STOCHASTIC ANALYSIS OF MICROSCOPIC CAR FOLLOW THE LEADER MODEL AND TRAFFIC MANAGEMENT

By

Tarekegn Dinku

A Thesis Submitted to

The Program of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

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Legesse Lemecha (PhD)

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As a thesis research advisor, I here by certify that I have read and evaluated this thesis prepared under my guidance by Tarekegn Dinku entitled **Stochastic Analysis of Microscopic Follow the Leader Model and Traffic Management**. I recommend that it will be submitted as fulfilling thesis the requirement.

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As members of the board of examiners of the M.Sc. thesis open defense examination, we certify that we have read and evaluated the thesis prepared by Tarekegn Dinku and examined the candidate. We recommend that the thesis be accepted as fulfilling the thesis requirements for the degree of Master of Science in Applied Mathematics.

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Abstract

We consider follow -the leader traffic model describing the dynamics of N cars on a single lane road, where each car driver chooses his acceleration according to a certain law.Traffic flow is a many-car system with complex and stochastic movement. It is difficult to describe the system dynamics solely using deterministic models which describe average system behavior. Therefore, a stochastic headway distribution is presented as further step forward to over come the well known a drawbacks of deterministic models. Modeling results will show that by taking car of second order statistics (that is mean and variance) a stochastic speed-density model is measure of the fluctuation of velocity and headway of the vehicle on single lane. Finally,we consider the fluctuation of velocity by calculating probability density from Fokker-Plank equation.

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Chapter 1

Introduction

Traffic flow problem has been extensively studied from physical point of view since 1950's. Car following is one of the sub-task involved in "driving task" which needs to be understood. It is defined as one vehicle following another on the single lane of road way or on a circular ring. In this dynamics, one aspect of interest is the average spacing, that one vehicle would follow another at given speed, which could estimate the capacity. In microscopic follow the leader model, we assume that the acceleration of each car depends on the difference between the car velocity and an optimal velocity, which models the velocity in an equilibrium situation for example as function of headway (distance of a car to the car in front). Due to high number of cars in real relevant traffic situation these models consist typically of big systems of nonlinear ordinary differential equations. For realistic optimal velocity functions it is easy to see that these models allow a quasi-stationary state, that is a solution where all cars have the same velocity and constant headway. A stability analysis shows that this solution is stable for certain parameter regimes. In other parameter regimes, such as high traffic density, this solution is unstable [6]. So far, the dynamics beyond the stable region of the quasi-stationary state has been an interesting and still widely open question. In general, based on the scale of the independent variables, level of detail, representation of the process, operationalization and scale of application; traffic flow models are classified into three main categories. The classifications are:

1 Microscopic Traffic Flow Model

2 Macroscopic Traffic Flow Model and

3 Mesoscopic Traffic Flow Models.

To briefly describe each of these model we have the following:

Macroscopic Traffic Flow Model

At the macroscopic level of traffic flow model, it is not necessary to look at the vehicles as separate entities but rather visualize the cars and the individual drivers as a coupled system. The macroscopic model is obtained from the conservation of mass and momentum equations by assuming that the collection of cars that flows on the road section is similar to the flow of blood in an artery or water in a home piping system [17].

Thus, it is a model which is expressed in terms of three gross or average variables for a whole lane of traffic. These are: flow rate which is defined as the number of vehicles passing a fixed point per unit time; traffic density that is defined as the number of vehicles in a traffic lane of a given length; and the distance covered per unit time which is identified as speed of traffic flow. Each of these variables are related to each other by the conservation law for cars like conservation of mass and momentum in physics[18].

Microscopic Traffic Flow Model

Microscopic modeling generally starts with and focuses on individual car movement. Most microscopic models are known as "Headway Models" because the individual car movement relates to the headway between the two cars. The other may be called "Interacting models", since for intersections or roundabouts, for example, individual car movements may be inter-dependent.

The microscopic traffic flow models describe the dynamics of individual vehicles and their interactions for the formulation of 'follow the leader traffic theory through which certain characteristics of a stream of interacting vehicles can be described by means of drivers-vehicle parameters [6] . It describes how an individual follower car responds to an individual leader car by modeling its acceleration as a function of various perceived stimuli, which might be the distance between the leader and the follower car. It is useful for the evaluation effects of new traffic control measures or for predicting the development of traffic flow in the possible situations.

Microscopic simulation models simulate traffic in a very detailed way by simulating the movement and the characteristics of each vehicle in contrast to a macroscopic model which is based on fluid dynamic equations and which simulates traffic flow in a highly aggregated way. Macroscopic Models result in less computational effort because it considers all vehicles as a system.

Mesoscopic traffic models

Mesoscopic traffic modeling is a consistent link between microscopic and macroscopic modelings. The link is reached by a transition from a microscopic to a macroscopic modeling. The usual transition with the theory of stochastic processes is very extensive [23]. Thus, a mesoscopic modeling is developed from the plain rule based microscopic modeling. Mesoscopic modeling bases on a distance density relation establishing a link between the distance of two sequenced vehicles in a microscopic modeling and the density in a macroscopic modeling. Apart from microscopic and macroscopic models it is also possible to use mixed dynamics, where individual vehicles are moved according to dynamic laws that are governed by macroscopic quantities . These models combine the computational efficiency of macroscopic models with the opportunity to derive properties that refer to individual vehicles, like emissions, probability distributions for accelerations, individual travel times and so on.

Stochastic traffic model

A stochastic process is a collection of random variables $\{X_t, t\epsilon T\}$, defined on some probability space (S,F,P). Where a three-tuple (S, F, P) whose components are sample space (S), event space (F) and probability function (P) is called a probability space . We call the values of X_t as state space denoted by Ω. The index set T from where t takes its value is called a parameter set or a time set. A stochastic process may be discrete or continuous according to whether the index set T is discrete or continuous. Traffic states that are commonly observed by researchers at the microscopic level are quite complex and at times even chaotic [22]. Some of the major traffic states that are commonly observed at the microscopic level are free flow, congested flow, synchronized traffic, and wide moving jams. Free flow is easily recognized by the ability of drivers to attain their desired speeds under very little, if any, interaction with other vehicles. Congested flow, in contrast is characterized by heavy vehicular interactions and usually very low flow.

Car-following (or follow-the-leader) models [26, 27, 28] appeared as away of obtaining equations which can be used in a wider context than their(empirical) predecessors. As the name denotes, car-following models describe traffic behavior based on the vehicle leading closely up front. In some of these model vehicle try to converge to their preferred (following) distance, relative to the vehicle in the front, thus creating an oscillatory behavior due to imperfect perception [2,9]. Road traffic flow is influenced by various random factors, including both external factors such as the weather, and internal factors such as transportation facilities, vehicle characteristics, driver behaviors, etc. These stochastic factors make the deterministic approaches difficult to accurately estimate or predict dynamic traffic evolutions. To overcome this problem, numerous stochastic approaches were developed for continuous traffic flow modeling [28]

1.1 Statement of the problem

Microscopic follow-the-leader car following model has been intensively studied deterministically as stated above. However, it yet studied stochastically. Hence, the leading questions are:

- How can stochastic fluctuation in the traffic flow be modeled microscopically?
- Which Variables have a strong influence on stochasticity in traffic dynamics?
- How can the distribution of the stochasticity variables in microscopic traffic flow be quantified?

1.2 The objective of the study

1.2.1 General objective

The general objective of this study is to analyze microscopic car follow the leader model stochastically and traffic management.

1.2.2 Specific objective

The specific objective of the proposed study are:

- Develop a microscopic mathematical model that describe the stochastic fluctuation of traffic flow.
- Investigating traffic variables that strongly influence traffic flow stochasticity.
- Analysis uncertainty and the fluctuation in traffic.
- study stability analysis and its application on traffic management.

1.3 Methodology

In this study we develop the mathematical model that describe microscopic flow of the vehicles on single lane. We will develop a system of non-linear differential equation in deterministic case and non-linear stochastic differential equation in stochastic case. After formulation, we can solve the system of non-linear differential equation analytically to understand the physical meaning of the governing equation. Further more, we study the stability analysis of the system by bifurcating parameters. And the stochastic differential equation is analyzing by calculating the expected value(mean) and the Variance of the fluctuating variables. Finally, analytic solution are supplemented by numerical simulation by choosing appropriate parameter values using MATLAB.

1.4 Significance of the model

The present study is highly important to understand traffic flow in the single lane of finite length or in the circular road way. Further, it gives useful information about interaction between cars and the fluctuation observed in traffic flow, even in macroscopic space which play fundamental role in traffic management. It also improve our understanding of microscopic car follow the leader and traffic management. Also, initiates other researchers to under take further extension and rigorous stochastic analysis on microscopic follow the leader model. used to clear characterization of the different regimes of traffic flow that can be made identified in the data using empirical results to enhance microscopic traffic flow model.

1.5 Expected out outcomes

The expected outcomes of the study are:

- stochastic model that governs the dynamics of traffic evolution.
- identify traffic variables that strongly influence traffic flow.

- reduce traffic congestion and improve traffic flow.

1.6 Mathematical preliminaries

1.6.1 Dynamical System

There are two main types of dynamical time system. The first is differential equation's which deal with continuous time systems, and other is difference equation which deal with discrete time. We will only focusing on differential equations because the problem that we are dealing is one in a continuous time space. The types of the differential equation , that we are dealing with is ordinary differential equation and The general structure of these is given as order differential system

$$x_1'(t) = f_1(x_1(t), x_2, ..., x_n(t))$$
$$\vdots$$
$$x_n'(t) = f_n(x_1(t), x_2, ..., x_n(t)).$$

where $x_i' = \frac{dx_i}{dt}$, $(x_1(t), x_2, ..., x_n(t)$ are time dependent variable at time t. $f_1, ..., f_n$ are functions that are determined by the problem at hand.[29]

1.6.2 Equilibrium points

In dynamical systems, only the solutions of linear systems may be found explicitly. The problem is that in general real life problems may only be modeled by nonlinear systems. The main idea is to approximate a nonlinear system by a linear one (around the equilibrium point). Of course, we do hope that the behavior of the solutions of the linear system will be the same as the nonlinear one. At point x^*, where $f(x^*) = 0$, there is no flow and these are called fixed points. A stable fixed point is denoted as black dot on the graph and the local flow is going to ward the fixed point. On other hand, if the local flow is moving away from the fixed point, this would be considered as unstable fixed point. An equilibrium is stable if all small disturbances away from it decrease over time and unstable if they grow in time.

An other way to determine the stability of fixed point in more qualitative measure is to linearize about fixed point. Let x^* a fixed point and $y = x(t) - x^*$ be small perturbation grows or decay. Here, we have

$$y' = \frac{d}{dt}(x - x^*) = x'.$$

Thus, $y' = x' = f(x) = f(x^* + y)$. Recall Taylor's expansion for arbitrary function $f(x)$ about point a is:

$$f(x) \simeq f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n.$$

So performing a Taylor's expansion about $a = x^*$ and $y = x - x^*$ so $x = x^* + y$ we get:

$$y' = f(x^* + y) = f(x^*) + yf'(x^*) + O(y^2),$$

where $O(y^2)$ denotes quadratically small terms in y and x^* is a fixed point so $f(x^*) = 0$. This requires $y' = yf'(x^*) + O(y^2)$. If $f'(x) \neq 0$ then $O(y^2)$ is negligible and

$$y' \simeq yf'(x) + O(y^2).$$

This is linearization about x^* and nows that the perturbation about $y(t) = 0$ grows exponentially if $f'(x^*) > 0$ and decays if $f'(x^*) < 0$.

Theorem 1.6.1. *(Existence and Uniqueness Theorem): Assume D is an open subset of $R \times R^n$, $f : D \to R^n$ is a continuous. Then for each $t_o \in R$ and $x_o \in R^n$, the initial value problem $x' = f(t,x), x(t_o) = 0$ has solution x. If in addition, f is continuous first order partial derivatives with respect to $x_1, x_2, ..., x_n$ on R^n, then initial value problem has unique solution.*

The following lemma is useful to prove the above theorem.[29]

Lemma 1.6.2. *Assume that f is continuous n-dimensional vector function on a rectangle*

$$R := \{(t,x) : t_0 \leq t \leq t_0 + a, \qquad ||x - x_o|| \leq b\}$$

and assume that $f(t,x)$ satisfies a uniform Lipschitz condition with respect to x on R. Let $M := max\{|f(t,x)| : (t,x) \in R\}$ and $\alpha := min\{a, \frac{b}{M}\}$. Then the IVP has unique solution x on $[t_o, t_o + \alpha]$.

Proof. (Theorem 3.1.1) (i) Existence: Given $f(t,x)$ as well as (t_o, x_o)it demarcate a neighborhood N around the central point and use to define the constants a', b', picard mapping p and h_o a point of contraction mapping A. The function $g : R \times R^n \to R_n$ given by $g(t,x) = x + h_o(t,x)$ is therefore always defined in neighborhood of (t_o, x_o). Applying the Picard mapping, $(Pg)(t,x) = x + (Ah_o)(t,x) = x + h_o(t,x) = g(t,x)$ which proves that g is solution of the differential equation which satisfies the initial condition $g(t_o, x_o = x)$ as long as t is in the neighborhood of the point to defined by $|t - t_o| leqa'$ and x is any point such that $|x - x_o| \leq b'$. (ii) Uniqueness: Let $(t_o, x_o) \in D$; there are positive numbers a and b such that the rectangle

$$R := \{(t,x) : |t - t_o| \leq a, ||x - x_o|| \leq b\} \subset D$$

By hypothesis of the theorem , $f(t,x)$ satisfies a Lipschitz condition with respect to x on R, with Lipschitz constant $k := max\{||D_x f(t,x)|| : (t,x) \in R\}$. Where $||.||$ is a matrix norm corresponding to the norm $||.||_1$. Let $M := max\{||f(t,x)|| : (t,x) \in R\}$ and $\alpha := min\{a, \frac{b}{M}\}$, then by Picard-Lindelof theorem the IVP has a unique solution on $[t_o - \alpha, t_o + \alpha]$.

□

Definition 1.6.1 (Equilibrium Point)**.** :Consider a nonlinear differential equation

$$x'(t) = f(x(t), u(t))$$

where f is a mapping from $R^n \times R^n \to R^n$. A point $\overline{x}$ is called an equilibrium point if there is a specific $\overline{u} \epsilon R^n$ such that $f(\overline{x}(t), \overline{u}(t)) = 0_n$

Definition 1.6.2. Let x_0 be an equilibrium point for $x' = f(x)$. A continuously differentiable function V defined on an open set $U \subset R^n$ with $x_0 \in U$ is called a Liapunov function for $x' = f(x)$ on U provided $V(x_0) = 0, V(x) > 0 for x \neq x_0, x \in U$, and

$$grad V(x).f(x) \leq 0, \tag{1.1}$$

for $x \in U$. If the inequality (1.1) is strict for $x \in U, x \neq x_0$, then V is called a strict Liapunov function for $x' = f(x)$ on U.

Note that (1.1) implies that if $x \in U$, then

$$\frac{d}{dt} V(\phi(t,x)) = grad V(\phi(t,x)).f(\phi(t,x)) \leq 0,$$

as long as $\phi(t,x)$ remains in U, so V is decreasing along orbits as long as they stay in U.

Here is Liapunov's stability theorem:

Theorem 1.6.3. *(Liapunov's Stability Theorem) If V is a Liapunov function for $x' = f(x)$ on an open set U containing an equilibrium point x_0, then x_0 is stable. If V is a strict Liapunov function, then x_0 is asymptotically stable .*

Proof. Assume V is a liapunov function for $x' = f(x)$ on an open set U containing an equilibrium point x_0. Pick $r > 0$ sufficiently small so that $B(x_0, r) \subset U$ and define

$$m \equiv min\{V(x) : |x - x_0| = r\} > 0$$

Now

$$W \equiv \{x : V(x) < \frac{m}{2}\} \cap B(x_0, r)$$

is open and contain x_0. Choose $x > 0$ so that $B(x_0), s \subset W$. For $x \in B(x_0), s$,

$$V(\phi(t, x)) < \frac{m}{2},$$

as long as $\phi(t, x)$ remains in W since $V(\phi(t, x))$ is decreasing. Thus $\phi(t, x)$ cannot intersect the boundary of $B(x_0, r)$ for $t \geq 0$, so $\phi(t, x)$remains in $B(x_0, r)$ for $t \geq 0$, and x_0 is stable. Now suppose V is a strict liapunov function, but x_0 is not asymptotically stable. Then there is an $x \in B(x_0, s)$ so that $\phi(t, x)$ does not go to x_0 as $t \to \infty$. Since the orbit is bounded, there is an $x_1 \neq x_0$ and a sequence $t_k \to \infty$ so that $\phi(t_k, x) \to x_1$ as $k \to \infty$. Note that by semigroup property for orbits

$$\phi(t_k + 1, x) = \phi(1, \phi(t_k, x))$$

As $k \to \infty$,

$$V(\phi(t_k + 1, x)) = V(\phi(1, \phi(t_k, x))) \to V(\phi(1, x_1)) < V(x_1),$$

So there is an integer N for which

$$V(\phi(t_N + 1, x)) < V(x_1).$$

Choose k so that $t_k > t_N + 1$. Then

$$V(x_1) \leq V(\phi(t_k, x)) < V(\phi(t_N + 1, x)),$$

a contradiction. We conclude that x_0 is asymptotically stable. □

1.6.3 Bifurcation analysis

In dynamical systems,a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative" or topological change in its behavior. Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets changes. It has two types; **Local bifurcations**, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds;

Global bifurcations, which often occur when larger invariant sets of the system collide with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria.

When parameter λ crosses some point $\lambda = \lambda_0$, the phase portrait of the system for $\lambda > \lambda_0$ is topologically different from the phase portrait of the system for $\lambda < \lambda_0$. The point $\lambda < \lambda_0$ is called a bifurcation point at which the system undergoes a bifurcation.

Hopf Bifurcation consider a two dimensional system of the form

$$\frac{dx}{dt} = f(x, y, \lambda)$$

$$\frac{dy}{dt} = g(x, y, \lambda)$$

where λ is a parameter. The system has a fixed point (x^*, y^*), which may depend on λ Let the eigenvalues of the linearized system about this fixed point be given by

$\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\mu(\lambda) = \alpha(\lambda) - i\beta(\lambda)$

Hopf bifurcation of the fixed point of the two dimensional system occurs at some critical value of the parameter, $\lambda = \lambda_0$, if the following conditions are satisfied:

i. $f(x^*, y^*, \lambda_0) = 0$ and $g(x^*, y^*, \lambda_0) = 0$

ii. The Jacobian matrix $\begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}$ has a pair of purely imaginary eigenvalues $\pm\omega i$ at (x^*, y^*, λ_0)

iii $\frac{d\alpha(\lambda)}{d\lambda} \neq 0$ at $\lambda = \lambda_0$

1.6.4 Stochastic Differential Equation(SDE)

Stochasticity is some thing that was randomly determined.

Definition 1.6.3. The random variable X can be described well by its distribution function (F_x). This is real valued function defined as $F_x(s) = P(x \leq s)$ on $\Re$, where $(x \leq s)$ is the event of all experments ω satisfying $X(\omega) \leq s$. Stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more in puts over time. The random variation is usually based on fluctuation observed in historical data for selected period using standard time serious technique [30].

Let us consider an ordinary differential equation

$$\frac{dx}{dt} = f(t, x)$$

or $dx(t) = f(x, t)dt$ with initial condition $x(0) = x_o$. Then the solution of the given ordinary differential equation is

$$x(t) = x_o + \int_0^t f(s, x(s))ds \tag{1.2}$$

where $x(t) = x(t, x_o, t_o)$ is solution with initial conditions $x(t_o) = x_o$.

Example 1.6.1. Let

$$\frac{dx(t)}{dt} = a(t)x(t) \tag{1.3}$$

and $x(0) = x_o$. When we take ordinary differential equation (1.3) and $a(t)$ is not deterministic parameter but rather stochastic parameter , we get a SDE. And the stochastic parameter $a(t)$ is given as:[31]

$$a(t) = f(t) + h(t)\xi(t) \tag{1.4}$$

where $\xi(t)$ denotes white noise process. Thus, we obtain

$$\frac{dx(t)}{dt} = f(t)x(t) + h(t)\xi(t) \tag{1.5}$$

when we write (1.5) as differential form of the Brownian motion, we obtain

$$dx(t) = f(t)x(t)dt + h(t)dW(t) \tag{1.6}$$

.

In general case, SDE is given as:

$$dX(t, \omega) = f(t, X(t, \omega))dt + g(t, X(t, \omega))dW(t) \tag{1.7}$$

where ω denotes that $X = X(t, \omega)$ is random variable and posses the initial condition $X(0, \omega) = X_o$ with probability one. And taking integration:

$$X(t, \omega) = X_o + \int_0^t f(s.X(s, \omega))ds + \int_0^t g(s, X(s, \omega))dW(s, \omega). \tag{1.8}$$

For calculation of stochastic integral $\int_0^T g(t,\omega)dW(t,\omega)$, we assume that $g(t,\omega)$ is only changed at discrete time points $t_i(i = 1, 2, ..., N-1)$ where $0 = t_o < t_1 < t_2 < ... < t_{N-1} < t_N < T$. We define the integral

$$S = \int_o^T g(t,\omega)dW(t,\omega)$$

as Riemann sum

$$S_N = \sum_{i=1}^{N} g(t_{i-1},\omega)(W(t_i,\omega) - W(t_{i-1}))$$

with $N \to \infty$ the random variable S is called the Ito integral of stochastic process $g(t,\omega)$ with respect to the Brownian motion $W(t,\omega)$ on the interval$[0,T]$ if

$$\lim_{N\to\infty} E[(S - \sum_{i=1}^{N} g(t_{i-1}(W(t_i,\omega) - W(t_{i-1},\omega))] = 0$$

Example 1.6.2. The simplest possible example if $g(t) = c \quad \forall t$. This is still a stochastic process, but a simple one. Taking definition above, we get

$$\int_0^T cdW(t,\omega) = c \lim_{N\to\infty} \sum_{i=1}^{N} (W(t,\omega) - W(t_{i-1},\omega))$$

$$= c \lim_{N\to\infty} [(W(t,\omega) - W(t_o,\omega)) + (W(t_2,\omega) - W(t_1,\omega)) + ... + (W(t_N,\omega) - W(t_{N-1},N)]$$

$$= c(W(T,\omega) - W(0,\omega)).$$

Where $W(T,\omega)$ and $W(0.\omega)$ are standard Gaussian random variables with $W(0,\omega) = 0$, the last results becomes

$$\int_0^T cW(T,\omega) = cW(T,\omega)$$

1.6.5 Characteristics of random variable

When we are studying the random variable the following results are very essential for understanding and manipulating these variables [30].

Expectation

Expectation, which is mean value of the random variable. Denoted by$E(x)$ and defined as

$$E(x) = \begin{cases} \sum_{o}^{\infty} X_k p(x = X_k), \quad X \quad is \quad discrite \\ \int_{-\infty}^{\infty} x f(x) dx, \quad X \quad is \quad continuous \end{cases}.$$

where $P(x = x_k)$ and $f(x)$ denotes probability function and probability density function respectively.

Variance

Variance, which is measure of spread of the sample paths around the mean value. Denoted by $V(X)$ and defined by

$$Var(X) = E[(X - E(X))^2] = E(X^2) - E^2(X) = \sigma_x^2.$$

Where $\sqrt{Var(X)} = \sqrt{\sigma_x^2} = \sigma_x$ is called standard deviation.

Covariance

Covariance, which is a measure of dependence of random process. Denoted by $Cov(X, Y)$ and defined as

$$Cov(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$$

Chapter 2

Literature Review

The mathematical modeling of traffic flow has a long tradition. Various approaches can be found in literature. A very common class of traffic models is the macroscopic one, where the traffic flow is described in terms of density and velocity distributions. One of the earliest models of this kind is the classical Light-hill- Whit-ham model [17]. A second class of approaches consists of the kinetic Boltzmann-like models, where probability distribution functions of the traffic flow are considered [16]. In the eighties cellular automate models became popular and are still used [20]. Yet another idea was that of regarding the formation of a traffic jam as a clustering phenomenon linked to a Markov process. This stochastic approach leads to the study of the master-equation for a particular probability distribution and can be found in works such as [18]. A very important class of models are the so called microscopic models, where the dynamics of the single cars is described. The earliest microscopic models where proposed in the early fifties in [22]. The models considered in this thesis are special class of microscopic models, the so called follow-the-leader models, where the dynamics of every car depends mainly on its distance to the car in front (called headway) and on the relative velocity with respect to the car in front. A historical overview on follow-the leader models is given in [4].

In car following state, a vehicle follow its predecessor at speed that is generally less than a maximum desired speed. The situation is such that drivers are led to adapt their behavior as function of the neighboring vehicles. The two fundamental variables regarding a vehicle i and its predecessors $i+1$ quickly emerged the difference of position $x_{i+1} - x_i$ and the speed difference $v_{i+1} - v_i$.

One of major difficulties of the model is to meet the frontier condition both in low and high densities, it seems necessary to distinguish between the free cases(less densities) and the interactive cases(above a certain critical density thresholds). Reciprocally, a microscopic safety function of distances,time or speed can associated with its density-velocity relationship. In [3] author states that microscopic models describing the spacing of the constrained vehicles in the traffic flow. The author also states that the over all reaction time T consists of:

- Perception time: time needed by the driver to recognize that there is an obstacle.
- Decision time:the time needed to make decision to decelerate.
- Braking time:time needed to apply the brakes.

In [24] it was stated that stimulus-response car-following models

$$a_\alpha(t) = \lambda(v_{(\alpha-1)}(t) - v_\alpha(t))$$

where $v_\alpha(t)$ and $a_\alpha(t)$ respectively denote velocity and acceleration of vehicle α at time t. And λ denotes the drivers sensitivity. The response does not depend on the headway or the spacing between the vehicles and thus, the model is unrealistic and it can not describe the desired headway. Most existing car following models are deterministic and do not consider the uncertainty and fluctuation of human perception and behavior [23]. Therefore, safety-relevant aspects may differ from reality in simulations which could also be seen by the fact that car following models are designed to be accident free. This constraint reduces the application of these models for traffic safety research significantly. The earliest work focused on the principle that vehicle separation is governed by safety consideration by which distance or time headway between vehicle is the function of the relative vehicles speed. In [22] it was developed that a car following model that assumes that drivers control their speed to maintain desired spacing. The analytically criteria for different stability region were also derived for group of car following models that describe vehicle motion through simple ordinary differential equation. That is the vehicle acceleration is a function of spacing, speed, and speed difference. The stability region depend on parameters and vary across models.

Traffic flow has been considered as a stochastic process at least since Adams [16]. He formulated the idea of arrivals as a random (that is, Poisson) process and verified good agreement with theory and observations. Since then many more sophisticated models have been proposed. The properties of headway have been extensively studied, especially in the 1960s. Some of this earlier work is now drawn together and evaluated in the light of recent data and more powerful statistical methods. No unified framework has been developed for traffic flow theory. Accordingly, the vehicle headway studies have been concentrated on the statistical analysis of headway data. There have been, however, some notable exceptions. In [8] author states that a connection between the microscopic traffic flow theory and headway distribution.

2.1 Models with optimal velocity function

Different from the models raised earlier car following model that focus on acceleration, some models define vehicle motion through an optimal velocity function. The car following model proposed by Newell's [27] and the optimal velocity model proposed by [1] are typical ones. Newells model defines unique speed-spacing relation ship. The main idea is that after reaction time, T, drivers will reach the optimal velocity, $v(s)$, which depends on the current spacing and desired speed. This model eliminates the possibility of collision, but requirement that drivers reach optimal velocity in reaction time, T, requires high acceleration and deceleration rates which is un realistic. Additionally, since the reaction time is the magnitude of several seconds, this suffers the same problem with the early car-following models. The velocity of v after reaction time τ depends on present spacing is given us:

$$v(t+T) = v(s(t))$$

The spirit of the Newell's model was incorporated with later in the optimal velocity model [1] and ensuring modified optimal velocity model. The Optimal Velocity model also defines a optimal velocity function, but it replaces the reaction time with relaxation time τ.

$$a(t) = \frac{v(s)-v}{\tau}$$

Now the governing equation of the thesis is:

$$x_j''(t) = \frac{V(x_{j+1}-x_j)-x_j'}{\tau}$$

Which describe the acceleration of j^{th} car on a circular road. where v is the optimal velocity function and x_j' is j^{th} car velocity.

This model is first time formulated in [1]. Latter in [6] the author was studied the stability analysis and instability due to Hopf bifurcation. In these models they assumed that the acceleration of each car depends on the difference between the car velocity and an optimal velocity. The authors also, showed with mathematical rigor that the loss of stability of the quasi-stationary solution due to Hopf bifurcation. Moreover they analyzed the loss of stability of corresponding quasi-stationary solution in optimal velocity models with non-equal drivers. In one year later those researchers with T.Seidel extended the model in two directions:

- including non-constant reaction time
- and aggressiveness of the driver.

Here again the loss of stability of stationary solution is due to Hopf bifurcation is addressed. Depending on their work, in present study we will analyze the stochastic analysis of microscopic car follow the leader model .

Chapter 3

Model Formulation and Analysis

3.1 Follow The Leader Model

The car following model imitates real traffic on a one-lane roads with out cross roads. This model takes into account mainly the interaction of the neighboring cars governing their dynamics and ignore over takings. Despite this interaction being locally in nature, it gives rise the cooperative phenomena(their dependence) that manifests the formation dissolution and the joining of large clusters. The cooperative phenomena are rather intricate depending up on specific values of the control parameters.

3.1.1 Assumption of the model

The assumption of the model mainly includes:

- All Vehicles move in the single lane.
- The model assumes no over taking of the Vehicle.
- vehicles maintain the same or nearly the same velocity.
- Vehicles try to maintain the same headway.

The basic element of this approach is the description of car pair where the lead car moves at a given velocity (that is changing with time). The governing equation is written for the following car. With in standard approximation,the motion of the following car is specified by its coordinate $x_n(t)$ and its velocity $v_n(t)$. The equation for the acceleration including the deterministic force and Langevin source(that is stochastic force).

$$\begin{cases} x_n' = v_n \\ m v_n' = F_{det}(x_n, x_{n+1}, v_n, v_{n+1}) + F_{stoch}(x_n, v_n, t). \end{cases} \tag{3.1}$$

where m is mass of each car. The right hand side of the equation of acceleration contains total force which represents the sum of all forces acting up on n^{th} vehicle.

The driver collects information about motion mainly through visual perception. The information consists of car velocities, acceleration, vehicle spacing, and relative velocities. It should be mentioned that driver should be sensitive only some of these characteristics. According to the information obtained, driver should make decision about his driving strategy. In this manner, the equation of acceleration interpreted as response-stimulus relationship. In other words, the driver responds according to a given stimulus derived from the current driving characteristics.

Usually deterministic force is responsible for optimal safe motion. More over, there are two stimuli affecting the driver's behavior.

- One of them is the wish to move as fast as possible (that is with the speed $v_{n+1}(t)$) of leading vehicle.In this manner driver should control the speed following car.

- The other is the necessity to maintain the headway difference $\Delta x_n = x_{n+1} - x_n$ between the two successive cars with position x_n and x_{n+1} respectively.

In particular, the earliest follow the leader models[2,3] takes into account the former stimulus only and leave out headway x_n completely. In contrast to that, the optimal velocity models [5,6] directly relates the acceleration to the difference between current velocity V_n and a certain optimal velocity $V_{opt}(\Delta x_n)$. The dependence of optimal motion is headway can be written applying the rather generalizations:

- when $\Delta x_n \longmapsto 0$,$V_{opt}(\Delta x_n \longrightarrow 0) \longrightarrow 0$.
- on the empty road, $(\Delta x_n \longmapsto \infty) \longrightarrow V_{max}$

3.1.2 Bando's Optimal Velocity Model

The idea proposed [3,4] consists of the introduction of smooth and sigmoid function V_{opt} satisfying the conditions above. The example of such function extensively used by Manhke[20] is given by

$$V_{opt}(\Delta x_n) = V_{max} * \frac{(\Delta x_n)^2}{D^2+(\Delta x_n)^2}.$$

Optimal velocity relationship contains two parameters that is, V_{max} and interaction distance D.

In order to explain the structure of right hand side of equation let us, first consider the pure deterministic model where $F_{stoch}(x, v, t) = 0$.

Under this condition, the deterministic force $F_{det}(x, v, t)$ can be interpreted as the sum of the acceleration force and depending on velocity and deceleration force depending on headway.

$$F_{det}(v, \Delta x) = F_{acc}(v) + F_{dec}(\Delta x)$$

.

The acceleration force is linear function with domain $V\epsilon[0, V_{max}]$ when $V = V_{max}$ further acceleration is not possible and $F_{acc}(V = V_{max}) = 0$. That is why, the following expression has been proposed.

$$F_{acc}(v, \Delta x) = m\frac{V_{max}-V}{\tau_1} \geq 0$$

.

where the time τ_1 characterizes the driver's response.

Following similar arguments, the headway dependence of deceleration is written as:

$$F_{dec}(\Delta x) = m\frac{V_{opt}(\Delta x)-V_{max}}{\tau_2} \leq 0.$$

Since $v_{opt} \leq v_{max}$

For simplicity, the case of $\tau = \tau_1 = \tau_2$, will be discussed further on.

$$\begin{cases} x'_n = v_n \\ v'_n = \frac{V_{opt}(\Delta x_n)-v_n}{\tau}. \end{cases} \tag{3.2}$$

$n = 1, 2, ..., N$ and N is the total number of cars.

The sign of term $V_opt(\Delta x_n) - v_n$ determines whether the driver will accelerate when $V_{opt}(\Delta x_n) > v_n$ or decelerate when $V_{opt}(\Delta x_n) < v_n$.

To compute the description of car dynamics, we consider a circle road of length L or which is equivalent, the periodic boundary conditions will be used as $\sum_{i=1}^{N} \Delta x_n$=0.

Stability of the system

We will first take look at the steady flow of the dynamical system:

$$x'' = a(V(\Delta x) - x') \tag{3.3}$$

where $a = \frac{1}{\tau}$. When we rewrite system as

$$\begin{cases} x'_n = y_n \\ y'_n = a(V(\Delta x_n) - y_n) \end{cases}.$$

Lemma 3.1.1. *There is a unique solution x_n of (3.3) with constant velocity c of all cars.*

Proof. The existence and uniqueness theorem guarantees , since all function and its partial derivatives are continuous on R^n. In this model the velocity of each driver depends only on the headway to the car in front. A solution with constant velocity must also have constant headways. There fore, we transform as:

$$\begin{cases} \xi'_n = \eta_n \\ \eta'_n = V(\xi_{n+1}) - V(\xi_n) - \eta_n \end{cases}. \tag{3.4}$$

Here, $\xi_n = x_{n+1} - x_n$ is headway between drivers $n+1$ and n, and $\eta_n = \xi'_n$ is corresponding relative speed. Now looking for stationary solution

$$\begin{cases} 0 = \eta_n \\ 0 = V(\xi_{n+1}) - V(\xi_n) - \eta_n \end{cases} . \tag{3.5}$$

A stationary solution (ξ^0, η^0) is given as $\xi^0 = \frac{L}{N}$, $\eta^0 = 0$. For each $n = 1, 2, 3, ..., N$. Thus stationary corresponds

$$x_n(t) = V(\frac{L}{N})t + \frac{L}{N}n$$

The equilibrium solution of the model is :

$$x_n^0 = bn + ct, \tag{3.6}$$

where $b = \frac{L}{N}$ and represents the constant spacing between two successive vehicles, $c = V(b)$ represents the constant velocity of the steady state traffic flow. □

Remark 3.1.1. From the above lemma we observe that velocity is not a further independents of L and N.

Equation (3.3) is our system with out congestion and the vehicles are all distributed uniformly along the single lane of length L all driving at constant velocity c with constant spacing b between each vehicle.

Note that we have defined the steady state flow for the system, we will look at the original non-linear equation (3.3) and linearize about the steady state (3.6) to determine its stability. We will consider y_n small perturbation from the steady state flow such that

$$x_n = x_n^0 + y_n, |y_n| << 1. \tag{3.7}$$

We will use equation (3.7) and take its derivation with respect to time and substitute it into equation (3.3), we get the following:

$$x'_n = x'^0_n + y'_n = V(b) + y'_n \tag{3.8}$$

$$x''_n = x''^0_n + y''_n = y''_n. \tag{3.9}$$

Now let's reconsider equation (3.3)

$$x''_n = a(V(\Delta x_n) - x'_n) \tag{3.10}$$

Substituting equation (3.8) and (3.9) we get

$$y_n'' = a[V(\Delta x_n^0 + \Delta y_n) - (V(b) + y_n')] \tag{3.11}$$

we will do Taylor series expansion of $V(\Delta x_n^0 + \Delta y_n)$ about b because it is non linear term that we want to linearize:

$$V(\Delta x_n^0 + \Delta y_n) = V(b) + V'(b)(\Delta x_n^0 + \Delta y_n - b) + \frac{V''(b)}{2!}(\Delta x_n^0 + \Delta y_n - b)^2 + ... + Higher \quad Order \quad Terms. \tag{3.12}$$

Note that

$$\Delta x_n^0 = x_{n+1}^0 - x_n^0 = [b(n+1) + ct] - [bn + ct] = bn + b + ct - bn - ct = b$$

Also, we will only take the linear terms and neglect the rest. This gives:

$$V(\Delta x_n^0 + \Delta y_n) = V(b) + V'(b)\Delta y_n \tag{3.13}$$

Now we will combine equations (3.11)and (3.13):

$$y_n'' = a[V(b) + V'(b)\Delta y_n - y_n'] \tag{3.14}$$

Simplifying the above equation, we will arrive at the linearized system which matches the result:

$$y_n'' = a(f\Delta y_n - y_n') \tag{3.15}$$

$N = 1, 2, ..., N$ and $f = V'(b)$

Stability in linearized system

Now let's look at the system form a different approach the solution to the equation (3.3) we obtain setting :

$$y_k(n,t) = \exp(i\alpha_k n + zt) \tag{3.16}$$

where $\alpha_k = \frac{2\pi k}{N}$,$k = 0, 1, ..., N-1$ and $z = u + iv$ $(u, v) \in R$. Let's test the solution (3.16) with our equation (3.3). To do this we will differentiate equation (3.16) with respect time to twice.

$$y_k(n,t) = \exp(i\alpha_k n + zt), \tag{3.17}$$

$$y'_k(n,t) = z\exp(i\alpha_k n + zt), \tag{3.18}$$

$$y''_k(n,t) = z^2\exp(i\alpha_k n + zt), \tag{3.19}$$

Considering equation (3.3), $y''_n = a(f\Delta y_n - y'_n)$ and we will substitute in the above equations we just obtain:

$$z^2\exp(i\alpha_k + zt) = a[f(\exp(i\alpha k)(n+1)) - \exp(i\alpha_k n + zt)) - z\exp(i\alpha_k n + zt)]. \tag{3.20}$$

$$z^2 = a(f(\exp(i\alpha_k) - 1) - z. \tag{3.21}$$

$$z^2 = af(\exp(i\alpha_k) - 1) - az. \tag{3.22}$$

$$z^2 + az - af(\exp(i\alpha_k) - 1) = 0 \tag{3.23}$$

We see that our solution (3.16) to the linearized system must satisfy equation (3.23). Now let's reconsider equation (3.16)

$$y_k(n,t) = \exp(i\alpha_k n + zt) = \exp(i\alpha_k n + ut + ivt) = \exp(ut)\exp(i(\alpha_k n + vt)) \tag{3.24}$$

Notice that $Re(z) = u$ determines whether the perturbations, y_k grow or decay.

- if$u < 0$, then y_k decay and system will be stable.
- if $u > 0$, then y_k grows and system will become unstable.
- $u = 0$, then this is marginal case.

If we substitute $u = 0$ in (3.23) become

$$(iv)^2 + a(iv) - af(\exp(i\alpha_k) - 1) = 0 \tag{3.25}$$

$$-v^2 + aiv = af(\cos(\alpha_k) + i\sin(\alpha_k) - 1) \tag{3.26}$$

Collecting real and imaginary part separately give as:

$$-v^2 = af(\cos(\alpha_k) - 1) \tag{3.27}$$

$$av = af(\sin(\alpha_k)). \tag{3.28}$$

$$-af(\cos(\alpha_k)-1) = f^2\sin^2(\alpha_k)$$

This implies that

$$\frac{f}{a} = \frac{1}{1+\cos(\frac{2\pi k}{N})}$$

where $k = 0, 1, ..., N-1$ and $f = V'(b)$.

Lemma 3.1.2. *If $k = N$, then $(0,0)$ and $(-a,0)$ are solution to the system (3.23).*

Proof. Letting $z = u + iv$, in the equation (3.3), we find that

$$(u+iv)^2 + a(u+iv) = af(\exp(\frac{i*2\pi k}{N}) - 1).$$

This equivalent to when collecting real and imaginary separately

$$\begin{cases} u^2 + au = af(c_k - 1) \\ 2uv + av = afs_k \end{cases}$$

where $c_k = \cos(\frac{2\pi k}{N})$ and $s_k = \sin(\frac{2\pi k}{N})$. If $k = N$, then $c_K = 1$ and $s_k = 0$. This leads to :

$$\begin{cases} u^2 + au = 0 \\ 2uv + av = 0 \end{cases}$$

this results in $u = 0$ and $-a$, hence when $k = N$, $(0,0)$ and $(-a,0)$ are solution of the system for independent of f. □

Theorem 3.1.3. *:*

$$f = V'(\frac{L}{N}) = \frac{1}{1+\cos(\frac{2\pi}{N})}$$

holds then the system undergoes Hopf bifurcation.

Proof. : Suppose $V'(\frac{L_H}{N})$ is verified for some L_H. Where $z(L_H) = u(L_H) + iv(L_H)$ is solution to the characteristic equation (3.23), such that $u(L_H) = 0$. Now, it is sufficient to proof $u'(L_H) \neq 0$. But from lemma(3.2), differentiating equation in the lemma with respect to f, results $u'(L_H) \neq 0$. Hence, the theorem of Hopf comes. □

When the parameters are such that critical value $V'(\frac{L}{N}) = \frac{1}{1+\cos(\frac{2\pi}{N})}$ is reached, then a qualitative change in dynamics occurs. This is called bifurcation. In this context critical parameter correspond to a critical mean density on single lane of circular road. When the critical density is exceeded the simple solutions can not be observed any more. In our case so called Hopf bifurcation occurs. Hopf bifurcation generates periodic solutions for parameters close to the critical one, which show oscillation in headway and velocity such that the congestion travels more. So, this stability border . Where c is dimensionless car density. If we choose Bando optimal velocity $V = \frac{x^2}{1+x^2}$ with $V_{max} = 1$ and taking derivative with respect to x and valuating at mean density c we have $f = \frac{2c^3}{(1+c^2)^2}$ has maximum at critical concentration $C_{cr} = \sqrt{3}$

3.2 Stochastic Approach Microscopic Car Following Model

3.2.1 Acceleration with random term

We define desired acceleration as the driver imposes to the vehicle when traveling at a speed $v(t)$ at time t under free flow conditions(that is when the vehicle is unbiased by the leading vehicle). Let us consider linear optimal velocity model with random term.

$$a(t) = (v(c) - v)\beta$$

where $v(c)$ is target speed , $v(t)$ is velocity at time t and β is inverse relaxation time. Now we add to the above in form of white noise process with diffusion coefficient σ. Now, using change of variable that is letting: $\xi' = v$ and $v' = a(t)$ and again randomizing the acceleration with diffusive term σ, we change ordinary to stochastic differential equation. Here it follows that to obtain desired vehicle position $\xi(t)$, one has to solve the following:

$$\begin{cases} d\xi(t) = v(t)dt \\ dv(t) = (v_c - v(t))\beta dt + \sigma dW(t), v(0) = v_0 \end{cases} \tag{3.29}$$

where $W(t)$ is standard Brownian. Integrating the second stochastic differential equation we obtain:

$$v(t) = v_o \exp(-\beta t) + v_c(1 - \exp(-\beta t) + \sigma \int_0^t \exp(-\beta t) dw(s)$$

Which has normal distribution with

$$E[v(t)] = v_o \exp(-\beta t) + v_c(1 - \exp(-\beta t))$$

$$V[v(t)] = \frac{\sigma^2}{2\beta}(1 - \exp(-2\beta t))$$

while the distribution of position $\xi(t) = \int_0^t v(s)ds$ is also normal distribution with

$$E[\xi(t)] = v_c t - (1 - \exp(-\beta t))(\frac{v_c - v_o}{\beta})$$

$$V[\xi(t)] = \frac{\sigma^2}{2\beta^3}(\exp(-\beta t)(4 - \exp(-\beta t)).$$

Here we conclude that :

- the mean speed of the vehicle converges to desired speed at large time(taking limit to its expectation as time goes larger and larger) that is to v_c.
- the variance of vehicle is $\frac{\sigma^2}{2\beta}$ at time goes to infinity.
- the mean and variance of the Vehicle position varies with time.

Let us now consider the stochastic description of traffic flow, the multiplicative Gaussian white noise has been added to the dynamics of non-linear optimal velocity function case described by the set of equations:

$$\begin{cases} \frac{dx_n}{dT} = y_n \\ \frac{dy_n}{dT} = a * (V_{opt}(\Delta x_n) - y_n) \end{cases} \tag{3.30}$$

to become $\begin{cases} dx_n = y_n dT \\ dy_n = a * (V_{opt}(\Delta x_n) - y_n)d(T) + \sigma dW_i(T) \end{cases}$.

Here, new parameter σ is the dimensionless noise amplitude.For our case system above can written in vector form as:

$$dr(T) = A(r, T)dT + B(r, T)dW(T) \tag{3.31}$$

where $r(T) \in \Re^N$ and $A(r, T) \in \Re^{2N}$ dimensional vectors

$$r(T) = \begin{pmatrix} y_1(T) \\ \vdots \\ y_N(T) \\ u_1(T) \\ \vdots \\ u_N(T) \end{pmatrix}, \ A(r,T) = \begin{pmatrix} u_1(T) \\ \vdots \\ u_N(T) \\ (u_{opt}(\Delta y_1) - u_1) \\ (u_{opt}(\Delta y_N) - u_N) \end{pmatrix}$$

The diffusion matrix $B(r,T) \in \Re^{2N\times 2N}$ has block structure

$$B(r) = \begin{pmatrix} 0 & 0 \\ 0 & u(r) \end{pmatrix}$$

where each block is $N \times N$ matrix and $0 \in \Re^{N\times N}$ consists of zero element due to fact that stochasticity term accounts for acceleration only.

Hence, $u(r) = \begin{pmatrix} \sigma u_1(T) & 0 & \cdots & 0 \\ 0 & \sigma u_2(T) & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \sigma u_N(T) \end{pmatrix}$

The Multiplicative noise has been added to equation for the acceleration and has been considered in order to control velocity and to neglect its negative value. Solving such stochastic equation is difficult in sense of deterministic analysis. Now we use Fokker-Plank equation for solving such kind of multidimensional with diffusive term.

3.3 Fokker-plank Equation

This equation gives the time evolution of the probability density for system governed by the mult-dimensional stochastic differential equation. So, the vector description of stochastic differential equation takes form:

$$\frac{\partial p(r,t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial r_i} A_i(r)p(r,t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial r_i \partial r_j} B_{ij}(r)p(r,t) \tag{3.32}$$

with

$$P(r,t|r_o,t_o) = \delta(r - r_o)$$

and

$$\int P(r,t)dr = 1$$

This equation written in the form of continuity equation.

$$\frac{\partial P(r,t)}{\partial t} = -\sum \frac{J_i(r,t)}{\partial r_i} \tag{3.33}$$

where

$$J_i(r,t) = A_i(r)P(r,t) - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial}{\partial r_j} B_{ij}(r)P(r,t)$$

Hence,stationary solution P^{st} can be found from the identity:

$$\sum_{i=1}^{n} \frac{\partial J_i(r,t)}{\partial r_i} = 0$$

When $A(r,t) = 0$ and $\sigma = \sqrt{2D}$ for $n = 1$, we obtain one-dimensional diffusion equation:

$$\begin{cases} dx = \sqrt{2D}dW \\ x(t = t_o) = x_o \end{cases}$$

with diffusion constant D is Fokker-Plank equation takes simplest form:

$$\frac{\partial p(r,t)}{\partial t} = D\frac{\partial^2 p(r,t)}{\partial x^2}$$

with initial condition:

$$p(x,t|x_o,t_o) = \delta(x - x_o)$$

and normalization condition

$$\int_{-\infty}^{\infty} p(x,t)dx = 1$$

The Gauss or normal distribution is solution of the the problem

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{(x - x_o)^2}{4Dt}$$

Similarly, for n=2, taking

$$A(r,t) = \begin{pmatrix} v \\ -\gamma v \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 \\ \sqrt{2B} \end{pmatrix}$$

we have the following system,

$$\begin{cases} dx = vdt \\ dv = -\gamma vdt + \sqrt{2B}dW(t) \end{cases}$$

which is written in Fokker-Plank form as

$$\frac{\partial p(x,v,t)}{\partial t} = \frac{-\partial}{\partial x}(vp(x,v,t)) + \gamma\frac{\partial}{\partial v}(vp(x,v,t)) + B\frac{\partial^2}{\partial v^2}p(x,v,t)$$

with initial condition:

$$p(x,v,t) = \delta(x - x_o)\delta(v - v_o).$$

Fokker-Plank velocity distribution. the equation can be considered 1D Fokker-Plank equation for velocity distribution $p_v(v,t)$ after transformation is $p_v(v,t) = \int_{-\infty}^{\infty} p(x,v,t)dt$ and probability density function $p_v(v,t)$ is the solution of the problem:

$$\frac{\partial p_v(v,t)}{\partial t} = \gamma\frac{\partial}{\partial v}(vp_v(v,t)) + B\frac{\partial^2}{\partial v^2}p(v,t)$$

with initial condition

$$p_v(v, t = 0) = \delta(v - v_o).$$

The stationary probability density p_v^{st} satisfies the equality

$$\frac{d}{dv}(\gamma vP^{st}(v) + B\frac{d}{dv}p^{st}(v)) = 0$$

Taking into account $p^{st}(v \to \infty) = 0$, the following identity holds

$$\gamma p^{st}(v) + B\frac{d}{dv}p^{st}(v) = 0.$$

Using the transformation $u(v) = \ln(p^{st}(v))$ simplifies the integration and the equation for$u(v)$ Leads to

$$\frac{du}{dv} = \frac{-\gamma}{B}v$$

allows to get solution for $u(v)$

$$u(v) = -\aleph\frac{\gamma}{2B}v^2$$

with an accuracy of the integration constant $\aleph$. Taking int account the inverse transformation $p^{st} = \exp(u(v))$ together with initial condition $p_v(v, t = o) = \delta(v - v_o)$ and $\int_{-\infty}^{\infty} p_v(v,t) = 1$, stationary solution for the velocity distribution function p^{st} has the form

$$p^{st}(v) = \sqrt{\frac{\gamma}{2\pi B}} \exp(\frac{-\gamma}{2B} v^2)$$

Theorem 3.3.1. *Consider non-linear Fokker-Plank equation*

$$\frac{\partial p}{\partial t} = \frac{\partial^2}{\partial v^2} p + \frac{\partial}{\partial v}(vp(1 + kp), \forall v \in \Re$$

and

$$P(v, 0) = p_o(v)$$

Let F be an integrable, strictly positive, stationary solution for above equation. Then

$$F(v) = \frac{1}{\beta \exp(\frac{v^2}{2} - k)}$$

Proof. We consider the stationary version:

$$\frac{\partial^2}{\partial v^2} p + \frac{\partial}{\partial v}(vp(1 + kp)) = 0$$

can be written in the form of

$$\frac{\partial}{\partial v}[p(1 + kp)[\frac{1}{p(1 + kp)} \frac{\partial p}{\partial v} + \frac{\partial}{\partial v} \frac{v^2}{2}]] = 0$$

equivalently,

$$\frac{\partial}{\partial v}[p(1 + kp) \frac{\partial}{\partial v}(\log(\frac{p}{1 + kp}) + \frac{v^2}{2})] = 0$$

since the solution is fast-decaying and less than one,then it implies

$$\frac{\partial}{\partial v}(\log(\frac{p}{1 + kp}) + \frac{v^2}{2})] = 0$$

from which we have analytically obtain stationary solution to the equation

$$\frac{\partial^2}{\partial v^2} p + \frac{\partial}{\partial v}(vp(1 + kp)) = 0$$

is $F(v) = \frac{1}{\beta \exp(\frac{v^2}{2} - k)}$,with $\beta \geq 0$ □

Now, equation(3.32) can be rewritten as :

$$\begin{cases} \dfrac{v_i}{dt} = \frac{1}{\tau}(V_{opt}(\Delta x_i) - v_i) + \sigma W_i(t) \\ \dfrac{\Delta x_i}{dt} = v_{i+1} - v_i \end{cases} \tag{3.34}$$

where $W_i(t)$ is a white independent and δ-correlated noise term,

$$< W_i(t) >= 0$$

and

$$< W_i(t)W_j(t) >= \sigma^2\delta_{ij}(t - t_o)$$

where the standard deviation σ denotes the fluctuation strength and Kronecker delta

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

and substituting in the equation (5.4) it takes form:

$$\frac{P(r,t)}{\partial t} = \sum_{i=1}^{n}[-\frac{\partial}{\partial \Delta x_i}(v_{i+1} - v_i) - \frac{\partial}{\partial v_i}(\frac{1}{\tau}(v_{opt}(\Delta x_i) - v_i)) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial (v_i)^2}]P(r,t). \quad (3.35)$$

Then we can simplify $P(\Delta x, v, t)$ into marginal distribution $P(\Delta x, t)$ or$P(v, t)$, because each of them is related to the macroscopic characteristics of traffic flow. It is not possible for each Δx_i to reach infinity or zero (if $\Delta x_i = \infty$, a car following relationship does not exist and if $\Delta x_i = 0$, a car crash happens.) Thus, a second order differential equation with respect to $P(v, t)$ is obtained:

$$\frac{\partial P(v,t)}{\partial t} = \sum_{i=1}^{n}[-\frac{\partial}{\partial v_i}(\frac{1}{\tau}E\{v_{opt}(\Delta x_i) - v_i\}) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial (v_i)^2}]P(r,t). \quad (3.36)$$

where $E\{v_{opt}(\Delta x_i)\} = \int_0^\infty v_{opt}(\Delta x_i)P(x_i)d\Delta x_i$.

For stationary solutions, the probability current of the equation

$$\frac{\partial P(v,t)}{\partial t} = \sum_{i=1}^{n}[-\frac{\partial}{\partial v_i}(\frac{1}{\tau}E\{v_{opt}(\Delta x_i) - v_i\}) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial (v_i)^2}]P(r,t). \quad (3.37)$$

must be constant. Which is :

$$\sum_{i=1}^{n}[\frac{\partial}{\partial v_i}(\frac{1}{\tau}E\{v_{opt}(\Delta x_i) - v_i\}) - \frac{\sigma^2}{2}\frac{\partial^2}{\partial (v_i)^2}]P(r,t) = constant \quad (3.38)$$

When $i = 1$

$$\frac{-d}{dv}(\frac{1}{\tau}(Ev_{opt}(\Delta x) - v)P^{st}(v,t) + \frac{\sigma^2}{2}\frac{\partial}{\partial v}P^{st}(v,t) = 0$$

Which gives

$$\frac{-1}{\tau}(E\{v_{opt}(\Delta x)\} - v)P^{st}(v,t) + \frac{d}{dv}P^{st}(v,t)$$

Let $u(v) = \ln(P^{st})$ then

$$\frac{du}{dv} = \frac{2}{\sigma^2\tau}(E\{v_{opt}(\Delta x)\} - v)$$

This implies

$$\int du = \int \frac{2}{\sigma^2\tau}(E\{v_{opt}(\Delta x)\} - v)dv$$

which gives

$$u(v) = \aleph\frac{2}{\sigma^2\tau}(E\{v_{opt}(\Delta x)\}v - \frac{v^2}{2})$$

This gives

$$P_v(v,t) = \frac{1}{\sqrt{\pi\tau\sigma^2}}\exp(-\frac{1}{\tau\sigma^2}(v - v_{opt}(\frac{L}{N}))^2)$$

Using the marginal distribution (that is in continues case)

$$P_v((v_1, v_2, ..., v_n), t) = \prod_{i=1}^{n} P_v(v_i, t)$$

Hence, when parameters keep the stable,stationary solution is :

$$P(v) = \prod_{i=1}^{n} \frac{1}{\sqrt{\pi\tau\sigma^2}}\exp(-\frac{1}{\tau\sigma^2}(v_i - v_{opt}(\frac{L}{N}))^2). \tag{3.39}$$

We infer that speed distribution for the N vehicles are Gaussian distribution as $\aleph(v_{opt}(\frac{L}{N}), \frac{\tau\sigma^2}{2})$. Because the speed distributions of N vehicle are independent and identical, we can deduce that the distribution of the flow is with $\aleph(v_{opt}(\frac{L}{N}), \frac{\tau\sigma^2}{2})$. However, for unstable case in which the parameters satisfy the constraint in (3.28), there is no stable headway Δx_i and so it is impossible to form probability current as constant. In the unstable cases, the Δx_i are different for each vehicle some of them are large enough that the vehicles can be treated as if they were in free way but some of them are small enough and these vehicles form congestion. As result, the distribution of the flow speed is the mixture of free flow and congestion speed.

Chapter 4

Numerical Simulation

This chapter will create the models presented in previous chapter that will help us model traffic congestion in cause of traffic fluctuation. our dynamical system

$$x_n'' = a(V(\Delta x_n) - x_n')$$

for $n = 1, 2, ..., N$ and $\Delta x_n = x_{n+1} - x_n$. We must write this system of second order differential equation into frist order dynamical system. In the mathematical preliminary it was shown that it is possible to rewrite using change of variable or transformation. This technique that will be utilized here to rewrite our system of N second order differential equation into frist order system. So we let $z_i = x_i, z_{N+i} = x_i'$. Differentiating z_i with respect to time for $i = 1, 2, ..., N$

$$
\begin{aligned}
z_1' &= z_{N+1},\\
z_2' &= z_{N+2},\\
&\vdots\\
z_{N-1}' &= z_{2N-1},\\
z_N' &= z_{2N},\\
z_{N+1}' &= a(V(\Delta z_1) - z_{N+1}),\\
z_{N+2}' &= a(V(\Delta z_2) - z_{N+2}),\\
&\vdots\\
z_{2N-1}' &= a(V(\Delta z_{N-1}) - z_{2N-1}),\\
z_{2N}' &= a(V(\Delta z_N) - z_{2N})
\end{aligned}
$$

This is $2N$ dimensional non-linear first order system , where V is non-linear function and $N \in Z$. If we choose $N = 3$, then the dynamic result in (6) system of ordinary differential equation.

$$
\begin{aligned}
z_1' &= z_4,\\
z_2' &= z_5,\\
z_2' &= z_5,\\
z_4' &= a(V(\Delta z_1) - z_4),\\
z_5' &= a(V(\Delta z_2) - z_5),\\
z_6' &= a(V(\Delta z_3) - z_6)
\end{aligned}
$$

choosing appropriate parameter value and using matlab of Runge kutta order 4 solver to help us to solve this system.

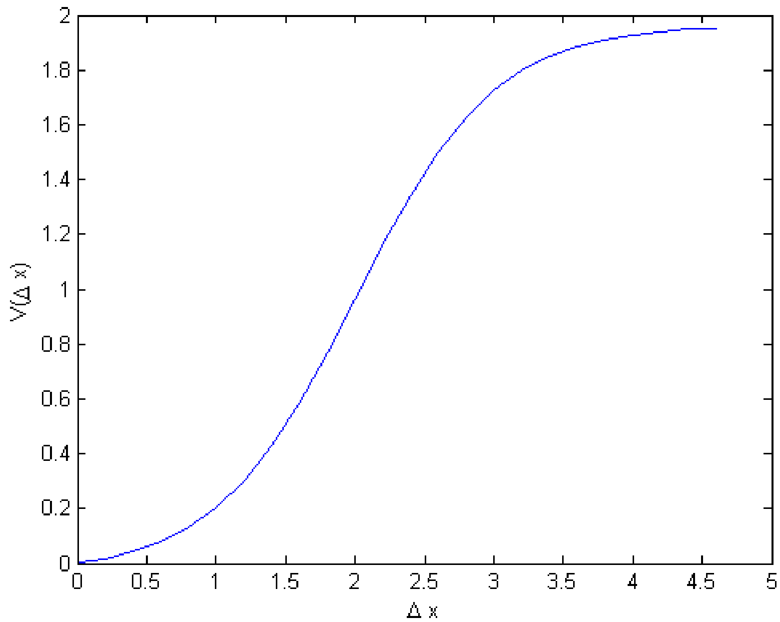

Figure 4.1: The graph of optimal velocity function

4.1 Simple Model

For the simple model we will use the velocity function v that satisfies our criteria where this function must be bounded and monotonically increasing.

$$V(\Delta x) = tanh(\Delta x)$$

Letting $a = 1$ and $N = 100$ for numerical simulation reasons.

We will looking at two different cases for the simple model: a stable and unstable case.

- Stable case: let $L = 200,\ N = 100$, this gives us $b = \dfrac{L}{N} = 2,\ f = V'(b) = 1 - tanh(2) = 0.077 < \dfrac{a}{2} = 0.5$.
- unstable case:Let$L = 50,\ N = 100$, so $b = \dfrac{L}{N} = 0.5,\ f = V'(b) = 1 - tanh(0.5) = 0.786 > \dfrac{a}{2} = 0.5$.

The optimal velocity function shows that the velocity of the vehicle increases with the increasing headway. As headway increases further the velocity function tends to the

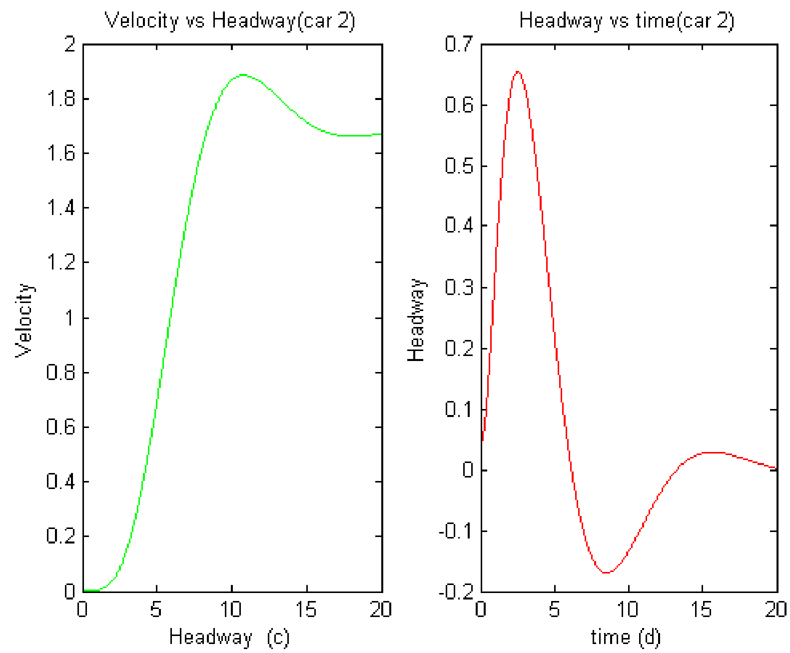

Figure 4.2: The graph of following car velocity versus time and distance versus time respectively

maximal velocity that the driver desires to maintain.

In the Figure 4.2 we compared the distance and velocity distribution with respect to time for the vehicles. In velocity versus time graph we see that velocity for stable case starts off slowly and reachs its optimal velocity. The velocity for unstable case changes as car speed up and slow down as time increases. In figure (4.2) and (4.3), it represents that both cars are trying to maintain optimal velocity. As we see when cars reach their critical density the velocities of the car become constant. But if we increase further the number of car may they create oscillation in velocity distribution , and similarly in headway. In the figure (4.3), the velocity of leading car tries to get optimal velocity depending on the headway. It is seen that vehicle gaining headway as time evolves(for stable case). Also we see that there are no longer any negative velocity, but negative headway which indicates the occurrence cars collision or car crash. A stable traffic flow corresponds to a stable periodic orbit for motion of cars. Hence, the only stable traffic flow in our model is that corresponding to the stable oscillations for $k = 1$, one traffic travels along the single lane.

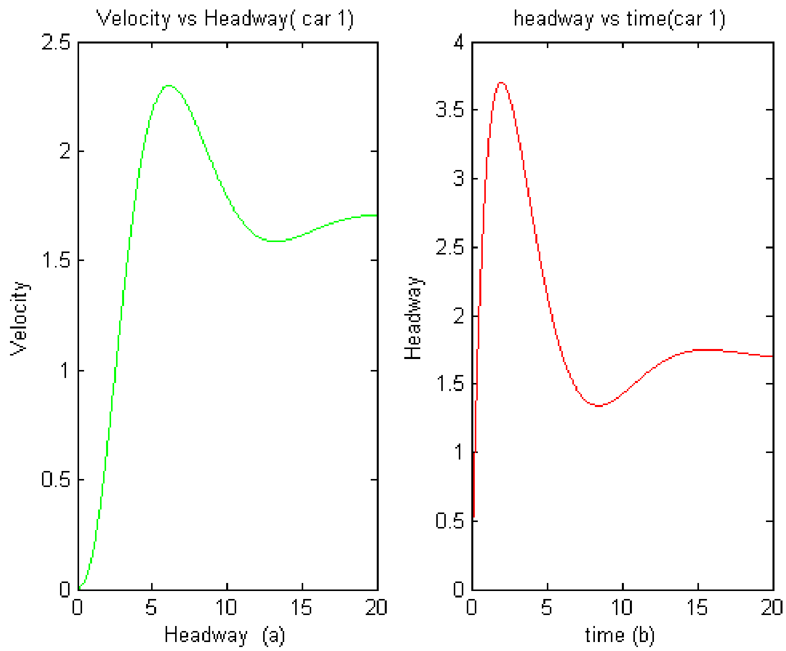

Figure 4.3: The graph of leading car velocity versus headway and headway versus cars respectively

Similarly, if the flow is nearly stable, a car almost has the same velocity and headway.

Chapter 5

Conclusion and Discussion

This study steps for ward to explore dynamics of simple follow the leader models. The main objective of the study was to provide detail analysis of the model deterministicaly and stochastically, using both analytical and numerical analysis. The microscopic description of vehicular ensemble governed by Bando's optimal velocity model for motion on circle of one lane road has been examined in detail. The analysis admits the existence of two phase of traffic flow only, that is either free or congested. The free flow is characterised by high velocities which approximatly equal to maximal allowed speed. The congested traffic occurs with the increase of the car concentration. Such a movement is represented by the different numbers of moving clusters with the same velocity close to zero. In this sense, the probability densities of velocities and headway distance have been calculated for different value of concentration.

According to linear analysis, we get a constraint between the parameters and unstable characteristic of the traffic flow. In addition, we verify that the driver characteristic influence the range of unstable flow densities. Finally, the stochastic approach validates how the constraint influence the unstable flow. The stability analysis results show that driver velocity and headway fluctuation impact the formation of congestion. Short interaction distance,high free speeds and long relaxation time all causes traffic congestion.

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Declaration

I here under signed declare that this thesis is my original work and has not been presented for a degree in any other university, and that all sources of material used for the thesis have been fully acknowledged.

(Name) Signiture