2}})\\ &+ \sqrt{2\kappa}(\frac{\delta\hat{a}_{in}^\dagger + \delta\hat{a}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.142}$$

Thus, by comparing Eq. (3.134), Eq. (3.135)and Eq. (3.136) with Eq. (3.142) one can get

$$\delta\dot{\hat{x}} = -(\kappa - 2G\cos\theta)\delta\hat{x} + (\Delta + 2G\sin\theta)\delta\hat{y} + \sqrt{2\kappa}\delta\hat{x}_{in}. \tag{3.143}$$

Again let's derivate Eq. (3.135) on both sides with respect to time

$$\delta\dot{\hat{y}} = i(\frac{\delta\dot{\hat{a}}^\dagger - \delta\dot{\hat{a}}}{\sqrt{2}}). \tag{3.144}$$

After inserting Eq. (3.132) and Eq. (3.133) into Eq. (3.144) we get

$$\begin{aligned}\delta\dot{\hat{y}} =&i(\frac{i\Delta\delta\hat{a}^\dagger - ig\alpha_s\delta\hat{q} + 2Ge^{-i\theta}\delta\hat{a} - \kappa\delta\hat{a}^\dagger + \sqrt{2\kappa}\delta\hat{a}_{in}^\dagger + i\Delta\delta\hat{a} - ig\alpha_s\delta\hat{q}}{\sqrt{2}})\\ &- i(\frac{2Ge^{i\theta}\delta\hat{a}^\dagger + \kappa\delta\hat{a} - \sqrt{2\kappa}\delta\hat{a}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.145}$$

Now, after inserting Eq. (3.123) and Eq. (3.124) into Eq. (3.145) and rearranging we get

$$\begin{aligned}\delta\dot{\hat{y}} =& -\Delta(\frac{\delta\hat{a}^\dagger + \delta\hat{a}}{\sqrt{2}}) + G_0\delta\hat{q} + 2iG\cos\theta(\frac{\delta\hat{a} - \delta\hat{a}^\dagger}{\sqrt{2}}) + 2G\sin\theta(\frac{\delta\hat{a}^\dagger + \delta\hat{a}}{\sqrt{2}})\\ &+ i\kappa(\frac{\delta\hat{a} - \delta\hat{a}^\dagger}{\sqrt{2}}) + \sqrt{2\kappa}(\frac{\delta\hat{a}_{in}^\dagger - \delta\hat{a}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.146}$$

Thus, by comparing Eq. (3.134), Eq. (3.135) and Eq. (3.137) with Eq. (3.146) one can get

$$\delta\dot{\hat{y}} = G_0\delta\hat{q} - (\Delta - 2G\sin\theta)\delta\hat{x} - (\kappa + 2G\cos\theta)\delta\hat{y} + \sqrt{2\kappa}\delta\hat{y}_{in}. \tag{3.147}$$

Finally, from Eq. (3.130), Eq. (3.139), Eq. (3.143) and Eq. (3.147) the linearized QLE with fluctuation and dissipation terms of our system are written as

$$\delta\dot{\hat{q}} = \hbar\omega_m\delta\hat{p}, \tag{3.148}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} + \hbar G_0\delta\hat{x} - \gamma_m\delta\hat{p} + \xi, \tag{3.149}$$

$$\delta\dot{\hat{x}} = -(\kappa - 2G\cos\theta)\delta\hat{x} + (\Delta + 2G\sin\theta)\delta\hat{y} + \sqrt{2\kappa}\delta\hat{x}_{in} \tag{3.150}$$

and

$$\delta\dot{\hat{y}} = G_0\delta\hat{q} - (\Delta - 2G\sin\theta)\delta\hat{x} - (\kappa + 2G\cos\theta)\delta\hat{y} + \sqrt{2\kappa}\delta\hat{y}_{in}. \tag{3.151}$$

The above Eq. (3.148), Eq. (3.149), Eq. (3.150) and Eq. (3.151) can be written in matrix form as:

$$\dot{u}(t) = Au(t) + n(t) \tag{3.152}$$

where $u(t) = (\delta\hat{q}, \delta\hat{p}, \delta\hat{x}, \delta\hat{y})^T$ is the vector of quadrature fluctuation operators, A is the so-called drift matrix (contains information about the system), which written on Eq.(3.154) and $n(t) = (0, \xi(t), \sqrt{2\kappa}\delta\hat{x}_{in}(t), \sqrt{2\kappa}\delta\hat{y}_{in}(t))^T$ is the vector of noise quadrature operators associated with the noise terms in Eq. (3.148) up to Eq. (3.151) and the exponent T represent the transpose. Here we assumed that $\hbar = 1$.

$$\begin{pmatrix} \delta\dot{\hat{q}} \\ \delta\dot{\hat{p}} \\ \delta\dot{\hat{x}} \\ \delta\dot{\hat{y}} \end{pmatrix} = \begin{pmatrix} 0 & \omega_m & 0 & 0 \\ -\omega_m & -\gamma_m & G_0 & 0 \\ 0 & 0 & -\kappa + 2G\cos\theta & \Delta + 2G\sin\theta \\ G_0 & 0 & -\Delta + 2G\sin\theta & -\kappa - 2G\cos\theta \end{pmatrix} \begin{pmatrix} \delta\hat{q} \\ \delta\hat{p} \\ \delta\hat{x} \\ \delta\hat{y} \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \\ \sqrt{2\kappa}\delta\hat{x}_{in} \\ \sqrt{2\kappa}\delta\hat{y}_{in} \end{pmatrix} \tag{3.153}$$

and

$$A = \begin{pmatrix} 0 & \omega_m & 0 & 0 \\ -\omega_m & -\gamma_m & G_0 & 0 \\ 0 & 0 & -\kappa + 2G\cos\theta & \Delta + 2G\sin\theta \\ G_0 & 0 & -\Delta + 2G\sin\theta & -\kappa - 2G\cos\theta \end{pmatrix}. \tag{3.154}$$

Equation (3.152) can be written in compact form as $\dot{u}(t) = Au(t) + n(t)$, whose solution is $u(t) = M(t)u(0) + \int_0^\infty M(s)n(t-s)ds$, where $M(t) = exp(At)$. The system is stable and reaches its steady state when all of the values of A have negative real parts so that $M(\infty) = 0$. The stability refers to a time independent state of motion while dynamic stability refers to a time dependent state of motion. This dynamic stability is quantified by the Routh-Hurwitz criteria (Schumacher, 2013; 202, 2021) as follows

$$2\kappa(\kappa^2 - 4G^2 + \Delta^2 + 2\kappa\gamma_m) + \gamma_m(2\kappa\gamma_m + \omega_m^2) > 0, \tag{3.155}$$

$$\begin{aligned}(2\kappa + \gamma_m)^2(\omega_m G_0^2\Delta + 2G_0^2\omega_m G\sin\theta) + 2\kappa\gamma_m((\kappa^2 - 4G^2 + \Delta^2)^2 + (2\kappa\gamma_m + \gamma_m^2) \\ - (\kappa^2 - 4G^2 + \Delta^2) + \omega_m^2[2(\kappa^2 - 4G^2 + \Delta^2) + \omega_m^2 + 2\kappa\gamma_m]) > 0,\end{aligned} \tag{3.156}$$

and

$$\omega_m^2(\kappa^2 - 4G^2 + \Delta^2) - \omega_m G_0^2\Delta - 2G_0^2\omega_m G\sin\theta > 0. \tag{3.157}$$

Therefore, all the external parameters must be chosen to satisfy the stability conditions described above. Due to the Gaussian nature of the quantum noise terms in Eq.(3.152) and the linearized dynamics, the steady-state quantum fluctuations of the system is a bipartite Gaussian state of one optical mode and one mechanical mode with OPA, fully characterized by the 4×4 covariance matrix V with its entries defined as

$$V_{ij} = \Big(\langle v_i(\infty)v_j(\infty) + v_j(\infty)v_i(\infty)\rangle\Big)/2. \tag{3.158}$$

In the stable system such a CM is given by

$$V = \int_0^\infty M(s)DM^T(s)ds. \tag{3.159}$$

If the system satisfied stability conditions $M(\infty) = 0$, since Eq.(3.159) is equivalent to the following Lyapunov equation (Anderson et al., 1986) for the steady-state CM

$$AV + VA^T = -D, \tag{3.160}$$

where V is the system CM contains all information about the steady state and D is the diffusion matrix stemming from the noise correlations, which is defined as

$$\frac{1}{2}\langle n_i(t)n_j(s) + n_j(s)\rangle = D_{ij}\delta(t-s). \tag{3.161}$$

The diffusion matrix is a diagonal matrix, that is $D = diag[0, \gamma_m(2\bar{n}+1), \kappa, \kappa]$. Here we have assumed the mechanical resonator is of high quality factor $Q = \omega_m/\gamma_m \gg 1$, which is typically satisfied under the current experiment conditions (Vitali et al., 2007a).

Thus, the bipartite entanglement of our system can be quantified numerically using the correlation matrix 4×4 of the two bipartite entanglement

$$V = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}. \tag{3.162}$$

Finally, we employ on logarithmic negativity E_N a quantity which has been already proposed as a measure of entanglement in the continuous variable case as defined in Eq. (2.21) Eq. (2.22) and Eq. (2.23). Based on those framework we can quantify the bipartite entanglement dynamics behavior of an optomechanical system with OPA.

4

RESULT AND DISCUSSION

In this section, we present the numerical results of the quantification of the bipartite entanglement dynamics behavior of an an optomechanical system with OPA. The entanglement properties can be verified by experimentally measuring the corresponding CM through logarithmic negativity. To measure E_N at the steady state, one has to measure all the determined independent entries of the correlation matrix V. This has been experimentally realized (Yang et al., 2017) for the case of two entangled optical modes at the output of a parametric oscillator.

The parameters used from (Vitali et al., 2007a; Yang et al., 2017) are optical cavity length $L = 10 \times 10^{-3} m$, the speed of light $c = 3 \times 10^{8} m/s$, Plancks constant $h = 6.626 \times 10^{-34} J.s$, Boltzmanns constant $K_B = 1.38 \times 10^{-23} J/K$, wavelength of driven laser $\lambda = 810 \times 10^{-9} m$, the power of driven laser $P = 50 \times 10^{-3} W$, frequency of mechanical oscillator $\omega_m = 10\pi MHz$ and damping rate $\gamma_m = 200\pi Hz$. Accordingly, we study effects of detuning, phase θ of the driving field on the OPA, temperatures and the optimal nonlinear gain G of OPA on the formation and enhancement of entanglement are investigated by plotting graphs as follows.

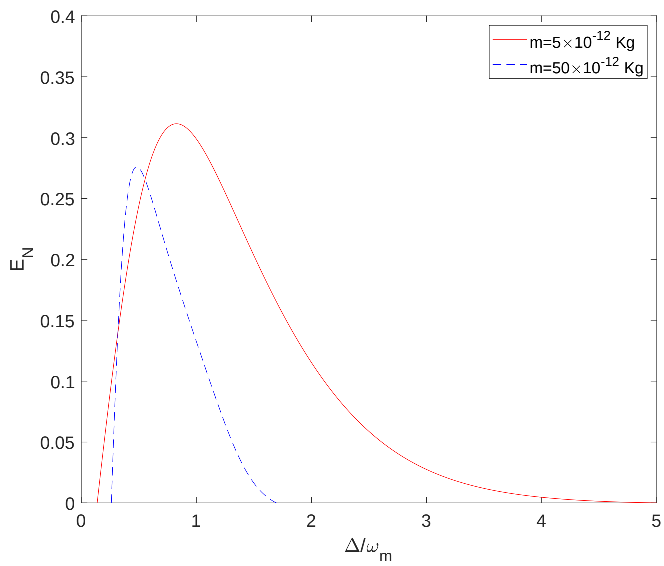

Fig. 4.1: Plot of the logarithmic negativity E_N as a function of the normalized detuning $\triangle/\omega_m$: in the case of an optical cavity of length $L = 10 \times 10^{-3} m$, driven by a laser with wavelength $\lambda = 810 \times 10^{-9} m$ and power $P = 50 \times 10^{-3} W$. The mechanical oscillator has a frequency $\omega_m = 10\pi MHz$, a damping rate $\gamma_m = 200\pi Hz$, and its temperature is $T = 400 \times 10^{-3} K$. The solid line refers to a mass $m = 5 \times 10^{-12} Kg$ and finesse $F = 1.07 \times 10^4$, while the dashed line refers to a mass $m = 50 \times 10^{-12} Kg$ and finesse $F = 3.4 \times 10^4$.

Cavity mode is driven at the blue sideband associated with the mechanical resonator, and we assume also that the system is in the resolved sideband limit, i.e., $\omega_m \gg \kappa \gg \gamma_m$, which requires that the cavity of high finesse. Under these conditions, the stationary continuous-variable entanglement can be generated (enhanced). Fig.(4.1) shows E_N versus the normalized detuning $\triangle/\omega_m$ for two different masses, $m = 5 \times 10^{-12} Kg$ and $m = 50 \times 10^{-12} Kg$ and for zero optimal nonlinear gain G=0, which means system without OPA; optomechanical entanglement is present only within a finite interval of values of $\triangle$ around $\triangle \simeq \omega_m$. The comparison between two different masses,

$m = 5 \times 10^{-12} Kg$ and $m = 50 \times 10^{-12} Kg$ illustrates that mirrors with $m = 5 \times 10^{-12} Kg$ has maximum optomechanical entanglement than mass of $m = 50 \times 10^{-12} Kg$. This indicates that small masses of mirrors have better optomechanical entanglement than bigger ones. The mechanical quality factor Q of this system is around $Q = 5 \times 10^4$ shows that the mirrors used for the analysis have high finesse (reflectivity).

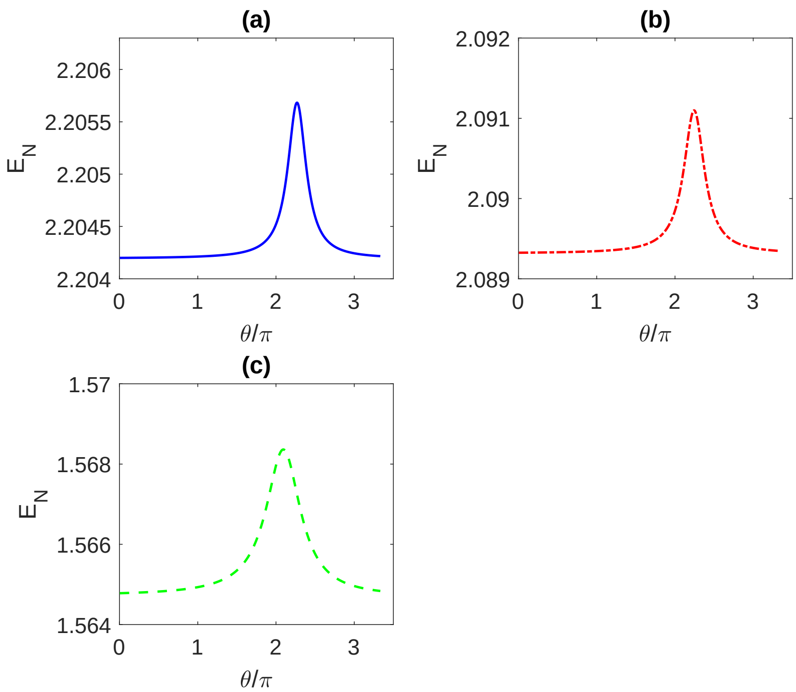

Fig. 4.2: Plot of the logarithmic negativity E_N as function of the phase θ for a specified temperature and the corresponding optimal value of the nonlinear gain: in the case finesse of $F = 1.07 \times 10^4$, a mass of $m = 5 \times 10^{-12} Kg$ and $\triangle = \omega_m$; (a) optimal value of the nonlinear gain $G = 5.6\kappa$ and temperature $T = 10 \times 10^{-3} K$, (b) optimal value of the nonlinear gain $G = 5.0\kappa$ and temperature $T = 100 \times 10^{-3} K$ and (c) optimal value of the nonlinear gain $G = 3\kappa$ and temperature $T = 1K$. The other parameters are same as Fig.(4.1)

Fig.(4.2) illustrates that the logarithmic negativity E_N as a function of the phase θ

for the mirrors environmental temperatures and the corresponding optimal value of the nonlinear gain of OPA. From this figure, plot(a) with the maximum optimal value of the nonlinear gain, $G = 5.6\kappa$ and minimum temperature $T = 10 \times 10^{-3}K$ have greater logarithmic negativity E_N as a function of the phase θ when compared with the plot (b) and (c). This indicates that as the values of the optimal value of the nonlinear gain G increases and the mirrors environmental temperatures decreases, the optomechanical entanglement increases.

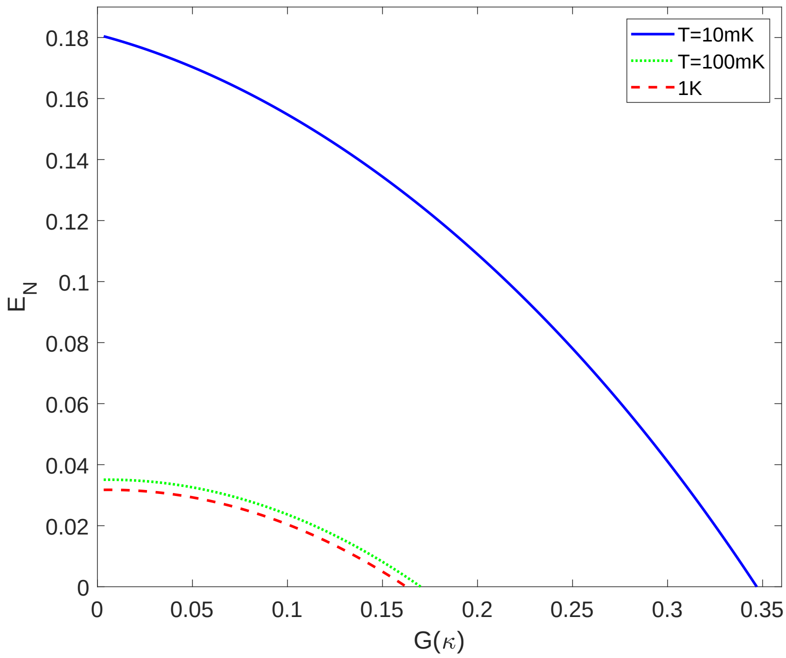

Fig. 4.3: Plot of the entanglement E_N versus the nonlinear gain G for various system temperatures: in the case finesse of $F = 1.07 \times 10^4$, $\theta = 2\pi$, a mass of $m = 5 \times 10^{-12}Kg$, $\triangle = \omega_m$ and temperatures are $T = 1K$, $T = 100mK$ and $T = 10mK$. The other parameters are same as Fig.(4.1)

This happen if the optimal phase for the entanglement is around $\theta = 2\pi$ for $\theta \in [0, 3\pi]$. It indicates that OPA generates squeezing of the optical field and the degree of squeezing is proportional to the nonlinear gain of the OPA and squeezed light enhances entanglement. The enhancement decreases as the temperature rises.

Fig.(4.3) indicates that the entanglement E_N versus the nonlinear gain G for various system temperatures; when G is large the optical noise becomes a significant effective thermal bath for the mechanical mode (accounting for the mechanical mode momentum fluctuation), leading to the degradation of entanglement.
Therefore, an optimal G exists corresponding to maximum entanglement as a result of the existence of the OPA between the cavity mode and mechanical mode, in addition to this entanglement enhancement at moderate values of G and entanglement degradation at large values of G.

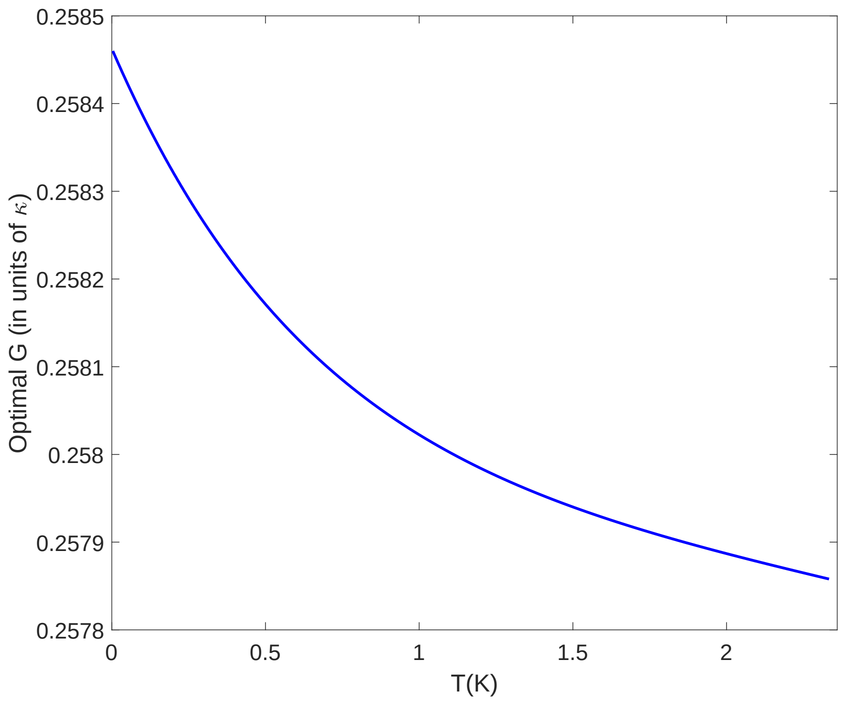

Fig. 4.4: Plot of the optimal nonlinear gain G for the entanglement as a function of temperature T: in the case finesse of $F = 1.07 \times 10^4$, $\theta = \pi/2$, a mass of $m = 5 \times 10^{-12} Kg$ and $\triangle = \omega_m$. The other parameters are same as Fig.(4.1)

Fig.(4.4) is the optimal nonlinear gain G for the entanglement as a function of temperature T. The larger values of optimal nonlinear gain G have the smallest temperature. This indicates the enhancement of bipartite optomechanical entanglement decreases as the temperatures of the system increases. This means that our scheme is preferred to work at cryogenic temperatures.

5

CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

In conclusion, we have studied the bipartite entanglement dynamics in an optomechanical system with OPA to generates the stationary continuous-variable entanglement. Particularly, we have studied how to generates (enhances) the stationary entanglement of an optomechanical system with OPA and quantification of their entanglement by using logarithmic negativity. The nonlinear crystal placed between a partially reflected mirror (fixed mirror) and fully reflected mirror (movable mirror) have pumped at a frequency of ω_L to generates and amplifies squeezed light, then the squeezed light enhances bipartite entanglement of an optomechanical system.

At the blue sideband regime the OPA enhanced the stationary continuous-variable entanglement. The results we obtained show that an optimal G exists corresponding to maximum entanglement as a result of the existence of the OPA between the cavity mode and mechanical mode, in addition to this entanglement enhancement at moderate values of G and entanglement degradation at large values of G. The larger values of optimal nonlinear gain G has the smallest temperature. This indicates the enhancement of bipartite optomechanical entanglement decreases as the temperatures of the system increases. This means that our scheme is preferred to work at cryogenic temperatures.

5.2 RECOMMENDATIONS

In this work, we have studied enhancement of the stationary continuous-variable entanglement at resolved sideband regime and at minimum temperature. Further studies will be conducted on the enhancement of entanglement at unresolved sideband regime by using short pulses of light and at warm temperatures.

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