ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

THESIS ON

NUMERICAL SOLUTION OF WAVE EQUATION IN 1-D AND 2-Ds USING FINITE DIFFERENCE METHOD (FDM)

A Thesis Submitted in Partial Fulfillment of Requirements for the Degree of Master's in Applied Mathematics (Numerical Analysis)

BY: Jewaro Gemedo

ID.NO GSU/0433/06

January, 2018

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Not forgetting to family members and all friends who had given encouragement and suggestions in completing this thesis.

ABBREVIATIONS AND ACRONYMS

FDM..Finite Difference Method

FEM……………………………………..Finite Element Method

FVM………………………………………Finite Volume Method

FD…………………………………………Finite Difference

EDM………………………………………Explicit Difference Method

EFDM……………………………………Explicit Finite Difference Method

PDE..Partial Differential Equation

1D...one Dimension

2D...Two Dimensions

3D..Three Dimensions

MATLAB..Matrix Laboratory

Hz…………………………………………...Hertz

BC…………………………………………...Boundary Condition

IC……………………………………………Initial Condition

ABC…………………………………………Absorbing Boundary Condition

Obs…………………………………………Observe

SYMBOLS AND ACRONOMY

∂ ………………………………………………Partial differential

∇ ………………………………………………Nabla (inverted delta or backward delta)

Δ ……………………………………………….Increment (forward delta)

i……………………………………………..index

c……………………………………………..the speed of wave propagation

f……………………………………………..frequency

T……………………………………………Period of wave oscillation

ω ……………………………………………Angular velocity

π ……………………………………………pi

δ ……………………………………………delta

ϕ ……………………………………………phi variant

ψ ……………………………………………psi

α ……………………………………………alpha

ABSTRACT

In this thesis finite difference method is discussed and defined to be used mainly in partial differential equation (PDE), particularly to solve numerical solution of wave equation which is a classical example of hyperbolic partial differential equation. In solving 1-D and 2-D wave equation we discuss some criteria such as discretizing the formula explicitly, constructing grid-points, initial condition (IC) and boundary condition (BC). Consequently, some problem examined using Explicit Finite Difference Method (EFDM) for second order wave equation in 1-D and 2-D.The approximate solution and the analytical (exact) solution are performed and finally, we investigate the absolute error (Ea) between the approximation (computed) value and the exact value. Finite difference scheme numerical technique was used to simulate the 1-dimensional wave equation and 2-dimensional wave equations. The 1-D and 2-D wave equations were solved in the MATLAB software. One dimensional wave equation is a physical phenomenon that happens in vibrating string, where as two-dimensional wave equations happen in vibrating membrane. The complete program of numerical simulation of one-dimensional wave equation and two-dimensional wave equations were analyzed.

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CHAPTER ONE

1 INTRODUCTION

A finite difference method is a technique for obtaining approximate solutions of many types of science and engineering problems. The need for finite difference methods arises from the fact that for most practical engineering problems their analytical solutions do not exist. While the governing equations and boundary conditions can usually be written for these problems, difficulties introduced by either irregular geometry or other discontinuities are difficult to be solved analytically. Finite Element Method (FEM) and Finite Difference Method (FDM) are examples of numerical method that can be used to solve this kind of problems. Finite Element Method (FEM) is a numerical method for solving problems of engineering and mathematical physics. It is useful for problems with complicated geometries, leadings and material properties where analytical solutions cannot be obtained. Problems related to wave equation can be solved by using finite difference method (FDM). Generally, finite difference method (FDM) is a simple method to use for common problems defined on regular geometries.

1.1 BACKGROUND OF THE STUDY

The purpose of this thesis is to look at the different finite difference schemes for first and second order wave equations in 1-D and 2-D.

This thesis is focusing on solving a wave equation by applying the finite difference method (FDM). Normally, computations of this method have been performed using MATLAB program since it is impossible to do the calculations manually for large scale problem and also in order to assure the accuracy of the solutions. Finite difference method (FDM) involves tedious and complex calculations. Thus, it was deal to be applied numerically in MATLAB program. A very wide range of physical processes lead to wave equation, where signals are propagated through a medium in space and time, normally with little or no permanent movement of the medium itself. The shape of the signals may undergo changes as they travel through matter, but usually not so much that signals cannot be recognized at some later point in space and time. In chapter 2 researchers had been treating the finite difference method in detail. Many types of wave equation can be described by the equation $u_{tt= \nabla(c^2\nabla u)+f}$ which the researcher has solved for the coming text by finite difference methods. The wave equation in three spatial variables (x, y, z) and time t is:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

The function u represents the displacement at time t of a particle whose position at rest is (x, y, z) with appropriate boundary conditions. This equation governs vibrations of a three –dimensional elastic body. The wave equation with one space variable $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ governs the vibration of string. The boundary conditions u = 0 makes u change sign at the boundary, while the conditions $u_x = 0$ perfectly reflects the wave. The boundary conditions $u_x = 0$, which is more complicated to express numerically and also to implement a given values of u. Researcher was present two methods for implementing $u_x = 0$ in finite difference scheme, one based on driving a modified stencil at the boundary and another one based on extending the mesh with ghost cells and ghost points, when a wave hits a boundary and is to be reflected back, one applies the condition $\frac{\partial_u}{\partial_n} \equiv n.\nabla u = 0.$ The derivative $\frac{\partial_u}{\partial_n}$ is in the outward normal direction from a general boundary conditions that specify the value of $\frac{\partial_u}{\partial_n}$ or shorter u_n, are known as Neumann conditions, while Dirichlet conditions refer to specifications of u. When the values are zero ($\frac{\partial_u}{\partial_n} = 0\ oru = 0$), researcher was speaking about homogeneous or Dirichlet conditions. $\frac{\partial_u}{\partial_n} \equiv n.\nabla u = 0$ applied at x = L is discretized by the central difference $\frac{U_{N_x}^n - U_{N_{x-1}}^n}{2\Delta x} = 0$ combined with the scheme for i = N_x. Researcher was get a modified scheme for the boundary value $U_{N_x}^{n+1}$: $U_i^{n+1} = -U_i^{n-1} + 2U_i^n + 2c^2(U_{i-1}^n - U_i^n), i = N_x.$ The modification of the scheme for the boundary is also required for the special formulas for the first time step. The implementation of the special formulas for the boundary points can benefit from using the general formula for the interior points also at the boundaries. A variable wave velocity C: C= c(x), usually motivated by wave motion in a domain composed of different physical media with different properties for propagating waves. A generalization of the Fourier series idea to non-periodic functions defined on the real line is the Fourier transform:

I(x) = $\int_{-\infty}^{\infty} A(k)e^{ikx}\, dk$

A(k) = $\int_{-\infty}^{\infty} I(x)e^{-ikx}\,dx$

The function A(k) reflects the weight of each wave component e^{ikx} in an infinite sum of such wave components. That is, A(k) reflects the frequency content in the function I(x).

The wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ has solutions of the form u(x, t) =$g_R(x - ct) + g_L(x + ct)$.

For any functions g_R and g_L sufficiently smooth to be differentiated twice. The approximation of the spatial derivatives generally leads to a system of ordinary differential equations in time that is commonly referred to as the semi-discrete form of the wave equation. The numerical solution of the equations of motion is one of the defining characteristics of full wave form inversion and an indispensable tool in seismic ground motion studies. Seismic wave motion in the earth is governed by the wave equation that relates the displacement field to external force and to the distributions of density and elastic parameters. In the finite-difference method (chapter 2), the spatial derivatives are approximated by difference quotients. The approximation leads, as in any other numerical method, to a dispersion error that depends, among other factors, one the number of grid points per wave length. The feasibility of finite-difference modeling in 3D rests on the definition of a staggered grid where the field variables are evaluated at different grid positions. Finite difference method (FDM) for wave equation is conducted by Hanspetter Langtangen[1] center for Biomedical computing. Finite difference method belongs to the so called grid-point methods. In the grid-point methods a computational domain is covered by a space-time grid and each function is represented by its values at grid points. The space-time distribution of the grid points may be, in principle, arbitrary, but it significantly affects the accuracy of the approximation. A derivative of a function is approximated by the so called finite difference formula which uses values of the function at a specified set of the grid points. The time dependence of a function is often approximated by the finite difference formulas also in other numerical methods. Due to its definition, the FDM is the most important numerical methods in seismology and certainly still dominant method in earth quake ground motion modeling. In particular, this is due to the fact, that, formally it is applicable to complex models, relatively accurate and computationally efficient. To introduce the basic finite-difference concepts, researchers will start with the one-dimensional scalar wave equation. This simple example also allows us to study the stability and the numerical dispersion of the discretize equations. Based on the Staggered-grid approach researchers then make the transition to the three –dimensional elastic case. The simulation is based on a 4^{th} –order finite-difference discretization of the elastic wave equation. The horizontal displacement velocity is largest above the densely populated sedimentary basin. Numerical modeling in nearly all physical sciences started with the finite-difference method, and seismology is no exception. The general form of hyperbolic type linear PDE having one, two and three space variables are as follows:

$$u_{tt} = c^2 u_{xx}\ ,\ u_{tt} = c^2 \left(u_{xx} + u_{yy}\right),\ u_{tt} = c^2 \left(u_{xx} + u_{yy} + u_{zz}\right)$$

However, the focus of this thesis is goes to solve one as well as two dimensional wave equations.

1.2. Statement of the problem

Grid or mesh based numerical methods such as finite difference methods (FDMs) has been widely applied to various wave motion such as wave on a membrane, seismic waves, waves on a string, sound waves in liquids and gases, waves in blood vessels, and acoustic waves. Problems related to wave equation can be solved by using Finite difference method (FDM). However, this approach will cause a few flows in the findings mainly because the method itself is low in accuracy. Although FDM is easier to compute as well as to code, some problems that involved complex geometry is either difficult or completely cannot be solved by finite difference method (FDM).

Generally, FDM is a simple method to use for common problems defined on regular geometry. The problem of regular geometry in wave equation has been solved using finite difference method. A derivation of function in wave equation has also been approximated using a finite difference formula which makes use of function values at a specified set of grid points. These methods are currently the dominant methods in solving problems of engineering and science. The questions concerning this title have been asked as follows:

- What is the effect if mesh point changed?
- Is the value obtained by explicit finite difference method similar to the exact value?
- What is the effect if the ratio of step size is too large?
- Is the graph of exact value and approximate value are the similar using MATLAB?

1.3. Objective of the study

1.3.1. General objectives of the study

The general objective of this study was to establish and to compare known analytical solution of guided finite difference method for wave equation.

1.3.2. Specific objectives of the study

The specific objectives of this study were:

- To verify the effect if the mesh point changed.
- To compare approximation value to the exact value.
- To determine the effect on the ratio of step size.
- To indicate the graph of exact value and approximate value using MATLAB.

1.4 Significance of the study

The significance of this study was to benefit other researcher whose study would be on this area and it will a good exposure to develop the concept of solving numerical solution of wave equation using finite difference method (FDM). The result of this thesis has been used for further thesis in related areas. One of the goals of this thesis is that, this thesis will give clearer view of FDM, where it will finally lead to further understanding on applying the FDM in solving wave equation and related problems especially in the science and engineering fields. However, finite difference method (FDM) was not use in irregular solid geometry, rather than in regular geometry.

1.5 Delimitation/Scope of the Study

This study was mainly delimit to finite difference method (FDM) and will focus on the application of FDM to solve wave equation problems in 1D and 2D.

1.6. Limitation of the Study

The limitation of this study was:

- Lack of sufficient related literature review.
- Lack of enough internet service at work place.

1.7 Organization

In this organization of thesis, there are five chapters which are introduction, literature review, method of solution, results and discussion, conclusion and recommendation. Chapter 1 has 10 subtopics such as introduction, background of the study, statement of the study, objective of the study, significant of the study, scope \ delimitation of the study, limitation of the study, waves, hyperbolic PDEs and organization of thesis. Chapter two introduces a literature review on FDM for 1D and 2D wave equation and it consists four subtopics namely, basic concepts in one dimension, finite difference method (FDM), simulation of wave equation and definition of wave equation. The formulation and the methods that will be used are discussed in chapter three. All results that are obtained from MATLAB will be discussed in chapter four and then the results are presented in figures and tables. Finally, chapter five is for the conclusion of this thesis and recommendation.

CHAPTER TWO

2. Related Literature Review

2.1. Basic concepts in one dimension

The one-dimensional scalar wave equation is particularly well suited not only for an introduction to the finite difference method itself but also for the illustration of fundamental concepts in numerical analysis, including stability and grid dispersion. Applications of finite-difference modeling include studies of seismic ground motion in densely populated areas. A.Saadatmandi and M. Dehghan[2], numerical solution of the one –dimensional wave equation with An integral condition, Numerical methods for Partial Differential Equations. The researcher will start his development with the description of two methods for the construction of finite difference approximations. These are then used to replace the exact derivatives in the wave equation. The result of this procedure is an iterative scheme that allows us to advance a discrete representation of the wave field in time. Both numerical experiments and rigorous analysis reveal the properties of the iterative scheme in general and its stability requirements in particular. Early implementations of the finite-difference method in a seismological context used a conventional grid where all field variables (e.g. displacement, stress, strain, etc.) are defined at the same grid positions. Dessa et al. [3]; Bleibinhaus et al, [3], the simulation of wave propagation through random media and full waveform inversion.

Hans Petter Langtanger [4], finite difference methods for wave motion, center for Biomedical computing, Simula Research Laboratory Department of Informatics, University of Oslo Dec 12, In the Staggered grid, field variables are defined at different gird positions, which reduce the effective grid spacing compared to the conventional grid. Further developments in finite-difference wave propagation focused, for instance, on the modeling of the free surface. Hong Kong, [6], proceedings of the International Multi Conference of Engineers and Computer Scientists , numerical method for solving Wave Equation with Non Local Boundary Conditions.

Kristek et al., [9], further developments in finite-difference wave propagation focused, for instance, on the modeling of the free surface.

Moczo et al. [12], For an excellent review of finite-difference methods for wave's propagation, Wang et al., [19]; Moczo et al., [19]; Applications of finite-difference modeling include studies of seismic ground motion in densely populated areas. Moczo, P.; J. Kristek, L. Halada, [13], 3D 4th order Staggered -grid finite difference schemes: Stability and grid dispersion. Richard et al. [14], Wave motion is energy transfer from one point to another. Sewell, Granville, John Wiley & Sons, Inc. Hoboken, NJ: [16]. The Numerical Solution of Ordinary and Partial Differential Equations, Smith, Julius O. "Physical Audio Signal Processing: for Virtual Musical Instruments and Digital Audio Effects, December [] Edition", 2D Mesh and the Wave Equation.

To illustrate the finite-difference concept, we consider a generic function, f (x), that represents the dynamic fields that appear in the wave equation (e.g. stress, strain, displacement,). The fundamental idea is to consider only a finite number of evenly spaced grid points, xi (i=1, N), and to replace the derivative ∂x f (xi) at grid point xi by a finite-difference approximation that involves f evaluated at neighboring grid points. The best-known example is the second-order central finite-difference approximation. $\partial x\ f(xi) = \frac{1}{2\Delta x}[f(xi + \Delta x) - f(xi - \Delta x)] + O(\Delta x^2)$, where Δx is the grid spacing. The feasibility of finite-difference modeling in three dimensions rests on the definition of a staggered grid where the dynamic fields are defined at different grid positions. Wakefield, M.A. Wave2dmovie: 2D Wave Equation - Finite Difference. [18].

The staggered grid results in a reduced average grid spacing that greatly reduces the numerical dispersion, i.e., the artificial dispersion introduced by the discretization of the original equations of motion. The popularity of finite-difference modeling is largely due to the comparatively low computational costs and the accuracy especially in the modeling of body wave propagation.

2.2 Finite Difference Method (FDM)

In the finite difference method each function is represented by its values at grid points. The space-time distributions of grid points may be, in principle, arbitrary but it significantly affects properties of the resulting numerical approximations. A derivation of a function is approximated using a finite difference formula which makes use of function values at specified set of grid points. It is based on the approximation of an exact derivative, $\partial_x f(x_i)$, at a grid position x_i, in terms of the function f evaluated at a finite number of neighboring grid points. The finite difference (FD) method transforms a differential equation or PDE into a difference equation that can be solved numerically. Finite difference methods have a long history going back to Leonhard

Euler [11] who began to study the calculus of finite differences. Finite difference methods were the first numerical methods to be used to solve the wave equation. The first Finite difference method that was used for the wave equation, was based on centered second- order approximations of the second-order derivatives in time as well of the Laplace operator, leading to a fully explicit scheme. Such an approximation has good stability and dispersion features needed for the wave equation, and is also a non dissipative scheme. The disadvantage of the Finite difference scheme is that there is difficulty in extending it to more general domains. The numerical solutions, based on finite differences, provide us with the values at discrete points in the domain which are known as grid points. The grid points are identified by an index i which increases in the positive x-direction, and an index j, which increases in the positive y-direction. If (i, j) is the index of point P in figure 1.1, then the point immediately to the right is designated as (i+1, j) and the point immediately to the left is (i-1, j). Similarly the point directly above is (i, j+1), and the point directly below is (i, j-1).

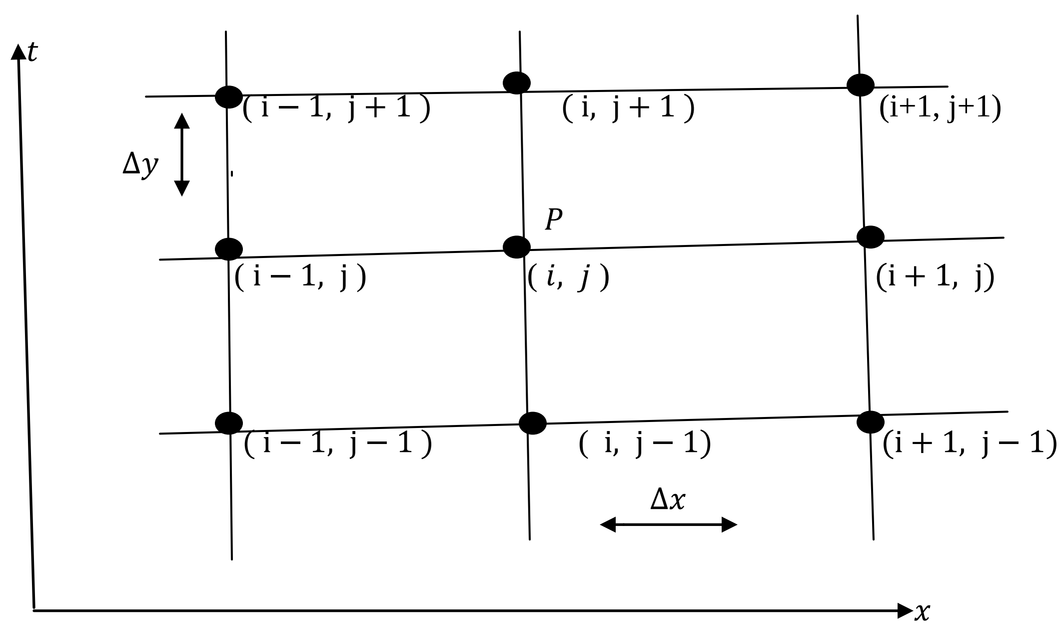

Figure 1.1 Discrete grid points.

The basic philosophy of finite difference method is to replace the derivatives of the governing equations with algebraic difference quotients. This will result in a system of algebraic equations which can be solved for the dependent variables at the grid points in the flow field. According to Allaire [1], numerical modeling is a combination of two distinct aspects of mathematical modeling and numerical simulation. According to Goering [6], the

MATLAB (Matrix Laboratory) is a numerical computing environment and fourth-generation programming language.

2.3 Waves

2.3.1 Definition of wave

Wave is a transfer of energy from one point to another or a disturbance /oscillation that travels or moves through space-time and a medium. A medium here is a substance or material that carries the disturbance from one point or from one place to another. A wave has the same sense as a particle. “Wave motion is an energy transfer from one point to another, as mentioned by [8]”. so, the study of wave is important. There are two basic types of wave: a wave can be transverse or longitudinal depending on the direction of its oscillation. The transverse wave is the wave that causes the medium to vibrate at a right angle to the direction of the wave. The transverse wave has crest (highest point) and through (lowest point) of the wave.

Transverse wave is a wave where a disturbance creates oscillations that are perpendicular to the propagation of energy transfer. An example of transverse wave is light wave (electromagnetic wave).

Longitudinal wave is a wave where the oscillations are **parallel** to the direction of energy propagation. The longitudinal wave (also known as 1-wave) is the wave has the same direction of vibration as their direction of travel. According to Richard, [15], the characteristic of longitudinal wave is same with compression and extension wave in spring. But, any kinds of waves have same properties either its longitudinal or transverse wave such as amplitude, wave length, period, frequency and speed. In physics, a wave is an oscillation accompanied by a transfer of energy that travels through a medium (space or mass). Frequency refers to the addition of time.

2.3.2 Properties of wave

The main properties of a wave are defined below.

i, **Amplitude**: the height of the wave from equilibrium position line to crest, measured in meters.

ii, **Wave length:** the distance between a crest to another crest, measured in meters.

iii, **Period:** the time it takes for one complete wave to pass a given point, measured in seconds means that wave length over time where the unit is cycle/ second.

iv, **Frequency:** the number of crest that a point in one second, measured in inverse seconds or Hertz(Hz).

v, **Speed:** the horizontal speed of a point on a wave as it propagates or the number of crest passes per unit time, measured in meters/ second.

The **Period T** is the time for one complete of an oscillation of a wave.

The **Frequency F** is the number of periods per unit time (per second) and is typically measured in hertz denoted as Hz. These are related by: $f = \frac{1}{T}$

In other words, the frequency and period of a wave are reciprocals.

The **Angular Frequency** $\boldsymbol{\omega}$ represents the frequency in radians per second. It is related to the frequency or period by: $\omega = 2\pi f = \frac{2\pi}{T}$

The **wave length** $\boldsymbol{\lambda}$ of a sinusoidal wave form traveling at constant speed v is given by: $\lambda = \frac{v}{f}$

Where v is called the phase speed (magnitude of the phase velocity) of the wave and f is the wave's frequency.

2.3.3 Dimensions of waves

There are three dimensions of wave that known as 1-dimension, 2-dimensions and 3-dimensions of waves.

i **One-Dimensional wave**

Examples of wave in one dimension are a wave that travels along string or sound waves going down a narrow tube. The energy of the wave motion of 1-dimensional wave has only one dimension in which to travel. The waves that have only one dimension to travel are transverse and longitudinal wave.

ii **Two-Dimensional wave**

By referring to Kneub$\ddot{u}$hl [8] stated that the energy of the wave motion of a 2-dimensional wave has the ability to travel around corners. The wave has a point of source and a path of the wave.

iii **Three-Dimensional wave**

The 3-dimensional wave is a wave that shows point sources, line sources and plane of wave such as the wave moving into different medium also called, interference between sources is the definition gave by Kneub$\ddot{u}$hl [8].

2.4 Hyperbolic PDEs

PDE methods for hyperbolic problems are FDM, FVM, and sometimes FEM. Among these FDM is applied on this thesis project. The simplest hyperbolic PDE is the 1D wave equation. In order to apply the finite difference method to the wave equation in 1D and 2D, difference approximations for the derivatives are required.

Initial Condition

Because the wave equation involves time, part of the boundary condition required for the finite difference approach is actually a boundary condition in time, or an initial condition. Since the PDE involves a second order time derivative, initial conditions at two time steps are required.

Boundary Conditions

The simplest boundary condition is Dirichlet: u(0; t) = 0 and u(d; t) = 0, where [0; d] is the simulation region. An absorbing boundary condition can be obtained by discretizing the one-way wave equation at the endpoints of the region. This is known as the Mur boundary condition.

The basic elements of FD are as follows:

1. **Grid.** This is a set of points at which the unknown function in the PDE is sampled.
2. **Stencil**. This is a difference approximation for the derivative of a quantity at one grid point in terms of values at neighboring points.
3. **Boundary Conditions**. At the edges of the grid, the stencil applied at points in the interior of the grid typically cannot be used.

- **Mixed:** A linear combination of f and its normal derivative are set to a constant.

- **Absorbing boundary condition (ABC):** This type of boundary condition is very important in applications of the finite difference method. Most electromagnetic problems involve unbounded regions, which cannot be modeled computationally. A better approach is to use a boundary condition that absorbs waves and reflects as little energy as possible. There are several types of ABCs, including: One-way wave equation. These are easy to implement but imperfect in 2D and higher dimensions. Boundary conditions may also be required at material interfaces inside the simulation region.

4. **Solution method**. Applying a stencil at each grid points leads to a system of difference equations which can be solved for the unknown sample values. Solution methods can be grouped into two categories:

- **Explicit:** update the value at one grid point at a time in terms of neighboring values.
- **Implicit:** Arrange the difference equations into a linear system and solve for all of the unknowns at one time.

2.5 Simulation of waves on string

Researcher will begin his study of wave equations by the simulating one dimensional wave on a string, say on guitar or violin string. Let the string in the deformed state coincide with the interval [0, L] on the x-axis, and let u(x, t) be the displacement at time t in the y-direction of a point initially at x. The displacement function u is governed by the mathematical model.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L], \quad t \in (0, t) \text{------} 2.1$$

$$u(x, 0) = I(x), \quad x \in [0, L] \text{------} 2.2$$

$$\frac{\partial^2 u}{\partial t^2}(x, 0) = 0, \quad x \in [0, L] \text{------} 2.3$$

$$u(0, t) = 0, \quad t \in (0, T] \text{------} 2.4$$

$$u(L, t) = 0, \quad t \in (0, T] \text{------} 2.5$$

The constant c and the function I(x) must be prescribed. Equation (2.1) is known as the one-dimensional wave equation. Since this PDE contains a second –order derivative in time, researcher needs two initial conditions. Here (2.2) specifying the initial shape of the string, I(x) and (2.3) reflecting that the initial velocity of the string is constant. In addition, PDEs need boundary conditions, here (2.4) and (2.5), specifying that the string is fixed at the ends, i.e, that the displacement u is zero. The solution u(x, t) varies in space and time and describes wave that are moving with velocity c to the left and right. Sometimes the researchers will use a more compact notation for the partial derivatives to save space. $u_t = \frac{\partial u}{\partial t} \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}$-----------------2.6

Then, the wave equation can be written comp actively as $u_{tt} = c^2 u_{xx}$.

2.6 Properties of the finite difference equation (FDE)

The most important properties of the finite difference equation are consistency, stability and convergence. These notions cover different aspects of the relation between the partial differential equation and finite difference equation.

Consistency: A finite difference of a partial differential equation is consistent, if the difference between partial differential equation (PDE) and finite differential equation (FDE) vanishes as the space and time step size approach zero. i.e.:

Partial Differential Equation-Finite Difference Equation (PDE-FDE) = 0

Stability: For as stable numerical scheme, the errors from any source will not grow unboundedly with time.

Convergence: It means that the solution to a finite difference equation approaches the true solution to the partial differential equation as both grid interval and time step sizes are reduced. The necessary and sufficient conditions for convergence are consistency and stability. i.e.:

Convergence = Stability + Consistency, this means the existence of stability and consistency shows us there is convergency.

These three factors that characterize a numerical scheme are linked together by lax equivalence theorem Frankel and Du Fort, [5] which states that given a well-posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence.

CHAPTER THREE

3 Methodology of the study

In this chapter the researcher mainly focus on the study design and the study procedures.

3.1 Study Design

The design of this study is to find numerical approximation solution by finite-difference method (FDM) to get the approximate solution which converges to the exact solution of wave equation.

3.2 Study Procedure

For this study, the researcher will follow the following procedures.

1. Construct the grid-points.

2. Apply the finite-difference method.

3. Determine the $u_{i,\ j+1}$ for 1D, $u_{i,\ j,\ n+1}$ for 2D wave equation.

4. Using the determined difference equation and the first approximation $u_{0,\ 1}$, $u_{0,\ 0,\ 1}$, then the solution of the wave equation will be obtained. The researchers were used finite- difference method to solve problems that deal with wave equation and gather data from internet

CHAPTER FOUR

4 Result and Discussions

4.1 Multi-dimensional wave equations

The governing wave equations are given by:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \quad or \quad \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \text{ --}4.1$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \text{ --}4.2$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\text{---}4.3$$

Where, c is the phase velocity of the wave field u. In the homogeneous case considered c is constant. Equation 4.1, 4.2 and 4.3 are known as 1D, 2D and 3D wave equations respectively.

4.2 Discretizing the domain

The temporal domain [0, T] is represented by a finite number of mesh points.

$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t-1} < t_{N_t=T}$.

Similarly, the spatial domain [0, T] is replaced by a set of mesh points.

$0 = x_0 < x_1 < x_2 < \cdots < x_{N_t-1} < x_{N_x=L}$

One may view the mesh as two dimensional in the x, t plane, consisting of points (x_i, t_n) with i = 0, N_x and n = 0, N_t.

4.3 Discretizing the Boundary conditions and the Initial conditions

Let's first deal with boundary conditions from u(0, t) = 0, u(L, t) = 0, t > 0.

$\boldsymbol{u_{0,\ j}}$= u(0, jk) = 0, j > 0 …………………………………………………………………...4.3 a

$\boldsymbol{u_{n,\ j}}$= u(nh, jk) = u(L, jk) = 0, j > 0…………………………………………………………4.3 b

And also the first initial condition yields $\boldsymbol{u_{i,\ 0}}$ = u(ih, 0) = f(ih), 0≤ $\boldsymbol{i} \le \boldsymbol{n}$…………………4.3 c

We now use the initial data f and g to generate a second set of initial value of u. For this purpose, it is best to use the centered first difference approximation.

$\boldsymbol{u_t}$ (x, t) ≈ $\frac{\boldsymbol{u(x,\ t+k)-u(x,\ t-k)}}{\boldsymbol{2k}}$, the order of accuracy is first order.

Setting t = 0, we find g(x) = $\boldsymbol{u_t}$ (x, t) ≈ $\frac{\boldsymbol{u(x,\ t+k)-u(x,\ t-k)}}{\boldsymbol{2k}}$, evaluating at the points x = ih, we get: $\boldsymbol{u_{i,\ 1} - u_{i,-1}}$ =2kg(ih)…………………………………………………………..4.3 d

4.4 Mesh

We have been introduced a mesh in time and in space. The mesh in time consists of time points $\boldsymbol{t_0 = 0 < t_1 < \cdots < t_{N_x}}$, often with a constant spacing $\boldsymbol{\Delta t = t_{n+1} - t_n, n \ni I_t^-}$. Finite difference methods are easy to implement on simple rectangle –or box-shaped domains. More complicated shapes of the domain require substantially more advanced techniques and implementation efforts. On a rectangle-or box-shaped domain mesh points are introduced separately in the various space direction.

4.5 Governing equation for 1-D and 2-D wave equation

4.5.1 One dimensional (1-D) wave equation

The wave equation in one dimension has the following forms.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{------------------------------------}4.4$$

Subject to the initial conditions;

u = f(x), $\frac{\partial u}{\partial t}$ = g(x), $0 \leq x \leq 1$ at t = 0--4.5

And the boundary conditions; u(0, t) = ϕ(t), u(1, t) = $\psi(t)$--4.6

Where, c is a constant propagation of speed.

4.6 First order one-way wave equation

The first order wave equation in one-dimensional space is as follows:

$u_t = cu_x$ ………………………………………………………………………...4.7

Where, c is a positive constant.

4.7 Analytical solutions to the 1-D wave equation

4.7.1 1-D wave equation

The wave equation in one dimension has the following forms.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \ldots\ldots\ldots (4.5)$$

4.8 Explicit difference method (EDM) for 1-D wave equation

To find the wave equation in the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ for } 0 \leq x \leq 1 \; and \; 0 \leq t \leq 1 \ldots\ldots\ldots 4.6$$

Consider the boundary conditions (BCs) and initial condition (ICs) as follows:-

Boundary condition (BC): u(0, t) = u(1, t) = 0……………………………………………4.7

Initial condition (IC): $\left.\begin{array}{l} u(x, 0) = f(x) = \sum_{k=1}^{\infty} a_k \sin k\pi x \\ u_t(x, 0) = g(x) = \sum_{k=1}^{\infty} b_k \sin k\pi x \end{array}\right\}$ ……………………………...4.8

Where u (x, 0) = f(x) = $\sum_{k=1}^{\infty} a_k \sin k\pi x$ and u_t(x, 0) = g(x) = $\sum_{k=1}^{\infty} b_k \sin k\pi x$ are known as initial position and velocity respectively.

The analytical solution of equation (4.6) as given by d'Alembert formula as found in [Hildebrand - 1968] is:

$$u(x, t) = \frac{1}{2} \left(f(x - ct) + f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)dy \quad \ldots\ldots\ldots 4.9\right.$$

4.9 Discretizing the solution of wave equation in one dimension

Consider a rectangular mesh in the x-t plane spacing h along x direction, with $h = \Delta x$ and k along time t- direction with $k = \Delta t$. Denoting a mesh point (x, t) = (ih, jk) as simply i, j, we have:

$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$ and $\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j-1}}{k^2}$, replacing the derivatives in Eq(4.5) by their above approximations, we obtain the following:

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2 k^2}{h^2} (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$$

or

$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = c^2\alpha^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$, where $\alpha = {}^{k}/_{h}$

$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = c^2\alpha^2 u_{i-1,j} - 2c^2\alpha^2 u_{i,j} + c^2\alpha^2 u_{i+1,j}$

or

$u_{i,j+1} = c^2\alpha^2 u_{i-1,j} - 2c^2\alpha^2 u_{i,j} + c^2\alpha^2 u_{i+1,j} + 2u_{i,j} - u_{i,j-1}$

or

$u_{i,j+1} = -2c^2\alpha^2 u_{i,j} + 2u_{i,j} + c^2\alpha^2 u_{i-1,j} + c^2\alpha^2 u_{i+1,j}$

or

$u_{i,j+1} = 2(1-c^2\alpha^2)u_{i,j} + c^2\alpha^2\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1}$ ------------------------------- (4.10)

Obs: 1.The coefficient of $u_{i,j}$ in Eq (4.10) will vanish if αc = 1 or k = ${}^{h}/_{c}$. Then, Eq (4.10) reduces to the simple form $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ ----------------- (4.11)

1. For $\alpha = {}^{1}/_{c}$, the solution of Eq(4.10) is stable and coincides with the solution of Eq(4.5).
2. For $\alpha < {}^{1}/_{c}$, the solution is stable but, inaccurate.
3. For $\alpha > {}^{1}/_{c}$, the solution is unstable.
4. The formula Eq(4.10) converges for $\alpha \le 1 \quad i.e.\ k \le h$.

4.10 Problem solved examples of wave equations in 1-D and 2-D

Example 3: consider the hyperbolic problem $\frac{\partial^2 u}{\partial t^2}(x,t) - 4\frac{\partial^2 u}{\partial t^2}(x,t) = 0$, subject to the following conditions.

Boundary condition (BC): u (0, t) = u (1, t) = 0 for 0 < t

Initial condition (IC): u (x, 0) = sin (πx), $\frac{\partial u}{\partial t}$ (x, 0) = 0 for all x in 0$\leq x \leq 1$

The actual (exact) solution of this problem is given by u (x, t) = $\sin(\pi x)\cos(2\pi t)$…………... (i)

We use the explicit formula given below: $u_{i,j+1} = 2(1\text{-}\alpha^2)u_{i,j} + \alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$ or

$u_{i,j+1} = -u_{i,j-1} + \alpha^2(u_{i-1,j} + u_{i+1,j}) + 2(1\text{-}\alpha^2)u_{i,j}$ where $\alpha = \frac{ck}{h} < 1$……………………………… (ii)

Let h = 0.2, k = 0.05 and $\alpha = \frac{ck}{h} = \frac{2x0.05}{0.2} = 0.5 < 1$, so that the stability condition is satisfied.

Let $u_{i,j}$ = u (ih, jk), so that the boundary conditions become:

$u_{0,\ j} = 0$ …… (iii)

$u_{5,\ j} = 0$ …… (iv)

$u_{i,\ 0}$ = $\sin(\pi ih)$, i =1, 2, 3, 4, 5 ………………………………………………………………………………………(v)

And $\frac{\partial u}{\partial t}$ (x, 0) = 0 by central difference $\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$ becomes:

$u_{i,1} - u_{i,-1}$ = 0 so that $u_{i,1} = u_{i,-1}$…………………………………………………………………………………(vi)

Substituting α = 0.5, in Eq (ii) becomes:

$u_{i,j+1} = -u_{i,j-1} + 0.25(u_{i-1,j} + u_{i+1,j}) + 2(0.75)u_{i,j}$……………………………………………………… (vii)

From the initial condition u (x, 0) = sin (πx) for j = 0, i = 0, 1, 2, 3, 4, 5 the $u_{i,\ 0} = \sin(\pi ih)$ where size step h = 0.2 is iterated as follows: $u_{0,\ 0}$ = 0.0000

$u_{1,\ 0}$ = sin ($\pi x 1 x 0.2$) = sin (0.2π) = sin (36) = 0.587785252≈ 0.5878

$u_{2,\ 0}$ = sin ($\pi x 2 x 0.2$) = sin (0.4π) = sin (72) = 0.951056516≈ 0.9511

$u_{3,\ 0}$ = sin ($\pi x 3 x 0.2$) = sin (0.6π) = sin (108) = 0.951056516≈ 0.9511

$u_{4,\ 0}$ = sin ($\pi x 4 x 0.2$) = sin (0.8π) = sin (144) = 0.587785252≈ 0.5878

$u_{5,\ 0}$ = sin ($\pi x 5 x 0.2$) = sin (1.0π) = sin (180) = 0 (Boundary condition)

Remark: $u_{0,\ 0} = u_{5,\ 0}$, $u_{1,\ 0} = u_{4,\ 0}$, $u_{2,\ 0} = u_{3,\ 0}$

At the first step, **j = 0** and the above Eq(vii) becomes:

Using Eq (vi) the Eq (vii) yields: $u_{i,\ 1} = \frac{1}{2}(0.25\left(u_{i-1,\ 0} + u_{i+1,\ 0}\right) + 2(0.75)u_{i,\ 0})$

$u_{i,\ 1} = 0.125\left(u_{i-1,\ 0} + u_{i+1,\ 0}\right) + 0.75u_{i,\ 0}$.. (viii)

Consequently, for **i = 1** Eq (viii) gives the following iterations.

$u_{1,\ 0}$ = sin ($\pi x 1 x 0.2$) ≈ 0.5878, $u_{2,\ 0}$ = sin ($\pi x 2 x 0.2$)≈ 0.9511 and successively, we obtain the left iteration as follows: $u_{3,\ 0}$ = sin ($\pi x 3 x 0.2$) ≈ 0.9511, $u_{4,\ 0}$ ≈ 0.5878, $u_{5,\ 0}$ = 0 (BC)

Remark: $u_{0,\ 0} = u_{5,\ 0}$, $u_{1,\ 0} = u_{4,\ 0}$, $u_{2,\ 0} = u_{3,\ 0}$

At the first step, **j = 0** and the above Eq(vii) becomes:

Using Eq (vi) the Eq (vii) yields: $u_{i,\ 1} = \frac{1}{2}(0.25\left(u_{i-1,\ 0} + u_{i+1,\ 0}\right) + 2(0.75)u_{i,\ 0})$

$u_{i,\ 1} = 0.125\left(u_{i-1,\ 0} + u_{i+1,\ 0}\right) + 0.75u_{i,\ 0}$.. (viii)

Consequently, for **i = 1** Eq (viii) gives the following iterations.

$u_{1,\ 1} = 0.125\left(u_{0,\ 0} + u_{2,\ 0}\right) + 0.75u_{1,\ 0} \approx 0.5597$

Exact: u (0.2, 0.05) = $\sin(\pi x 0.2)\cos(2\pi x 0.05)$ ≈0.5590 Error = |Exact − Approximate| ≈0.0007 and successively, we obtain the left iteration as follows:

$u_{2,\ 1}$ ≈ 0.9057, Exact: $u_{2,\ 1}$ ≈ 0.9045 and error ≈ 0.0011

$u_{3,\ 1}$ ≈ 0.9057, Exact: $u_{3,\ 1}$ ≈ 0.9045 and error ≈ 0.0011

$u_{4,\ 1}$ ≈ 0.5597, Exact: $u_{4,\ 1}$ ≈ 0.5590 and error ≈ 0.0007

$u_{5,\ 1}$ = 0.0000 (Boundary condition), Exact: $u_{5,\ 1}$ = 0.0000 and Error = 0.0000

At the second step, **i = j = 1**, then Eq (viii) iterated as follows:

$u_{1,\ 2} = 0.125(u_{0,\ 1} + u_{2,\ 1}) + 0.75u_{1,\ 1} \approx 0.5330$

Exact: u (0.2, 0.1) = $\sin(\pi x 0.2)\cos(2\pi x 0.1) \approx 0.47552$, Error ≈0.0575

And successively, we obtain the left iteration as follows:$u_{2,\ 2} \approx 0.8624$, Exact: $u_{2,\ 2} \approx 0.7694$ and error ≈ 0.0930, $u_{3,\ 2} \approx 0.8624$, Exact: $u_{3,\ 2} \approx 0.7694$ and error ≈ 0.0930

$u_{4,\ 2} \approx 0.5330$, Exact: $u_{4,\ 2} \approx 0.4755$ and error ≈ 0.0575, $u_{5,\ 2} = 0.0000$ (Boundary condition), Exact: $u_{5,\ 2} = 0.0000$ and error = 0.0000. At the third step, for **i = 1** and **j = 2**, then Eq (viii) iterated as follows: $u_{1,\ 3} = 0.125(u_{0,\ 2} + u_{2,\ 2}) + 0.75\, u_{1,\ 2} \approx 0.5076$

Exact: u (0.2, 0.15) = $\sin(\pi x 0.2)\cos(2\pi x 0.15) \approx 0.3455$ and error ≈ 0.1621, and successively, we obtain the left iteration as follows: $u_{2,\ 3} \approx 0.8212$, Exact:$u_{2,\ 3} \approx 0.5590$ and error ≈ 0.2622, $u_{3,\ 3} \approx 0.8212$, Exact: $u_{3,\ 3} \approx 0.5590$ and error ≈ 0.2622

$u_{4,\ 3} \approx 0.5075$, Exact: $u_{4,\ 3} \approx 0.3455$ and error ≈ 0.1621, $u_{5,\ 3} = 0.0000$ (Boundary condition), Exact: $u_{5,\ 3} = 0.0000$ and error= 0.0000. At the fourth step, for **i = 1** and **j = 3**, then Eq (viii) iterated as follows: $u_{1,\ 4} = 0.125(u_{0,\ 3} + u_{2,\ 3}) + 0.75u_{1,\ 3} \approx 0.4833$, Exact: u (0.2, 0.2) = $\sin(\pi x 0.2)\cos(2\pi x 0.2) \approx 0.1816$ and error ≈ 0.3017. And successively, we obtain the left iteration as follows: $u_{2,\ 4} \approx 0.7820$, Exact: $u_{2,\ 4} \approx 0.2938$ and error ≈ 0.4881, $u_{3,\ 4} \approx 0.7820$, Exact: $u_{3,\ 4} \approx 0.2939$ and error ≈ 0.4881, $u_{4,\ 4} \approx 0.4833$, Exact: $u_{4,\ 4} \approx 0.1816$ and error ≈ 0.3017, $u_{5,\ 4} = 0.0000$ (Boundary condition), Exact: $u_{5,\ 4} = 0.0000$ and Error = 0.0000. At the fifth step, for **i = 1** and **j = 4**, then Eq (viii) iterated as follows: $u_{1,\ 5} = 0.125(u_{0,\ 4} + u_{2,\ 4}) + 0.75u_{1,\ 4} \approx 0.4602$, Exact: u (0.2, 0.25) = $\sin(\pi x 0.2)\cos(2\pi x 0.25) = 0.0000$ and error ≈ 0.4602. And successively, we obtain the left iteration as follows: $u_{2,\ 5} \approx 0.7447$, Exact: $u_{2,\ 5} = 0.0000$ and Error ≈ 0.7447, $u_{3,\ 5} \approx 0.7447$, Exact: $u_{3,\ 5} = 0.0000$ and Error ≈ 0.7447

$u_{4,\ 5} \approx 0.4602$, Exact: $u_{4,\ 5} = 0.0000$ and error ≈ 0.4602, $u_{5,\ 5} = 0.0000$ (Boundary condition), Exact: $u_{5,\ 5} = 0.0000$ and Error = 0.0000

Entry	Iteration	Approximate Solution	Entry	Iteration	Approximate Solution
1^{st}	$u_{1,\ 1}(0.2,\quad 0.05)$	0.5590	4^{th}	$u_{1,\ 4}(0.2,\quad 0.2)$	0.1816
	$u_{2,\ 1}(0.4,\quad 0.05)$	0.9045		$u_{2,\ 4}(0.4,\quad 0.2)$	0.2939
	$u_{3,\ 1}(0.6,\quad 0.05)$	0.9045		$u_{3,\ 4}(0.6,\quad 0.2)$	0.2939
	$u_{4,\ 1}(0.8,\quad 0.05)$	0.5590		$u_{4,\ 4}(0.8,\quad 0\ .2)$	0.1816
	$u_{5,\ 1}(1.0,\quad 0.05)$	0.0000		$u_{5,\ 4}(1.0,\quad 0.2)$	0.0000
2^{nd}	$u_{1,\ 2}(0.2,\quad 0.1)$	0.4755	5^{th}	$u_{1,\ 5}(0.2,\quad 0.25)$	0.0000
	$u_{2,\ 2}(0.4,\quad 0.1)$	0.7694		$u_{2,\ 5}(0.4,\quad 0.25)$	0.0000
	$u_{3,\ 2}(0.6,\quad 0.1)$	0.7694		$u_{3,\ 5}(0.6,\quad 0.25)$	0.0000
	$u_{4,\ 2}(0.8,\quad 0.1)$	0.4755		$u_{4,\ 5}(0.8,\quad 0.25)$	0.0000
	$u_{5,\ 2}(1.0,\quad 0.1)$	0.0000		$u_{5,\ 5}(1.0,\quad 0.25)$	0.0000
3^{rd}	$u_{1,\ 3}(0.2,\quad 0.15)$	0.3455			
	$u_{2,\ 3}(0.4,\quad 0.15)$	0.5590			
	$u_{3,\ 3}(0.6,\quad 0.15)$	0.5590			
	$u_{4,\ 3}(0.8,\quad 0.15)$	0.3455			
	$u_{5,\ 3}(1.0,\quad 0.15)$	0.0000			

Table 1 Entry and Exact Solution for wave Equation in 1D.

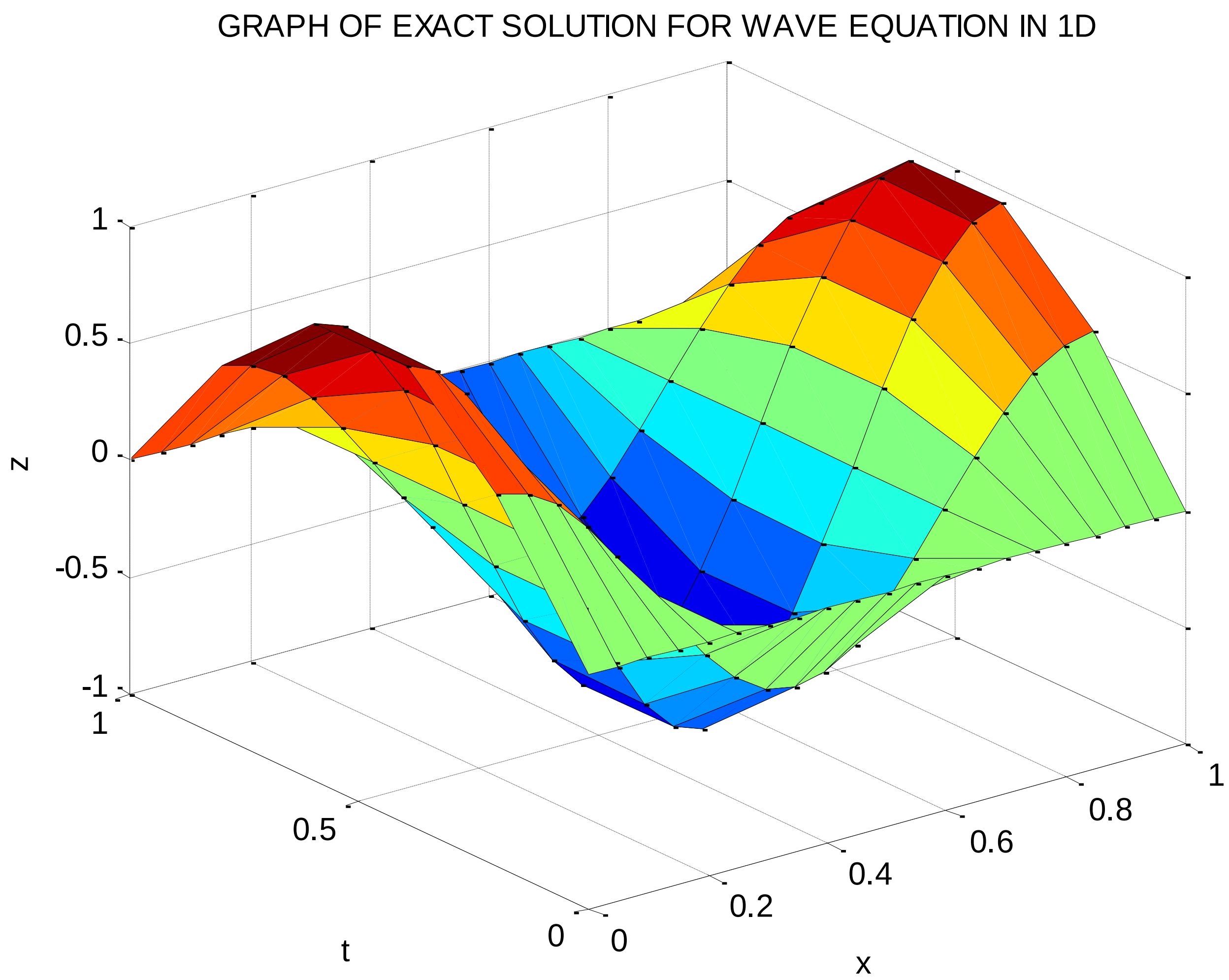

Figure 1 graph of exact solution for 1D wave equation for step size h = 0.2 and k = 0.05

Entry	Iteration	Approximate Solution	Entry	Iteration	Approximate Solution
1^{st}	$u_{1,\ 1}$	0.5597	4^{th}	$u_{1,\ 4}$	0.4834
	$u_{2,\ 1}$	0.9057		$u_{2,\ 4}$	0.7822
	$u_{3,\ 1}$	0.9057		$u_{3,\ 4}$	0.7838
	$u_{4,\ 1}$	0.5597		$u_{4,\ 4}$	0.4887
	$u_{5,\ 1}$	0.0000		$u_{5,\ 4}$	0.0000
2^{nd}	$u_{1,\ 2}$	0.5330	5^{th}	$u_{1,\ 5}$	0.4604
	$u_{2,\ 2}$	0.8624		$u_{2,\ 5}$	0.7451
	$u_{3,\ 2}$	0.8624		$u_{3,\ 5}$	0.7468
	$u_{4,\ 2}$	0.5422		$u_{4,\ 5}$	0.4645
	$u_{5,\ 2}$	0.0000		$u_{5,\ 5}$	0.0000
3^{rd}	$u_{1,\ 3}$	0.5076			
	$u_{2,\ 3}$	0.8212			
	$u_{3,\ 3}$	0.8224			
	$u_{4,\ 3}$	0.5145			
	$u_{5,\ 3}$	0.0000			

Table 2 entry, iteration and approximate solution of wave equation in 1D.

Entry	Iteration	Exact Solution	Approximate Solution	Error
1^{st}	$u_{1,\ 1}$	0.5590	0.5597	0.0007
	$u_{2,\ 1}$	0.9045	0.9057	0.0012
	$u_{3,\ 1}$	0.9045	0.9057	0.0012
	$u_{4,\ 1}$	0.5590	0.5597	0.0007
	$u_{5,\ 1}$	0.0000	0.0000	0.0000
2^{nd}	$u_{1,\ 2}$	0.4755	0.5330	0.0575
	$u_{2,\ 2}$	0.7694	0.8624	0.0930
	$u_{3,\ 2}$	0.7694	0.8624	0.0930
	$u_{4,\ 2}$	0.4755	0.5422	0.0667
	$u_{5,\ 2}$	0.0000	0.0000	0.0000
3^{rd}	$u_{1,\ 3}$	0.3455	0.5076	0.1621
	$u_{2,\ 3}$	0.5590	0.8212	0.2622
	$u_{3,\ 3}$	0.5590	0.8224	0.2634
	$u_{4,\ 3}$	0.3455	0.5145	0.1690
	$u_{5,\ 3}$	0.0000	0.0000	0.0000
4^{th}	$u_{1,\ 4}$	0.1816	0.4834	0.3018
	$u_{2,\ 4}$	0.2939	0.7822	0.4883
	$u_{3,\ 4}$	0.2939	0.7838	0.4899
	$u_{4,\ 4}$	0.1816	0.4887	0.3071
	$u_{5,\ 4}$	0.0000	0.0000	0.0000
5^{th}	$u_{1,\ 5}$	0.0000	0.4604	0.4604
	$u_{2,\ 5}$	0.0000	0.7451	0.7451
	$u_{3,\ 5}$	0.0000	0.7468	0.7468
	$u_{4,\ 5}$	0.0000	0.4645	0.4645
	$u_{5,\ 5}$	0.0000	0.0000	0.0000

Table 4 Entry, Iteration, Exact solution, Approximation Solution and Error of wave Equation in 1D.

Solution of 1D wave equation for c = 2, h = 0.1, k = 0.05 and $\alpha = \frac{ck}{h} = \frac{2x0.05}{0.1} = 1$ concerning **example1** above. $u_{1,\ 0} = \sin(\pi x 0.1) = \sin(0.1\pi) = \sin 18 = 0.3090$

$u_{2,\ 0} = \sin(\pi x 0.2) = \sin(0.2\pi) = 0.5878$, $u_{3,\ 0} = \sin(\pi x 0.3) = \sin(0.3\pi) = 0.8090$

$u_{4,\ 0}$ = sin(πx0.4) = sin(0.4π)= 0.9511, $u_{5,\ 0}$ = sin(πx0.5) = sin(0.5π) =1.0000

$u_{6,\ 0}$ = sin(πx0.6) = sin(0.6π) = 0.9511, $u_{7,\ 0}$ = sin(πx0.7) = sin(0.7π) =0.8090

$u_{8,\ 0}$ = sin(πx0.8) = sin(0.8π) = 0.5878, $u_{9,\ 0}$ = sin(πx0.9) = sin(0.9π) = 0.3090 and $u_{10,\ 0} = \sin(10\pi) = 0.0000$ (BC). Applying Eq. (4.10) we have $u_{i,\ 1} = \frac{\alpha^2}{2}\left(u_{i+1,\ 0} + u_{i-1,\ 0}\right) + (1-\alpha^2)u_{i,\ 0}$

For i = 1, then $u_{1,\ 1} = 0.5\left(u_{2,\ 0} + u_{0,\ 0}\right) \approx 0.2939$

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 1} \approx 0.5590$, $u_{3,\ 1} \approx 0.7695$, $u_{4,\ 1} \approx 0.9045$, $u_{5,\ 1} \approx 0.9511$, $u_{6,\ 1} \approx 0.9045$

$u_{7,\ 1} \approx 0.7695$, $u_{8,\ 1} \approx 0.5590$, $u_{9,\ 1} \approx 0.2939$ and $u_{10,\ 1} = 0.0000$ (BC)

For i = j = 1, then $u_{1,\ 2} \approx 0.2939$, $u_{2,\ 2} \approx 0.5317$, $u_{3,\ 2} \approx 0.7318$, $u_{4,\ 2} \approx 0.8603$

$u_{5,\ 2} \approx 0.9045$, $u_{6,\ 2} \approx 0.8603$, $u_{7,\ 2} \approx 0.7318$, $u_{8,\ 2} \approx 0.5317$, $u_{9,\ 2} \approx 0.2939$ and $u_{10,\ 2} = 0.0000$, **For j = 2**, $u_{1,\ 3} \approx 0.2659$, $u_{2,\ 3} \approx 0.5057$, $u_{3,\ 3} \approx 0.6960$, $u_{4,\ 3} \approx 0.8182$, $u_{5,\ 3} \approx 0.8603$, $u_{6,\ 3} \approx 0.8182$, $u_{7,\ 3} \approx 0.6960$, $u_{8,\ 3} \approx 0.5057$, $u_{9,\ 3} \approx 0.2659$ and $u_{10,\ 3} = 0.0000$, **For j = 3**, $u_{1,\ 4} = 0.5\left(u_{2,\ 3} + u_{0,\ 3}\right) \approx 0.2529$

$u_{2,\ 4} \approx 0.4810$, $u_{3,\ 4} \approx 0.6620$, $u_{4,\ 4} \approx 0.7782$, $u_{5,\ 4} \approx 0.8182$, $u_{6,\ 4} \approx 0.7782$

$u_{7,\ 4} \approx 0.6620$, $u_{8,\ 4} \approx = 0.4810$, $u_{9,\ 4} \approx 0.2529$ and $u_{10,\ 4} = 0.0000$

For j = 4, $u_{1,\ 5} = 0.5\left(u_{2,\ 4} + u_{0,\ 4}\right) \approx 0.2405$, $u_{2,\ 5} \approx 0.4575$, $u_{3,\ 5} \approx 0.6296$, $u_{4,\ 5} \approx 0.7401$, $u_{5,\ 5} \approx 0.7782$, $u_{6,\ 5} \approx 0.7401$, $u_{7,\ 5} \approx 0.6296$, $u_{8,\ 5} \approx 0.4575$, $u_{9,\ 5} \approx 0.2405$ and $u_{10,\ 5} = 0.0000$, **For j = 5**, $u_{1,\ 6} = 0.5\left(u_{2,\ 5} + u_{0,\ 5}\right) = 0.2289$

$u_{2,\ 6} \approx 0.4351$, $u_{3,\ 6} \approx 0.5988$, $u_{4,\ 6} \approx 0.7039$, $u_{5,\ 6} \approx 0.7401$, $u_{6,\ 6} \approx 0.7039$

$u_{7,\ 6} \approx 0.5988$, $u_{8,\ 6} \approx 0.4351$, $u_{9,\ 6} \approx 0.2289$ and $u_{10,\ 6} = 0.0000$, **For j = 6**

$u_{1,\ 7} \approx 0.2176$, $u_{2,\ 7} \approx 0.4139$, $u_{3,\ 7} \approx 0.5695$, $u_{4,\ 7} \approx 0.6695$, $u_{5,\ 7} \approx 0.7039$, $u_{6,\ 7} \approx 0.6695$, $u_{7,\ 7} \approx 0.5695$, $u_{8,\ 7} \approx 0.4139$, $u_{9,\ 7} \approx 0.2176$ and $u_{10,\ 7} = 0.0000$

For j = 7, $u_{1,\ 8}$ $\approx$ 0.2070, $u_{2,\ 8} \approx$ 0.3936, $u_{3,\ 8} \approx$ 0.5417, $u_{4,\ 8} \approx$ 0.6367, $u_{5,\ 8} \approx$ 0.6695, $u_{6,\ 8} \approx$ 0.6367, $u_{7,\ 8} \approx$ 0.5417, $u_{8,\ 8} \approx$ 0.3936, $u_{9,\ 8} \approx$ 0.2070 and $u_{10,\ 8}$ = 00.0000, **For j = 8**, $u_{1,\ 9} \approx$ 0.1968, $u_{2,\ 9} \approx$ 0.3744, $u_{3,\ 9} \approx$ 0.5152, $u_{4,\ 9} \approx$ 0.6056, $u_{5,\ 9} \approx$ 0.6367, $u_{6,\ 9} \approx$ 0.6056, $u_{7,\ 9} \approx$ 0.5152, $u_{8,\ 9} \approx$ 0.3744, $u_{9,\ 9} \approx$ 0.1968 and $u_{10,\ 9}$ = 0.0000, **For j = 9**, $u_{1,\ 10} \approx$ 0.1872, $u_{2,\ 10} \approx$ 0.3560, $u_{3,\ 10} \approx$ 0.4900, $u_{4,\ 10} \approx$ 0.5760, $u_{5,\ 10} \approx$ 0.6056, $u_{6,\ 10} \approx$ 0.5760, $u_{7,\ 10} \approx$ 0.4900, $u_{9,\ 10} \approx$ 0.1872 and $u_{10,\ 10}$ = 0.0000

Table 5 Entry, Iteration, Exact solution, Approximation Solution and Error for wave Equation in 1D.

Entry	Iteration	Exact	Approximate	Error	Entry	Iteration	Exact	Approximate	Error
1^{st}	$u_{1,\ 1}$	0.2939	0.2939	0.0000		$u_{7,\ 3}$	0.4755	0.6960	0.2205
	$u_{2,\ 1}$	0.5590	0.5590	0.0000		$u_{8,\ 3}$	0.3455	0.5057	0.1602
	$u_{3,\ 1}$	0.7694	0.7695	0.0001		$u_{9,\ 3}$	0.1816	0.2659	0.0843
	$u_{4,\ 1}$	0.9045	0.9045	0.0000		$u_{10,\ 3}$	0.0000	0.0000	0.0000
	$u_{5,\ 1}$	0.9511	0.9511	0.0000	4^{th}	$u_{1,\ 4}$	0.0955	0.2529	0.1574
	$u_{6,\ 1}$	0.9045	0.9045	0.0000		$u_{2,\ 4}$	0.1816	0.4810	0.2994
	$u_{7,\ 1}$	0.7694	0.7695	0.0001		$u_{3,\ 4}$	0.2500	0.6620	0.4120
	$u_{8,\ 1}$	0.5590	0.5590	0.0000		$u_{4,\ 4}$	0.2939	0.7782	0.4843
	$u_{9,\ 1}$	0.2939	0.2939	0.0000		$u_{5,\ 4}$	0.3090	0.8182	0.5092
	$u_{10,\ 1}$	0.0000	0.0000	0.0000		$u_{6,\ 4}$	0.2939	0.7782	0.4843
2^{nd}	$u_{1,\ 2}$	0.2500	0.2795	0.0295		$u_{7,\ 4}$	0.2500	0.6620	0.4120
	$u_{2,\ 2}$	0.4755	0.5317	0.0562		$u_{8,\ 4}$	0.1816	0.4810	0.2994
	$u_{3,\ 2}$	0.6545	0.7318	0.0773		$u_{9,\ 4}$	0.0955	0.2529	0.1574
	$u_{4,\ 2}$	0.7694	0.8603	0.0909		$u_{10,\ 4}$	0.0000	0.0000	0.0000
	$u_{5,\ 2}$	0.8090	0.0.9045	0.0955	5^{th}	$u_{1,\ 5}$	0.0000	0.2405	0.2405
	$u_{6,\ 2}$	0.7694	0.8603	0.0909		$u_{2,\ 5}$	0.0000	0.4575	0.4575
	$u_{7,\ 2}$	0.6545	0.7318	0.0773		$u_{3,\ 5}$	0.0000	0.6296	0.6296
	$u_{8,\ 2}$	0.4755	0.5317	0.0562		$u_{4,\ 5}$	0.0000	0.7401	0.7401
	$u_{9,\ 2}$	0.2500	0.2795	0.0295		$u_{5,\ 5}$	0.0000	0.7782	0.7782
	$u_{10,\ 2}$	0.0000	0.0000	0.0000		$u_{6,\ 5}$	0.0000	0.7401	0.7401
3^{rd}	$u_{1,\ 3}$	0.1816	0.2659	0.0843		$u_{7,\ 5}$	0.0000	0.6296	0.6296
	$u_{2,\ 3}$	0.3455	0.5057	0.1602		$u_{8,\ 5}$	0.0000	0.4575	0.4575
	$u_{3,\ 3}$	0.4755	0.6960	0.2205		$u_{9,\ 5}$	0.0000	0.2405	0.2405
	$u_{4,\ 3}$	0.5590	0.8182	0.2592		$u_{10,\ 5}$	0.0000	0.0000	0.0000
	$u_{5,\ 3}$	0.5878	0.8603	0.2725		$u_{1,\ 6}$	-0.0955	0.2289	0.3244
	$u_{6,\ 3}$	0.5590	0.8182	0.2592		$u_{2,\ 6}$	-0.1816	0.4351	0.6167

Table 6 Entry, Iteration, Exact solution, Approximation Solution and Error for wave Equation in 1D.

Entry	Iteration	Exact	Approximate	Error	Entry	Iteration	Exact	Approximate	Error
6^{th}	$u_{3,\ 6}$	-0.2500	0.5988	0.8488		$u_{10,\ 8}$	0.0000	0.0000	0.0000
	$u_{4,\ 6}$	-0.2939	0.7039	0.9978	9^{th}	$u_{1,\ 9}$	-0.2939	0.1968	0.4907
	$u_{5,\ 6}$	-0.3090	0.7401	1.0491		$u_{2,\ 9}$	-0.5590	0.3744	0.9334
	$u_{6,\ 6}$	-0.2939	0.7039	0.9978		$u_{3,\ 9}$	-0.7694	0.5152	1.2846
	$u_{7,\ 6}$	-0.2500	0.5988	0.8488		$u_{4,\ 9}$	-0.9045	0.6056	1.5101
	$u_{8,\ 6}$	-0.1816	0.4351	0.6167		$u_{5,\ 9}$	-0.9511	0.6367	1.5878
	$u_{9,\ 6}$	-0.0955	0.2289	0.3244		$u_{6,\ 9}$	-0.9045	0.6056	1.5101
	$u_{10,\ 6}$	0.0000	0.0000	0.0000		$u_{7,\ 9}$	-0.7694	0.5152	1.2846
7^{th}	$u_{1,\ 7}$	-0.1816	0.2176	0.3992		$u_{8,\ 9}$	-0.5590	0.3744	0.9334
	$u_{2,\ 7}$	-0.3455	0.4139	0.7594		$u_{9,\ 9}$	-0.2939	0.1968	0.4907
	$u_{3,\ 7}$	-0.4755	0.5695	1.045		$u_{10,\ 9}$	0.0000	0.0000	0.0000
	$u_{4,\ 7}$	-0.5590	0.6695	1.2285	10^{th}	$u_{1,\ 10}$	-0.3090	0.1872	0.4962
	$u_{5,\ 7}$	-0.5878	0.7039	1.2917		$u_{2,\ 10}$	-0.5878	0.3560	0.9438
	$u_{6,\ 7}$	-0.5590	0.6695	1.2285		$u_{3,\ 10}$	-0.8090	0.4900	1.2990
	$u_{7,\ 7}$	-0.4755	0.5695	1.045		$u_{4,\ 10}$	-0.9511	0.5760	1.5271
	$u_{8,\ 7}$	-0.3455	0.4139	0.7594		$u_{5,\ 10}$	-1.0000	0.6056	1.6056
	$u_{9,\ 7}$	-0.1816	0.2176	0.3992		$u_{6,\ 10}$	-0.9511	0.5760	1.5271
	$u_{10,\ 7}$	0.0000	0.0000	0.0000		$u_{7,\ 10}$	-0.8090	0.4900	1.2990
8^{th}	$u_{1,\ 8}$	-0.2500	0.2070	0.4570		$u_{8,\ 10}$	-0.5878	0.3560	0.9438
	$u_{2,\ 8}$	-0.4755	0.3936	0.8691		$u_{9,\ 10}$	-0.3090	0.1872	0.4962
	$u_{3,\ 8}$	-0.6545	0.5417	1.1962		$u_{10,\ 10}$	0.0000	0.0000	0.0000
	$u_{4,\ 8}$	-0.7694	0.6367	1.4061					
	$u_{5,\ 8}$	-0.8090	0.6695	1.4785					
	$u_{6,\ 8}$	-0.7694	0.6367	1.4061					
	$u_{7,\ 8}$	-0.6545	0.5417	1.1962					
	$u_{8,\ 8}$	-0.4755	0.3936	0.8691					
	$u_{9,\ 8}$	-0.2500	0.2070	0.4570					

Figure 3 graph of approximate solution for 1D wave equation for step size h = 0.1 and k = 0.05

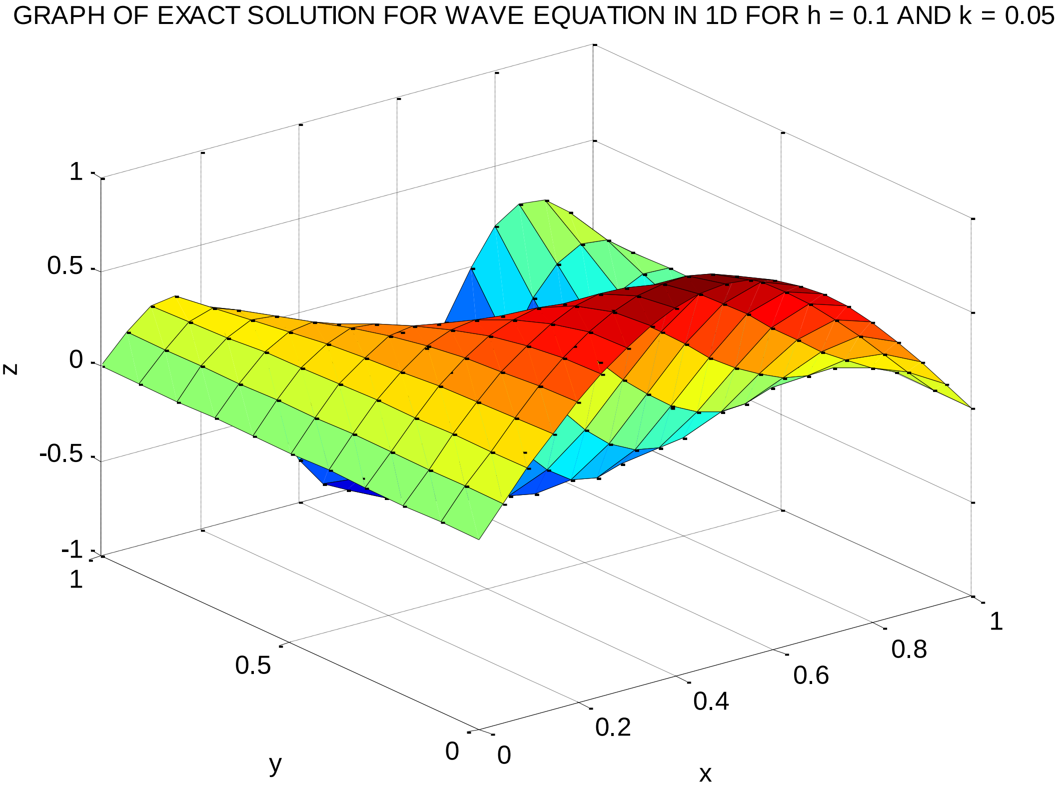

Figure 4 Graph of Exact Solution for 1D Wave Equation for step size h = 0.1 and k = 0.05

4.11 Two (2-D) Wave equations

The propagation of waves in a two dimensional vibrating membrane of length a and width b is governed by the following initial-boundary value problem.

PDE**:** $0 < x < a, \quad 0 < t < b, \quad b > 0$……………………………………4.12

Consider the boundary conditions (BCs) and initial conditions (ICs) as follows:-

Boundary Condition (BC): $u\,(0, y, t) = u\,(a, y, t) = 0$

$$u\,(x, 0, t) = u\,(x, b, t) = 0$$

Initial Condition (IC): $u\,(x, y, 0) = f\,(x, y)$

$$\boldsymbol{u_t(x, y,\ 0)} = g\,(x, y)$$

Where u (x, y, t) is the displacement function of any point located at the position (x, y) of vibrating membrane at any time t, and c is related to the elasticity of the material of the rectangular plate. The solution in the t direction, in the x space or in the y space will lead to identical results. However, the solution in the t direction reduces the size of calculations compared with the other space solutions because it uses the initial conditions only.

4.12 Homogeneous Wave Equations

The Finite Difference Method (FDM) will be used to solve the following homogeneous wave equations in two dimensional vibrating membranes with homogeneous boundary conditions.

The wave equations in two dimensions have the following forms.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad \ldots\ldots 4.19$$

4.13 Discretization of 2D wave equation

The methods employed for the solution of one dimensional wave equation can be readily extended to the solution of (4.19). Consider a square region $0 \le x \le y \le \alpha$ and assume that u is known at all points within and on the boundary of this square.

If h be the step-size, then a mesh point (x, y, t) = (ih, jk, nℓ) may be denoted as simply i, j, n. Replacing the derivatives in equation (4.19) by their finite difference approximations, we get:

$$\frac{u_{i,j,n+1}-2u_{i,j,n}+u_{i,j,n-1}}{\ell^2} = c^2\left(\frac{u_{i-1,j,n}-2u_{i,j,n}+u_{i+1,j,n}}{h^2} + \frac{u_{i,j-1,n}-2u_{i,j,n}+u_{i,j+1,n}}{k^2}\right) =$$

$$\frac{c^2}{h^2}\{(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}) + (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})\}$$

i.e $u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} + \frac{\ell^2 c^2}{h^2}\{(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}) + (u_{i,j-1,n} - 2u_{i,j,n} + u_{i,j+1,n})\}$

$$\Rightarrow u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} + c^2\alpha^2(u_{i-1,j,n} - 2u_{i,j,n} + u_{i+1,j,n}) + c^2\alpha^2 u_{i,j-1,n} - 2c^2\alpha^2 u_{i,j,n} + c^2\alpha^2 u_{i,j+1,n}$$

$$\Rightarrow u_{i,j,n+1} = 2u_{i,j,n} - 4c^2\alpha^2 u_{i,j,n} - u_{i,j,n-1} + c^2\alpha^2 u_{i-1,j,n} + c^2\alpha^2 u_{i+1,j,n} + c^2\alpha^2 u_{i,j-1,n} + c^2\alpha^2 u_{i,j+1,n}$$

$\Rightarrow u_{i,j,n+1} = 2 \quad (1 - 2c^2\alpha^2)u_{i,j,n} + c^2\alpha^2\left(u_{i-1,j,n} + u_{i+1,j,n} + u_{i,j-1,n} + u_{i,j+1,n}\right) - u_{i,j,n-1}$ ..4.34

Where $\alpha = \ell/_{h}$. A sufficient condition for stability is: $\alpha = \frac{4c^2\Delta t^2}{\Delta x^2 + \Delta y^2} \leq 1$.

The objective of the MATLAB routine "wave2 ( )" is to implement the two-dimensional wave equation.

Example: A hyperbolic PDE: Two-Dimensional Wave (Vibration) over a Square Membrane. Consider a two-dimensional hyperbolic PDE $\frac{\partial^2 u(x,\ y,\ t)}{\partial t^2} = 2(\frac{\partial^2 u(x,\ y,\ t)}{\partial x^2} + \frac{\partial^2 u(x,\ y,\ t)}{\partial y^2})$

For $0 < x < \pi, 0 < y < \pi\ and\ 0 < t < \pi$, then we have the zero boundary conditions (BCs) and initial conditions (ICs) as follows:-

Boundary Condition (BC) u (0, y, t) = u (π, y, t) = 0, u (x, 0, t) = u (x, π, t) = 0 and

Initial Condition (IC) u (x, y, 0) = sin (x) sin (y), $\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial t}}(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0})$= 0, for t = 0.

Exact problem is given by: u(x, y, t) = sin(x) sin(y) cos(2t).

Note that we can be sure of stability, since we have $r = \frac{4c^2\Delta t^2}{\Delta x^2 + \Delta y^2} \leq 1 = \frac{4(2)(\pi/_{20})^2}{(\pi/_5)^2 + (\pi/_5)^2} = \frac{1}{4} \leq 1$

$c^2 = 2$, h = $\pi/_5$ = 36, k = $\pi/_5$ = 36, ℓ = $\pi/_{20}$ = 9

$r_x = \frac{c^2\Delta t^2}{\Delta x^2} = \frac{2x(\pi/_{20})^2}{(\pi/_5)^2} = 0.125$ and $r_y = \frac{c^2\Delta t^2}{\Delta y^2} = \frac{2x(\pi/_{20})^2}{(\pi/_5)^2} = 0.125$

$u_{i,\ j,\ k+1} = r_x\ (u_{i,\ j+1,\ k} + u_{i,\ j-1,\ k}) + r_y\ (u_{i+1,\ j,\ k} + u_{i-1,\ j,\ k})$ +2 $(1 - r_x - r_y)\, u_{i,\ j,\ k} -$..(i)

$u_{i,\ j,\ 0} = \sin(ih)\sin(jk)$, i =1, 2, 3, 4, 5 ....(ii)

And $\frac{\partial u}{\partial t}$ (x, y, 0) = 0 by central difference $\frac{u_{i,\ j,\ k+1} - u_{i,\ j,\ k-1}}{2\Delta t} = 0$ for k = 0 becomes:

$u_{i,\ j,\ 1} - u_{i,\ j,\ -1}$ = 0 so that $u_{i,\ j,\ 1} = u_{i,\ j,\ -1}$...(iii)

From the initial condition u (x, y, 0) = sin (x) sin (y) for **j = 0,   i = 0, 1, 2, 3, 4, 5** the $u_{i,\ j,\ 0}$ = sin (ih) sin (jk) where size step h = 36, k = 36 is iterated as follows:

$u_{1,\ 1,\ 0}$ = sin (36) sin (36) ≈ 0.3455, $u_{2,\ 1,\ 0}$ = sin (72) sin (36) ≈ 0.5590

$u_{3,\ 1,\ 0}$ = sin (108) sin (36) ≈ 0.5590, $u_{4,\ 1,\ 0}$ = sin (144) sin (36) ≈ 0.3455

$u_{5,\ 1,\ 0}$ = sin (180) sin (36) = 0.0000, **For j = 2,** $u_{1,\ 2,\ 0}$ = sin (36) sin (72) ≈ 0.5590,
$u_{2,\ 2,\ 0}$ = sin (72) sin (72) ≈ 0.9045, $u_{3,\ 2,\ 0}$ = sin (108) sin (72) ≈ 0.9045,

$u_{4,\ 2,\ 0}$ = sin (144) sin (72) ≈ 0.5590, $u_{5,\ 2,\ 0}$ = sin (180) sin (72) = 0.0000

For j = 3, $u_{1,\ 3,\ 0}$ = sin (36) sin (108) ≈ 0.5590, $u_{2,\ 3,\ 0}$ = sin (72) sin (108) ≈ 0.9045

$u_{3,\ 3,\ 0}$ = sin (108) sin (108) ≈ 0.9045, $u_{4,\ 3,\ 0}$ = sin (144) sin (108) ≈ 0.5590

$u_{5,\ 3,\ 0}$ = sin (180) sin (108) = 0.0000, **For j = 4,** $u_{1,\ 4,\ 0}$ = sin (36) sin (144) ≈ 0.3455,
$u_{2,\ 4,\ 0}$ = sin (72) sin (144) ≈ 0.5590, $u_{3,\ 4,\ 0}$ = sin (108) sin (144) ≈ 0.5590,
$u_{4,\ 4,\ 0}$ = sin (144) sin (144) ≈ 0.3455, $u_{5,\ 4,\ 0}$ = sin (180) sin (144) = 0.0000

For j = 5, $u_{1,\ 5,\ 0}$ = sin (36) sin (180) = 0.0000, $u_{2,\ 5,\ 0}$ = sin (72) sin (180) = 0.0000

$u_{3,\ 5,\ 0}$ = sin (108) sin (180) = 0.0000, $u_{4,\ 5,\ 0}$ = sin (144) sin (180) = 0.0000

$u_{5,\ 5,\ 0}$ = sin (180) sin (180) = 0.0000

Solve the approximation value for 2D wave equation by substituting r_x = 0.125 and r_y = 0.125 in Eq (i) becomes: $u_{i,\ j,\ k+1} = 0.125\ (u_{i,\ j+1,\ k} + u_{i,\ j-1,\ k}) + 0.125\ (u_{i+1,\ j,\ k} + u_{i-1,\ j,\ k}) + 2\ (1 - 0.125)\ u_{i,\ j,\ k} - u_{i,\ j,\ k}$ and from (iii) we obtain the following formula for k = 0

$u_{i,\ j,\ 1} = 0.0625\ (u_{i,\ j+1,\ 0} + u_{i,\ j-1,\ 0}) + 0.0625\ (u_{i+1,\ j,\ 0} + u_{i-1,\ j,\ 0})$
$+0.75\ u_{i,\ j,\ 0}$, For **i = j = 1** we have obtained the following iteration by varying the value of i and j,

$u_{1,\ 1,\ 1} = 0.0625\ (u_{1,\ 2,\ 0} + u_{1,\ 0,\ 0}) + 0.0625\ (u_{2,\ 1,\ 0} + u_{0,\ 1,\ 0}) + 0.75\ u_{1,\ 1,\ 0} \approx$ 0.3289, Exact value: $u_{1,\ 1,\ 1}$ = sin (36) sin (36)cos $2(9)$ = 0.3286 and Error = 0.0003

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 1,\ 1} \approx 0.5323$, Exact value: $u_{2,\ 1,\ 1} = 0.5317$ and Error = 0.0006, $u_{3,\ 1,\ 1} \approx 0.5323$, Exact value: $u_{3,\ 1,\ 1} = 0.5317$ and Error = 0.0006, $u_{4,\ 1,\ 1} \approx 0.3289$

Exact value: $u_{4,\ 1,\ 1} = 0.3286$ and Error = 0.00000, $u_{5,\ 1,\ 1} \approx 0.0216$

Exact value: $u_{5,\ 1,\ 1} = 0.0000$ and Error = 0.0216 and **for j = 2**

$u_{1,\ 2,\ 1} \approx 0.5384$**,** Exact value: $u_{1,\ 2,\ 1} = 0.5317$ and Error = 0.0067**,** $u_{2,\ 2,\ 1} \approx 0.8614$

Exact value: $u_{2,\ 2,\ 1} = 0.8602$ and Error = 0.0012, $u_{3,\ 2,\ 1} \approx 0.8614$

Exact value: $u_{3,\ 2,\ 1} = 0.8602$ and Error = 0.0012, $u_{4,\ 2,\ 1} \approx 0.5323$

Exact value: $u_{4,\ 2,\ 1} = 0.5317$ and Error = 0.0067, $u_{5,\ 2,\ 1} \approx 0.0349$

Exact value: $u_{5,\ 2,\ 1} = 0.0000$ and Error = 0.0349 and **for j = 3** $u_{1,\ 3,\ 1} \approx 0.5323$

Exact value: $u_{1,\ 3,\ 1} = 0.5317$ and Error = 0.0067, $u_{2,\ 3,\ 1} \approx 0.8614$

Exact value: $u_{2,\ 3,\ 1} = 0.8602$ and Error = 0.0012, $u_{3,\ 3,\ 1} \approx 0.8614$

Exact value: $u_{3,\ 3,\ 1} = 0.8602$ and Error = 0.0012, $u_{4,\ 3,\ 1} \approx 0.5384$

Exact value: $u_{4,\ 3,\ 1} = 0.5317$ and Error = 0.0067

$u_{5,\ 3,\ 1} \approx 0.0349$, Exact value: $u_{5,\ 3,\ 1} = 0.0000$ and Error = 0.0349, and **for j = 4**, $u_{1,\ 4,\ 1} \approx 0.3289$, Exact value: $u_{1,\ 4,\ 1} = 0.3286$ and Error = 0.0003

$u_{2,\ 4,\ 1} \approx 0.5323$, Exact value: $u_{2,\ 4,\ 1} = 0.5317$ and Error = 0.0006

$u_{3,\ 4,\ 1} \approx 0.5384$, Exact value: $u_{3,\ 4,\ 1} = 0.5317$ and Error = 0.0067

$u_{4,\ 4,\ 1} \approx 0.3289$, Exact value: $u_{4,\ 4,\ 1} = 0.3286$ and Error = 0.0003

$u_{5,\ 4,\ 1} \approx 0.0216$, Exact value: $u_{5,\ 4,\ 1} = 0.0000$ and Error = 0.0216, and **for j = 5**

$u_{1,\ 5,\ 1} \approx 0.0216$, Exact value: $u_{1,\ 5,\ 1} = 0.0000$ and Error = 0.0216

$u_{2,\ 5,\ 1} \approx 0.0349$, Exact value: $u_{2,\ 5,\ 1} = 0.0000$ and Error = 0.0216

$u_{3,\ 5,\ 1} \approx 0.0349$, Exact value: $u_{3,\ 5,\ 1} = 0.0000$ and Error = 0.0349

$u_{4,\ 5,\ 1} \approx 0.0216$, Exact value: $u_{4,\ 5,\ 1} = 0.0000$ and Error = 0.0216

$u_{5,\ 5,\ 1}$ = 0.0000, Exact value: $u_{5,\ 5,\ 1}$ = 0.0000 and Error = 0.0000

For k = 1 and for i = j = 1

$u_{1,\ 1,\ 2} \approx$ 0.3137, Exact value: $u_{1,\ 1,\ 2}$ = 0.2795 and Error = 0.0342

$u_{2,\ 1,\ 2} \approx$ 0.5068, Exact value: $u_{2,\ 1,\ 2}$ = 0.4523 and Error = 0.0545

$u_{3,\ 1,\ 2} \approx$ 0.5068, Exact value: $u_{3,\ 1,\ 2}$ = 0.4523 and Error = 0.0545

$u_{4,\ 1,\ 2} \approx$ 0.3146, Exact value: $u_{4,\ 1,\ 2}$ = 0.2795 and Error = 0.0351

$u_{5,\ 1,\ 2} \approx$ 0.0390, Exact value: $u_{5,\ 1,\ 2}$ = 0.0000 and Error = 0.0390, and **for j = 2**

$u_{1,\ 2,\ 2} \approx$ 0.5114, Exact value: $u_{1,\ 2,\ 2}$ = 0.4523 and Error = 0.0591

$u_{2,\ 2,\ 2} \approx$ 0.5114, Exact value: $u_{2,\ 2,\ 2}$ = 0.7318 and Error = 0.0884

$u_{3,\ 2,\ 2} \approx$ 0.8203, Exact value: $u_{3,\ 2,\ 2}$ = 0.7318 and Error = 0.0885

$u_{4,\ 2,\ 2} \approx$ 0.5094, Exact value: $u_{4,\ 2,\ 2}$ = 0.4523 and Error = 0.0884

$u_{5,\ 2,\ 2} \approx$ 0.0630, Exact value: $u_{5,\ 2,\ 2}$ = 0.0000 and Error = 0.0630 and **for j = 3**

$u_{1,\ 3,\ 2} \approx$ 0.5072, Exact value: $u_{1,\ 3,\ 2}$ = 0.4523 and Error = 0.0549

$u_{2,\ 3,\ 2} \approx$ 0.8203, Exact value: $u_{2,\ 3,\ 2}$ = 0.7318 and Error = 0.0885

$u_{3,\ 3,\ 2} \approx$ 0.8211, Exact value: $u_{3,\ 3,\ 2}$ = 0.7318 and Error = 0.0893

$u_{4,\ 3,\ 2} \approx$ 0.5136, Exact value: $u_{4,\ 3,\ 2}$ = 0.4523 and Error = 0.0613

$u_{5,\ 3,\ 2} \approx$ 0.0634, Exact value: $u_{5,\ 3,\ 2}$ = 0.0000 and Error = 0.0634 and **for j = 4**

$u_{1,\ 4,\ 2} \approx$ 0.3146, Exact value: $u_{1,\ 4,\ 2}$ = 0.2795 and Error = 0.0351

$u_{2,\ 4,\ 2} \approx$ 0.5094, Exact value: $u_{2,\ 4,\ 2}$ = 0.4523 and Error = 0.0571

$u_{3,\ 4,\ 2} \approx$ 0.5136, Exact value: $u_{3,\ 4,\ 2}$ = 0.4523 and Error = 0.0613

$u_{4,\ 4,\ 2} \approx$ 0.3167, Exact value: $u_{4,\ 4,\ 2}$ = 0.2795 and Error = 0.0372

$u_{5,\ 4,\ 2} \approx$ 0.0390, Exact value: $u_{5,\ 4,\ 2}$ = 0.0000 and Error = 0.0390 and f**or j = 5**

$u_{1,\ 5\ ,\ 2} \approx 0.0390$, Exact value: $u_{1,\ 5\ ,\ 2}$ = 0.0000 and Error = 0.0390

$u_{2,\ 5\ ,\ 2} \approx 0.0630$, Exact value: $u_{2,\ 5\ ,\ 2}$ = 0.0000 and Error = 0.0630

$u_{3,\ 5\ ,\ 2} \approx 0.0634$, Exact value: $u_{3,\ 5\ ,\ 2}$ = 0.0000 and Error = 0.0634

$u_{4,\ 5\ ,\ 2} \approx 0.0390$, Exact value: $u_{4,\ 5\ ,\ 2}$ = 0.0000 and Error = 0.0390

$u_{5,\ 5\ ,\ 2} \approx 0.007$, Exact value: $u_{5,\ 5\ ,\ 2}$ = 0.0000 and Error = 0.007

For n = 2 and for i = j =1, $u_{1,\ 1\ ,\ 3}$ = 0.0625 ($u_{1,\ 2,\ 2} + u_{1,\ 0,\ 2}$) + 0.0625 ($u_{2,\ 1,\ 2} + u_{0,\ 1,\ 2}$) $+0.75\, u_{1,\ 1,\ 2} \approx 0.2990$**,** Exact value: $u_{1,\ 1\ ,\ 3}$ = 0.2031 and Error = 0.0959

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 1\ ,\ 3} \approx 0.4827$, Exact value: $u_{1,\ 1\ ,\ 3}$ = 0.3286 and Error = 0.1541

$u_{3,\ 1\ ,\ 3} \approx 0.4827$, Exact value: $u_{3,\ 1\ ,\ 3}$ = 0.3286 and Error = 0.1541

$u_{4,\ 1\ ,\ 3} \approx 0.3019$, Exact value: $u_{4,\ 1\ ,\ 3}$ = 0.2031 and Error = 0.0988

$u_{5,\ 1\ ,\ 3} \approx 0.0260$, Exact value: $u_{5,\ 1\ ,\ 3}$ = 0.0000 and Error = 0.0260 and **for j = 2**

$u_{1,\ 2\ ,\ 3}$ = 0.0625 ($u_{1,\ 3,\ 2} + u_{1,\ 1,\ 2}$) + 0.0625 ($u_{2,\ 2,\ 2} + u_{0,\ 2,\ 2}$) $+0.75\, u_{1,\ 2,\ 2}) \approx$ 0.4862, Exact value: $u_{1,\ 2\ ,\ 3}$ = sin (36) sin (72)cos $2(27)$ = 0.3286 and Error = 0.1576

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 2\ ,\ 3} \approx 0.7813$, Exact value: $u_{2,\ 2\ ,\ 3}$ = 0.5317 and Error = 0.2496

$u_{3,\ 2\ ,\ 3} \approx 0.7813$, Exact value: $u_{3,\ 2\ ,\ 3}$ = 0.5317 and Error = 0.2496

$u_{4,\ 2\ ,\ 3} \approx 0.4891$, Exact value: $u_{4,\ 2\ ,\ 3}$ = 0.3286 and Error = 0.1605

$u_{5,\ 2\ ,\ 3} \approx 0.0855$, Exact value: $u_{5,\ 2\ ,\ 3}$ = 0.0000 and Error = 0.**0855** and **for j = 3**

$u_{1,\ 3\ ,\ 3} \approx 0.4833$, Exact value: $u_{1,\ 3\ ,\ 3}$ = 0.3286 and Error = 0.1547

$u_{2,\ 3\ ,\ 3} \approx 0.7813$, Exact value: $u_{2,\ 3\ ,\ 3}$ = 0.5317 and Error = 0.2496

$u_{3,\ 3\ ,\ 3} \approx 0.7826$, Exact value: $u_{3,\ 3\ ,\ 3}$ = 0.5317 and Error = 0.2509

$u_{4,\ 3\ ,\ 3} \approx 0.4921$, Exact value: $u_{4,\ 3\ ,\ 3}$ = 0.3286 and Error = 0.1635

$u_{5,\ 3,\ 3} \approx 0.0861$, Exact value: $u_{5,\ 3,\ 3}$ = 0.0000 and Error = 0.0861 and **for j = 4**

$u_{1,\ 4,\ 3} \approx 0.3019$, Exact value: $u_{1,\ 4,\ 3}$ = 0.2031 and Error = 0.0988

$u_{2,\ 4,\ 3} \approx 0.4891$, Exact value: $u_{2,\ 4,\ 3}$ = 0.3286 and Error = 0.1605

$u_{3,\ 4,\ 3} \approx 0.4921$, Exact value: $u_{3,\ 4,\ 3}$ = 0.3286 and Error = 0.1635

$u_{4,\ 4,\ 3} \approx 0.3065$, Exact value: $u_{4,\ 4,\ 3}$ = 0.2031 and Error = 0.1034

$u_{5,\ 4,\ 3} \approx 0.0535$, Exact value: $u_{5,\ 4,\ 3}$ = 0.0000 and Error = 0.0535 and **for j = 5**

$u_{1,\ 5,\ 3} = 0.0625\,(u_{1,\ 6,\ 2} + u_{1,\ 4,\ 2}) + 0.0625\,(u_{2,\ 5,\ 2} + u_{0,\ 5,\ 2}) + 0.75\,u_{1,\ 5,\ 2} \approx$ 0.0529, Exact value: $u_{1,\ 5,\ 3}$ = sin (36) sin (180)cos $2(27)$ = 0.0000 and Error = 0.0529

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 5,\ 3} \approx 0.0855$, Exact value: $u_{2,\ 5,\ 3}$ = 0.0000 and Error = 0.0855

$u_{3,\ 5,\ 3} \approx 0.0861$, Exact value: $u_{3,\ 5,\ 3}$ = 0.0000 and Error = 0.0861

$u_{4,\ 5,\ 3} \approx 0.0535$, Exact value: $u_{4,\ 5,\ 3}$ = 0.0000 and Error = 0.0535

$u_{5,\ 5,\ 3} \approx 0.0102$, Exact value: $u_{5,\ 5,\ 3}$ = 0.0000 and Error = 0.0102 and **for n = 3**

$u_{1,\ 1,\ 4} = 0.0625\,(u_{1,\ 2,\ 3} + u_{1,\ 0,\ 3}) + 0.0625\,(u_{2,\ 1,\ 3} + u_{0,\ 1,\ 3}) + 0.75\,u_{1,\ 1,\ 3} \approx$ 0.2849, Exact value: $u_{1,\ 1,\ 4}$ = sin (36) sin (36)cos $2(36)$ = 0.1068 and Error = 0.1781
Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 1,\ 4} \approx 0.4597$, Exact value: $u_{2,\ 1,\ 4}$ = 0.1728 and Error = 0.2869

$u_{3,\ 1,\ 4} \approx 0.4598$, Exact value: $u_{3,\ 1,\ 4}$ = 0.1728 and Error = 0.2870

$u_{4,\ 1,\ 4} \approx 0.2888$, Exact value: $u_{4,\ 1,\ 4}$ = 0.1068 and Error = 0.1820

$u_{5,\ 1,\ 4} \approx 0.0437$, Exact value: $u_{5,\ 1,\ 4}$ = 0.0000 and Error = 0.0437 and **for j = 2**

$u_{2,\ 2,\ 4} = 0.0625\,(u_{2,\ 3,\ 3} + u_{2,\ 1,\ 3}) + 0.0625\,(u_{3,\ 2,\ 3} + u_{1,\ 2,\ 3}) + 0.75\,u_{2,\ 2,\ 3} \approx$ 0.7442, Exact value: $u_{2,\ 2,\ 4}$ = sin (36) sin (72)cos $2(36)$ = 0.4647 and Error = 0.2795

Similarly, following this steps the left iteration is approximated as follows:

$u_{3,\ 2,\ 4} \approx 0.7445$, Exact value: $u_{3,\ 2,\ 4}$ = 0.2795 and Error = 0.4650

$u_{4,\ 2,\ 4} \approx 0.4453$, Exact value: $u_{4,\ 2,\ 4}$ = 0.1728 and Error = 0.2725

$u_{5,\ 2,\ 4} \approx 0.1017$, Exact value: $u_{5,\ 2,\ 4}$ = 0.0000 and Error = 0.1017 and **for j = 3**

$u_{1,\ 3,\ 4} = 0.0625\ (u_{1,\ 4,\ 3} + u_{1,\ 2,\ 3}) + 0.0625\ (u_{2,\ 3,\ 3} + u_{0,\ 3,\ 3}) + 0.75\, u_{1,\ 3,\ 3} \approx$ 0.4606, Exact value: $u_{1,\ 3,\ 4}$ = sin (36) sin (108)cos $2(36)$ = 0.1728 and Error = 0.2878

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 3,\ 4} \approx 0.7455$, Exact value: $u_{2,\ 3,\ 4}$ = 0.2795 and Error = 0.4660

$u_{3,\ 3,\ 4} \approx 0.7462$, Exact value: $u_{3,\ 3,\ 4}$ = 0.2795 and Error = 0.4667

$u_{4,\ 3,\ 4} \approx 0.4731$, Exact value: $u_{4,\ 3,\ 4}$ = 0.1728 and Error = 0.3003

$u_{5,\ 3,\ 4} \approx 0.1041$, Exact value: $u_{5,\ 3,\ 4}$ = 0.0000 and Error = 0.1041 and **for j = 4**

$u_{1,\ 4,\ 4} = 0.0625\ (u_{1,\ 5,\ 3} + u_{1,\ 3,\ 3}) + 0.0625\ (u_{2,\ 4,\ 3} + u_{0,\ 4,\ 3}) + 0.75\, u_{1,\ 4,\ 3} \approx$ 0.2905, Exact value: $u_{1,\ 4,\ 4}$ = sin (36) sin (144)cos $2(36)$ = 0.1068 and Error = 0.1837

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 4,\ 4} \approx 0.4905$, Exact value: $u_{2,\ 4,\ 4}$ = 0.1728 and Error = 0.3177

$u_{3,\ 4,\ 4} \approx 0.4731$, Exact value: $u_{3,\ 4,\ 4}$ = 0.1728 and Error = 0.3003

$u_{4,\ 4,\ 4} \approx 0.2981$, Exact value: $u_{4,\ 4,\ 4}$ = 0.1068 and Error = 0.1913

$u_{5,\ 4,\ 4} \approx 0.0653$, Exact value: $u_{5,\ 4,\ 4}$ = 0.0000 and Error = 0.0653

For j = 5, $u_{1,\ 5,\ 4} = 0.0625\ (u_{1,\ 6,\ 3} + u_{1,\ 4,\ 3}) + 0.0625\ (u_{2,\ 5,\ 3} + u_{0,\ 5,\ 3})$ $+0.75\, u_{1,\ 5,\ 3} \approx 0.0642$**,** Exact value: $u_{1,\ 5,\ 4}$ = sin (36) sin (180)cos $2(36)$ = 0.0000 and Error = 0.0642

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 5,\ 4} \approx 0.1034$, Exact value: $u_{2,\ 5,\ 4}$ = 0.0000 and Error = 0.1034

$u_{3,\ 5,\ 4} \approx 0.1041$, Exact value: $u_{3,\ 5,\ 4}$ = 0.0000 and Error = 0.1041

$u_{4,\ 5,\ 4} \approx 0.0653$, Exact value: $u_{4,\ 5,\ 4}$ = 0.0000 and Error = 0.0653

$u_{5,\ 5,\ 4} \approx 0.0142$, Exact value: $u_{5,\ 5,\ 4}$ = 0.0000 and Error = 0.0142

For n = 4, $u_{1,\ 1,\ 5} = 0.0625\ (u_{1,\ 2,\ 4} + u_{1,\ 0,\ 4})$ + 0.0625 $(u_{2,\ 1,\ 4} + u_{0,\ 1,\ 4})$ $+0.75\ u_{1,\ 1,\ 4} \approx 0.2713$**,** Exact value: $u_{1,\ 1,\ 5}$ = 0.0000 and Error = 0.2713

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 1,\ 5} \approx 0.4378$, Exact value: $u_{2,\ 1,\ 5}$ = 0.0000 and Error = 0.4378

$u_{3,\ 1,\ 5} \approx 0.4382$, Exact value: $u_{3,\ 1,\ 5}$ = 0.0000 and Error = 0.4382

$u_{4,\ 1,\ 5} \approx 0.2759$, Exact value: $u_{4,\ 1,\ 5}$ = 0.0000 and Error = 0.2759

$u_{5,\ 1,\ 5} \approx 0.0573$, Exact value: $u_{5,\ 1,\ 5}$ = 0.0000 and Error = 0.0573

For j = 2, $u_{1,\ 2,\ 5} = 0.0625\ (u_{1,\ 3,\ 4} + u_{1,\ 1,\ 4})$ + 0.0625 $(u_{2,\ 2,\ 4} + u_{0,\ 2,\ 4})$ $+0.75\ u_{1,\ 2,\ 4} \approx 0.4400$**,** Exact value: $u_{1,\ 2,\ 5}$ = sin (36) sin (72)cos $2(45)$ = 0.0000 and Error = 0.4400

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 2,\ 5} \approx 0.7089$, Exact value: $u_{2,\ 2,\ 5}$ = 0.0000 and Error = 0.7089

$u_{3,\ 2,\ 5} \approx 0.6904$, Exact value: $u_{3,\ 2,\ 5}$ = 0.0000 and Error = 0.6904

$u_{4,\ 2,\ 5} \approx 0.4345$, Exact value: $u_{4,\ 2,\ 5}$ = 0.0000 and Error = 0.4345

$u_{5,\ 2,\ 5} \approx 0.1133$, Exact value: $u_{5,\ 2,\ 5}$ = 0.0000 and Error = 0.1133

For j = 3, $u_{1,\ 3,\ 5} = 0.0625\ (u_{1,\ 4,\ 4} + u_{1,\ 2,\ 4})$ + 0.0625 $(u_{2,\ 3,\ 4} + u_{0,\ 3,\ 4})$ $+0.75\ u_{1,\ 3,\ 4} \approx 0.4392$**,** Exact value: $u_{1,\ 3,\ 5}$ = sin (36) sin (108)cos $2(45)$ = 0.0000 and Error = 0.4392

$u_{2,\ 3,\ 5} \approx 0.7117$, Exact value: $u_{2,\ 3,\ 5}$ = 0.0000 and Error = 0.7117

$u_{3,\ 3,\ 5} \approx 0.7120$, Exact value: $u_{3,\ 3,\ 5}$ = 0.0000 and Error = 0.7120

$u_{4,\ 3,\ 5} \approx 0.4544$, Exact value: $u_{4,\ 3,\ 5}$ = 0.0000 and Error = 0.4544

$u_{5,\ 3,\ 5} \approx 0.1181$, Exact value: $u_{5,\ 3,\ 5}$ = 0.0000 and Error = 0.1181

For j = 4 $u_{1,\ 4,\ 5} = 0.0625\,(u_{1,\ 5,\ 4} + u_{1,\ 3,\ 4}) + 0.0625\,(u_{2,\ 4,\ 4} + u_{0,\ 4,\ 4}) + 0.75\,u_{1,\ 4,\ 4} \approx 0.2814$**,**

Exact value: $u_{1,\ 4,\ 5} = \sin(36)\sin(144)\cos 2(45) = 0.0000$ and Error = 0.2814

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 4,\ 5} \approx 0.4687$, Exact value: $u_{2,\ 4,\ 5} = 0.0000$ and Error = 0.4687

$u_{3,\ 4,\ 5} \approx 0.4572$, Exact value: $u_{3,\ 4,\ 5} = 0.0000$ and Error = 0.4572

$u_{4,\ 4,\ 5} \approx 0.2878$, Exact value: $u_{4,\ 4,\ 5} = 0.0000$ and Error = 0.2878

$u_{5,\ 4,\ 5} \approx 0.0981$, Exact value: $u_{5,\ 4,\ 5} = 0.0000$ and Error = 0.0981

For j = 5, $u_{1,\ 5,\ 5} = 0.0625\,(u_{1,\ 6,\ 4} + u_{1,\ 4,\ 4}) + 0.0625\,(u_{2,\ 5,\ 4} + u_{0,\ 5,\ 4}) + 0.75\,u_{1,\ 5,\ 4} \approx 0.0729$

Exact value: $u_{1,\ 5,\ 5} = \sin(36)\sin(180)\cos 2(45) = 0.0000$ and Error = 0.0729

Similarly, following this steps the left iteration is approximated as follows:

$u_{2,\ 5,\ 5} \approx 0.1188$, Exact value: $u_{2,\ 5,\ 5} = 0.0000$ and Error = 0.1188

$u_{3,\ 5,\ 5} \approx 0.1182$, Exact value: $u_{3,\ 5,\ 5} = 0.0000$ and Error = 0.1182

$u_{4,\ 5,\ 5} \approx 0.0750$, Exact value: $u_{4,\ 5,\ 5} = 0.0000$ and Error = 0.0750

$u_{5,\ 5,\ 5} \approx 0.0189$, Exact value: $u_{5,\ 5,\ 5} = 0.0000$ and Error = 0.0189

Table 7 Entry, Iteration, Exact solution, Approximation Solution and Error for wave Equation in 2D.

Entry	Iteration	Exact	Approximate	Error	Entry	Iteration	Exact	Approximate	Error
1^{st}	$u_{1,\ 1,\ 1}$	0.3286	0.3289	0.0003		$u_{2,\ 1,\ 2}$	0.4523	0.5068	0.0545
	$u_{2,\ 1,\ 1}$	0.5317	0.5323	0.0006		$u_{3,\ 1,\ 2}$	0.4523	0.5068	0.0545
	$u_{3,\ 1,\ 1}$	0.5317	0.5323	0.0006		$u_{4,\ 1,\ 2}$	0.2795	0.3146	0.0351
	$u_{4,\ 1,\ 1}$	0.3286	0.3289	0.0003		$u_{5,\ 1,\ 2}$	0.0000	0.0390	0.0390
	$u_{5,\ 1,\ 1}$	0.0000	0.0216	0.0216	7^{th}	$u_{1,\ 2,\ 2}$	0.4523	0.5114	0.0591
2^{nd}	$u_{1,\ 2,\ 1}$	0.5317	0.5384	0.0067		$u_{2,\ 2,\ 2}$	0.7318	0.8202	0.0884
	$u_{2,\ 2,\ 1}$	0.8602	0.8614	0.00012		$u_{3,\ 2,\ 2}$	0.7318	0.8203	0.0885
	$u_{3,\ 2,\ 1}$	0.8602	0.8614	0.00012		$u_{4,\ 2,\ 2}$	0.4523	0.5094	0.0571
	$u_{4,\ 2,\ 1}$	0.5317	0.5323	0.0006		$u_{5,\ 2,\ 2}$	0.0000	0.0630	0.0630
	$u_{5,\ 2,\ 1}$	0.0000	0.0349	0.0349	8^{th}	$u_{1,\ 3,\ 2}$	0.4523	0.5072	0.0549
3^{rd}	$u_{1,\ 3,\ 1}$	0.5317	0.5323	0.0006		$u_{2,\ 3,\ 2}$	0.7318	0.8203	0.0885
	$u_{2,\ 3,\ 1}$	0.8602	0.8614	0.00012		$u_{3,\ 3,\ 2}$	0.7318	0.8211	0.0893
	$u_{3,\ 3,\ 1}$	0.8602	0.8614	0.00012		$u_{4,\ 3,\ 2}$	0.4523	0.5136	0.0613
	$u_{4,\ 3,\ 1}$	0.5317	0.5384	0.0067		$u_{5,\ 3,\ 2}$	0.0000	0.0634	0.0634
	$u_{5,\ 3,\ 1}$	0.0000	0.0349	0.0349	9^{th}	$u_{1,\ 4,\ 2}$	0.2795	0.3146	0.0351
4^{th}	$u_{1,\ 4,\ 1}$	0.3286	0.3289	0.0003		$u_{2,\ 4,\ 2}$	0.4523	0.5094	0.0571
	$u_{2,\ 4,\ 1}$	0.5317	0.5323	0.0006		$u_{3,\ 4,\ 2}$	0.4523	0.5136	0.0613
	$u_{3,\ 4,\ 1}$	0.5317	0.5384	0.0067		$u_{4,\ 4,\ 2}$	0.2795	0.3167	0.0372
	$u_{4,\ 4,\ 1}$	0.3286	0.3289	0.0003		$u_{5,\ 4,\ 2}$	0.0000	0.0390	0.0390
	$u_{5,\ 4,\ 1}$	0.0000	0.0216	0.0216	10^{th}	$u_{1,\ 5,\ 2}$	0.0000	0.0390	0.0390
5^{th}	$u_{1,\ 5,\ 1}$	0.0000	0.0216	0.0216		$u_{2,\ 5,\ 2}$	0.0000	0.0630	0.0630
	$u_{2,\ 5,\ 1}$	0.0000	0.0349	0.0349		$u_{3,\ 5,\ 2}$	0.0000	0.0634	0.0634
	$u_{3,\ 5,\ 1}$	0.0000	0.0349	0.0349		$u_{4,\ 5,\ 2}$	0.0000	0.0390	0.0390
	$u_{4,\ 5,\ 1}$	0.0000	0.0216	0.0216		$u_{5,\ 5,\ 2}$	0.0000	0.0000	0.0000
	$u_{5,\ 5,\ 1}$	0.0000	0.0000	0.0000	11^{th}	$u_{1,\ 1,\ 3}$	0.2031	0.2990	0.0959
6^{th}	$u_{1,\ 1,\ 2}$	0.2795	0.3137	0.0342		$u_{2,\ 1,\ 3}$	0.3286	0.4827	0.1541

Entry	Iteration	Exact	Approximate	Error	Entry	Iteration	Exact	Approximate	Error
	$u_{3,\ 1,\ 3}$	0.3286	0.4827	0.1541		$u_{4,\ 1,\ 4}$	0.1068	0.2888	0.1820
	$u_{4,\ 1,\ 3}$	0.2031	0.3019	0.0988		$u_{5,\ 1,\ 4}$	0.0000	0.0437	0.0437
	$u_{5,\ 1,\ 3}$	0.0000	0.0260	0.0260	17^{th}	$u_{1,\ 2,\ 4}$	0.1728	0.4625	0.2897
12^{th}	$u_{1,\ 2,\ 3}$	0.3286	0.4862	0.1576		$u_{2,\ 2,\ 4}$	0.2795	0.7442	0.4647
	$u_{2,\ 2,\ 3}$	0.5317	0.7813	0.2496		$u_{3,\ 2,\ 4}$	0.2795	0.7445	0.4650
	$u_{3,\ 2,\ 3}$	0.5317	0.7813	0.2496		$u_{4,\ 2,\ 4}$	0.1728	0.4453	0.2725
	$u_{4,\ 2,\ 3}$	0.3286	0.4891	0.1605		$u_{5,\ 2,\ 4}$	0.0000	0.1017	0.1017
	$u_{5,\ 2,\ 3}$	0.0000	0.0855	0.0855	18^{th}	$u_{1,\ 3,\ 4}$	0.1728	0.4606	0.2878
13^{th}	$u_{1,\ 3,\ 3}$	0.3286	0.4833	0.1547		$u_{2,\ 3,\ 4}$	0.2795	0.7455	0.4660
	$u_{2,\ 3,\ 3}$	0.5317	0.7813	0.2496		$u_{3,\ 3,\ 4}$	0.2795	0.7462	0.4667
	$u_{3,\ 3,\ 3}$	0.5317	0.7826	0.2509		$u_{4,\ 3,\ 4}$	0.1728	0.4731	0.3003
	$u_{4,\ 3,\ 3}$	0.3286	0.4921	0.1635		$u_{5,\ 3,\ 4}$	0.0000	0.1041	0.1041
	$u_{5,\ 3,\ 3}$	0.0000	0.0861	0.0861	19^{th}	$u_{1,\ 4,\ 4}$	0.1068	0.2905	0.1837
14^{th}	$u_{1,\ 4,\ 3}$	0.2031	0.3019	0.0988		$u_{2,\ 4,\ 4}$	0.1728	0.4905	0.3177
	$u_{2,\ 4,\ 3}$	0.3286	0.4891	0.1605		$u_{3,\ 4,\ 4}$	0.1728	0.4731	0.3003
	$u_{3,\ 4,\ 3}$	0.3286	0.4921	0.1635		$u_{4,\ 4,\ 4}$	0.1068	0.2981	0.1913
	$u_{4,\ 4,\ 3}$	0.2031	0.3065	0.1034		$u_{5,\ 4,\ 4}$	0.0000	0.0653	0.0653
	$u_{5,\ 4,\ 3}$	0.0000	0.0535	0.0535	20^{th}	$u_{1,\ 5,\ 4}$	0.0000	0.0642	0.0642
15^{th}	$u_{1,\ 5,\ 3}$	0.0000	0.0529	0.0529		$u_{2,\ 5,\ 4}$	0.0000	0.1034	0.1034
	$u_{2,\ 5,\ 3}$	0.0000	0.0855	0.0855		$u_{3,\ 5,\ 4}$	0.0000	0.1041	0.1041
	$u_{3,\ 5,\ 3}$	0.0000	0.0861	0.0861		$u_{4,\ 5,\ 4}$	0.0000	0.0653	0.0653
	$u_{4,\ 5,\ 3}$	0.0000	0.0535	0.0535		$u_{5,\ 5,\ 4}$	0.0000	0.0142	0.0142
	$u_{5,\ 5,\ 3}$	0.0000	0.0102	0.0102	21^{th}	$u_{1,\ 1,\ 5}$	0.0000	0.2713	0.2713
16^{th}	$u_{1,\ 1,\ 4}$	0.1068	0.2849	0.1781		$u_{2,\ 1,\ 5}$	0.0000	0.4378	0.4378
	$u_{2,\ 1,\ 4}$	0.1728	0.4597	0.2869		$u_{3,\ 1,\ 5}$	0.0000	0.4382	0.4382
	$u_{3,\ 1,\ 4}$	0.1728	0.4598	0.2870		$u_{4,\ 1,\ 5}$	0.0000	0.2759	0.2759

Table 8 Entry, Iteration, Exact solution, Approximation Solution and Error for wave Equation in 2D.

Entry	Iteration	Exact	Approximate	Error
	$u_{5,\ 1,\ 5}$	0.0000	0.0573	0.0573
22^{th}	$u_{1,\ 2,\ 5}$	0.0000	0.4400	0.4400
	$u_{2,\ 2,\ 5}$	0.0000	0.7089	0.7089
	$u_{3,\ 2,\ 5}$	0.0000	0.6904	0.6904
	$u_{4,\ 2,\ 5}$	0.0000	0.4345	0.4345
	$u_{5,\ 2,\ 5}$	0.0000	0.1133	0.1133
23^{th}	$u_{1,\ 3,\ 5}$	0.0000	0.4392	0.4392
	$u_{2,\ 3,\ 5}$	0.0000	0.7117	0.7117
	$u_{3,\ 3,\ 5}$	0.0000	0.7120	0.7120
	$u_{4,\ 3,\ 5}$	0.0000	0.4544	0.4544
	$u_{5,\ 3,\ 5}$	0.0000	0.1181	0.1181
24^{th}	$u_{1,\ 4,\ 5}$	0.0000	0.2814	0.2814
	$u_{2,\ 4,\ 5}$	0.0000	0.4687	0.4687
	$u_{3,\ 4,\ 5}$	0.0000	0.4572	0.4572
	$u_{4,\ 4,\ 5}$	0.0000	0.2878	0.2878
	$u_{5,\ 4,\ 5}$	0.0000	0.0981	0.0981
25^{th}	$u_{1,\ 5,\ 5}$	0.0000	0.0729	0.0729
	$u_{2,\ 5,\ 5}$	0.0000	0.1182	0.1188
	$u_{3,\ 5,\ 5}$	0.0000	0.1182	0.1182
	$u_{4,\ 5,\ 5}$	0.0000	0.0750	0.0750
	$u_{5,\ 5,\ 5}$	0.0000	0.0189	0.0189

Table 9 Entry, Iteration, Exact solution, Approximation Solution and Error for wave Equation in 2D.

CHAPTER FIVE

5.1 Conclusions

In this thesis, we have covered first order 1-D, second order 1-D, and second order 2-D wave equations. We saw explicit finite difference schemes for second order 1-D and 2-D wave equations. The 1-D and 2-D wave equation are well discretized using finite difference method. The grid points (mesh), initial condition (IC) and boundary condition are discussed as criteria in order to solve problem aroused on 1-D and 2-D wave equation. Some problems have examined by explicit finite difference method concerning one dimensional (1D) and two dimensional (2D) wave equations. We have discussed the effect of changing the grid points in the given examples. This implies that the error found/exist between the actual /exact solution and approximate solution is certainly changed as we change the grid points that is either the step size 'h' or the step size 'k' and we have also observed that the error increases or decreases as the step size become large or small respectively. The large error shows us the approximate solution/the numerical solution in the given examples is very far apart from exact solution. The result and the graph of actual solution and approximate solution were obtained using MATLAB. The result of the exact solution and the approximate solution are quite different as the step size 'h' and 'k' increases. However, we conclude that the graph of actual solution and approximate solution were almost similar in shape or in feature. In order to write a complete solution using MATLAB, several steps were needed for examples function declaration, global parameters, definition, stability condition setting, derived parameter declaration and graph plotting sections. As a conclusion, the 1-dimensional wave equation and 2-dimensional wave equations can be derived by using finite difference scheme. Changing the coordinate such as x, t, y and u on surf create different graphs that are the position and the thickness of graphs completely changed. Finally, we also conclude that the center finite difference approximation is used to derive 1-D and 2-D wave equations.

5.2 Recommendation

We recommended that discussions of higher order finite difference schemes for second order hyperbolic equations of 3D wave equation for future work. The researcher also recommended that to use other methods such as finite element method (FEM) and finite volume method (FVM) in order to overcome the problem aroused on non-linear hyperbolic partial differential equation (PDE) particularly either on 2D wave equation or 3D wave equation.

CHAPTER SIX

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MATLAB on 1D wave Equation

Appendices

Appendix A

Graph of Exact solution for 1D wave equation (Figure 1)

```
>> x=0:0.2:1;

>> t=0:0.05:1;

>> [x,t]=meshgrid(x,t);

>> u=sin(pi.*x).*cos(2.*pi.*x.*t);

>> Surf(x, t);

>> Title ('GRAPH OF EXACT SOLUTION FOR 1D WAVE EQUATION');

>> X-label ('x')

>> Y-label ('t')

>> Z-label ('z')

>> 'font size', 15,'fontweight','b'
```

Appendix B

Graph of exact solution for 1D wave equation (Figure 3)

```
>> x=0:0.1:1;

>> t=0:0.05:1;

>> [x, t]=meshgrid(x, t);

>> u=sin (pi.*x).*cos(2.*pi.*x.*t);

>> Surf(x, t);

>> Title ('GRAPH OF EXACT SOLUTION FOR 1D WAVE EQUATION');
```

>> X-label ('x')

>> Y-label ('t')

>> Z-label ('z')

>> 'font size', 15,'fontweight','b'

Appendix C

Table of exact solution for 1D wave equation using MATLAB

>> syms t x

>> u = inline (sin (pi.*x).*cos(2.*pi.*t));

>>u(0.05,0.2),u(0.05,0.4),u(0.05,0.6),u(0.05,0.8),u(0.05,1.0),u(0.1,0.2),u(0.1,0.4),u(0.1,0.6),u(0.1,0.8),u(0.1,1.0),u(0.15,0.2),u(0.15,0.4),u(0.15,0.6),u(0.15,0.8),u(0.15,1.0),u(0.2,0.2),u(0.2,0.4),u(0.2,0.6),u(0.2,0.8),u(0.2,1.0),u(0.25,0.2),u(0.25,0.4),u(0.25,0.6),u(0.25,0.8),u(0.25,1.0)

Appendix D

Table of exact solution for 2D wave equation using MATLAB

>> syms t x y

>> u = inline (cos (2.*t).* sin (x).*sin(y));

>>u(9,36,36),u(9,72,36),u(9,108,36),u(9,144, 36),u(9,180,36),u(9,36,72),u(9,72,72),…