An Implicit Compact Finite Difference Method for Partial Integro-Differential Diffusion Equations

By

Mekash Zewdu

A Thesis Submitted to

The Program of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

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Certification

This is to certify that the thesis prepared by Mekash Zewdu, entitled: **An Implicit Compact Finite Difference Method for Partial Integro-Differential Diffusion Equations** and submitted in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics(Specialization in Numerical Analysis). As a member of the board of examiners of the MSc. Thesis open defense examination, we certify that we have read, evaluated the thesis prepared by Mekash Zewdu and examined the candidate. We recommended that the thesis be accepted as fulfilling the thesis requirement for the degree of Master's of Science in Applied Mathematics.

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Abstract

This thesis deals with numerical solution of some partial integro-differential equations which occur in diffusion partial differential equations. The numerical solution described in this thesis is based on the compact finite difference method (CFDM), a numerical method based on second order finite difference method, is presented for the solution of partial integro-differential equation (PIDE). A composite weighted trapezoidal rule is manipulated to handle the numerical integrations which results in a closed-form difference scheme. We also consider the effect of the integral part of partial integro-differential equation (PIDE).

Compact finite difference method (CFDM) is based on domain discretization. Second replacing the given equations and compact finite difference approximations in space. First we discretize the domain of the problem and substitute the forward, the backward and second order central difference. Using the discretized mesh points we construct systems of linear equations and form tri-diagonal system. Using iterative methods we obtain the solutions and also estimate the errors at each mesh points. The error analysis of compact finite difference method is based on Taylor's theory. Finally we discuss convergence, stability, consistency and accuracy applying basic theorems with proofs, depending on the various mesh lengths. Numerical experiments are used to compare and test the compact finite difference method for accuracy.

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Adama Science and Technology University **Mekash Zewdu**

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Notations and Abbreviations

DEs	Differential Equations
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
OIDE	Ordinary Integro-Differential Equation
PIDE	Partial Integro-Differential Equation
IDE	Integro-Differential Equation
$O(h) = O(\triangle\ x)$	Errors of order h
$O(h^n) = O(\triangle\ x^n)$	Errors of order h^n
$\triangle\ x = h$	The length of mesh points
$\triangle\ k = t$	The time level of grid points
RDE	Reaction Diffusion Equation
FWE	Forward Euler
BWE	Backward Euler
FDM	Finite Difference Method
CFDM	Compact Finite Difference Method
LTE	Local truncation error
$U(x,t)$	Is the analytical solution of the PDE
$U(x_i, t_j)$	Continuous solution evaluated at the mesh point
U_i^j	Approximate numerical solution
ICs	Initial Conditions
BCs	Boundary Conditions
IVP	Initial Value Problem
BVP	Boundary Value Problem

Chapter 1

Introduction

1.1 Background of Study

Partial integro-differential equation is an equation that the unknown function appears under the sign of integration and contains the derivatives of the unknown function. Most of integro-differential equations are usually difficult to solve analytically. In many methods it is required to obtain the approximate solution. Numerical methods are mathematical techniques used for solving mathematical problems that can not be solved or difficult to be solved analytically. Analytical solution is an exact solution in the form of a mathematical expression in terms of the variables associated with the problem that is being solved. A numerical solution is an approximate numerical value (a number) for the solution. Although numerical solutions are approximations, they can be very accurate.

Numerical methods provide approximations to the problems in question. No matter how accurate they are they do not, in most cases, provide the exact solution. In some instances, working out the exact solution by different approachs may not be possible or may be too time consuming and it is in these cases where numerical methods are most often used.

The compact finite difference approximations for derivatives are one of the most important methods to solve differential equations. To approximate the model equations by finite differences divide the closed domain by a set of lines parallel to the spatial and time axes to form a grid or a mesh.

Many modern numerical schemes for transport phenomena trace their origins to finite difference approximations developed in the late **1950s** through early **1980s** and their development was stimulated by the emergence of computers that offered a convenient framework for dealing with complex problems of science and engineering.

Theoretical results have been obtained during the last five decades regarding the accuracy, stability and convergence of the finite difference methods for partial differential equations. The standard and compact techniques in finite difference methods are usually used to obtain the numerical solutions of differential equations.

Compact finite difference method is a special finite difference method which uses the values of the function and its derivatives. High order schemes are that they tend to be more complicated to implement and more expensive to calculate than lower order schemes.
High order time discretization schemes are often less stable. High order accurate time discretization method that does not fall under the same category as the methods used so far in the class of time compact schemes.
The analytical solutions of most of the partial differential equations (PDEs) can not be explicitly expressed and the theoretical analysis of PDEs is also difficult but solved by numerical methods.
Reaction diffusion equation is an interaction between two or more objects. For the solution reaction diffusion equation problem, we need the differential equations, some initial conditions, and boundary conditions. The initial conditions will be initial values of the concentrations over the domain of the problem. A reaction often refers to an interaction of some sort between two or more objects sometimes to express partial integro-differential equation(PIDE).
Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient numerical solution. The principal aim of this thesis is to present an approximate solution for a parabolic partial integro-differential equation representing diffusion equation. There are several methods for solving integro-differential equations **(E. G. Yanik and G. Fairweather 1988)** used finite element methods for solving integro-differential equation of parabolic type . In this, study we present compact finite difference method to the solution of partial integro-differential equation, which often occur in the mathematical physics, contain derivative with respect to different variables called partial integro-differential equation (PIDE). The form of the PIDE is:

$$h(x)\frac{\partial f}{\partial t} = g(x) + \frac{\partial^2 f}{\partial x^2} + \lambda \int_a^b k(x,y)f(y)dy$$

where λ is known parameter, h and g are known function of $x, k(x,y)$ is the known function of x and y called the kernel of the integro-differential equation.
(Sloan et al 1986; Branco 2010) numerically studied it by the Backward Euler and Crank Nicolson, method for the analysis of traveling waves in a one-dimensional reaction-diffusion system. Recently, **Khuri et al (1996)** concentrated on the finite difference methods for solving partial integro-differential equations by approximating them with difference equations, in which finite difference approximates the derivatives.
Finite difference methods are one of the most widely used numerical schemes to solve differential equations.
The general concepts of finite difference methods concern with stability, convergence conditions, have been presented in literature for solving differential equations. Finite difference methods are thus discretization methods.

(D.Liang 2003) studied the structure of the solution to the reaction-diffusion equations from both the theoretical and numerical points of view. **(J. Ferreira, P. de Oliveira, 2010)** constructed a second-order linear finite difference scheme for partial integro-differential equation.

Finite difference schemes are a technique by which derivatives of a function are approximated by differences in the value of the function. Later then, **(Wright W.M 2003)** considered a mesh less method on delay differential equations flows with mixing length growth in porous media based on radial basis functions. However, most of the numerical methods are no more than second-order accuracy, while there are a large number of scenarios where higher order accurate schemes are a necessity due to the desired accuracy of the simulations.

On the other hand, the higher order schemes allow one to approximate a solution with fewer grid points, while maintaining the same accuracy as a low order scheme. In certain circumstances, the desired grid point size is based on the ability to resolve the structure of the solution, and not on the accuracy of computation.

But, the higher order finite difference schemes, typically achieved by computing derivatives with a wider matrix stencil, because some difficulties near the boundary, just as one must be able to calculate the inner point near the boundary with the same accuracy as the internal scheme, which should be complicated to implement. Based on such a reason, there is a great interest in the higher order finite difference schemes.

In **1950s**, the compact finite difference schemes have been applied to solve partial differential equations more and more frequently, and more recently, the compact finite difference schemes have been extended to partial differential equation, for instance **U. M. Ascher and L. R. Petzold**, by proposing a compact multi scheme for the nonlinear delay convection reaction-diffusion equations, and **Numerical Solution of Partial Differential Equations** by applying the compact difference scheme to reaction diffusion equations based on Crank-Nicolson scheme in temporal direction, and **(S. Abarbanel,1993)** by employing the compact difference scheme combined with extrapolation techniques to solve a class of parabolic differential equations.

The main advantage is that the compact finite difference schemes exhibit higher order accuracy but still only rely on the closest neighboring points for computations. The resulting algebraic systems are tri-diagonal which can be inverted easily by Thomas algorithm. To the best of our knowledge, there are few works using the numerical methods to higher order compact finite difference scheme and the other is based on standard second order central difference approximation is the higher order. High-order finite difference schemes require additional numerical boundary schemes at grid points near the boundaries of the computational domain.

The main purpose of this study is to solve partial integro-differential equation using compact finite difference method with higher-order accurate scheme for solving partial integro-differential equations with non-homogeneous-Dirichlet boundary .

And deal with a combination of the compact finite difference method and the composite trapezoidal rule for calculating the integral term. Finally the obtained system of algebraic equations is solved by iterative methods. We study effect of the integral part of partial integro-differential equation on the solution of compact finite difference method (CFDM).

1.2 Statement of Problem

In the past, many researchers have solved the partial integro-differential equations using different types of numerical techniques, for instance:
FDMs(implicit, explicit, crank-nicolson method,...,) and so on. In particular**(Branco et al 2012)**, for solving the higher order reaction-diffusion equations with compact finite difference method from both the theoretical and numerical points of view. However, in his investigation, he didn't consider the rate and the region of convergence of the approximate solution of the partial integro-differential equations. Therefore, in this study, we will try to investigate on the rate and the region of convergence of the approximate solution of the partial integro-differential equations is considered.
Also we study effect of the integral part of partial integro-differential equation (PIDE) on the solution.
So, this study is intended to answer the following basic questions:

- What is the effect of grid refinement on the solution of compact finite difference method?
- How to analysis convergence, consistence and stability of compact finite difference method?
- How to reduce computational error?
- what is the effect of the integral part of partial integro-differential equation on the solution of compact finite difference method?

1.3 Objective

1.3.1 General Objective

The general objective of this study is for solving numerical solution of partial integro-differential equations(PIDEs) using compact finite difference method.

1.3.2 Specific Objectives

The specific objectives of this study are :

- To investigate the convergence of PIDE.
- To determine stability of higher order finite difference methods.
- To study consistence and error analysis.

1.4 Significance of Study

- This study provides a reliable information how can we solve partial integro-differential equations using compact finite difference method.
- For the growth of research in the area.
- To develop our knowledge of higher order partial integro-differential equations using compact finite difference approximation and applying the method.
- Moreover, it provides several guides to some textbook, teachers, friends and anyone who works on this area.

1.5 Scope of the Study

This study focuses only on numerical solution of partial integro-differential equations by compact finite difference method with grid refinement and compare numerical solution of diffusion equation with finite difference method. And observe effect of the integral part of partial integro-differential equation on the solution.

Chapter 2

Literature Review

A. Arajo, J. Ferreira, P. de Oliveira,(2005) Proposed reaction-diffusion equations with rate terms are widely used as models for biological, physical and control system. Over the past several years, such equations have been extensively investigated. Reaction-advection-diffusion equations are of the mixed type, both hyperbolic and parabolic with advection do minted reaction-diffusion equations are abundant throughout science and nature.

It turns out that the net effect of the two processes is just the sum of the individual rates of change. The governing reaction or set of reactions leads to reaction rate terms R, then the general form of the partial differential equation model is

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R \tag{2.0.1}$$

where, D is a diagonal matrix of diffusion coefficient , R is rate of terms and c is dependent variable. For obvious reasons, this is called a reaction-diffusion equation. Reaction-diffusion equations are members of a more general class known as partial differential equations (PDEs), so called because they involve the partial derivatives of functions of many variables. In the case of a reaction diffusion equation, c depends on t and on the spatial variables. One of the most popular examples of which are the chemical (RDE) systems that mimics the development of a leopards spots developed initially by **(Alan Turing 1998)** and solved by using methods of numerical mathematics also complex geometries numerical solution and then expanded. Time dependent partial differential equations are solved numerically by discretizing both the space and time.

In its early stages of development, the theory of second order linear partial differential equations was concentrated on applications to mechanics and physics. All such equations can be classified into three basic categories**:** the wave equation, the heat equation, and the Laplace equation (or potential equation). Also the study of these three different kinds of equations yields much information about more general second order linear partial differential equations.

Jean d'Alembert (1717) first derived the one dimensional wave equation for vibration of an elastic string and solved this equation in **1746**. His solution is now known as the d'Alembert solution.

The wave equation is one of the oldest equations in mathematical physics. Some form of this equation, or its various generalizations, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. Higher order schemes lead to an increase in the amount of work per grid point in time. Especially when using implicit time schemes, the number of grid points in time can however, often be dramatically reduced.

G. Gao and Z. Sun, (2011) have presented a high order compact finite difference scheme for the fractional sub diffusion equations . They first transformed the problem using the definition and other analytical theories then the problem discretization was applied to approximate the temporal fractional derivative. Then the stability and convergence of the method were analyzed by the energy method to obtained periodic solutions for high order reaction equations.

The approximate solutions obtained from compact methods are compared with exact solution by using numerical example. The solution of the numerical scheme should be a large approximation to the exact solution.

Zhang and Vandewalle (2013) studied the general linear methods for solving the nonlinear equations. Wang proved that the proposed numerical methods are sufficient to preserve stability of the underlying systems. Besides if we discretize the diffusion term by using the centered difference operator, the energy stability of full discrete reaction diffusion equation with a variable reaction can also be derived. Finite difference method consists of approximating the differential operator by replacing the derivatives in the equation using differential quotients.

The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points.

In this study we apply higher order compact finite difference method with grid refinement for partial integro -differential equation .

The refinement is expected to faster convergence and reduce error of computation.

We also study effect of the integral part of partial integro-differential equation on the solution.

2.1 Higher order finite difference method

Zhang et al(1986) constructed a second-order linear finite difference scheme. The order of the partial differential equation is the order of the highest order derivative that appears in the equation. In general, a k^{th} order partial differential equation is an equation which can be written in the form: $F(x, u, Du, D^2u, ..., D^K u) = 0$. The unknown in differential equation is a function and not a number.

The fundamental problems on linear second order partial differential equations of parabolic type with different boundary condition have been substantially investigated in [**I.Solan, V.Thome , Numer.Anal,23(1986)**] and others. Recently, Bange [constructive existence theorem for a non linear parabolic equation] has reported certain results on the existence and uniqueness of solutions of quasilinear parabolic partial differential equations of second order. The problem of the existence and uniqueness of solution for system governed by partial integro-differential equations of parabolic has been considered in [**I.H.Solan and V.Thome, Time discretization of an integro-differential equation of parabolic type**]. In a linear differential equation, all terms involving the unknown function are linear in the unknown functions. Linear equation is one in which the equation and any boundary or initial conditions do not include any product of the dependent variables or their derivatives. Linear multistep methods constitute a power full class of numerical procedures for solving second order differential equations.

General second order linear partial differential equation**:**

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$ where A, B, C, D, E, F, G are functions of x and y.

The second order spatial derivative in the problem was approximated with fourth order accuracy. The stability the convergence of proposed method were investigated. Multistep structure with the use of grid points, we seek a method that are multistage and multi value, because it will be convenient to extend the general linear method formulation to the high order case **(Butcher 2003)**. Multistep methods make use of information about the solution and its derivative at more than one point in order to extrapolate to the next point.

Chapter 3

Research Methodology

This section consists of methods and materials used to undertake the study. These are study design and procedure of the study.

3.1 Study Design

Descritization $\Rightarrow$ replacing the partial differential equation $\Rightarrow$ solving the linear system of equations using matlab$\Rightarrow$ the resulting linear system is solved by iterative methods

3.2 Procedures of the Study

To attain the objective of the study, the following procedures are undertaken:

- Defining the problem/formulating the method.
- Discretizing the domain/partial integro-differential equation.
- Replacing the given equation by the finite difference equations.
- Solving system of linear equation.
- Writing a code using matlab language.
- Validating the methods using numerical examples.
- Discussing the methods from the graphs and tables

Chapter 4

Result and Discussion

4.1 Partial Integro-Differential Equations (PIDEs)

An integro-differential equation (IDE) is defined as a functional equation involving unknown function f(x), together with both derivatives of the function and integral operations on f(x). While solution techniques for many types of these problems are well known, there is a large class of problems that lack standard solution methods, namely, partial integro-differential equations. In general, partial integro-differential equations are difficult to solve analytically and, as a result, one has to resort to numerical approximation of the solution. A differential equation is a relation between an unknown function and its derivatives. An equation containing the derivative of one or more dependent variables with respect to one or more independent variables.

To talk about them, we shall classify differential equations by type, order and linearity. When derivatives are taken with respect to one variable, the integro-differential equation called ordinary. The ordinary/partial differential equation along with the weighted integral of unknown function gives rise to an integro-differential equation (IDE) or a partial integro-differential equation (PIDE) respectively. If an equation contains one or more than dependent variable with respect to two or more than two independent variable it is said to be a partial integro-differential equations (PIDEs). The general form of the linear partial integro-differential equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \lambda \int_a^b k(x,y)u(y)dy + f(x,t)$$

where a known function of x, k(x,y) is a known function of x and y called the kernel of the integro-differential equation where a and λ are known parameters and f is the unknown function that must be determined.

A differential equation that contains in addition to the dependent variable and the independent variables, one or more partial derivatives of the dependent variable is called a partial differential equation. Partial differential equations are equations that involves

unknown functions of several variables and their partial derivatives. An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called partial differential equation (PDE). Partial differential equations are relations between the values of an unknown function and its derivatives of different orders. Partial differential equation is said to be linear if it is linear in the unknown function and all its derivatives with coefficients depending only the independent variables .
In general it my be written in the form :

$$F(x, y, ..., u, u_x, u_y, ..., u_{xx}, u_{xy}, ...,) = 0$$

Involving several independent variables x,y,....,an unknown function u of these variables and the partial derivative:

$$u_x, u_y, ..., u_{xx}, u_{xy}, ...,$$

Note : If the integral part $\int_0^{t_{i+1}} kudy = 0$ and $f(x,t) = 0$, the partial integro-differential equation becomes one dimensional heat equation of the form**:**

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

where $u(x,t) \in [a,b] \times [0,T]$ the boundary condition associated with (1) is given by

$$u|_{x=a} = g_1(t), u|_{x=b} = g_2(t), t \geqslant 0, \tag{2}$$

and the initial conditions are specified by

$$u|_{t=0} = u_0(x), a \leq x \leq b. \tag{3}$$

Using finite difference method for the boundary value problem (1)-(3), we set the maximum time T, choose two positive integers n and m and find the mesh constants h and k such as $h = \dfrac{b-a}{n}$ and $k = \dfrac{T}{m}$. Hence the grid points are (x_i, t_j), where $x_i = a + ih$ for $i = 0, 1, ..., n$ and $t_j = jk$ for $j = 0, 1, ..., m$.
The finite difference approximations to the various derivatives are derived as Taylor's series for each interior grid points

$$u(x_{i+1}, t_j) = u(x_i, t_j) + \frac{u'(x_i, t_j)h}{1!} + \frac{u''(x_i, t_j)h^2}{2!} + \frac{u'''(x_i, t_j)h^3}{3!} + ... \tag{4}$$

from which to obtain

$$u'(x_i, t_j) = \frac{u(x_{i+1}, t_j) - u(x_i, t_j)}{h} - \frac{h\ u''(x_i, t_j)}{2} - ....$$

Thus to have

$$u'(x_i, t_j) = \frac{u(x_{i+1}, t_j) - u(x_i, t_j)}{h} + O(h)$$

Which is the forward difference approximation of $u'(x_i, t_j)$. Similarly expansion of $u(x_{i+1}, t_j)$ in Taylor's series gives

$$u(x_{i-1}, t_j) = u(x_i, t_j) - \frac{u'(x_i, t_j)h}{1!} + \frac{u''(x_i, t_j)h^2}{2!} - \frac{u'''(x_i, t_j)h^3}{3!} + ... \quad (5)$$

from which to obtain

$$u'(x_i, t_j) = \frac{u(x_i, t_j) - u(x_{i-1}, t_j)}{h} + O(h)$$

Which is the backward difference approximation of $u'(x_i, t_j)$. A central difference approximation for $u'(x_i, t_j)$ can be obtained by subtracting equation (5)$-$ from equation (4) to get

$$u'(x_i, t_j) = \frac{u(x_{i+1}, t_j) - u(x_{i-1}, t_j)}{2h} + O(h^2)$$

This is central difference approximation of $u'(x_i, t_j$. And adding equation (5) and equation (4) to get

$$u''(x_i, t_j) = \frac{u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)}{h^2} + O(h^2)$$

Which is the central difference approximation of $u''(x_i, t_j)$. From the above we obtain the forward difference formula for $\dfrac{\partial u}{\partial t}(x_i, t_j)$ in the form

$$\frac{\partial u}{\partial t}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} + O(k) \quad (6)$$

and the central difference formula for

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2) \quad (7)$$

Then substituting (6) and (7) into (1) gives

$$\frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{k} + O(k) - \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} + O(h^2)$$

$$\Rightarrow u(x_i, t_{j+1}) = u(x_i, t_j) + (\frac{k}{h^2})(u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j)) + O(k + h^2) \quad (8)$$

with the initial boundary conditions

$$u(x_i, 0) = u_0(x_i), \;\; i = 0, 1, ..., n,$$

$$u(a, t_j) = g_1(t_j), u(b, t_j) = g_2(t_j), \;\; j = 1, 2, ..., m. \quad (9)$$

Now let us denote $\lambda = (\frac{k}{h^2})$ and let $u_{i,j}$ approximate $u(x_i, t_j)$. If we neglect the error term then from (8) we get the backward difference method with local truncation error $O(k + h^2)$ of the form:

$$u_i^{j+1} = \lambda u_{i-1}^j + u_i^j(1 - 2\lambda) + \lambda u_{i+1}^j,\ i = 1, 2, ..., n-1,\ \ j = 1, 2, ..., m. \tag{10}$$

where

$$u_i = u(x_i),\ i = 0, 1, ..., n$$

,

$$u(a, t_j) = g_1(t_j), u(b, t_j) = g_2(t_j),\ j = 1, 2, ..., m. \tag{11}$$

has also the matrix representation as follows:

$$A'u^{(j)} = u^{(j+1)},\ j = 1, 2, ..., m, \tag{12}$$

where

A'=

$$\begin{pmatrix} 1-2\lambda & \lambda & & & & \\ \lambda & 1-2\lambda & \lambda & & & \\ & \lambda & 1-2\lambda & \lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda & 1-2\lambda & \lambda \\ & & & & \lambda & 1-2\lambda \end{pmatrix} \tag{13}$$

$$u^j = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{n-3,j} \\ u_{n-2,j} \\ u_{n-1,j} \end{pmatrix}, u^{j+1} = \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{n-3,j+1} \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{pmatrix} \tag{14}$$

Note that the matrix A is tridiagonal and symmetric. Since $\lambda < 0$, then A is also positive defined and strictly diagonally dominant. It can also be shown that if $u(x_i, 0) \in u_{i,0}$, $i = 0, 1, ..., n$ and the assumptions given above are met then $u(x_i, t_j) \in u_{i,j}$, $i = 0, 1, ..., n,\ j = 1, 2, ..., m$

4.2 Integro-Differential Equation and Solution Methods(Diffusion equation)

The partial integro-differential equation (PIDE) is an integro-differential equation such that the unknown function depends on more than one independent variable . We shall focus on the partial integro-differential equation which often occur in the mathematical physics, contain derivative with respect to different variables, are called partial integro-differential equation (PIDE) of the form **:**

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_0^t k(t, s)u(x, s)ds + f(x, t) \tag{15}$$

where $k(t, s)$ is the kernel of integro-differential equation, $u(x, s)$ is the unknown real function that must be determined and $f(x, t)$ is the unknown function thought for :
$x \in [0, 1], t \in [0, T]$ and are positive constants and x = space-position, t = time.
With the boundary condition associated with (15) is given by

$$u|_{x=a} = g_1(t), u|_{x=b} = g_2(t), t \geqslant 0, \tag{16}$$

and the initial conditions are specified by

$$u|_{t=0} = u_0(x), a \leq x \leq b. \tag{17}$$

The solution of the linear problem of the form (15) is usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Equation (15) is called reaction-diffusion equations.
The numerical solutions of partial integro-differential equations were introduced using some of the numerical techniques such as, compact finite difference methods. Partial differential equation containing many independent variables are harder than ordinary differential equation containing few independent variables.

4.3 Compact finite difference method for solving partial integro-differential equations

Compact finite difference method is a special finite difference method which uses the values of the function and its derivatives. Compact finite difference approximation gives

an efficient method for solving a partial integro-differential equation. Compact finite difference method is designed to obtain quick and accurate solution to partial differential equations. Compact schemes are based on a fourth-order accurate approximation to the derivative calculated from ordinary differential equation and partial differential equation. To developed the scheme for one-dimensional uniform Cartesian grids with spacing $\triangle x = h$, let us introduce the following notations [**J.C. Strikwerda, 2004**]: If $u_i \equiv u(x_i)$, then we use notations and applying Taylor's series. Consider the fourth order compact finite difference method to solve problem (15)-(17). To construct a numerical solution, first consider the nodal points (x_i, t_j) defined in the region $[a, b] \times [0, T]$ where $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, $x_{i+1} - x_i = h$, and $0 = t_0 < t_1 < ... < t_j < ... < T$, $t_{j+1} - t_j = \tau$. In such a case we have $x_i = a + ih$ for $i = 0, 1, 2, ..., n$, and $t_j = j\tau$ for $j = 0, 1, 2, ...$
The approximate solutions at the point (x_i, t_j) are denoted by u_j . Also, we define the following operators:
$\delta_{x_-} u_j = \dfrac{u_j - u_{j-1}}{h}$, $\delta_x^2 u_j = \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$, $\delta_{t_+} u_j = \dfrac{u_{j+1} - u_j}{\tau}$, $\delta_{t_-} u_j = \dfrac{u_j - u_{j-1}}{\tau}$
$\delta_0 u_j = \dfrac{u_{j+1} - u_{j-1}}{2\tau}$
$\delta_t^2 u_j = \delta_{t_+}\delta_{t_-} u_j = \dfrac{1}{\tau^2}(u_{j+1} - 2u_j + u_{j-1})$
$Ahu_j = \dfrac{1}{12}(u_{j-1} + 10u_j + u_{j+1})$ where $j = 1, 2, ..., n-1$.
The initial condition in equation (16) is approximated as follows:

$$u(x, 0) = u_0 = u(x, t_0),\ a \le x \le b. \tag{18}$$

Next, the two-point Euler backward differentiation formula is manipulated to approximate u_t, given in equation (15), at the time-level t_{j+1} for $j = 0, 1, 2, ...$ There for, we have

$$\frac{u_{j+1}(x) - u_j(x)}{\tau} - \frac{d^2 u_{j+1}(x)}{dx^2} = \int_0^{t_{j+1}} k_{i+1}(s) u(x, s) ds + f_{j+1}(x), \tag{19}$$

where $f_{j+1}(x) = f(x, t_{j+1})$, $k_{j+1}(s) = k(t_{j+1}, s)$ and $u_{j+1}(x) = u(x, t_{j+1})$. Equivalently, we can rewrite equation (19) as**:**

$$u_{j+1}(x) - u_j(x) - \tau \frac{d^2 u_{j+1}(x)}{dx^2} = \tau \int_0^{t_{j+1}} k_{j+1}(s) u(x, s) ds + \tau f_{j+1}(x),$$

Again equivalently, to rewrite equation (19) as **:**

$$u''_{j+1}(x) = \frac{u_{j+1}(x) - u_j(x)}{\tau} - \int_0^{t_{j+1}} k_{j+1}(s) u(x, s) ds - f_{j+1}(x) \tag{20}$$

putting $x = x_i$, $i = 1, ..., n-1$ in (20), then

$$u''_{i+1,j} = \frac{u_{i+1,j} - u_{i,j}}{\tau} - \int_0^{t_{j+1}} k_{j+1}(s) u(x_i, s) ds - f_{i+1,j},\ \ j = 0, ..., n, \tag{21}$$

where $u''_{i+1,j} = u''(x_i, t_{j+1})$, $u_{i+1,j} = u(x_i, t_{j+1})$, $u_{i,j} = u(x_j, t_j)$, and $f_{i+1}, j = f(x_i, t_{j+1})$. The fourth order accurate finite difference estimate for $u''(x)$ is used from (7) to give

$$\delta_x^2 u_{i+1,j} = u''_{i+1,j} + (\frac{h^2}{12}\delta_x^2)(u''_{i+1,j}) + O(h^4). \tag{22}$$

Then, a compact (implicit) approximation for $u''(x)$ with fourth-order accuracy will be given as**:**

$$u''_{i+1,j} = \frac{\delta_x^2 u_{i+1,j}}{(1 + \frac{h^2}{12}\delta_x^2)} + O(h^4). \tag{23}$$

Using this estimate and considering the discrete solution of equation (21) which satisfies the approximation, to get **:**

$$[1 - \frac{h^2}{12\tau}]\delta_x^2 u_{i+1,j} - \frac{u_{i+1,j}}{\tau} + \int_0^{t_{j+1}} k_{j+1}(s)u(x_i, s)ds + \frac{h^2}{12}\int_0^{t_{j+1}} k_{j+1}(s)\delta_x^2 u(x_i, s)ds$$

$$= -\frac{u_{i,j}}{\tau} - \frac{h^2}{12\tau}\delta_x^2 u_{i,j} - f_{i+1,j} - \frac{h^2}{12\tau}\delta_x^2 f_{i+1,j} \tag{24}$$

$$\Rightarrow [\frac{1}{h^2} - \frac{1}{12\tau}](u_{i+1,j+1} + u_{i+1,j-1}) + [\frac{-2}{h^2} - \frac{5}{6\tau}]u_{i+1,j}$$
$$+\frac{5}{6}\int_0^{t_{j+1}} k_{i+1}(s)u_j(s)ds + \frac{1}{12}\int_0^{t_{i+1}} k_{j+1}(s)u_{j+1}(s)ds$$
$$+\frac{1}{12}\int_0^{t_{j+1}} k_{j+1}(s)u_{j-1}(s)ds = -\frac{1}{12\tau}(u_{i,j+1} + u_{i,j-1})$$
$$-\frac{5}{6\tau}u_{i,j} - \frac{5}{6}(f_{i+1,j+1} + f_{i+1,j-1}) - \frac{1}{12}f_{i+1,j}. \tag{25}$$

The later integral will be handled numerically using the composite weighted trapezoidal rule given by:

$$\int_0^{t_{j+1}} f(s)ds \approx \tau\sum_{m=0}^{i}[wf(t_m) + (1-w)f(t_{m+1})]$$

$$= \tau[wf(t_0) + (1-w)f(t_{j+1}) + \sum_{m=1}^{i} f(t_m)] \tag{26}$$

Using (26) to get

$$\int_0^{t_{j+1}} k_{j+1}(s)u(x, s)ds \approx \tau w k_{i+1}(0)u_0(x)$$

$$+\tau(1-w)k_{j+1}(t_{j+1})u_{j+1}(x)+\tau\sum_{m=1}^{i}k_{j+1}(t_m)u_{i+1}-m(x) \qquad (27)$$

The substitutions of this equation into equation (25) yields

$$\begin{aligned}
&[\frac{1}{h^2}-\frac{1}{12\tau}](u_{i+1,j+1}+u_{i+1,j-1})+[\frac{-2}{h^2}-\frac{5}{6\tau}]u_{i+1,j}\\
&\quad+\frac{5\tau}{6}[wk_{j+1}(0)u_{0,j}+(1-w)k_{j+1}(t_{j+1})u_{i+1,j}]\\
&\quad+\frac{5\tau}{6}\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j}+\frac{\tau}{12}wk_{j+1})(0)u_{0,j+1}\\
&\quad+\frac{\tau}{12}[(1-w)k_{i+1}(t_{i+1})u_{i+1,j+1}+\sum_{m=1}^{i}k_{j+1}(t_m)u_{i+1-m,j+1}]\\
&\quad+\frac{\tau}{12}[wk_{j+1}(0)u_{0,j-1}+(1-w)k_{j+1}(t_{j+1})u_{i+1,j-1}]\\
&\quad+\frac{\tau}{12}\sum_{m=1}^{i}k_{i+1}(t_m)u_{i+1-m,j-1}=-\frac{1}{12\tau}(u_{i,j+1}+u_{i,j-1})\\
&-\frac{5u_{i,j}}{6\tau}-\frac{5}{6}(f_{i+1,j+1}+f_{i+1,j-1})-\frac{f_{i+1,j}}{12}. \qquad (28)
\end{aligned}$$

Let $U_j(x)$ be a function that approximates $u(x,t_j)$ for the time-level $t_j=j\tau$ and is a linear combination of $n+1$ shape functions which is expressed as:

$$U_j(x)=\sum_{m=0}^{n}c_{mi}\phi_m(x), \qquad (29)$$

where $\{c_{mi}\}_{m=0}^{n}$ are the unknown real coefficients, to be evaluated, and the $\phi_m(x)$ are any knowing basis functions. Shape function is the function which interpolates the solution between the discrete values obtained at the mesh node.

The approximate solutions $u_j(x)$ for different time-levels are determined iteratively as follows. Starting with the time-level $t_0=0$, the value of $u_0(x_i), u_0(x_{i+1})$, and $u_0(x_{i-1})$ for $i=1,2,...,n-1$ are found from equation (16). Next, we will approximate the solution u_{i+1} for $i=0$ in equation (25) by the shape functions U_1, as is given in equation (29). Hence equation (25) is approximated by:

$$\begin{aligned}
&[\frac{1}{h^2}-\frac{1}{12\tau}](U_{1,j+1}+U_{1,j-1})+[\frac{-2}{h^2}-\frac{5}{6\tau}]U_{1,j}\\
&\quad+\frac{5\tau}{6}[wk_1)(0)u_{0,j}+(1-w)k_1(t_1)U_{1,j}]\\
&\quad+\frac{\tau}{12}[wk_1)(0)u_{0,j+1}+(1-w)k_1(t_1)U_{1,j+1}]
\end{aligned}$$

$$+\frac{\tau}{12}[wk_1)(0)u_{0,j-1}+(1-w)k_1(t_1)U_{1,j-1}]=-\frac{f_{1,j}}{12}$$

$$=\frac{-1}{12\tau}[u_{0,j+1}+u_{0,j-1}]-\frac{5u_{0,j}}{6\tau}-\frac{5}{6}[f_{1,j+1}+f_{1,j-1}] \qquad (30)$$

Replacing U_1 by the approximate solution given by equation (29) yields the following linear system of $n-1$ equations

$$[\frac{1}{h^2}-\frac{1}{12\tau}](\sum_{m=0}^{n}c_{m1}\phi_{mj+1}+\sum_{m=0}^{n}c_{m1}\phi_{mj-1})$$

$$+[\frac{-2}{h^2}-\frac{5}{6\tau}]\sum_{m=0}^{n}c_{m1}\phi_{mj}+\frac{5\tau(1-w)k_1(t_1)}{6}\sum_{m=0}^{n}c_{m1}\phi_{mj}$$

$$+\frac{\tau(1-w)k_1(t_1)}{12}(\sum_{m=0}^{n}c_{m1}\phi_{mj+1}+\sum_{m=0}^{n}c_{m1}\phi_{mj-1})$$

$$=-\frac{5}{6}[\tau wk_1(0)+\frac{1}{\tau}]u_{0,j}-\frac{5}{6}(f_{1,j+1}+f_{1,j-1})$$

$$-[\frac{\tau wk_1(0)}{12}+\frac{1}{12\tau}](u_{0,j+1}+u_{0,j-1})-\frac{f_{1,j}}{12}, \qquad (31)$$

where $\sum_{m=0}^{n}c_{m1}\phi_{mj+1}=\sum_{m=0}^{n}c_{m1}\phi_m(x_{j+1})$.
Rewrite equation (31) as

$$\sum_{m=0}^{n}c_{m1}[a_1\phi_{mj+1}+a_2\phi_{mj}+a_1\phi_{mj-1}]$$

$$=a_3u_{0,j}+a_4(u_{0,j+1}+u_{0,j-1})-\frac{1}{12}f_{1,j}-\frac{5}{6}(f_{1,j+1}+f_{1,j-1}) \qquad (32)$$

where
$a_1=\frac{1}{h^2}-\frac{1}{12\tau}+\frac{\tau}{12}(1-w)k_1(t_1)$, $a_2=\frac{-2}{h^2}-\frac{5}{6\tau}+\frac{5\tau}{6}(1-w)k_1(t_1)$,

$$a_3=\frac{-5}{6}(\tau wk_1(0)+\frac{1}{\tau}),\ \ a_4=-[\frac{\tau wk_1(0)}{12}+\frac{1}{12\tau}] \qquad (33)$$

The system (32) consists of $(n+1)$ equation in the $(n+1)$ unknowns $\{c_m\}_{m=0}^{n}$. To get a solution of this system we need two additional conditions. These conditions are obtained from the boundary conditions (16)

$$\sum_{m=0}^{n}c_{m1}\phi_m(a)=g_1(t_j), j=0,...,n \qquad (34)$$

$$\sum_{m=0}^{n}c_{m1}\phi_m(b)=g_2(t_j), j=0,...,n \qquad (35)$$

Since f and u_0 are known at every grid point, the right hand side of equation (32) is known for all nodes. The system (32), equations (34) and (35) consist of $(n+1)$ equations in $(n+1)$ unknowns; this system is of the form

$$AC = F \tag{36}$$

Upon solving the system (36), the function $u_1(x)$ is approximated by the sum:

$$u_1(x_i) = \sum_{m=0}^{n} c_{m1}\phi_m(x_i), i = 0, 1, 2..., n. \tag{37}$$

Next, to find the approximate solution at time levels $t_2, t_3, ...$ recursively by solving the following system for $j = 2, 3, ...$

$$\begin{aligned}
&\sum_{m=0}^{n} c_{mi}(a_1\phi_{mj+1} + a_2\phi_{mj} + a_1\phi_{mj-1}) \\
&= \frac{-1}{12\tau}(u_{i-1,j+1} + u_{i-1,j-1}) - \frac{5}{6\tau}u_{i-1,j} \\
&-\frac{1}{12\tau}(u_{0,j+1} + u_{0,j-1}) - \frac{5}{6}(f_{i,j+1} + f_{i,j-1}) \\
&-\frac{1}{12}f_{i,j} - \frac{5}{6\tau}u_{0,j} - \frac{5\tau}{6}\sum_{m=1}^{i} k_j(t_m)u_{i-m,j} \\
&-\frac{\tau}{12}\sum_{m=1}^{i} k_j(t_m)(u_{i-m,j+1} - u_{i-m,j-1}),
\end{aligned} \tag{38}$$

$$\sum_{m=0}^{n} c_{m1}\phi_m(a) = g_1(t_j), j = 0, ..., n \tag{39}$$

$$\sum_{m=0}^{n} c_{m1}\phi_m(b) = g_2(t_j), j = 0, ..., n \tag{40}$$

Note: The relation ship between equation (32) and (12) both are approximate solution of the parabolic equation and (12) is the solution of heat equation with the integral part and the function of $f(x,t) = 0$, for the partial integro-differential equation.

4.4 Stability, consistence and convergence analysis of the compact scheme

Stability: The error caused by a small perturbation in the numerical method remains bounded. A method is said to be stable if it produces a bounded solution which imitates the exact solution otherwise it is said to be unstable. When the difference between the computed numerical solution and the exact solution of the discrete equation remains bounded as time grows. A numerical method is said to be stable if numerical errors, e.g. due to round off, are not amplified, and the approximate solution remains bounded.
Consistence: The finite difference approximation is called consistent with the differential equation if the truncation error goes to zero when the numerical parameters are made arbitrarily small. A numerical method is said to be consistent if

$$\lim_{\triangle x \to 0} \|\psi_n\| = 0$$

where ψ_n is the discretization error. According to the interpolation of the discretization error stated after that equation, any consistent numerical method closely matches the original differential equation when the step size $\triangle x$ is sufficiently small. Given a partial differential equation $Au = f$, approximation by:
$A_{i,j}u = f$. The finite difference scheme $A_{i,j}u = f$ is consistent with the partial differential equation if $A\psi - A_{i,j}\psi \to 0$ as $\triangle x, \triangle k \to 0$, where ψ unknown function. Therefor a scheme approximating a partial differential equation may be stable but has a solution that converges to the solution of a different equation as the mesh length $\to \infty$ in consistent. According to the interpolation of the discretization error stated after that equation, any consistent numerical method closely matches the original differential equation when the mesh size is sufficiently small.
Proof: Let

$$A = \frac{\partial}{\partial k} + \alpha \frac{\partial}{\partial x}$$
$$A\psi = \frac{\partial \psi}{\partial k} + \alpha \frac{\partial \psi}{\partial x}$$

where $\psi = \psi(x,k)$ is the dependent variable and α is a constant coefficient
$A_{i,j}$: forward space, forward time

$$A_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j}}{\triangle k} + \alpha \frac{\psi_{i+1,j} - \psi_{i,j}}{\triangle x}$$

$$\psi_{i,j+1} = \psi_{i,j} + \triangle k \frac{\partial \psi}{\partial k}|_{(x_i,k_j)} + \frac{(\triangle k)^2}{2}\frac{\partial^2 \psi}{\partial k^2}| + O(k^2)$$
$$\psi_{i+1,j} = \psi_{i,j} + \triangle x \frac{\partial \psi}{\partial x}|_{(x_i,k_j)} + \frac{(\triangle x)^2}{2}\frac{\partial^2 \psi}{\partial x^2}| + O(x^2)$$

$$A_{i,j}\psi = \frac{\partial\psi}{\partial k} + \alpha\frac{\partial\psi}{\partial x} + \frac{1}{2}\triangle k\frac{\partial^2\psi}{\partial k^2} + \frac{\alpha}{2}\triangle x\frac{\partial^2\psi}{\partial x^2} + O(\triangle k^2) + O(\triangle x^2)$$

$$\therefore A\psi - A_{i,j}\psi = -\frac{1}{2}\triangle k\frac{\partial^2\psi}{\partial k^2} - \frac{\alpha}{2}\triangle x\frac{\partial^2\psi}{\partial x^2} + O((\triangle x^2) + (\triangle k^2)) \to 0$$

as $\triangle x, \triangle k \to 0$. There for the scheme is consistent.

Convergence : A one step finite difference system approximating a partial differential equation is a convergent scheme if the solution of the finite difference scheme $\overline{u}_{i,j}$ converges to $\overline{u}(x,t)$ any solution of the partial differential equation us $\triangle x, \triangle t \to 0$.

Which means as a method is continually refined by taking smaller and smaller step-sizes, the sequence of approximate solutions must converge to the exact solution.

Assume that the solution of Eq.(32) at the grid point (ih, jk) is :

$$u(i,j) = \lambda^j e^{ip\theta} \tag{41}$$

where $p = \sqrt{-1}$, θ is the areal number and λ is a complex number. Substituting equation (41) into equation (32) gives**:**

$$\sum_{m=0}^{n} c_{m1}[a_1\phi_{mj+1} + a_2\phi_{mj} + a_1\phi_{mj-1}]$$

$$= a_3\lambda^j e^{(i-1)p\theta} + a_4(\lambda^{j+1}e^{(i-1)p\theta} + \lambda^{j-1}e^{(i-1)p\theta}) - \frac{1}{12}f_{1,j} - \frac{5}{6}(f_{1,j+1} + f_{1,j-1})$$

for $i = 1, ..., n-1$ and $j = 1, ..., m-1$. Let $f_{1,j}, f_{1,j+1}, f_{1,j-1} \to 0$

$$\Rightarrow \lambda^j(a_3e^{ip\theta}e^{-p\theta}) + (\lambda^{j+1} + \lambda^{j-1})(a_4e^{ip\theta}e^{-p\theta} + a_4e^{ip\theta}e^{-p\theta})$$

$$\Rightarrow \lambda(a_3e^{p\theta}) + \lambda^2(a_4e^{2p\theta} + a_4) = 0$$

$$a\lambda^2 + b\lambda = 0 \tag{42}$$

where $a = a_4e^{2p\theta} + a_4$, $b = a_3e^{p\theta}$

By using Routh Hurtiz criteria and using the transformation [Numerical Solution of One-Dimensional parabolic equation], $\lambda = \dfrac{1+z}{1-z}$ in equation (42) we have :

$$(a-b)\lambda^2 + 2(a)\lambda + (a+b) = 0 \tag{43}$$

If $|\lambda| < 0$, then the difference scheme of equation (32) is stable. It is sufficient to show that: $a > 0$ and $b < 0$, from equation (42)

$$a = a_4e^{2p\theta} + a_4 = -[\frac{\tau wk_1(0)}{12} + \frac{1}{12\tau}]e^{2p\theta} - [\frac{\tau wk_1(0)}{12} + \frac{1}{12\tau}]$$

$$= -\frac{\tau wk_1(0)}{12}e^{2p\theta} - \frac{1}{12\tau}e^{2p\theta} - \frac{\tau wk_1(0)}{12} - \frac{1}{12\tau}$$

$$b = a_3 e^{p\theta} = -\frac{5}{6}(\tau w k_1(0) + \frac{1}{\tau})e^{p\theta}$$

Clearly, for sufficiently small k_1, $a > 0$ and $b < 0$. Thus, the compact finite difference scheme given equation (32) is absolutely stable for partial integro-differential equation. Now, expand equation equation (19) in Taylor series and replace the derivatives involving τ and x for the relation,

$$\frac{u_{j+1}(x) - u_j(x)}{\tau} - \frac{d^2 u_{j+1}(x)}{dx^2} = \int_0^{t_{j+1}} k_{j+1}(s)u(x,s)ds + f_{j+1}(x) \tag{44}$$

and then to drive the local truncation error. The principal part of the local truncation error of the proposed method using equations (21), (22) and (31) for the partial integro-differential equation is:

$$\begin{aligned}
U_1 &= [\frac{1}{h^2} - \frac{1}{12\tau}](\sum_{m=0}^{n} c_{m1}\phi_{mj+1} + \sum_{m=0}^{n} c_{m1}\phi_{mj-1}) \\
&+[\frac{-2}{h^2} - \frac{5}{6\tau}]\sum_{m=0}^{n} c_{m1}\phi_{mj} + \frac{5\tau(1-w)k_1(t_1)}{6}\sum_{m=0}^{n} c_{m1}\phi_{mj} \\
&+\frac{\tau(1-w)k_1(t_1)}{12}(\sum_{m=0}^{n} c_{m1}\phi_{mj+1} + \sum_{m=0}^{n} c_{m1}\phi_{mj-1}) \\
&= -\frac{5}{6}[\tau w k_1(0) + \frac{1}{\tau}]u_{0,j} - \frac{5}{6}(f_{1,j+1} + f_{1,j-1}) \\
&-[\frac{\tau w k_1(0)}{12} + \frac{1}{12\tau}](u_{0,j+1} + u_{0,j-1}) - \frac{f_{1,j}}{12}
\end{aligned}$$

Thus, $U_1 \to 0$ as $h \to \infty$, $\tau \to \infty$ or $\to 0$ and $f_{1,j}$, $f_{1,j+1}$, $f_{1,j-1} \to 0$. So that the scheme is consistent with order of $O(h^4)$. Hence the scheme is convergent.

Lax-Richkmyer Equivalence Theorem (Fundamental Theorem of Numerical Analysis).
Consistency + Stability $\Leftrightarrow$ Convergence.
If a linear finite difference scheme is consistent with a well defined linear initial value problem (IVP) the stability guarantees convergence as mesh length $\to 0$

Proof: Let $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$, be approximated by

$$\frac{u_{i,j+1}}{2k} - \frac{u_{i+1,j} - (3/2u_{i,j+1} + 1/2u_{i,j-1}) + u_{i-1,j}}{h^2} = 0$$

$$\begin{aligned}
u_{i,j+1} - u_{i,j-1} &= u + k\frac{\partial u}{\partial k} + \frac{k^2}{2}\frac{\partial^2 u}{\partial k^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial k^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial k^4} + ... \\
&-(u - k\frac{\partial u}{\partial k} + \frac{k^2}{2}\frac{\partial^2 u}{\partial k^2} - \frac{k^3}{6}\frac{\partial^3 u}{\partial k^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial k^4} + ...)
\end{aligned}$$

$$= 2k\frac{\partial u}{\partial k} + \frac{k^2}{3}\frac{\partial^3 u}{\partial k^3} + \frac{2k^5}{120}\frac{\partial^5 u}{\partial k^5} + ...$$

$$\frac{3}{2}u_{i,j+1} + \frac{1}{2}u_{i,j-1} = \frac{3}{2}(u + h\frac{\partial u}{\partial t} + \frac{h^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial t^3} + ...)$$

$$+\frac{1}{2}(u - h\frac{\partial u}{\partial t} + \frac{h^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial t^3} + ...)$$

$$= 2u + h\frac{\partial u}{\partial t} + 2h^2\frac{\partial^2 u}{\partial t^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial t^3} + ...$$

$$u_{i+1,j} + u_{i-1,j} = 2u + h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + ...$$

$$T_{i,j} = \frac{\partial u}{\partial k} + \frac{k^2}{6}\frac{\partial^3 u}{\partial k^3} + \frac{k^5}{120}\frac{\partial^5 u}{\partial t^5}$$

$$-\frac{1}{h^2}\{2u + h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + ... - 2u - k\frac{\partial u}{\partial k} - 2k^2\frac{\partial^2 u}{\partial k^2}....\}$$

$$= (\frac{\partial u}{\partial k} - \frac{\partial^2 u}{\partial x^2}) + \frac{k}{h^2}\frac{\partial u}{\partial k} + \frac{2k^2}{h^2}\frac{\partial^2 u}{\partial k^2} + \frac{k^2}{6}\frac{\partial^3 u}{\partial k^3} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + ...$$

case one $k = \lambda h$, $T_{i,j} = (\frac{\partial u}{\partial k} - \frac{\partial^2 u}{\partial x^2}) + \frac{\lambda}{h}\frac{\partial u}{\partial k} + 2\lambda^2\frac{\partial^2 u}{\partial k^2}\frac{\lambda^2 h^2}{6}\frac{\partial^3 u}{\partial k^3}...$ as $h \to 0$, $\frac{\lambda}{h}\frac{\partial u}{\partial k} \to \infty$

$\therefore$ Inconsistent

case two $k = \lambda h^2$

$$T_{i,j} = (\frac{\partial u}{\partial k} - \frac{\partial^2 u}{\partial x^2}) + \lambda\frac{\partial u}{\partial k} + \lambda^2\frac{\partial^2 u}{\partial k^2} + ... \to 0$$

as $h \to 0$

$\therefore$ consistent

Convergence of differential equation for the error (Direct method)

Consider: $\frac{\partial \overline{u}}{\partial k} = \frac{\partial^2 \overline{u}}{\partial x^2}$, $0 < x < 1$, $k > 0$, initial condition $\overline{u}(x, 0)$ = given and boundary condition $\overline{u}(a, k)$ =given, $\overline{u}(b, k)$ = given, $\forall k$, $a = 0$, $b = 1$ $(*)$. Then the finite difference approximating $(*)$ by $\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$ at mesh point $u_{i,j} = \overline{u}_{i,j} - e_{i,j}$

$$u_{i,j+1} = \overline{u}_{i,j} - e_{i,j+1}...$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

where $\lambda = \frac{k}{h^2}$

$$\Rightarrow u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i+1,j}$$

$$e_{i,j+1} = \lambda e_{i-1,j} + (1 - 2\lambda)e_{i,j} + \lambda e_{i+1,j}$$

$$+\overline{u}_{i,j+1} - \overline{u}_{i,j} + \lambda(2\overline{u}_{i,j} - \overline{u}_{i-1,j} - \overline{u}_{i+1,j})$$

To expand Taylor series

$$\overline{u}_{i+1,j} = \overline{u}(x_i + h, k_j) = \overline{u}_{i,j} + h\frac{\partial \overline{u}}{\partial x}(x_i, k_j) + \frac{h^2}{2}\frac{\partial k^2 \overline{u}}{\partial x^2}(x_i + \theta_1 h, k_j)$$

$$\overline{u}_{i-1,j} = \overline{u}(x_i - h, k_j) = \overline{u}_{i,j} - h\frac{\partial \overline{u}}{\partial x}(x_i, k_j) - \frac{h^2}{2}\frac{\partial^2 \overline{u}}{\partial x^2}(x_i - \theta_2 h, k_j)$$

$$\overline{u}_{i,j+1} = \overline{u}_{i,j} + k\frac{\partial \overline{u}}{\partial k}(x_i, k_j + \theta_3 k), \;\; 0 < \theta_1, \theta_2, \theta_3 < 1$$

$$e_{i,j+1} = \lambda e_{i-1,j} + (1-2\lambda)e_{i,j} + \lambda e_{i+1,j}$$

$$+k\{\frac{\partial \overline{u}}{\partial k}(x_i, k_j + \theta_3 k) - \frac{\partial^2 \overline{u}}{\partial x^2}(x_i + \theta_4 h, k_j)\} \quad -1 < \theta_4 < 1.....(**)$$

$(**)$ is a difference equation for $e_{i,j}$, let $E_j = max(e_{i,j})$, j - level

$$M = max(\frac{\partial \overline{u}}{\partial k} - \frac{\partial^2 \overline{u}}{\partial x^2}) \qquad \forall_{i,j}$$

for $\lambda \leq \frac{1}{2}$, coefficients of $e_{i,j}$ in $(**)$ are positive of zero.

$$\therefore |e_{i,j+1}| \leq \lambda|e_{i-1,j}| + (1-2\lambda)|e_{i,j}| + \lambda|e_{i+1,j}| + kM$$

$$\leq \lambda E_j + (1-2\lambda)E_j + \lambda E_j + kM = E_j + kM$$

$|e_{i,j+1}| \leq E_j + kM$, which is true $\forall i$, true for max $|e_{i,j+1}|$
$\therefore E_{j+1} \leq E_j + kM \leq E_{j-1} + kM + kM \leq E_{j-2} + 3kM \leq E_0 + jkM = jkM \; (E_0 = 0) = kM(\therefore jk = k)$ no error
further when $h \to 0$, $k = \lambda h^2 \to 0$, $M(\frac{\partial \overline{u}}{\partial k} - \frac{\partial^2 \overline{u}}{\partial x^2}|_{i,j} \to 0$, $\therefore E_{j+1} \to 0$ as $|\overline{u}_{i,j+1} - u_{i,j}| \leq E_j$, $u \to \overline{u}$ as $h \to 0$ where $\lambda \leq \frac{1}{2}$, k-finite; $\lambda > \frac{1}{2}$: The errors blows up.

Let us discuss some useful notations and lemmas.

Let $V = \{v \mid v = (v_0, v_1, ..., v_M),\; v_0 = v_M = 0\}$. For any grid function $v, w \in V$, we define the following inner product and corresponding norms as follows.

$$\langle v,\; w\rangle = h\sum_{i=1}^{M-1} v_i w_i,\; \|\, v \,\|^2 = \langle v,\; v\rangle,\; \|\, v \,\| = \sqrt{h\sum_{i=1}^{M-1} \mid v_i \mid^2}, \quad \|\, \delta_x v \,\| = \sqrt{h\sum_{i=0}^{M-1} \mid \delta_x v_i \mid^2}$$

$$\mid v \mid_1 = \sqrt{(\delta_x v, \delta_x v)}, \quad \|\, v \,\|_\infty = \max_{1 \leqslant i \leqslant M-1} \mid v_i \mid .$$

$$\|\, v \,\| = \sqrt{(v, v)},$$

$$\|\, \delta_x v \,\|^2 = h\sum_{i=1}^{M}(\delta_x v_i)^2$$

Lemma 4.3.1. Let $v \in V$ then

$$\|\delta_x^2 v\|^2 \leq \frac{4}{h^2}\|\delta_x v\|^2,$$

where

$$\|\delta_x^2 v\|^2 = h\sum_{i=1}^{M-1}(\delta_x^2 v_i)^2.$$

Proof.

$$\|\delta_x^2 v\| = \frac{1}{h^4}h\sum_{i=1}^{M-1}[(v_{i+1}-v_i)+(v_{i-1}-v_i)]^2.$$

$$= \frac{1}{h^2}h\sum_{i=1}^{M-1}[(\delta_x v_{i+1})^2 + (\delta_x v_i)^2 - 2(\delta_x v_{i+1})(\delta_x v_i)]$$

$$\le \frac{2}{h^2}h\sum_{i=2}^{M}(\delta_x v_i)^2 + \frac{2}{h^2}h\sum_{i=1}^{M-1}(\delta_x v_i)^2 \le \frac{4}{h^2}\|\delta_x v\|^2.$$

which completes the proof.

Lemma 4.3.2. Suppose that $u(x) \in C^6[x_{i-1}, x_{i+1}]$, then a compact finite difference form of the following differential equation

$$\begin{cases} u'' + \beta u' = f(x), x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{45}$$

Is

$$(r_1 + r_2)u_{i+1} - 2r_1 u_i + (r_1 - r_2)u_{i-1} = (r_3 + r_4)f_{i+1} + 10r_3 f_i + (r_3 - r_4)f_{i-1} \tag{46}$$

where

$$r_1 = \frac{1}{h^2} + \frac{\beta^2}{12},\ \ r_2 = \frac{\beta}{2h},\ \ r_3 = \frac{1}{12},\ \ r_4 = \frac{\beta h}{24}.$$

Proof: The well-know central difference approximation for the first and second derivatives of $u(x)$ are applied which result in the following discrete form of equation (45) at the x_i point :

$$\delta_x^2 u_i + \beta\delta_x u_i - \tau_i = f_i, \tag{47}$$

where

$$\delta_x^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{h^2}{12}u^{(4)},$$

$$\delta_x u_i = \frac{u_{i+1} - u_{i-1}}{2h} + \frac{h^2}{6}u''', \tag{48}$$

$$\tau_i = \frac{h^2}{12}(\frac{d^4u}{dx^4} + 2\beta\frac{d^3u}{dx^3}) + O(h^4).$$

In order to take a fourth-order compact difference scheme for equation (45), the third and fourth derivatives of $u(x)$, in (48) should be approximated [**Austin, TX.(1995)**]. Considering (45), one has:

$$(\frac{d^3u}{dx^3})_i = (\frac{df}{dx} - \beta\frac{d^2u}{dx^2})_i = (\delta_x f_i - \beta\delta_x^2 u_i) + O(h^2), \tag{49}$$

$$(\frac{d^4u}{dx^4})_i = (\frac{d^2f}{dx^2} - \beta\frac{d^3u}{dx^3})_i = (\delta_x^2 f_i - \beta\delta_x f_i + \beta^2\delta_x^2 u_i) + O(h^2).$$

By substituting (49) in to the third equation of (48), truncation error can be written as the following:

$$\tau_i = \frac{h^2}{12}[(\delta_x^2 f_i + \beta\delta_x f_i - \beta^2\delta_x^2 u_i) + O(h^4)]. \tag{50}$$

By substituting (50) in (47), a fourth-order compact finite difference form of equation (45) can be obtained.

Lemma (4.3.3) :[Chen H.2012]. Suppose U_h be the space of grid functions as follows: $U_h = \{U \mid U = (U_0, U_1, ..., U_{M-1}, U_M),\ U_0 = U_M = 0\}$. Defining the grid function $U_{i,j} = u(x_i, t_j),\ 0 \le i \le M,\ 0 \le j \le N,$ for any two grid functions $U, W \in U_h$,
one gets the followings;

$\delta_t U_{i,j} = \frac{1}{k}(U_{i,j+1} - U_{i,j}),\ \delta_x U_i = \frac{1}{2h}(U_{i+1} - U_{i-1}),\ \delta_x^2 U_i = \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1}),$

$\overline{U}_{i,j} = \frac{1}{2}(U_{i,j+1} + U_{i,j}),$

$(UW)_i = U_i W_i, \| U \|_\infty = \max_{1\leqslant i\leqslant M-1} \mid U_i \mid, \langle U, W\rangle = h\sum_{i=1}^{M-1} U_i W_i, \| U \|^2 = \langle U, U\rangle.$ (51)

(A) If

$$U_{\cdot,k}\,,\ \ U_{\cdot,1} \in U_h,$$

then

$$\mid \langle(\delta_x^2 + \beta\delta_x)U_{\cdot,k}\,, U_{\cdot,1}\,\rangle \mid \le \frac{4+\beta}{h^2} \| U_{\cdot,k} \| \| U_{\cdot,1} \| .$$

Proof

(A): When $N \geqslant 1$, one has**:**

$$\begin{aligned}\mid \langle(\delta_x^2 + \beta\delta_x)U_{\cdot,k}\,, U_{\cdot,1}\,\rangle \mid &\le \mid \langle{\delta_x}^2 U_{\cdot,k}\,, U_{\cdot,1}\,\rangle\mid + \beta \mid \langle\delta_x U_{\cdot,k}\,, U_{\cdot,1}\,\rangle\mid \\ &= \mid \sum_{i=1}^{M-1} h\delta_x^2 U_{i,k} U_{i,1} \mid + \beta \mid \sum_{i=1}^{M-1} h\delta_x U_{i,k} U_{i,1} \mid,\end{aligned} \tag{52}$$

each term in (52) can be estimated. First, it can be written as the following:

$$(|\sum_{i=1}^{M-1} hU_{i+1,k}U_{i,1}|)^2 \leq (\sum_{i=1}^{M-1} h \mid U_{i+1,k} \mid\mid U_{i,1} \mid)^2 \leq (\sum_{i=1}^{M-1} h(U_{i+1,k})^2)(\sum_{i=1}^{M-1} h(U_{i,1})^2)$$

$$\leq \| U._{,k} \|^2 \| U._{,1} \|^2, \tag{53}$$

by using Cauchy-Schwarz inequality one gets:

$$\| \delta_x U._{,k} \|^2 = h \sum_{i=0}^{M-1} \mid \delta_x U_{i,k} \mid^2 = h \sum_{i=1}^{M-1} \mid \frac{U_{i+1,k} - U_{i-1k}}{2h} \mid^2$$

$$\leq \frac{1}{4h}[\sum_{i=0}^{M-1} \mid U_{i+1,k} \mid^2 + 2\sum_{i=1}^{M-1} \mid U_{i+1,k}U_{i-1,k} \mid^2 + \sum_{i=0}^{M-1} \mid U_{i-1,k} \mid^2]$$

$$\leq \frac{1}{h^2} \| U._{,k} \|^2 . \tag{54}$$

Consequently, inequality (52) can be written the following:

$$\mid \langle (\delta_x^2 + \beta\delta_x)U._{,k}\, , U._{,1} \rangle \mid \leq \mid \sum_{i=1}^{M-1} h\delta_x^2 U_{i,k}U_{i,1} \mid + \beta \mid \sum_{i=1}^{M-1} h\delta_x U_{i,k}U_{i,1} \mid$$

$$= \frac{1}{h} \mid \sum_{i=1}^{M-1} (U_{i+1,k} - 2U_{i,k} + U_{i-1,k})U_{i,1} \mid + \frac{\beta}{2} \mid \sum_{i=1}^{M-1} (U_{i+1,k} - U_{i-1,k})U_{i,1} \mid,$$

Using (53) and (54), yields to :

$$\mid \langle (\delta_x^2 + \beta\delta_x)U._{,k}\, , U._{,1} \rangle \mid \leq \frac{4+\beta}{h^2} \| U._{,k} \| \| U._{,1} \| . \tag{55}$$

Which completes the proof.

(4.3.4.) Compact Scheme for One Dimensional Problems

We apply fourth-order compact finite difference schemes for the following general form of two point boundary value problems

$$-u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \;\; x \in [a, d] \tag{56}$$

$$v_0 = u_a \text{ and } \; v_N = u_d,$$

where $u'(x) = \dfrac{du(x)}{dx}$, $u''(x) = \dfrac{d^2u(x)}{dx^2}$,and the coefficient functions, $b(x)$ and $c(x)$ in equation (56) are given.

Denote the first and second-order central difference approximations by δ_x and δ_x^2 on a uniform mesh with spacing h. We use central differencing scheme on (56) yielding:

$$-\delta_x^2 v_i + b_i\delta_x v_i + c_i v_i = f_i,\ \ i = 1, 2, ..., N-1 \tag{57}$$

and the truncation error is :

$$R = \frac{h^2}{12}(-2b_i v_i^{(3)} + v_i^{(4)}) + \frac{h^4}{360}(-3b_i v_i^{(5)} + v_i^{(6)}) + O(h^6) \tag{58}$$

Here $v^{(n)} = \dfrac{\partial^n v(x)}{\partial x^n}$, $n \in N$. If we try to obtain higher order of accuracy, we need scheme (57) to include the lower order term $R_2 = \dfrac{h^2}{12}(-2b_i v_i^{(3)} + v_i^{(4)})$ in the truncation error R. For this propose, we can differentiate equation (56), then we obtain

$$v_i^{(3)} = b_i' v_i' + b_i v_i'' + c_i' v_i + c_i v_i' - f_i'. \tag{59}$$

If we differentiate again, it yields :

$$v_i^{(4)} = b_i'' v_i' + 2b_i' v_i'' + b_i b_i' v_i' + b_i^2 v_i'' + b_i c_i' v_i + b_i c_i v_i' + c_i'' v_i + 2c_i' v_i' + c_i v_i''$$
$$-b_i f_i' - f_i''. \tag{60}$$

Substitute $v_i^{(3)}$ and $v_i^{(4)}$ in the lower order truncation error R_2 by (59) and (60), then replace v_i' and v_i'' by central differencing schemes and add them to scheme (57), we obtain the following fourth order accurate formula

$$\begin{aligned}
&(-\frac{1}{12}b_i^2h^2 - 1 + \frac{1}{12}h^2 c_i + \frac{1}{6}h^2 b_i')\delta_x^2 v_i \\
&+(-\frac{1}{12}b_i h^2 b_i' + b_i - \frac{1}{12}b_i h^2 c_i + \frac{1}{6}h^2 c_i' + \frac{1}{12}h^2 b_i'')\delta_x v_i \\
&+(c_i - \frac{1}{12}b_i h^2 c_i + \frac{1}{12}h^2 c_i'')v_i \\
&= -\frac{1}{12}b_i h^2 f_i' + \frac{1}{12}h^2 f_i'' + f_i.
\end{aligned} \tag{61}$$

Rearrange the above equation, we know equation (56) can be approximated with fourth-order of accuracy by the compact finite difference scheme:

$$A_i v_{i-1} + B_i v_i + C_i v_{i+1} = F_i, i = 1, 2, ..., N-1, \tag{62}$$

$$A_i = 2hc_i' + hb_i'' - hb_i c_i - hb_i b_i' + 2b_i^2 - 4b_i' - 2c_i + \frac{12b_i}{h} + \frac{24}{h^2}$$

$$B_i = 2h^2 b_i c_i' - 2h^2 b_i c_i'' - 4b_i^2 + 8b_i' - 20c_i - \frac{48}{h^2}$$

$$C_i = -2hc_i' - hb_i'' + hb_i c_i + hb_i b_i' + 2b_i^2 - 4b_i' - 2c_i + \frac{12b_i}{h} + \frac{24}{h^2}$$

$$F_i = 2h^2 f_i'' - 2h^2 b_i f_i' + 24 f_i.$$

Or write them into matrix form: $DV = F$ where

$D=$

$$\begin{pmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-2} & B_{N-2} & C_{N-2} \\ & & & A_{N-1} & B_{N-1} \end{pmatrix}$$

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-2} \\ v_{n-1} \end{pmatrix}, and\ F = \begin{pmatrix} F_1 - A_1 v_0 \\ F_2 \\ \vdots \\ F_{N-2} \\ F_{N-1} - C_{N-1} v_N \end{pmatrix}$$

Theorem (4.1) Error Estimates for One Dimensional Compact Scheme.
When we choose boundary values $v_0 = u_a$ and $v_N = u_d$, the compact finite difference scheme
$(A_i v_{i-1} + B_i v_i + C_i v_{i+1} = F_i, i = 1, 2, ..., N-1)$ and this can be approximated with fourth order of accuracy by the compact finite difference scheme: can obtain the following discrete norm error estimates

$$\|U - V\| \leq Ch^4 \tag{63}$$

where U is the vector of accurate solutions, V is the vector of approximate solution and C is a positive constant .
Proof. Let vector T stands for the truncation error of numerical approximation
$(A_i v_{i-1} + B_i v_i + C_i v_{i+1} = F_i, i = 1, 2, ..., N-1,)$ so we know that $\|T\| = O(h^4)$, and it holds:

$$DU = F + T \tag{64}$$

here $U = (u_1, ..., u_{N-1})^T$. The difference between equation $(DV = F)$ and $(DU = F + T)$ yields

$$-DE = T \tag{65}$$

where vector $E = V - U = (e_1, ..., e_{N_1})^T$. Consider the scalar product

$$(-hDE, E) = (hT, E). \tag{66}$$

Let $\{\sum_{i=1}^{N-1} \lambda_i^h\}$ denote the real parts of eigenvalues of matrix $-D$, and d_0 is the smallest among them. By $(-D \to D^{+\infty})$. The smallest eigenvalue of the symmetric matrix $D^{+\infty}$ is $24\pi^2$, we know

$$d_0 = \lim_{h\to 0} \min_i \lambda_i^h = 24\pi^2. \tag{67}$$

Since the rank of matrix $D^{+\infty}$ is full, we know that there exists full orthonormal eigen vector basis for matrix $-D$ when h is small . To use $\{\sum_{=1}^{N-1} \eta_i\}$ to stand for it when h is small for the left side of equation (66) to obtain

$$(-hDE, E) = Re(-hDE, E)$$

$$= Re(-hD\sum_{i=1}^{N-1} a_i\eta_i,\ \sum_{i=1}^{N-1} a_i\eta_i)$$

$$= Re(h\sum_{i=1}^{N-1} a_i\lambda_i^h\eta_i,\ \sum_{i=1}^{N-1} a_i\eta_i)$$

$$= h\sum_{i=1}^{N-1} a_i^2 Re(\lambda_i^h)$$

$$> \frac{d_0}{2}\sum_{i=1}^{N-1} a_i^2 h = 12\pi^2\|E\|^2. \tag{69}$$

By the equation (69) and (66) to obtain

$$\|E\| \le \frac{1}{12\ \pi^2}\|T\|. \tag{70}$$

This completes the proof.

4.5 Numerical Examples and Results

In this section we present test examples.

Example -1: Consider the problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in (0,1)$ and $t \in (0,1)$ write a MATLAB code implementing the finite difference method

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{(\ u_{i-1}^j - 2u_i^j + u_{i+1}^j)}{h^2} \qquad (I)$$

Boundary condition $u(0,t) = u(1,t) = 0$, and Initial condition $u(x,0) = sin\pi x$, $f = 0$, $\triangle t = 0.6$
where $i = 2, ..., N$, $j = 1, ..., M$, $h = \triangle x = \dfrac{xmax}{N}$
$u_1^j = u_{N+1}^j = 0, u_i^1 = 4x - 4x^2$, taking $k = \dfrac{h}{2}$, to ensure stability

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{(\ u_{i-1}^j - 2u_i^j + u_{i+1}^j)}{h^2}$$

$$u_i^{j+1} - u_i^j = \frac{k}{h^2}(u_{i-1}^j - 2u_i^j + (u_{i+1}^j)$$

let $p = \dfrac{k}{h^2}$

$$\Rightarrow u_i^{j+1} - u_i^j = p(u_{i-1}^j - 2u_i^j + u_{i+1}^j)$$

$$u_i^{j+1} = u_i^j + pu_{i-1}^j - 2pu_i^j + pu_{i+1}^j$$

$$= u_i^{j+1} = pu_{i-1}^j + u_i^j(1-2p) + pu_{i+1}^j$$

let $q = (1-2p)$ $\therefore$ The linear system of the problem is

$$u_i^{j+1} = pu_{i-1}^j + u_i^j q + pu_{i+1}^j \qquad (II)$$

At first time level: putting $i = 2, ..., 7$ and $j = 1$ in (II) to get $u_1 = u_{1,1} = 0, u_2 = u_{2,1} = 0.2959, u_3 = u_{3,1} = 0.5306, u_4 = u_{4,1} = 0.6633, u_5 = u_{5,1} = 0.6633, u_6 = u_{6,1} = 0.5306, u_7 = u_{7,1} = 0.2959, u_8 = u_{8,1} = 0$.
At the second time level: substitute $i = 2, ..., 7$ and $j = 2$ in (II) to get $u_1 = u_{1,2} = 0, u_2 = u_{2,2} = 0.3265, u_3 = u_{3,2} = 0.5918, u_4 = u_{4,2} = 0.7347, u_5 = u_{5,2} = 0.7347, u_6 = u_{6,2} = 0.5918, u_7 = u_{7,2} = 0.3265, u_8 = u_{8,2} = 0$.

At the third time level: substitute $i = 2, ..., 7$ and $j = 3$ in (II) to get $u_1 = u_{1,3} = 0, u_2 = u_{2,3} = 0.3673, u_3 = u_{3,3} = 0.6531, u_4 = u_{4,3} = 0.8163, u_5 = u_{5,3} = 0.8163, u_6 = u_{6,3} = 0.6531, u_7 = u_{7,3} = 0.3673, u_8 = u_{8,3} = 0.$

At fourth time level: substitute $i = 2, ..., 7$ and $j = 4$ in (II) to get $u_1 = u_{1,4} = 0, u_2 = u_{2,4} = 0.4082, u_3 = u_{3,4} = 0.7347, u_4 = u_{4,4} = 0.8980, u_5 = u_{5,4} = 0.8980, u_6 = u_{6,4} = 0.7347, u_7 = u_{7,4} = 0.4082, u_8 = u_{8,4} = 0.$

Table 4.1: The comparison of the first, second, third and fourth time level

	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	Error
	x=0	x=0.1429	x=0.2857	x=0.4286	x=0.5714	x=0.7143	x=0.8571	x=1	
j=1	0	0.2959	0.5306	0.6633	0.6633	0.5306	0.2959	0	0.4388
j=2	0	0.3265	0.5918	0.7347	0.7347	0.5918	0.3265	0	0.5307
j=3	0	0.3673	0.6531	0.8163	0.8163	0.6531	0.3673	0	0.3265
j=4	0	0.4082	0.7347	0.8980	0.8980	0.7347	0.4082	0	0

The results obtained by the first, second, third and forth time level respectively.

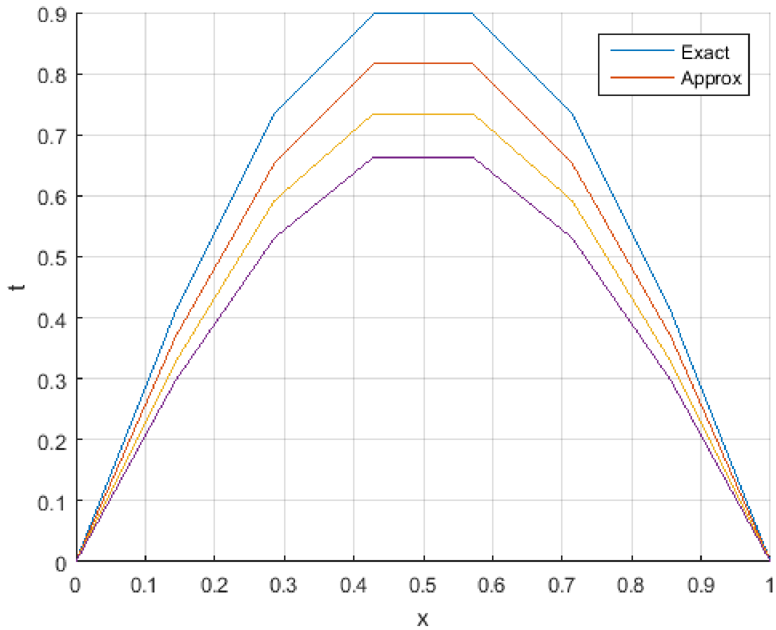

Figure 4.1: The graph of both analytical and approximate solution of the problem two by FDM with the first time level

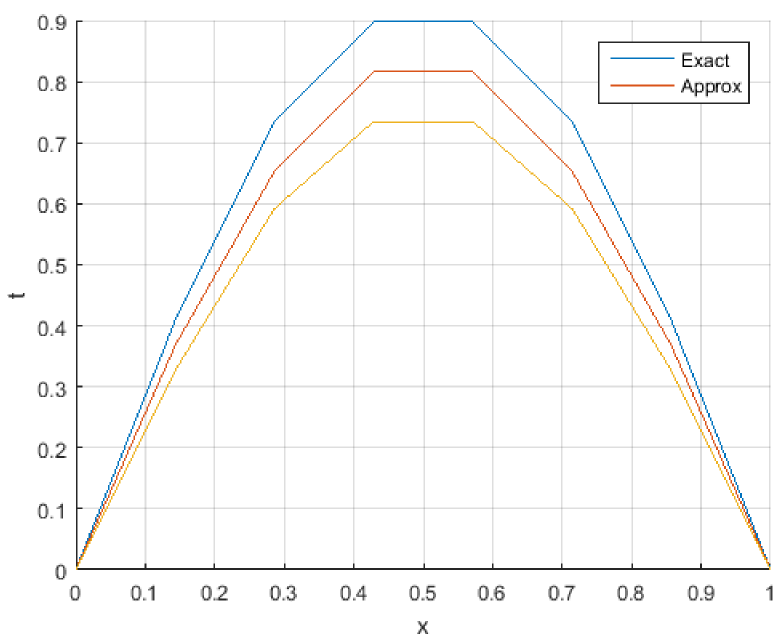

Figure 4.2: The graph of both analytical and approximate solution of the problem two by FDM with the second time level

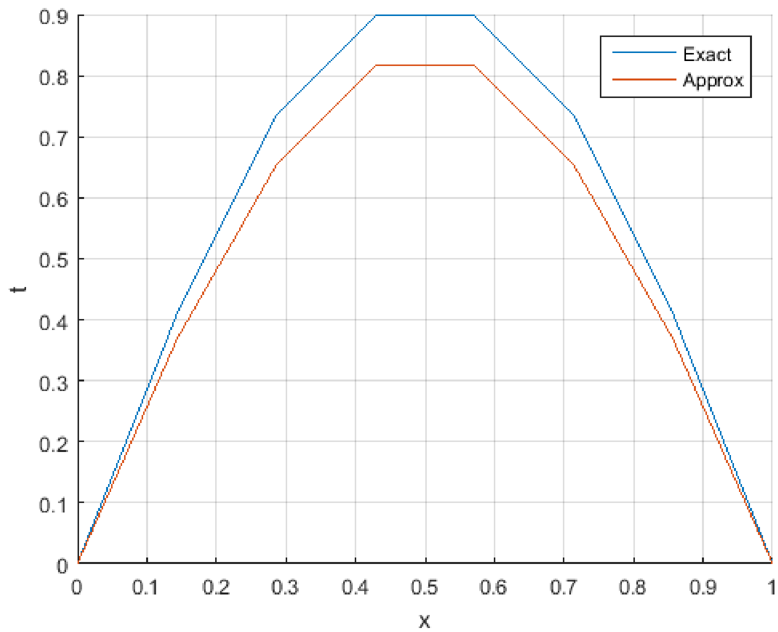

Figure 4.3: The graph of both analytical and approximate solution of the problem two by FDM with the third time level

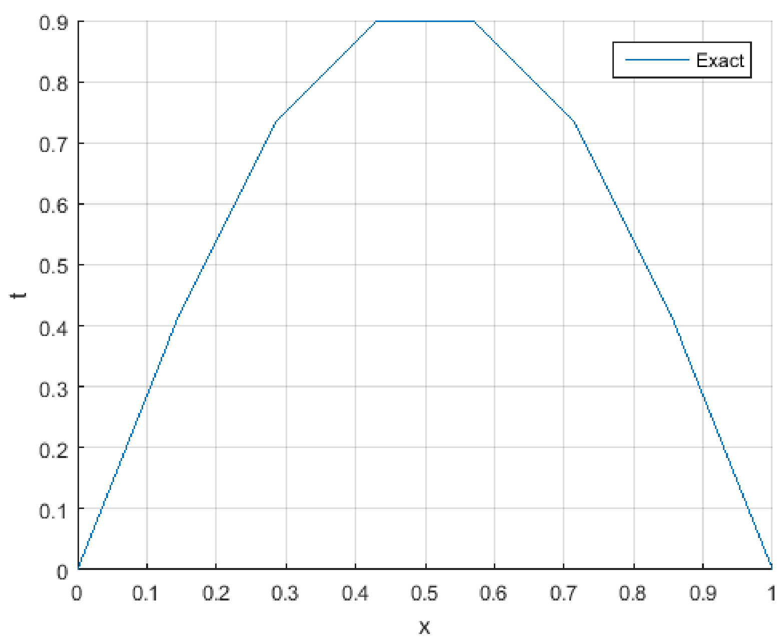

Figure 4.4: The graph of analytical solution of the problem two by FDM with the fourth time level

Example -2: In this section, we solve the integro-differential equation $(1) - (3)$ in $(0,1) \times (0,T)$ with $k(t,s) = s\,t$, $f(x,t) = sin(\pi x)(-t\dfrac{e^{-\pi^2 t}}{\pi^2} - \dfrac{e^{-\pi^2 t}}{\pi^4} + \dfrac{1}{\pi^4})$ $u_0 = sin(\pi x)$, and $g_1(t) = g_2(t) = 0$. The theoretical solution of this problem is $u(x,t) = e^{-\pi t} sin(\pi x)$. We employ a compact difference scheme for the space derivative so that we get a full discretization scheme with error estimation $O(h^4) + O(\tau)$. We compare the results obtained by the suggested approximation scheme with the exact solution.

Table 4.2: Comparison between exact and numerical solution, where $t = 0.2$, $n = 10$, $h = 0.1$

	t=0.2,n=10		
x	Exact solution	Present method $\tau =$ 0.001	Error
0.0	0	0	0
0.1	4.2925E-002	4.2931E-002	0.0006
0.2	8.1649E-002	8.1659E-002	0.001
0.3	1.1238E-001	1.1239E-001	0.0001
0.4	1.3211E-001	1.3212E-001	0.0001
0.5	1.3891E-001	1.3892E-001	0.0001
0.6	1.3211E-001	1.3212E-001	0.0001
0.7	1.1238E-001	1.1239E-001	0.0001
0.8	8.1649E-002	8.1659E-002	0.001
0.9	4.2925E-002	4.2931E-002	0.0006
1.0	0	0	0

Table 4.3: Comparison between exact and numerical solution, where $t = 0.6$, $n = 20$, $h = 0.5$

	t=0.6,n=20		
x	Exact solution	Present method $\tau =$ 0.001	Error
0.0	0	0	0
0.1	8.2831E-004	8.3369E-004	0.0538
0.2	1.5755E-003	1.5857E-003	0.0102
0.3	2.1685E-003	2.1826E-003	0.0141
0.4	2.5492E-003	2.5658E-003	0.0166
0.5	2.6804E-003	2.6978E-003	0.0174
0.6	2.5492E-003	2.5658E-003	0.0166
0.7	2.1685E-003	2.1826E-003	0.0141
0.8	1.5755E-003	1.5857E-003	0.0102
0.9	8.2831E-004	8.3369E-004	0.0538
1.0	0	0	0

Compare the errors in the above tables.

Table 4.2 is less error than table, 4.1 and 4.3

Table 4.1 is higher error than table, 4.2 and 4.3

$\star$ Less errors $\rightarrow$ more accurate result

$\star$ Higher errors $\rightarrow$ less accurate result

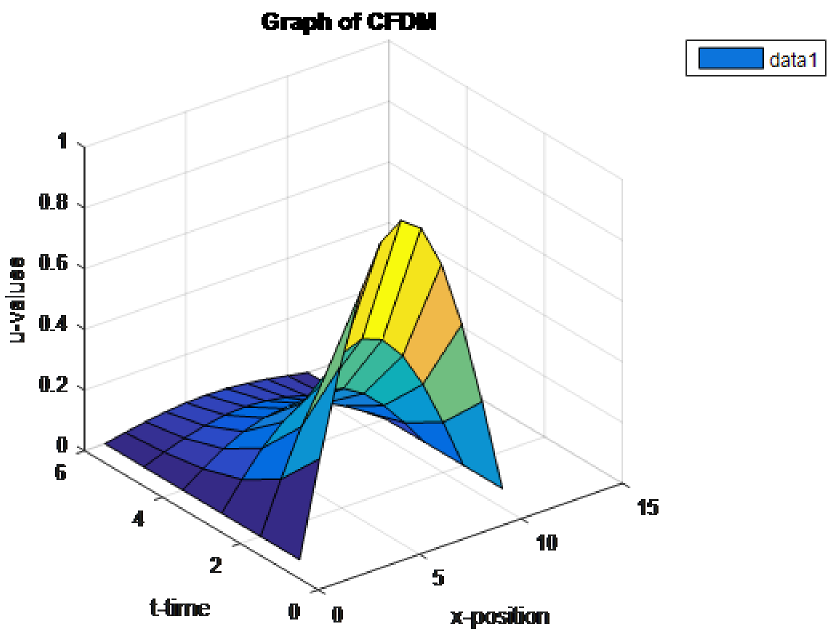

Figure 4.5: The graph of exact solution

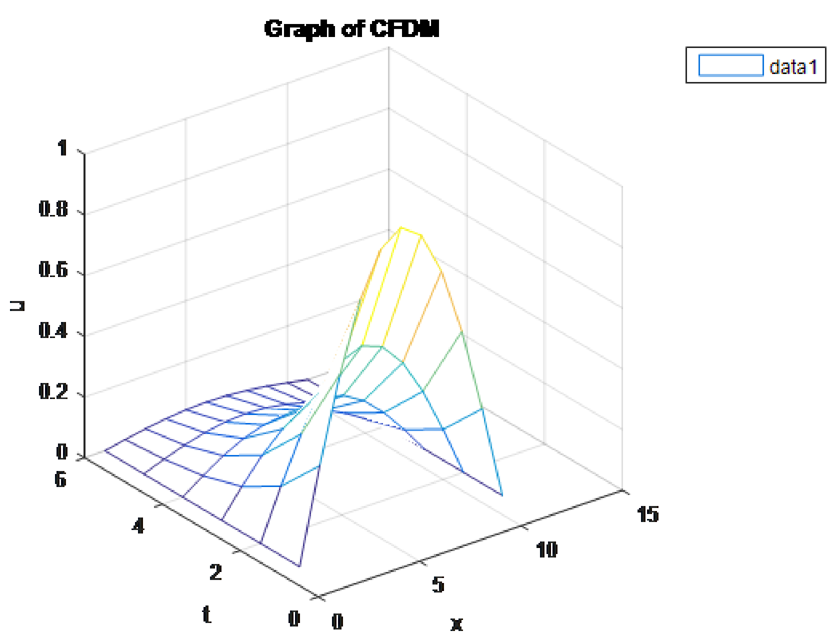

Figure 4.6: The graph of compact finite difference method

4.6 Discussions

In this study we applied fourth-order accurate compact finite difference scheme for partial integro-differential equation. The standard and compact techniques in finite difference methods are usually used to obtain the numerical solution of differential equations. Therefore, the combination of the compact finite difference approximation and composite trapezoidal method gives an efficient method for solving partial integro-differential equation (1).

We developed the linear system of equation used for the matlab simulation with exact and approximate solution. From the system of equation to under stand the behavior of numerical and exact solution at different mesh size. The method reduces the underlying problem to linear system of algebraic equations, which can be solved successively to obtain a numerical solution at varied time-levels.

From the above tables (4.2) and (4.3) different mesh size value have different exact and approximate solution but some mesh size value have the same exact and approximate solution when the solution is convergent. If the number of mesh size increase also the number of discretization error increase. Numerical experiments which were shown in the above tables and graphs are good agreement with the exact ones. The tables and graphs shown that compact finite difference method gives good numerical results. From the second example one also easily observes how the removal of the integral part of partial integro-differential equation reducing to the heat equation (PDE) affects the solution by compact finite difference method (CFDM).

4.7 Conclusions and Recommendations

4.7.1 Conclusion

This thesis presents the basic idea of compact finite difference method to solve partial integro-differential equation with boundary value problem. We applied Taylor expansion approximation to obtain the approximate solution at grid points $x = x_i$, on an interval $[a, b]$. Compact finite difference method has higher accuracy than the classical finite difference scheme. The product trapezoidal method is applied for discretization of the integral term. A fourth-order accurate compact finite difference scheme for partial integro-differential problems was developed. We compare the results of partial integro-differential equation with the integral part and unknown function $f(x, t)$ are zero. The method reduces to the underlying heat equation, which can be solved successively to obtain a numerical solution at varied time-levels. The consistency, stability and convergence of the method was shown by Lax-Richkmyer Equivalence Theorem. Thus as becomes $\triangle\ x$ closer and closer to zero, the numerical solution of compact finite difference method is more convergent to the exact solution . Therefore from tested examples one and two, the stability, consistency and convergence of the numerical solution of compact finite difference method depends on the condition of discretization of the domain of the boundary conditions of the partial integro-differential equation(PIDE). As number of discretization error is reduced ($\triangle\ x \to 0$), the numerical solution is faster convergent to the exact solution. Numerical experiments which are shown in the above scheme are good agreement with the exact ones. Moreover, the results in tables (4.2) and (4.3) confirm that the numerical solutions can be refined when the time-step τ is reduced. Numerical examples confirm the theoretical prediction and show that the combination of the compact finite difference approximaton and product trapezoidal method give efficient method for solving a partial integro-differential equation. The results of numerical experiments show the efficiency and applicability of method. Also we have illustrated that the computational orders of the introduced method are in good agreement with the theoretical ones.
We have shown the effect of the integral part and f in partial integro-differential equation (PIDE) to reduce the error .

4.7.2 Recommendation

The results obtained in this thesis on the compact finite difference method is for solving partial integro-differential equations with boundary value problem on stability, consistency and convergence for parabolic partial integro-differential equation.
We may suggest one to solve partial integro-differential equation using other methods such as finite element method (FEM)and alternating directions implicit method(ADIM).

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