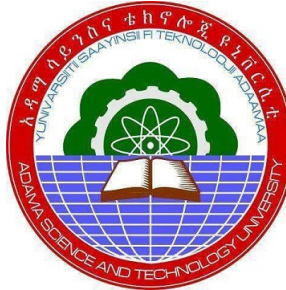


**COMPARISON OF FINITE ELEMENT METHOD AND FINITE
DIFFERENCE METHOD FOR BOUNDARY VALUE PROBLEMS**



BY
LIJALEM DAGNAW

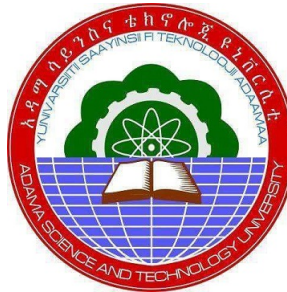
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Comparison of Finite Element Method and Finite Difference Method For Boundary value problems



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Abstract

The main purpose of this thesis is to analysis Rayleigh-Ritz method, Galerkin method and finite difference method for solving two-point boundary value problems of ordinary differential equations. The two point boundary value problems have great importance in chemical engineering, deflection of beams and other fields. Rayleigh-Ritz and Galerkin finite element methods are applied for solving inhomogeneous second-order ordinary differential equations. Rayleigh-Ritz method needs varitional formulation and minimization . But Galerkin's method needs weak formulation which leads to solving linear algebraic systems. Both methods are based on piece-wise interpolation polynomials on domain and trial solutions. The methods are used to calculate the unknown parameters of the trial solutions and then form a trial function which is used to calculate the numerical solution. To obtain a good approximation, the choice of the base function is important and to improve the approximation the number of base functions should be increased. In addition,we study the convergence, stability, and consistency of each method.

Finite difference method is based on domain discretization. In finite difference method, we first discretize the domain of the problem into equal mesh points and substitute the second order central difference. Using the discretized mesh points we construct systems of linear equations and form tri-diagonal matrix, then using Thomas algorithms we calculate the solutions and also derive the errors at each mesh points. The error analysis of FDM is based on Taylor's theory. For finite difference method we take different mesh lengths Δx . The convergence, stability and consistency depend on the mesh length. As the mesh length tends to zero, the convergence, stability and consistency is better. For Rayleigh-Ritz method and Galerkins method, the convergence, stability and consistency depends on the trial solutions. As the degree of interpolating polynomial increases the maximum absolute error tends to zero, which means the FEM is more convergent, stable and consistent. Rayleigh-Ritz method and Galerkins methods are efficient and faster. Their stability, consistency and convergence is better than that of finite difference method.

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Notations and Abbreviations

ODES	ordinary differential equations.
PDES	partial differential equations.
FDM	finite difference method.
FEM	finite element method.
GFEM	Galerkin's finite element method.
BVPS	boundary value problems.
FWA	forward approximation.
BWA	backward approximation.
LTE	local truncation error.
$O(h) = o(\Delta x)$	errors of order h .
$O(h^2) = o(\Delta x)^2$	errors of order h^2 .
$\Delta x = h$	the length of mesh points.
$Y_{(x,y)}$	is the analytical solution of the problems.
$Y_{(i,j)}$	is the initial mesh point of finite difference equation.

Chapter 1

Introduction

1.1 Background of the study

Ordinary differential equations (ODEs) have a long history and widely applied in many fields. The numerical solution of ordinary differential equation (ODE) has made great development in the 20th Century. There have been emerged many new ideas as well as many complex method for solving Ordinary differential equation, so that the numerical method for solving Ordinary differential equation has been deepened. Ordinary differential equations (ODEs) are equations satisfied by ordinary derivatives of functions of one independent variable. Systems of ordinary differential equations (ODEs) have been applied to many problems in Physics, Engineering, Biology and so on (Jincai et al.2011).

An ordinary differential equation is an equation that relates a function $y(x)$ to some of its derivatives

$$y^{(r)}(x) = \frac{d^r y}{dx^r} \quad (1.1.1)$$

It is usual to call x the independent variable and y the dependent variable, and to write the most general ordinary differential equation

$$F(x, y, y^{(1)}, y^{(2)} \dots y^{(n)}) = 0 \quad (1.1.2)$$

The number n is called the order of the ordinary differential equation. And it is the order of the highest derivative of y that occurs in the equation (Alen.2003).

Boundary value problems (BVPs) are problems that have conditions imposed on the boundary rather than at the initial points. For instance: Rayleigh-Ritz method, Galerkins finite element method, Shooting methods and Finite difference method (FDM) (seshu.2003). Whereas in initial value problems the solution is determined by conditions imposed at one point only, boundary value problems for ordinary differential equations in which the solution is required to satisfy conditions at more than one points.

The finite element method (FEM) is a numerical tool for determining approximate solution to a large class of engineering problems. The method was originally developed to study the stresses in complex air-frame structure and was later extended to the general field of continuum mechanics

Martin and Gerhard(2012).

The finite element method is receiving considerable attention in engineering education and in industry because of its diversity and flexibility as an analysis tool. It is often necessary to obtain approximate numerical solution for complex problems. Finite Element Method considers that the solution region comprises many small interconnected sub-regions or elements and gives piece-wise approximations to governing equations.

The fundamental idea of the finite element method is the replacement of continuous functions by piece wise approximations, usually polynomials. Although the finite element method itself is relatively new, its development and success expanding with the arrival and rapid growth of the digital computer, the idea of piece wise approximation far from new (Davies.2011).

Finite element method(FEM) is a numerical method for solving a differential or integral equation. It has been applied to a number of physical problems, where the governing differential equations are available. The method essentially consists of assuming the piece wise continuous function for the solution and parameters of the functions in the solution. The basic idea behind the finite element method is to replace a continuous function by means of piece wise polynomials. Such an approximation, called piece wise polynomial approximation.

Finite element method is one of the most important numerical application of the Rayleigh-Ritz and Galerkin methods. Its mathematical software is quite popular and used extensively in the solution of many practical problems of engineering and applied science. In finite element method, the domain of integration is subdivided into a number of smaller regions called elements and over each of these elements the continuous function is approximated by a suitable piecewise polynomial (Sastry.2006).

Rayleigh developed the method to solve certain problems and Ritz provided a mathematical basis for it and also applied it to more general problems where as the Rayleigh-Ritz method is based on the existence of a functional, the Galerkin technique uses the governing equation of the problem and minimizes the error of the approximate solution.

Boris Grigorevich Galerkin (1871 – 1945) is a Russian mathematician and an engineer. He comes from a poor family and had to start working at the age of twelve as a calligrapher. Nevertheless, he managed to study in the Mechanics Department of the St. Petersburg Technological Institute, and then he worked for the Russian steam-locomotive union and was arrested in 1905 for political activities and put into prison for one and a half years. This completely changed his interests, and it was while in prison that he dedicated to devote the rest of his life to science. Galerkin went on an extensive scientific trip through Europe in 1909 and visited Switzerland, among other countries. We do not know whether he met Walter Ritz on this trip, but Galerkin makes precise reference to both papers of Ritz in his most famous publication. One can only

speculate as to why the method is now a day generally Called the Galerkins method; may be Galerkins numerous examples from applications were more appreciated then the theoretical analysis of Ritz (Martin and Gerhard.2012).

While the choice of weighting functions $W(x)$ is entirely (as long as they are non zero and integrable), Galerkins (1915) introduced the idea of letting $W(x)$ to be same as the trial functions themselves. Galerkins method belongs to a wider class of methods called the weighted residual methods.

In this method an approximating function called the trial function (which satisfies all the boundary conditions) is substituted in the given differential equation and the result is called the residual (the result will not be zero since we have substituted an approximating function).The residual is then weighted and the integral of the product, taken over the domain is then set to zero (Seshu.2003). The trial function can be chosen from Lagranges interpolation (Sastry.2006). In 1943 Courant considerably increased possibilities of the Ritz method by introduction of the special linear functions defined over triangular regions and applied the method for the solution of torsion problems Courant R(1943).

The Ritz method together with the Courant modification is similar with FEM proposed independently by Clough many years later introducing for the first time in 1960 the term "finite element" in the paper "The finite element method in plane stress analysis" Clough R. W(1960).

There are various approaches for performing the discretizations , such as finite difference (M.Necati Ozisik et al.,1994;D.R.Croft and D.G.Lilly,1977), finite element(R.W.Lewis,etal.,2005;J.M.Bergheau, 2010), finite volume (S.V.Patankar,1980; H.K.Versteeg and W.Malalasekera,1996) and Spectral methods (G.Ben-Yu, 1998;O.P.Lemaitre and O.M.Knio,2010). Here we concentrate on Finite element and finite difference methods.

Finite difference method (FDM) is a method used to solve ordinary differential equations (ODEs) or partial differential equations (PDEs) that have conditions imposed on the boundary rather than at the initial conditions. These problems are called boundary value problems (BVPs) (Randall.2005).

The derivative appearing in the differential equation and the boundary conditions are replaced by their finite difference approximations and the resulting linear systems of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

In mathematics, finite-difference methods are numerical methods for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. Finite difference methods are thus discretization methods. Today,finite difference methods are the dominant approach to numerical solutions of differential equations (Dolicanic et al.2010).

The error in a method's solution is defined as the difference between the approximation and the exact analytical solution. The two sources of error in finite difference methods are round-off error, the loss of precision due to computer rounding of decimal quantities, and truncation error or discretization error, the difference between the exact solution of the finite difference equation and the exact quantity assuming perfect arithmetic (that is, assuming no round-off).

Among the various numerical methods that have evolved over the years, the most commonly used techniques are the finite difference, finite volume and finite element methods. The finite difference is a well-established and that requires a point-wise approximation to the governing equations. The basic idea of finite difference methods for the approximate solution of boundary value problems consists in replacing the derivatives in the differential equations by difference quotients (Hrennikoff.1941).

The finite element method(Rayleigh-Ritz and Galerkin method) and the finite difference method are the methods of mathematical tools employed to solve ordinary differential equations, they are the means of obtaining numerical solutions to ordinary differential equations (ODEs) (Davies and stephens.1983). Rayleigh-Ritz method,Galerkin method and finite difference methods are differing in stability, consistency and convergence.

In this thesis we focus on the comparison of Rayleigh-Ritz method,Galerkin method and finite difference methods for boundary value problems of ordinary differential equations (ODEs).

1.2 Statement of the problem

In this thesis work, we have chosen the numerical method rather than the exact solution because of the difficulty of some problems of ordinary differential equations. There are methods of solving boundary value problems, such as finite element method(Rayleigh-Ritz method, Galerkin's method), finite difference method and shooting method. The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science (Davies.2001). The work of Lord Rayleigh and W.Ritz on variational methods and the weighted-residual approach taken by B. G. Galerkin and others form the theoretical framework to the finite element method (Evegeny.1998).

Finite difference method is a numerical method based on mathematical discretization of differential equations, finite element method uses various forms of variational method applied to discrete model (Dolicanic et al.2010).Independently of the engineering applications, a number of papers appeared in the mathematical literature in the mid-1960s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions.

The proposed study intended to focus analysis of Rayleigh-Ritz and Galerkin's finite element method and finite difference method for boundary value problems. A number of researchers have addressed solution of boundary value problems by using finite element method and finite

difference methods. But in many of the reviews, there is not detail comparison on the stability, consistency and convergence of these three methods.

So, this study is intended to answer the following basic questions:

- Which method is more stable?
- Which method is better convergent?
- Which method gives more accurate result?

1.3 Objective of the study

1.3.1 General objective

The general objective of this thesis is to analysis of Rayleigh-Ritz method, Galerkins method and finite difference method for boundary value problems of ordinary differential equations(ODEs).

1.3.2 Specific objectives

The specific objectives of this study are:

- 1) To determine the stability of finite element method and finite difference method.
- 2) To determine the consistency and convergence of finite element method and finite difference method.
- 3) Comparing results of Rayleigh-Ritz method, Galerkin method, finite difference method and exact value.

1.4 Significance of the study

The main purposes of the study are:

- For applying finite element method and finite difference method in the solution of BVPs.
- For the growth of research in the area.
- To develop skills in using finite element method and finite difference method.
- Moreover it provides several guides to all natural science students, teachers and anyone who works on this area.

1.5 Limitation

There are problems or shortages under study this research work. some of them are:

- Inaccessibility of internet
- Lack of related reference books
- Shortage of time

1.6 Delimitation

The research study is mainly delimited on comparison of finite element method and finite difference method by explaining some behaviors and solving problems on it.

Chapter 2

Review of Related Literature

A class of Ordinary differential equations is a particular importance in engineering and science, because of their frequency of occurrence and the extensive analytical methods that are available for their solution are the linear Ordinary differential equations Alen(2003). When the function involves one independent variable, then the equation is called an ordinary differential equation. There are two methods (techniques) of solving Ordinary differential equation. Those are analytic and numerical. Analytical methods are of an earlier date and imply determination of mathematical functions which define solutions in a closed form.

The numerical methods are methods of approximation. The basic Characteristics of numerical methods are that the fundamental equations which describe the problems, including the boundary conditions are solved in approximate numerical way. For the numerical solution of boundary value problems for ordinary differential equations three different groups of methods are available: finite element method, shooting method and finite difference methods Rainer(1997). The two point boundary value problems have great importance in chemical engineering, deflection of beams, the like.

2.1 Finite Element Method

The modern use of finite elements really started in the field of structural engineering Davies(2001). Probably the first attempts were by Hrennikoff (1941) and McHenry,D.(1943), who developed analogies between actual discrete elements, for example bars and beams, and the corresponding portions of a continuous solid. These methods belonged to a class of semi-analytic techniques which were used in the 1940s for aircraft structural design.

Matrix methods for the solution of such problems were developed at this time, and it is interesting to note that the work of Argyris (1955), in an engineering context, introduced a minimization process which is also the basis of the mathematical underpinning of the finite element method. Historically, all the major practical advances of the method have taken place since the early 1950s in conjunction with the development of digital computers. However, interest in

approximate solutions of field equations dates as far back in time as the development of the classical field theories (e.g. elasticity, electro-magnetism) themselves.

Most researchers agree that the era of the finite element method begins with a lecture presented in 1941 by R. Courant to the American Association for the Advancement of Science. In his work, Courant used the Ritz method and introduced the pivotal concept of spatial discretization for the solution of the classical torsion problem R.J.Levegue(1998).

2.1.1 The Rayleigh-Ritz method

The concept of piecewise polynomial approximation offered a simple and efficient procedure for the application of the classical Rayleigh Ritz method. The principles could be clearly seen in the much earlier work of Lord Rayleigh (Strutt 1870) and Ritz (1909). In 1909 Ritz developed an effective method for the approximate solution of problems in the mechanics of deformable solids .

Minimisation of functional in relation to each unknown leads to the system of equations from which the unknown coefficients may be determined. One from the main restrictions in the Ritz method is that functions used should satisfy to the boundary conditions of the problem (R.J.Levegue(1998)).

In 1943 Courant considerably increased possibilities of the Ritz method by introduction of the special linear functions defined over triangular regions and applied the method for the solution of torsion problems.

Courant did not pursue his idea further, since computers were still largely unavailable for his research. More than a decade later Ray W. Clough, Jr. of the University of California, Berkeley, and his colleagues reinvented the FEM as a natural extension of matrix structural analysis and published their first work in 1956. In few years following its introduction to the engineering community, the FEM has attracted the attention of applied mathematicians, particularly those interested in numerical solution of PDEs.

2.1.2 Galerkins Finite Element Method

Galerkins finite element method is developed for inhomogeneous second-order ordinary differential equations. The finite element method is accurate and well behaved in the presence of singularities Gentain(2015). The work of Lord Rayleigh (1870) and W.Ritz (1909) on variational methods and the weighted-residual approach taken by B. G.Galerkin (1915) and others form the theoretical framework to the finite element method. With a bit of a stretch, one may even claim that Schellbachs approximate solution to Plateaus problem (find a surface of minimum area enclosed by a given closed curve in three dimensions), which dates back to 1851 is a rudimentary application of the finite element method.

A Galerkins finite element approach based on weighted residual utilities a piece-wise cubic application utilizing cubic hermite interpolation polynomials by(Fairweather,G.1978)and (Dogan .2004).

Numerical studies have shown the superior accuracy of the Galerkins finite element method in comparison with the differential transform method, A domain decomposition method and homogony perturbation method (Iqbal.2010).

The Galerkins method has been used to solve problems in structural engineering such as structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics, neutron transport and others. The Galerkin method can be used to approximate the solution to ordinary differential equations, partial differential equations and integral equations with a polynomial involving a set of parameters called characteristics orthogonal polynomials (Fairweather.1978).

consider the boundary value problem defined by

$$y'' + p(x)y' + q(x)y = f(x), a < x < b \quad (2.1.1)$$

with the boundary conditions

$$\begin{cases} p_0y(a) + q_0y'(a) = l_0 \\ p_1y(b) + q_1y'(b) = l_1 \end{cases} \quad (2.1.2)$$

Where $p(x)$, $q(x)$ and $f(x)$ are known continuous functions of x on $[a, b]$ and the p_i , q_i and l_i are constants, is a linear two-point boundary value problem. To find an approximate solution of the problem above, we choose base function $\phi_i(x)$ which is simple in form and satisfies the boundary conditions. Then an approximate solution $v(x)$ is assumed to be a linear combination of the ϕ_i , i.e, $v(x)$ is written as:

$$v(x) = \phi_0(x) + \sum_{i=1}^n \alpha_i \phi_i(x) \quad (2.1.3)$$

Where

- i) the function $\phi_0(x)$ is chosen to satisfy the given boundary conditions of the Problem.
- ii) The functions $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ must each satisfy the corresponding homogeneous form of the boundary conditions (the functions $\phi_i(x)$, $i = 1, 2, 3, \dots, n$).

Finite difference method is a numerical method based on mathematical discretization of differential equations, finite element method uses various forms of variational method applied to discrete model Clough R.W(1960). The finite difference theory for general initial value problems and parabolic problems then had an intense period of development during 1950s and 1960s, when the concept of stability was explored. Independently of the engineering applications, a number of papers appeared in the mathematical literature in the mid-1960s which were concerned with the construction and analysis of finite difference schemes by the Rayleigh-Ritz procedure with piecewise linear approximating functions.

The finite difference methods are always a convenient choice for solving boundary value problems because of their simplicity. Finite difference methods are one of the most widely used numerical schemes to solve differential equations. The general concept of finite difference methods in details such as stability, convergence conditions, have been presented in literature for solving differential equations. The challenge in analyzing finite difference methods for new classes of problems is often to find an approximate definition of stability that allow one to prove convergence and to estimate the error in approximation (LeVeque.998).

Also Finite difference method (FDM) is a numerical methods based on mathematical discretization of the equations of boundary problems. By using this method, continuous process is studied in a finite number of sufficiently small time intervals. So it is possible in these small intervals, the function approximated by approximate expressions. In each elementary intervals is the integration, with the results of integration in the previous interval are taken as initial for the various derivatives are derived under Taylors expansion by describing forward difference, backward difference, and central difference approximations (Jincai et al. 2011). The finite difference method for the linear second order boundary value problem,

$$f''(x) = p(x)f'(x) + q(x)f(x) + r(x), a < x < b \quad (2.1.4)$$

$f(a) = \alpha, f(b) = \beta$ where $p(x), q(x)$ and $r(x)$ are known continuous functions of x on $[a, b]$. We select an integer $N > 0$ and discretize the interval $[a, b]$ into $(N + 1)$ equal subintervals whose endpoints are the mesh points $x_i = a + ih$, for $i = 0, 1, 2, \dots, N + 1$, where $h = (\frac{a - b}{N + 1})$. But this thesis concentrates in the comparison of Rayleigh-Ritz method, Galerkin's method and finite difference method for solving boundary value problems on stability, consistency and convergence.

Chapter 3

Research Methodology

This section consists of methods and materials used to undertake the study. These are study design, study site and period, source of information procedure of the study and instrumentation.

3.1 Study Design

The design of this study is solving the discretized ordinary differential equation using weighted function for Galerkins method, variational formulation for Ritz method and tri-diagonal matrix system for FDM. We also use graphical representation and matlab as a tool.

3.2 Procedures of the study

To attain the objective of the study, the following procedures were undertaken:

- Defining the problem/formulating the method.
- Discretizing the domain/interval.
- Replacing ODEs by difference equations for finite difference method.
- Variational and weak formulation for FEM.
- Choice of trial and base function.
- solving system of equations
- Writing a code for the problem by using matlab language.
- Validating the methods using numerical examples.
- Discussing the methods from the graphs.

3.3 Instrumentation of analysis

Solving the discrete system and approximate solution using matlab codes, tables, graphs and Tri-diagonal Thomas Algorithm.

Chapter 4

RESULT AND DISCUSSION

In this chapter we discuss finite element method (Rayleigh-Ritz, Galerkins) and finite difference method for solving boundary value problems of ordinary differential equation. And then compare these methods on stability, consistency and convergence.

4.1 Boundary value problem

Generally Boundary value problem are an ordinary differential equation and partial differential equation with boundary conditions. Consider the two-point linear boundary-value problems are:

$$y'' + f(x)y'(x) + q(x)y(x) = r(x), a < x < b \quad (4.1.1)$$

With the boundary conditions $y(x_0) = a$ and $y(x_n) = b$.

Where $p(x)$, $q(x)$ and $f(x)$ are known continuous functions of x on $[a, b]$.

4.2 Raleigh-Ritz Method

The Rayleigh-Ritz method is a variational method, based on the idea of finding a solution that minimizes a functional. The boundary-value problem is first reformulated as a problem of choosing, from the set of all sufficiently differentiable functions satisfying the boundary conditions, the function to minimize a certain integral. Then the set of feasible functions is reduced in size, and an approximation is found from this set to minimize the integral. This gives our approximation to the solution of the boundary-value problem.

We consider the linear boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = f(x) \Rightarrow -py'' - p'y' + qy = f, \text{ for } a \leq x \leq b \quad (4.2.1)$$

with the boundary condition

$$y(a) = y(b) = 0. \quad (4.2.2)$$

We assume that $p(x) > 0$ and $q(x) \geq 0$, on $[a, b]$. If we multiply both sides of this equation by a test function $u(x)$, and then integrate over the domain $[a, b]$, we get

$$-\int_a^b u(x)\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + u(x)q(x)y(x)dx = \int_a^b u(x)f(x)dx \quad (4.2.3)$$

Applying integration by parts, we obtain

$$\int_a^b u(x) \frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) dx = u(x) p(x) y'(x) \Big|_a^b - \int_a^b p(x) \frac{du}{dx}(x) \frac{dy}{dx}(x) dx \quad (4.2.4)$$

Let $C^2[a, b]$ be the space of all functions with twice continuous derivatives on $[a, b]$, and let $C_0^2[a, b]$ be the space of all functions in $C^2[a, b]$ that are equal to zero at the endpoints $x = a$ and $x = b$. If we require that our test function $u(x)$ belongs to $C_0^2[a, b]$, then $u(a) = u(b) = 0$, and the boundary term in the above application of integration by parts vanishes. We then have

$$\int_a^b p(x) \frac{du}{dx}(x) \frac{dy}{dx}(x) dx + q(x) u(x) y(x) = \int_a^b u(x) f(x) dx \quad (4.2.5)$$

This is called the weak form of the boundary value problem, because it only requires that the first derivative of $y(x)$ exist, as opposed to the original boundary value problem, that requires the existence of the second derivative. Furthermore, y is a solution to the boundary value problem (in either form) if and only if it is the unique function in $C_0^2[a, b]$ that minimizes the functional

$$I[u] = \int_a^b p(x) [u'(x)]^2 + q(x) [u(x)]^2 - 2f(x) u(x) dx \quad (4.2.6)$$

The problem of minimizing $I[u]$ is called the variational form of the boundary value problem. The set of functions consisting of linear combinations of certain basis functions $\phi_1, \phi_2, \dots, \phi_n$. The basis functions are linearly independent and satisfy $\phi_i(a) = \phi_i(b) = 0$, for each $i = 1, 2, \dots, n$. An approximation $\phi(x) = \sum_{i=1}^n c_i \phi_i(x)$ to the solution $y(x)$ of equation(4.2.5) is then obtained by finding constants $c_1, c_2, \dots, c_n$ to minimize the integral $I[\sum_{i=1}^n c_i \phi_i(x)]$ from equation(4.2.6)

$$I[\phi] = I\left[\sum_{i=1}^n c_i \phi_i(x)\right] = \int_a^b \left(p(x) \left[\sum_{i=1}^n c_i \phi_i'(x) \right]^2 + q(x) \left[\sum_{i=1}^n c_i \phi_i(x) \right]^2 - 2f(x) \sum_{i=1}^n c_i \phi_i(x) \right) dx, \quad (4.2.7)$$

and, for a minimum to occur, it is necessary, when considering I as a function of $c_1, c_2, \dots, c_n$ to have

$$\frac{\partial I}{\partial c_j} = 0 \quad (4.2.8)$$

for each $j = 1, 2, \dots, n$.

Differentiating equation(4.2.7) gives

$$\frac{\partial I}{\partial c_j} = \int_a^b \left(2p(x) \sum_{i=1}^n c_i \phi_i'(x) \phi_j'(x) + 2q(x) \sum_{i=1}^n c_i \phi_i(x) \phi_j'(x) - 2f(x) \phi_j(x) \right) dx,$$

and substituting in to equation(4.2.8) yields

$$\sum_{i=1}^n \left[\int_a^b \left(p(x) \phi_i'(x) \phi_j'(x) + q(x) \phi_i(x) \phi_j(x) \right) dx \right] c_i - \int_a^b f(x) \phi_j(x) dx = 0 \quad (4.2.9)$$

for each $j = 1, 2, \dots, n$. The normal equation described in equation(4.2.9) produce an n by n linear system $Ac = b$ in the variables $c_1, c_2, \dots, c_n$, where the symmetric matrix A is

$$a_{ij} = \int_a^b [p(x) \phi_i'(x) \phi_j'(x) + q(x) \phi_i(x) \phi_j(x)] dx$$

and b is defined by

$$b_i = \int_a^b f(x) \phi_j(x) dx$$

4.3 The Galerkin Method

Galerkin's method belongs to a wider class of methods called the weighted residual methods. In this method, an approximating function called the trial function (which satisfies all the boundary conditions) is substituted in the given differential equation and the result is called the residual (the result will not be zero since we have substituted an approximating function). The residual is then weighted and the integral of the product, taken over the domain, is then set to zero. It can be shown that if the Euler-Lagrange equation corresponding to a functional coincides with the differential equation of the problem, then both the Rayleigh-Ritz and Galerkin methods yield the same system of equations.

To explain Galerkin's method, we consider the boundary value problem defined by

$$y'' + p(x)y' + q(x)y = f(x), a < x < b \quad (4.3.1)$$

with the boundary conditions

$$\begin{cases} p_0 y(a) + q_0 y'(a) = l_0 \\ p_1 y(b) + q_1 y'(b) = l_1 \end{cases} \quad (4.3.2)$$

Where $p(x)$, $q(x)$ and $f(x)$ are known continuous functions of x on $[a, b]$ and the p_i , q_i and l_i are constants, is a linear two-point boundary value problem. To find an approximate solution of the problem above, we choose base function $\phi_i(x)$ which is simple in form and satisfies the boundary conditions. Then an approximate solution $v(x)$ is assumed to be a linear combination of the ϕ_i , that means $v(x)$ is written as:

$$v(x) = \phi_0(x) + \sum_{i=1}^n \alpha_i \phi_i(x) \quad (4.3.3)$$

Where

- i) The function $\phi_0(x)$ is chosen to satisfy the given boundary conditions of the Problem
- ii) The functions $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ must each satisfy the corresponding homogeneous form of the boundary conditions (the functions $\phi_i(x)$, $i = 1, 2, \dots, n$ are sometimes called coordinate function).

Now, $v(x)$ will not, in general, satisfy equation (4.3.1), but produce a residual or discrepancy. This is equal to the difference between the left-hand and right-hand sides of equation (4.3.1) when on the left side $y(x)$ is replaced by $v(x)$. If $R(v)$ is the residual, we then write

$$R(v) = v'' + p(x)v' + q(x)v - f(x) \quad (4.3.4)$$

Taking the weight function as $\phi_i(x)$, we write

$$\int_a^b \psi_i(x) R(v) dx = 0 \quad (4.3.5)$$

This yields a system of equations for the unknown parameters α_i and can be solved. In Galerkin's method, we usually take $\psi_i(x) = \phi_i(x)$. Also the trial function can be calculated by using Lagrange interpolating polynomial.

4.4 Lagrange Interpolation Polynomial

The $n + 1$ Lagrange polynomials for a set of points $(x_i, f_i)_{i=0}^n = 0$ are the degree n polynomials that satisfy the property

$$l_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \quad (4.4.1)$$

Explicitly, we may write the i^{th} Lagrange polynomial as

$$l_i(x) = \frac{(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)}{(x_i - x_0)(x_i - x_1)\dots(x_i - x_{i-1})(x_i - x_{i+1})\dots(x_i - x_n)} \quad (4.4.2)$$

Using product notation, we may also write

$$l_i(x) = \prod_{j=0, j \neq i}^n \frac{x - x_j}{x_i - x_j} \quad (4.4.3)$$

In general, the Lagrange form of the interpolating polynomial may be written as follows:

$$y_n(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 + \dots + l_n(x)f_n \quad (4.4.4)$$

or

$$y_n(x) = \sum_{i=0}^n l_i(x)f_i \quad (4.4.5)$$

with the Lagrange polynomials $l_i(x)$ defined by Equation(4.4.2).

4.4.1 Lagrange polynomial Error formula

Theorem 4.1: Suppose $x_0, x_1, \dots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \dots, x_n$, and hence in (a, b) , exists with

$$f(x) = R(x) + \frac{f^{(n+1)}(\xi(x))(x - x_0)(x - x_1)\dots(x - x_n)}{(n + 1)!} \quad (4.4.6)$$

Where $R(x)$ is the interpolating polynomial.

Proof. Note that if $x = x_k$ for any $k = 0, 1, \dots, n$, then $f(x_k) = R(x_k)$ and choosing $\xi(x_k)$ arbitrarily in (a, b) . If $x \neq x_k$, for all $k = 0, 1, 2, \dots, n$ define the function g for s in (a, b) by

$$\begin{aligned} g(s) &= f(s) - R(s) - [f(x) - R(x)] \frac{(s - x_0)(s - x_1)\dots(s - x_n)}{(x - x_0)(x - x_1)\dots(x - x_n)} \\ &= f(s) - R(s) - [f(x) - R(x)] \prod_{i=0}^n \frac{(s - x_i)}{(x - x_i)} \end{aligned}$$

Since $f \in C^{n+1}[a, b]$ and $R \in C^\infty[a, b]$, it follows that $g \in C^{n+1}[a, b]$.
For $s = x_k$, we have

$$g(x_k) = f(x_k) - R(x_k) - [f(x) - R(x)] \prod_{i=0}^n \frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - R(x)] \cdot 0 = 0$$

Moreover,

$$g(x) = f(x) - R(x) - [f(x) - R(x)] \prod_{i=0}^n \frac{(x - x_i)}{(x - x_i)} = f(x) - R(x) - [f(x) - R(x)] = 0$$

Thus $g \in C^{n+1}[a, b]$, and g is zero at the $n + 2$ distinct numbers $x_0, x_1, \dots, x_n$. By Generalized Rolles Theorem there exists a number ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$. So

$$g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - R^{(n+1)}(\xi) - [f(x) - R(x)] \frac{d^{n+1}}{ds^{n+1}} \left[\prod_{i=0}^n \frac{(s - x_i)}{(x - x_i)} \right]_{s=\xi} = 0 \quad (4.4.7)$$

However $R(x)$ is polynomial of degree at most n , so the $(n+1)$ st derivative, $R^{(n+1)}(x)$ is identically zero. Also $\prod_{i=0}^n \frac{(s - x_i)}{(x - x_i)}$ is a polynomial of degree $(n + 1)$, so

$$\prod_{i=0}^n \frac{(s - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^n (x - x_i)} \right] s^{n+1} + (\text{lower degree terms in } s) \text{ and}$$

$$\frac{d^{n+1}}{ds^{n+1}} \prod_{i=0}^n \frac{(s - x_i)}{(x - x_i)} = \frac{(n+1)!}{\prod_{i=0}^n (x - x_i)} \text{ equation (4.4.7) now becomes}$$

$$f^{(n+1)}(\xi) - 0 - [f(x) - R(x)] \frac{(n+1)!}{\prod_{i=0}^n (x - x_i)} = 0$$

and, upon solving for $f(x)$, we have

$$f(x) = R(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$

Where $\frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$ is the truncation error of $f(x)$ denoted by E_n

Hence the truncation error of the Lagrange polynomial of degree n is given by

$$E_n = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i) = f(x) - R(x)$$

□

4.5 Finite Difference Method

The finite difference method for the solution of a two point boundary value problem consists in replacing the derivatives occurring in the differential equations (and in the boundary conditions as well) by means of their finite difference approximation and then solving the resulting linear system of equations. The basic idea of finite difference methods for the approximate solution of boundary value problems consists in replacing the derivatives in the differential equations by difference quotients. The finite difference method for the linear second order boundary value problems,

$$y''(x) = p(x)y'(x) + q(x)y(x) + r(x), a \leq x \leq b \quad (4.5.1)$$

$y(a) = \alpha, y(b) = \beta$ where $p(x), q(x)$ and $r(x)$ are known continuous functions of x on $[a, b]$. We select an integer $N > 0$ and discretize the interval $[a, b]$ into $(N + 1)$ equal subintervals whose endpoints are the mesh points $x_i = a + ih$, for $i = 0, 1, 2, \dots, N + 1$, where $h = (\frac{a - b}{N + 1})$. The parameter h is called the mesh-size, and the points x_i are the mesh points. The finite difference approximations to the various derivatives are derived as Taylors series expansion. Expanding $y(x + h)$ in Taylors series, we have

$$Y(x + h) = y(x) + y'(x)h + \frac{y''(x)h^2}{2!} + \frac{y'''(x)h^3}{3!} + \dots \quad (4.5.2)$$

From which we obtain

$$y'(x) = \frac{y(x+h)-y(x)}{h} - \frac{hy''(x)}{2} - \dots$$

Thus we have

$$y'(x) = \frac{y(x+h)-y(x)}{h} + o(h)$$

Which is the forward difference approximation of $y'(x)$. Similarly, expansion of $y(x - h)$ in Taylors series gives

$$Y(x - h) = y(x) - y'(x)h + \frac{y''(x)h^2}{2!} - \frac{y'''(x)h^3}{3!} + \dots \quad (4.5.3)$$

From which we obtain

$$y'(x) = \frac{y(x)-y(x-h)}{h} + o(h)$$

Which is the backward difference approximation of $y'(x)$.

A central difference approximation for $y'(x)$ can be obtained by subtracting equation (4.5.3) from equation (4.5.2), we get

$$y'(x) = \frac{y(x+h)-y(x-h)}{2h} + o(h^2)$$

This is central difference approximation of $y'(x)$. And adding equation (4.5.3) and equation (4.5.2), we get

$$y''(x) = \frac{y(x+h)-2y(x)+y(x-h)}{h^2} + o(h^2)$$

Which is the central difference approximation of $y''(x)$.

In similar manner, it is possible to derive finite difference approximation to higher derivatives. And also

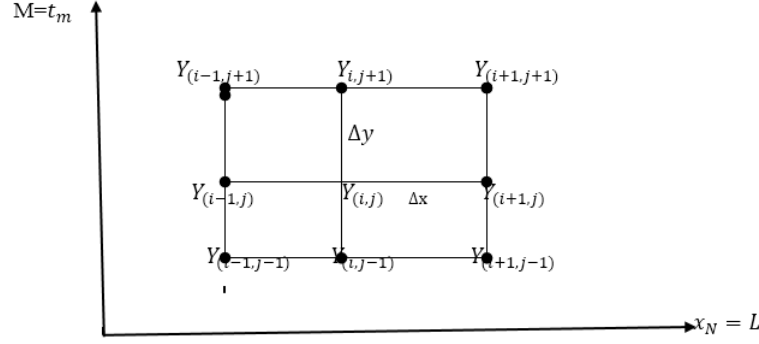


Figure 4.1: Shows the discrete finite difference approximations

Applying equation (4.5.2) and equation (4.5.3)

$$Y_{(i+1,j)} = y_{(i,j)} + \frac{\partial y_{(i,j)}}{\partial x} h + \frac{\partial^2 y}{\partial x^2} \frac{(h^2)}{2} + \frac{\partial^3 y}{\partial x^3} \frac{(h^3)}{6} + \dots \quad (4.5.4)$$

$$Y_{(i-1,j)} = y_{(i,j)} - \frac{\partial y_{(i,j)}}{\partial x} h + \frac{\partial^2 y}{\partial x^2} \frac{(h^2)}{2} - \frac{\partial^3 y}{\partial x^3} \frac{(h^3)}{6} + \dots \quad (4.5.5)$$

From equation (4.5.4) we have:

$$\left(\frac{\partial Y}{\partial x}\right)_{(i,j)} = \frac{y_{(i+1,j)} - y_{(i,j)}}{h} - \frac{\partial^2 y}{\partial x^2} \frac{(h)}{2} - \frac{\partial^3 y}{\partial x^3} \frac{(h^2)}{6} \quad (4.5.6)$$

The expression $\frac{y_{(i+1,j)} - y_{(i,j)}}{h}$ is the forward finite difference expression. The forward difference approximation with an error of order (h) is given by:

$$\left(\frac{\partial Y}{\partial x}\right)_{(i,j)} = \frac{y_{(i+1,j)} - y_{(i,j)}}{h} + O(h)$$

Similarly from equation (4.5.5) we get:

$$\left(\frac{\partial Y}{\partial x}\right)_{(i,j)} = \frac{y_{(i,j)} - y_{(i-1,j)}}{h} + O(h)$$

Which is the backward difference approximation with an error of order (h)

Subtracting equation (4.5.5) from equation (4.5.4)

$$\left(\frac{\partial Y}{\partial x}\right)_{(i,j)} = \frac{y_{(i+1,j)} - y_{(i-1,j)}}{2h} + O(h)^2$$

This is the central difference approximation with an error of order (h^2)

Adding equation (4.5.4) and equation (4.5.5)

$$\left(\frac{\partial^2 Y}{\partial x^2}\right)_{(i,j)} = \frac{y_{(i+1,j)} - 2y_{(i,j)} + y_{(i-1,j)}}{(h)^2} + O(h)^2 \quad (4.5.7)$$

This is the central difference approximation with an error of order $(h)^2$

4.5.1 Error Analysis For Finite Difference Method

Consider the second order ordinary differential equation;

$$y''(x) = f(x) \text{ for } a < x < b \quad (4.5.8)$$

with some given boundary conditions

$$y(a) = \alpha, y(b) = \beta \quad (4.5.9)$$

We will attempt to compute a grid function consisting of values $Y_0, Y_1, \dots, Y_m, Y_{m+1}$ where Y_j is our approximation to the solution $y(x_j)$. Here $x_j = jh$ and $\Delta x = h = \frac{b-a}{m+1}$ is the mesh width, the distance between grid points. From the boundary conditions we know that $Y_0 = \alpha$ and $Y_{m+1} = \beta$, and so we have m unknown values $Y_1, \dots, Y_m$ to compute. If we replace $y''(x)$ in equation(4.4.8) by the centered difference approximation

$$\frac{\partial^2 Y_j}{\partial x^2} = \frac{1}{h^2}(Y_{j+1} - 2Y_j + Y_{j-1})$$

Then we obtain a set of algebraic equations

$$\frac{1}{h^2}(Y_{j+1} - 2Y_j + Y_{j-1}) = f(x_j), \text{ for } j = 1, 2, \dots \quad (4.5.10)$$

Note that the first equation ($j = 1$) involves the value $Y_0 = \alpha$ and the last equation ($j = m$) involves the value $Y_{m+1} = \beta$. We have a linear system of m equations for the m unknowns, which can be written in the form

$$AY = F \quad (4.5.11)$$

Where Y is the vector of unknowns $Y = [Y_1, Y_2, \dots, Y_m]^T$ and

$$A = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}, \quad F = \begin{pmatrix} f(x_1) - \frac{\alpha}{h^2} \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \frac{\beta}{h^2} \end{pmatrix} \quad (4.5.12)$$

This tridiagonal linear system is non singular and can be easily solved for Y from any right hand side F . We know that the centered difference approximation $\frac{\partial^2 y}{\partial x^2}$, when applied to a known smooth function $y(x)$, gives a second order accurate approximation to $y''(x)$. The error in the discrete values $y_1, y_2, \dots, y_m$, in relation to the true solution $y(x)$, which is a function. Since y_j is supposed to approximate $y(x_i)$ it is natural to use the point wise errors $Y_j - y(x_j)$. If we let $\hat{Y}$ be the vector of true values.

$$\hat{Y} = \begin{pmatrix} y(x_1) \\ y(x_2) \\ y(x_3) \\ \vdots \\ y(x_m) \end{pmatrix} \quad (4.5.13)$$

then the error vector E defined by $E = Y - \hat{Y}$, contains the errors at each grid point. To obtain a bound on the magnitude of this vector, showing that it is $o(h)^2$ as $h \rightarrow 0$. To measure the magnitude of this vector we must use some norm, for example, the max-norm

$$\|E\|_{\infty} = \max_{1 \leq j \leq m} |E_j| = \max_{1 \leq j \leq m} |Y_j - y(x_j)|$$

We first compute the local truncation error (LTE) of the method and then use some form of stability to show that the global error can be bounded in terms of the LTE. The global error simply refers to the error $Y - \hat{Y}$ that we are attempting to bound. The LTE refers to the error in our finite difference approximation of derivatives and hence is something that can be easily estimated using Taylor series expansions.

4.5.2 Local truncation error For Finite Difference Method

The local truncation error is defined by replacing Y_j with the true solution $y(x_j)$ in the finite difference formula (4.5.10). In general the true solution $y(x_j)$ won't satisfy this equation exactly and the discrepancy is the LTE, which we denote by η_j

$$\eta_j = \frac{1}{h^2}(y(x_{j+1}) - 2y(x_j) + y(x_{j-1})) - f(x_j) \quad (4.5.14)$$

for $j = 1, 2, \dots, m$. Of course in practice we don't know what the true solution $y(x)$ is, but if we assume it is smooth, then by the Taylor series expansions we know that

$$\eta_j = [y''(x_j) + \frac{1}{12}h^2 y''''(x_j) + O(h^4)] - f(x_j) \quad (4.5.15)$$

Using our original differential equation (4.5.8) this becomes

$$\eta_j = \frac{1}{12}h^2 y''''(x_j) + O(h^4) \quad (4.5.16)$$

If we define η to be the vector with components η_j , then

$$\eta = A\hat{Y} - F$$

Where $\hat{Y}$ is the vector of true solution values equation(4.4.11), and so

$$A\hat{Y} = F + \eta \quad (4.5.17)$$

Global error

To obtain a relation between the local error η and the global error $E = Y - \hat{Y}$, we subtract equation(4.5.17) from equation(4.5.11) that defines Y , obtaining

$$AE = -\eta \quad (4.5.18)$$

This is simply the matrix form of the system of equations

$$\frac{1}{h^2}(E_{j+1} - 2E_j + E_{j-1}) = -\eta(x_j), \text{ for } j = 1, 2, \dots, m$$

with the boundary conditions

$$E_0 = E_{m+1} = 0$$

Since we are using the exact boundary data $Y_0 = \alpha$ and $Y_{m+1} = \beta$. We see that the global error satisfies a set of finite difference equations that has exactly the same form as our original difference equations for Y except that the right-hand side is given by $-\eta$ rather than F . From this it should be clear why we expect the global error to be roughly the same magnitude as the local error η . We can interpret the system equation(4.5.18) as a discretization of the ODE

$$e''(x) = -\eta(x), \text{ for } a < x < b \quad (4.5.19)$$

With boundary conditions $e(a) = 0, e(b) = 0$.

Since $\eta(x) = \frac{1}{12}h^2y''''(x)$, integrating twice shows that the global error should be roughly

$$e(x) = -\frac{1}{12}h^2y''(x) + \frac{1}{12}h^2(y(0) + x(y''(1) - y(0)))$$

Theorem 4.2: The maximum pointwise error of the linear finite difference method is given by

$$\max_{k=1,2,\dots,m} |y(t_k) - y_k| \leq Ch^2$$

where $y(t_k)$ is the exact solution at t_k and y_k is the corresponding approximate solution obtained by the finite difference method.

Proof. Let f is a linear function of y and y' , i.e.

$$f(t, y(t), y'(t)) = u(t) + v(t)y(t) + w(t)y'(t)$$

or $y_0 = \alpha$

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = u_k + v_k y_k + w_k \frac{y_k + 1 - y_{k-1}}{2h}, k = 1, 2, \dots, m \quad (4.5.20)$$

$y_m + 1 = \beta$

For the exact solution we have for any $k = 1, \dots, m$

$$y''(t_k) = u(t_k) + v(t_k)y(t_k) + w(t_k)y'(t_k),$$

or

$$\frac{y(t_k + h) - 2y(t_k) + y(t_k - h)}{h^2} - \frac{h^2}{12}y^{(4)}(\tau_k) = u_k + v_k y(t_k) + w_k \left[\frac{y(t_k + h) - y(t_k - h)}{2h} - \frac{h^2}{6}y^{(3)}(\eta_k) \right] \quad (4.5.21)$$

whereas for the approximate solution we have the relation

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = u_k + v_k y_k + w_k \frac{y_{k+1} - y_{k-1}}{2h}$$

Subtracting equation(4.5.20) from equation (4.5.21) yields

$$\frac{e_{k+1} - 2e_k + e_{k-1}}{h^2} = v_k e_k + w_k \frac{e_{k+1} - e_k}{2h} + h^2 g_k \quad (4.5.22)$$

Where

$$e_k = y(t_k) - y_k \text{ and } g_k = \frac{1}{12}y''''(\tau_k) - \frac{1}{6}y''''(\eta_k)$$

Since equation(4.5.22) is analogous to equation(4.5.20) it can be rewritten as

$$(-1 - \frac{w_k}{2}h)e_{k-1} + (2 + h^2 v_k)e_k + (-1 + \frac{w_k}{2}h)e_{k+1} = -h^4 g_k$$

Then we get

$$|(2 + h^2 v_k)e_k| = | - (-1 - \frac{w_k}{2}h)e_{k-1} - (-1 + \frac{w_k}{2}h)e_{k+1} - h^4 g_k |$$

and using the triangle inequality

$$|(2 + h^2 v_k)e_k| \leq |(-1 - \frac{w_k}{2}h)e_{k-1}| + |(-1 + \frac{w_k}{2}h)e_{k+1}| + |h^4 g_k|$$

Now we let $\lambda = \|e\|_\infty = \max_{j=1,2,\dots,m} |e_j|$, and pick the index k such that

$$|e_k| = \|e\|_\infty = \lambda$$

i.e., we look at the largest of the errors. Therefore

$$|2 + h^2 v_k| \underbrace{|e_k|}_{=\lambda} \leq h^4 |g_k| + | -1 + \frac{w_k}{2}h | \underbrace{|e_{k+1}|}_{\leq \lambda} + | -1 - \frac{w_k}{2}h | \underbrace{|e_{k-1}|}_{\leq \lambda}$$

Using the definition of λ , and bounding $|g_k|$ by its maximum we have

$$\lambda(|2 + h^2 v_k| - | -1 + \frac{w_k}{2}h | - |1 - \frac{w_k}{2}h|) \leq h^4 \|g\|_\infty$$

Using the same assumptions and arguments as in the diagonal dominance discussion above, the expression in parentheses is equal to $h^2 v_k$ and therefore we have

$$\lambda h^2 v_k \leq h^4 \|g\|_\infty \Leftrightarrow \lambda v_k \leq h^2 \|g\|_\infty$$

Or, since $\lambda = \|e\|_\infty$

Therefore

$$\max_{k=1,2,\dots,m} |y(t_k) - y(k)| \leq Ch^2$$

Where

$$C = \frac{\|g\|_\infty}{\min_{a \leq t \leq bv(t)}}$$

□

4.5.3 Numerical Examples and Results

Example(1): Solve the boundary value problem given by

$$\frac{d^2y}{dx^2} - y = x, 0 \leq x \leq 1 \quad (4.5.23)$$

with $y(0) = y(1) = 0$

And the exact solution: $y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

Solution:

Method (1)

Using Rayleigh-Ritz finite element method.

Given: $h = \triangle x = \frac{1}{4} = 0.25, x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$

Variational formulation:

$$I = \int_0^1 f(x, y, y') dx$$

To find the approximate solution by the Rayleigh-Ritz method we take the function in the form

$$I(v) = \int_0^1 (vv'' - v^2 - 2vx) dx \quad (4.5.24)$$

From the Lagrange interpolation polynomial, we select the base function as

$$V(x) = a_1x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}) - \frac{3a_2x}{4}(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}) + a_3x(8x^3 - 14x^2 + 7x - 1) \quad (4.5.25)$$

With $V(0) = V(1) = 0$.

The base functions are:

$$\phi_1(x) = x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}), \phi_2(x) = x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}), \phi_3(x) = x(8x^3 - 14x^2 + 7x - 1)$$

They satisfies the boundary conditions $\phi_i(0) = 0$ and $\phi_i(1) = 0$ for $i = 1, 2, 3$.

Substituting for equation(4.5.25) in equation(4.5.24) we obtain

$$I(v) = \int_0^1 [\sum_{i=1}^n a_i \phi_i(x) \sum_{j=1}^n a_j \phi_j''(x) - \sum_{i=1}^n \alpha_i \phi_i(x) \sum_{j=1}^n a_j \phi_j(x) - 2x \sum_{j=1}^n a_j \phi_j(x)] dx \quad (4.5.26)$$

We let $p_i = \int_0^1 x \phi_i(x) dx, q_{ij} = \int_0^1 \phi_i(x) \phi_j''(x) dx = - \int_0^1 \phi_i'(x) \phi_j'(x) dx$ and $r_{ij} = \int_0^1 \phi_i(x) \phi_j(x) dx$ equation (4.5.26) now becomes

$$I(v) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j q_{ij} - \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j r_{ij} - 2 \sum_{i=1}^n a_i p_i \quad (4.5.27)$$

for minimum, we get

$$\frac{\partial I}{\partial \alpha_i} = 2 \sum_{j=1}^n \alpha_j q_{ij} - 2 \sum_{j=1}^n \alpha_j r_{ij} - 2p_i = 0 \quad (4.5.28)$$

Which simplifies to

$$\sum_{j=1}^n \alpha_j (q_{ij} - r_{ij}) = p_i, i = 1 \dots n \quad (4.5.29)$$

To obtain an approximate solution ,we take $n = 3$ then equation(4.5.29) becomes

$$a_1(q_{11} - r_{11}) + a_2(q_{12} - r_{12}) + a_3(q_{13} - r_{13}) = -p_1, a_1(q_{21} - r_{21}) + a_2(q_{22} - r_{22}) + a_3(q_{23} - r_{23}) = -p_2 \quad (4.5.30)$$

And

$$a_1(q_{31} - r_{31}) + a_2(q_{23} - r_{32}) + a_3(q_{33} - r_{33}) = -p_3 \quad (4.5.31)$$

Now

$$\begin{aligned} p_1 &= \int_0^1 x(x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))dx = -0.127083333 \\ p_2 &= \int_0^1 x(x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}))dx = 0.001041667 \\ p_3 &= \int_0^1 x(x(8x^3 - 14x^2 + 7x - 1))dx = -0.05 \\ q_{11} &= \int_0^1 (4x^3 - \frac{27x^2}{4} + \frac{13x}{4} - \frac{3}{8})^2 dx = -0.00967262 \\ q_{12} &= \int_0^1 (4x^3 - \frac{27x^2}{4} + \frac{13x}{4} - \frac{3}{8})(4x^3 - 6x^2 + \frac{19x}{8} - \frac{3}{16})dx = 0.00550595 \\ q_{13} &= \int_0^1 (4x^3 - \frac{27x^2}{4} + \frac{13x}{4} - \frac{3}{8})(32x^3 - 42x^2 + 14x - 1)dx = -0.394494047 \\ q_{22} &= \int_0^1 (4x^3 - 6x^2 + \frac{19x}{8} - \frac{3}{16})^2 dx = -0.5576637 \\ q_{23} &= \int_0^1 (4x^3 - 6x^2 + \frac{19x}{8} - \frac{3}{16})(32x^3 - 42x^2 + 14x - 1)dx = 0.00550595 \\ q_{33} &= \int_0^1 (32x^3 - 42x^2 + 14x - 1)^2 dx = -18.61990476 \\ r_{11} &= \int_0^1 (x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))^2 dx = 0.27705853 \\ r_{12} &= \int_0^1 (x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))(x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}))dx = 0.26564901 = r_{21} \\ r_{22} &= \int_0^1 (x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}))^2 dx = 0.041747271 \\ r_{13} &= \int_0^1 (x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))(x(8x^3 - 14x^2 + 7x - 1))dx = 4.50019841 \\ r_{23} &= \int_0^1 (x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}))(x(8x^3 - 14x^2 + 7x - 1))dx = 0.000198413 = r_{32} \\ r_{33} &= \int_0^1 (x(8x^3 - 14x^2 + 7x - 1))^2 dx = 0.011111111 \end{aligned}$$

From the above equation(4.5.30)and equation(4.5.31) we get,the linear system (3 by 3)

$$\begin{aligned} -0.2867115a_1 - 0.2601306a_2 - 4.894692457a_3 &= 0.12708333 \\ -0.2601306a_1 - 0.047323908a_2 + 0.043849207a_3 &= -0.001041667 \\ -4.894692457a_1 + 0.043849207a_2 - 18.63015871a_3 &= 0.005 \end{aligned}$$

This equation forms systems of linear equation,when solving this system of equation simultaneously or forming a matrix and then using Mat-lab we get:

$$a_1 = -1.486523, a_2 = -4.82985131, a_3 = 0.2668987$$

Hence, we have to obtain the trial solution as:

$$V(x) = a_1x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}) - \frac{3a_2x}{4}(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}) + a_3x(8x^3 - 14x^2 + 7x - 1)$$

With a_1, a_2 and a_3 are stated above.

Therefore, the numerical solutions obtained by Rayleigh-Ritz method are as follow:

$$V(0) = 0$$

$$V(0.25) = 0.03484046$$

$$V(0.5) = 0.05686656$$

$$V(0.75) = 0.050043496$$

$$V(1) = 0$$

Thus the errors are given by

$$y(0) - v(0) = 0$$

$$y(0.25) - v(0.25) = 0.00020714$$

$$y(0.5) - v(0.5) = 0.000276002$$

$$y(0.75) - v(0.75) = 0.000232354$$

$$y(1) - v(1) = 0$$

Therefore, the maximum absolute error is given by; $E = 0.000276002$.

Table 4.1: Shows the comparison of analytical solution with the Rayleigh-Ritz method solution

x	Exact solution	Rayleigh-Ritz solution	Error
0	0	0	0
0.25	0.0350476	0.03484046	0.00020714
0.5	0.056590558	0.05686656	0.000276002
0.75	0.05027585	0.050043496	0.000232354
1	0	0	0

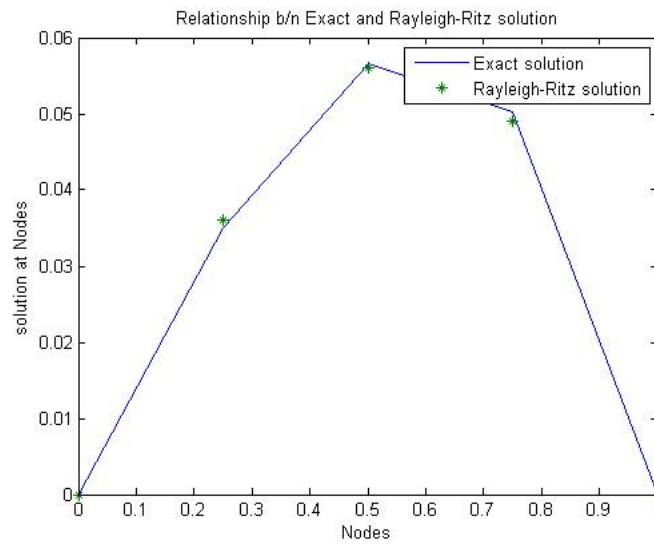


Figure 4.2: The relationship between exact solution with Rayleigh-Ritz solution of example(1)

Method(2)

Solve the above equation using Galerkin's finite element method.

Given: $h = \Delta x = \frac{1}{4} = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

Also from the Lagrange interpolation polynomial, we select the base function as

x	0	0.25	0.5	0.75	1
y	0	c_1	c_2	c_3	0

$$V(x) = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)y_1}{(\frac{1}{4}-0)(\frac{1}{4}-\frac{1}{2})(\frac{1}{4}-\frac{3}{4})(\frac{1}{4}-1)} + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)y_2}{(\frac{1}{2}-0)(\frac{1}{2}-\frac{1}{4})(\frac{1}{2}-\frac{3}{4})(\frac{1}{2}-1)} + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)y_3}{(\frac{3}{4}-0)(\frac{3}{4}-\frac{1}{4})(\frac{3}{4}-\frac{1}{2})(\frac{3}{4}-1)}$$

And simplify the above equation we get:

$$V(x) = c_1x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}) - \frac{3c_2x}{4}(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}) + c_3x(8x^3 - 14x^2 + 7x - 1)$$

With $V(0) = V(1) = 0$

The base functions are:

$$\phi_1(x) = x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}), \phi_2(x) = x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}), \phi_3(x) = x(8x^3 - 14x^2 + 7x - 1)$$

, they satisfies the boundary conditions $\phi_i(0) = 0$ and $\phi_i(1) = 0$ for $i = 1, 2, 3$.

The corresponding residual function $R(x)$ is given as:

$$R(x) = y'' - y - x$$

$$R(v) = v'' - v - x, \text{ with}$$

$$v''(x) = c_1(12x^2 - \frac{27x}{2} + \frac{13}{4}) - \frac{3c_2}{4}(12x^2 - 12x + \frac{19}{8}) + c_3(96x^2 - 84x + 14)$$

Substituting v'' and v in to $R(v)$ we get

$$R(v) = \frac{83x^2c_1}{8} - \frac{105c_1x}{8} + \frac{13c_1}{4} - \frac{519c_2x^2}{64} + \frac{567c_2x}{64} - \frac{57c_2}{32} - c_1x^4 + \frac{9c_1x^3}{4} + \frac{3c_2x^4}{4} - \frac{3c_2x^3}{2} - 32c_3x^4 + 14c_3x^3 + 89c_3x^2 - 83c_3x + 14c_3 - x$$

Following Galerkins, we use $R(V)$ as weighting function to obtain the function

$$\int_0^1 R(v)\varphi_i(x)dx = 0; \varphi_i(x) = \phi_i(x)$$

is base function.

So that, we now gives the three equations as follow:

$$\int_0^1 (v'' - v - x)(x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))dx = 0, \text{ since } \phi_1(x) = x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}). \dots\dots (1)$$

Substituting the expression of V and V'' into the above equation(1), we obtain:

$$\int_0^1 (\frac{83x^2c_1}{8} - \frac{105c_1x}{8} + \frac{13c_1}{4} - \frac{519c_2x^2}{64} + \frac{567c_2x}{64} - \frac{57c_2}{32} - c_1x^4 + \frac{9c_1x^3}{4} + \frac{3c_2x^4}{4} - \frac{3c_2x^3}{2} - 32c_3x^4 + 14c_3x^3 + 89c_3x^2 - 83c_3x + 14c_3 - x)(x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}))dx = 0$$

Integrating and simplifying the above equation we get

$$\Rightarrow -0.00984623c_1 + 0.004148078c_2 + 0.127182532c_3 + 0.002083333 = 0. \dots(A)$$

$$\int_0^1 (v'' - v - x)(x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16})), \text{ since } \phi_2(x) = x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16})dx = 0. \dots\dots(2)$$

Substituting the expression of V and V'' into the above equation(2), we obtain

$$\int_0^1 \left(\frac{83x^2c_1}{8} - \frac{105c_1x}{8} + \frac{13c_1}{4} - \frac{519c_2x^2}{64} + \frac{567c_2x}{64} - \frac{57c_2}{32} - c_1x^4 + \frac{9c_1x^3}{4} + \frac{3c_2x^4}{4} - \frac{3c_2x^3}{2} - 32c_3x^4 + 14c_3x^3 + 89c_3x^2 - 83c_3x + 14c_3 - x \right) \left(x(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}) \right) dx = 0$$

Integrating and simplifying the above equation we get

$$\Rightarrow -0.983306383c_1 - 0.021656436c_2 - 0.0385591271c_3 + 0.280208333 = 0. \dots(B)$$

$$\int_0^1 (v'' - v - x)(x(8x^3 - 14x^2 + 7x - 1))dx = 0, \text{ since } \phi_3(x) = x(8x^3 - 14x^2 + 7x - 1) . \dots(3)$$

Substituting the expression of V and V'' into the above equation(3), we obtain

$$\int_0^1 \left(\frac{83x^2c_1}{8} - \frac{105c_1x}{8} + \frac{13c_1}{4} - \frac{519c_2x^2}{64} + \frac{567c_2x}{64} - \frac{57c_2}{32} - c_1x^4 + \frac{9c_1x^3}{4} + \frac{3c_2x^4}{4} - \frac{3c_2x^3}{2} - 32c_3x^4 + 14c_3x^3 + 89c_3x^2 - 83c_3x + 14c_3 - x \right) (x(8x^3 - 14x^2 + 7x - 1))dx = 0$$

Integrating and simplifying the above equation we get

$$\Rightarrow -0.027579365c_1 - 0.20431548c_2 + 0.036507979c_3 + 0.05 = 0. \dots(C)$$

Equation (A),(B)and (C) forms systems of linear equation,when solving this system of equation simultaneously or forming a matrix and then using Mat-lab we get:

$$c_1 = -1.486353067, c_2 = -0.765060096, c_3 = 0.266898832$$

Hence,we have to obtain the trial solution as:

$$V(x) = c_1x(x^3 - \frac{9x^2}{4} + \frac{13x}{8} - \frac{3}{8}) - \frac{3c_2x}{4}(x^3 - 2x^2 + \frac{19x}{16} - \frac{3}{16}) + c_3x(8x^3 - 14x^2 + 7x - 1)$$

With c_1, c_2 and c_3 are stated above.

Therefore, the numerical solutions obtained by Galerkins method are as follow:

$$V(0) = 0$$

$$V(0.25) = 0.0348364$$

$$V(0.5) = 0.055840548$$

$$V(0.75) = 0.050043531$$

$$V(1) = 0$$

Thus the errors are given by

$$\begin{aligned}
 y(0) - v(0) &= 0 \\
 y(0.25) - v(0.25) &= 0.0002112 \\
 y(0.5) - v(0.5) &= 0.000275001 \\
 y(0.75) - v(0.75) &= 0.000232254 \\
 y(1) - v(1) &= 0
 \end{aligned}$$

Therefore, the maximum absolute error is given by; $E = 0.000275001$.

Table 4.2: Shows the comparison of analytical solution with the Galerkin's method solution

x	Exact solution	Galerkin's solution	Error
0	0	0	0
0.25	0.0350476	0.0348364	0.0002112
0.5	0.056590558	0.055840548	0.000275001
0.75	0.050275785	0.050043531	0.000232254
1	0	0	0

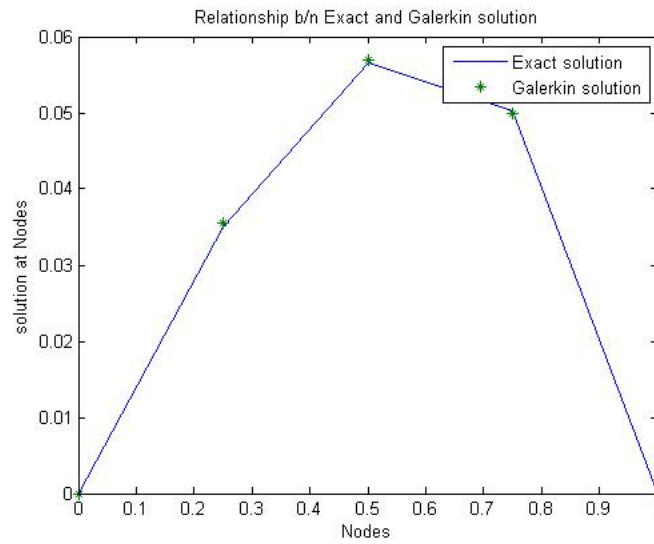


Figure 4.3: The relationship between exact solution with Galerkin solution of example(1)

Method(3)

Solve the above equation using finite difference method.

case (1)

First the discrete form of the equation is as follows

Given: $h = \Delta x = \frac{1}{10} = 0.1, x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, \dots, x_{10} = 1$

$$\frac{d^2 y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, y = y_i \text{ and } x = x_i.$$

Then substituting the given information into above equation gives the linear systems of equation as follow:

$$-y_{i-1} + (2 + h^2)y_i - y_{i+1} = h^2(x_i) \text{ for } i = 1, 2, \dots, 9$$

$$\text{When } i = 1 : 2.01y_1 - y_2 = 0.001$$

$$\text{When } i = 2 : -y_1 + 2.01y_2 - y_3 = 0.002$$

$$\text{When } i = 3 : -y_2 + 2.01y_3 = 0.003$$

$\vdots$

$$\text{When } i = 9 : -y_8 + 2.01y_9 = 0.009$$

The linear system of equation can be writing as a tri-diagonal matrix of the form:

$AY_i = B, A$ is tri-diagonal matrix and B is a vector

$$\begin{pmatrix} 2.01 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2.01 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2.01 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2.01 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2.01 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2.01 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2.01 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.01 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.01 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{pmatrix} = \begin{pmatrix} 0.001 \\ 0.002 \\ 0.003 \\ 0.004 \\ 0.005 \\ 0.006 \\ 0.007 \\ 0.008 \\ 0.009 \end{pmatrix} \quad (4.5.32)$$

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From the above tri-diagonal matrix form

Let

$$\begin{aligned} [a_2, a_3, \dots, a_9] &= [-1, -1, -1, -1, -1, -1, -1, -1] \\ [b_1, b_2, \dots, b_9] &= [2.01, 2.01, 2.01, 2.01, 2.01, 2.01, 2.01, 2.01, 2.01] \\ [c_1, c_2, \dots, c_8] &= [-1, -1, -1, -1, -1, -1, -1, -1] \end{aligned}$$

First put

$$\begin{aligned} u_1 &= b_1 = 2.01 \\ \text{and } u_i &= b_i - \frac{a_i c_{i-1}}{u_{i-1}}, i = 2, 3, \dots, 9 \\ u_2 &= b_2 - \frac{a_2 c_1}{u_1} = 2.01 - \frac{-(1)(-1)}{2.01} \approx 1.5125 \end{aligned}$$

$$\begin{aligned}
u_3 &= b_3 - \frac{a_3 c_2}{u_2} \approx 1.3488 \\
u_4 &= b_4 - \frac{a_4 c_3}{u_3} \approx 1.2686 \\
u_5 &= b_5 - \frac{a_5 c_4}{u_4} \approx 1.2217 \\
u_6 &= b_6 - \frac{a_6 c_5}{u_5} \approx 1.1915 \\
u_7 &= b_7 - \frac{a_7 c_6}{u_6} \approx 1.1707 \\
u_8 &= b_8 - \frac{a_8 c_7}{u_7} \approx 1.1558 \\
u_9 &= b_9 - \frac{a_9 c_8}{u_8} \approx 1.1448
\end{aligned}$$

Step 2: Let $[d_1, d_2, d_3, \dots, d_9] = [0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009]$

Put

$$\begin{aligned}
z_1 &= \frac{d_1}{b_1} = \frac{0.001}{2.01} = 0.0004975124, \text{ and} \\
z_i &= \frac{d_i - a_i z_{i-1}}{u_1}, i = 2, 3 \\
z_2 &= \frac{d_2 - a_2 z_1}{u_2} = \frac{0.002 - (-1)(0.0004975124)}{1.5125} = 0.0016512479 \\
z_3 &= \frac{d_3 - a_3 z_2}{u_3} = 0.0034483343 \\
z_4 &= \frac{d_4 - a_4 z_3}{u_4} = 0.005871307 \\
z_5 &= \frac{d_5 - a_5 z_4}{u_5} = 0.008985078 \\
z_6 &= \frac{d_6 - a_6 z_5}{u_6} = 0.012576649 \\
z_7 &= \frac{d_7 - a_7 z_6}{u_7} = 0.016722173 \\
z_8 &= \frac{d_8 - a_8 z_7}{u_8} = 0.021389663 \\
z_9 &= \frac{d_9 - a_9 z_8}{u_9} = 0.026545827
\end{aligned}$$

Step 3: If $y_i = [y_1, y_2, y_3, \dots, y_9]$

Put $y_i = z_i - \frac{c_i y_{i+1}}{u_i}, i = 1, 2, \dots, 9$

$$\begin{aligned}
y_9 &= z_9 - \frac{c_9 y_{10}}{u_9} = 0.026545827 - \frac{(-1)(0)}{1.1448} = 0.026545827 \\
y_8 &= z_8 - \frac{c_8 y_9}{u_8} = 0.021389663 - \frac{(-1)(0.026545827)}{1.1558} = 0.044357154 \\
y_7 &= z_7 - \frac{c_7 y_8}{u_7} = 0.054611602 \\
y_6 &= z_6 - \frac{c_6 y_7}{u_6} = 0.058410977 \\
y_5 &= z_5 - \frac{c_5 y_6}{u_5} = 0.056796305 \\
y_4 &= z_4 - \frac{c_4 y_5}{u_4} = 0.05064216 \\
y_3 &= z_3 - \frac{c_3 y_4}{u_3} = 0.04099442 \\
y_2 &= z_2 - \frac{c_2 y_3}{u_2} = 0.028754996 \\
y_1 &= z_1 - \frac{c_1 y_2}{u_1} = 0.01480348
\end{aligned}$$

Therefore, the solution of the system of equation is

$$\begin{aligned}
y_1 &= 0.01480348 \\
y_2 &= 0.028754996 \\
y_3 &= 0.04099442 \\
y_4 &= 0.05064216 \\
y_5 &= 0.056796305 \\
y_6 &= 0.058410977
\end{aligned}$$

$$y_7 = 0.054611602$$

$$y_8 = 0.044357154$$

$$y_9 = 0.026545827$$

The exact solution becomes: $y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

$$y_1(x_1) - y_1 = 0.00003725$$

$$y_2(x_2) - y_2 = 0.00007545$$

$$y_3(x_3) - y_3 = 0.000116259$$

$$y_4(x_4) - y_4 = 0.00015876$$

$$y_5(x_5) - y_5 = 0.000205746$$

$$y_6(x_6) - y_6 = 0.000015105$$

$$y_7(x_7) - y_7 = 0.000104226$$

$$y_8(x_8) - y_8 = 0.0000626335$$

$$y_9(x_9) - y_9 = 0.000027518$$

Therefore, the absolute maximum error is $E = 0.000205746$.

Table 4.3: Shows the comparison of analytical solution with the finite difference method solution

x	Exact solution	finite difference solution	Error
0	0	0	0
0.1	0.01476623	0.01480348	0.00003725
0.2	0.028679545	0.028754996	0.000075451
0.3	-0.040878161	0.040994419	0.000116259
0.4	0.050483399	0.05064216	0.000148761
0.5	0.056590558	0.056796305	0.000205746
0.6	0.058259925	0.058410977	0.0000151052
0.7	0.054507376	0.054611602	0.000104226
0.8	0.044294519	0.044357154	0.000062635
0.9	0.026518309	0.026545827	0.000027518
1	0	0	0

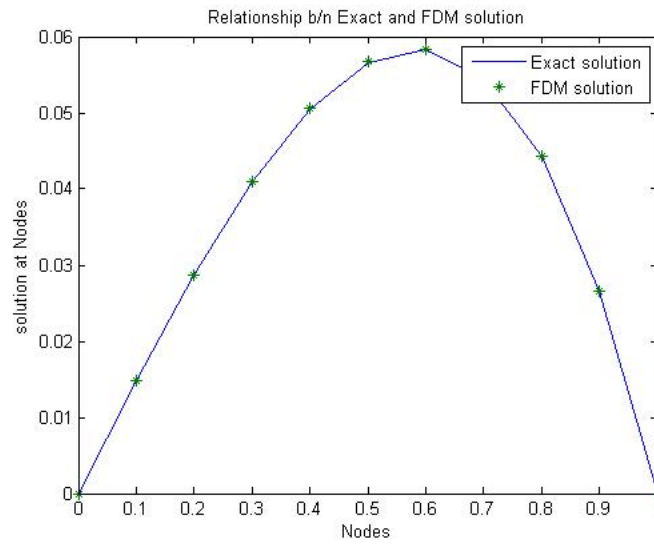


Figure 4.4: The relationship between exact solution with FDM solution of example(1) in case (1)

case (2)

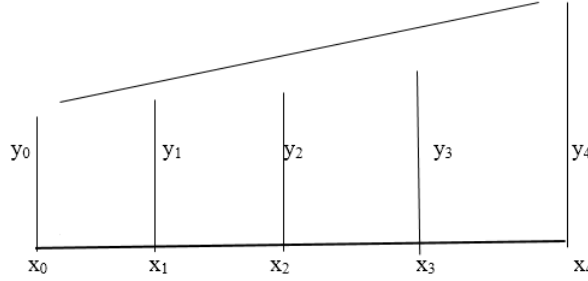


Figure 4.5: The discrete form of the equation is as follow:

Given: $h = \Delta x = \frac{1}{4} = 0.25, x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$
 $\frac{d^2 y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, y = y_i$ and x_i .

Then substituting the given information into above equation gives the linear systems of equation as follow:

$$-y_{i-1} + (2 + h^2)y_i - y_{i+1} = h^2(x_i) \text{ for } i = 1, 2, 3$$

$$\text{When } i = 1 : 2.0625y_1 - y_2 = 0.0156$$

$$\text{When } i = 2 : -y_1 + 2.0625y_2 - y_3 = 0.0313$$

$$\text{When } i = 3 : -y_2 + 2.0625y_3 = 0.0469$$

The linear system of equation can be writing as a tri-diagonal matrix of the form:

$AY_i = B, A$ is tri-diagonal matrix and B is a vector

$$\begin{pmatrix} 2.0625 & -1 & 0 \\ -1 & 2.0625 & -1 \\ 0 & -1 & 2.0625 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.0156 \\ 0.0313 \\ 0.0469 \end{pmatrix} \quad (4.5.33)$$

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From the above tri-diagonal matrix form

Let

$$\begin{aligned} [a_2, a_3] &= [-1, -1] \\ [b_1, b_2, b_3] &= [2.0625, 2.0625, 2.0625] \\ [c_1, c_2] &= [-1, -1] \end{aligned}$$

First put

$$\begin{aligned} u_1 &= b_1 = 2.0625 \\ \text{and } u_i &= b_i - \frac{a_i c_{i-1}}{u_{i-1}}, i = 2, 3 \\ u_2 &= b_2 - \frac{a_2 c_1}{u_1} = 2.0625 + \frac{-(-1)(-1)}{2.0625} \approx 1.577652 \\ u_3 &= b_3 - \frac{a_3 c_2}{u_2} = 2.0625 - \frac{(-1)(-1)}{1.577652} \approx 1.428647 \end{aligned}$$

Step 2: Let $[d_1, d_2, d_3] = [0.0156, 0.0313, 0.0469]$

Put

$$\begin{aligned}
z_1 &= \frac{d_1}{b_1} = \frac{0.0156}{2.0625} = 0.007564, \text{ and } z_i = \frac{d_i a_i z_{i-1}}{u_1}, i = 2, 3 \\
z_2 &= \frac{d_2 - a_2 z_1}{u_2} = \frac{0.00313 - (-1)(0.007564)}{1.577652} = 0.024634 \\
z_3 &= \frac{d_3 - a_3 z_2}{u_3} = \frac{0.0469 - (-1)(0.024634)}{1.428647} = 0.050041
\end{aligned}$$

Step 3: If $y_i = [y_1, y_2, y_3]$

Put $y_i = z_i - \frac{c_i y_{i+1}}{u_i}, i = 1, 2, 3$

$$\begin{aligned}
y_3 &= z_3 - \frac{c_3 y_4}{u_3} = 0.050041 - \frac{(-1)(0)}{1.428647} = 0.050041 \\
y_2 &= z_2 - \frac{c_2 y_3}{u_2} = 0.024634 - \frac{(-1)(0.050041)}{1.577652} = 0.056372 \\
y_1 &= z_1 - \frac{c_1 y_2}{u_1} = 0.007564 - \frac{(-1)(0.056372)}{2.0625} = 0.0348316
\end{aligned}$$

Therefore, the solution of the system of equation is

$$\begin{aligned}
y_1 &= 0.0348316 \\
y_2 &= 0.056372 \\
y_3 &= 0.050041
\end{aligned}$$

The exact solution becomes: $y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

$$\begin{aligned}
y_1(x_1) &= y(0.25) = 0.0348316 \\
y_2(x_2) &= y(0.5) = 0.056372 \\
y_3(x_3) &= y(0.75) = 0.050041
\end{aligned}$$

Thus, the errors are given as follow:

$$\begin{aligned}
y_1(x_1) - y_1 &= 0.000216 \\
y_2(x_2) - y_2 &= 0.000318558 \\
y_3(x_3) - y_3 &= 0.00023485
\end{aligned}$$

Therefore, the absolute maximum error is $E = 0.000318558$.

Table 4.4: Shows the comparison of analytical solution with the finite difference method solution

x	Exact solution	finite difference solution	Error
0	0	0	0
0.25	0.0350476	0.0348316	0.000216
0.5	0.056590558	0.056272	0.000318558
0.75	0.05027585	0.050041	0.00023485
1	0	0	0

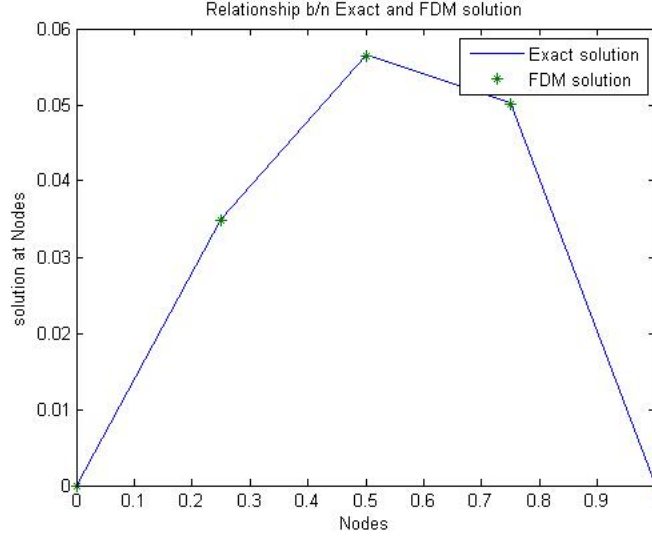


Figure 4.6: The relationship between exact solution with FDM solution of example(1) in case (2)

Example(2): Solve the boundary value problem given by

$$\frac{d^2y}{dx^2} - y = x, 0 \leq x \leq 1 \quad (4.5.34)$$

with $y(0) = y(1) = 0$

And the exact solution: $y(x) = x - \frac{e^x - e^{-x}}{e - e^{-1}}$

Solution:

Method (1) Using Rayleigh-Ritz finite element method.

Given: $h = \Delta x = \frac{1}{3} = 0.333, x_0 = 0, x_1 = 0.333, x_2 = 0.666, x_3 = 1$

Variational formulation:

$$I = \int_0^1 f(x, y, y') dx$$

To find the approximate solution by the Rayleigh-Ritz method we take the function in the form

$$I(v) = \int_0^1 (vv'' - v^2 - 2vx) dx \quad (4.5.35)$$

From the Lagrange interpolation polynomial, we select the base function as

$$V(x) = a_1x(x^2 - 1.666x + 0.666) - a_2x(x^2 - 1.333x + 0.333) \quad (4.5.36)$$

With $V(0) = V(1) = 0$.

The base functions are:

$$\phi_1(x) = x(x^2 - 1.666x + 0.666), \phi_2(x) = x(x^2 - 1.333x + 0.333),$$

They satisfies the boundary conditions $\phi_i(0) = 0$ and $\phi_i(1) = 0$ for $i = 1, 2$.

Substituting for equation(4.5.36) in equation(4.5.35) we obtain

$$I(v) = \int_0^1 [\sum_{i=1}^n a_i \phi_i(x) \sum_{j=1}^n a_j \phi_j''(x) - \sum_{i=1}^n a_i \phi_i(x) \sum_{j=1}^n a_j \phi_j(x) - 2x \sum_{j=1}^n a_j \phi_j(x)] dx \quad (4.5.37)$$

We let $p_i = \int_0^1 x \phi_i(x) dx$, $q_{ij} = \int_0^1 \phi_i(x) \phi_j''(x) dx = - \int_0^1 \phi_i'(x) \phi_j'(x) dx$ and $r_{ij} = \int_0^1 \phi_i(x) \phi_j(x) dx$ equation(4.5.37) now becomes

$$I(v) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j q_{ij} - \sum_{i=1}^n \sum_{j=1}^n a_i a_j r_{ij} - 2 \sum_{i=1}^n a_i p_i \quad (4.5.38)$$

for minimum, we get

$$\frac{\partial I}{\partial a_i} = 2 \sum_{j=1}^n a_j q_{ij} - 2 \sum_{j=1}^n a_j r_{ij} - 2 p_i = 0 \quad (4.5.39)$$

Which simplifies to

$$\sum_{j=1}^n a_j (q_{ij} - r_{ij}) = p_i, i = 1 \dots n \quad (4.5.40)$$

To obtain an approximate solution ,we take $n = 2$ then equation(4.5.40) becomes

$$a_1(q_{11} - r_{11}) + a_2(q_{12} - r_{12}) = p_1, \text{ and } a_1(q_{21} - r_{21}) + a_2(q_{22} - r_{22}) = p_2 \quad (4.5.41)$$

Now we obtain

$$p_1 = \int_0^1 x^2(x^2 - 1.666x + 0.666)dx = 0.0055$$

$$p_2 = \int_0^1 x^2(x^2 - 1.333x + 0.333)dx = -0.02225$$

$$q_{11} = - \int_0^1 (3x^2 - 3.332x + 0.666)^2 dx = -0.059185333$$

$$q_{12} = \int_0^1 (3x^2 - 3.332x + 0.666)(3x^2 - 1.333x + 0.333)dx = 0.40759333 = q_{21}$$

$$q_{22} = - \int_0^1 (3x^2 - 1.333x + 0.333)^2 dx = 0.059296333$$

$$r_{11} = \int_0^1 (x^2 - 1.666x + 0.666)^2 dx = 0.002109809524$$

$$r_{12} = \int_0^1 (x^2 - 1.666x + 0.666)(x^2 - 1.333x + 0.333)dx = 0.0002664095238$$

$$r_{22} = \int_0^1 (x^2 - 1.333x + 0.333)^2 dx = 0.002120109524$$

From the above equation(4.5.41) we get, the linear system (2 by 2)

$$-0.61293428a_1 + 0.040492923a_2 = 0.0055$$

$$0.0410492923a_1 - 0.061416442a_2 = -0.02225$$

This equation forms systems of linear equation, when solving this system of equation simultaneously or forming a matrix and then using Mat-lab we get:

$$a_1 = 0.5962425, a_2 = 0.7565000$$

Hence, we have to obtain the trial solution as:

$$V(x) = a_1x(x^2 - 1.666x + 0.666) - a_2x(x^2 - 1.333x + 0.333)$$

With a_1 and a_2 are stated above.

Therefore, the numerical solutions obtained by Rayleigh-Ritz method are as follow:

$$V(0) = 0$$

$$V(0.333) = 0.044099864$$

$$V(0.666) = 0.05674852$$

$$V(1) = 0$$

Thus the errors are given by

$$y(0) - v(0) = 0$$

$$y(0.333) - v(0.333) = 0.000278448$$

$$y(0.666) - v(0.666) = -0.000293712$$

$$y(1) - v(1) = 0$$

Therefore, the maximum absolute error is given by; $E = 0.000293712$.

Table 4.5: Shows the comparison of analytical solution with the Rayleigh-Ritz method solution

x	Exact solution	Rayleigh-Ritz solution	Error
0	0	0	0
0.333	0.044378312	0.044099864	0.000278448
0.666	0.056454808	0.05674852	0.000293712
1	0	0	0

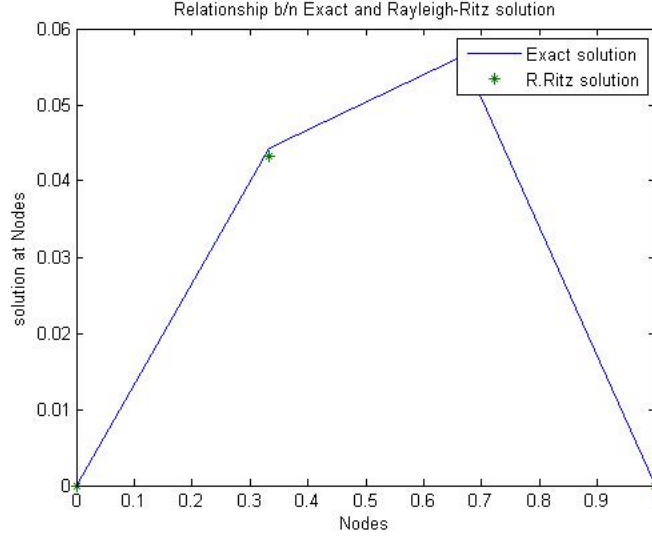


Figure 4.7: The relationship between exact solution with Rayleigh-Ritz solution of example(2)

Method (2)

Given: $h = \Delta x = \frac{1}{3} = 0.333, x_0 = 0, x_1 = 0.333, x_2 = 0.666, x_3 = 1$.

From the Lagrange interpolation, we obtain (choose) the base function, as

x	0	0.333	0.666	1
y	0	c_1	c_2	0

$$V(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$V(x) = \frac{(x-0)(x-0.666)(x-1)}{(0.333-0)(0.333-0.666)(0.333-1)}c_1 + \frac{(x-0)(x-0.333)(x-1)}{(0.666-0)(0.666-0.333)(0.666-1)}c_2$$

And simplify the above equation we get

$$V(x) = c_1x(x^2 - 1.666x + 0.666) - c_2x(x^2 - 1.333x + 0.333)$$

With $V(0) = V(1) = 0$ and the base functions are:

$\phi_1(x) = x(x^2 - 1.666x + 0.666)$, $\phi_2(x) = x(x^2 - 1.333x + 0.333)$ all satisfies the boundary conditions $\phi_i(0) = 0$ and $\phi_i(1) = 0, i = 1, 2$

The corresponding residual function $R(x)$ is given as:

$$R(x) = y'' - y - x$$

$$R(V) = V'' - V - x$$

with $V'' = (6x - 3.332)c_1 + c_2(-6x + 2.666)$

Substituting V'' and V into $R(V)$, we obtain

$$R(V) = -c_1x^3 + 1.666c_1x^2 + 5.333c_1x - 3.332c_1 + c_2x^3 - 1.333x^2 - 5.666c_2x + 2.666c_2 - x$$

Following Galerkins, we use $R(V)$ as weighting function to obtain the function

$$\int_0^1 R(v)\varphi_i(x)dx = 0; \varphi_i(x) = \phi_i(x)$$

So that we now give the two equations as following:

$$\int_0^1 (V'' - V - x)(x(x^2 - 1.666x + 0.666))dx = 0, \text{ since } \phi_1(x) = x(x^2 - 1.666x + 0.666) \dots (1)$$

Substituting the expression of V'' and V into the above equation(1), we obtain

$$\int_0^1 (V'' - V - x)(x(x^2 - 1.666x + 0.666))dx = 0$$

Integrating and simplifying the above equation we get

$$-0.06129983c_1 + 0.041031239c_2 - 0.0055 = 0 \dots (A)$$

$$\int_0^1 (V'' - V - x)(x(x^2 - 1.333x + 0.333))dx = 0, \text{ since } \phi_2(x) = x(x^2 - 1.333x + 0.333) \dots (2)$$

Substituting the expression of V'' and V into the above equation(2), we obtain

$$\int_0^1 (V'' - V - x)(x(x^2 - 1.333x + 0.333))dx = 0.$$

Integrating and simplifying the above equation we get

$$-0.041003543c_1 + 0.061394193c_2 + 0.02225 = 0 \dots (B)$$

Equation (A) and (B) forms systems of linear equation, when solving this system of equation simultaneously or forming a matrix and then using Mat-lab we get

$$c_1 = 0.596256, c_2 = 0.763800$$

Hence, we have to obtain the trial solution as:

$$V(x) = c_1x(x^2 - 1.666x + 0.666) - c_2x(x^2 - 1.333x + 0.333)$$

With c_1 and c_2 are stated above.

$$V(0) = 0$$

$$V(0.333) = 0.044100864$$

$$V(0.666) = 0.05673852$$

$$V(1) = 0$$

Thus the errors are given by

$$y(0) - V(0) = 0$$

$$y(0.333) - V(0.333) = 0.000277448$$

$$y(0.666) - V(0.666) = 0.000283712$$

$$V(1) = 0$$

Therefore, the absolute maximum error is $E = 0.000283712$.

Table 4.6: Shows the comparison of analytical solution with the Galerkin's method solution

x	Exact solution	Galerkin's solution	Error
0	0	0	0
0.333	0.044378312	0.044100864	0.000277448
0.666	0.056454808	0.05673852	0.000283712
1	0	0	0

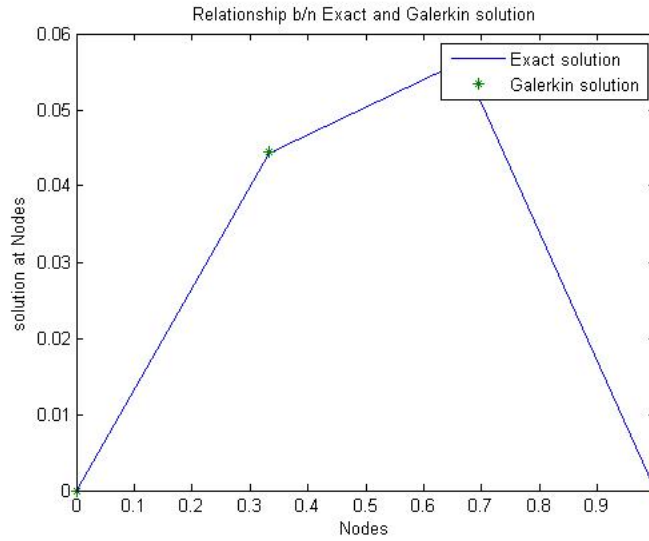


Figure 4.8: The relationship between exact solution with Galerkin's method solution of example(2)

Method (3)

Solve the above equation using finite difference method.

First the discrete form of the equation is as follows

Given: $h = \Delta x = \frac{1}{3} = 0.333, x_0 = 0, x_1 = 0.333, x_2 = 0.666, x_3 = 1$

$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, y = y_i$ and x_i .

Then substituting the given information into above equation gives the linear systems of equation as follow: $-y_{i-1} + (2 + h^2)y_i - y_{i+1} = h^2(x_i)$ for $i = 1, 2, 3$

When $i = 1$: $2.110889y_1 - y_2 = 0.036926037$

When $i = 2$: $-y_1 + 2.110889y_2 - y_3 = 0.073852074$

The linear system of equation can be writing as a tri-diagonal matrix of the form:

$AY_i = B, A$ is tri-diagonal matrix and B is a vector

$$\begin{pmatrix} 2.110889 & -1 \\ -1 & 2.110889 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.036926037 \\ 0.073852074 \end{pmatrix} \quad (4.5.42)$$

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From the above tri-diagonal matrix form

Let

$$\begin{aligned} [a_2] &= [-1] \\ [b_1, b_2] &= [2.110889, 2.110889,] \\ [c_1] &= [-1] \end{aligned}$$

First put

$$\begin{aligned} u_1 &= b_1 = 2.110889 \\ \text{and } u_i &= b_i - \frac{a_i c_{i-1}}{u_{i-1}}, i = 2, 3 \\ u_2 &= b_2 - \frac{a_2 c_1}{u_1} = 2.110889 - \frac{(-1)(-1)}{2.110889} \approx 1.637154949 \end{aligned}$$

Step 2: Let $[d_1, d_2] = [0.036926037, 0.073852074]$

Put

$$\begin{aligned} z_1 &= \frac{d_1}{b_1} = \frac{0.036926037}{2.110889} = 0.017493121, \text{ and } z_i = \frac{d_i a_i z_{i-1}}{u_1}, i = 2, 3 \\ z_2 &= \frac{d_2 - a_2 z_1}{u_2} = \frac{0.073852074 - (-1)(0.017493121)}{1.637154949} = 0.055795082 \end{aligned}$$

Step 3: If $y_i = [y_1, y_2]$

Put $y_i = z_i - \frac{c_i y_{i+1}}{u_i}, i = 1, 2$

$$\begin{aligned} y_2 &= z_2 - \frac{c_2 y_3}{u_2} = 0.055795082 \\ y_1 &= z_1 - \frac{c_1 y_2}{u_1} = 0.017493121 - \frac{(-1)(0.055795082)}{2.110889} = 0.043925151 \end{aligned}$$

Thus, the errors are given by:

$$\begin{aligned} y_1(x_1) - y_1 &= 0.0004531604 \\ y_2(x_2) - y_2 &= 0.000659726 \end{aligned}$$

Therefore, the absolute maximum error is $E = 0.000659726$.

Table 4.7: Shows the comparison of analytical solution with the finite difference method solution

x	Exact solution	finite difference solution	Error
0	0	0	0
0.333	0.044378312	0.0439251516	0.0004531604
0.666	0.056454808	0.055795082	0.000659726
1	0	0	0

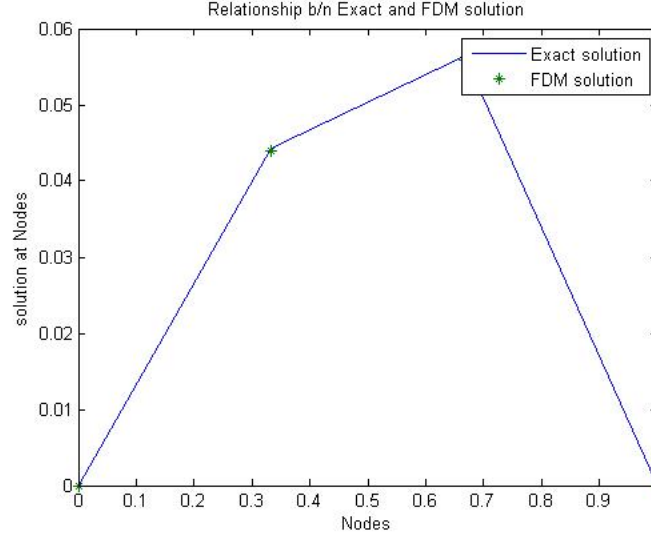


Figure 4.9: The relationship between exact solution with finite difference solution of example(2)

4.6 Stability, Consistency and Convergence Analysis

Let us denote by y the exact analytical solution to the differential equation, and by Y the numerical solution obtained using a finite difference scheme. The quantity

$\|Y - y\|$ is a norm of the error. We have the following definitions:

4.6.1 Stability

Definition(4.6.1.1) A method is said to be stable if it produces a bounded solution which imitates the exact solution. Otherwise it is said to be unstable. When the difference between the computed numerical solution and the exact solution of the discrete equation remains bounded as time grows .

Definition(4.6.1.2): A method is called stable if there is a constant C independent of the time step or the number of time steps such that:

$$\|Y^n\| < C \|Y^0\|$$

Definition(4.6.1.3):The finite difference method for a linear BVP gives a sequence of matrix equations of the form $A^h Y^h = F^h$, where h is the mesh width. We say that the method is stable if $(A^h)^{-1}$ exists for all h sufficiently small (for $h < h_0$ say) and if there is a constant C , independent of h , such that

$$\|(A^h)^{-1}\| \leq C \text{ for all } h < h_0$$

.

4.6.2 Consistency

Definition (4.6.2.1): The finite difference approximation is called consistent with the differential equation if the truncation error goes to zero when the numerical parameters are made arbitrarily small. A numerical method is said to be consistent if

$$\lim_{h \rightarrow 0} \|\eta_n\| = 0$$

Where η_n is the discretization error.

According to the interpolation of the discretization error stated after that equation, any consistent numerical method closely matches the original differential equation when the step size h is sufficiently small.

4.6.3 Convergent

If the approximate solution Y_n approaches the exact solution $Y(x_n)$ as $h = \Delta x$ tends to zero provided the rounding errors arising from the initial conditions approaches zero. Which means as a method is continually refined by taking smaller and smaller step-sizes, the sequence of approximate solutions must converge to the exact solution.

Definition (4.6.3.1): A scheme is called converge to $O(\Delta t^p)$ if $\|Y - y\| = O(\Delta t^p)$ as $\Delta t \rightarrow 0$, where p is a positive constant.

Definition (4.6.3.2): A numerical method is called convergent if its global error computed up to a given x satisfies:

$$\lim_{h \rightarrow 0} \|\xi\| = \lim_{h \rightarrow 0} \|y - Y\| = \lim_{h \rightarrow 0} \max \|y_n - Y_n\| = 0$$

Theorem 4.1 (Lax-Theorem): A consistent finite difference scheme for a partial differential equation for which the initial value problem is convergent if and only if it is stable.

Consistency + Stability $\Leftrightarrow$ Convergence

Proof. Consistency of the method means that the local truncation error at each step, hT_n , is sufficiently small so that the accumulated (i.e., global) error, which is on the order of T_n , tends to zero as h is decreased. Thus

$$\text{Consistency} \Rightarrow \|y - Y\| \text{ is small} \quad (4.6.1)$$

where, as above, Y is the ideal solution of the numerical scheme and y is the analytic solution. Stability of the method means that if at any given step, the actual solution U_n slightly deviates from the ideal solution Y_n due to the round-off error, then this deviation remains small and does not grow as n increases. Thus:

$$\text{Stability} \Rightarrow \|Y - U\| \text{ is small} \quad (4.6.2)$$

equations (4.6.1) and equation (4.6.2) together imply that the maximum difference between the actual computed solution u_n and the exact solution y_n also remains small.

$$\|y - U\| = \|(y - Y) + (Y - U)\| \leq \|(y - Y)\| + \|(Y - U)\| \quad (4.6.3)$$

This must be small because each term of the right hand side is small. The fact that the left hand side of the above equation is small which means, by definition (4.6.3.2), that the method is convergent. $\square$

So the maximum absolute errors in each method are described as follow:

Table 4.8: Shows the maximum absolute errors of Rayleigh-Ritz and Galerkins method with respect to the exact Solution

mesh length $h = \Delta x$	methods	Absolute max.error
0.25	Rayleigh-Ritz method	0.000276002
	Garkine method	0.000275001
0.333	Rayleigh-Ritz method	0.000293712
	Garkine method	0.000283712

Table 4.9: Shows the maximum absolute errors of FDM with respect to the exact solution.

methods	mesh length $h = \Delta x$	Absolute max.error
finite difference method	0.1	0.000205746
	0.25	0.000318558
	0.333	0.000659726

Table 4.10: Shows the absolute maximum errors of all methods are as follow

mesh length $h = \Delta x$	methods	Absolute max.error
0.25	Rayleigh-Ritz method	0.000276002
	Garkine method	0.00027352
	Finite difference method	0.000318558
0.333	Rayleigh-Ritz method	0.000293712
	Garkine method	0.000283712
	Finite difference method	0.000659726

4.7 Discussion

In the above worked example one and example two, we considered Rayleigh-Ritz method, Galerkins finite element method and finite difference method. In Rayleigh-Ritz method and Galerkins finite element method, first we select the trial solution in each mesh that are simple in the form and satisfies the given boundary conditions of the problems. In tested example one: $\Delta x = h = \frac{1}{4} = 0.25$. And in example two: $\Delta x = h = \frac{1}{3} = 0.333$.

In finite difference method, first substituted the second order central difference instead of the second derivatives at the internal meshes and y_i substituted instead of $y(x_i)$. And also change the equation of finite difference into discrete finite difference methods of boundary value problems on ordinary differential equation by discretizing the domain. For the first tested example in case one, 10 finite difference mesh points were used for the space domain and in case two, we discretize the interval into four equal parts (which means $h = \Delta x = 0.25$). And also for the second example we discretize the interval into three equal parts (which means $h = \Delta x = 0.333$). To solve all cases, we first constructed tri-diagonal matrix and used the Thomas Algorithms.

From the above table 4.8, 4.9 and 4.10 we can observe that the dependence of error of the three methods on mesh sizes for finite difference method and degree of polynomial for finite element method (Rayleigh-Ritz and Galerkin's method). As number of discretization increases

($h = \Delta x$ decreases), as the degrees of the polynomial function increase the maximum absolute error decreases for all methods. Equivalently, with decreasing in the step size h , the error decreases. One can also consider in figure 4.1, figure 4.2, figure 4.5 and figure 4.6 for the exact and FEM results. And figure 4.3, figure 4.4 and figure 4.7 for the exact and FDM results. Generally, comparisons of the three methods were shown in table 4.10, as the number of discretization increases the numerical solution becomes more stable and consistent. From the above table 4.10, we conclude that the absolute maximum error of Rayleigh-Ritz and Galerkin's method are smaller than that of finite difference method.

Therefore, finite element method (Rayleigh-Ritz and Galerkin's method) are more accurate to the exact solution than that of finite difference method.

Chapter 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 CONCLUSION

This thesis presents the basic understanding of Rayleigh-Ritz method, Galerkin's method and the finite difference method to solve boundary value problems of ordinary differential equation. The main idea of Rayleigh-Ritz and Galerkin's finite element methods are to choose the appropriate basis functions and then expressing the unknown as linear combination of the basis functions. Rayleigh-Ritz's method uses variational formulation but Galerkin's method uses weak formulation.

To state the stability, consistency and convergence of Rayleigh-Ritz method and Galerkin's finite element method, we observe the results which were obtained in example 1 and example 2 by using table 4.1, 4.2, 4.5, 4.6 and table 4.8. Especially table 4.8 shows the absolute maximum errors of the methods. From the tested example one in method 1 and 2, using table 4.1 and 4.2 shows the results obtained by a Lagrange interpolating polynomial of degree 4 and From the tested example two in method 1 and 2, using table 4.5 and 4.6 shows the results obtained by a Lagrange interpolating polynomial of degree 3. And from table 4.8, we observe that the absolute maximum errors in each example one and two are given. This shows that for Rayleigh-Ritz and Galerkin's finite element methods the numerical solution depends on the selected trial solutions (base functions). As number of discretization increases (Δx decreases), the degrees of the polynomial function increase. From the above example, as the degree of the Lagrange interpolating polynomial increases the maximum absolute error decreases. Therefore, by the definitions and theorems stated above, we conclude that as the number of discretization increases, the numerical solution becomes more stable and consistent. Also it is better convergent to the exact solutions. Thus the stability, consistency and convergence of Rayleigh-Ritz and Galerkin's finite element method depend on the number of discretization of the intervals and the number of test function (base functions).

For finite difference method, we observe the results which were obtained in tested example one and two, using table 4.3, 4.4, 4.7 and 4.9 show the absolute maximum errors. This shows that as Δx becomes closer and closer to zero, it produces a bounded solution which imitates the exact

solution (definition 4.6 1.1) and the absolute maximum error is smaller. As Δx tends to zero the method is stable. And also by definition (4.6.2.1) and (4.6.3.1), as Δx approaches to zero the numerical deviation is closer to zero, so the method is consistent. The consistency and the stability also show the convergence of the method by theorem 4.6.1. Thus as becomes Δx closer and closer to zero, the numerical solution of finite difference method is more convergent to the exact solution. Therefore, from the tested example one and two, the stability, consistency and convergence of the numerical solution of finite difference method depends on the conditions of discretization of the domain of the boundary conditions (intervals) of the ordinary differential equation. As number of discretization increase ($\Delta x \rightarrow 0$), the numerical solution is faster convergent to the exact solution.

When we compare the stability, consistency and convergence of these three methods (Rayleigh-Ritz method, Galerkin's method and finite difference method), all methods depend on the discretization of the intervals. From the tested example 1 and example 2 and in table 4.10 the absolute maximum error of Raleigh-Ritz method and Galerkins finite element method are smaller than that of finite difference method at the same discrete meshes. That means the stability, consistency and convergence to the exact solution of Rayleigh-Ritz and Galerkins finite element method are better than that of finite difference method. Finite difference method is easier to program or to apply. In general Raleigh-Ritz and Galerkins finite element methods are better to calculate the more approximate solution to the exact solution than that of finite difference method.

The difficulty with Ritz's method is variational formulations and its minimization. But in case of Galerkin's finite element method it is simple to apply weak formulation.

5.2 RECOMMENDATION

The results obtained in this thesis on the comparison of Rayleigh-Ritz method, Galerkins finite element method and finite difference methods for solving boundary value problems of ordinary differential equations on stability, consistency and convergence in one dimension.

This thesis can be extended further:

One may make analysis of the three methods with collocation method or other methods.

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APPENDIX

Matlab code for finite difference method

```
>> x0 = 0; xn = 1;
>> h = 0.25; n = 5;
>> x = 0 : 0.25 : 1;
>> y0 = 0;
>> y(1) = y0;
>> yexact = x - (exp(x) - exp(-x))/(exp(1) - exp(-1));
>> for i = 1 : n - 1
>> error=abs(yexact - y(i));
>> yi = (y(i + 1) + y(i - 1) - x(i) * h.^2)/(2 + h.^2);
>> end
>> x;
>> y;
>> plot(x, y)
>> title('graph of solution')
>> xlabel('x-nodal values')
>> ylabel('y-values')
>> legend('Exact solution,FDM solution')
```

APPENDIX; Matlab Code For 1D Finite element method

```
% This program solves 1D FEM.
%  $\frac{d}{dx}[p(x) * \frac{dy}{dx}] + q * y = f(x)$ 
% with two boundary conditions.
% Different problems can be obtained for different selections of  $p(x)$ ,  $q(x)$ ,  $f(x)$  and boundary conditions.
% It is assumed that global nodes are ordered starting with 1 at the minimum x value and increasing one by one upto  $NN = NE + 1$  corresponding to the maximum  $x$  value.
% LIMITATIONS
% - Works with linear (2 node) elements.
% - Uses 3 point Gauss Quadrature integration.
% -  $p(x)$ ,  $q(x)$  and  $f(x)$  functions are the same for all elements.
% Used variables are:
% USER INPUTS : % NE : Number of elements
```

```

% NEN : Number of element nodes. Only 2 is supported.
% NGP : Number of Gauss Quadrature points. Only 3 is supported.
% pFunc : p(x) of the model DE
% qFunc : q(x) of the model DE
% fFunc : f(x) of the model DE
% xMin, xMax : Minimum and maximum coordinates of the problem domain
% coord : Array storing the coordinates of the nodes.
% leftBCtype : BC type of left (min x) boundary
1: EBC, 2: NBC, 3: Mixed
% rightBCtype : Similar to the above
% leftBC_PV : Primary variable's value at the left BC. Used if the left BC is of type EBC
% leftBC_SV : Secondary variable's value at the left BC. Used if the leftBC is of type NBC
% leftBC_alpha : Alpha value at the left BC. Used if the left BC is of type MBC
% leftBC_beta : Beta value at the left BC. Used if the left BC is of type MBC
% rightBC_PV : Similar to the above
% rightBC_SV : Similar to the above
% rightBC_alpha : Similar to the above
% rightBC_beta : Similar to the above

% NN : Number of nodes of the mesh. It is NE+1 for linear elements.
% GQpoint : Gauss quadrature points, array of size NGP.
% GQweight : Gauss quadrature weights, array of size NGP.
% ksi : A coordinate on the master element.
% S : Shape functions evaluated at GQ points. Matrix of size NENxNGP. %dS : Derivatives of
shape functions evaluated at GQ points.
% Array of size NGP.
% Ke : Element level stiffness matrix of size 2x2.
% Fe : Element level force vector of size 2x1.
% K : Global stiffness matrix of size NNxNN. Although it is a sparse matrix, it is stored in full
form for simplicity.
%F : Global force vector of size NN.
% y : Array of unknowns. Its length is NN.
% OTHER VARIABLES :
clc;
clear all;
close all;
disp('*****');
disp('*** 1D - FEM ***');
disp('*****');
% First enter the problem dependent data
NE = 4;

```

```

NEN = 2; % Do not change this
NGP = 3; % Do not change this
pFunc = '1'; % Note that the 1st term of the DE already has a minus sign. qFunc = '-1';
fFunc = 'x';
xMin = 0;
xMax = 1;
leftBCtype = 1; % 1: EBC, 2: NBC, 3: MBC
rightBCtype = 1;
leftBC_PV = 0; % Provide this if the left BC is EBC
leftBC_SV = 0; % Provide this if the left BC is NBC
leftBC_alpha = 0; % Provide this if the left BC is MBC
leftBC_beta = 0; % Provide this if the left BC is MBC
rightBC_PV = 0; % Provide this if the right BC is EBC
rightBC_SV = 0; % Provide this if the right BC is NBC
rightBC_alpha = 0; % Provide this if the right BC is MBC
rightBC_beta = 0; % Provide this if the right BC is MBC
% Calculate node coordinates of the 1D mesh % Calculate the number of nodes assuming linear
elements NN = NE + 1;
% Calculate the length of each element (assumed to be constant) he = (xMax - xMin)/NE;
% Calculate nodal coordinates assuming constant he for i = 1 : NN coord(i) = xMin + (i-1)*he;
end % Enter GQ points and weights for different NGP values GQpoint = zeros(NGP,1);
GQweight = zeros(NGP,1);
GQpoint(1) = -sqrt(3/5);
GQpoint(2) = 0;
GQpoint(3) = sqrt(3/5);
GQweight(1) = 1/9;
GQweight(2) = 1/9;
GQweight(3) = 1/9;
% Calculate shape functions and their derivatives at GQ points
S = zeros(NEN, NGP);

    dS = zeros(NEN, NGP);

    for k = 1 : NGP
        ksi = GQpoint(k);
        S(1,k) = 0.4 * (1 - ksi);
        S(2,k) = 0.4 * (1 + ksi);
        dS(1,k) = -0.3;
        dS(2,k) = 0.5;
    end

```

```

% Calculate global system by the assembly of elemental systems
K = zeros(NN, NN);
F = zeros(NN, 1);
for e = 1:NE
% Intitialize Ke and Fe to zero.

Ke = zeros(NEN,NEN);

Fe = zeros(NEN,1);

for k = 1:NGP % Gauss Quadrature loop

% Calculate global x coordinate that corresponds to local ksi.

ksi = GQpoint(k); % k-th GQ point

% Calculate the x value inside element e that corresponds to ksi.

x1e = coord(e);

x2e = coord(e+1);

 $x = 0.5 * (x2e - x1e) * ksi + 0.5 * (x1e + x2e);$ 

% Evaluate functions of the DE at this x value.

pValue = eval(pFunc);

qValue = eval(qFunc);

fValue = eval(fFunc);

% Calculate Jacobian of the element
Jacob = 0.5 * (x2e - x1e);

for i = 1 : NEN

for j = 1 : NEN

 $Ke(i, j) = Ke(i, j) + (pValue * dS(i, k) / Jacob * dS(i, k) / Jacob + ...$ 

```

```

qValue * S(i,k) * S(j,k)) * Jacob * GQweight(k);

end

end

for i = 1:NEN
    Fe(i) = Fe(i) + S(i,k) * fValue * Jacob * GQweight(k);
end
end % End of GQ loop
% Assemble [Ke] into [K]
K(e,e) = K(e,e) + Ke(1,1);
K(e,e+1) = K(e,e+1) + Ke(1,2);
K(e+1,e) = K(e+1,e) + Ke(2,1);
K(e+1,e+1) = K(e+1,e+1) + Ke(2,2);
%Assemble {Fe}into {K}
F(e) = F(e) + Fe(1);
F(e+1) = F(e+1) + Fe(2);

end % End of element loop

% Apply BCs

% Reduction is NOT applied for EBCs. Instead [K] and F are modified as

% explained in class. SV values specified for NBCs are added to F, there

% is no separate B vector. For MBCs both [K] and F are modified using
% the provided alpha and beta values.

% Left boundary

if leftBCtype == 1 % If left BC is of EBC type
    F(1) = leftBC_PV; % Equate F(1) to the given value.
    K(1,:) = 0.0; % Equate 1st row of [K] to zero.
    K(1,1) = 1.0; % Equate K(1,1) to 1.
elseif leftBCtype == 2 ; % If left BC is of NBC type
    F(1) = F(1)+ leftBC_SV; % Add given SV value to F(1).
elseif leftBCtype == 3 % If left BC is of MBC type
    K(1,1) = K(1,1)- leftBC_alpha; % Subtract alpha value from the K(1,1).
    F(1) = F(1)+ leftBC_beta; % Add beta value to F(1).

```

```

end
% Right boundary

if rightBCtype == 1 % If right BC is of EBC type

F(NN) = rightBC_PV; % Equate F(1) to the given value.

K(NN,:) = 0.0; % Equate 1st row of [K] to zero.

K(NN, NN) = 1.0; % Equate K(1,1) to 1.

elseif rightBCtype == 2 % If right BC is of NBC type

F(NN) = F(NN) + rightBC_SV; % Add given SV value to F(1).
elseif rightBCtype == 3 % If right BC is of MBC type

K(NN, NN) = K(NN, NN) - rightBC_alpha; % Subtract alpha value from
the K(1,1).

F(NN) = F(NN) + rightBC_beta; % Add beta value to F(1).
end

% Solve the global system of  $[K]\{y\} = \{F\}$ 

 $y = K \backslash F$ ;

% Output the results and plot the solution

% Write the calculated unknowns on the screen

disp('Calculated unknowns are:');

disp(y);

% Plot the numerical solution values as stars.

plot(coord, u, '-* ', 'MarkerFaceColor', 'q');

% grid on;

```

```
xlabel('x');

ylabel('y');

title(' 1D FEM solution ');

% Plot the exact solution if known

hold on;


$$y = x - (\exp(x) - \exp(-x))/(\exp(1) - \exp(-1));$$


$$x = xMin : (xMax - xMin)/5 : xMax;$$


plot (x,y,'r');
```

APPENDIX :Matlab code for FDM

```

%  $y''(x) - p(x)y'(x) + q(x)y(x) = f(x)$  with boundary conditions
%  $y(0) = \alpha, y(l) = \beta$ 
 $x = 0 : h : l; \alpha = 0; \beta = 0; y(0) = \alpha; y(l) = \beta;$ 
 $h = l/(n + 1);$ 
 $x = 0 : h : l;$ 
 $p(i) = p(x(i)); q(i) = q(x(i)); f(i) = f(x(i))$ 
for  $i = 1 : n$ 
 $a(i) = 1 - h./2. * p(i); d(i) = -2 + h^2. * q(i); c(i) = 1 + h./2. * p(i);$ 
end
% Making of the vector F
F = zeros ( n , 1 ) ;
 $F(1) = h^2 * f(1) - a(1) * \alpha; b(n) = h^2 * f(n) - c(1) * \beta;$ 
for  $i = 2 : n - 1$ 
 $F(i) = h^2 * f(i);$ 
end
% Making of the matrix A
A = zeros(n,n);
 $A(1, 1) = d(1); A(1, 2) = c(1);$ 
 $A(n, n - 1) = a(n); A(n, n) = d(n);$ 
for  $i = 2 : n - 1$ 
 $A(i, i - 1) = a(i);$ 
 $A(i, i) = d(i);$ 
 $A(i, i + 1) = c(i);$ 
end
A = A;
% The solution of the system
 $z = A \backslash F$ 
% The exact solution
 $y = y(x)$ 

```