Enhancing Bipartite Entanglement of Opto-electromechanical System Assisted by Three Level Laser

Abebe Senbeto Kussia

A Thesis Submitted to Department of Applied Physics School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Physics (Quantum Optics)

Office of Graduate Studies

Adama Science and Technology University

June, 2023
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Declaration

I hereby declare that this Master Thesis entitled "Enhancing Bipartite Entanglement of Opto-electromechanical System Assisted by Three Level Laser" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation

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I, the advisor of this thesis, hereby certify that I have read the revised version of the thesis entitled "Enhancing Bipartite Entanglement of Opto-electromechanical System Assisted by Three Level Laser" prepared under my guidance by Abebe Senbeto submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Quantum Optics. Therefore, I recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-Advisor	Signature	Date

Approval Page of M.Sc. Thesis

I, the advisors of the thesis entitled "Enhancing Bipartite Entanglement of Opto-electromechanical System Assisted by Three Level Laser" and developed by Abebe Senbeto, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	**Signature**	**Date**
Co-Advisor	**Signature**	**Date**

We, the undersigned, members of the Board of Examiners of the thesis by Abebe Senbeto have read and evaluated the thesis entitled" Enhancing Bipartite Entanglement of Opto-electromechanical System Assisted by Three Level Laser" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Physics (Quantum Optics)

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Finally, I must express my very profound gratitude to my family for providing me with unfailing support and continuous encouragement through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

Contents

List of Abbreviations

MR	Mechanical Resonator
MC	Microwave Cavity
OC	Optical Cavity
QLEs	Quantum Langevin Equations
OEMS	Opto-electromechanical System
NOEMS	Nano-Opto-Electro-Mechanical Systems

List of Figures

Abstract

In this research, the theoretically bipartite entanglement dynamics in an opto-electromechanical system with a degenerate three-level laser were studied. The system consists of a microwave cavity, optical cavity, mechanical resonator, and a degenerate three-level atom within the optomechanical arrangement. To this aim, we constructed the model Hamiltonian for the system. The system's dynamics have been determined using the nonlinear quantum-Langevin equations. By applying the linearization approximation, the bipartite entanglement can be assessed through the logarithmic negativity. The results showed that the entanglement in an opto-electromechanical system is enhanced by degenerate three-level atoms compared to a standard opto-electromechanical system. Interestingly, we observed that as the coupling rate g *is increased, the entanglement between optical cavity-three level laser against normalized optical cavity detuning also increases. In addition our study suggested that the entanglement between each subsystem increases as the atom injection rate increases because the larger number of atoms injection rate leads to a more laser emission. Moreover, we showed that as temperature increases, particles in the environment have more thermal energy and are more likely to interact with the particles in the system. This increased interaction leads to a decrease in entanglement between the subsystems, as their states become more independent due to the influence of the environment.*

1

INTRODUCTION

1.1. Background of the Study

For the last few decade, there has been outstanding growth in the field of quantum optics, which deals with the interactions between matter and the radiation field where quantum effects are very important (Aspelmeyer et al. (2016)). Such interaction is caused by the mechanical effect of light, also known as optical force. In the field of quantum optics and information optomechanics is an emerging field exploring the interaction between light and mechanical motion (Liu et al. (2013)). This coupling of mechanical and optical degrees of freedom via radiation pressure has been attracted a great attention of researchers due to its applications in both theoretical and experimental such as entangling microscopic and macroscopic system (De Martini et al. (2008)), generation of bipartite entanglement (Tesfahannes (2020)), Gaussian states correlations (Adesso and Datta (2010)), performing high-precision measurements (Mueller et al. (2001)), and processing quantum information (Jaeger (2007)). Micro and nanomechanical resonators can be efficiently coupled with a wide range of different devices, which makes them a key tool for the realization of quantum interfaces that are able to store and redistributing quantum information (Johansson et al. (2014)).

Cavity quantum optomechanics combines non-classical states of light and mechanical motion to manipulate and detect motion in the quantum regime. These devices are the basis for applications in quantum information processing, where they could serve as coherent light-matter interfaces, for instance, to interconvert information from solid-state qubits into flying photonic qubits. Another illustration is the capability to create hybrid quantum systems that combine otherwise incompatible degrees of freedom of different physical systems (Aspelmeyer et al. (2014)). The combination of optomechanical and electromechanical systems brings together the advantages of both technologies, enabling the conversion of quantum information between photons and solid-state qubits. This conversion process, also known as quantum transduction, is necessary for the development of distributed quantum networks and long-distance quantum communication. Most recently, the coupling of opto-electromechanical systems offer considerable attention for their potential opportunities in both theoretical and experimental (Chu and Gröblacher (2020)). Electromechanical systems, in which a mechanical resonator (MR) can be coupled capacitively or inductively with an electric circuit, or optomechanical systems, in which either radiation pressure or gradient force allows strong coupling with an optical cavity mode, are popular examples (Barzanjeh et al. (2011)). Nano-optoelectromechanical systems (NOEMS), are new platforms for studying electronic and mechanical freedoms in the field of nano photonics. NOEMS provide exciting opportunities to manipulate information carriers using optical, electrical, and mechanical degrees of freedom, where light flow, electron dynamics, and mechanical vibration modes can be explored in both classical and quantum domains (Xu et al. (2021)).

One of the main milestones in the study of opto and electromechanical systems is to certify entanglement between a mechanical resonator and an optical or microwave mode of a cavity field (Mari and Eisert (2012)). On the other hand optomechanical systems interacts with two or three level laser leads to hybrid system. A three-level laser is a quantum optical system in which light is generated by three-level atoms within a cavity, which is typically coupled to a vacuum reservoir via a single-port mirror (Tesfa (2006)). The statistical and squeezing properties of the light generated by such a three level laser have been investigated by several authors (Darge and Kassahun (2010)Alemu (2020)).

The squeezing properties in such lasers is caused by atomic coherence (Huang et al. (2008)), which can be introduced by either initially preparing the three-level atoms in a coherent superposition of the top and bottom levels or coupling these levels with external coherent light. When a three-level atom makes a transition from the top to bottom level through the intermediate level, two highly correlated photons are generated. If these photons are identical, the laser is referred to as a degenerate three-level laser, otherwise it is a non-degenerate three-level laser (Biyazn (2019)). Quantum entanglement is a phenomenon in quantum mechanics where two or more particles become interconnected in such a way that the state of one particle cannot be described independently of the other(s), even when the particles are separated by a large distance. This interconnection means that the properties of one particle are dependent on the properties of the other, and a change in one particle's state will result in a corresponding change in the other particle's state instantly, no matter how far apart they are (Vedral (2014)). One of the primary goals of quantum information science is to create, manipulate, and maintain quantum entanglement in various physical systems. In recent years, opto-electromechanical systems have emerged as potential candidates for generating and studying quantum entanglement. These systems typically consist of an optical cavity, an electron beam, and a micromechanical resonator, and have been used to investigate a wide range of quantum phenomena, including quantum state transfer, optomechanical entanglement, and quantum teleportation (Cai et al. (2019)).

1.2. Statement of the Problem

Entangling hybrid quantum systems has indeed become an exciting area of research in recent years. Currently, significant progress has been reported on the study of the bipartite entanglement between subsystems of a hybrid optomechanical system. Bipartite entanglement is a special type of quantum entanglement involving two subsystems, where the state of one subsystem is directly related to the state of the other. Accordingly enhancing the bipartite entanglement of quantum system is crucial for improving the performance of quantum communication and information processing tasks. However, there are various challenges associated with the generation, manipulation, and preservation of bipartite entanglement, such as decoherence and loss, which can severely limit the entanglement quality and its usefulness in practical applications. In order to overcome these challenges, different techniques have been proposed to enhance bipartite entanglement in hybrid optomechanical systems. One such approach involves using a three-level laser as an external control to manipulate the entanglement between the optical cavity and the mechanical resonator. The three-level laser system offers unique advantages, since it allows for a more precise and efficient control of the optomechanical interactions, and can potentially lead to a significant enhancement in the quality of the entanglement. Therefore, in this thesis the bipartite entanglement dynamics in an opto-electromechanical system assisted by three level laser will be studied. Particularly, we study the effect of three level laser on entanglement generated between any subsystems of opto-electromechanical system.

1.3. Objective of the study

1.3.1 General Objective

The general objective of this thesis is to enhance bipartite entanglement of opto-electromechanical system assisted by three level laser.

1.3.2 Specific Objectives

The specific objectives of the thesis are:

- to design the model and formulate Hamiltonian for the system.
- to obtain the quantum Langevin equations of the system.
- to quantify bipartite entanglement between OC-MC,OC-MR and MC-MR subsystems in the absence of three level laser and compared to the bipartite entanglement between OC-MC,OC-MR and MC-MR in the presence of three level laser.

1.4. Significance of the study

The significance of the study on generating enhanced bipartite entanglement of opto-electromechanical system lies in its potential applications and benefits in various fields such as quantum communication and quantum computing. This research will contribute significantly to the understanding and advancement of quantum optics and how it can be harnessed to improve quantum systems. The ability to create and maintain a high degree of entanglement can lead to secure transmission of information via quantum channels. In addition to the benefits in quantum communication, this study has significant implications for the development of quantum computing. Entangled quantum states form the core of many quantum algorithms, and the ability to efficiently generate and manipulate entangled states is a key aspect of scaling up quantum computers from experimental prototypes to practical devices. Enhanced bipartite entanglement in OEMS may enable advanced control, storage, and conversion of quantum information among different physical carriers, making these systems more powerful for scalable and practical quantum technologies. The insights gained from this research will help guide the design of future experimental setups and devices that rely on bipartite entanglement, fostering innovation and growth in quantum systems.

1.5. Outline of the Thesis

In general, the thesis organized in five chapters. The first chapter describes an overview of the thesis background, statement of the problem, objective of the thesis and the significance of the study. Chapter two focuses on the review from previous studies that was made helpful to gain knowledge and to understand the thesis better and discusses about the hybrid optomechanical systems, radiation pressure force and optomechanical coupling, degenerate three-level lasers in a cavity, basic tools and techniques of optomechanics and entanglement quantification. Chapter three gives detail description about the model Hamiltonian and dynamics of the system. Chapter four presents result and discussion. Finally, chapter five provides conclusion and recommendations of the thesis.

2

REVIEW LITERATURE

2.1. Overview

In this paper, we utilize the brief and basic introduction to the theory used to describe the hybrid optomechanical systems. Particularly in this chapter we describe the over view of hybrid optomechanical systems such as opto-electromechanical system, where radiation pressure plays a vital role in coupling electrical and mechanical systems through optical means, degenerate three-level lasers in a cavity, which exploit the interaction between a three-level laser system and an optical cavity for enhanced control. In addition we explain radiation pressure force and Optomechanical Coupling, which studies the effects of radiation pressure force in the coupling between mechanical and optical systems. Moreover, we describe the Hamiltonian formulation, Heisenberg-Langevin equation, Hamiltonian linearization and logarithmic negativity approach as the basic tools and techniques to analyse entanglement optomechanical system.

2.2. Hybrid Optomechanical Systems

In order to investigate new phenomena, a hybrid quantum system combines two or more physical components or subsystems. The combination of optomechanics with mechanical elements, like nano-mechanical membranes leads to hybrid optomechanics to study coherent dynamics in microscopic and macroscopic domains, quantum transfer protocol or exploiting entanglement. A large number of schemes have been recently proposed for entangling hybrid systems involving a MR (Mari and Eisert (2012)).

2.2.1 Opto-Electromechanical System

Opto-electromechanical system Fig. (2.1) is a hybrid optomechanical system in witch on the one side a MR is capacitively coupled to the field of a superconducting microwave cavity (MC) of resonant frequency ω_m and, on the other side, coupled to a driven optical cavity (OC) with resonant frequency ω_c. This scheme offers an alternative platform for controlling the optical signal via an electric field leads to a variety of applications such as convenient control of optomechanical devices single-photon transport, investigation quantum entanglement, quantum ground-state cooling, quantum coherent coupling between light and mechanical resonator, squeezing optomechanical memory and telecommunication (Sohail et al. (2020)). The mechanical oscillator in the opto-electromechanical system (OEMS) acts as a quantum or classical interface between the optical and microwave modes. Theoretically, it is possible to realize the electric-controlled optomechanically induced transparency, optical non-linearity and quantum state transfer between an optical and microwave modes based on this phonon-interface in the hybrid OEMS (Wang et al. (2016)).

A microwave cavity is essentially a resonant structure designed to contain and manipulate electromagnetic fields at specific frequencies. In essence, it is a generalization of the basic LC (inductor-capacitor) circuit to the microwave frequency region. To understand how the Hamiltonian of a microwave cavity is derived from an LC circuit, we first need to understand the Hamiltonian of an LC circuit and then extend the concepts to the microwave cavity.

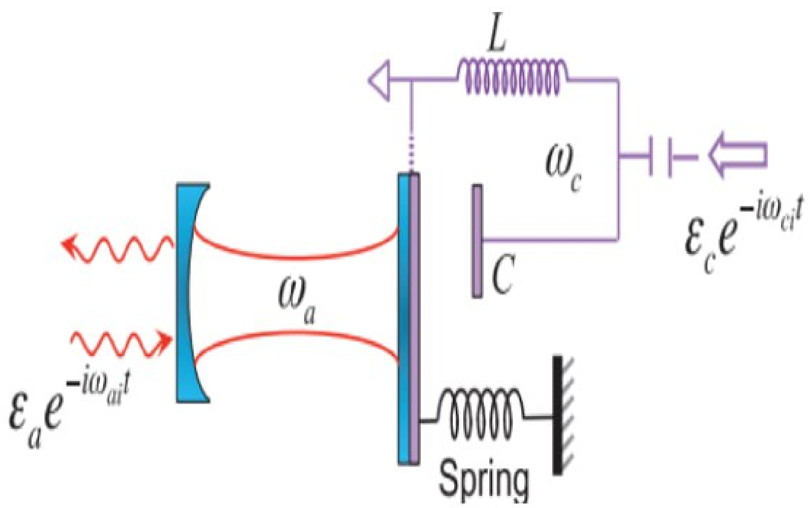

Figure 2.1: Schematic description of hybrid opto-electro-mechanical system (Wang et al. (2016)).

Hamiltonian is a function representing the total energy of a system, including both kinetic and potential energy components. For the LC circuit, the energy is stored in the electric and magnetic fields of the capacitor and inductor, respectively. Thus the Hamiltonian of LC circuit can be expressed as (Clemente-Gallardo and Scherpen (2003)):

$$H = \frac{1}{2}LI^2 + \frac{Q^2}{2C}. \tag{2.1}$$

From the fact that $I = \frac{\Phi}{L}$, we have

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C}, \tag{2.2}$$

where Φ is the flux through an equivalent inductor L and I is current. Due to the mechanical oscillation the energy stored in a capacitor is position $C_o(x)$ dependent as

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}2}{2[C + C_o(x)]}. \tag{2.3}$$

The MR is coupled to the microwave cavity because the capacity of the latter is a function of the resonator displacement, C_o. We expand this function around the equilibrium position of the resonator corresponding to a separation d between the plates of the capacitor, with corresponding bare capacitance, C_o. $C_o(x) = C_o[\frac{1+x(t)}{d}]$, expanding the capacitive energy as a Taylor series, we find to first order

$$\frac{\hat{Q}^2}{2[C + C_o(x)]} = \frac{\hat{Q}^2}{2C_\Sigma} - \frac{\mu}{2dC_\Sigma}x(t)\hat{Q}^2, \tag{2.4}$$

where $C_\Sigma = C + C_o$ and $\mu = \frac{C_o}{C_\Sigma}$ (Vitali et al. (2007)). The Hamiltonian of Eq. (2.3) becomes

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C_\Sigma} - \frac{\mu}{2dC_\Sigma}x(t)\hat{Q}^2. \tag{2.5}$$

The first two terms term of Eq. (2.5) are free Hamiltonian of microwave cavity and the third term describes electromechanical interaction Hamiltonian. Now let's calculate free Hamiltonian as

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{1}{2}L\omega_w^2\hat{Q}^2 = \frac{1}{2L}\left[L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2\right], \tag{2.6}$$

where $\omega_w = \frac{1}{\sqrt{LC_\Sigma}}$. Now, let by trick

$$
\begin{aligned}
(L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi}) &= L^2\omega_w^2\hat{Q}^2 + iL\omega_w\hat{Q}\hat{\Phi} - iL\omega_w\hat{\Phi}\hat{Q} + \hat{\Phi}^2 \\
&= L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2 + iL\omega_w(\hat{Q}\hat{\Phi} - \hat{\Phi}\hat{Q}), \\
&= L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2 + iL\omega_w[\hat{Q}, \hat{\Phi}], \\
&= L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2 + iL\omega_w(i\hbar), \\
&= L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2 - \hbar L\omega_w.
\end{aligned}
$$

$$
L^2\omega_w^2\hat{Q}^2 + \hat{\Phi}^2 = (L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi}) + \hbar L\omega_w. \tag{2.7}
$$

Inserting Eq. (2.7) into Eq. (2.6) we get

$$
\begin{aligned}
\hat{H} &= \frac{1}{2L}\left[(L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi}) + \hbar L\omega_w\right], \\
&= \frac{1}{2L}\left[(L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi})\right] + \frac{\hbar\omega_w}{2}, \\
&= \frac{\hbar\omega_w}{\hbar\omega_w}\frac{1}{2L}\left[(L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi})\right] + \frac{\hbar\omega_w}{2}, \\
&= \frac{\hbar\omega_w}{2L\hbar\omega_w}\left[(L\omega_w\hat{Q} - i\hat{\Phi})(L\omega_w\hat{Q} + i\hat{\Phi})\right] + \frac{\hbar\omega_w}{2}, \\
&= \hbar\omega_w\left[(\frac{L\omega_w\hat{Q} - i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}})(\frac{L\omega_w\hat{Q} + i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}})\right] + \frac{\hbar\omega_w}{2}.
\end{aligned}
$$

Therefore,

$$
\hat{H} = \hbar\omega_w\left[(\frac{L\omega_w\hat{Q} - i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}})(\frac{L\omega_w\hat{Q} + i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}}) + \frac{1}{2}\right]. \tag{2.8}
$$

From Eq. (2.8), we define creation operator $\hat{b}^\dagger = \left(\frac{L\omega_w\hat{Q} - i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}}\right)$ and $\hat{b} = \left(\frac{L\omega_w\hat{Q} + i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}}\right)$ is annihilation operator satisfying the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{b}, \hat{b}] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0$. Therefore, the harmonic Hamiltonian of microwave field is written as

$$
\hat{H} = \hbar\omega_w(\hat{b}^\dagger\hat{b} + \frac{1}{2}) - \frac{\mu}{2dC_\Sigma}x(t)Q^2, \tag{2.9}
$$

$$
\hat{H} = \hbar\omega_w\hat{b}^\dagger\hat{b} + \frac{\hbar\omega_w}{2} - \frac{\hbar}{2}\frac{\mu\omega_w}{2d}\sqrt{\frac{\hbar}{m\omega_m}}q(\hat{b} + \hat{b}^\dagger)^2, \tag{2.10}
$$

where $\hat{Q}^2 = \frac{\hbar}{2L\omega_w}(\hat{b} + \hat{b}^\dagger)^2$ and $x(t) = q\sqrt{\frac{\hbar}{m\omega_m}}$. We have neglected the offset term $\frac{\hbar\omega_w}{2}$, in Eq. (2.10) corresponding to the zero-point energy, since it makes no contribution to the Hamiltonian dynamics, and will not affect our results. Finally the Hamiltonian of Eq. (2.3) becomes

$$\hat{H} = \hbar\omega_w \hat{b}^\dagger \hat{b} - \frac{\hbar G_{ow}}{2} q(\hat{b} + \hat{b}^\dagger)^2, \tag{2.11}$$

where $G_{ow} = \frac{\mu\omega_w}{2d}\sqrt{\frac{\hbar}{m\omega_m}}$ is electromechanical coupling rate.

2.2.2 Degenerate Three-Level Laser in a Cavity

Combination of optomechanical systems with atoms gives rise to hybrid atom-cavity optomechanical systems, which provides the coherent manipulation of light with additional degrees of freedom and has achieved significant progress in the past few years. This emerging subject leads to a variety of applications, including convenient control of optomechanical devices, single-photon transport, and resolving the vacuum fluctuations. Recently, it was demonstrated that the fast and slow light effects can be enhanced and mutually switched in a hybrid atom-cavity optomechanical system, which may pave the way for real-world applications such as optomechanical memory and telecommunication (Liu et al. (2018)).
All of these effects involve the linearized description of optomechanical interactions, which are easily explained by the linearized Heisenberg-Langevin equations. Degenerate three-level cascade atom injected to cavity is one example of the hybrid optomechanical system in witch the cavity mode is interact with three level atom. A three-level laser is a quantum optical system in which light is produced by three-level atoms inside a cavity that is typically

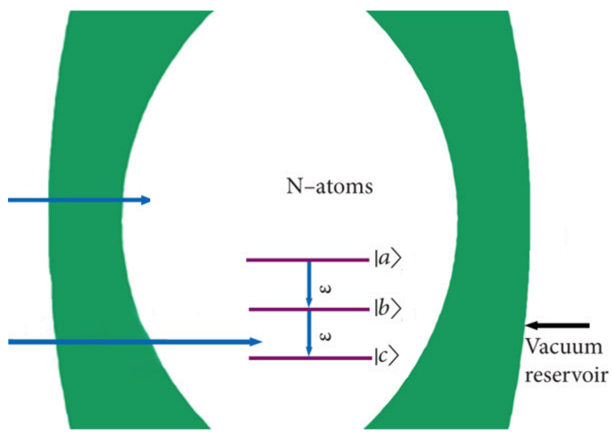

Figure 2.2: Schematic representation of a coherently driven N number of degenerate three-level atom in a cascade configuration with parametric amplifier(Abebe et al. (2020a)).

coupled to a vacuum reservoir (Alebachew and Fesseha (2006)). The injected atoms may be prepared in a coherent superposition of the top and bottom levels initially, and then coupled by strong coherent light after being injected into the cavity.

The interesting non-classical features of the generated light are induced by the superposition or coupling of the top and bottom levels (Darge and Kassahun (2010)). When a three-level atom in cascade configuration makes a transition from the upper to the intermediate level and then from the intermediate to the bottom level, two photons are emitted. If the frequencies of the two photons are the same, the three-level atom is referred to as a degenerate three-level atom; otherwise, it is termed a non-degenerate three-level atom (Abebe et al. (2020a)). Here we consider a degenerate three-level atom available in a closed cavity coupled to a vacuum reservoir and driven by coherent light. The top, middle and bottom levels of the atom denoted by $|a\rangle$, $|b\rangle$ and $|c\rangle$ respectively. The transitions $|a\rangle\rightarrow|b\rangle$ and $|b\rangle\rightarrow|c\rangle$ are assumed to be dipole permitted, while the transition $|a\rangle\rightarrow|c\rangle$ is assumed to be dipole forbidden. Consider the case where the cavity mode is in resonance with both the $|a\rangle\rightarrow|b\rangle$ and $|b\rangle\rightarrow|c\rangle$ transitions . Thus, the interaction of a degenerate three-level atom with a cavity mode can be described by the Hamiltonian

$$H = \hbar\omega_c a^\dagger a + \sum_{i=a,b,c} \hbar E_i \sigma_{ii} + ig(\sigma_{ab} a + a^\dagger \sigma_{ba} + \sigma_{bc} a + a^\dagger \sigma_{cb}), \tag{2.12}$$

where a annihilation operator of cavity field, σ_{ba} and σ_{cb} are atomic annihilation operators (Asili Firouzabadi and Tavassoly (2021)).

2.3. Radiation Pressure Force and Optomechanical Coupling

The opto-mechanical coupling between a moving mirror and the radiation pressure of light has first observed in the context of interferometric gravitational wave experiments. One of the best way to interpret the principle of optomechanical cavities is by using the concept of radiation pressure. According to the quantum theory of light, every photon with wave number k carries a momentum $p = \hbar k$, where $\hbar$ is the Planck constant . This means that a single photon reflected off a mirror surface transfers a momentum $\Delta p = 2\hbar k$ onto the mirror due to the conservation of momentum (Kippenberg and Vahala (2007)).

This is a very small effect that cannot be seen on most everyday objects. When the mass

of the mirror is very small and/or the number of photons is very large, it becomes more significant (i.e. high intensity of the light). Because photon momentum is extremely low and insufficient to significantly change the position of a suspended mirror, the interaction need to be improved. The possible way to do this is by using engineered optical cavities. If a photon is trapped between two mirrors, one of which is an oscillator and the other is a heavy fixed mirror, it will bounce off the mirrors many times, transferring momentum each time.
The number of times that a photon can transfer its momentum is proportional to the cavity's finesse, and can be increased by using highly reflective mirror surfaces. The photon radiation pressure simply shifts the position of the suspended mirror. When the mirror is moved, the length of the cavity changes, which affects the cavity resonance frequency. As a result, the detuning between the changed cavity and the unchanged laser driving frequency, which determines the light amplitude inside the cavity, is modified (KJ (2008)).

Because the light amplitude, or the number of photons inside the cavity, causes the radiation pressure force and thus the mirror displacement, the loop is closed: the radiation pressure force effectively depends on the mirror position. Another advantage of optical cavities is that the modulation of cavity length via an oscillating mirror can be seen directly in the cavity spectrum. (Marquardt et al. (2008)).
In our discussion the fundamental mechanism that couples the properties of the cavity radiation field to the mechanical motion is the momentum transfer of photons, i.e. radiation pressure. The simplest form of radiation-pressure coupling is the momentum transfer due to reflection that occurs in a Fabry-Perot cavity. For a single photon it transfers the momentum $\Delta P = \frac{2h}{\lambda}$($\lambda$ is the photon wavelength and h is Planck's constant).

2.4. Basic Tools and Techniques of Optomechanics

In this section, we will go over some commonly used concepts and tools in cavity optomechanical systems. In particular, we focus on the Hamiltonian formulation, Heisenberg quantum Langevin equations, the Linearization approximation approach, and then discuss the logarithmic negativity to measure generated entanglement in the optomechanical system, in that order.

2.4.1 Hamiltonian Formulation

In this subsection we consider a simple and commonly used cavity optomechanical system with a single optical cavity mode coupled to a mechanical mode, which is canonically modelled as a Fabry-perot cavity with one fixed mirror and movable mirror mounted on a spring as shown Fig. (2.3).

The Hamiltonian describing the coupled system of a cavity mode interacting with a mechanical mode driven by a single-mode laser field is

$$\hat{H} = \hat{H}_{free} + \hat{H}_I + \hat{H}_{drive}. \tag{2.13}$$

The first term $\hat{H}_{free}$ is the free Hamiltonian of the optical and mechanical resonator with formalism described as (Sonar et al. (2018))

$$\hat{H}_{free} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}^\dagger \hat{b}. \tag{2.14}$$

Here both of the optical and the mechanical modes are represented by quantum harmonic oscillators, where $\hat{a}(\hat{a}^\dagger)$ is the annihilation (creation) operator of the optical cavity mode with its resonance frequency ω_c and $\hat{b}(\hat{b}^\dagger)$ annihilation (creation) operator of the mechanical mode, with ,mechanical mode frequency ω_m. The Bosonic operators $\hat{a}$ and $\hat{b}$ satisfies the commutation relations $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^\dagger] = 0$.

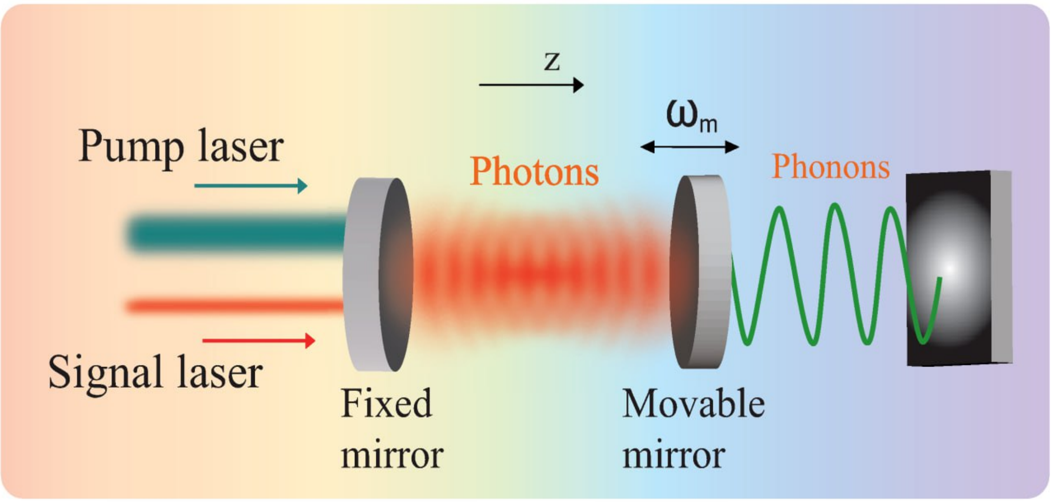

Figure 2.3: Schematic of a generic optomechanical system, both in the optical domain, with a laser-driven optical cavity and a vibrating end mirror (Li et al. (2013)).

The Hamiltonian in Eq. (2.13) can be derived in such away that the radiation pressure of the light field eventually moves the mirror, which changes the cavity length and made displacement dependent cavity resonance frequency.

$$\hat{H} = \hbar\omega_c(x)\hat{a}^\dagger\hat{a} + \hbar\omega_m\hat{b}^\dagger\hat{b} + \hat{H}_{driv}. \tag{2.15}$$

Applying a Taylor expansion on ω_c as :

$$\omega_c(x) = \omega_c(0) + x\frac{\partial\omega_c}{\partial x} + \frac{1}{2}x^2\frac{\partial^2}{\partial x^2} + ..., \tag{2.16}$$

where the quadratic and higher-order coupling terms are usually neglected. This leads to

$$\omega_c(x) = \omega_c(0) + x\frac{\partial\omega_c}{\partial x}. \tag{2.17}$$

We define the optical frequency shift per displacement as $G = -\frac{\partial\omega_c}{\partial x}$, so

$$\hbar\omega_c(x)\hat{a}^\dagger\hat{a} \approx \hbar(\omega_c - G\hat{x})\hat{a}^\dagger\hat{a}, \tag{2.18}$$

where $x = x_{ZPF}(\hat{b} + \hat{b}^\dagger)$. Thus, the interaction part of the Hamiltonian can be written as:

$$\hat{H}_I = -\hbar g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger), \tag{2.19}$$

where $g_0 = Gx_{ZPF}$ is the vacuum optomechanical coupling strength and $x_{ZPF} = \sqrt{\hbar/2m_{eff}\omega_m}$. It quantifies the interaction between a single phonon and a single photon. Under intense driving field applied on the cavity, having energy

$$\hat{H}_{drive} = i\hbar E(\hat{a}^\dagger e^{-i\omega_L t} - \hat{a}e^{i\omega_L t}), \tag{2.20}$$

where $|E| = \sqrt{\frac{2P_k}{\hbar\omega_L}}$ is the driving amplitude, related to the driving power P and optical decay rate κ, the total hamiltonian of an optomechanical system become,

$$\hat{H} = \hbar\omega_c\hat{a}^\dagger\hat{a} + \hbar\omega_m\hat{b}^\dagger\hat{b} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b} + \hat{b}^\dagger) + i\hbar E(\hat{a}^\dagger e^{-i\omega_L t} - \hat{a}e^{i\omega_L t}). \tag{2.21}$$

The annihilation and creation of phonon operators in the above equation can be defined as:

$$\hat{b} = \frac{m\omega_m \hat{x} - i\hat{p}}{\sqrt{2\hbar m\omega_m}}, \tag{2.22a}$$

$$\hat{b}^\dagger = \frac{m\omega_m \hat{x} + i\hat{p}}{\sqrt{2\hbar m\omega_m}}. \tag{2.22b}$$

In order to write phonon operators in terms of dimensionless position and momentum operators

$$\hat{b}^\dagger \hat{b} = (\frac{m\omega_m \hat{x} - i\hat{p}}{\sqrt{2\hbar m\omega_m}})(\frac{m\omega_m \hat{x} + i\hat{p}}{\sqrt{2\hbar m\omega_m}}). \tag{2.23}$$

It then follows that

$$\hat{b}^\dagger \hat{b} = \frac{(m\omega_m \hat{x})^2 + im\omega_m(\hat{x}\hat{p} - \hat{p}\hat{x}) + \hat{p}^2}{2\hbar m\omega_m}. \tag{2.24}$$

From position and momentum commutation relation we do have

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar. \tag{2.25}$$

After inserting Eq. (2.25) into Eq. (2.24) we can get

$$\hat{b}^\dagger \hat{b} = \frac{(m\omega_m \hat{x})^2}{2\hbar} + \frac{\hat{p}^2}{2\hbar m\omega_m} - \frac{1}{2}. \tag{2.26}$$

Now, lets define $\hat{x}$ and $\hat{p}$ as

$$\hat{x} = q\sqrt{\frac{\hbar}{m\omega_m}}, \tag{2.27}$$

and

$$\hat{p} = p\sqrt{\hbar m\omega_m}. \tag{2.28}$$

By inserting Eq. (2.27) and Eq. (2.28) into Eq. (2.26) we get

$$\hat{b}^\dagger \hat{b} = \frac{q^2}{2} + \frac{p^2}{2} - \frac{1}{2}. \tag{2.29}$$

Again the term $\hat{b} + \hat{b}^\dagger$

$$\hat{b} + \hat{b}^\dagger = (\frac{m\omega_m \hat{x} - i\hat{p}}{\sqrt{2\hbar m\omega_m}}) + (\frac{m\omega_m \hat{x} + i\hat{p}}{\sqrt{2\hbar m\omega_m}}) = \frac{2m\omega_m \hat{x}}{\sqrt{2\hbar m\omega_m}}. \tag{2.30}$$

Now, afrer inserting Eq. (2.27) into Eq. (2.30) with $\frac{2}{\sqrt{2}} \approx 1$, we get

$$\hat{b}+\hat{b}^{\dagger}=q. \tag{2.31}$$

Finally, by inserting Eq. (2.29) and Eq. (2.31) into Eq. (2.22) and omitted the offset term $\frac{1}{2}\hbar\omega_m$, we get the Hamiltonian for the cavity and mechanical mode with their interaction

$$\hat{H}=\hbar\omega_c\hat{a}^{\dagger}\hat{a}+\frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2)-\hbar g_0 q\hat{a}^{\dagger}\hat{a}+i\hbar E(\hat{a}^{\dagger}e^{-i\omega_L t}-\hat{a}e^{i\omega_L t}). \tag{2.32}$$

Later, we also mention G_i, which is often-used measure for the effective optomechanical coupling in the linearized regime. It is enhanced compared to g_0 by the amplitude of the photon field (Sonar et al. (2018)).

2.4.2 Heisenberg-Langevin Equation

Heisenberg-Langevin Equation is one of the powerful technique for calculating the dynamics of an optomechanical system's operators. The Heisenberg equation of motion for an arbitrary operator of a given system is found from the Hamiltonian of the system, and has the form

$$\frac{d}{dt}\hat{O}=\frac{1}{i\hbar}[\hat{O},\hat{H}]-\beta\hat{O}+\sqrt{2\beta}\hat{O}_{in}, \tag{2.33}$$

where $\hat{O}$ is an arbitrary operator of the system. Langevin equations provide method to study the dynamics of general quantum systems interacting linearly with their environment (Kassahun (2011)).

2.4.3 Linearization Approximation Approach

The linearized approximation approaches gives us a simple way to understand the dynamics of coupled optomechanical system since it is always possible to exact solve the linear dynamics in principle. The linearized Langevin equations could be obtained using the following linearized approximation description of cavity optomechanics. In the dynamics of the system, the Heisenberg operator has a steady state value plus an additional fluctuation operator with zero mean value. This is done by increasing the drive power. With a strong enough drive, the dynamics of the system can be considered as quantum fluctuations around a classical steady state (Meystre (2013))

$$\hat{a} = \alpha + \delta\hat{a}, \tag{2.34}$$

where α is the mean light field amplitude and $\delta\hat{a}$ denotes the fluctuations. Expanding the photon number $\hat{a}^\dagger\hat{a}$, the term α^2 can be omitted as it leads to a constant radiation pressure force which simply shifts the resonator's equilibrium position. The driving term from the standard Hamiltonian is not part of the linearized Hamiltonian, since it is the source of the classical light amplitude α around which the linearization was executed (Aspelmeyer et al. (2014)).

2.4.4 Logarithmic Negativity Approach

Logarithmic negativity is the most popular tool to measure the degree of the entanglement between two systems of any size. Researchers use this tool to compute the entanglement of mixed states to investigate its distribution in space, to capture quantum correlation, to quantify the continuous-variable Gaussian type bipartite state because it is the only one which can always be explicitly computed, and it is also additive (Liu et al. (2018)). Therefore, we use this tool to studied the three possible bipartite entanglement generated the interaction between mechanical resonator-optical cavity, optical cavity-microwave cavity and mechanical resonator-microwave cavity subsystem. The generated CV bipartite entanglement measure in terms of logarithmic negat ivity E_N defined as (Liu et al. (2018))

$$E_N = max[0, -ln2\eta^-], \tag{2.35}$$

where $\eta^-$is the minimum symplectic eigenvalue of partial transposed cavity mode corresponding to the bipartite system of interest which is given by

$$\eta^- = \sqrt{\frac{\Sigma(V) - \sqrt{\Sigma(V)^2 - 4detV}}{2}}, \tag{2.36}$$

where Vcan be expressed in a form of block matrix as

$$V = \begin{pmatrix} B & C \\ C^T & B' \end{pmatrix} \tag{2.37}$$

and $\Sigma(V) = detB + detB^{'} - 2detC$, where B,C and $B^{'}$ are 2×2 matrices extracted from the correlation matrix. E_N is entangled if $detC < 0$ and $\eta^{-} \leq 1$ (Abebe et al. (2020a)).

2.5. Entanglement Quantification

When two physical systems have an interaction with each other, some form of correlation with a quantum nature is created between them, which remains even when the two systems get specially separated (Horodecki et al. (2009)). This suggests that, if one performs a measurement on a local observable on the first system, the state of the second system,no matter where it is, is modified instantaneously. Quantum entanglement is a physical phenomenon that occurs when two or more particles cannot be described independently; instead, a quantum state for the system as a whole can be given (Gebru and Getahun (2015)Abebe et al. (2020b)).

Physical property measurements on entangled particles, such as position, momentum, spin polarization, and so on, are found to be appropriately correlated. As a particular resource of the quantum world, entanglement plays an important role in fundamental quantum mechanics and high-speed development of quantum technologies, such as quantum internet, quantum communication, quantum sensing and quantum computer. The realization of the above techniques must rely on the generation and distribution of entanglement between different quantum systems, including atoms, spins, photons, ions, phonons and superconducting circuits (Mosisa (2021)).

3

METHODOLOGY

3.1. Overview of the Study

In this chapter, we investigate theoretically bipartite entanglement between subsystems of our hybrid model opto-electro-mechanical system in witch the degenerate three level atom is injected in to the cavity. Moreover, the dynamics of the system in terms of quantum langevin equation for relevant operators are studied. The system is driven by a laser field to decompose the field operator as a sum of their steady state value and small quantum fluctuations around the steady state. Finally the correlation matrix of the system is obtained by linearized quantum langevin equation and using logarithmic negativity measure to quantify the generated entanglement between the subsystem of the system.

3.2. Theoretical Model and Hamiltonian Formulation

In this thesis, we consider a model of opto-electromechanical system in witch the degenerate three level atom is injected into the cavity at constant rate r_a as shown in Fig. (3.1). We denote the upper, intermediate, and bottom levels of a three-level atom (in a cascade configuration) by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively. When the atom makes a transition from the upper to the intermediate level and then from the intermediate to the bottom level, two photon modes with the same frequency are emitted. We assume the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. On the other hand we assume a MR which on the one side is capacitively coupled to the field of a superconducting microwave cavity (MC) of resonant frequency ω_w and, on the other side, coupled to a driven optical cavity (OC) with resonant

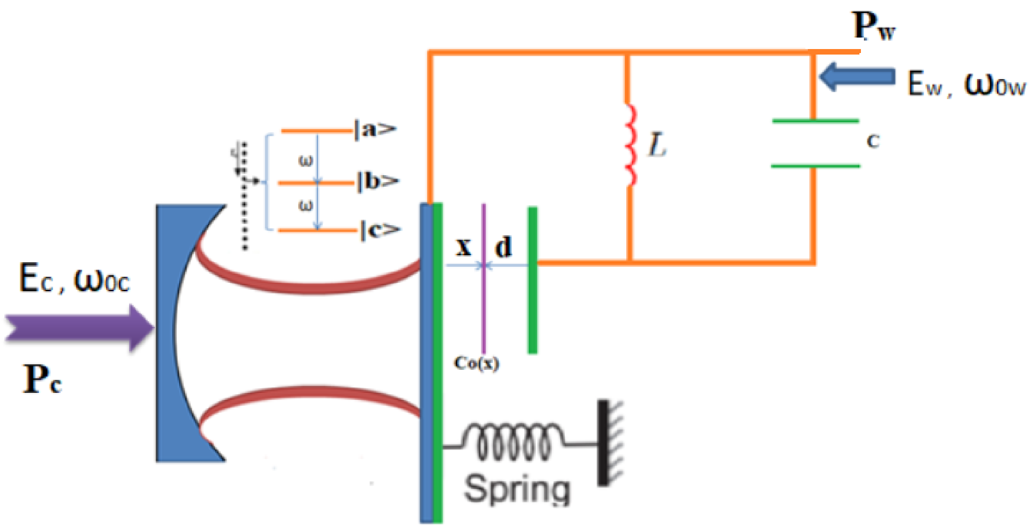

Figure 3.1: Schematic diagram of hybrid opto-electro-mechanical system with degenerate three-level atom

frequency ω_c.

Thus the total Hamiltonian of the system is then

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega_m^2\hat{x}^2}{2} + \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2[C+C_o(x)]} - e(t)\hat{Q} + \hbar\omega_c\hat{a}^\dagger\hat{a} + \sum_{i=a,b,c}\hbar E_i\hat{\sigma}_{ii}$$
$$-\hbar G_{oc}\hat{a}^\dagger\hat{a}\hat{x} + \hbar g(\hat{\sigma}_{ba}\hat{a}^\dagger + \hat{\sigma}_{cb}\hat{a}^\dagger + \hat{a}\hat{\sigma}_{ab} + \hat{a}\hat{\sigma}_{bc}) + i\hbar E_c(\hat{a}^\dagger e^{-i\omega_{oc}t} - \hat{a}e^{i\omega_{oc}t}), \tag{3.1}$$

where (x, p_x) are the canonical position and momentum of a MR having commutation relation $[x, p_x] = i\hbar$ with frequency ω_m, (Φ, Q) are the canonical coordinates for the MC with commutation relation $[Q, \Phi] = i\hbar$ describing the flux through an equivalent inductor L and the charge on an equivalent capacitor C, respectively, $(\hat{a}, \hat{a}^\dagger)$ show the annihilation and creation operators of the OC mode with $[\hat{a}, \hat{a}^\dagger] = 1$ and $|E_c| = \sqrt{\frac{2P_c\kappa_c}{\hbar\omega_{oc}}}$ is related to input driving laser, where p_c is the power of the input laser and κ_c describes the damping rate of the optical cavity. $G_{oc} = (\frac{\omega_c}{\ell})\sqrt{\frac{\hbar}{m\omega_m}}$ is the optomechanical coupling rate with m the effective mass of mechanical mode, and ℓ the length of the optical Fabry-Perot cavity, while the coherent driving of the MC with damping rate κ_w is given by $e(t) = -i\sqrt{2\hbar\omega_w L}E_w(e^{i\omega_{ow}t} - e^{-i\omega_{ow}t})$. The microwave resonator is driven by a tunable electric field, which acts as a control part of the system and is interacts with cavity field through the mediator of MR (Wang et al. (2016)).

By using Eq. (2.11), Eq. (2.12) and Eq. (2.32) the Hamiltonian of Eq. (3.1) rewritten in the terms of the raising and lowering operators of the MC field $\hat{b}, \hat{b}^\dagger$ where $[\hat{b}, \hat{b}^\dagger] = 1$ and the dimensionless position and momentum operators of the MR, $\hat{q}, \hat{p}$ where $[\hat{q}, \hat{p}] = i\hbar$ as

$$H = \hbar\omega_w \hat{b}^\dagger \hat{b} + \hbar\omega_c \hat{a}^\dagger \hat{a} + \sum_{i=a,b,c} \hbar E_i \hat{\sigma}_{ii} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \frac{\hbar G_{ow}}{2}\hat{q}(\hat{b} + \hat{b}^\dagger)^2 - \hbar G_{oc} q a^\dagger a$$
$$-i\hbar E_w (e^{i\omega_{ow}t} - e^{-i\omega_{ow}t})(\hat{b} + \hat{b}^\dagger) + i\hbar E_c(\hat{a}^\dagger e^{-i\omega_{oc}t} - \hat{a} e^{i\omega_{oc}t}) + \hbar g(\hat{\sigma}_{ba}\hat{a}^\dagger + \hat{\sigma}_{cb}\hat{a}^\dagger + \hat{a}\hat{\sigma}_{ab} + \hat{a}\hat{\sigma}_{bc}), \tag{3.2}$$

where

$$\hat{a} = \frac{m\omega_m \hat{x} + i\hat{p}}{\sqrt{2\hbar m\omega_m}}, \tag{3.3a}$$
$$\hat{b} = \frac{L\omega_w \hat{Q} + i\hat{\Phi}}{\sqrt{2L\hbar\omega_w}}, \tag{3.3b}$$
$$\hat{q} = \sqrt{\frac{m\omega_m}{\hbar}}\hat{x}, \tag{3.3c}$$
$$\hat{p} = \frac{\hat{p}_x}{\sqrt{\hbar m\omega_m}}, \tag{3.3d}$$
$$G_{ow} = \frac{\mu\omega_w}{2d}\sqrt{\frac{\hbar}{m\omega_m}}. \tag{3.3e}$$

To obtain the standard Hamiltonian of the system in terms of microwave, optical, and atom detuning, we first apply the rotating wave approximation on Eq. (3.2) to eliminate fast oscillating terms. The rotating wave approximation assumes that the slowly varying terms of the Hamiltonian have a significant contribution to the dynamics, while the rapidly oscillating terms do not. So that the final Hamiltonian of the system becomes

$$\begin{aligned}\hat{H} = \hbar\Delta_{ow}\hat{b}^\dagger\hat{b} + \hbar\Delta_{oc}\hat{a}^\dagger\hat{a} + \hbar\Delta_{a1}\hat{\sigma}_{aa} + \hbar\Delta_{a2}\hat{\sigma}_{cc} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) - \hbar G_{ow}\hat{q}\hat{b}^\dagger\hat{b} \\ -\hbar G_{oc}\hat{q}\hat{a}^\dagger\hat{a} - i\hbar E_w(\hat{b} - \hat{b}^\dagger) + i\hbar E_c(\hat{a}^\dagger - \hat{a}) + \hbar g(\hat{\sigma}_{ba} a^\dagger + \hat{\sigma}_{cb}\hat{a}^\dagger + \hat{a}\hat{\sigma}_{ab} + \hat{a}\hat{\sigma}_{bc}).\end{aligned} \tag{3.4}$$

The first term describes free Hamiltonian of microwave where $\Delta_{ow} = \omega_w - \omega_{ow}$ is microwave detuning with deriving amplitude $|E_w| = \sqrt{\frac{2P_w\kappa_w}{\hbar\omega_{ow}}}$. The second term describes free Hamiltonian of optical cavity where $\Delta_{oc} = \omega_c - \omega_{oc}$ is optical cavity detuning. The third and fourth terms describes free Hamiltonian of three level atom where $\Delta_{a1} = \omega - \omega_{oc}$ and $\Delta_{a2} = \omega - \omega_{oc}$ are the detuning between atomic transition frequencies and driving laser frequency with ω is atomic transition frequencies. The fifth terms describes free Hamiltonian of MR, the six term describes coupling between microwave and MR with $G_{ow} = (\frac{\mu\omega_w}{2d})\sqrt{\frac{\hbar}{m\omega_m}}$ is the micromechanical coupling rate, with m the effective mass of mechanical mode, and d is separation between the plates of the capacitor. The seventh term describes coupling between optical cavity and MR, the eighth and ninth terms describes input driving terms of microwave and laser respectively and the last term describes the coupling between cavity mode and three level laser with interaction strength g.

3.3. Dynamics of the System

The dynamics of the opto-electro-mechanical system assisted by three-level atoms can be described by a set of nonlinear quantum langevin equations in which the dissipation and fluctuation terms are added to the Heisenberg equations of motion. Therefore, in order to obtain the non-linear QLEs for our system operators $\hat{q}, \hat{p}, \hat{a}, \hat{b}, \hat{\sigma}_{ba}$ and $\hat{\sigma}_{cb}$ we use the Langevin-Heisenberg equation of motion in Eq. (2.33). Applying the following commutation relation below on Eq. (3.5) and Eq. (3.6).

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C} \tag{3.5}$$

and

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}. \tag{3.6}$$

We get, the QLEs, setting $\hbar = 1$ as:

$$\dot{\hat{q}} = \omega_m \hat{p}, \tag{3.7}$$

$$\dot{\hat{p}} = -\omega_m \hat{q} + \hbar G_{ow} \hat{b}^{\dagger}\hat{b} + \hbar G_{oc} \hat{a}^{\dagger}\hat{a} - \gamma_m \hat{p} + \zeta, \tag{3.8}$$

$$\dot{\hat{a}} = -(\kappa_c + \Delta_{oc})\hat{a} + iG_{oc}\hat{q}\hat{a} + E_c - ig(\hat{\sigma}_{ba} + \hat{\sigma}_{cb}) + \sqrt{2\kappa_c}\hat{a}^{in}, \tag{3.9}$$

$$\dot{\hat{b}} = -(\kappa_w + \Delta_{ow})\hat{b} + iG_{ow}\hat{q}\hat{b} + E_w + \sqrt{2\kappa_w}\hat{b}^{in}, \tag{3.10}$$

$$\dot{\hat{\sigma}}_{ba} = -(\kappa_a + i\Delta_{a1})\hat{\sigma}_{ba} + ig\hat{\sigma}_{ca}\hat{a}^{\dagger} - ig\hat{a}(\hat{\sigma}_{bb} - \hat{\sigma}_{aa}) + \sqrt{2\kappa_a}\hat{\sigma}_{ba}^{in}, \tag{3.11}$$

$$\dot{\hat{\sigma}}_{cb} = -(\kappa_a - i\Delta_{a2})\hat{\sigma}_{cb} - ig\hat{\sigma}_{ca}\hat{a}^{\dagger} - ig\hat{a}(\hat{\sigma}_{cc} - \hat{\sigma}_{bb}) + \sqrt{2\kappa_a}\hat{\sigma}_{cb}^{in}, \tag{3.12}$$

where γ_m, κ_c, κ_w are mechanical damping rate, cavity decay rate and microwave cavity decay rate respectively. The atomic coherence between the states $|a\rangle$, $|b\rangle$ and $|c\rangle$ decays at rate κ_a for both transitions. The mechanical resonator is affected by the random Brownian stochastic force $\xi(t)$ acting on the mechanical mode and it is also connected to a thermal bath at a damping rate of γ_m possessing the correlation function (Abdi et al. (2011))

$$\langle \xi(t)\xi(t')\rangle = \frac{\gamma_m}{\omega_m}\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\omega \left[coth\left(\frac{\hbar\omega}{2K_BT}\right)+1\right]. \tag{3.13}$$

The optical and microwave cavity amplitude decay at a rate κ_c and κ_w is affected by the radiation input noise $a^{in}(t)$ and $b^{in}(t)$ whose correlation function given by (Gardiner and Zoller (2004))

$$\langle \hat{a}^{in}(t)\hat{a}^{\dagger in}(t')\rangle = [N(\omega_c)+1]\delta(t-t'), \tag{3.14}$$

$$\langle \hat{a}^{\dagger in}(t)\hat{a}^{in}(t')\rangle = N(\omega_c)\delta(t-t'), \tag{3.15}$$

$$\langle \hat{b}^{in}(t)\hat{b}^{\dagger in}(t')\rangle = [N(\omega_w)+1]\delta(t-t'), \tag{3.16}$$

$$\langle \hat{b}^{\dagger in}(t)\hat{b}^{in}(t')\rangle = N(\omega_w)\delta(t-t'), \tag{3.17}$$

where $N_{(\omega_c)} = [exp(\hbar\omega_c/\kappa_B T)-1]^{-1}$ and $N_{(\omega_w)} = [exp(\hbar\omega_w/\kappa_B T)-1]^{-1}$ are the equilibrium mean thermal photon numbers of the optical and microwave fields respectively. One can safely assume $N_{(\omega_c)} \approx 0$ since $\hbar\omega_c/\kappa_B T \gg 1$ at optical frequencies, while thermal microwave photons cannot be neglected in general, even at very low temperatures. The correlation function of atomic input noise operators depends on the specifics of the system under consideration. In our case we assume that the orrelation function of atomic input noise operators is considered as Markovian Gaussian noise, meaning it only depends on the current state and has no memory of past states which is described by delta function as

$$\langle \hat{\sigma}^{in}_{ba}(t)\hat{\sigma}^{in}_{ab}(t')\rangle = [N(\omega)+1]\delta(t-t'), \tag{3.18}$$

$$\langle \hat{\sigma}^{in}_{ab}(t)\hat{\sigma}^{in}_{ba}(t')\rangle = N(\omega)\delta(t-t'), \tag{3.19}$$

$$\langle \hat{\sigma}^{in}_{cb}(t)\hat{\sigma}^{in}_{bc}(t')\rangle = [N(\omega)+1]\delta(t-t'), \tag{3.20}$$

$$\langle \hat{\sigma}^{in}_{bc}(t)\hat{\sigma}^{in}_{cb}(t')\rangle = N(\omega)\delta(t-t'), \tag{3.21}$$

where $N_{(\omega)} = [exp(\hbar\omega/\kappa_B T) - 1]^{-1}$ and $N_{(\omega)} = [exp(\hbar\omega/\kappa_B T) - 1]^{-1}$. We assuming that the initial state of a three-level atom $\rho_a = \rho^0_{aa} |a\rangle \langle a| + \rho^0_{cc} |c\rangle \langle c| + \rho^0_{ca}(|c\rangle \langle a| + |a\rangle \langle c|)$ to be injected in to the cavity at constant rate of r_a (Zhou et al. (2011)). As the result we can rewrite Eq. (3.11) and Eq. (3.12) as

$$\dot{\hat{\sigma}}_{ba} = -(\kappa_a + i\Delta_{a1})\hat{\sigma}_{ba} + igr_a\rho^0_{ca}\hat{a}^\dagger + igr_a\rho^0_{aa}\hat{a} + \sqrt{2\kappa_a}\hat{\sigma}^{in}_{ba}, \tag{3.22}$$

and

$$\dot{\hat{\sigma}}_{cb} = -(\kappa_a - i\Delta_{a2})\hat{\sigma}_{cb} - igr_a\rho^0_{ca}\hat{a}^\dagger - igr_a\rho^0_{cc}\hat{a} + \sqrt{2\kappa_a}\hat{\sigma}^{in}_{cb}. \tag{3.23}$$

3.4. Linerazation of the Quantum Langevin Equations

The Quantum Langevin Equations (QLEs) describe the dynamics of open quantum systems in interaction with a reservoir, which is typically modeled as a collection of harmonic oscillators that are linearly coupled to the system. Linearization is a common technique used to simplify the description of the system dynamics by considering small perturbations around a steady-state solution. Linearization can be helpful when dealing with complex or nonlinear quantum systems while still capturing the essential physics of the problem. In this section we proceed to linearize the above nonlinear Quantum langevin equations of the system operators. This is done by assuming that the light mode is strongly driven leading to the amplitude of its mean value $|\alpha_s| \gg 1$. This allows us to linearize the dynamics of our system around the steady-state values by expanding our system operators $\hat{q}$, $\hat{p}$, $\hat{a}$, $\hat{b}$, $\hat{\sigma}_{ba}$, and $\hat{\sigma}_{cb}$ as a sum of its coherent amplitudes semi classical steady-state value plus an additional small fluctuation operator with zero mean values. Applying Eq.(2.34) one can write

$$\hat{q} = q_s + \delta\hat{q}, \tag{3.24a}$$

$$\hat{p} = p_s + \delta\hat{p}, \tag{3.24b}$$

$$\hat{a} = \alpha_s + \delta\hat{a}, \tag{3.24c}$$

$$\hat{b} = \beta_s + \delta\hat{b}, \tag{3.24d}$$

$$\hat{\sigma}_{ba} = \sigma_{ba_s} + \delta\hat{\sigma}_{ba}, \tag{3.24e}$$

$$\hat{\sigma}_{cb} = \sigma_{cb_s} + \delta\hat{\sigma}_{cb}, \tag{3.24f}$$

where p_s, q_s, α_s, β_s, σ_{ba_s} and σ_{cb_s} represent steady-state (amplitude) of the system operators, where as $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$, $\delta\hat{b}$, $\delta\hat{\sigma}_{ba}$ and $\delta\hat{\sigma}_{cb}$ are the fluctuations with zero-mean values.

3.5. Steady State Solutions of the Equations of Motion

The steady state solution of the optoelectromechanical system refers to finding a stable equilibrium condition in which the optoelectromechanical system (OMEMS) exhibits constant behavior over time, with no net changes of dynamic parameters The steady state solution is important for understanding the long-term behavior, stability, and performance of the system, and for designing devices that can efficiently convert, manipulate, or exchange energy or information between different physical domains. To get the steady state solution of the system operators setting the time derivatives zero and calculate the mean values of the operators. The steady state never fluctuate with time. Inserting Eq. (3.24) in to the Langevin equations of Eqs. (3.7)-(3.10) and Eqs. (3.22)-(3.23) since the the mean values of input noise terms and fluctuate terms are equal to zero, steady state solutions of the system operators become

$$p_s = 0, \tag{3.25a}$$

$$q_s = \frac{G_{oc}|\alpha_s|^2 + G_{ow}|\beta_s|^2}{\omega_m}, \tag{3.25b}$$

$$\alpha_s = \frac{E_c - ig(\sigma_{ba_s} + \sigma_{cb_s})}{i\Delta_c + \kappa_c}, \tag{3.25c}$$

$$\beta_s = \frac{E_w}{i\Delta_w + \kappa_w}, \tag{3.25d}$$

$$\sigma_{ba_s} = \frac{igr_a(\rho^0_{ca} + \rho^0_{aa})}{\kappa_a + i\Delta_{a1}}\alpha_s, \tag{3.25e}$$

$$\sigma_{cb_s} = -\frac{igr_a(\rho^0_{ca} + \rho^0_{cc})}{\kappa_a - i\Delta_{a2}}\alpha_s, \tag{3.25f}$$

where $\Delta_c = \Delta_{oc} - G_{oc}q_s$ and $\Delta_w = \Delta_{ow} - G_{ow}q_s$ describe the effective detuning of the optical and microwave cavities field respectively. The dynamics of the fluctuation of the operators in the system is obtained by decomposing each operators as a sum of its steady-state value and a small fluctuation. Now in order to obtain the fluctuation of the operators of the system, substituting Eq. (3.24) in to Eqs. (3.7)-(3.10) and Eqs. (3.22)-(3.23) and neglect the second order fluctuations (small terms), $\delta\hat{q}\delta\hat{a} \approx \delta\hat{q}\delta\hat{b} \approx \delta\hat{q}\delta\widehat{a^\dagger} \approx \delta\hat{q}\delta\widehat{b^\dagger} \approx 0$ (Firomsa

(2021)). The fluctuation of the system operators can be written as:

$$\delta\dot{\hat{q}} = \omega_m \delta\hat{p}, \tag{3.26}$$

$$\delta\dot{\hat{p}} = -\omega_m \delta\hat{q} - \gamma_m \delta\hat{p} + G_{oc}\alpha_s(\delta\hat{a}^\dagger + \delta\hat{a}) + G_{ow}\beta_s(\delta\hat{b}^\dagger + \delta\hat{b}) + \xi, \tag{3.27}$$

$$\delta\dot{\hat{a}} = -(i\Delta_c + \kappa_c)\delta\hat{a} + iG_{oc}\alpha_s\delta\hat{q} - ig(\delta\hat{\sigma}_{ba} + \delta\hat{\sigma}_{cb}) + \sqrt{2\kappa_c}\hat{a}^{in}, \tag{3.28}$$

$$\delta\dot{\hat{b}} = -(i\Delta_w + \kappa_w)\delta\hat{b} + iG_{ow}\beta_s\delta\hat{q} + \sqrt{2\kappa_w}b^{in}, \tag{3.29}$$

$$\dot{\hat{\sigma}}_{ba} = -(\kappa_a + i\Delta_{a1})\hat{\sigma}_{ba} + igr_a\rho^0_{ca}\hat{a}^\dagger + igr_a\rho^0_{aa}\hat{a} + \sqrt{2\kappa_a}\hat{\sigma}^{in}_{ba}, \tag{3.30}$$

$$\dot{\hat{\sigma}}_{cb} = -(\kappa_a - i\Delta_{a2})\hat{\sigma}_{cb} - igr_a\rho^0_{ca}\hat{a}^\dagger - igr_a\rho^0_{cc}\hat{a} + \sqrt{2\kappa_a}\hat{\sigma}^{in}_{cb}, \tag{3.31}$$

where α_s and β_s are real and positive the chosen phase reference for the optical and microwave field.

3.6. The quantum fluctuations of the system

In order to study the entanglement properties of opto-electromechanical system the quantum correlations between appropriate quadratures of the two intracavity fields and the position and momentum of the resonator are studied. Hence we introduce the dimensionless quadrature fluctuation for each system operator. The dimensionless fluctuation quadrature of the optical cavity

$$\delta\hat{X}_c = \frac{(\delta\hat{a} + \delta\hat{a}^\dagger)}{\sqrt{2}}, \delta\hat{Y}_c = \frac{(\delta\hat{a} - \delta\hat{a}^\dagger)}{i\sqrt{2}}, \tag{3.32}$$

the microwave field dimensionless fluctuation quadrature

$$\delta\hat{X}_w = \frac{(\delta\hat{b} + \delta\hat{b}^\dagger)}{\sqrt{2}}, \delta\hat{Y}_w = \frac{(\delta\hat{b} - \delta\hat{b}^\dagger)}{i\sqrt{2}}, \tag{3.33}$$

and atomic field dimensionless fluctuation quadrature

$$\delta\hat{X}_{a_1} = \frac{(\delta\hat{\sigma}_{ba} + \delta\hat{\sigma}_{ab})}{\sqrt{2}}, \delta\hat{Y}_{a_1} = \frac{(\delta\hat{\sigma}_{ba} - \delta\hat{\sigma}_{ab})}{i\sqrt{2}}, \tag{3.34}$$

$$\delta\hat{X}_{a_2} = \frac{(\delta\hat{\sigma}_{cb} + \delta\hat{\sigma}_{bc})}{\sqrt{2}}, \delta\hat{Y}_{a_2} = \frac{(\delta\hat{\sigma}_{cb} - \delta\hat{\sigma}_{bc})}{i\sqrt{2}}. \tag{3.35}$$

Next we introduce quadrature fluctuation of the corresponding Hermitian input noise operators of the system. Hermitian input noise operators are mathematical operators in quantum physics that describe the noise or fluctuations present in a system. These operators are specifically related to open quantum systems, where the system of interest is interacting with an outside environment. The noise operators play a significant role in determining how the system evolves and can cause decoherence or loss of information in the system. The quadrature fluctuation of the corresponding Hermitian input noise operators are

$$\hat{X}_c^{in} = \frac{(\hat{a}^{in} + \hat{a}^{\dagger in})}{\sqrt{2}}, \hat{Y}_c^{in} = \frac{(\hat{a}^{in} - \hat{a}^{\dagger in})}{i\sqrt{2}}, \tag{3.36}$$

$$\hat{X}_w^{in} = \frac{(\hat{b}^{in} + \hat{b}^{\dagger in})}{\sqrt{2}}, \hat{Y}_w^{in} = \frac{(\hat{b}^{in} - \hat{b}^{\dagger in})}{i\sqrt{2}}, \tag{3.37}$$

$$\hat{X}_{a_1}^{in} = \frac{(\hat{\sigma}_{ba}^{in} + \hat{\sigma}_{ab}^{in})}{\sqrt{2}}, \hat{Y}_{a_1}^{in} = \frac{(\hat{\sigma}_{ba}^{in} - \hat{\sigma}_{ab}^{in})}{i\sqrt{2}}, \tag{3.38}$$

$$\hat{X}_{a_2}^{in} = \frac{(\hat{\sigma}_{cb}^{in} + \hat{\sigma}_{bc}^{in})}{\sqrt{2}}, \hat{Y}_{a_2}^{in} = \frac{(\hat{\sigma}_{cb}^{in} - \hat{\sigma}_{bc}^{in})}{i\sqrt{2}}. \tag{3.39}$$

The linearized QLEs describing the system quadrature fluctuations for our proposed model can be obtained by differentiating both sides of Eqs. (3.32 - 3.35) and substituting Eqs. (3.26 - 3.31) and separating real and imaginary parts. The linearized QLEs become

$$\delta\dot{\hat{q}} = \omega_m \delta\hat{p}, \tag{3.40a}$$

$$\delta\dot{\hat{p}} = -\omega_m \delta\hat{q} - \gamma_m \delta\hat{p} + G_c \delta\hat{X}_c + G_w \delta\hat{X}_w + \xi, \tag{3.40b}$$

$$\delta\dot{\hat{X}}_c = -\kappa_c \delta\hat{X}_c + \Delta_c \delta\hat{Y}_c + g\delta\hat{Y}_{a_1} + \delta\hat{Y}_{a_2} + \sqrt{2\kappa_c}\hat{X}_c^{in}, \tag{3.40c}$$

$$\delta\dot{\hat{Y}}_c = G_c \delta\hat{q} - \Delta_c \delta\hat{X}_c - \kappa_c \delta\hat{Y}_c - g\delta\hat{X}_{a_1} - g\delta\hat{X}_{a_2} + \sqrt{2\kappa_c}\hat{Y}_c^{in}, \tag{3.40d}$$

$$\delta\dot{\hat{X}}_w = -\kappa_w \delta\hat{X}_w + \Delta_w \delta\hat{Y}_w + \sqrt{2\kappa_w}\hat{X}_w^{in}, \tag{3.40e}$$

$$\delta\dot{\hat{Y}}_w = G_w \delta\hat{q} - \Delta_w \delta\hat{X}_w - \kappa_w \delta\hat{Y}_w + \sqrt{2\kappa_w}\hat{Y}_w^{in}, \tag{3.40f}$$

$$\delta\dot{\hat{X}}_{a_1} = gr_a(\rho_{ca}^0 - \rho_{aa}^0)\delta\hat{Y}_c - \kappa_a \delta\hat{X}_{a_1} + \Delta_{a1}\delta\hat{Y}_{a_1} + \sqrt{2\kappa_a}\hat{X}_{a_1}^{in}, \tag{3.40g}$$

$$\delta\dot{\hat{Y}}_{a_1} = gr_a(\rho_{ca}^0 + \rho_{aa}^0)\delta\hat{X}_c - \Delta_{a1}\delta\hat{X}_{a_1} - \kappa_a \delta\hat{Y}_{a_1} + \sqrt{2\kappa_a}\hat{Y}_{a_1}^{in}, \tag{3.40h}$$

$$\delta\dot{\hat{X}}_{a_2} = gr_a(\rho_{cc}^0 - \rho_{ca}^0)\delta\hat{Y}_c - \kappa_a \delta\hat{X}_{a_2} - \Delta_{a2}\delta\hat{Y}_{a_2} + \sqrt{2\kappa_a}\hat{X}_{a_2}^{in}, \tag{3.40i}$$

$$\delta\dot{\hat{Y}}_{a_2} = gr_a(\rho_{ca}^0 + \rho_{cc}^0)\delta\hat{X}_c + \Delta_{a2}\delta\hat{X}_{a_2} - \kappa_a \delta\hat{Y}_{a_2} + \sqrt{2\kappa_a}\hat{Y}_{a_2}^{in}, \tag{3.40j}$$

where

$$G_c = \sqrt{2}G_{oc}\alpha_s = \frac{\omega_c}{\ell}\sqrt{\frac{2\hbar}{m\omega_m}}\left[\frac{E_c - ig(\sigma_{ba_s} + \sigma_{cb_s})}{\kappa_c + i\Delta_c}\right] \tag{3.41}$$

is the effective coupling of the optical cavity fluctuation with mechanical resonator and

$$G_w = \sqrt{2}G_{ow}\beta_s = \frac{\mu\omega_w}{d}\sqrt{\frac{P_w\kappa_w}{m\omega_m\omega_{ow}(\kappa_w^2+\Delta_w^2)}} \tag{3.42}$$

is the effective coupling of microwave cavity fluctuation with mechanical resonator.
The above Eq. (3.40) can be written in the following covariance matrix form

$$\dot{u}(t) = Au(t) + n(t). \tag{3.43}$$

The steady-state dynamics of the covariance matrix for an optoelectromechanical system is concerned with understanding how the statistical properties of the system reach equilibrium over time. The covariance matrix for an optoelectromechanical system is a mathematical tool used to understand and analyze the relationships between different components of the system. Analyzing the covariance matrix provides insights into the correlations and entanglement between different parts of the OMEMS, which are crucial for understanding its quantum behavior, performance, and potential applications. The above Eq. (3.43) can be rewritten in matrix form as:

$$\begin{pmatrix} \delta\dot{\hat{q}} \\ \delta\dot{\hat{p}} \\ \delta\dot{\hat{X}}_c \\ \delta\dot{\hat{Y}}_c \\ \delta\dot{\hat{X}}_w \\ \delta\dot{\hat{Y}}_w \\ \delta\dot{\hat{X}}_{a_1} \\ \delta\dot{\hat{Y}}_{a_1} \\ \delta\dot{\hat{X}}_{a_2} \\ \delta\dot{\hat{Y}}_{a_2} \end{pmatrix} = (A) \begin{pmatrix} \delta\hat{q} \\ \delta\hat{p} \\ \delta\hat{X}_c \\ \delta\hat{Y}_c \\ \delta\hat{X}_w \\ \delta\hat{Y}_w \\ \delta\hat{X}_{a_1} \\ \delta\hat{Y}_{a_1} \\ \delta\hat{X}_{a_2} \\ \delta\hat{Y}_{a_2} \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \\ \sqrt{2\kappa}\hat{X}_c^{in} \\ \sqrt{2\kappa}\hat{Y}_c^{in} \\ \sqrt{2\kappa}\hat{X}_w^{in} \\ \sqrt{2\kappa}\hat{Y}_w^{in} \\ \sqrt{2\kappa_a}\hat{X}_{a_1}^{in} \\ \sqrt{2\kappa_a}\hat{Y}_{a_1}^{in} \\ \sqrt{2\kappa_a}\hat{X}_{a_2}^{in} \\ \sqrt{2\kappa_a}\hat{Y}_{a_2}^{in} \end{pmatrix}, \tag{3.44}$$

where $u(t)=(\delta\hat{q}(t),\delta\hat{p}(t),\delta\hat{X}_c(t),\delta\hat{Y}_c(t),\delta\hat{X}_w(t),\delta\hat{Y}_w(t),\delta\hat{X}_{a_1}(t),\delta\hat{Y}_{a_1}(t),\delta\hat{X}_{a_2}(t),\delta\hat{Y}_{a_2})^T$, is quadrature fluctuation vector, $n(t)=(0,\xi(t),\sqrt{2\kappa}\hat{X}_c^{in}(t),\sqrt{2\kappa}\hat{Y}_c^{in}(t),\sqrt{2\kappa}\hat{X}_w^{in}(t),$ $\sqrt{2\kappa}\hat{Y}_w^{in}(t),\sqrt{2\kappa_a}\hat{X}_{a_1}^{in},\sqrt{2\kappa_a}\hat{Y}_{a_1}^{in},\sqrt{2\kappa_a}\hat{X}_{a_2}^{in},\sqrt{2\kappa_a}\hat{Y}_{a_2}^{in})^T$ is the noise vector with the exponent T represent the transpose and A is the so-called drift matrix (contains information about the system) and given by

$$A=\begin{pmatrix} 0 & \omega_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega_m & -\gamma_m & G_c & 0 & G_w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa_c & \Delta_c & 0 & 0 & 0 & g & 0 & g \\ G_c & 0 & -\Delta_c & -\kappa_c & 0 & 0 & -g & 0 & -g & 0 \\ 0 & 0 & 0 & 0 & -\kappa_w & \Delta_w & 0 & 0 & 0 & 0 \\ G_w & 0 & 0 & 0 & -\Delta_w & -\kappa_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & gr_a(\rho_{ca}^0-\rho_{aa}^0) & 0 & 0 & -\kappa_a & \Delta_{a1} & 0 & 0 \\ 0 & 0 & gr_a(\rho_{ca}^0+\rho_{aa}^0) & 0 & 0 & 0 & -\Delta_{a1} & -\kappa_a & 0 & 0 \\ 0 & 0 & 0 & gr_a(\rho_{cc}^0-\rho_{ca}^0) & 0 & 0 & 0 & 0 & -\kappa_a & -\Delta_{a2} \\ 0 & 0 & gr_a(\rho_{cc}^0+\rho_{ca}^0) & 0 & 0 & 0 & 0 & 0 & \Delta_{a2} & -\kappa_a \end{pmatrix}. \tag{3.45}$$

Due to the linearized dynamics and the Gaussian nature of the quantum noises in Eq. (3.43), the steady state of the quantum fluctuations of the system is a continuous variable bipartite Gaussian state, which is completely characterized by the 10×10 correlation matrix V with its entires are defined as (Jiao et al. (2022)):

$$V_{ij}=\frac{1}{2}\langle u_i(t)u_j(t')+u_j(t')u_i(t)\rangle, \tag{3.46}$$

where $(i,j=1,2,...,10)$. Eq. (3.43) can be written in compact form as $\dot{u}(t)=Au(t)+n(t)$ whose solution is

$$u_i(t)=M(t)u(0)+\int_0^t dt' M(t')n_k(t-t'), \tag{3.47}$$

where $M(t) = exp(At)$.

According to the Routh-Hurwitz criterion (Schumacher (2013)), the system reaches around steady state and stable. i.e; when all the eigen values of the coefficient matrix A have negative real parts. Routh-Hurwitz criterion is a method in control theory used to determine the stability of a linear system by examining the roots of its characteristic polynomial. As $t \to \infty$ so that $M(\infty) = 0$ and Eq. (3.47) can be rewritten as

$$u_i(\infty) = \int_0^\infty dt' \sum_k M(t')n_k(t-t'). \tag{3.48}$$

Consequently Eq. (3.46) also rewritten as

$$V_{ij} = \frac{1}{2}\langle u_i(\infty)u_j(\infty) + u_j(\infty)u_i(\infty)\rangle. \tag{3.49}$$

By using the steady-state solutions in Eqs. (3.47 - 3.48) and the fact that the ten components of $n(t)$ are uncorrelated with each other, one gets

$$V_{ij} = \sum_{l,k} \int_0^\infty dt \int_0^\infty dt' M_{ik}(t)M_{jl}(t')n_{kl}(t-t'), \tag{3.50}$$

where $n_{kl}(t-t') = \frac{1}{2}\langle n_k(t)n_l(t') + n_l(t')u_k(t)\rangle$ is the matrix of stationary noise correlation functions. The Brownian noise term $\xi(t)$ is in general a non-Markovian Gaussian noise as shown in Eq. (3.13), but in the limit of large mechanical quality factor $Q_m = \omega_m/\gamma_m \gg 1$, becomes with a good approximation Markovian (Genes et al. (2008)) with symmetrized correlation function

$$\langle \xi(t)\xi(t') + \xi(t')\xi(t)\rangle = 2\gamma_m(2n+1)\delta(t-t'), \tag{3.51}$$

where $n = [exp(\frac{\hbar\omega_m}{\kappa_B T}) - 1]^{-1}$ is the mean thermal excitation number of the resonator (Abdi et al. (2011)). As a consequence, $n_{kl}(t-t') = D_{kl}\delta(t-t')$, where $D_{kl} = Diag[0, \gamma_m(2n+1), \kappa_c, \kappa_c, \kappa_w(2N(\omega_w)+1), \kappa_w(2N(\omega_w)+1), \kappa_a, \kappa_a, \kappa_a, \kappa_a]$ is the diffusion matrix stemming from the noise correlations. So that Eq. (3.50) becomes

$$V_{ij} = \int_0^\infty d_s M_{(s)} D M_{(s)}^T. \tag{3.52}$$

When the stability condition is fulfilled, the steady-state CM in Eq. (3.52) is determined by the Lyapunov equation

$$AV + VA^T = -D, \tag{3.53}$$

which is linear in V and can be straightforwardly solved. However, its explicit solution is cumbersome and will not be reported here. The linearized QLEs of the system can be expressed by 10×10 correlation matrix:

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} & V_{16} & V_{17} & V_{18} & V_{19} & V_{110} \\ V_{21} & V_{22} & V_{23} & V_{24} & V_{25} & V_{26} & V_{27} & V_{28} & V_{29} & V_{210} \\ V_{31} & V_{32} & V_{33} & V_{34} & V_{35} & V_{36} & V_{37} & V_{38} & V_{39} & V_{310} \\ V_{41} & V_{42} & V_{43} & V_{44} & V_{45} & V_{46} & V_{47} & V_{48} & V_{49} & V_{410} \\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55} & V_{56} & V_{57} & V_{58} & V_{59} & V_{510} \\ V_{61} & V_{62} & V_{63} & V_{64} & V_{65} & V_{66} & V_{67} & V_{68} & V_{69} & V_{610} \\ V_{71} & V_{72} & V_{73} & V_{74} & V_{75} & V_{76} & V_{77} & V_{78} & V_{79} & V_{710} \\ V_{81} & V_{82} & V_{83} & V_{84} & V_{85} & V_{86} & V_{87} & V_{88} & V_{89} & V_{810} \\ V_{91} & V_{92} & V_{93} & V_{94} & V_{95} & V_{96} & V_{97} & V_{98} & V_{99} & V_{910} \\ V_{101} & V_{102} & V_{103} & V_{104} & V_{105} & V_{106} & V_{107} & V_{108} & V_{109} & V_{1010} \end{pmatrix}. \tag{3.54}$$

We have five mode Gaussian state characterized by covariance matrix V in Eq. (3.54) which can also be expressed in the form of block matrix as:

$$V = \begin{pmatrix} V_m & V_{ma} & V_{mb} & V_{m\sigma_{ba}} & V_{m\sigma_{cb}} \\ V_{ma}^T & V_a & V_{ab} & V_{a\sigma_{ba}} & V_{a\sigma_{cb}} \\ V_{mb}^T & V_{ab}^T & V_b & V_{b\sigma_{ba}} & V_{b\sigma_{cb}} \\ V_{m\sigma_{ba}}^T & V_{a\sigma_{ba}}^T & V_{b\sigma_{ba}}^T & V_{\sigma_{ba}} & V_{\sigma_{ba}\sigma_{cb}} \\ V_{m\sigma_{cb}}^T & V_{a\sigma_{cb}}^T & V_{b\sigma_{cb}}^T & V_{\sigma_{cb}\sigma_{cb}}^T & V_{\sigma_{cb}} \end{pmatrix}, \tag{3.55}$$

where each block represents 2×2 matrix. The blocks on the diagonal indicate the variance within each subsystem, while the off-diagonal blocks indicate covariance across different subsystems (Bai et al. (2016)).

We are interested in the entanglement properties of the steady state of the opto-electromechanical system under study and therefore we shall focus onto the entanglement of the three possible bipartite subsystems that can be formed by tracing over the remaining degree of freedom. The sub matrix representing the covariance of cavity and mechanical subsystem is determined by the first four rows and columns of V. Accordingly the correlation between optical cavity-mechanical oscillator (V_s), microwavecavity-mechanical oscillator (V_r), optical cavity-microwavecavity (V_z) subsystem can be represented by:

$$V_s = \begin{pmatrix} V_m & V_{ma} \\ V_{ma}^T & V_a \end{pmatrix}, V_r = \begin{pmatrix} V_m & V_{mb} \\ V_{mb}^T & V_b \end{pmatrix}, V_z = \begin{pmatrix} V_a & V_{ab} \\ V_{ab}^T & V_b \end{pmatrix}, \tag{3.56}$$

where the index m, a and b represents MR, OC and MC subsystem respectively. The correlation between optical cavity-atom operators are

$$V_t = \begin{pmatrix} V_a & V_{a\sigma_{ba}} \\ V_{a\sigma_{ba}}^T & V_{\sigma_{ba}} \end{pmatrix}, V_y = \begin{pmatrix} V_a & V_{a\sigma_{cb}} \\ V_{a\sigma_{cb}}^T & V_{\sigma_{cb}} \end{pmatrix}, \tag{3.57}$$

where the index σ_{ba} and σ_{cb} represents atomic operators subsystem. In order to quantify bipartite entanglement among different subsystems, we use logarithmic negativity (Xiong et al. (2017)).

$$E_N = max[0, -ln2\eta^-], \tag{3.58}$$

where $\eta^- = 2^{-\frac{1}{2}} \left[\Sigma(V_k) - \sqrt{\Sigma(V_k)^2 - 4detV_k}\right]^2$ is the lowest symplectic eigenvalue of the partial transpose of the 4×4 CM, associated with the selected bipartition, obtained by neglecting the rows and columns of the uninteresting mode, where V_k $(k = s, r, z, t, y)$ can be expressed in a form of block matrix as Eq. (3.56) and Eq. (3.57). Each block matrix V_k mentioned above can be in general quantified according to Eq. (2.37). Based on the framework described above, we carry out numerical stimulation to investigate the bipartite entanglement in an opto-electro-mechanical system composed of three subsystems using MATLAB code.

4

RESULT AND DISCUSSION

In this chapter, we present the numerical results of the analaysis of bipartite entanglement between subsystems of an opto-electromechanical system assisted by three level laser. The entanglement properties can be verified by experimentally measuring the corresponding CM through logarithmic negativity. To measure E_N at the steady state, one has to measure all the determined independent entries of the correlation matrix V . We have taken the parameters analogous to the values of the parameter used in experimental work from (Teufel et al. (2011)Zhou et al. (2011)Barzanjeh et al. (2012)) listed as follow.

The speed of light $c = 3 \times 10^8 m/s$, $\hbar = h/2\pi$, Plancks constant $h = 6.626 \times 10^{-34} J.s$, Boltzmanns constant $K_B = 1.38 \times 10^{-23} J/K$, the optical cavity length $L = 1 \times 10^{-3} m$, the spacing of capacitor plate $d = 100 \times 10^{-9} m$, permeability $\mu = 0.008$, mechanical resonator mass $m = 10 \times 10^{-12} Kg$, temperature $T = 15 \times 10^{-3} K$, mechanical resonator frequency $\omega_m = 2\pi \times 10^7 Hz$, Microwave cavity frequency $\omega_\omega = 2\pi \times 10^7 Hz$, mechanical quality factor $Q = 5 \times 10^4$, mechanical decay rate $\gamma_m = \omega_m/Q$, $F = 4.07 \times 10^4$, optical cavity decay rate $\kappa_c = 0.08\omega_m$, microwave cavity decay rate $\kappa_\omega = 0.02\omega_m$, optical cavity detuning $\Delta_w = \omega_m$, optical cavity beam wavelength $\lambda_{oc} = 810 \times 10^{-9} m$, optical cavity beam frequency $\omega_{oc} = 2\pi c/\lambda_{oc}$, optical cavity driven power $P_c = 30 \times 10^{-3} W$, the microwave cavity driven power $P_\omega = 30 \times 10^{-3} W$, and internal energy of the atoms are assumed to be $\rho^0_{aa} = \rho^0_{cc} = \rho^0_{ca} = 0.5$, $\Delta_{a1} = \Delta_{a2} = 2\pi \times 10^7$ are atom transitions detuning.
Accordingly, we study effects of injected atom on the entanglement between subsystems of opto-electromechanical system and compare with standard opto-electromechanical system without three level atom.

Fig. (4.1) shows that the plot of logarithmic negativity E_N between OC-MR versus nor-

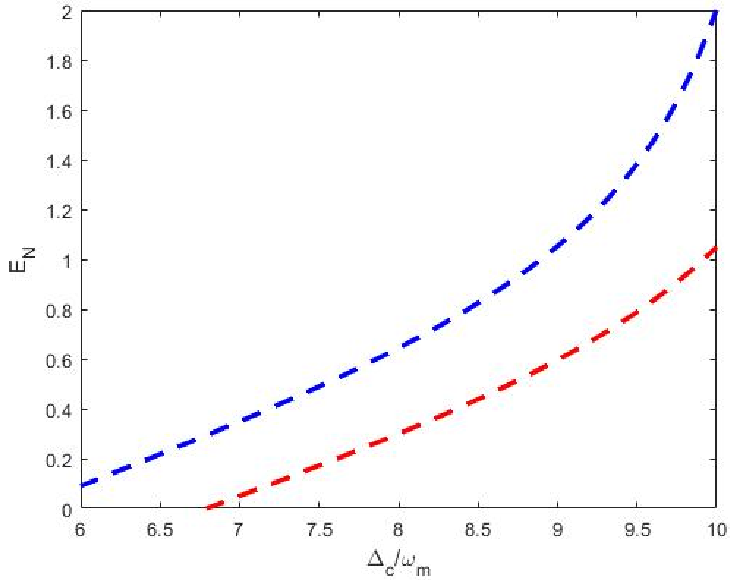

Figure 4.1: The plot of logarithmic negativity E_N between OC-MR versus normalize optical cavity detuning Δ_c/ω_m, the parameters we used are $L = 1 \times 10^{-3}m$, $d = 100 \times 10^{-9}m$, $\mu = 0.008$, $m = 10 \times 10^{-12}Kg$, $T = 15 \times 10^{-3}K$, $\omega_m = 2\pi \times 10^7 Hz$, $\omega_\omega = 2\pi \times 10^7 Hz$, $\gamma_m = 200\pi$, $\kappa_c = 0.1\omega_m$, $\kappa_w = 0.08\omega_m$, $\Delta_w = \omega_m$, $\lambda_{oc} = 810 \times 10^{-9}m$, $\omega_{oc} = 2\pi c/\lambda_{oc}$, $P_c = 30 \times 10^{-3}W$, $P_\omega = 30 \times 10^{-3}W$, $r_a = 1.6 \times 10^5$, $g = 2\pi \times 8 \times 10^5$, $\Delta_{a1} = 2\pi \times 10^{10}$, $\Delta_{a2} = 2\pi \times 10^7$, $\rho^0_{aa} = \rho^0_{cc} = \rho^0_{ca} = 0.5$.

malize optical cavity detuning Δ_c/ω_m. The red dashed line illustrates the entanglement level within the original OC-MR, whereas the blue dashed line presents the entanglement level of the OC-MR enhanced by a three-level laser. By introducing three-level atoms into the opto-electromechanical system, higher degrees of entanglement can be achieved due to the added three level laser-field interaction. In this case, the entanglement between the optical and mechanical components can be improved by the atomic degree of freedom. This enhancement enables more robust and efficient entanglement generation, as well as more control over the entangled state.

In Fig. (4.2), we can see two different entanglement scenarios regarding the interaction between a MC and a MR, one without the injection of a three-level atom (red dashed line) and one where a three-level atom is injected into the cavity (blue dashed line), increasing the entanglement between MC-MR. In the first scenario (red dashed line), the entanglement between MC and MR is generated solely from the interaction between the two subsystems, without injection of atom. This curve thus depicts the baseline level of entanglement generated by the interaction of the two subsystems. The entanglement arises due to the radiation pressure of the optical cavity on the MR, which results in a change in the length of the ca-

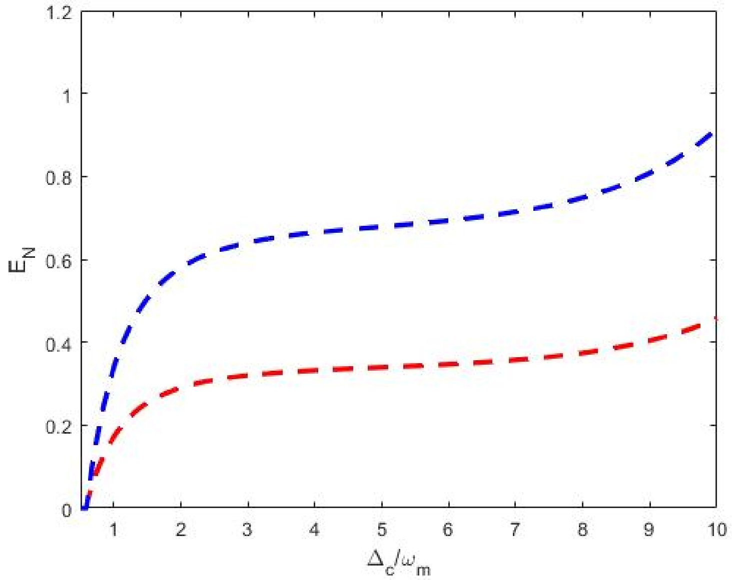

Figure 4.2: The plot of logarithmic negativity E_N between MC-MR versus normalize optical cavity detuning Δ_c/ω_m, the parameters we used are $\kappa_c = 0.08\omega_m$, $r_a = 2000$, $\Delta_{a1} = \Delta_{a2} = 2\pi \times 10^7$, $\gamma_m = \omega_m/Q$, $g = 2\pi \times 1.0 \times 10^5$, $\kappa_a = 2\pi \times 10^6$, $\Delta_w = \omega_m$ and the others parameters are the same as Fig. (4.1).

pacitor connecting the two subsystems. On the other hand, the blue dashed line represents the entanglement between the MC and the MR when a three-level atom is injected into the optical cavity at the rate r_a. By comparing the red and blue dashed lines, one can observe that the presence of a three-level laser indeed enhances the entanglement between the MC and the MR, especially for certain values of the normalized detuning parameter, Δ_c/ω_m.

Fig. (4.3) illustrates an entanglement setup involving an OC and MC subsystems. In this case, the red dashed line represents the entanglement between the optical and microwave cavities, while the blue dashed line signifies the entanglement between OC-MC, with the three level atom being injected into the optical cavity at a rate of r_a. It is important to highlight that the entanglement between OC-MC is not achieved through direct interaction but is mediated by the mechanical resonator.

To discuss the entanglement between OC-MC represented by the red dashed line, let's assume an initial state. On the other hand, the blue dashed line represents the entanglement between OC-MC when the three-level laser is present. As the atom is injected into the OC, the emitted laser experiences a coupling between the optical cavity field mode. This coupling induces an indirect interaction between the OC and MC, promoting the generation of entanglement between them. When we comparing the two situations, it is clear that the presence

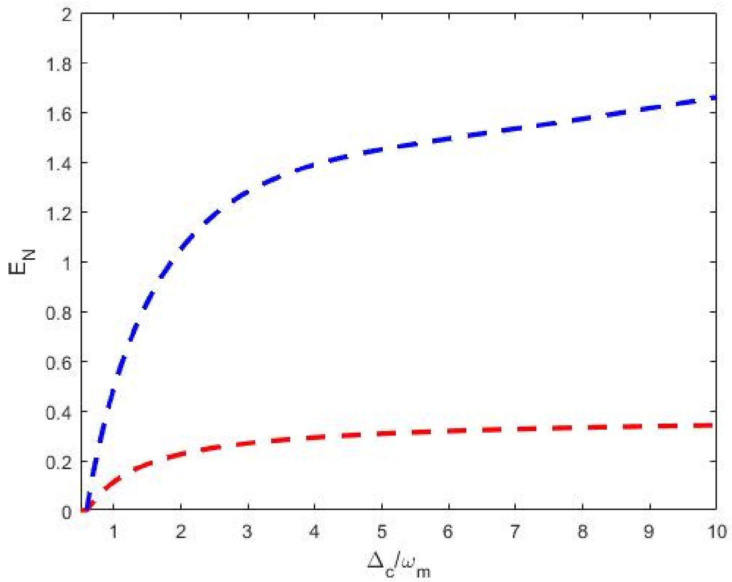

Figure 4.3: The plot of logarithmic negativity E_N between OC-MC versus normalize optical cavity detuning Δ_c/ω_m, the parameters we used are $r_a = 1.6 \times 10^6$, $\Delta_{a1} = 2\pi \times 10^{10}$, $\Delta_{a2} = 2\pi \times 10^6$, $g = 2\pi \times 1.5 \times 10^6$, $\kappa_c = 0.08\omega_m$ and the others parameters are the same as Fig. (4.2).

of the three-level laser significantly improves the entanglement between the optical and microwave cavities. This suggests that the injection of the atom alters the dynamics of the OC, MC, and MR system, which leads to the enhancement of the entanglement process. Moreover, by controlling the injection rate, the level of entanglement can be adjusted according to desired parameters.

The inclusion of three-level laser can also increase the system's coherence and stability, which in turn can improve performance in a range of quantum applications like quantum information processing and communication, where strong entanglement between quantum systems is desirable. In general in the above three bipartite entanglement between subsystems OC-MR, OC-MC, and MR-MC, it is shown that the subsystem containing OC has higher logarithmic negativity due to the atom injected in to the optomechanical arrangement and emitted laser which interact with OC. Also it is shown that the entanglement between subsystems of opto-electromechanical system discussed above are indeed enhanced compared to the entanglement of opto-electromechanical studied by (Barzanjeh et al. (2011)) and (Tesfahannes and Alemu (2021)).

Fig. (4.4) is a plot of logarithmic negativity (E_N) between OC-three level laser against normalized optical cavity detuning Δ_c/κ_c for different three level laser-cavity field coupling

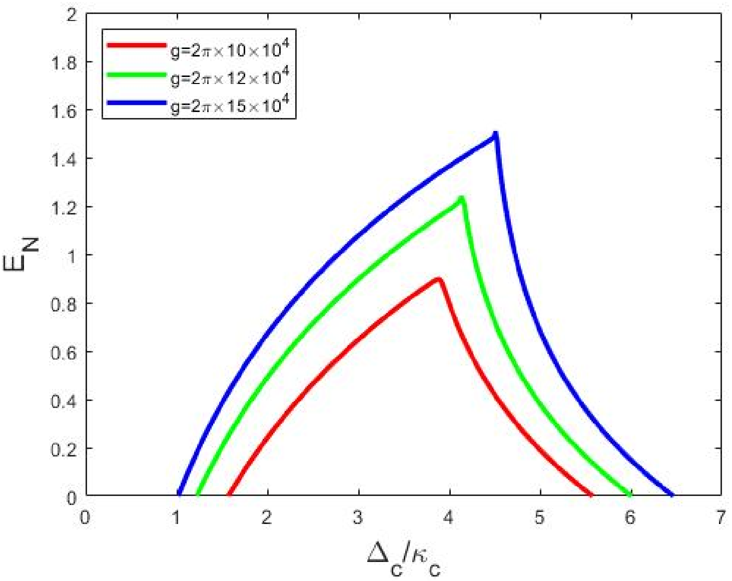

Figure 4.4: The plot of logarithmic negativity E_N between OC-three level laser versus normalize optical cavity detuning Δ_c/κ_c at different three level laser-cavity field coupling rate, the parameters we used are $r_a = 1.6 \times 10^6$, $\Delta_{a1} = \Delta_{a2} = 2\pi \times 10^6$, $\kappa_a = 2\pi \times 10^5$, $\kappa_c = 0.02$ and the others parameters are the same as Fig. (4.3).

rates g. Three scenarios are presented in the plot, distinguished by different colored dashed lines: red ($g = 2\pi \times 5 \times 10^4$), green ($g = 2\pi \times 8 \times 10^4$), and blue ($g = 2\pi \times 15 \times 10^4$). The purpose of these three scenarios is to demonstrate how the entanglement between the optical cavity and three level laser varies as the coupling rate is increased. As the coupling rate is increased, represented by the color change from red to blue, the entanglement (logarithmic negativity) also increases. This implies that the degree of correlation between the two subsystems, the optical cavity and three level laser, becomes stronger as the three level laser-cavity field coupling rate is increased. The dependence of logarithmic negativity on the normalized optical cavity detuning (Δ_c/κ_c) can be observed as well. The entanglement increases as the detuning is increased but reaches a peak at a certain value of Δ_c/κ_c and then starts to decrease. This indicates that there is an optimal detuning value that maximizes the entanglement between the optical cavity and three level laser for a given coupling rate. The peak entanglement value, i.e., the maximum logarithmic negativity, as well as the optimal detuning value at which this maximum entanglement occurs, both increase as the coupling rate is increased.

Fig. (4.5) shows the plot of logarithmic negativity (E_N) for three bipartite sub-systems (OC-MR, OC-MC, and MC-MR) as a function of the normalized optical cavity detuning

Δ_c/ω_m. In the first sub-plot (a), the entanglement between the optical cavity and mechanical resonator subsystems is depicted for three different atom injection rates. The red line corresponds to $r_a = 0.2 \times 10^4$, the green line corresponds to $r_a = 0.3 \times 10^4$, and the blue line corresponds to $r_a = 0.4 \times 10^4$. The plot shows that as the atom injection rate increases, the entanglement between the OC-MR subsystem also increases.

The second sub-plot (b) shows the entanglement between the optical cavity and microwave cavity subsystems for three different atom injection rates. In this case, the red line corresponds to $r_a = 1.0 \times 10^4$, the green line corresponds to $r_a = 2.0 \times 10^4$, and the blue line corresponds to $r_a = 4.0 \times 10^4$. Similar to the first sub-plot, an increase in atom injection rate leads to an enhancement in entanglement between the OC-MC subsystem. Finally, the third sub-plot (c) presents the entanglement between the microwave cavity and mechanical resonator subsystems for three different atom injection rates. The red line corresponds to $r_a = 0.2 \times 10^4$, the green line corresponds to $r_a = 0.5 \times 10^4$, and the blue line corresponds to $r_a = 1.5 \times 10^4$. As observed in the previous sub-plots, an increase in atom injection rate

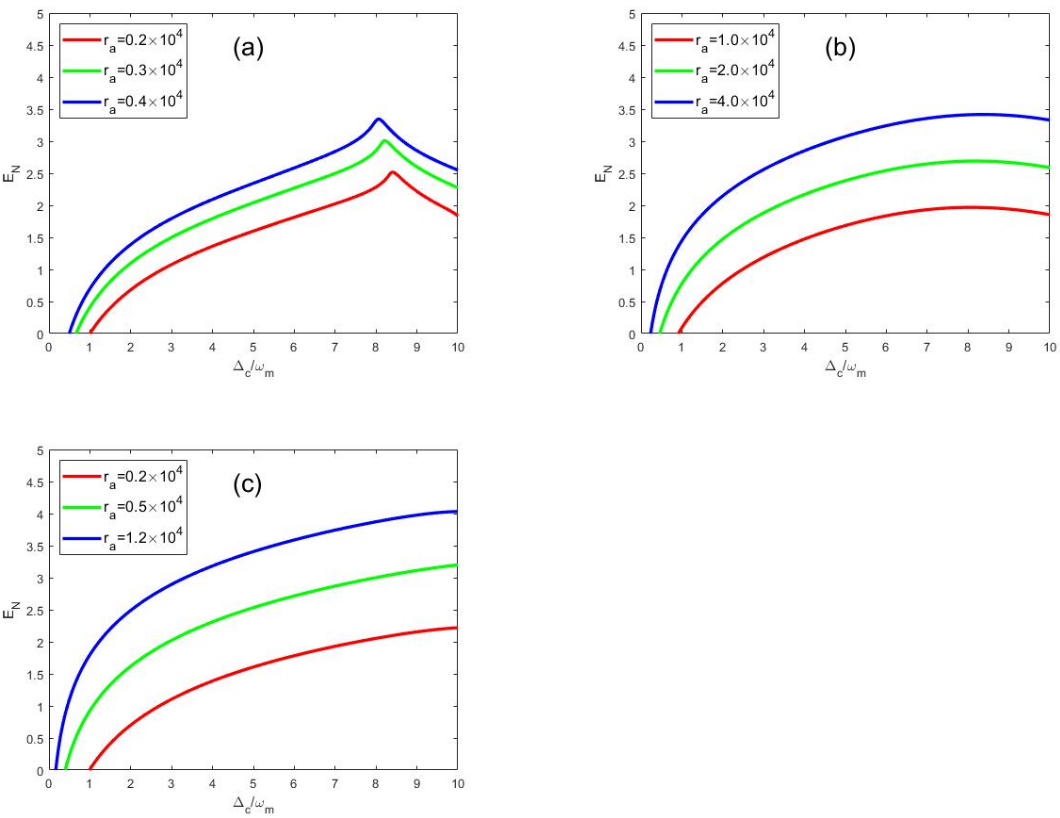

Figure 4.5: The plot of E_N of the three bipartite sub-systems versus the normalized optical cavity detuning Δ_c/ω_m at three different atom injection rate : OC-MR (a), OC-MC (b), MC-MR (c), the parameters we used are $g = 2\pi \times 15 \times 10^5$, $\Delta_w = -\omega_m$ and the others parameters are the same as Fig. (4.2).

results in an increase in entanglement between the MC-MR subsystem. In general, the figure demonstrates that increasing the atom injection rate leads to an enhancement in entanglement between each subsystem of the tripartite system (OC-MR, OC-MC, and MC-MR). This insight could be valuable for optimizing entanglement in such systems, particularly for applications in quantum communication and computation.

In the following numerical results, we investigate the three possible stationary entanglement of the system versus the detuning of the optical cavity, for different values of the temperature.

As shown in Fig. (4.6), the logarithmic negativity E_N for three bipartite entanglement versus the normalized optical cavity detuning Δ_c/ω_m at three different temperatures: $T = 5 \times 10^{-3} K$) (a), $T = 250 \times 10^{-3} K$ (b) and $T = 350 \times 10^{-3} K$ (c).

The entanglement between the different subsystems is represented by different colored lines - red dashed line represents the entanglement between OC-MR, green solid line for OC-MC, and the blue dashed line for MC-MR.

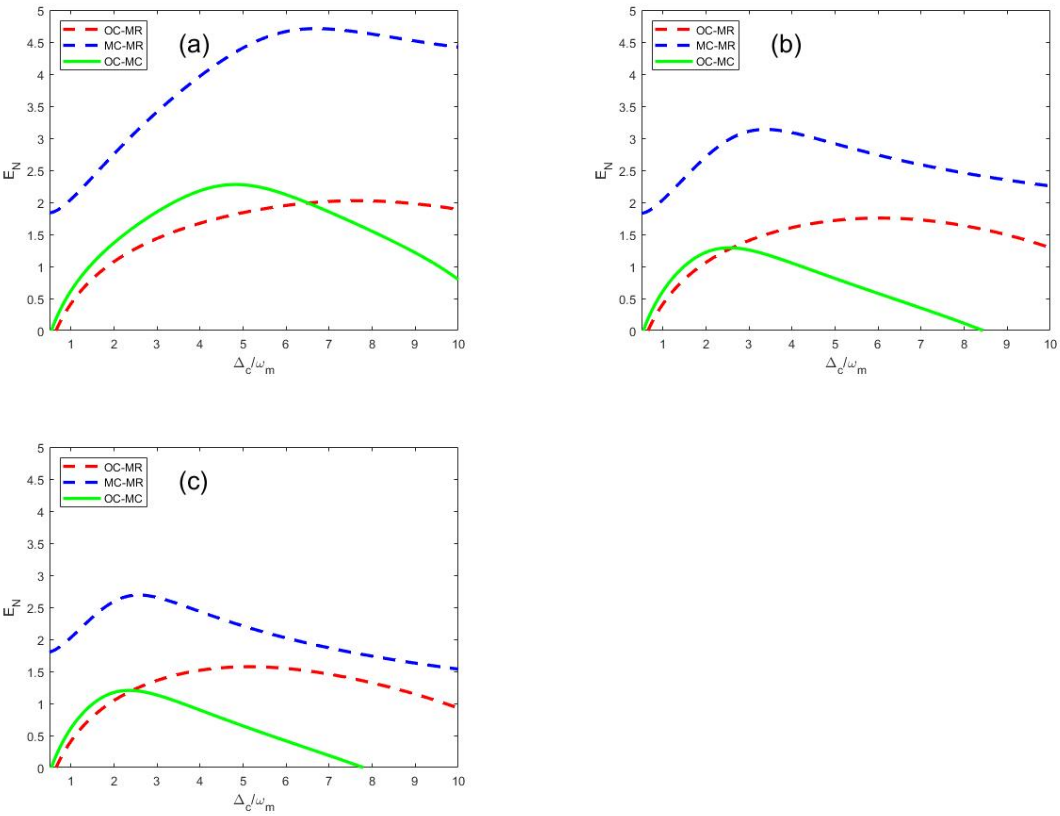

Figure 4.6: The plot of E_N of the three bipartite sub-systems versus the normalized optical cavity detuning Δ_c/ω_m at three different temperatures: $T = 5 \times 10^{-3} K$ (a), $T = 250 \times 10^{-3} K$ (b), $T = 350 \times 10^{-3} K$ (c), the parameters we used are $r_a = 1.6 \times 10^6$, $\Delta_{a1} = 2\pi \times 10^{10}$, $\Delta_{a2} = 2\pi \times 10^6$, $\gamma_m = 200\pi$, $F = 4.07 \times 10^4$, $\kappa_c = \pi c/(F \times L)$, $g = 2\pi \times 1.0 \times 10^5$, $\kappa_a = 2\pi \times 10^6$, $\Delta_w = -\omega_m$ and the others parameters are the same as Fig. (4.5).

From the plot, it can be observed that the entanglement between each subsystem decreases as the temperature increases from $T = 5 \times 10^{-3} K$ (a) to $T = 350 \times 10^{-3} K$ (c). This can be attributed to the fact that entanglement is a quantum mechanical property that tends to become fragile with increasing temperature. As the environment becomes warmer, the increased thermal fluctuations lead to greater decoherence and noise, which negatively impact the entanglement between the various sub-systems.

Additionally, by altering the parameter, one can potentially control the level of entanglement in the system. This tunability of entanglement may have implications on designing robust entangled systems for various practical applications. Moreover, this relationship may indicate that the behavior of the entangled sub-system pairs is sensitive to temperature with higher temperatures leading to decay entanglement between the sub-systems. When we compare the peaks of the plotted lines for each temperature, it is evident that the entanglement between the OC-MC pair is significantly decay at higher temperatures than the entanglement in the OC-MR and MC-MR pairs. This is because the entanglement between OC-MC is increase due to the OC-MR and MC-MR entanglement.

5

CONCLUSION AND RECOMMENDATION

5.1. Conclusion

In this thesis, the bipartite entanglement dynamics in an opto-electromechanical system assisted by three level laser was studied. The dynamics of a system have been studied by using the QLEs and linearization approximation, leading to a correlation matrix containing block matrices describing entanglement between each subsystems. Accordingly the bipartite entanglement is quantified through logarithmic negativity. Specifically, we investigate the entanglement properties generated between various subsystems, including the OC-MC, OC-MR, and MC-MR subsystems. The numerical results indicated that the introduction of three-level atoms into opto-electromechanical systems significantly enhances the entanglement between each subsystems. In addition, we showed that the degree of correlation between optical cavity and three level laser, becomes stronger as the three level laser-cavity field coupling rate is increased. Interestingly, it is so happened that as the atom injection rate increases, more atoms are introduced into the system, and more opportunities for entanglement arise from the interactions between the three level laser and cavity field. This typically results in a more complex quantum state, where the subsystems become more tightly entangled with each other, and their properties become increasingly interconnected. Moreover, the results demonstrated that temperature has a significant impact on entanglement, with higher temperatures leading to a reduction in entanglement between the sub-systems. This increased interaction leads to a decrease in entanglement between the subsystems, as their states become more independent due to the influence of the environment. Thus, generating enhanced bipartite entanglement in opto-electromechanical systems has significant in the advancement of quantum communication, and information processing control in hybrid quantum systems.

5.2. Recommendation

In this thesis we studied on the enhancement of bipartite entanglement of opto-electromechanical system by injection of three level atom in to the system. Further studies will be conducted on the enhancement of opto-electromechanical system by putting optical parametric amplifier in an optomechanical arrangement.

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