SOLVING (2+1) DIMENSIONAL ABLOWITZ-KAUP-NEWELL-SEGUR EQUATION USING HOMOTOPY ANALYSIS METHOD

Muhammed Ahmed Ibrahim

A Thesis Submitted to The Department of
Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the
Degree of Master's in Applied Mathematics
(Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

June, 2022

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Declaration

I hereby declare that this Master Thesis entitled ***" Solving (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation using Homotopy Analysis Method"*** is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

Name of Student	Signature	Date

Recommendation

We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Solving (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation using Homotopy Analysis Method"*** prepared under our guidance by **Muhammed Ahmed Ibrahim** submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics. Therefore, we recommend that the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled ***"Solving (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation using Homotopy Analysis Method"*** and developed by **Muhammed Ahmed Ibrahim**, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by **Muhammed Ahmed Ibrahim**, have read and evaluated the thesis entitled ***"Solving (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation using Homotopy Analysis Method"*** and examined the candidate during the open defense. This is, therefore, to certify that the thesis is accepted for partial fulfilment of the requirement of the degree of Master of Science in Applied Mathematics.

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List of Acronyms

ADM	Adomian decomposition method
AKNS	Ablowitz-Kaup-Newell-Segur
HAM	Homotopy analysis method
HPM	Homotopy perturbation method
KdV	Korteweg-de Vries
NPDE	Nonlinear partial differential equation.
ODE	Ordinary differential equation
PDE	Partial dIfferential equation
PT	Perturbation technique

Abstract

This research focus on the homotopy analysis method for solving nonlinear partial differential equations. In this study, we have used the homotopy analysis method (HAM) to obtain approximate analytical solutions of (2+1) dimensional Ablowitz-kaup-newell-segur equations. Furthermore, we compared the HAM results and the results obtained by the well-known numerical techniques(FDM). One example(2-dimensional wave equations) and one implementation ((2+1) dimensional Ablowitz-kaup-newell-segur-equations) took to show the efficiency and power of this method with plotted graphs using python code.

Keywords: Homotopy Analysis Method, Ablowitz-Kaup-Newell-Segur-equation, Homotopy Perturbation Techniques.

CHAPTER 1

INTRODUCTION

1.1 Background of Study

Many of the models that describe physical phenomena (related to physics, engineering and applied sciences) are usually expressed as a nonlinear differential equations (Chidozie, 2015). Very few of these equations can be solved directly. In the similar context, there has been a dire need to develop appropriate analytical and numerical techniques to tackle such problems. The approximation techniques have been introduced in order to perform approximate solution to nonlinear differential problems. One such techniques is the homotopy analysis techniques (Abbasbandy, 2006).

The homotopy analysis method, introduced first by Liao in 1992, (Liao, 2003) is a general approximate analytic approach used to obtain series solutions of nonlinear equations of various types, including algebraic equations, ordinary differential equations, partial differential equations and differential–integral equations (Słota et al., 2019). This method is valid no matter whether a nonlinear problem contains small/large physical parameters or not, which is essentially required in perturbation techniques. More importantly, unlike all perturbation and traditional non-perturbation methods. The homotopy analysis method provides us with both the freedom to choose proper initial guess for approximating a nonlinear problem and a simple way to ensure the convergence of the series solution (Abbas, 2014).

Perturbation technique is widely used to investigate physical systems that can be exactly solved but contain small perturbation parameters (Roozi et al., 2011). When applying Perturbation technique to such a system, expansion around the perturbation parameter is involved, and approximants are expressed as power series of these parameters. On the other hand, non perturbative techniques have been established to explore physical problems which do not a small physical parameter to be used as the perturbation parameter (Liao, 2005).

Unlike other analytic approximation methods, the homotopy analysis method provides us great freedom and flexibility to choose equation type and solution expression of high-order approximation equations, and especially a simple way to guarantee the convergence of solution series. Thus, the HAM provides us a large possibility and chance to give something news or different, and to attack some difficult nonlinear problems. The HAM has been successfully employed to solve some complicated nonlinear PDEs. The multiple equilibrium-states of resonant waves in deep water (Yin et al., 2015) and in finite water depth were discovered by means of the HAM for the first time. All of these successful applications show the originality, validity and generality of the HAM for nonlinear problems and encourage us to apply the HAM to attack some famous, challenging nonlinear problems (Liao & Campo, 2002).

The basic motivation of present study is the extension of homotopy analysis method (HAM) for partial differential equations. It is observed that proposed technique is highly effective. Moreover, this method (HAM) is more user-friendly and it overcomes the complexities of selection of initial value. Several examples are given which reveal the efficiency and reliability of the proposed algorithm.

In pure mathematics, the homotopy is a fundamental concept in topology and differential geometry. The concept of homotopy can be traced back to Jules Henri Poincare (1854–1912), a French mathematician(Roozi et al., 2011). A homotopy describes a kind of continuous variation (or deformation) in mathematics. For example, a circle can be continuously deformed into a square or an ellipse, the shape of a coffee cup can deform continuously into the shape of a doughnut but cannot be distorted continuously into the shape of a football. Essentially, a homotopy defines a connection between different things in mathematics, which contain same characteristics in some aspects. In pure mathematics, the homotopy is widely used to investigate existence and uniqueness of solutions of some equations, and so on.

In applied mathematics, the concept of homotopy was used long ago to develop some numerical techniques for nonlinear algebraic equations. The so-called “differential arc length homotopy continuation method” were proposed in 1970s by Keller (Keller, 1978). However, the global homotopy methods can be traced as far back as the work of Lahaye (Lahaye, 1934) in 1934. To solve a nonlinear algebraic equation $f(x) = 0$ by means of the homotopy continuation method, by constructing a homotopy.

1.2 Ablowitz-Kaup-Newell-Segur Equation

During the early 1970s, motivated by the applications to nonlinear optics, Ablowitz, Kaup, Newell, and Segur derived the AKNS equations on the basis of the generalized Zakharov-Shabat spectral problem (Shabat & Zakharov, 1972). The AKNS equations are very important because they can be reduced to some well-known nonlinear evolution equations such as the KdV, the mKdV, the nonlinear Schrödinger, and the sine-Gordon equations, and others, which have many applications in physics and other nonlinear sciences.

Some solutions of AKNS equation have been obtained by using Hirota bilinear method, tanh-coth method, Exp-function, Darboux transformation method and multi-linear variable separation approach (Wazwaz, 2008). In additional, (Lü et al., 2010) have studied the (2+1) dimensional AKNS equation with variable coefficients and have got some new explict solutions by applying Lie symmetry method.

Many analytical methods have been added to the literature to find the exact solution of the nonlinear patial differential equations. We know that each method obtain different types of solutions. For example, sub equation method produces dark solitons, Hirota bilinear form lump solitons. In this study, we aim to obtain the approximate analytical solution of hyperbolic type wave solutions using the homotopy analysis method. For this, we have dealt with the AKNS equation with strong nonlinearity

$$\begin{cases} 4u_{xt} + u_{xxxt} + 8u_{xy}u_x + 4u_y u_{xx} = 0, \\ u(x,y,0) = cosh(x+y), \end{cases}$$

where, $u = u(x,y,t)$

1.3 Statement of the Problem

We consider (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation(Ablowitz et al., 1973)

$$\begin{cases} 4u_{xt} + u_{xxxt} + 8u_{xy}u_x + 4u_y u_{xx} = 0, \\ u(x, y, 0) = cosh(x + y). \end{cases} \tag{1.1}$$

Then, the HAM approximate analytical solution of equation (1.1) may then be written as

$$u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t).$$

Hence, in this study we aim to apply homotopy analysis method on equation (1.1). Thus, this study is intended to answer the following basic questions:

- How to apply HAM on equation (1.1)?
- How to obtain approximate analytical solutions of equation (1.1)?
- Is this method is efficient to obtain approximate analytical solutions of nonlinear partial differential equation?

1.4 Objectives of Study

1.4.1 General Objective

The general objective of this study is to solve (2+1)dimensional Ablowitz-kaup-newell-segur equations by homotopy analysis method.

1.4.2 Specific Objectives

The specific objectives of this study are to:

- apply HAM on Ablowitz-Kaup-Newell-Segur equations.
- approximate analytical solution of Ablowitz-Kaup-Newell-Segur equation by using HAM?
- write python code for the proposed equation and plot the graph.

1.5 Significance of Study

The significance of this study help:

- some individual to understand how HAM applied to nonlinear differential equation to get approximate analytical solutions.
- to understand how HAM solves Ablowtz-kaup-newell-segur equations.
- the researcher by serving as a resource for those who need to conduct research in this field.

1.6 Scope of the Study

This thesis is mainly delimited to apply homotopy analysis method (HAM) for solving (2+1) dimensional Ablowitz-Kaup-Newell-Segur equation.

CHAPTER 2

LITERATURE REVIEW

The homotopy is a fundamental concept in topology, which can be traced back to Jules Henri Poincar's (1854–1912), a French mathematician. Based on the homotopy, two methods have been developed(Abbasbandy, 2008).

Iba et al. (1991) in their article showed that the homotopy continuation method dating back to 1930s, which is a global convergent numerical method mainly for nonlinear algebraic equations. The other is homotopy analysis method proposed in 1992 by Shijun Liao, which is an analytic approximation method with guarantee of convergence, mainly for nonlinear differential equations(Liao, 2003).

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (Alomari et al., 2008). This method has been successfully applied to solve many types of nonlinear problems. The homotopy analysis method offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power. The HAM is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time. Unlike the homotopy continuation method that is a global convergent numerical method mainly for nonlinear algebraic equations, the homotopy analysis method is a kind of analytic approximation method mainly for nonlinear differential equations. Note that the HAM uses much more complicated homotopy equation than for the homotopy continuation method. Furthermore, the homotopy analysis method provides larger freedom to choose the auxiliary linear operator L.

Most importantly, the so-called convergence-control parameter is introduced for the first time, to the best of our knowledge, in the homotopy equation so that the HAM provides us a convenient way to guarantee the convergence of series(Al-Hayani et al., 2019).

Chidozie (2015); Duan et al. (2012) in their article showed that higher dimensional initial boundary value problems of variable coefficients can be solved by means of an analytic technique, namely the homotopy analysis method (HAM). There also comparisons were made between the Adomian decomposition method (ADM), the exact solution and the homotopy analysis method. They discovered that all the examples solved shows that the results obtained from HAM was in perfect agreement with those obtained by Adomian decomposition method (ADM)(Babolian et al., 2004). It was apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wider classes of linear and non linear problems. Hence they also found that the homotopy analysis method HAM offers certain advantages over routine numerical methods(Izadian et al., 2012). HAM is better since it does not involve discretization of the variables hence is free from rounding off errors and does not require large computer memory or time. Results reveal that the proposed method is very effective and simple(Al-Hayani et al., 2019).

Abbasbandy (2008) applied homotopy analysis method to non linear equation arising in heat transfer. In his work, the basic idea of the homotopy analysis method HAM was introduced and then applied to obtain the solution of nonlinear equations arising in heat transfer. He explained further that the nonlinear equations of the heat radiation and conduction equations of a fin in the steady state and in the free space are solved through HAM. Comparisons were made between HAM, perturbation method and HPM (homotopy perturbation method). The HAM provided a convenient way to control the convergence of approximation series which is a fundamental qualitative difference in analysis between HAM and other methods. Also it has been shown that the HPM and perturbation method are valid only for small parameter, but in HAM we can choose auxiliary parameter h in an appropriate way. This shows the validity and great potential of HAM for non linear problem in science and engineering.

Ghoreishi et al. (2011) applied homotopy analysis method to solve a model for HIV Infection of $CD4^+$ T-cells. In their work, the homotopy analysis method (HAM) was investigated to give an approximate solution of a model for HIV infection of $CD4^+$ T-cells. This method allows for the solution of the governing differential equation to be calculated in the form of an infinite series with components which can be easily calculated. The HAM utilizes a simple method to adjust and control the convergence region of the infinite series solution by using an auxiliary parameter. They further explained that this model of virus spread has three variables: the population sizes of the infested cells, infected cells and free virus particle.

Gupta and Gupta (2012) applied homotopy analysis method to solve Non-linear Cauchy problem. They showed by of means HAM, that the solutions of some linear Cauchy problems of parabolic-hyperbolic type are exactly obtained in the form of convergent Taylor series. The HAM contains the auxiliary parameter, h, that provide a convenient way of controlling the convergent region of series solutions.

Liao and Tan (2007) employed homotopy analysis method to solve nonlinear differential equation governing Jeffery–Hamel flow. In their work they showed that the Jeffery–Hamel flow has been studied and its nonlinear ordinary differential equation has been solved through homotopy analysis method (HAM). The obtained solution in comparison with the numerical ones represents a remarkable accuracy. The results also indicate that HAM can provide us with a convenient way to control and adjust the convergence region.

Applied homotopy analysis method is olso used to solve the Convection-Diffusion equation. (Mohamed & Hamed, 2016) in their work, the homotopy analysis method was used to find series solution of linear convection diffusion (CD) equation. They further explained that convection diffusion equation is a combination of the diffusion and convection (advection) equations and describes physical phenomena where particles, energy or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection.

From the above literature survey it is very relevant to point out that no attempt has been made to study on approximate analytical solution of Ablowitz-Kaup-Newell-Segur equations. Hence this thesis focus on approximate analytical solution of Ablowitz-Kaup-Newell-Segur equations by homotopy analysis method.

CHAPTER 3

PRELIMINARIES

In this chapter we gather some general preliminaries that we will use throughout this thesis. We shall not go into details but cite references in which the interested reader can find further details.

3.1 Differential Equation

Definition 3.1.1 *A differential equation is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable). A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity (Braun & Golubitsky, 1983).*

3.1.1 Ordinary and Partial Differential Equation

Definition 3.1.2 *An ordinary differential equation (ODE) is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function (Birkhoff & Rota, 1978).*

$$F(t, u(t), u^{t'}, u^{(2)}, u^{(3)}, u^{(4)}, ..., u^{(m)}) = 0$$

Definition 3.1.3 *A partial differential equation is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function (Greiner et al., 1997).*

$$F(x_1, ..., x_n; u, \frac{du}{dx_1}, ..., \frac{du}{dx_n}; \frac{d^2u}{dx_1x_2}, ..., \frac{d^2u}{dx_1x_n}; ...) = 0$$

3.1.2 Linear and Nonlinear Partial Differential Equation

Definition 3.1.4 *If the dependent variable and all its partial derivatives occur linearly in any partial differential equation then such an equation is called linear partial differential.*

Two-dimensional wave equation is an example of linear PDEs (Abbas, 2014).

$$\begin{cases} u_{tt} - \frac{1}{2}(u_{yy} + u_{xx}) = 0, \\ \\ u(x, y, 0) = sin(\pi x)cos(\pi y), \\ u_t(x, y, 0) = \pi sin(\pi x)cos(\pi y). \end{cases}$$

Definition 3.1.5 *A non-linear partial differential equation is a partial differential equation that is not a linear equation in the unknown function and its derivatives.*

Ablowitz-Kaup-Newell-Segur equation is one examples of nonlinear PDEs (Ablowitz et al., 1973).

$$\begin{cases} 4u_{xt} + u_{xxxt} + 8u_{xy}u_x + 4u_y u_{xx} = 0, \\ u_0(x, y, t) = cosh(x + y), \end{cases}$$

3.2 Taylor series

The essential tool in the development of approximate analytical solution is Taylor's series. Taylor's series will enable us to approximate a function with a polynomial, and polynomials are easy to compute.

Definition 3.2.1 *(Moreno et al., 1996) Let $f(x)$ be a function which is analytic at $x = x_0$. Then we can write $f(x)$ as the power series, called the Taylor series of $f(x)$ at $x = x_0$.*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 ...,$$

If we write the nth derivative of $f'(x)$ as $f^n(x)$, this becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n.$$

3.3 Homotopy Analysis

3.3.1 Definitions

Definition 3.3.1 *(Al-Hayani et al., 2019) Suppose maps f, $g : X \to Y$, we say that f is homotopic to g ($f \sim g$), if there is a continuous function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ $\forall x \in X$.*

Definition 3.3.2 *(Al-Hayani et al., 2019) The embedding parameter $q \in [0, 1]$ in a homotopy of functions or equations is called homotopy-parameter.*

Definition 3.3.3 *(Liao, 2012)Given an equation denoted by f_1, which has at least one solution u. let f_0 denote a proper simpler equation, called the initial equation, whose solution u_0 is known. If one can construct a homotopy of equation $f(q) : f_0 \sim f_1$ such that, as the homotopy parameter $q \in [0, 1]$ increases from 0 to 1, $f(q)$ deforms (or varies) continuously from the the initial equation f_0 to the the original equation f_1, while its solution varies continuously from the known solution u_0 of f_0 to the unknown solution u of f_1, then this kind of homotopy of equations is called the zeroth-order deformation equation.*

In general, if a continuous function $f(x) \in C[a, b]$ can be deformed continuously into another function $g(x) \in C[a, b]$ one can construct a homotopy $H : f(x) \sim g(x)$ in the way

$$H(x, q) = (1 - q)f(x) + qg(x), \quad q \in [0, 1]$$

clearly, if $p = 0$ and $p = 1$,
$H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The changing process of q from 0 to 1 is just that of $H(x, q)$ from $f(x)$ to $g(x)$. In topology this is called deformation and $f(x)$, g(x) are homotopic.

3.3.2 Properties of Homotopy Derivative

Definition 3.3.4 *(Liao, 2012) Let ϕ be a function of the homotopy parameter q, then*

$$D_m(\phi) = \left[\frac{1}{m!}\frac{d^m\phi}{dq^m}\right]_{q=0},$$

is called the mth-order homotopy-derivative of ϕ, where $m \geq 0$ is an integer, and D_m is called the operator of the mth-order homotopy-derivative.

Theorem 3.3.1 *(Liao, 2012) For two arbitrary homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \psi = \sum_{k=0}^{+\infty} w_k q^k,$$

where ϕ and ψ are analytic in $q \in [0, a)$, it holds

1. $D_m(\phi) = u_m,$
2. $D_m(q^k\phi) = D_{m-k}(\phi),$
3. $D_m(\phi\psi) = \sum_{k=0}^{m} D_k(\phi) D_{m-k}(\psi) = \sum_{k=0}^{m} u_k w_{m-k},$

where $m \geq 0$ and $0 \leq k \leq m$ are integers.

Proof. (1), According to Taylor theorem (Moreno et al., 1996), the unique coefficient u_m of the Maclaurin series

$$\phi = \sum_{m=0}^{+\infty} u_m q^m$$

with respect to the homotopy-parameter q is given by

$$D_m(\phi) = \left[\frac{1}{m!}\frac{d^m\phi}{dq^m}\right]_{q=0},$$

which gives theorem (3.3.1) by means of the definition (3.3.4) of $D_m(\phi)$.

(2) It holds

$$q^k\phi = q^k \sum_{j=0}^{+\infty} u_j q^j = \sum_{j=0}^{+\infty} u_j q^{j+k} = \sum_{m=k}^{+\infty} u_{m-k} q^m,$$

which gives by means of theorem (3.3.1) that

$$D_m(q^k\phi) = u_{m-k} = D_{m-k}(\phi), \quad when \ \ 0 \leq k \leq m,$$

and

$$D_m(q^k\phi) = 0 \quad when \ \ k > m.$$

(3) According to Leibnitz's rule for derivatives of product, it holds

$$\frac{\partial^m(\phi\psi)}{\partial q^m} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}\frac{\partial^k\phi}{\partial q^k}\frac{\partial^{m-k}\psi}{\partial q^{m-k}} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!}\frac{\partial^k\psi}{\partial q^k}\frac{\partial^{m-k}\phi}{\partial q^{m-k}},$$

which gives according to definition (3.3.4) and theorem (3.3.1) that

$$D_m(\phi\psi) = \frac{1}{m!}\frac{\partial^m(\phi\psi)}{\partial q^m} = \sum_{k=0}^{m}\left(\frac{1}{m!}\frac{\partial^m(\phi\psi)}{\partial q^m}\right)\left[\frac{1}{(m-1)!}\frac{\partial^{m-k}\phi}{\partial q^{m-k}}\right]$$
$$\sum_{k=0}^{m} D_k(\phi)D_{m-k}(\psi) = \sum_{k=0}^{m} u_k w_{m-k}.$$

Similarly, it holds

$$D_m(\phi\psi) = \sum_{k=0}^{m} D_k(\phi)D_{m-k}\psi = \sum_{k=0}^{m} u_{m-k}w_k.$$

Theorem 3.3.2 *(Liao, 2012) If*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k \quad \textit{and} \quad \psi = \sum_{k=0}^{+\infty} w_k q^k$$

are two homotopy Maclaurin series, where ϕ and ψ analytic in $q \in [0, a)$, f and g are independent of the homotopy-parameter $q \in [0, 1]$, then it holds

$$D_m(f\phi + g\psi) = fD_m(\phi) + gD_m(\psi) = fu_m + gw_m.$$

Proof. Because D_m defined by theorem (3.3.1) is a linear operator, and besides f and g are independent of q, it holds

$$D_m(f\phi + g\psi) = D_m(f\phi) + D_m(g\psi) = fD_m(\phi) + gD_m(\psi)$$

According to (3.3.1), we have $D_m(\phi) = u_m$ and $D_m(\psi) = w_m$.

Theorem 3.3.3 *(Liao, 2012) Let L denote a linear operator independent of the homotopy-parameter $q \in [0, 1]$. For two homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \psi = \sum_{k=0}^{+\infty} w_k q^k,$$

where ϕ and ψ are analytic in $q \in [0, a)$, it holds

$$D_m[L(\phi)] = L[D_m(\phi)] = L(u_m),$$

and

$$D_m[\psi L(\phi)] = \sum_{n=0}^{m} D_{m-n}(\psi)L[D_n(\psi)] = \sum_{n=0}^{m} w_{m-n}L(u_n).$$

where $m \geq 0$ is an integer.

Proof. Since L is linear and independent of q, using theorem (3.3.1), we have

$$\phi = L(\sum_{k=0}^{+\infty} u_k q^k) = \phi = \sum_{k=0}^{+\infty} L(u_k)q^k.$$

Using the theorem (3.3.1), one has $D_m[L(\phi)]= L(u_m)$ on the other side, according to theorem (3.3.1), it holds $L[D_m(\phi)]= L(u_m)$. Thus,

$$D_m[L(\phi)] = L[D_m(\phi)] = L(u_m)$$

holds. Then, according to theorem (3.3.1), we have

$$\begin{aligned} D_m[\psi L(\phi)] &= \sum_{n=0}^{m} D_{m-n}(\psi)D_n[L(\phi)] \\ &= \sum_{n=0}^{m} D_{m-n}(\psi)L[D_n(\phi)] = \sum_{n=0}^{m} w_{m-n}L(u_n). \end{aligned}$$

3.3.3 Deformation Equations

Definition 3.3.5 *(Liao, 2003)For a given nonlinear differential equation $N[u(x, y, t)] = 0$, where N is a nonlinear differential operator, $u(x, y, t)$ is an unknown function, x, y, denotes a vector of spatial independent-variables, t denotes the temporal independent-variable. Then the zeroth-order deformation equation is given by*

$$(1 - q)L[\phi(x, y, t; q) - u_0(x, y, t)] + qN[\phi(x, y, t; q)] = 0. \qquad q \in [0, 1]$$

Lemma 3.3.4 *(Liao, 2003) let*

$$\phi = \sum_{m=0}^{+\infty} u_m q^m,$$

denote a homotopy-Maclaurin series, where $q \in [0, 1]$ is the homotopy-parameter, and L denotes an auxiliary linear operator which has the property $L[0] = 0$ and is independent of q. It holds

$$D_m[(1 - q)L(\phi - u_0)] = L(u_m - \chi_m u_{m-1}),$$

where

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases}$$

Proof. Since L is a linear operator independent of q, it holds

$$(1-q)L(\phi-u_0)=L(\phi-q\phi+u_0q-u_0).$$

According to theorem (3.3.1), (3.3.2) and (3.3.3), we have

$$\begin{aligned}D_m[(1-q)L(\phi-u_0)] &= D_m[L(\phi-q\phi+u_0q-u_0)]\\ &= L[D_m(\phi-q\phi+u_0q-u_0)]\\ &= L[D_m(\phi)-D_m(q\phi)+u_0D_m(q)]\\ &= L[u_m-u_{m-1}+u_0D_m(q)],\end{aligned}$$

which equals to $L(u_m)$ when $m=1$, and $L(u_m-u_{m-1})$ when $m>1$, respectively. Thus, according to the definition (3.3.4) of χ_m, it holds

$$D_m[(1-q)L(\phi-u_0)]=L(u_m-\chi_m u_{m-1}).$$

Theorem 3.3.5 *(Abbas, 2014) Let L denote an auxiliary linear operator which has the property $L(0)=0$ and is independent of the homotopy-parameter $q\in[0,1]$, N denote a nonlinear operator, $u_0(x,y,t)$ an initial approximation of the original equation $N(u)=0$, h is the convergence-control parameter independent of q, and $H(x,y,t)$ an auxiliary function independent of q, respectively, where x, y denotes a vector of the spatial independent variable, and t is temporal independent variable. If*

$$\phi=\sum_{m=0}^{+\infty}u_m(x,y,t)q^m,$$

is the homotopy-Maclaurin series of the zeroth-order deformation equation

$$(1-q)L(\phi-u_0)=qhH(x,y,t)N(\phi),$$

then the corresponding mth-order deformation equation reads

$$L[u_m(x,t)-\chi_m u_{m-1}]=hH(x,y,t)D_{m-1}[N(\phi)].$$

3.4 Homotopy Analysis Method(HAM)

We consider the differential equation(Chidozie, 2015)

$$N[u(x, y, t)] = 0, \tag{3.1}$$

where N is a nonlinear operator, x, y, and t denote the independent variables and u is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of HAM we construct the zero order deformation equation.

$$(1 - q)L[\phi(x, y, t; q) - u_0(x, y, t)] = qhH(x, y, t)N[\phi(x, y, t; q)], \tag{3.2}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x, y, t; q)$ is unknown function, $u_0(x, y, t)$ is an initial guess of $u(x, y, t)$ and $H(x, y, t)$ denotes a nonzero auxiliary function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, equation (3.2) becomes

$$\phi(x, y, t; 0) = u_0(x, y, t) \text{ and } \phi(x, y, t : 1) = u(x, y, t),$$

respectively. Thus as q increases from 0 to 1, the solution varies from the initial guess $u_0(x, y, t)$ to the solution $u(x, y, t)$. Assuming that L, $H(x, y, t)$, h and $u_0(x, y, t)$ are properly chosen, so that the solution $\phi(x, y, t; q)$ of equation (3.1) exist for $0 < q < 1$. Besides this $\phi(x, y, t; q)$ analytic at $q = 0$. Thus, we can write $\phi(x, y, t; q)$ as Taylor series with respect to q at $q = 0$.

$$\phi(x, y, t; q) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t)q^m, \tag{3.3}$$

where

$$u_m(x, y, t) = \left[\frac{1}{m!}\frac{\partial^m \phi[x, y, t; q)]}{\partial q_m}\right]_{q=0}.$$

The convergence of the series depends upon the auxiliary parameter h. If it is convergent at $q = 1$, the HAM approximate analytical solution may then be written as

$$u(x, y, t) = u_0(x, y, t) + \sum_{m=1}^{\infty} u_m(x, y, t), \tag{3.4}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao (Liao, 1997). Define the vectors

$$\vec{u}_n(x, y, t) = (u_1(x, y, t), u_2(x, y, t), u_3(x, y, t), ...u_n(x, y, t)).$$

Differentiate the zeroth-order deformation equation m-times with respect to q and then dividing them by $m!$ and finally setting $q = 0$ and choosing $H(x, y, t) = 1$, we get the mth-order deformation equation:

$$L[u_m(x, y, t) - \chi_m u_{m-1}(x, y, t)] = hR_m(\vec{u}_{m-1}), \tag{3.5}$$

where

$$R_m(\vec{u}_{m-1}) = \left[\frac{1}{m!}\frac{\partial^{m-1} N[\phi(x, y, t; q)]}{\partial q_{m-1}}\right]_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

Taking L^{-1} of the above equation

$$u_m(x, y, t) = \chi_m u_{m-1}(x, y, t) + hL^{-1}[R_m(\vec{u}_{m-1})]. \tag{3.6}$$

To get the higher order of this system of model is

$$u(x, y, t) = \sum_{m=1}^{\infty} u_m(x, y, t).$$

It should be emphasized that $u_m(x, y, t)$ for $m \leq 1$ is governed by the linear equation (3.4) with linear boundary conditions that comes form the original problem, which can be solved by the symbolic computation software such as Matlab or python. For the convergence of the above method we refer the reader to Liao(Liao, 2012). If equation (3.1) admits unique solution, then this method will produce the unique solution. If equation (3.1) does not posses a unique solution, the HAM will give a solution among many other possible solutions.

3.5 Solution Expression

For a given nonlinear equation, the key of the HAM is to construct a good enough zeroth-order deformation equation by means of choosing a proper initial approximation u_0 and a proper auxiliary linear operator L we have extremely large freedom to choose the initial approximation u_0 and the auxiliary linear operator L, it is such kind of fantastic freedom that differs the HAM from other analytic techniques.

However, anything has its bright and dark sides. The extremely large freedom of constructing a zeroth-order deformation equation makes it difficult for a beginner to apply the HAM. Thus, for multifarious applications of the HAM in science and engineering, we need some rules to guide the choice of the initial approximation u_0 and the auxiliary linear operator L(Liao, 2012).

Definition 3.5.1 *(Liao, 2012) Let $u_m(x,y,t)$ denote the base function, $u(x,y,t)$ be a solution of a nonlinear equation $N(u)=0$, respectively. Then,*

$$u(x,y,t) \approx \sum_{m=1}^{+\infty} a_m u_m(x,y,t), \tag{3.7}$$

is called the solution expression of $u(x,y,t)$, if

$$\lim_{k\to\infty} ||u(x,y,t) - \sum_{m=1}^{+\infty} a_m u_m(x,y,t)|| \longrightarrow 0. \tag{3.8}$$

3.5.1 Choice of Initial Approximation

Assume that a solution expression (3.7) is chosen for a given equation $N(u)=0$. Our aim is to find a proper initial approximation u_0 and a proper auxiliary linear operator L such that the corresponding homotopy-series converge(Liao & Tan, 2007).
Obviously, the initial approximation u_0 must obey the so-called solution expression (3.7). Since we have freedom to choose the initial approximation, we could choose such a kind of initial approximation

$$u_0(x,y,t) \approx \sum_{m=1}^{n_0} a_m u_m(x,y,t),$$

where $u_m(x,y,t)$ is the base function, a_m is unknown constant, and n_0 equal to or greater than the number of linear boundary/initial conditions denoted by m. Then, enforcing the above initial approximation to satisfy the m boundary/initial linear conditions, we have $n_0 - m$ unknown coefficients left. So, when $n_0 = m$, then all coefficients of the initial approximation are known so that it is completely determined.

3.5.2 Choice of Auxiliary Linear Operator

To obey the so-called solution-expression (3.8) of a given equation $N(u)=0$, the initial guess u_0 and the auxiliary linear operator L must be chosen so that the solution u_m of the high-order deformation equation exists and obeys, and besides the homotopy series

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{+\infty} u_m(x,y,t),$$

converges.
The choice of the auxiliary linear operator L is mainly determined by the solution-expression (3.7), but sometimes also by the boundary/initial conditions.
In particular, there are three methods to choose auxiliary linear operators(Motsa et al., 2012).

- Method of the highest order differential matching: This approach defines the auxiliary linear operator using only the highest order derivative. This approach is the most widely used approach in the HAM solution of nonlinear differential equations defined in finite domains.

$$L_1(u) = \frac{d^p u}{dt^p}, \quad \text{when } \ p > 0.$$

- Method of linear partition matching: This approach defines the auxiliary linear operator to be the collection of all linear terms in the governing equation.

$$L_2(u) = \sum_{j=0}^{p} \alpha_j(t) \frac{d^j u}{dt^j}, \quad \text{when } \ p, j > 0.$$

- Method of complete differential matching: In this approach, the collection of all linear operators and some nonlinear factors of the governing nonlinear equations are used to define the linear operator. The aim is to ensure that all the terms of the governing nonlinear differential equation contribute to the auxiliary linear operator selected(Van Gorder & Vajravelu, 2009).

$$L_3(u) = \sum_{j=0}^{p} \alpha_j(t) \frac{d^j u}{dt^j} + \sum_{r=0}^{p} \frac{d^r u}{dt^r} \sum_{j=0}^{r} \beta_{r,j}, \quad \text{when } \ p, j, r > 0.$$

3.6 Example (Two-dimensional Wave Equation)

Consider the two-dimensional wave equation(Abbas, 2014)

$$\begin{cases} u_{tt} - \frac{1}{2}(u_{yy} + u_{xx}) = 0, \\ \\ u(x, y, 0) = sin(\pi x) cos(\pi y), \\ u_t(x, y, 0) = \pi sin(\pi x) cos(\pi y). \end{cases} \tag{3.9}$$

Where, $u = u(x, y, t)$.
To solve equation (3.9) by means of the homotopy analysis method let us consider the

linear operator

$$L[\phi(x,y,t;q)] = \frac{\partial^2 \phi(x,y,t;q)}{\partial t^2},$$

with the property that

$$L(c + c_1 t) = 0,$$

which implies that

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt.$$

We now define the nonlinear operator as

$$N[\phi(x,y,t;q)] = \frac{\partial^2 \phi(x,y,t;q)}{\partial t^2} - \frac{1}{2}\frac{\partial^2 \phi(x,y,t;q)}{\partial x^2} - \frac{1}{2}\frac{\partial^2 \phi(x,y,t;q)}{\partial y^2}. \tag{3.10}$$

Construct the zeroth-order deformation equation by

$$(1-q)L[\phi(x,y,t;q) - u_0(x,y,t)] = qhH(x,y,t)N[\phi(x,y,t;q)], \tag{3.11}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,y,t)$ is an unknown function, $u_0(x,y,t)$ is an initial guess of $u(x,y,t)$ and $H(x,y,t)$ denotes a nonzero auxiliary function. When the embedding parameter $q = 0$ and $q = 1$, equation (3.11) becomes

$$\phi(x,y,t;0) = u_0(x,y,t), \quad \phi(x,y,t;1) = u(x,y,t). \tag{3.12}$$

We choose auxiliary function $H(x,y,t) = 1$, then we obtain the m-th order deformation equation

$$\begin{aligned} L[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] &= h\Re_m(u_{m-1}) \\ u_m(x,y,t) &= \chi_m u_{m-1}(x,y,t) + hL^{-1}(\Re_m(u_{m-1}), \end{aligned} \tag{3.13}$$

where

$$\Re_m(u_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial y^2}, \tag{3.14}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

when, $m = 0, 1, 2, 3, \ldots$

Then

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + hL^{-1}(\frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial y^2}),$$

since

$$L^{-1} = \int_0^1 \int_0^1 (.)dtdt.$$

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + h\int_0^t \int_0^t (\frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u_{m-1}}{\partial y^2})dtdt \quad (3.15)$$

For solving equation (3.9) under the initial condition $u(x,y,0) = sin(\pi x)cos(\pi y)$ and $u_t(x,y,0) = \pi sin(\pi x)cos(\pi y)$, we choose initial guess

$$u_0(x,y,t) = sin(\pi x)cos(\pi y)(1+\pi t)$$

$$\begin{aligned}
u_0(x,y,t) &= sin(\pi x)cos(\pi y)(1+\pi t).\\
u_1(x,y,t) &= \chi_1 u_0(x,y,t) + h\int_0^t \int_0^t \left(\frac{\partial^2 u_0}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u_0}{\partial y^2}\right)dtdt,\\
u_1(x,y,t) &= \frac{h}{2}\left(\int_0^t \int_0^t \pi^2 sin\pi x cos\pi y(1+\pi t)dtdt + \int_0^t \int_0^t \pi^2 sin\pi x cos\pi y(1+\pi t))dtdt\right)\\
&= hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}),\\
u_1(x,y,t) &= hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}).
\end{aligned}$$

$$\begin{aligned}
u_2(x,y,t) &= \chi_2 u_1(x,y,t) + h\int_0^t \int_0^t (\frac{\partial^2 u_1}{\partial t^2} - \frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u_1}{\partial y^2})dtdt,\\
&= hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}) + h\int_0^t \int_0^t \frac{\partial^2 hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!})}{\partial t^2}dtdt\\
&- h\int_0^t \int_0^t \frac{1}{2}\frac{\partial^2 hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!})}{\partial x^2}dtdt\\
&- h\int_0^t \int_0^t \frac{1}{2}\frac{\partial^2 hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!})}{\partial y^2})dtdt,\\
&= hsin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}) + h^2 sin\pi x cos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!})\\
&+ h^2 sin\pi x cos\pi y(\frac{(\pi t)^4}{4!} + \frac{(\pi t)^5}{5!}),
\end{aligned}$$

$$u_2(x,y,t) = hsin\pi xcos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}) + h^2 sin\pi xcos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}) + h^2 sin\pi xcos\pi y(\frac{(\pi t)^4}{4!} + \frac{(\pi t)^5}{5!}).$$

Similarly, solving for $u_3(x,y,t)$, $u_4(x,y,t)$, $u_5(x,y,t)$,...
Now, we choose h=-1, then we obtain the following recurrence relation:

$$\begin{aligned}
u_0(x,y,t) &= sin\pi xcos\pi y(1+\pi t),\\
u_1(x,y,t) &= sin\pi xcos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}),\\
u_2(x,y,t) &= sin\pi xcos\pi y(\frac{(\pi t)^4}{4!} + \frac{(\pi t)^5}{5!}),\\
u_3(x,y,t) &= sin\pi xcos\pi y(\frac{(\pi t)^6}{6!} + \frac{(\pi t)^7}{7!})\\
. &\qquad\qquad .\\
. &\qquad\qquad .\\
. &\qquad\qquad .
\end{aligned}$$

From homotopy series, the approximate analytical solutions of given equation (3.9) is given as the form of

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t). \tag{3.16}$$

Then

$$u(x,y,t) = sin\pi xcos\pi y(1+\pi t) + sin\pi xcos\pi y(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!}) + sin\pi xcos\pi y(\frac{(\pi t)^4}{4!} + \frac{(\pi t)^5}{5!}) + sin\pi xcos\pi y(\frac{(\pi t)^6}{6!} + \frac{(\pi t)^7}{7!}) + ...$$

Hence

$$u(x,y,t) = sin\pi xcos\pi y(1+\pi t+\frac{(\pi t)^2}{2!}+\frac{(\pi t)^3}{3!}+\frac{(\pi t)^4}{4!}+\frac{(\pi t)^5}{5!}+\frac{(\pi t)^6}{6!}+\frac{(\pi t)^7}{7!}+...) \tag{3.17}$$

By using Taylor series the equation (3.17) converges to

$$u(x,y,t) = sin\pi xcos\pi y(e^{\pi t}) = e^{\pi t} sin\pi xcos\pi y$$

So the result shows that the method provides an excellent approximations.

Comparison of HAM and Numerical Solution of 2D Wave Equation

Table 3.1: Approximate values and numerical solution for 2D wave equation

t	x	y	HAM Solution	Numerical Solution	Error=$\lvert HAM_{sol} - Num_{sol}$
0.01	0	0	0	0	0
0.01	0.1	0.1	0.0056579	0.0054662	0.0001917
0.01	0.2	0.2	0.0113153	0.0112042	0.0001111
0.01	0.3	0.3	0.0169712	0.0167933	0.0001779
0.01	0.4	0.4	0.0203638	0.0202253	0.0001385
0.01	0.5	0.5	0.0254511	0.0251246	0.0003265
0.01	0.6	0.6	0.0305359	0.0301350	0.0004009
0.01	0.7	0.7	0.0356177	0.0346811	0.0009366
0.01	0.8	0.8	0.0406960	0.0394513	0.0012447
0.01	0.9	0.9	0.0457704	0.0446793	0.0010911
0.01	1	1	0.0508403	0.0508403	0

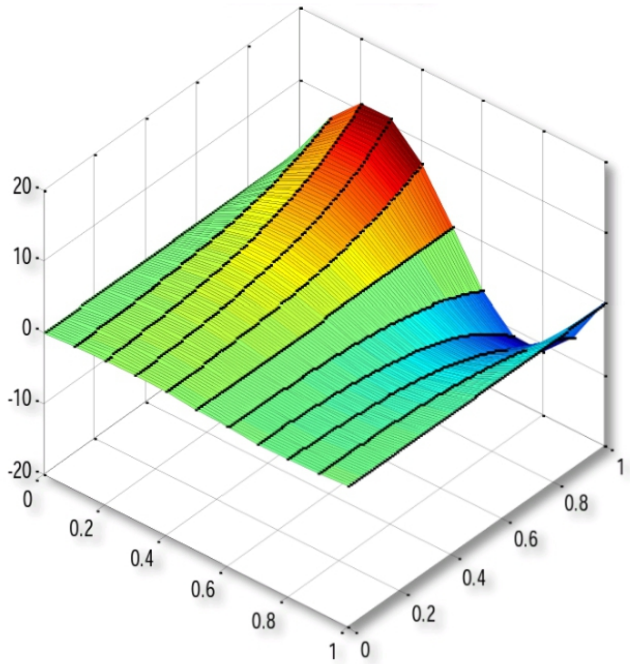

Figure 3.1: HAM solution for 2D wave equation at $t = 0.01$ and $h = -1$

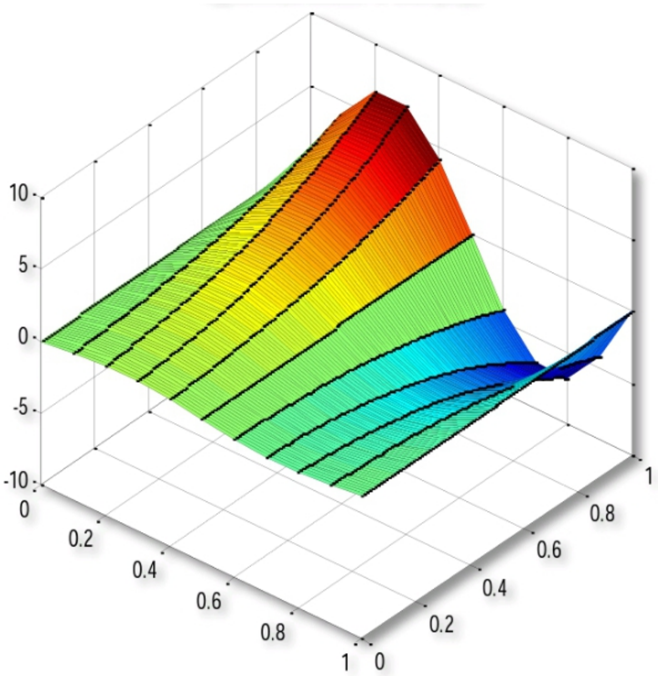

Figure 3.2: Numerical solution for 2D wave equation at $t = 0.01$ and $h = -1$

CHAPTER 4

RESULT AND DISCUSSION

4.1 Application

In this section, we give some computational results of numerical experiments with methods for solving Ablowitz-kaup-newell-segur equations.

4.1.1 Problem (Ablowitz-Kaup-Newell-Segur Equations)

Consider Ablowitz-Kaup-Newell-Segur equations (Ablowitz et al., 1973)

$$\begin{cases} 4u_{xt} + u_{xxxt} + 8u_{xy}u_x + 4u_y u_{xx} = 0, \\ u(x, y, 0) = cosh(x + y), \end{cases} \tag{4.1}$$

where $u = u(x, y, t)$.

To solve equation (4.1) by means of the homotopy analysis method let us consider the linear operator

$$L[\phi(x, y, t; q)] = \frac{\partial \phi(x, y, t; q)}{\partial t},$$

with the property that

$$L(c_1) = 0,$$

which implies that

$$L^{-1}(.) = \int_0^t (.)dt.$$

We now define the nonlinear operator as

$$\begin{aligned} N[\phi(x, y, t; q)] =& 4\frac{\partial^2 \phi(x, y, t; q)}{\partial x \partial t} + \frac{\partial^4 \phi(x, y, t; q)}{\partial x^3 \partial t} + 8\frac{\partial^2 \phi(x, y, t; q)}{\partial x \partial y}\frac{\partial \phi(x, y, t; q)}{\partial x} \\ &+ 4\frac{\partial \phi(x, y, t; q)}{\partial y}\frac{\partial^2 \phi(x, y, t; q)}{\partial x^2}. \end{aligned} \tag{4.2}$$

Construct the zeroth-order deformation equation by

$$(1-q)L[\phi(x,y,t;q)-u_0(x,y,t)] = qhH(x,t)N[\phi(x,y,t;q)], \tag{4.3}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,y,t)$ is an unknown function, $u_0(x,y,t)$ is an initial guess of $u(x,y,t)$ and $H(x,y,t)$ denotes a nonzero auxiliary function. When the embedding parameter $q=0$ and $q=1$, equation (4.3) becomes

$$\phi(x,y,t;0) = u_0(x,y,t), \quad \phi(x,y,t;1) = u(x,y,t). \tag{4.4}$$

We choose auxiliary function $H(x,y,t)=1$, then we obtain the m-th order deformation equation

$$\begin{aligned} L[u_m(x,y,t)-&\chi_m u_{m-1}(x,y,t)] = h\Re_m(u_{m-1}), \\ &u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + hL^{-1}(\Re_m(u_{m-1}), \end{aligned}$$

where

$$\Re_m(u_{m-1}) = 4\frac{\partial^2 u_{m-1}}{\partial x\partial t} + \frac{\partial^4 u_{m-1}}{\partial x^3\partial t} + 8\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x\partial y}\frac{\partial u_{m-1-i}}{\partial x} + 4\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial y}\frac{\partial^2 u_{m-1-i}}{\partial x^2}.$$

Then

$$\begin{aligned} u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + hl^{-1}(4\frac{\partial^2 u_{m-1}}{\partial x\partial t} + \frac{\partial^4 u_{m-1}}{\partial x^3\partial t} + 8\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x\partial y}\frac{\partial u_{m-1-i}}{\partial x} \\ + 4\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial y}\frac{\partial^2 u_{m-1-i}}{\partial x^2}), \end{aligned}$$

$$\begin{aligned} u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + h\int_0^t(4\frac{\partial^2 u_{m-1}}{\partial x\partial t} + \frac{\partial^4 u_{m-1}}{\partial x^3\partial t} + 8\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x\partial y}\frac{\partial u_{m-1-i}}{\partial x} \\ + 4\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial y}\frac{\partial^2 u_{m-1-i}}{\partial x^2})dt, \end{aligned}$$

$$\begin{aligned} u_m(x,y,t) =& \chi_m u_{m-1}(x,y,t) + h\int_0^t 4\frac{\partial^2 u_{m-1}}{\partial x\partial t} + \frac{\partial^4 u_{m-1}}{\partial x^3\partial t}dt + h\int_0^t(8\sum_{i=0}^{m-1}\frac{\partial^2 u_i}{\partial x\partial y}\frac{\partial u_{m-1-i}}{\partial x} \\ &+ 4\sum_{i=0}^{m-1}\frac{\partial u_i}{\partial y}\frac{\partial^2 u_{m-1-i}}{\partial x^2})dt, \end{aligned}$$

since

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

and $m = 0, 1, 2, 3, ...$

For solving equation (4.1) by using HAM, we choose the initial approximation(guess) from initial condition

$$u_0(x, y, t) = cosh(x + y)$$

$$\begin{aligned}
u_0(x,y,t) &= cosh(x+y) \\
u_1(x,y,t) &= \chi_1 u_0(x,y,t) + h\int_0^t \left(4\frac{\partial^2 u_0}{\partial x \partial t} + \frac{\partial^4 u_0}{\partial x^3 \partial t} + 8\sum_{i=0}^{0} \frac{\partial^2 u_0}{\partial x \partial y}\frac{\partial u_0}{\partial x} + 4\sum_{i=0}^{0}\frac{\partial u_0}{\partial y}\frac{\partial^2 u_0}{\partial x^2}\right) dt \\
&= \chi_1 u_0(x,y,t) + h\int_0^t \left(4\frac{\partial^2 u_0}{\partial x \partial t} + \frac{\partial^4 u_0}{\partial x^3 \partial t} + 8\frac{\partial^2 u_0}{\partial x \partial y}\frac{\partial u_0}{\partial x} + 4\frac{\partial u_0}{\partial y}\frac{\partial^2 u_0}{\partial x^2}\right) dt \\
&= 0.cosh(x+y) + h\int_0^t \left(4\frac{\partial^2 cosh(x+y)}{\partial x \partial t} + \frac{\partial^4 cosh(x+y)}{\partial x^3 \partial t}\right) dt \\
&+ h\int_0^t \left(8\frac{\partial^2 cosh(x+y)}{\partial x \partial y}\frac{\partial cosh(x+y)}{\partial x} + 4h\frac{\partial cosh(x+y)}{\partial y}\frac{\partial^2 cosh(x+y)}{\partial x^2}\right) dt \\
&= 0 + 0 + 8h\int_0^t \left(\frac{\partial^2 cosh(x+y)}{\partial x \partial y}\frac{\partial cosh(x+y)}{\partial x}\right) dt \\
&+ 4h\int_0^t \left(\frac{\partial cosh(x+y)}{\partial y}\frac{\partial^2 cosh(x+y)}{\partial x^2}\right) dt \\
&= 8h\int_0^t \frac{\partial^2 cosh(x+y)}{\partial x \partial y}\frac{\partial cosh(x+y)}{\partial x} dt \\
&+ 4h\int_0^t \frac{\partial cosh(x+y)}{\partial y}\frac{\partial^2 cosh(x+y)}{\partial x^2} dt \\
&= 8h\int_0^t cosh(x+y)sinh(x+y)dt \\
&+ 4h\int_0^t sinh(x+y)cosh(x+y)dt \\
&= 12h\int_0^t cosh(x+y)sinh(x+y)dt \\
&= 12htcosh(x+y)sinh(x+y)
\end{aligned}$$

$$\begin{aligned}
u_1(x,y,t) &= 12htcosh(x+y)sinh(x+y) \\
&= 6htsinh2(x+y)
\end{aligned}$$

$$u_2(x,y,t) = \chi_2 u_1(x,y,t) + h\int_0^t \Big(4\frac{\partial^2 u_1}{\partial x \partial t} + \frac{\partial^4 u_1}{\partial x^3 \partial t}$$
$$+ 8\sum_{i=0}^{1} \frac{\partial^2 u_i}{\partial x \partial y}\frac{\partial u_{1-i}}{\partial x} + 4\sum_{i=0}^{1} \frac{\partial u_i}{\partial y}\frac{\partial^2 u_{1-i}}{\partial x^2}\Big)dt$$
$$u_2(x,y,t) = \chi_2 u_1(x,y,t) + h\int_0^t (4\frac{\partial^2 u_1}{\partial x \partial t} + \frac{\partial^4 u_1}{\partial x^3 \partial t})dt$$
$$+ h\int_0^t (8\sum_{i=0}^{1} \frac{\partial^2 u_i}{\partial x \partial y}\frac{\partial u_{1-i}}{\partial x} + 4\sum_{i=0}^{1} \frac{\partial u_i}{\partial y}\frac{\partial^2 u_{1-i}}{\partial x^2})dt$$
$$= \chi_2 u_1(x,y,t) + \int_0^t 4\frac{\partial^2 u_1}{\partial x \partial t} + \frac{\partial^4 u_1}{\partial x^3 \partial t}dt + 8\int_0^t (\frac{\partial^2 u_0}{\partial x \partial y}\frac{\partial u_1}{\partial x} + \frac{\partial^2 u_1}{\partial x \partial y}\frac{\partial u_0}{\partial x})dt$$
$$+ 4\int_0^t (\frac{\partial u_0}{\partial y}\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial y}\frac{\partial^2 u_0}{\partial x^2})dt$$
$$u_2(x,y,t) = 6htsinh2(x+y) + h\int_0^t \left(4\frac{\partial^2(6htsinh2(x+y))}{\partial x \partial t} + \frac{\partial^4 6htsinh2(x+y)}{\partial x^3 \partial t}\right)dt$$
$$+ 8h\int_0^t \left(\frac{\partial^2(cosh(x+y))}{\partial x \partial y}.\frac{\partial(6htsinh2(x+y))}{\partial x}\right)dt$$
$$+ 8h\int_0^t \left(\frac{\partial^2(6htsinh2(x+y))}{\partial x \partial y}.\frac{\partial cosh(x+y)}{\partial x}\right)dt$$
$$+ 4h\int_0^t \left(\frac{\partial^2(cosh(x+y))}{\partial x^2}.\frac{\partial(6htsinh2(x+y))}{\partial y}\right)dt$$
$$+ 4h\int_0^t \left(\frac{\partial^2(6htsinh2(x+y))}{\partial x^2}.\frac{\partial(cosh(x+y))}{\partial y}\right)dt$$
$$u_2(x,y,t) = 6htsinh2(x+y) + h\left(\int_0^t 4\frac{\partial^2(6htsinh2(x+y))}{\partial x \partial t}dt + \int_0^t \frac{\partial^4 6htsinh2(x+y)}{\partial x^3 \partial t}dt\right)$$
$$+ 8h\Big(\int_0^t \left(\frac{\partial^2(cosh(x+y))}{\partial x \partial y}.\frac{\partial(6htsinh2(x+y))}{\partial x}\right)dt$$
$$+ 8h\int_0^t \left(\frac{\partial^2 6htsinh2(x+y)}{\partial x \partial y}.\frac{\partial cosh(x+y)}{\partial x}\right)dt\Big)$$
$$+ 4h\Big(\int_0^t \left(\frac{\partial^2(cosh(x+y))}{\partial x^2}.\frac{\partial 6htsinh2(x+y)}{\partial y}\right)dt$$
$$+ 4h\int_0^t \left(\frac{\partial^2 6htsinh2(x+y)}{\partial x^2}.\frac{\partial(cosh(x+y))}{\partial y}\right)dt\Big)$$
$$u_2(x,y,t) = 6htsinh2(x+y) + h\int_0^1 (96hcosh2(x+y))$$
$$+ 8h\left(\int_0^t 12htcosh(x+y)cosh2(x+y)dt\right)$$
$$+ 8h\left(\int_0^t 24htsinh2(x+y)sinh(x+y)dt\right)$$
$$+ 4h\left(\int_0^t 12htcosh(x+y)cosh2(x+y)dt\right)$$
$$+ 4h\left(\int_0^t 24htsin2h(x+y)sinh(x+y)dt\right)$$

$$
\begin{aligned}
u_2(x,y,t) &= 6htcosh2(x+y) + h\int_0^1 (96hcosh2(x+y)) \\
&+ 12h\left(\int_0^t 12htcosh(x+y)cosh2(x+y)dt\right) \\
&+ 12h\left(\int_0^t 24htsin2(x+y)sinh(x+y)dt\right) \\
u_2(x,y,t) &= 6htcosh2(x+y) + 96h^2tcosh2(x+y) + 12h(6ht^2cosh(x+y)sinh2(x+y)) \\
&+ 12h(12ht^2cosh2(x+y)sinh(x+y)) \\
u_2(x,y,t) &= 6htcosh2(x+y) + 96h^2tcosh2(x+y) + 72h^2t^2cosh(x+y)cosh2(x+y) \\
&+ 144h^2t^2sinh2(x+y)sinh(x+y))
\end{aligned}
$$

Therefore

$$
\begin{aligned}
u_2(x,y,t) &= 6htcosh2(x+y) + 96h^2tcosh2(x+y) + 72h^2t^2cosh(x+y)cosh2(x+y) \\
&+ 144h^2t^2sinh2(x+y)sinh(x+y)
\end{aligned}
$$

So the components are

$$
\begin{aligned}
u_0(x,y,t) &= cosh(x+y), \\
u_1(x,y,t) &= 6htsinh2(x+y), \\
u_2(x,y,t) &= 6htcosh2(x+y) + 96h^2tcosh2(x+y) + 72h^2t^2cosh(x+y)cosh2(x+y) \\
&+ 144h^2t^2sinh2(x+y)sinh(x+y), \\
&. \qquad\qquad\qquad .\\
&. \qquad\qquad\qquad .\\
&. \qquad\qquad\qquad .
\end{aligned}
$$

where c is positive constant.
From homotopy series, the approximate analytical solutions of given nonlinear equation is given as

$$
u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,t,y).
$$

Therefore, at $h = -1$ the 3rd iteration HAM approximate analytical solution is given by

$$
\begin{aligned}
u(x,y,t) &= cosh(x+y) + 6htsinh2(x+y) + 6htcosh2(x+y) + 96h^2tcosh2(x+y) \\
&+ 72h^2t^2cosh(x+y)cosh2(x+y) + 144h^2t^2sinh2(x+y)sinh(x+y)
\end{aligned}
$$

$$u(x,y,t) = cosh(x+y) + 12htsinh2(x+y) + 96h^2tcosh2(x+y) \\ + 72h^2t^2cosh(x+y)cosh2(x+y) + 144h^2t^2sinh2(x+y)sinh(x+y)$$

$$u(x,y,t) = cosh(x+y) - 12tsinh2(x+y) + 96tcosh2(x+y) \\ + 72t^2cosh(x+y)cosh2(x+y) + 144t^2sinh2(x+y)sinh(x+y)$$

Comparison of HAM and Numerical solution of AKNS Equation

Table 4.1: Approximate value and numerical solution for AKNS equation

t	x	y	HAM Solution	Numerical Solution	Error=$\lvert HAM_{sol} - Num_{sol}\rvert$
0.01	0	0	1.9672	1.9672	0
0.01	0.1	0.1	2.0177	2.0165	0.0012
0.01	0.2	0.2	2.2741	2.2688	0.0053
0.01	0.3	0.3	2.7719	2.7580	0.0139
0.01	0.4	0.4	3.5819	3.5516	0.0303
0.01	0.5	0.5	4.8227	4.7614	0.0613
0.01	0.6	0.6	6.6806	6.5618	0.1188
0.01	0.7	0.7	9.4429	9.2183	0.2246
0.01	0.8	0.8	13.5501	13.1312	0.4189
0.01	0.9	0.9	19.6779	18.9032	0.7747
0.01	1	1	28.8683	27.4430	1.4253

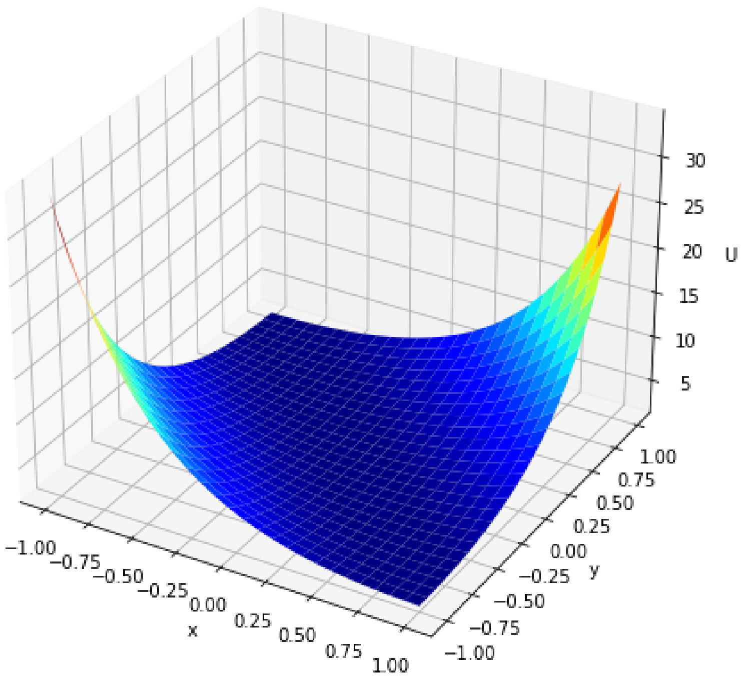

Figure 4.1: HAM solution for AKNS equation at $t = 0.01$ and $h = -1$

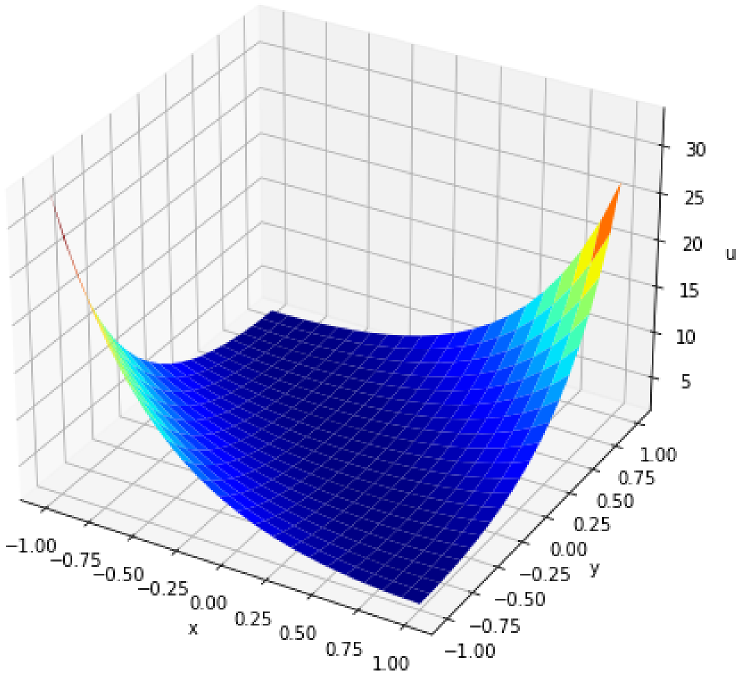

Figure 4.2: Numerical solution for AKNS equation at $t = 0.01$ and $h = -1$

4.2 Discussion

In this thesis we used homotopy analysis method to solve AKNS equation. First, by choosing initial approximation (u_0), linear operator (L), auxiliary function $(H(x, y, t) \neq 0)$ and convergence control parameter $(h \neq 0)$ obtain HAM approximate analytical solution of AKNS equations. Then, by using finite difference method (FDM) obtain approximate numerical solution graphically. After that, the comparison between HAM and the FDM solutions was made for both equations and it was found that the error between them is small. Generally, the approximation of HAM is in excellent agreement with the numerical (FDM) solutions. Therefore, based on these present results, we can say that HAM is effective and powerful method, at least for this particular problem.

CONCLUSION AND RECOMMENDATION

Conclusion

In this thesis, the homotopy analysis method has been successfully applied to obtain the solution of Ablowitz-Kaup-Newell-Segur equation. The solution obtained by the homotopy analysis method is an infinite power series for appropriate initial conditions. HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. In comparison with HAM and FDM method solutions we find the better approximations. The results show that the homotopy analysis method is a powerful Mathematical tool for solving Ablowitz-Kaup-Newell-Segur equations. This paper also illustrated the validity and the great potential of the HAM for approximate analytical solution of nonlinear problems in science and engineering. It is also worth mentioning to this end that the advantage of this method is the fast convergent of the solutions by means of the auxiliary parameter. Python has been used for computations in this paper.

Recommendation

This thesis specifically focused on application of homotopy analysis method to solve approximate analytic solution of Ablowitz-Kaup-Newell-Segur equations. Homotopy analysis method is very interesting and easy to find approximate analytical solution on nonlinear partial differential equations. Hence, we recommended that, future researchers should try to compare homotopy analysis method with other method to generalize the efficiency, validity of this method. Also one can extend the homotopy analysis method to nonlinear fifth order PDE, (3+1)-dimensional PDE and etc.

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