# Quantum Dynamics of V Laser Coupled to Squeezed Vacuum Reservoir

**By**

**Temesgen Fikadu Yadeta**

A Thesis Submitted to

the Department of Applied Physics Program

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement

for the Degree of Masters in Physics (Quantum Optics)

Office of Graduate Studies

Adama Science and Technology University

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**Approved by the Examination Committee**

External Examiner ————————————- ————————-

Internal Examiner ————————————- ————————

Advisor ————————————- ————————-

Chairman ————————————- ————————

## Declaration

I hereby declare that this Msc thesis is my original work and has not been presented for a degree in any other universities, and all sources of material used for the thesis have been duly acknowledged.

Name: Temesgen Fikadu

Signature: ——————————————

This Msc thesis has been submitted to for examination with my approval as university advisor supervisor.

Name: Dr. Tewodros Yirgashewa

Signature:——————————————

Date of submission:——————————————

# Acknowledgements

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# Abstract

The quantum dynamics of light produced by a two-mode v laser coupled to squeezed vacuum reservoir have been analyzed with the aid of the solutions of c-number Langevin equations associated with the normal order along with the correlation relation. We determined the quadrature variances and photon statistics of the cavity modes. The results showed that the light produced by the system under consideration is in squeezed state and the degree of squeezing increases with squeeze parameter(r). Furthermore, it was found that the mean and variance of photon number increases with the linear gain coefficient. In addition, we have calculated the variance of the sum of EPR-operators with the aid of the solutions of c-number Langevin equation and Duan et al. criteria. The results showed that the entanglement of the cavity mode increases with squeeze parameter(r) and decreases with linear gain coefficient.

# Contents

# 1

# Introduction

## 1.1 Background of the study

Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Classically light consists of waves with well-defined amplitude and phase such is not case when we treat the field quantum mechanically. The quantum property of light are largely determined by the state of the mode and the well-known quantum states are; the number state, the coherent state, the chaotic state and the squeezed state.

Quantum optics is the subject that deals with optical phenomena that can only be explained by treating light as a stream of photons than as electromagnetic waves. It is the field of quantum physics that studies specifically with the interaction of photon with matter. In the progressive development of the theory to light, three general approaches can be clearly identified, namely the classical, semi-classical and quantum theories. In classical physics, the light is conceived as electromagnetic waves. Semi-classical theories are quite adequate for most purposes, when the theory of absorption of light by atoms is first considered, it is usual to apply quantum mechanics to the atoms, but treat the light as a classical electromagnetic waves, but in quantum optics, the quantum nature of the light is included by treating the light as photons.

Squeezing is one of the interesting non-classical features of light that has been attracting attention [1-17]. In squeezed light the noise in one quadrature is below the vacuum or coherent level at the expense of enhanced fluctuation in the other quadrature, with the product of uncertainties in the two quadratures satisfying the uncertainties relations. Squeezed light can be generated as a result of quantum optical process, such as; sub harmonic generation, second harmonic generation and three level laser under certain condition. Squeezing light can be generated by a three level laser under certain condition[12, 13, 17]. Squeezed light has potential applications in low-noise communication, weak signal detection and optical amplification measurement[11, 14, 18].

The photon statistics and quantum noise of a two-level laser have been investigated by several authors [1,3,8]. It has been found that a two level laser generates chaotic light when operating below threshold and coherent light when operating well above threshold.

B. Petros [8] had studied squeezing and statistical properties of light produced the two-level atom coupled to squeezed vacuum reservoir. He was found that the light produced by the system under consideration is a squeezed state for the case if the atom are initially prepared in the lower level and the squeezing occurred in the plus quadrature. His result shows that squeezed vacuum reservoir increases the mean photon number.

K. Fesseha [12] has analyzed the dynamics of the two level atoms coupled to squeezed vacuum reservoir. He has carried out his analysis after driving the master equation for a two level atom directly coupled to squeezed vacuum reservoir. The result indicates that the atom oscillates back and forth between the upper and lower levels.

G. Seyoum [15] has studied a two mode v laser dynamics with atomic states whose cavity contains coherent light coupled to ordinary vacuum. He was found by applying stochastic differential equation, if the atom injected into the cavity are initially prepared in a coherent superposition of the upper states and the cavity modes are initially in a vacuum states, there is no squeezing in the light generated by a two mode v laser.

M. Ayele [16] has studied the squeezing property of the cavity modes produced by two-mode coherent and squeezed light applying the solution of stochastic differential equations. He has found that the squeezing increases with increasing the squeeze parameter r.

A coherently prepared two-mode v laser has been studied by some authors [19, 20]. This type of laser is defined as a quantum optical system in which three level atoms in v configuration and initially prepared in a coherent superposition of the two upper levels are injected at a certain rate into a cavity coupled to vacuum reservoir via a single-port mirror.They found that the degree of squeezing increases with linear gain coefficient.

Another interesting non-classical features of light is Entanglement, which has been studied by many authors [21,22,23]. Research in to quantum entanglement was initiated, by Einstein-Podolsky-Rosen describing the EPR paradox and several papers by Erwin Schrodinger shortly.

There after this area is an area of extremely active research by physics community despite the fact that this behavior isn't demonstrated experimentally. Quantum entanglement is a physical phenomenon which is created usually by direct interaction between sub-atomic particles such as photon, electron and molecules and then separated. Before interaction each particle is described by its own quantum state.

The two-mode squeezing state itself is an entangled state which entitles the idler mode and signal mode in the frequency domain [21]. Recently, it has been shown that the non degenerate parametric oscillator can generate macroscopic entangled state. It is believed that the key in gradient of quantum information is entanglement which has been recognized as the essential resource for quantum teleportation, quantum decoding,quantum computation and quantum cryptography[23]. Recently, much attention has been paid to the generation and detection of continuous variable entanglement as it might be easier to manipulate than the discrete counter parts, quantum bits, in order to perform quantum information processing. On the other hand, the efficiency of the quantum information processing highly depends on the degree of entanglement.

## 1.2 Statement of the problem

In previous works, different Authors have studied about quantum dynamics of two mode v laser coupled to vacuum reservoir by using different methods.

In this thesis we seek to study the quantum dynamics of two-mode light produced by a v laser coupled to squeezed vacuum reservoir by using C-number Langevin equation and their correlation relations.

## 1.3 Objective of the Study

### 1.3.1 General objective of the study

The general objective of this thesis is to study quantum dynamics of a two-mode v laser coupled to squeezed vacuum reservoir.

### 1.3.2 Specific objectives of the study

The specific objectives of this thesis are:

- to calculate the c-number Langevin equation and their solutions of two mode v laser coupled to squeezed vacuum reservoir.
- to determine the quadrature variance
- to calculate the mean and variance of photon number.
- to calculate the sum of EPR-operators.

# 2

# Review of the Literature

Quantum optics deals mainly with quantum properties of the light generated by various quantum optical system, such as; Laser and maser with their effect on the dynamics of atoms. Laser is an acronym for Light Amplification by Stimulated Emission of Radiation. It is the powerful source of light having extraordinary properties which are not found in the normal light sources like tungsten lumps, mercury lamps, etc. The unique property of laser is that its light waves travel very long distances with a very little divergence. Maser is an acronym for Microwaves Amplifications by Stimulated Emission of Radiation. It emitted coherent microwaves.

The three-level atomic systems are: V-type, $\lambda$-type and Cascade-type.

The V-type atoms have three states: $\mid a\rangle, \mid b\rangle and \mid c\rangle$. The energy levels of $\mid a\rangle, \mid b\rangle$ are higher than that of $\mid c\rangle$.

In $\lambda$-type atoms, there are also three states: $\mid a\rangle, \mid b\rangle and \mid c\rangle$. However, in this type, $\mid a\rangle$ is at the highest energy level, while $\mid b\rangle, \mid c\rangle$ are at lower levels, respectively.

A three-level cascade laser is a quantum optical system in which three level atoms in a cascade configuration. In this system the top, intermediate and bottom levels of a three level cascade atom can be denoted by $\mid a\rangle, \mid b\rangle and \mid c\rangle$, respectively.

F. Gebremedhin [20] has studied that squeezing and statistical properties of light generated by a non-degenerate three-level laser whose cavity contains two parametric amplifiers coupled to an ordinary vacuum reservoir. He has found that parametric amplifiers increases the degree of squeezing.

Some authors have studied the squeezing and the statistical properties of light produced by three-level lasers in which the crucial role is played by the superposition of the top and bottom levels [24, 25]. Ansari [26] has studied the squeezing and statistical properties of a degenerate three level laser with the atoms initially prepared in a coherent superposition of the top and bottom levels. He has predicted that cavity modes of such laser is in squeezed state, when there are more atoms initially in the bottom level than in the top level.

In a cascade three level laser, three level atoms in a cascade configuration are injected into a cavity coupled to squeezed vacuum reservoir via a single port-mirror. The injected atoms may initially be prepared in the coherent superposition of the top and the bottom levels. When three level atom in a cascade configuration makes the transition from the top to the bottom level via the intermediate level, two photons are generated. If two photons have the same frequency, the three level atom is called degenerate otherwise it is called non-degenerate [27, 28].

Abraham [29] has studied the squeezing and entanglement of light produced by a nondegenerate three-level laser whose cavity contains two degenerate parametric amplifiers coupled to the squeezed vacuum reservoir. He has found that the degree of squeezed vacuum reservoir increases when the degree of squeezing and entanglement decreases, and the squeezing of the cavity mode is greater than output mode by some values. He also found that the maximum intra cavity of the entanglement of the cavity mode and the output mode is $99.44\%$ and $59.55\%$ respectively below coherent level.

B. Abiti [30] has studied the squeezing and entanglement of light produced by a non degenerate three-Level laser whose cavity contains two degenerate parametric amplifiers coupled to squeezed vacuum reservoir. He has found that the presence of parametric amplifier and squeezed vacuum reservoir on the three level laser is to enhance the degree of squeezing and the degree of entanglement.

# 3

# Methodology

## 3.1 Interaction between cavity mode and the squeezed vacuum reservoir

Here we wish to drive the equation of evolution of the reduced density operator for the cavity mode coupled to a two-mode squeezed vacuum reservoir. The interaction of the system with the reservoir can be described by hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{3.1}$$

where $\hat{H}_S$ is the hamiltonian for the interaction in the cavity and $\hat{H}_{SR}$ is the hamiltonian for the interaction between the cavity modes and reservoir. Now let $\hat{\rho}_{SR}$ be the density operator for the system plus the reservoir in the interaction picture. We can then write the time evolution of this operator as

$$\frac{d\hat{\rho}_{SR}(t)}{dt} = -i[\hat{H}_S(t) + \hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{3.2}$$

In order to obtain the evolution of the system alone, which we are interested in, we trace $\hat{\rho}_{SR}(t)$ over the reservoir variables, that is, the density operator for the system, $\hat{\rho}(t) = TrR(\hat{\rho}_{SR}(t))$.

In view of this, Eq.(3.2) takes the form

$$\frac{d\hat{\rho}_{SR}(t)}{dt} = -i[\hat{H}_S(t), \hat{\rho}(t)] - iTrR[\hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{3.3}$$

In order to obtain a manageable equation of evolution of the reduced density operator, we need to adopt some approximation scheme. We thus seek to obtain this equation of evolution applying the approximately valid relation.

$$\frac{d\hat{\rho}_{SR}(t)}{dt} = \frac{\hat{\rho}_{SR}(t) - \hat{\rho}_{SR}(t-h)}{h}, \tag{3.4}$$

where h is a a small parameter. Now in view of Eq.(3.2) and Eq.(3.4), we see that

$$\hat{\rho}_{SR}(t) = \hat{\rho}_{SR}(t-h) - ih[\hat{H}_{SR}(t) + \hat{H}_s(t), \rho_{SR}(t)]. \tag{3.5}$$

On substituting Eq.(3.5) into Eq.(3.3), we arrive at

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -i[\hat{H}_s(t), \hat{\rho}(t)] - iTrR[\hat{H}_{SR}(t), \hat{\rho}_{SR}(t-h)] - hTrR[\hat{H}_{SR}(t), [\hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]] \\ &- hTrR[\hat{H}_{SR}(t), [\hat{H}_S(t), \hat{\rho}_{SR}(t)]]. \end{aligned} \tag{3.6}$$

So as to proceed further, we need to introduce a certain approximation scheme. To this end, assuming that there is a weak interaction between the system and reservoir, we can write approximately valid relation

$$\hat{\rho}_{SR}(t) = \hat{\rho}(t), \tag{3.7}$$

where $\hat{R}$ is the operator for the reservoir assumed to be constant in time. With the aid of Eq.(3.7) into Eq.(3.6), we obtain

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -i[\hat{H}_s(t), \hat{\rho}(t)] - i[\langle \hat{H}_{SR}\rangle_R, \hat{\rho}(t-h)] - h[\langle \hat{H}_{SR}\rangle_R, [\hat{H}_S, \hat{\rho}(t)]] \\ &- hTrR[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\rho}(t)\hat{R}]], \end{aligned} \tag{3.8}$$

where the subscript $R$ indicates that the expectation value is to be calculated using the reservoir density operator $\hat{R}$.

The interaction of a cavity mode with a squeezed vacuum reservoir via of a single-port mirror can be described by the hamiltonian

$$\hat{H}_{SR} = i\lambda(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger), \tag{3.9}$$

where $\lambda$ is the coupling constant between a two-mode and $\hat{a}(\hat{b})$ is the annihilation operator for the signal(idler) mode.

Using Eq.(3.9) and taking into account the interaction of the cavity modes with a two-mode squeezed vacuum reservoir, we find the equation of evolution of the density operator coupled to two-mode squeezed vacuum reservoir to be[1]

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= -i[\hat{H}_S, \hat{\rho}(t)] + \frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b})\\ &+ \frac{\kappa N}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger + 2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger)\\ &+ \kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{b} + \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}),\end{aligned} \tag{3.10}$$

where $\kappa$ is the cavity damping constant, $N$ and $M$ are the reservoir parameter given by

$$N = Sinh^2 r \tag{3.11}$$

and

$$M = Sinhrcoshr, \tag{3.12}$$

with $r$ being the squeeze parameter.

## 3.2 The interaction between an atom and cavity mode

A two-mode v laser consists of a cavity into three-level atoms in v configuration are injected at a constant rate $r_a$ and removed from the cavity after a certain time $\tau$. we consider three-level atoms in v configuration as shown in fig.1.1 with the upper levels denoted by $\mid a\rangle$ and $\mid b\rangle$. The laser cavity is arranged such that it would be at resonance with the transition between $\mid a\rangle$ and $\mid c\rangle$ as well as between $\mid b\rangle$ and $\mid c\rangle$. The transition between $\mid a\rangle$ and $\mid b\rangle$ is assumed to be dipole forbidden. The transition from the upper levels $\mid a\rangle$ and $\mid b\rangle$ to the lower level $\mid c\rangle$ produce two photons of different frequencies[12].

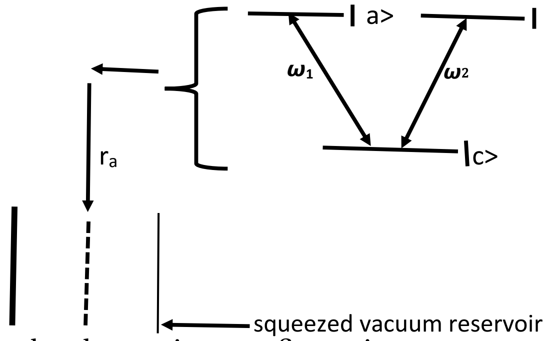


Fig.3.1: A three-level atom in v configuration

Interaction of a three-level atom with cavity mode can be described by Hamiltonian

$$\hat{H} = ig[\hat{a}^{\dagger} \mid c\rangle\langle a \mid -\hat{a} \mid a\rangle\langle c \mid +\hat{b}^{\dagger} \mid c\rangle\langle b \mid -\hat{b} \mid b\rangle\langle c \mid, \tag{3.13}$$

where g is the atom-radiation coupling constant, taken to be equal for both modes, $a$ and $b$ are the annihilation operators for the cavity modes. We take the initial state of a three -level atom to be

$$\mid \psi_A(0)\rangle = C_a(0) \mid a\rangle + C_b(0) \mid b\rangle \tag{3.14}$$

and hence the initial density operators of an atom is given by

$$\hat{\rho}_A^{(o)} = \rho_{aa}^{(0)} \mid a\rangle\langle a \mid +\rho_{bb}^{(0)} \mid b\rangle\langle b \mid +\rho_{ab}^{(0)} \mid a\rangle\langle b \mid +\rho_{ba}^{(0)} \mid b\rangle\langle a \mid, \tag{3.15}$$

in which

$$\rho_{aa}^{(0)} = \mid C_a(0) \mid^2, \rho_{bb}^{(0)} = \mid C_b(0) \mid^2, \rho_{ab}^{(0)} = C_a(0)C_b^*(0), \rho_{ba}^{(0)} = C_b(0)C_a^*(0).$$

Suppose $\hat{\rho}_{AR}(t, t_j)$ the density operator for a single atom plus the cavity modes at time t, with the atom injected at time $t_j$, such that $t - \tau \leq t_j \leq t$ the density operator for all atoms in the cavity plus the cavity modes at time t can then be written as [12]

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t, \tag{3.16}$$

where $r_a \Delta t$ represents the number of atoms injected into the cavity at time $t_j$. Now converting the summation into integration, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}(t, t^{'})dt^{'}. \tag{3.17}$$

Upon differentiating with respect to $t$ and using the relation

$$\int_{x+x_o}^{x} f(x, y)dy = f(x, x) - f(x, x + x_o) + \int_{x+x_o}^{x} \frac{d}{dx} f(x, y)dy,$$

there follows

$$\frac{d\hat{\rho}_{AR}(t)}{dt} = r_a(\hat{\rho}_{AR}(t, t) - \hat{\rho}_{AR}(t, t - \tau)) + r_a \int_{t-\tau}^{t} \frac{d\hat{\rho}_{AR}(t, t^{'})}{dt} dt^{'}, \tag{3.18}$$

where $\hat{\rho}_{AR}(t, t)$ is the density operator for the cavity modes and an atom, with the atom injected at time $t$. Since the cavity modes and the atom at time $t$ are independent, the density operator $\hat{\rho}_{AR}(t, t)$ can be written as

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(t, t)\hat{\rho}(t), \tag{3.19}$$

in which $\hat{\rho}(t)$ is the density operator for the cavity modes alone. Moreover, $\hat{\rho}_{AR}(t, t - \tau)$ is the density operator for the cavity modes plus an atom, with the atom injected at time $t - \tau$ and being removed from the cavity at time $t$. Therefore, the operator $\hat{\rho}_{AR}(t, t - \tau)$ can also be expressed as

$$\hat{\rho}_{AR}(t,t-\tau)=\hat{\rho}_A(t,t-\tau)\hat{\rho}(t). \tag{3.20}$$

Now we insert Eq. (3.19) and (20) into Eq. (3.18) to obtain

$$\frac{d\hat{\rho}_{AR}(t)}{dt}=r_a(\hat{\rho}_A(t,t)-\hat{\rho}_A(t,t-\tau))\hat{\rho}(t)+r_a\int_{t-\tau}^{t}\frac{d\hat{\rho}_{AR}(t,t^{'})}{\text{t}}dt^{'}. \tag{3.21}$$

In the absence of damping, the density operator $\hat{\rho}_{AR}(t,t^{'})$ evolves in time according to

$$\frac{d\hat{\rho}_{AR}(t,t^{'})}{dt}=-i[\hat{H},\hat{\rho}_{AR}(t,t^{'})], \tag{3.22}$$

so that on substituting Eq.(3.22) into Eq.(3.21) and using Eq. (3.17), we find

$$\frac{d\hat{\rho}_{AR}(t)}{dt}=r_a(\hat{\rho}_A(t,t)-\hat{\rho}_A(t,t-\tau))\hat{\rho}(t)-i[\hat{H},\hat{\rho}_{AR}(t)]. \tag{3.23}$$

Since we are mainly interested in the density operator for the cavity modes alone. We trace Eq. (3.23) over the atomic variables. Thus upon tracing over atomic variables and taking into account the damping of the cavity modes by a squeezed vacuum reservoir, the equation of evolution of the density operator for the cavity modes turnout to be

$$\begin{aligned}\frac{d\hat{\rho}}{dt}&=-iTr_A[\hat{H},\hat{\rho}_{AR}(t)]+\frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a}+2\hat{b}\hat{\rho}\hat{b}^{\dagger}-\hat{b}^{\dagger}\hat{b}\hat{\rho}-\hat{\rho}\hat{b}^{\dagger}\hat{b})\\&+\frac{\kappa N}{2}(2\hat{a}^{\dagger}\hat{\rho}\hat{a}-\hat{a}\hat{a}^{\dagger}\hat{\rho}-\hat{\rho}\hat{a}\hat{a}^{\dagger}+2\hat{b}^{\dagger}\hat{\rho}\hat{b}-\hat{b}\hat{b}^{\dagger}\hat{\rho}-\hat{\rho}\hat{b}\hat{b}^{\dagger})\\&+\kappa M(\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}+\hat{b}^{\dagger}\hat{\rho}\hat{a}^{\dagger}+\hat{a}\hat{\rho}\hat{b}+\hat{b}\hat{\rho}\hat{a}-\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}-\hat{a}\hat{b}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}-\hat{\rho}\hat{a}\hat{b}),\end{aligned} \tag{3.24}$$

where $\kappa$ is the cavity damping constant assumed to be the same for both modes. On substituting Eq.(3.13) into Eq.(3.24), we obtain

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= gTr_A(\hat{a}^\dagger \mid c\rangle\langle a \mid \hat{\rho}_{AR} - \hat{a} \mid a\rangle\langle c \mid \hat{\rho}_{AR} + \hat{b}^\dagger \mid c\rangle\langle b \mid \hat{\rho}_{AR} - \hat{b} \mid b\rangle\langle c \mid \hat{\rho}_{AR}) \\ &- gTr_A(\hat{\rho}_{AR}\hat{a}^\dagger \mid c\rangle\langle a - \hat{\rho}_{AR}\hat{a} \mid a\rangle\langle c \mid + \hat{\rho}_{AR}\hat{a}^\dagger \mid c\rangle\langle b \mid - \hat{\rho}_{AR}\hat{b} \mid b\rangle\langle c \mid) \\ &+ \frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ &+ \frac{\kappa N}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger + 2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ &+ \kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{b} + \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}),\end{aligned} \tag{3.25}$$

so that applying the cyclic property of trace operation, the master equation for the cavity modes can be put in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g[\hat{a}^\dagger\hat{\rho}_{ac} - \hat{\rho}_{ac}\hat{a}^\dagger + \hat{\rho}_{ca}\hat{a} - \hat{a}\hat{\rho}_{ca} + \hat{b}^\dagger\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{b}^\dagger + \hat{\rho}_{cb}\hat{b} - \hat{b}\hat{\rho}_{cb}] \\ &+ \frac{\kappa(N+1)}{2}(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\ &+ \frac{\kappa N}{2}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger + 2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\ &+ \kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{b} + \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}),\end{aligned} \tag{3.26}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha \mid \hat{\rho}_{AR} \mid \beta\rangle,$$

with $\alpha, \beta = a, b, c.$

On the other hand, we see from Eq.(3.23) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a(\langle\alpha \mid \hat{\rho}_A(t,t) \mid \beta\rangle - \langle\alpha \mid \hat{\rho}_A(t,t-\tau) \mid \beta\rangle)\hat{\rho}(t) - i\langle\alpha \mid [\hat{H}, \hat{\rho}_{AR}(t)] \mid \beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{3.27}$$

where the last term is included to account for the damping of the atoms due to spontaneous emission and $\gamma$ is the atomic decay rate considered to be the same for all three atoms. We assume that the atoms are removed from the cavity after they have decayed to a level other than level $\mid c\rangle$, so that

$$\langle \alpha \mid \hat{\rho}_A(t, t-\tau) \mid \beta \rangle = 0.$$

Hence Eq.(3.27) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a \langle \alpha \mid \hat{\rho}_A(t,t) \mid \beta \rangle \hat{\rho}(t) - i\langle \alpha \mid \hat{H}\hat{\rho}_{AR}(t) \mid \beta \rangle + i\langle \alpha \mid \hat{\rho}_{AR}(t)\hat{H} \mid \beta \rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{3.28}$$

where $\hat{\rho}_A(0) = \hat{\rho}_A(t,t)$. Then in view of Eq.(3.13) and Eq.(3.15), Eq.(3.28) takes the form

$$\begin{aligned}\frac{d\hat{\rho}_{\alpha\beta}}{dt} &= r_a[\rho_{aa}^{(0)}\sigma_{\alpha a}\sigma_{a\beta} + \rho_{ab}^{(0)}\sigma_{\alpha a}\sigma_{b\beta} + \rho_{ba}^{(0)}\sigma_{\alpha b}\sigma_{a\beta} + \rho_{bb}^{(0)}\sigma_{\alpha b}\sigma_{b\beta}]\hat{\rho}(t) \\ &+ g[\hat{a}^\dagger\hat{\rho}_{a\beta}\sigma_{\alpha c} - \hat{a}\hat{\rho}_{c\beta}\sigma_{\alpha a} + \hat{b}^\dagger\hat{\rho}_{b\beta}\sigma_{\alpha c} - \hat{b}\hat{\rho}_{c\beta}\sigma_{\alpha b} - \hat{\rho}_{\alpha c}\hat{a}^\dagger\sigma_{a\beta} + \hat{\rho}_{\alpha a}\hat{a}\sigma_{c\beta} - \hat{\rho}_{\alpha c}\hat{b}^\dagger\sigma_{b\beta} + \hat{\rho}_{\alpha b}\hat{b}\sigma_{c\beta}] - \gamma\hat{\rho}_{\alpha\beta}.\end{aligned} \tag{3.29}$$

From which we obtain

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} - g(\hat{a}\rho_{ca} + \rho_{ac}\hat{a}^\dagger) - \gamma\hat{\rho}_{aa}, \tag{3.30}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = r_a\rho_{bb}^{(0)}\hat{\rho} - g(\hat{b}\rho_{cb} + \rho_{bc}\hat{b}^\dagger) - \gamma\hat{\rho}_{bb}, \tag{3.31}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g(\hat{a}^\dagger\hat{\rho}_{ac} + \hat{b}^\dagger\hat{\rho}_{bc} + \hat{\rho}_{ca}\hat{a} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}, \tag{3.32}$$

$$\frac{d\hat{\rho}_{ab}}{dt} = r_a\rho_{ab}^{(0)}\hat{\rho} - g(\hat{a}\rho_{cb} + \rho_{ac}\hat{b}^\dagger) - \gamma\hat{\rho}_{ab}, \tag{3.33}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = g(\hat{\rho}_{aa}\hat{a} - \hat{a}\hat{\rho}_{cc} + \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{ac}, \tag{3.34}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{\rho}_{ba}\hat{a} - \hat{b}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{b}) - \gamma\hat{\rho}_{bc}. \tag{3.35}$$

Now we restrict ourselves to linear analysis of the system under consideration and this is achieved by dropping the g-terms in EqS.(3.30), (3.31), (3.32) and (3.33). Therefore, these equation reduce to

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(0)}\hat{\rho} - \gamma\hat{\rho}_{aa}, \tag{3.36}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = r_a\rho_{bb}^{(0)}\hat{\rho} - \gamma\hat{\rho}_{bb}, \tag{3.37}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = -\gamma\hat{\rho}_{cc}, \tag{3.38}$$

$$\frac{d\hat{\rho}_{ab}}{dt} = r_a\rho_{ab}^{(0)}\hat{\rho} - \gamma\hat{\rho}_{ab}. \tag{3.39}$$

We consider the case for which $\kappa << g, \gamma$(the good cavity limit) [12]. With this condition, we see that the atomic variables change rapidly while the cavity mode variables change slowly. This implies that the atomic variables reach steady state in the relatively short period of $\gamma^{-1}$. Consequently, we can replace the atomic variables by their steady state values(adiabatic approximate scheme). Thus applying the adiabatic approximate scheme, we get from EqS.(3.30) - (33) that

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(0)}\hat{\rho}}{\gamma}, \tag{3.40}$$

$$\hat{\rho}_{bb} = \frac{r_a\rho_{bb}^{(0)}\hat{\rho}}{\gamma}, \tag{3.41}$$

$$\hat{\rho}_{cc} = 0, \tag{3.42}$$

$$\hat{\rho}_{ab} = \frac{r_a\rho_{ab}^{(0)}\hat{\rho}}{\gamma}. \tag{3.43}$$

Applying once more the adiabatic approximation scheme to Eq.(3.34) and Eq.(3.35), we get

$$\hat{\rho}_{ac} = \frac{g}{\gamma}(\hat{\rho}_{aa}\hat{a} - \hat{a}\hat{\rho}_{cc} + \hat{\rho}_{ab}\hat{b}) \tag{3.44}$$

and

$$\hat{\rho}_{bc} = \frac{g}{\gamma}(\hat{\rho}_{ba}\hat{a} - \hat{b}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{b}). \tag{3.45}$$

Then combining Eq.(3.44) and Eq.(3.45) with (3.39)- (3.42), we easily find

$$\hat{\rho}_{ac} = \frac{r_a g}{\gamma^2}(\hat{\rho}^{(0)}_{aa}\hat{\rho}\hat{a} + \hat{\rho}^{(0)}_{ab}\hat{\rho}\hat{b}), \tag{3.46}$$

$$\hat{\rho}_{bc} = \frac{r_a g}{\gamma^2}(\hat{\rho}^{(0)}_{ba}\hat{\rho}\hat{a} + \hat{\rho}^{(0)}_{bb}\hat{\rho}\hat{b}). \tag{3.47}$$

Finally, on substituting Eqs.(3.46), (3.47) and their complex conjugate into Eq.(3.26), the equation of evolution of the density operator for the cavity modes takes the form

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} &= \frac{1}{2}A\hat{\rho}^{(0)}_{aa}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) + \frac{1}{2}A\hat{\rho}^{(0)}_{ab}(2\hat{a}^\dagger\hat{\rho}\hat{b} - \hat{\rho}\hat{b}\hat{a}^\dagger - \hat{b}\hat{a}^\dagger\hat{\rho}) \\
&+ \frac{1}{2}A\hat{\rho}^{(0)}_{ba}(2\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b}^\dagger) + \frac{1}{2}A\hat{\rho}^{(0)}_{bb}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\
&+ \frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}) \\
&+ \frac{\kappa}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger + 2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger) \\
&+ \kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger + \hat{a}\hat{\rho}\hat{b} + \hat{b}\hat{\rho}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho} - \hat{a}\hat{b}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger - \hat{\rho}\hat{a}\hat{b}),
\end{aligned} \tag{3.48}$$

where

$$A = \frac{2r_a g^2}{\gamma^2}. \tag{3.49}$$

is the linear gain coefficient. It is worth mentioning that the quantum properties of the light generated by two-mode v laser are determined by the master equation (3.48). In this equation, the terms involving $\hat{\rho}^{(0)}_{aa}$ and $\hat{\rho}^{(0)}_{bb}$ are the gain terms for the two modes. The terms proportional to $\hat{\rho}^{(0)}_{ab}$ and $\hat{\rho}^{(0)}_{ba}$ are phase sensitive as they arise due to the upper levels $\mid a\rangle$ and $\mid b\rangle$.

## 3.3 C-Number Langevin Equation

The dynamics of cavity mode coupled to squeezed vacuum reservoir can be also described using the c-number Langevin Equation. In this section we determine c-number Langevin Equations for the cavity mode employing the master equation in Eq.(3.48) to determine the equation of evolution for the expectation value of an operator $\hat{A}$ in Schrodinger picture can be expressed as [12]

$$\frac{d}{dt}\langle\hat{A}(t)\rangle = Tr(\frac{d}{dt}\hat{\rho}(t)\hat{A}). \tag{3.50}$$

on account of Eq.(3.50) and Eq.(3.48), the expectation value of the operator $\hat{a}$ can be written as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= Tr(\frac{1}{2}A\hat{\rho}^{(0)}_{aa}(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}) + \frac{1}{2}A\hat{\rho}^{(0)}_{ab}(2\hat{a}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a} - \hat{b}\hat{a}^\dagger\hat{\rho}\hat{a}))\\ &+ Tr(\frac{1}{2}A\hat{\rho}^{(0)}_{ba}(2\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}^\dagger\hat{a}) + \frac{1}{2}A\hat{\rho}^{(0)}_{bb}(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}))\\ &+ Tr(\frac{\kappa}{2}(N+1)(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}))\\ &+ Tr(\frac{\kappa}{2}N(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} + 2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}))\\ &+ Tr(\kappa M(\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a} + \hat{a}\hat{\rho}\hat{b}\hat{a} + \hat{b}\hat{\rho}\hat{a}\hat{a} - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a})).\end{aligned} \tag{3.51}$$

Applying the cyclic property of trace operation along with the fact $[\hat{a},\hat{a}^\dagger] = 1$ and $[\hat{a},\hat{b}] = [\hat{a}^\dagger,\hat{b}^\dagger] = [\hat{a},\hat{b}^\dagger] = 0$, we readily get

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}\mu\langle\hat{a}(t)\rangle + \frac{1}{2}A\hat{\rho}^{(0)}_{ab}\langle\hat{b}(t)\rangle, \tag{3.52}$$

where

$$\mu = \kappa - A\hat{\rho}^{(0)}_{aa}. \tag{3.53}$$

We can easily verify in a similar manner that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{1}{2}\eta\langle\hat{b}(t)\rangle + \frac{1}{2}A\hat{\rho}^{(0)}_{ba}\langle\hat{a}(t)\rangle, \tag{3.54}$$

where

$$\eta = \kappa - A\hat{\rho}_{bb}^{(0)}. \tag{3.55}$$

Furthermore, in view of Eq.(3.50) and Eq.(3.48), there follows

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}^2(t)\rangle &= \frac{1}{2}A\hat{\rho}_{aa}^{(0)}Tr((2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2) + \frac{1}{2}A\hat{\rho}_{ab}^{(0)}(2\hat{a}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a}^2 - \hat{b}\hat{a}^\dagger\hat{\rho}\hat{a}^2))\\ &+ \frac{1}{2}A\hat{\rho}_{ba}^{(0)}Tr((2\hat{b}^\dagger\hat{\rho}\hat{a}^3 - \hat{a}\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{b}^\dagger\hat{a}^2) + \frac{1}{2}A\hat{\rho}_{bb}^{(0)}(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2))\\ &+ \frac{\kappa}{2}(N+1)Tr((2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^2 - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{a}^3 + 2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a}^2 - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a}^2))\\ &+ \frac{\kappa}{2}NTr((2\hat{a}^\dagger\hat{\rho}\hat{a}^3 - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}^2 + 2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a}^2 - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a}^2))\\ &+ \kappa MTr((\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a}^2 + \hat{b}^\dagger\hat{\rho}\hat{a}^\dagger\hat{a}^2 + \hat{a}\hat{\rho}\hat{b}\hat{a}^2 + \hat{b}\hat{\rho}\hat{a}^3 - \hat{a}^\dagger\hat{b}^\dagger\hat{\rho}\hat{a}^2 - \hat{a}\hat{b}\hat{\rho}\hat{a}^2 - \hat{\rho}\hat{a}^\dagger\hat{b}^\dagger\hat{a}^2 - \hat{\rho}\hat{a}\hat{b}\hat{a}^2)).\end{aligned} \tag{3.56}$$

Applying the cyclic property and trace operation along with the relations

$$[\hat{a}, f(\hat{a}, \hat{a}^\dagger)] = \frac{\partial f(\hat{a}, \hat{a}^\dagger)}{\partial \hat{a}^\dagger}, \tag{3.57}$$

$$[\hat{a}^\dagger, f(\hat{a}, \hat{a}^\dagger)] = -\frac{\partial f(\hat{a}, \hat{a}^\dagger)}{\partial \hat{a}} \tag{3.58}$$

and

$$[\hat{a}^2, f(\hat{a}, \hat{a}^\dagger)] = 2\hat{a}, \tag{3.59}$$

One readily gets

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -\mu\langle\hat{a}^2(t)\rangle + A\hat{\rho}_{ab}^{(0)}\langle\hat{a}(t)\rangle\langle\hat{b}(t)\rangle. \tag{3.60}$$

It can also established in a similar way that

$$\frac{d}{dt}\langle\hat{b}^2(t)\rangle = -\eta\langle\hat{b}^2(t)\rangle + A\hat{\rho}_{ba}^{(0)}\langle\hat{a}(t)\rangle\langle\hat{b}(t)\rangle, \tag{3.61}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\frac{1}{2}(\mu+\eta)\langle\hat{a}(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{a}^2(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{b}^2(t)\rangle - \kappa M, \tag{3.62}$$

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle = -\frac{1}{2}(\mu+\eta)\langle\hat{a}(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}[\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \langle\hat{a}^\dagger(t)\hat{a}(t)\rangle] + A\rho_{ab}^{(o)}, \tag{3.63}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{b}^\dagger(t)\hat{a}(t)\rangle + \kappa N + A\rho_{aa}^{(o)}, \tag{3.64}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle = -\eta\langle\hat{b}^\dagger(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{b}(t)\hat{a}^\dagger(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle + \kappa N + A\rho_{bb}^{(o)}. \tag{3.65}$$

The c- number equations corresponding to Eqs. (3.52) and (3.54) are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu\langle\alpha(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\beta(t)\rangle, \tag{3.66}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}\eta\langle\beta(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha(t)\rangle. \tag{3.67}$$

On the basis of Eqs. (3.66) and (3.67), one can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + \frac{1}{2}A\rho_{ab}^{(o)}\beta(t) + f_\alpha(t), \tag{3.68}$$

$$\frac{d}{dt}\beta(t) = -\frac{1}{2}\eta\beta(t) + \frac{1}{2}A\rho_{ba}^{(o)}\alpha(t) + f_\beta(t), \tag{3.69}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces whose properties remain to be determined. Moreover, we se that Eq. (3.66) and the expectation values of Eq. (3.68) as well as Eq. (3.67) and the expectation value of Eq. (3.69) will have the same forms if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.70}$$

We note that

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = 2\langle\alpha(t)\frac{d\alpha(t)}{dt}\rangle,$$

so that on substituting (3.66) into this equation, we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + A\rho_{ab}^{(o)}\alpha(t)\beta(t)\rangle + 2\langle\alpha(t)f_\alpha(t)\rangle. \tag{3.71}$$

In a similar manner, we can verify that

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -\eta\langle\beta^2(t)\rangle + A\rho_{ba}^{(o)}\alpha(t)\beta(t)\rangle + 2\langle\beta(t)f_\beta(t)\rangle. \tag{3.72}$$

We also note that

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\frac{d\alpha^*(t)}{dt}\alpha(t)\rangle + \langle\alpha^*(t)\frac{d\alpha(t)}{dt}\rangle,$$

so that in view of Eqs. (3.68) and its complex conjugate, we get

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -\mu\langle\alpha^*(t)\alpha(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha^*(t)\beta(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha(t)\beta^*(t)\rangle + \langle f_\alpha^*(t)\alpha(t)\rangle + \langle f_\alpha(t)\alpha^*(t)\rangle. \tag{3.73}$$

One can also verify that

$$\frac{d}{dt}\langle\beta^*(t)\beta(t)\rangle = -\eta\langle\beta^*(t)\beta(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha^*(t)\beta(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\beta^*(t)\alpha(t)\rangle + \langle f_\beta^*(t)\beta(t)\rangle + \langle f_\beta(t)\beta^*(t)\rangle. \tag{3.74}$$

In addition, we see that

$$\frac{d}{dt}\alpha(t)\beta(t)\rangle = \langle\beta(t)\frac{d\alpha(t)}{dt}\rangle + \langle\alpha(t)\frac{d\beta(t)}{dt}\rangle.$$

It then follows from (3.68) and (3.69) that

$$\frac{d}{dt}\alpha(t)\beta(t)\rangle = -\frac{1}{2}(\mu+\eta)\langle\alpha(t)\beta(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\beta^2(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha^2(t)\rangle + \langle f_\alpha(t)\beta(t)\rangle + \langle f_\beta(t)\alpha(t)\rangle. \tag{3.75}$$

We can verify in a similar manner that

$$\frac{d}{dt}\alpha(t)\beta^*(t)\rangle = -\frac{1}{2}(\mu+\eta)\langle\alpha(t)\beta^*(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}[\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle] + \langle f_\alpha(t)\beta^*(t)\rangle + \langle f_\beta^*(t)\alpha(t)\rangle. \tag{3.76}$$

We observe that Eqs. (3.60) and (3.71) as well as Eqs. (3.61) and (3.72) will have the same forms provided that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0, \tag{3.77}$$

$$\langle\beta(t)f_\beta(t)\rangle = 0. \tag{3.78}$$

Moreover, Eqs.(3.64) and (3.73), (3.65) and (3.74), (3.62) and (3.75), (3.63) and (3.76) will have the same forms if

$$\langle f_\alpha^*(t)\alpha(t)\rangle + \langle f_\alpha(t)\alpha^*(t)\rangle = \kappa N + A\rho_{aa}^{(o)}, \tag{3.79}$$

$$\langle f_\beta^*(t)\beta(t)\rangle + \langle f_\beta(t)\beta^*(t)\rangle = \kappa N + A\rho_{bb}^{(o)}, \tag{3.80}$$

$$\langle f_\alpha(t)\beta(t)\rangle + \langle \alpha(t) f_\beta(t)\rangle = -\kappa M, \tag{3.81}$$

$$\langle f_\alpha(t)\beta^*(t)\rangle + \langle f_\beta^*(t)\alpha(t)\rangle = A\rho_{ab}^{(o)}. \tag{3.82}$$

The formal solution of (3.68) and (3.69) can be written as

$$\alpha(t) = \alpha(o)e^{-\frac{1}{2}\mu t} + \int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}[\frac{1}{2}A\rho_{ab}^{(o)}\beta(t^{'}) + f_\alpha(t^{'})]dt^{'} \tag{3.83}$$

and

$$\beta(t) = \beta(o)e^{-\frac{1}{2}\eta t} + \int_o^t e^{-\frac{1}{2}\eta(t-t^{'})}[\frac{1}{2}A\rho_{ba}^{(o)}\alpha(t^{'}) + f_\beta(t^{'})]dt^{'}. \tag{3.84}$$

Then in view of (3.83), we see that

$$\langle \alpha(t) f_\alpha(t)\rangle = \langle \alpha(o) f_\alpha(t)\rangle e^{-\frac{1}{2}\mu t} + \int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}[\langle \beta(t^{'}) f_\alpha(t)\rangle + \langle f_\alpha(t^{'}) f_\alpha(t)\rangle]dt^{'}. \tag{3.85}$$

Assuming the noise force f at time t does not affect the cavity mode variables at earlier times $t^{'}$ [30] and taking into account (3.77), we have

$$\int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}\langle f_\alpha(t^{'}) f_\alpha(t)\rangle dt^{'} = 0.$$

It then follows from this result that

$$\langle f_\alpha(t^{'}) f_\alpha(t)\rangle = 0. \tag{3.86}$$

Following a similar line of reasoning, one easily gets

$$\langle f_\beta(t^{'}) f_\beta(t)\rangle = 0. \tag{3.87}$$

Further more, employing (3.83) along with (3.79), one readily finds

$$\kappa N + A\rho_{aa}^{(o)} = \langle\alpha(o)f_\alpha^*(t)\rangle e^{-\frac{1}{2}\mu t} + \langle\langle f_\alpha(t)\alpha^*(o)\rangle e^{-\frac{1}{2}\mu t} + \frac{1}{2}A\rho_{ab}^{(o)}\int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}\langle f_\alpha^*(t)\beta(t^{'})\rangle dt^{'}$$
$$\frac{1}{2}A\rho_{ba}^{(o)}\int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}\langle f_\alpha(t)\beta^*(t^{'})\rangle dt^{'} + \int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}[\langle f_\alpha^*(t)f_\alpha(t^{'})\rangle + \langle f_\alpha(t)f_\alpha^*(t^{'})\rangle]dt^{'}, \tag{3.88}$$

so that on recalling the noise force f at time t does not affect the cavity mode variables at earlier times, the above expression reduces to

$$\int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}\langle f_\alpha^*(t)f_\alpha(t)\rangle dt^{'} = \int_o^t e^{-\frac{1}{2}\mu(t-t^{'})}\langle f_\alpha(t)f_\alpha^*(t)\rangle dt^{'} = \frac{\kappa N + A\rho_{aa}^{(o)}}{2}.$$

Now on the basis of the relation [12]

$$\int_o^t e^{-\frac{1}{2}a(t-t^{'})}\langle f(t)g(t^{'})\rangle dt^{'} = D. \tag{3.89}$$

We assert that

$$\langle f(t)g(t^{'})\rangle dt^{'} = 2D\delta(t-t^{'}), \tag{3.90}$$

where a is a constant and D is a constant or some function of the time t.

We then see that

$$\langle f_\alpha^*(t)f_\alpha(t^{'})\rangle = \langle f_\alpha(t)f_\alpha^*(t^{'})\rangle = (\kappa N + A\rho_{aa}^{(o)})\delta(t-t^{'}). \tag{3.91}$$

Following a similar procedure, one can easily verify that

$$\langle f_\beta^*(t)f_\beta(t^{'})\rangle = \langle f_\beta(t)f_\beta^*(t^{'})\rangle = (\kappa N + A\rho_{bb}^{(o)})\delta(t-t^{'}), \tag{3.92}$$

$$\langle f_\alpha(t) f_\beta(t^{'})\rangle = \langle f_\beta(t) f_\alpha(t^{'})\rangle = -\kappa M \delta(t - t^{'}), \tag{3.93}$$

$$\langle f_\alpha(t) f^*_\beta(t^{'})\rangle = \langle f^*_\beta(t) f_\alpha(t^{'})\rangle = A\rho^{(o)}_{ab} \delta(t - t^{'}). \tag{3.94}$$

Eqs. (3.86), (3.87), (3.90), (3.91),(3.92) and (3.93) describes the correlation properties of the noise forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

Now we seek to obtain the time dependent solution of $\alpha(t)$ and $\beta(t)$ by solving Eqs. (3.68) and (3.69) simultaneously. To solve such coupled differential equations, One can write a single matrix equation

$$\frac{d}{dt}Q(t) = MQ(t) + S(t), \tag{3.95}$$

in which

$$Q(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}, \tag{3.96}$$

$$M(t) = \begin{pmatrix} -\frac{\mu}{2} & \frac{A}{2}\rho^{(o)}_{ab} \\ \frac{A}{2}\rho^{(o)}_{ba} & -\frac{\eta}{2} \end{pmatrix}, \tag{3.97}$$

$$S(t) = \begin{pmatrix} f_\alpha(t) \\ f_\beta(t) \end{pmatrix}. \tag{3.98}$$

In order to solve (3.92), we need the eigen vectors and the eigen values of M such that

$$MV_i = \lambda_i V_i \qquad\qquad (i = 1, 2). \tag{3.99}$$

Let the egein vectors be

$$Vi = \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{3.100}$$

with the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{3.101}$$

It can be shown using(3.94) and (3.96) that

$$\begin{vmatrix} -\frac{\mu}{2} - \lambda & \frac{A}{2}\rho_{ab}^{(o)} \\ \frac{A}{2}\rho_{ba}^{(o)} & -\frac{\eta}{2} - \lambda \end{vmatrix} = 0. \tag{3.102}$$

Then one readily obtains the eigen values to be

$$\lambda_1 = \frac{-(\mu+\eta) + \sqrt{(\mu-\eta)^2 + 4A^2 \mid \rho_{ab}^{(o)} \mid^2}}{4}, \tag{3.103}$$

$$\lambda_2 = \frac{-(\mu+\eta) - \sqrt{(\mu-\eta)^2 + 4A^2 \mid \rho_{ab}^{(o)} \mid^2}}{4}, \tag{3.104}$$

so that in view of (3.53) and (3.55) along with the fact that

$$\mid \rho_{ab}^{(o)} \mid^2 = \rho_{aa}^{(o)} \rho_{bb}^{(o)} \tag{3.105}$$

and

$$\rho_{aa}^{(o)} + \rho_{bb}^{(o)} = 1. \tag{3.106}$$

Eqs. (3.103) and (3.104) reduce to

$$\lambda_1 = -\frac{\kappa - \lambda}{2}, \tag{3.107}$$

$$\lambda_2 = -\frac{\kappa}{2}, \tag{3.108}$$

with the aid of (3.99), (3.100) and (3.101), the corresponding eigen vectors are found to be

$$V_1 = \frac{1}{\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}}} \begin{pmatrix} \rho_{ab}^{(o)} \\ \rho_{bb}^{(o)} \end{pmatrix}, \tag{3.109}$$

$$V_2 = \frac{1}{\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}}} \begin{pmatrix} -\rho_{ab}^{(o)} \\ \rho_{aa}^{(o)} \end{pmatrix}, \tag{3.110}$$

we now construct a matrix V consisting of the eigen vectors of the matrix M as column matrices

$$V = (V_1 \qquad V_2),$$

that is

$$V = \begin{pmatrix} \frac{\rho_{ab}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{bb}^{(o)2}}} & \frac{-\rho_{ab}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{aa}^{(o)2}}} \\ \frac{\rho_{bb}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{bb}^{(o)2}}} & \frac{\rho_{aa}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{aa}^{(o)2}}} \end{pmatrix}. \tag{3.111}$$

The inverse of V turns out to be

$$V^{-1} = \frac{1}{\rho_{ab}^{(o)}} \begin{pmatrix} \rho_{aa}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}} & \rho_{ab}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}} \\ -\rho_{bb}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}} & \rho_{ab}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}} \end{pmatrix}. \tag{3.112}$$

Since $VV^{-1} = V^{-1}V = I$, we can rewrite Eq.(3.95) as

$$\frac{d}{dt}Q(t) = VV^{-1}MVV^{-1}Q(t) + S(t)$$

or

$$\frac{d(V^{-1}Q(t))}{dt} = DV^{-1}Q(t) + V^{-1}S(t), \tag{3.113}$$

where

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.114}$$

The formal solution of Eq.(3.113) is

$$V^{-1}Q(t) = e^{Dt}V^{-1}Q(0) + \int_o^t e^{D(t-t^{'})}V^{-1}S(t^{'})dt^{'}. \tag{3.115}$$

Now multiplying Eq.(3.115) on the left by V it follows that

$$Q(t) = Ve^{Dt}V^{-1}Q(0) + \int_o^t Ve^{D(t-t^{'})}V^{-1}S(t^{'})dt^{'}. \tag{3.116}$$

In view of the fact that D is diagonal, $e^{Dt}$ is simply equal to

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \tag{3.117}$$

and

$$e^{D(t-t^{'})} = \begin{pmatrix} e^{\lambda_1(t-t^{'})} & 0 \\ 0 & e^{\lambda_2(t-t^{'})} \end{pmatrix}. \tag{3.118}$$

From Eqs. (3.96), (3.111), (3.112), (3.117) and (3.118), we readily find

$$Ve^{Dt}V^{-1}Q(0) = \begin{pmatrix} [\rho_{aa}^{(o)}e^{\lambda_1(t)} + \rho_{bb}^{(o)}e^{\lambda_2(t)}]\alpha(o) + \rho_{ab}^{(o)}(e^{\lambda_1(t)} - e^{\lambda_2(t)})\beta(o) \\ \rho_{ba}^{(o)}(e^{\lambda_1(t)} - e^{\lambda_2(t)})\alpha(o) + [\rho_{bb}^{(o)}e^{\lambda_1(t)} + \rho_{aa}^{(o)}e^{\lambda_2(t)}]\beta(o) \end{pmatrix}, \tag{3.119}$$

and

$$\int_o^t Ve^{D(t-t^{'})}V^{-1}S(t^{'})dt^{'} =$$

$$\begin{pmatrix} \int_o^t[\rho_{aa}^{(o)}e^{\lambda_1(t-t^{'})} + \rho_{bb}^{(o)}e^{\lambda_2(t-t^{'})}]f_\alpha(t^{'})dt^{'} + \rho_{ab}^{(o)}\int_o^t(e^{\lambda_1(t-t^{'})} - e^{\lambda_2(t-t^{'})})f_\beta(t^{'})dt^{'} \\ \rho_{ba}^{(o)}\int_o^t(e^{\lambda_1(t-t^{'})} - e^{\lambda_2(t-t^{'})})f_\alpha(t^{'})dt^{'} + \int_o^t[\rho_{bb}^{(o)}e^{\lambda_1(t-t^{'})} + \rho_{aa}^{(o)}e^{\lambda_2(t-t^{'})}]f_\beta(t^{'})dt^{'} \end{pmatrix}. \tag{3.120}$$

Using Eqs. (3.116), (3.119) and (3.120), we obtain

$$\alpha(t) = A_1(t)\alpha(o) + B_1(t)\beta(o) + F_1(t) \tag{3.121}$$

and

$$\beta(t) = A_2(t)\alpha(o) + B_2(t)\beta(o) + F_2(t), \tag{3.122}$$

where

$$A_1(t) = \rho_{aa}^{(o)} e^{\lambda_1(t)} + \rho_{bb}^{(o)} e^{\lambda_2(t)}, \tag{3.123}$$

$$B_1(t) = \rho_{ab}^{(o)} (e^{\lambda_1(t)} - e^{\lambda_2(t)}), \tag{3.124}$$

$$A_2(t) = \rho_{ba}^{(o)} (e^{\lambda_1(t)} - e^{\lambda_2(t)}), \tag{3.125}$$

$$B_2(t) = \rho_{bb}^{(o)} e^{\lambda_1(t)} + \rho_{aa}^{(o)} e^{\lambda_2(t)}, \tag{3.126}$$

$$F_1(t) = \int_o^t [\rho_{aa}^{(o)} e^{\lambda_1(t-t')} + \rho_{bb}^{(o)} e^{\lambda_2(t-t')}] f_\alpha(t') dt',$$

$$+ \ \ \rho_{ab}^{(o)} \int_o^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}) f_\beta(t') dt', \tag{3.127}$$

$$F_2(t) = \rho_{ba}^{(o)} \int_o^t (e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}) f_\alpha(t') dt',$$

$$+ \ \ \int_o^t [\rho_{bb}^{(o)} e^{\lambda_1(t-t')} + \rho_{aa}^{(o)} e^{\lambda_2(t-t')}] f_\alpha(t') dt'. \tag{3.128}$$

These solutions along with the correlation relations are used to determine the mean photon number, the variance of photon number, the quadrature variance and the variance of sum of EPR-operator.

# 4

## Results and Discussion

### 4.1 Quadrature Variance

We next proceed to determine the quadrature variance for the cavity radiation produced by two-mode v laser coupled to squeezed vacuum reservoir. The variances of the quadrature operators, for a two mode radiation [31],

$$\hat{C} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}). \tag{4.1}$$

Use the commutation relation

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1. \tag{4.2}$$

It can easily be shown that

$$[\hat{c}, \hat{c}^{\dagger}] = 1. \tag{4.3}$$

The squeezing properties of a two-mode light are described by the quadrature operators

$$\hat{C_{+}} = \hat{C} + \hat{C}^{\dagger} \tag{4.4}$$

and

$$\hat{C_-} = i(\hat{C}^\dagger - \hat{C}). \tag{4.5}$$

With the aid of Eqs.(4.3), (4.4) and (4.5), One can readily verify that

$$[\hat{C_+}, \hat{C_-}] = 2i. \tag{4.6}$$

A two-mode light is said to be in a squeezed state if either $\Delta C_+ < 1$ and $\Delta C_- > 1$ or $\Delta C_+ > 1$ and $\Delta C_- < 1$ such that $\Delta C_+ \Delta C_- \geq 1$. The variance of the quadrature operators are defined by

$$\Delta c_\pm^2 = \langle \hat{C}_\pm^2(t) \rangle - \langle \hat{C}_\pm(t) \rangle^2. \tag{4.7}$$

Applying Eqs.(4.4) and (4.5), one can write (4.7) in the normal order as

$$\Delta c_\pm^2 = 1 \pm [\langle \hat{c}^{\dagger 2} \rangle + \langle \hat{c}^2 \rangle \pm 2\langle \hat{c}^\dagger \hat{c} \rangle - \langle \hat{c}^\dagger \rangle^2 - \langle \hat{c} \rangle^2 \mp 2\langle \hat{c} \rangle \langle \hat{c}^\dagger \rangle]. \tag{4.8}$$

This can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_\pm^2(t) = 1 \pm [\langle \gamma^{*2}(t) \rangle + \langle \gamma^2(t) \rangle \pm 2\langle \gamma^*(t)\gamma(t) \rangle - \langle \gamma^*(t) \rangle^2 - \langle \gamma(t) \rangle^2 \mp 2\langle \gamma(t) \rangle \langle \gamma^*(t) \rangle], \tag{4.9}$$

where $\gamma(t)$ is the c-number corresponding to the operator $\hat{c}$. Eq.(4.9) can be written as

$$\Delta c_\pm(t) = 1 \pm (\langle \gamma_\pm^2(t) - \langle \gamma_\pm(t) \rangle^2), \tag{4.10}$$

where

$$\gamma_\pm(t) = \gamma^*(t) \pm \gamma(t). \tag{4.11}$$

The c-number equation corresponding to Eq. (4.1) is expressible as

$$\gamma(t) = \frac{1}{\sqrt{2}}(\alpha(t) + \beta(t)). \tag{4.12}$$

With the aid of Eqs.(4.11) and (4.12), we get

$$\begin{aligned}\Delta c_{\pm}{}^2 = 1 \pm [&\frac{1}{2}(\langle\alpha^2(t)\rangle + \langle\alpha^{*2}(t)\rangle + \langle\beta^2(t)\rangle + \langle\beta^{*2}(t)\rangle + 2\langle\alpha(t)\beta(t)\rangle + 2\langle\alpha^*(t)\beta^*(t)\rangle) \\ &\pm (\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle + \langle\alpha^*(t)\beta(t)\rangle + \langle\beta^*(t)\alpha(t)\rangle)] \mp \langle\alpha^*(t) + \beta^*(t) \pm (\alpha(t) + \beta(t))\rangle^2.\end{aligned} \tag{4.13}$$

Now we proceed to find the expectation values involved in Eq. (4.13). With the aid of Eqs. (3.121), (3.122), (3.127), (3.128) and the assumption that the cavity modes initially in a vacuum state, one can verify that

$$\langle\alpha(t)\rangle = \langle\beta(t)\rangle = 0. \tag{4.14}$$

On the other hand, taking into account Eq. (3.121) and its complex conjugate along with the fact that the noise force does not affect the cavity modes to be initially in a vacuum state, we can write as

$$\langle\alpha^*(t)\alpha(t)\rangle = \langle F_1^*(t)F_1(t)\rangle. \tag{4.15}$$

On account of Eq. (3.127) and its complex conjugate, we see that

$$\begin{aligned}
\langle\alpha^*(t)\alpha(t)\rangle &= \rho_{aa}^{(o)2}\int_o^t e^{\lambda_1(2t-t'-t'')}\langle f_\alpha^*(t')f_\alpha(t'')\rangle dt'dt'' \\
&+ \rho_{bb}^{(o)2}\int_o^t e^{\lambda_2(2t-t'-t'')}\langle f_\alpha^*(t')f_\alpha(t'')\rangle dt'dt'' \\
&+ \rho_{ab}^{(o)2}\int_o^t (e^{\lambda_1(2t-t'-t'')} + e^{\lambda_2(2t-t'-t'')} - 2e^{\lambda_1(t-t')+\lambda_2(t-t'')})\langle f_\beta^*(t')f_\beta(t'')\rangle dt'dt'' \\
&+ 2\rho_{aa}^{(o)}\rho_{bb}^{(o)}\int_o^t e^{\lambda_1(t-t')+\lambda_2(t-t'')})\langle f_\alpha^*(t')f_\alpha(t')\rangle dt'dt'' \\
&+ \rho_{aa}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(2t-t'-t'')} - e^{\lambda_1(t-t')+\lambda_2(t-t'')})\langle f_\alpha^*(t')f_\beta(t'')\rangle dt'dt'' \\
&+ \rho_{bb}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(t-t')+\lambda_2(t-t'')} - e^{\lambda_2(2t-t'-t'')})\langle f_\alpha^*(t')f_\beta(t'')\rangle dt'dt'' \\
&+ \rho_{aa}^{(o)}\rho_{ba}^{(o)}\int_o^t (e^{\lambda_1(2t-t'-t'')} - e^{\lambda_1(t-t')+\lambda_2(t-t'')})\langle f_\beta^*(t')f_\alpha(t'')\rangle dt'dt'' \\
&+ \rho_{bb}^{(o)}\rho_{ba}^{(o)}\int_o^t (e^{\lambda_1(t-t')+\lambda_2(t-t'')} - e^{\lambda_2(2t-t'-t'')})\langle f_\beta^*(t')f_\alpha(t'')\rangle dt'dt''. \qquad (4.16)
\end{aligned}$$

Carrying out the integration gives

$$\begin{aligned}
\langle\alpha^*(t)\alpha(t)\rangle &= \frac{1}{2\lambda_1}[\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} + A\rho_{aa}^{(o)}\rho_{ab}^{(o)2}](e^{2\lambda_1 t} - 1) \\
&+ \frac{1}{2\lambda_2}[\rho_{bb}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{bb}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} - A\rho_{bb}^{(o)}\rho_{ab}^{(o)2}](e^{2\lambda_2 t} - 1) \\
&+ \frac{1}{\lambda_1+\lambda_2}[-2\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + 2\rho_{aa}^{(o)}\rho_{bb}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} \\
&+ A\rho_{bb}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} - A\rho_{aa}^{(o)}\rho_{ab}^{(o)2} + A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{(\lambda_1+\lambda_2)t} - 1). \qquad (4.17)
\end{aligned}$$

It can also established following a similar procedure that

$$\begin{aligned}
\langle\beta^*(t)\beta(t)\rangle &= \frac{1}{2\lambda_1}[\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{bb}^{(o)}\rho_{ba}^{(o)2} + A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{2\lambda_1 t} - 1) \\
&+ \frac{1}{2\lambda_2}[\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{aa}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{aa}\rho_{ba}^{(o)2} - A\rho_{aa}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{2\lambda_1 t} - 1) \\
&+ \frac{1}{\lambda_1+\lambda_2}[-2\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + 2A\rho_{aa}^{(o)}\rho_{bb}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{bb}^{(o)}\rho_{ba}^{(o)2} \\
&+ A\rho_{aa}^{(o)}\rho_{ba}^{(o)2} - A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)} + A\rho_{aa}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{(\lambda_1+\lambda_2)t} - 1), \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}\langle\alpha^*(t)\beta(t)\rangle &= \frac{1}{2\lambda_1}[\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{aa}^{(o)}\rho_{bb}^{(o)}\rho_{ba}^{(o)} + A\rho_{ab}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{2\lambda_1 t}-1)\\ &+ \frac{1}{2\lambda_2}[-\rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - \rho_{aa}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{bb}^{(o)}\rho_{aa}^{(o)}\rho_{ba}^{(o)} + A\rho_{ab}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}(e^{2\lambda_2 t}-1)\\ &+ \frac{1}{\lambda_1+\lambda_2}[-\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - A\rho_{bb}^{(o)}A\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})\\ &+ A\rho_{aa}^{(o)}A\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{aa}^{(o)2}\rho_{ba}^{(o)} + A\rho_{bb}^{(o)2}\rho_{ba}^{(o)} - 2A\rho_{ab}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}](e^{(\lambda_1+\lambda_2)t}-1),\end{aligned} \tag{4.19}$$

$$\begin{aligned}\langle\beta^*(t)\alpha(t)\rangle &= \frac{1}{2\lambda_1}[\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{ba}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} + A\rho_{bb}^{(o)}\rho_{aa}^{(o)}\rho_{ab}^{(o)}](e^{2\lambda_1 t}-1)\\ &+ \frac{1}{2\lambda_2}[-\rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - \rho_{aa}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{ba}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} + A\rho_{aa}^{(o)}\rho_{bb}^{(o)}\rho_{ab}^{(o)}(e^{2\lambda_2 t}-1)\\ &+ \frac{1}{\lambda_1+\lambda_2}[-\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - 2A\rho_{ba}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} + A\rho_{ab}^{(o)2}\rho_{ab}^{(o)}\\ &+ A\rho_{aa}^{(o)2}\rho_{ab}^{(o)} + \rho_{aa}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) - \rho_{bb}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})](e^{(\lambda_1+\lambda_2)t}-1),\end{aligned} \tag{4.20}$$

$$\begin{aligned}\langle\alpha(t)\beta(t)\rangle &= \frac{1}{2\lambda_1}[-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M](e^{2\lambda_1 t}-1)\\ &+ \frac{1}{2\lambda_2}[-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M](e^{2\lambda_2 t}-1)\\ &+ \frac{1}{\lambda_1+\lambda_2}[-\rho_{aa}^{(o)2}\kappa M - \rho_{bb}^{(o)2}\kappa M + 2\rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M](e^{(\lambda_1+\lambda_2)t}-1),\end{aligned} \tag{4.21}$$

$$\langle\alpha^2(t)\rangle = \frac{-2\rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M}{2\lambda_1}(e^{2\lambda_1 t}-1) + \frac{2\rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M}{2\lambda_2}(e^{2\lambda_2 t}-1) + \frac{2\rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M - 2\rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1), \tag{4.22}$$

$$\langle\beta^2(t)\rangle = \frac{-2\rho_{bb}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_1}(e^{2\lambda_1 t}-1) + \frac{2\rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_2}(e^{2\lambda_2 t}-1) + \frac{2\rho_{bb}^{(o)}\rho_{ba}^{(o)}\kappa M - 2\rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{\lambda_1+\lambda_2}(e^{(\lambda_1+\lambda_2)t}-1), \tag{4.23}$$

$$\langle\alpha(t)\beta(t)\rangle = \langle\alpha^*(t)\beta^*(t)\rangle, \tag{4.24}$$

$$\langle\alpha^2(t)\rangle = \langle\alpha^{*2}(t)\rangle, \tag{4.25}$$

$$\langle \beta^2(t) \rangle = \langle \beta^{*2}(t) \rangle. \tag{4.26}$$

Now with the aid of Eqs.(4.17)-(4.26), we arrive at

$$\begin{aligned}
\Delta c_{\pm}^2 &= 1 + [\frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A|\ \rho_{ab}^{(o)}\ |^2 + (\kappa N + A)(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2})}{2\lambda_1} \\
&+ \frac{4A|\ \rho_{ab}^{(o)}\ |^3 \cos\theta + 2\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + 2\rho_{bb}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})}{2\lambda_1}](e^{2\lambda_1 t} - 1) \\
&+ [\frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \kappa N(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2}) + 4A|\ \rho_{ba}^{(o)}\ |^3 \cos\theta - A|\ \rho_{bb}^{(o)}\ |^2}{2\lambda_2} \\
&- \frac{2\rho_{aa}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) - 2\rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)})}{2\lambda_2}](e^{2\lambda_2 t} - 1) \\
&+ [\frac{2|\ \rho_{ab}^{(o)}\ |^2(\kappa N + A) + 2\rho_{ab}^{(o)}(\kappa N + A\rho_{aa}^{(o)})(\rho_{bb}^{(o)} - \rho_{aa}^{(o)}) + 2\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})(\rho_{aa}^{(o)} - \rho_{bb}^{(o)})}{\lambda_1 + \lambda_2} \\
&\frac{-2\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + 2A\rho_{aa}^{(o)}|\ \rho_{ab}^{(o)}\ |^2 - A\rho_{aa}^{(o)}(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2}) + +A\rho_{bb}^{(o)}|\ \rho_{ab}^{(o)}\ |^2}{\lambda_1 + \lambda_2} \\
&+ \frac{2A|\ \rho_{ab}^{(o)}\ |(\rho_{aa}^{(o)2} + \rho_{bb}^{(o)2})\cos\theta - 2\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) - A\rho_{bb}^{(o)}\rho_{b}^{(o)2}}{\lambda_1 + \lambda_2}](e^{(\lambda_1 + \lambda_2)t} - 1) \\
&\pm [\frac{-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M - \rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M - \rho_{bb}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_1}](e^{2\lambda_1 t} - 1) \\
&\pm [\frac{-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M + \rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M + \rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_2}](e^{2\lambda_2 t} - 1) \\
&\pm [\frac{-\rho_{ab}^{(o)2}\kappa M - \rho_{bb}^{(o)2}\kappa M + 2\rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M + \rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M}{\lambda_1 + \lambda_2} \\
&- \frac{\rho_{bb}^{(o)}\rho_{ab}^{(o)}KM + \rho_{bb}^{(o)}\rho_{ba}^{(o)}KM - \rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{\lambda_1 + \lambda_2}](e^{(\lambda_1 + \lambda_2)t} - 1).
\end{aligned} \tag{4.27}$$

At steady state we have,

$$
\begin{aligned}
\Delta c_{\pm}^{2} &= 1 + \frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A|\ \rho_{ab}^{(o)}\ |^{2} + (\kappa N + A)(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2})}{2\lambda_1} \\
&+ \frac{4A|\ \rho_{ab}^{(o)}\ |^{3}\cos\theta + 2\rho_{aa}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) + 2\rho_{bb}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})}{2\lambda_1} \\
&+ \frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \kappa N(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2}) + 4A|\ \rho_{ba}^{(o)}\ |^{3}\cos\theta - A|\ \rho_{bb}^{(o)}\ |^{2}}{2\lambda_2} \\
&- \frac{2\rho_{aa}^{(o)}\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) - 2\rho_{bb}^{(o)}\rho_{ba}^{(o)}(\kappa N + A\rho_{aa}^{(o)})}{2\lambda_2} \\
&+ \frac{2|\ \rho_{ab}^{(o)}\ |^{2}(KN + A) + 2\rho_{ab}^{(o)}(\kappa N + A\rho_{aa}^{(o)})(\rho_{bb}^{(o)} - \rho_{aa}^{(o)}) + 2\rho_{ab}^{(o)}(\kappa N + A\rho_{bb}^{(o)})(\rho_{aa}^{(o)} - \rho_{bb}^{(o)})}{\lambda_1 + \lambda_2} \\
&\frac{-2\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + 2A\rho_{aa}^{(o)}|\ \rho_{ab}^{(o)}\ |^{2} - A\rho_{aa}^{(o)}(\rho_{ab}^{(o)2} + \rho_{ba}^{(o)2}) + +A\rho_{bb}^{(o)}|\ \rho_{ab}^{(o)}\ |^{2}}{\lambda_1 + \lambda_2} \\
&+ \frac{2A|\ \rho_{ab}^{(o)}\ |(\rho_{aa}^{(o)2} + \rho_{bb}^{(o)2})\cos\theta - 2\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) - A\rho_{bb}^{(o)}\rho_{b}^{(o)2}}{\lambda_1 + \lambda_2} \\
&\pm [\frac{-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M - \rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M - \rho_{bb}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_1}] \\
&\pm [\frac{-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M + \rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M + \rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{2\lambda_2}] \\
&\pm [\frac{-\rho_{ab}^{(o)2}\kappa M - \rho_{bb}^{(o)2}\kappa M + 2\rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M + \rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M - \rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M + \rho_{bb}^{(o)}\rho_{ba}^{(o)}\kappa M - \rho_{aa}^{(o)}\rho_{ba}^{(o)}\kappa M}{\lambda_1 + \lambda_2}].
\end{aligned}
\tag{4.28}
$$

Now for the sake of convenience, we introduce a new parameter defined by

$$
\rho_{aa}^{(o)} - \rho_{bb}^{(o)} = \epsilon, \tag{4.29}
$$

where $\epsilon$ is initial preparation of three level atom.

Now with the aid of Eqs. (3.105), (3.106), (3.107), (3.108) and (4.29), Eq. (4.28) can be rewritten as

$$\Delta c_{\pm}^{2} = \frac{\kappa + 2\kappa N - NA + A(1+N)(1-\epsilon^{2})^{\frac{1}{2}}e^{i\theta}}{\kappa - A}$$
$$\pm\,[\frac{8\kappa MA - 8\kappa^{2}M - 2MA^{2} + 2MA^{2}\epsilon^{2} + (2MA^{2} - 4\kappa MA(1-\epsilon^{2})^{\frac{1}{2}}e^{i\theta}}{4\kappa^{2} - 6\kappa A + 2A^{2}}]. \tag{4.30}$$

Eq.(4.30) represents the cavity quadrature variances for a two mode v laser coupled to squeezed vacuum reservoir.

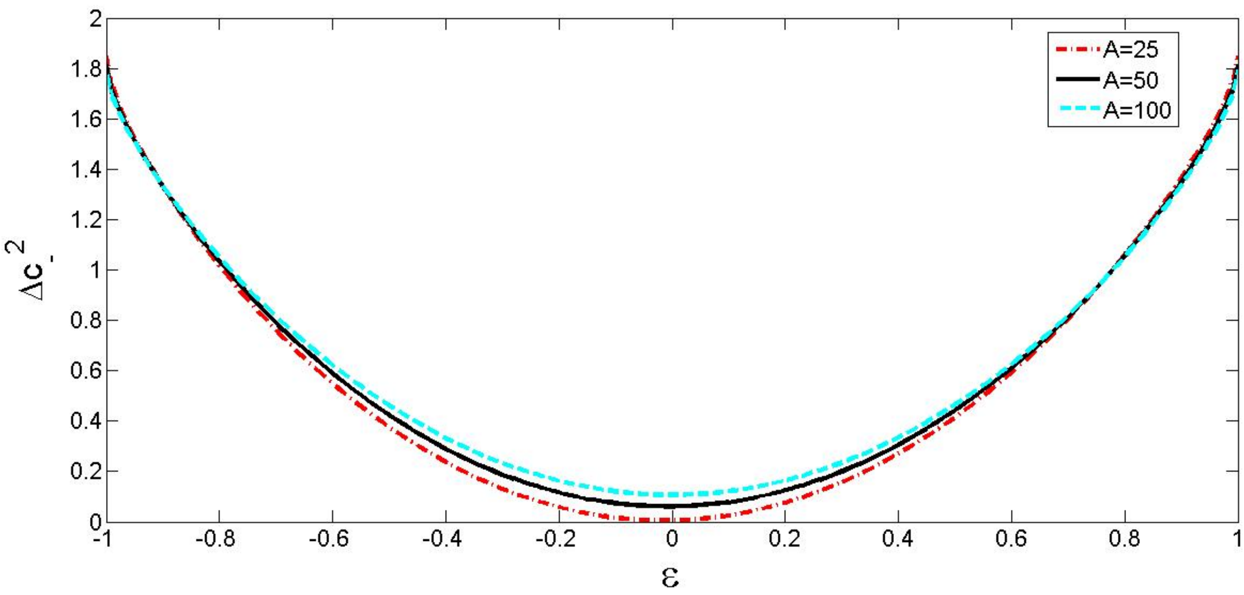


*Fig. 4.1:* Plots of a minus quadrature variance[Eq.(4.30)] versus epsilon for $\theta = 0, \kappa = 0.8, r = 1$, and different values of the linear gain coefficient for A = 25(dashed curve), A = 50(solid curve) and A = 100(dash-dot curve).

We see from the figure that the quadrature variance increases with increasing linear gain coefficient.

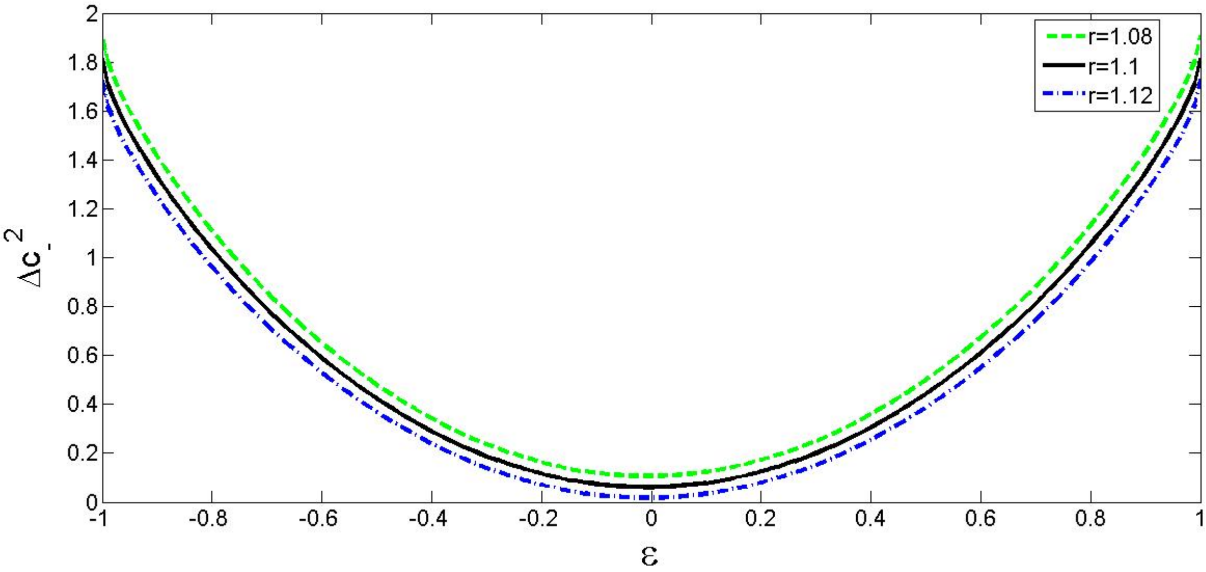


*Fig. 4.2:* Plots of a minus quadrature variance[Eq.(4.30)] versus epsilon for $\theta = 0, \kappa = 0.8, A = 50$, and different values of squeeze parameter for r = 1.08(dashed curve), r = 1.1(solid curve) and r = 1.12(dash-dot curve).

As we see from the figure the variance of the quadrature decrease with increasing squeeze parameter(r). One of the effects of squeezed vacuum reservoir is to decrease the minus quadrature variance of the cavity modes.

We now consider the case in which the cavity mode is coupled to a two-mode vacuum reservoir, upon setting $r = 0$ in Eq.(4.30) we get,

$$\Delta c_{\pm}^2 = \frac{\kappa + A(1-\epsilon^2)^{\frac{1}{2}} e^{i\theta}}{\kappa - A}. \qquad (4.30a)$$

Equation(4.30 a)represents quadrature variance of two-mode v laser coupled to vacuum reservoir.

We consider the special case in which no atoms are injected into the cavity coupled to a two-mode squeezed vacuum reservoir, upon setting $A = 0$ in Eq.(4.30), we get

$$\Delta c_{\pm}^2 = \frac{\kappa + 2\kappa N \mp 8\kappa^2 M}{4\kappa}. \qquad (4.30b)$$

Equation (4.30 b) represents the quadrature variance of squeezed vacuum reservoir.

## 4.2 Photon statistics

Photon statistics of light modes based on the relation between the variance and the mean of the photon number. Hence the photon statistics of a light mode for which $\Delta n^2 = \bar{n}$ is referred to as Poissonian and the photon statistics of a light mode for which $\Delta n^2 > \bar{n}$ is called super-Poissonian. Otherwise the photon statistics is said to be sub-Poissonian.

### 4.2.1 The mean photon number

The mean and variance of the photon number for the system under consideration can be studied using the number operator represented by

$$\hat{n} = \hat{n}_a + \hat{n}_b, \tag{4.31}$$

where $\hat{n}_a = \hat{a}^\dagger \hat{a}$ and $\hat{n}_b = \hat{b}^\dagger \hat{b}$ are the photon number operators for mode a and b. The mean photon number for the cavity mode can also be expressed in terms of c-number variables associated with the normal ordering as

$$\langle \hat{n} \rangle = \langle \alpha^*(t)\alpha(t) \rangle + \langle \beta^*(t)\beta(t) \rangle. \tag{4.32}$$

From Eq.(4.17) and (4.18), we have

$$\begin{aligned}
\langle \alpha^*(t)\alpha(t) \rangle = {} & [\frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} + A\rho_{aa}^{(o)}\rho_{ab}^{(o)2}}{2\lambda_1}](e^{2\lambda_1 t} - 1) \\
& + [\frac{\rho_{bb}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{bb}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)} - A\rho_{bb}^{(o)}\rho_{ab}^{(o)2}}{2\lambda_2}](e^{2\lambda_2 t} - 1) \\
& + [\frac{-2\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + 2\rho_{aa}^{(o)}\rho_{bb}^{(o)}(\kappa N + A\rho_{aa}^{(o)}) - A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)}}{\lambda_1 + \lambda_2} \\
& + \frac{A\rho_{bb}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)}}{\lambda_1 + \lambda_2}](e^{(\lambda_1 + \lambda_2)t} - 1)
\end{aligned} \tag{4.33}$$

and

$$\langle\beta^*(t)\beta(t)\rangle = [\frac{\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{bb}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) + A\rho_{bb}^{(o)}\rho_{ba}^{(o)2} + A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}}{2\lambda_1}](e^{2\lambda_1 t} - 1)$$
$$+ [\frac{\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + \rho_{aa}^{(o)2}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{aa}^{(o)}\rho_{ba}^{(o)2} - A\rho_{aa}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}}{2\lambda_2}](e^{2\lambda_2 t} - 1)$$
$$+ [\frac{-2\rho_{ba}^{(o)2}(\kappa N + A\rho_{aa}^{(o)}) + 2A\rho_{aa}^{(o)}\rho_{bb}^{(o)}(\kappa N + A\rho_{bb}^{(o)}) - A\rho_{bb}^{(o)2}}{\lambda_1 + \lambda_2}$$
$$+ \frac{A\rho_{aa}^{(o)}\rho_{ba}^{(o)2} - A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ba}^{(o)} + A\rho_{aa}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}}{\lambda_1 + \lambda_2}](e^{(\lambda_1 + \lambda_2)t} - 1). \tag{4.34}$$

From Eqs.(4.33) and (4.34), we obtain

$$\langle\hat{n}\rangle = \frac{2(\kappa N + A)}{(\kappa - A)}(1 - e^{-(\kappa - A)t}) + 2N(1 - e^{-\kappa t}). \tag{4.34a}$$

With the aid of (3.105), (3.106), (3.107), (3.108) and (4.29) on combining Eqs. (4.33), (4.34) and (4.34a) at steady state, we obtain

$$\langle\hat{n}\rangle_{ss} = \frac{2\kappa N + A - NA}{(\kappa - A)}, \tag{4.35}$$

where $ss$ stands for steady state. This represent the mean of the photon number due to the two-mode v laser coupled to squeezed vacuum reservoir.

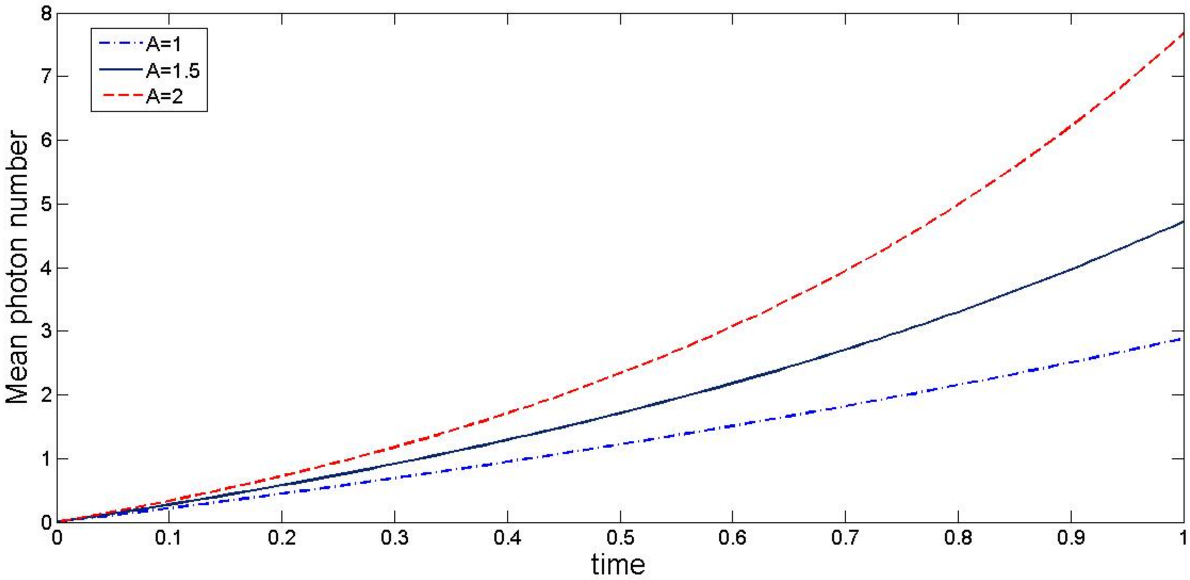


*Fig. 4.3:* Plots of the mean photon number [Eq.(4.34a)] versus time for $\kappa = 0.8, r = 1, \mathbf{e} = 2.72,$, and different values of the linear gain coefficient,for A = 1(dash-dot curve), A = 1.5(solid cure) and A = 2(dashed curve).

As we see from Fig(4.2) the mean photon number increases when time and linear gain coefficient increase.

### 4.2.2 Photon Number Variance

The variance of photon number can be expressed as

$$\triangle n^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{4.36}$$

Applying Eq.(4.31), the variance can be put in the form

$$\triangle n^2 = \triangle n_a^2 + \triangle n_b^2 + 2(\langle \hat{n}_a \hat{n}_b \rangle - \langle \hat{n}_a \rangle \langle \hat{n}_b \rangle). \tag{4.37}$$

On the other hand, the photon number variance for mode a can be written in the normal-order as

$$\triangle n_a^2 = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle \hat{a}^\dagger \hat{a}\rangle - \langle \hat{a}^\dagger \hat{a}\rangle^2. \tag{4.38}$$

This can also be expressed in terms of c-number variables associated in the normal-ordering as

$$\triangle n_a^2 = \langle \alpha^{*2}\alpha^2\rangle + \langle \alpha^*\alpha\rangle - \langle \alpha^*\alpha\rangle^2. \tag{4.39}$$

Following a similar procedure, the photon number variance for mode b can also be written as

$$\triangle n_b^2 = \langle \beta^{*2}\beta^2\rangle + \langle \beta^*\beta\rangle - \langle \beta^*\beta\rangle^2. \tag{4.40}$$

In addition, one can also write

$$\langle \hat{n}_a \hat{n}_b\rangle = \langle \alpha^*\alpha\beta^*\beta\rangle. \tag{4.41}$$

Hence substitution of Eqs.(4.39), (4.40) and (4.41), in to (4.37) yields

$$\triangle n^2 = \langle \alpha^{*2}\alpha^2\rangle + \langle \beta^{*2}\beta^2\rangle + \langle \alpha^*\alpha\rangle + \langle \beta^*\beta\rangle - (\langle \alpha^*\alpha\rangle + \langle \beta^*\beta\rangle)^2 + 2\langle \alpha^*\alpha\beta^*\beta\rangle. \tag{4.42}$$

In view of Eq.(4.32), we see that

$$\triangle n^2 = \langle \bar{n}\rangle - \langle \bar{n}\rangle^2 + \langle \alpha^{*2}\alpha^2\rangle + \langle \beta^{*2}\beta^2\rangle + 2\langle \alpha^*\alpha\beta^*\beta\rangle. \tag{4.43}$$

From Eq.(3.121) and (3.122), we have

$$\alpha(t) = A_1(t)\alpha(o) + B_1(t)\beta(o) + \eta_\alpha(t), \tag{4.44}$$

$$\beta(t) = A_2(t)\alpha(o) + B_2(t)\beta(o) + \eta_\beta(t), \tag{4.45}$$

where

$$\eta_\alpha(t) = \int_o^t [\rho_{aa}^{(o)} e^{\lambda_1(t-t^{'})} + \rho_{bb}^{(o)} e^{\lambda_2(t-t^{'})}] f_\alpha(t^{'})dt^{'} + \rho_{ab}^{(o)} \int_o^t (e^{\lambda_1(t-t^{'})} - e^{\lambda_2(t-t^{'})}) f_\beta(t^{'})dt^{'}, \quad (4.46)$$

$$\eta_\beta(t) = \rho_{ba}^{(o)} \int_o^t (e^{\lambda_1(t-t^{'})} - e^{\lambda_2(t-t^{'})}) f_\alpha(t^{'})dt^{'} + \int_o^t [\rho_{bb}^{(o)} e^{\lambda_1(t-t^{'})} + \rho_{aa}^{(o)} e^{\lambda_2(t-t^{'})}] f_\beta(t^{'})dt^{'}. \quad (4.47)$$

Employing Eq.(4.42) and its complex conjugate along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we have

$$\langle \alpha^{*2}(t)\alpha^2(t)\rangle = \langle \eta^2(t)\eta^{*2}(t)\rangle. \quad (4.48)$$

We recall that Gaussian variables with vanishing means satisfying the relation [32]

$$\langle ABCD\rangle = \langle AB\rangle\langle CD\rangle + \langle AC\rangle\langle BD\rangle + \langle AD\rangle\langle BC\rangle. \quad (4.49)$$

Next we need to show that $\eta_\alpha(t)$ is Gaussian variable, one can rewrite Eq.(4.46) as

$$\eta_\alpha(t) = \eta_+(t) + \eta_-(t), \quad (4.50)$$

in which

$$\eta_+(t) = \int_o^t [\rho_{aa}^{(o)} e^{\lambda_1(t-t^{'})} + \rho_{bb}^{(o)} e^{\lambda_2(t-t^{'})}] f_\alpha(t^{'})dt^{'}, \quad (4.51)$$

$$\eta_-(t) = \rho_{ab}^{(o)} \int_o^t (e^{\lambda_1(t-t^{'})} - e^{\lambda_2(t-t^{'})}) f_\beta(t^{'})dt^{'}. \quad (4.52)$$

Employing Eqs. (4.51) and (4.52), the time evolution of the expectation values of $\eta_+(t)$ and $\eta_-(t)$ can be expressed as

$$\frac{d}{dt}\langle\eta_+(t)\rangle = -(\lambda_1+\lambda_2)\langle\eta_+(t)\rangle \tag{4.53}$$

and

$$\frac{d}{dt}\langle\eta_-(t)\rangle = -(\lambda_1+\lambda_2)\langle\eta_-(t)\rangle. \tag{4.54}$$

Inspection of Eq. (4.51) and (4.52) indicates that $\eta_+(t)$ and $\eta_-(t)$ are Gaussian variables. Thus in view of Eq. (4.48), we see that $\eta_+(t)$ is also Gaussian variable. In addition, using Eq. (4.46), one can gets

$$\langle\eta_\alpha(t)\rangle = 0. \tag{4.55}$$

Hence on the basis of the relation Eq.(4.49) leads to

$$\langle\alpha^{*2}\alpha^2(t)\rangle = \langle{\eta_\alpha}^2\rangle\langle\eta_\alpha^{*2}\rangle + 2\langle\eta_\alpha\eta_\alpha^*\rangle^2. \tag{4.56}$$

Now with the help of Eq. (4.46), we can write

$$\begin{aligned}\langle{\eta_\alpha}^2\rangle &= \rho_{aa}^{(o)}\rho_{ab}^{(o)}\int_o^t(e^{\lambda_1(2t-t^{'}-t^{''})} - e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})})\langle f_\alpha(t^{'})f_\beta(t^{''})\rangle dt^{'}dt^{''}\\ &+ \rho_{aa}^{(o)}\rho_{ab}^{(o)}\int_o^t(e^{\lambda_1(t-t^{'}))+\lambda_2(t-t^{''})})\langle f_\alpha(t^{'})f_\beta(t^{''})\rangle dt^{'}dt^{''}\\ &+ \rho_{ab}^{(o)}\rho_{aa}^{(o)}\int_o^t(e^{\lambda_1(2t-t^{'}-t^{''})} - e^{\lambda_1(t-t^{'}))+\lambda_2(t-t^{''})})\langle f_\beta(t^{'})f_\alpha(t^{''})\rangle dt^{'}dt^{''}\\ &+ \rho_{ab}^{(o)}\rho_{bb}^{(o)}\int_o^t(e^{\lambda_1(t-t^{'}))+\lambda_2(t-t^{''})}) - e^{\lambda_2(2t-t^{'}-t^{''})}\langle f_\beta(t^{'})f_\alpha(t^{''})\rangle dt^{'}dt^{''}.\end{aligned} \tag{4.57}$$

Using Eq.(3.93), we get

$$\begin{aligned}\langle{\eta_\alpha}^2(t)\rangle &= \frac{[-2\rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M]}{2\lambda_1}(1-e^{2\lambda_1 t})\\ &+ \frac{[2\rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M]}{2\lambda_2}(1-e^{2\lambda_2 t})\\ &+ \frac{[2\rho_{aa}^{(o)}\rho_{ab}^{(o)}\kappa M + 2\rho_{bb}^{(o)}\rho_{ab}^{(o)}\kappa M]}{\lambda_1+\lambda_2}(1-e^{(\lambda_1+\lambda_2)})t,\end{aligned} \tag{4.58}$$

$$\langle {\eta_\alpha}^2(t)\rangle = U + V + Y, \tag{4.59}$$

where

$$U = \frac{-\kappa m - \kappa m\varepsilon}{4\lambda_1}\sqrt{1-\varepsilon^2}e^{i\theta}(1-e^{2\lambda_1 t}),$$

$$V = \frac{\kappa m - \kappa m\varepsilon}{4\lambda_2}\sqrt{1-\varepsilon^2}e^{i\theta}(1-e^{2\lambda_2 t}),$$

$$Y = \frac{\kappa m}{\lambda_1+\lambda_2}\sqrt{1-\varepsilon^2}e^{i\theta}(1-e^{(\lambda_1+\lambda_2)t}).$$

We also note that

$$\langle \eta^{*2}(t)\rangle = \langle \eta^2(t)\rangle. \tag{4.60}$$

Furthermore, applying Eq. (4.46) and its complex conjugate, we find

$$\langle \eta_\alpha\eta_\alpha^*\rangle = z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7, \tag{4.61}$$

in which

$$z_1 = \rho_{aa}^{(o)2}\int_o^t e^{\lambda_1(2t-t^{'}-t^{''})}\langle f_\alpha(t^{'})f_\alpha^*(t^{'})\rangle dt^{'}dt^{''} + \rho_{bb}^{(o)2}\int_o^t e^{\lambda_2(2t-t^{'}-t^{''})}\langle f_\alpha(t^{'})f_\alpha^*(t^{'})\rangle dt^{'}dt^{''}, \tag{4.62}$$

$$z_2 = 2\rho_{aa}^{(o)}\rho_{bb}^{(o)}\int_o^t e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})}\langle f_\alpha(t^{'})f_\alpha^*(t^{'})\rangle dt^{'}dt^{''}, \tag{4.63}$$

$$z_3 = \rho_{ab}^{(o)2}\int_o^t (e^{\lambda_1(2t-t^{'}-t^{''})} + e^{\lambda_2(2t-t^{'}-t^{''})} - 2e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})})\langle f_\beta(t^{'})f_\beta^*(t^{'})\rangle dt^{'}dt^{''}, \tag{4.64}$$

$$z_4 = \rho_{aa}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(2t-t^{'}-t^{''})} - e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})})\langle f_\alpha(t^{'})f_\beta^*(t^{'})\rangle dt^{'}dt^{''}, \tag{4.65}$$

$$z_5 = \rho_{bb}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})} - e^{\lambda_2(2t-t^{'}-t^{''})})\langle f_\alpha(t^{'})f_\beta^*(t^{'})\rangle dt^{'}dt^{''}, \tag{4.66}$$

$$z_6 = \rho_{aa}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(2t-t^{'}-t^{''})} - e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})})\langle f_\beta(t^{'})f_\alpha^*(t^{'})\rangle dt^{'}dt^{''} \tag{4.67}$$

and

$$z_7 = \rho_{bb}^{(o)}\rho_{ab}^{(o)}\int_o^t (e^{\lambda_1(t-t^{'})+\lambda_2(t-t^{''})} - e^{\lambda_2(2t-t^{'}-t^{''})})\langle f_\beta(t^{'})f_\alpha^*(t^{'})\rangle dt^{'}dt^{''}. \tag{4.68}$$

With the aid of Eq. (3.91), we obtain

$$z_1 = \rho_{aa}^{(o)2}(\kappa N + A\rho_{aa})\int_o^t e^{\lambda_1(2t-t^{'}-t^{''})}\delta(tt^{''})dt^{'}dt^{''}$$

$$+\rho_{bb}^{(o)2}(\kappa N + A\rho_{aa})\int_o^t e^{\lambda_2(2t-t^{'}-t^{''})}\delta(tt^{''})dt^{'}dt^{''}. \tag{4.69}$$

By carrying out the integration of Eq. (4.69), we get

$$z_1 = \frac{\rho_{aa}^{(o)2}(\kappa N + A\rho_{aa})}{2\lambda_1}(1 - e^{2\lambda_1 t}) + \frac{\rho_{bb}^{(o)2}(\kappa N + A\rho_{aa})}{2\lambda_2}(1 - e^{2\lambda_2 t}). \tag{4.70}$$

On account of Eqs. (3.101), (3.93),(3.94) and using similar procedure, we obtain

$$z_2 = \frac{2\rho_{aa}^{(o)}\rho_{bb}^{(o)}(\kappa N + A\rho_{aa})}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)t}), \tag{4.71}$$

$$z_3 = \frac{\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb})}{2\lambda_1}(1-e^{2\lambda_1 t})+\frac{\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb})}{2\lambda_2}(1-e^{2\lambda_2 t})-2\frac{2\rho_{ab}^{(o)2}(\kappa N + A\rho_{bb})}{\lambda_1 + \lambda_2}(1-e^{(\lambda_1+\lambda_2)t}), \tag{4.72}$$

$$z_4 = \frac{A\rho_{aa}^{(o)}\rho_{ab}^{(o)2}}{2\lambda_1}(1 - e^{2\lambda_1 t} - \frac{A\rho_{aa}^{(o)}\rho_{ab}^{(o)2}}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)t}), \tag{4.73}$$

$$z_5 = \frac{A\rho_{bb}^{(o)}\rho_{ab}^{(o)2}}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)t} - \frac{A\rho_{bb}^{(o)}\rho_{ab}^{(o)2}}{2\lambda_2}(1 - e^{2\lambda_2 t}), \tag{4.74}$$

$$z_6 = \frac{A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)}}{2\lambda_1}(1 - e^{2\lambda_1 t}) - \frac{A\rho_{aa}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)}}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)t}), \quad (4.75)$$

$$z_7 = \frac{A\rho_{bb}^{(o)}\rho_{ba}^{(o)}\rho_{ab}^{(o)}}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)t} - \frac{A\rho_{bb}^{(o)}\rho_{ab}^{(o)}\rho_{ba}^{(o)}}{2\lambda_2}(1 - e^{2\lambda_2 t}). \quad (4.76)$$

Hence substitution of Eqs.(4.70) - (4.76) into Eq.(4.61) results in

$$\langle \eta_\alpha \eta_\alpha^* \rangle = (S + T), \quad (4.77)$$

in which

$$S = \frac{\kappa N + A(A + \kappa N)\varepsilon}{2\lambda_1}(1 - e^{2\lambda_1 t}) \quad (4.78)$$

and

$$T = \frac{\kappa N - \kappa N\varepsilon}{2\lambda_2}(1 - e^{2\lambda_2 t}). \quad (4.79)$$

On account of Eq. (4.59) and (4.77), Eq.(4.56) takes the form

$$\langle \alpha^{*2}(t)\alpha^2(t) \rangle = U^2 + V^2 + Y^2 + 2UV + 2UY + 2VY + (S + T)^2. \quad (4.80)$$

On similar procedure

$$\langle \beta^{*2}(t)\beta^2(t) \rangle = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + (c_1 + c_2)^2, \quad (4.81)$$

in which

$$c_1 = \frac{(\kappa N + A - A\varepsilon - \kappa N\varepsilon)}{2\lambda_1}(1 - e^{2\lambda_1 t}), \quad (4.82)$$

$$c_2 = \frac{(\kappa N + \kappa N\varepsilon)}{2\lambda_2}(1 - e^{2\lambda_2 t}), \tag{4.83}$$

$$x_1 = \frac{(\kappa M - \kappa M\varepsilon)}{4\lambda_1}\sqrt{1-\varepsilon^2}e^{i\theta}(1 - e^{2\lambda_1 t}), \tag{4.84}$$

$$x_2 = \frac{(\kappa M - \kappa M\varepsilon)}{4\lambda_2}\sqrt{1-\varepsilon^2}e^{i\theta}(1 - e^{2\lambda_2 t}), \tag{4.85}$$

$$x_3 = (\frac{-KM\varepsilon}{\lambda_1+\lambda_2})\sqrt{1-\varepsilon^2}e^{i\theta}(1 - e^{(\lambda_1+\lambda_2)t}). \tag{4.86}$$

On the other hand, employing Eq.(4.44) and (4.45) along with the assumption that the cavity radiation is initially in a two-mode vacuum state, we can write

$$\langle\alpha^*\alpha\beta^*\beta\rangle = \langle\eta_\alpha\eta_\alpha^*\eta_\beta\eta_\beta^*\rangle. \tag{4.87}$$

We recall that $\eta_\alpha(t)$ and $\eta_\beta$ are Gaussian variables with vanishing means. Thus on the basis of the relation (4.49), we get

$$\langle\alpha^*\alpha\beta^*\beta\rangle = \langle\eta_\alpha\eta_\alpha^*\rangle\langle\eta_\beta\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta\rangle\langle\eta_\alpha^*\eta_\beta^*\rangle + \langle\eta_\alpha\eta_\beta^*\rangle\langle\eta_\alpha^*\eta_\beta\rangle. \tag{4.88}$$

Now with the help of Eqs. (4.46) and (4.47), one can write

$$\begin{aligned}\langle\eta_\alpha\eta_\beta\rangle = &\frac{[-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M]}{2\lambda_1}(1 - e^{2\lambda_1 t})\\ &+ \frac{[-\rho_{aa}^{(o)}\rho_{bb}^{(o)}\kappa M - \rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M]}{2\lambda_2}(1 - e^{2\lambda_2 t})\\ &+ \frac{[-\rho_{aa}^{(o)2}\kappa M - \rho_{bb}^{(o)2}\kappa M + 2\rho_{ab}^{(o)}\rho_{ba}^{(o)}\kappa M]}{\lambda_1+\lambda_2}(1 - e^{(\lambda_1+\lambda_2)}).\end{aligned} \tag{4.89}$$

With the help of (3.105), (3.106),(3.107),(3.108) and (4.29) equation above gives,

$$\langle\eta_\alpha\eta_\beta\rangle = \frac{(\kappa M\varepsilon^2 - \kappa M)}{4\lambda_1}(1 - e^{2\lambda_1 t})$$

$$+\frac{(\kappa M\varepsilon^2 - \kappa M)}{4\lambda_2}(1 - e^{2\lambda_2 t}) - \frac{(\kappa M\varepsilon^2)}{\lambda_1 + \lambda_2}(1 - e^{(\lambda_1+\lambda_2)}t). \tag{4.90}$$

On the similar procedure

$$\langle\eta_\alpha\eta_\beta^*\rangle = \frac{(\kappa N + A)}{4\lambda_1}\sqrt{1-\varepsilon^2}e^{i\theta}(1 - e^{2\lambda_1 t}) + \frac{(-\kappa N)}{4\lambda_2}\sqrt{1-\varepsilon^2}e^{i\theta}(1 - e^{2\lambda_2 t}), \tag{4.91}$$

$$\langle\eta_\alpha\eta_\alpha^*\rangle = \frac{(\kappa N + A + A\varepsilon + \kappa N\varepsilon)}{4\lambda_1}(1 - e^{2\lambda_1 t}) + \frac{(\kappa N - \kappa N\varepsilon)}{4\lambda_2}(1 - e^{2\lambda_2 t}), \tag{4.92}$$

$$\langle\eta_\beta\eta_\beta^*\rangle = \frac{(\kappa N + A - A\varepsilon - \kappa N\varepsilon)}{4\lambda_1}(1 - e^{2\lambda_1 t}) + \frac{(\kappa N + KN\varepsilon)}{4\lambda_2}(1 - e^{2\lambda_2 t}). \tag{4.93}$$

We also note that

$$\langle\eta_\alpha\eta_\beta^*\rangle = \langle\eta_\alpha^*\eta_\beta\rangle, \tag{4.94}$$

$$\langle\eta_\alpha\eta_\beta\rangle = \langle\eta_\alpha^*\eta_\beta^*\rangle. \tag{4.95}$$

In order to have a mathematical manageable let we use Eqs. (4.90) - (4.95) at steady state,we get

$$\langle\alpha^*\alpha\beta^*\beta\rangle = [\frac{-2\kappa^2 N - \kappa A + \kappa NA - \kappa A\varepsilon - \kappa NA\varepsilon}{\kappa^2 - \kappa A}][\frac{-2\kappa^2 N - \kappa A + \kappa NA - \kappa A\varepsilon + \kappa NA + \kappa NA\varepsilon}{\kappa^2 - \kappa A}]$$

$$+[\frac{4\kappa^2 MA - 4\kappa^3 M - \kappa MA^2 + (-2\kappa^2 MA + 2\kappa^3 M + \kappa MA^2)\varepsilon^2}{(\kappa A - \kappa^2)(A - 2\kappa)}]^2$$

$$+(\frac{\kappa^2 N^2 A + 2\kappa^2 N^2 A + \kappa^2 A^2}{\kappa^2 - \kappa A})(1 - \epsilon^2)e^{2i\theta}. \tag{4.96}$$

From Eq. (4.56) and (4.81), we obtain

$$\begin{aligned}\langle\alpha^{*2}\alpha^2\rangle &= [\frac{2\kappa^3M-2\kappa^2MA-4\kappa^3M\varepsilon+4\kappa^2MA\varepsilon+\kappa MA^2-\kappa MA^2\varepsilon-2\kappa^3MA}{(\kappa A-\kappa^2)(A-2\kappa)}]^2\\ &+[\frac{-4\kappa^2N-2\kappa A-2\kappa A\varepsilon+2\kappa NA-2\kappa NA\varepsilon}{\kappa^2-\kappa A}]^2,\end{aligned} \tag{4.97}$$

$$\begin{aligned}\langle\beta^{*2}\beta^2\rangle &= [\frac{4\kappa^3M-6\kappa^3M-4\kappa^2MA+6\kappa^2MA\varepsilon+\kappa MA^2-\kappa MA^2\varepsilon}{(\kappa A-\kappa^2)(A-2\kappa)}]^2\\ &+[\frac{-4\kappa^2N-2\kappa A+2\kappa A\varepsilon+2\kappa NA+2\kappa NA\varepsilon}{\kappa^2-\kappa A}]^2.\end{aligned} \tag{4.98}$$

By substitution of Eq.(4.96), (4.97) and (4.98) in Eq.(4.43), we get

$$\begin{aligned}\Delta n^2 = \langle\bar{n}\rangle_{ss} &- (\frac{2NA-4\kappa N-2A}{A-\kappa})^2\\ &+[\frac{2\kappa^3M-2\kappa^2MA-4\kappa^3M\varepsilon+4\kappa^2MA\varepsilon+\kappa MA^2-\kappa MA^2\varepsilon-2\kappa^3MA}{(\kappa A-\kappa^2)(A-2\kappa)}]^2\\ &+[\frac{-4\kappa^2N-2\kappa A-2\kappa A\varepsilon+2\kappa NA-2\kappa NA\varepsilon}{\kappa^2-\kappa A}]^2+[\frac{-4\kappa^2N-2\kappa A+2\kappa A\varepsilon+2\kappa NA+2\kappa NA\varepsilon}{\kappa^2-\kappa A}]^2\\ &+[\frac{4\kappa^3M-6\kappa^3M\varepsilon-4\kappa^2MA+6\kappa^2MA\varepsilon+\kappa MA^2-\kappa MA^2\varepsilon}{(\kappa A\kappa^2)(A-2\kappa)}]^2\\ &+[\frac{4\kappa^2MA-4\kappa^3M-\kappa MA^2+(-2\kappa^2MA+2\kappa^3M+\kappa MA^2)\varepsilon^2}{(\kappa A-\kappa^2)(A-2\kappa)}]^2\\ &+(\frac{-2\kappa^2N-\kappa A+\kappa NA-\kappa A\varepsilon-\kappa NA\varepsilon}{\kappa^2-\kappa A})(\frac{-2\kappa^2N-\kappa A+\kappa A\varepsilon+\kappa NA+\kappa NA\varepsilon}{\kappa^2-\kappa A})\\ &+(\frac{\kappa^2N^2A+2\kappa^2N^2A+\kappa^2A^2}{\kappa^2-\kappa A})(1-\varepsilon^2)e^{2i\theta}.\end{aligned} \tag{4.99}$$

We get the variance of a photon number by using Eq.(4.99),

$$\begin{aligned}
\Delta n^2 = &\frac{16N^2\kappa A - 4^2A^2 + 6NA^2 - 16N^2\kappa^2 - 14N\kappa A + 4N\kappa^2 + \kappa A - 2A^2}{2\kappa^3 - 3\kappa^2 A + \kappa A^2} \\
&+ [\frac{2\kappa^3 M - 2\kappa^2 MA - 4\kappa^3 M\varepsilon + 4\kappa^2 MA\varepsilon + \kappa MA^2 - \kappa MA^2\varepsilon - 2\kappa^3 MA}{(\kappa A - \kappa^2)(A - 2\kappa)}]^2 \\
&+ [\frac{-4\kappa^2 N - 2\kappa A - 2\kappa A\varepsilon + 2\kappa NA - 2\kappa NA\varepsilon}{\kappa^2 - \kappa A}]^2 + [\frac{-4\kappa^2 N - 2\kappa A + 2\kappa A\varepsilon + 2\kappa NA + 2\kappa NA\varepsilon}{\kappa^2 - \kappa A}]^2 \\
&+ [\frac{4\kappa^3 M - 6\kappa^3 M\varepsilon - 4\kappa^2 MA + 6^2 MA\varepsilon + KMA^2 - \kappa MA^2\varepsilon}{(\kappa A - \kappa^2)(A - 2\kappa)}]^2 \\
&+ [\frac{4\kappa^2 MA - 4\kappa^3 M - \kappa MA^2 + (-2\kappa^2 MA + 2\kappa^3 M + \kappa MA^2)\varepsilon^2}{(\kappa A - \kappa^2)(A - 2\kappa)}]^2 \\
&+ (\frac{-2\kappa^2 N - \kappa A + \kappa NA - \kappa A\varepsilon - \kappa NA\varepsilon}{\kappa^2 - \kappa A})(\frac{-2\kappa^2 N - \kappa A + \kappa A\varepsilon + \kappa NA + \kappa NA\varepsilon}{\kappa^2 - \kappa A}) \\
&+ (\frac{\kappa^2 N^2 A + 2\kappa^2 N^2 A + \kappa^2 A^2}{\kappa^2 - \kappa A})(1 - \varepsilon^2)e^{2i\theta}. \qquad (4.100)
\end{aligned}$$

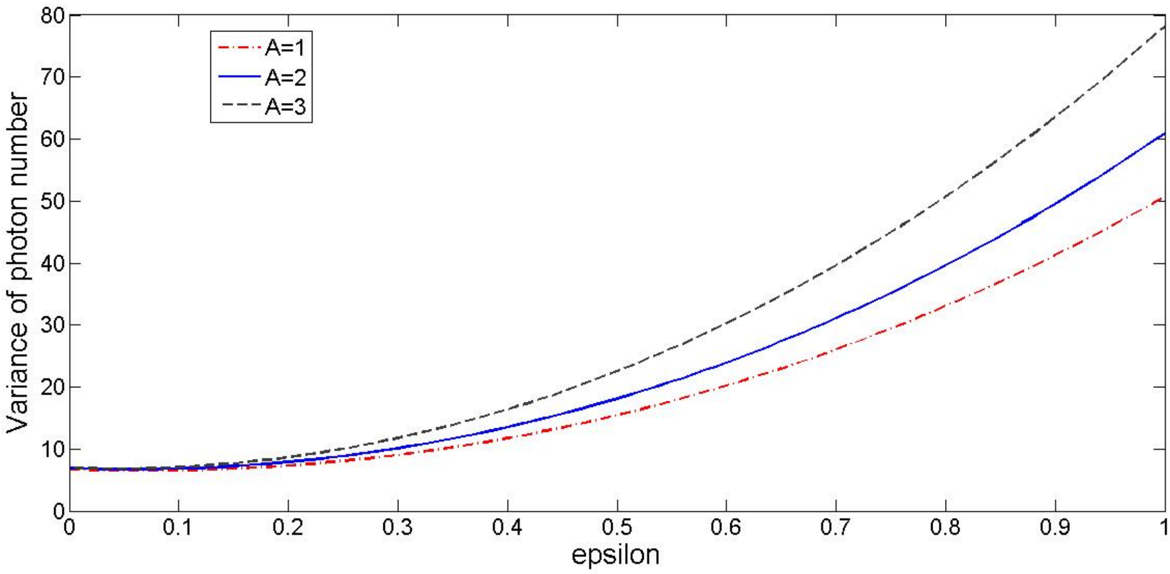


*Fig. 4.4:* Plots of the variance of photon number of the cavity a two-mode at steady state[Eq.(4.100)] versus $\varepsilon$ for $\kappa = 0.01, r = 1, \theta = 0$, and for different values of the linear gain coefficient,for A = 1(dash-dot curve), A = 2(solid curve) and A = 3(dashed curve).

we see from the figure that the variance of photon number increases with $\epsilon$ and linear gain coefficient.

## 4.3 Quantum Entanglement of the Cavity Modes

In this section, we wish to investigate the entanglement properties of two mode inside and outside the cavity coupled to squeezed vacuum reservoir. It is known that a state of a system $\hat{\rho}$ of a two modes $\hat{a}$ and $\hat{b}$ is said to be entangled or not separable if it is not possible to express in the form

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i{}^{(\hat{a})} \hat{\rho}_i{}^{(\hat{b})}, \tag{4.101}$$

where $\hat{\rho}_i{}^{(\hat{a})}$ and $\hat{\rho}_i{}^{(\hat{b})}$ are assumed to be the normalized density operators of modes $\hat{a}$ and $\hat{b}$ respectively, with $p_i \geq 0$ and $\sum_i p_i = 1$. A maximally entangled continuous variable state can be expressed as a co-eigen state of a pair of Einstein-Podolsky-Rosen (EPR)-type operators[ 27] such as $\hat{x}_a - \hat{x}_b$ and $\hat{p}_a + \hat{p}_b$. Thus the total variance of this operators reduced to zero for the maximally entangled continuous variable states. In addition, according to the criterion set by Duan et al.[33] quantum states of the system claimed to be entangled.

In order to verify the entanglement of the cavity modes, we apply the criteria presented in [30]. According to this section, the quantum state of a system is said to be entangled if the sum of the quantum fluctuations of the two Einstain- Podolsky- Rosen(EPR)-like operator $\hat{\upsilon}$ and $\hat{U}$ of the two modes satisfies the inequality

$$\Delta\upsilon^2 + \Delta U^2 < 2, \tag{4.102}$$

where

$$\hat{U} = \hat{x}_a - \hat{x}_b, \tag{4.103}$$

$$\hat{\upsilon} = \hat{p}_a + \hat{p}_b, \tag{4.104}$$

with

$\hat{x}_a = \frac{1}{\sqrt{2}}(a + a^{\dagger})$, $\hat{x}_b = \frac{1}{\sqrt{2}}(b + b^{\dagger})$,$\hat{p}_a = \frac{i}{\sqrt{2}}(a^{\dagger} - a)$ and $\hat{p}_b = \frac{i}{\sqrt{2}}(b^{\dagger} - b)$.

are the quadrature for the two modes of the cavity. But

$$\begin{aligned}\Delta U^2 &= \langle \hat{U}^2 \rangle - \langle \hat{U} \rangle^2 \\ &= \langle (\hat{x}_a - \hat{x}_b)^2 \rangle - \langle \hat{x}_a - \hat{x}_b \rangle^2 \\ &= \langle {x_a}^2 \rangle + \langle {x_b}^2 \rangle - (\langle x_a x_b \rangle + \langle x_b x_a \rangle) - [\langle x_a \rangle^2 + \langle x_b \rangle^2 - (\langle x_a x_b \rangle + \langle x_b x_a \rangle)],\end{aligned} \tag{4.105}$$

$$\begin{aligned}\Delta U^2 = 1 + \frac{1}{2}[&\langle a^2 \rangle + \langle a^{\dagger 2} \rangle + \langle b^2 \rangle + \langle b^{\dagger 2} \rangle + 2\langle a^{\dagger} a \rangle + 2\langle b^{\dagger} b \rangle \\ &- (\langle ab \rangle + \langle ba \rangle + \langle ab^{\dagger} \rangle + \langle b^{\dagger} a \rangle + \langle a^{\dagger} b \rangle + \langle ba^{\dagger} \rangle + \langle a^{\dagger} b^{\dagger} \rangle + \langle b^{\dagger} a^{\dagger} \rangle)].\end{aligned} \tag{4.106}$$

Following the same steps, one can readily find

$$\begin{aligned}\Delta \upsilon^2 &= \langle \hat{\upsilon}^2 \rangle - \langle \upsilon \rangle^2 \\ &= \langle (\hat{p}_a + \hat{p}_b)^2 \rangle - \langle \hat{p}_a + \hat{p}_b \rangle^2 \\ &= \langle p_a^2 \rangle + \langle p_b^2 \rangle + \langle p_a p_b \rangle + \langle p_b p_a \rangle - [\langle p_a \rangle^2 + \langle p_b \rangle^2 + \langle p_a \rangle \langle p_b \rangle + \langle p_b \rangle \langle p_a \rangle],\end{aligned} \tag{4.107}$$

$$\begin{aligned}\Delta \upsilon^2 = 1 - \frac{1}{2}[&\langle a^{\dagger 2} \rangle + \langle a^2 \rangle + \langle b^{\dagger 2} \rangle + \langle b^2 \rangle - 2\langle a^{\dagger} a \rangle - 2\langle b^{\dagger} b \rangle \\ &- \langle a^{\dagger} b^{\dagger} \rangle - \langle a^{\dagger} b \rangle - \langle ab^{\dagger} \rangle + \langle ab \rangle + \langle b^{\dagger} a^{\dagger} \rangle - \langle b^{\dagger} a \rangle - \langle ba^{\dagger} \rangle + \langle ba \rangle].\end{aligned} \tag{4.108}$$

Upon adding Eqs.(4.107) and (4.108), we get

$$\Delta U^2 + \Delta \upsilon^2 = 2 + 2\langle a^{\dagger} a \rangle + 2\langle b^{\dagger} b \rangle - \langle ab \rangle - \langle ba \rangle - \langle b^{\dagger} a^{\dagger} \rangle. \tag{4.109}$$

The c- number equations corresponding to Eq. (4.109), can be written as

$$\Delta U^2 + \Delta \upsilon^2 = 2 + 2\langle \alpha^* \alpha \rangle + 2\langle \beta^* \beta \rangle - \langle \alpha \beta \rangle - \langle \beta \alpha \rangle - \langle \beta^* \alpha^* \rangle. \tag{4.110}$$

On account of Eqs. (4.17), (4.18),(4.19), (4.20),(4.21) and its complex conjugates, Eq.(4.110) gives,

$$\Delta U^2 + \Delta \upsilon^2 = \frac{2\kappa + 4\kappa N - 2NA}{\kappa - A} - [\frac{4\kappa^2 M - 4\kappa MA + MA^2 + (8\kappa^2 M - MA^2)\epsilon^2}{4\kappa^2 - 6\kappa A + 2\kappa A^2}]. \tag{4.111}$$

This is the variance of the sum of EPR-operators for two-mode v laser coupled to squeezed vacuum reservoir.

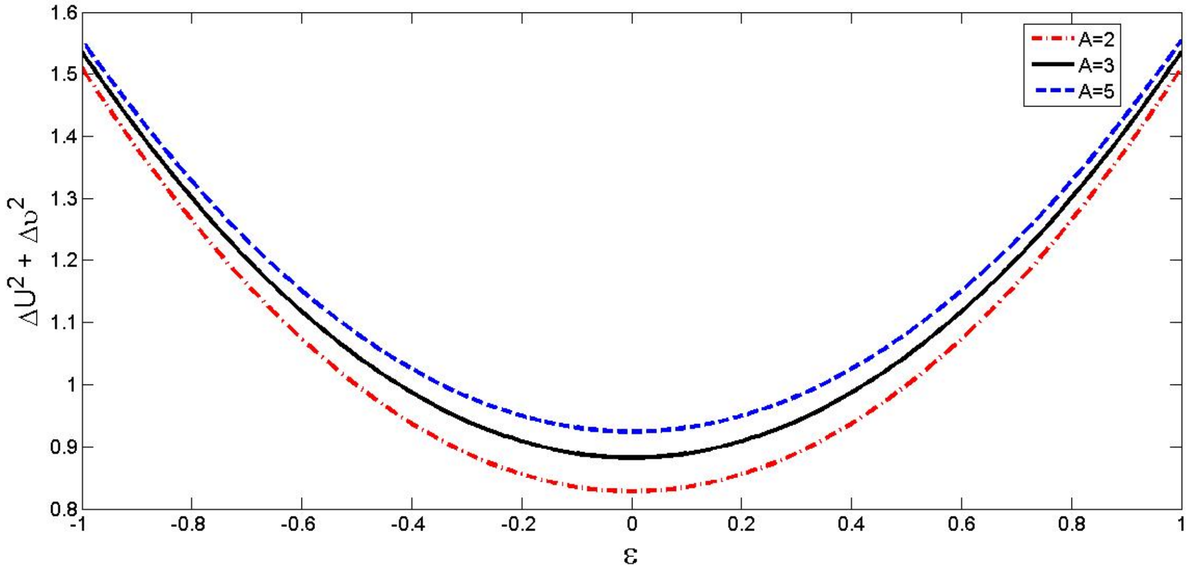


*Fig. 4.5:* Plots of variance of the sum of EPR-operators [Eq.(4.111)] versus epsilon for $\theta = 0, \kappa = 0.1, r = 0.8$,, and different values of the linear gain coefficient,for A = 2(dash-dot curve), A = 3(solid curve) and A = 5(dotted curve).

Fig(4.5) shows that the variance of the sum of the EPR-operators is increasing with linear gain coefficient.

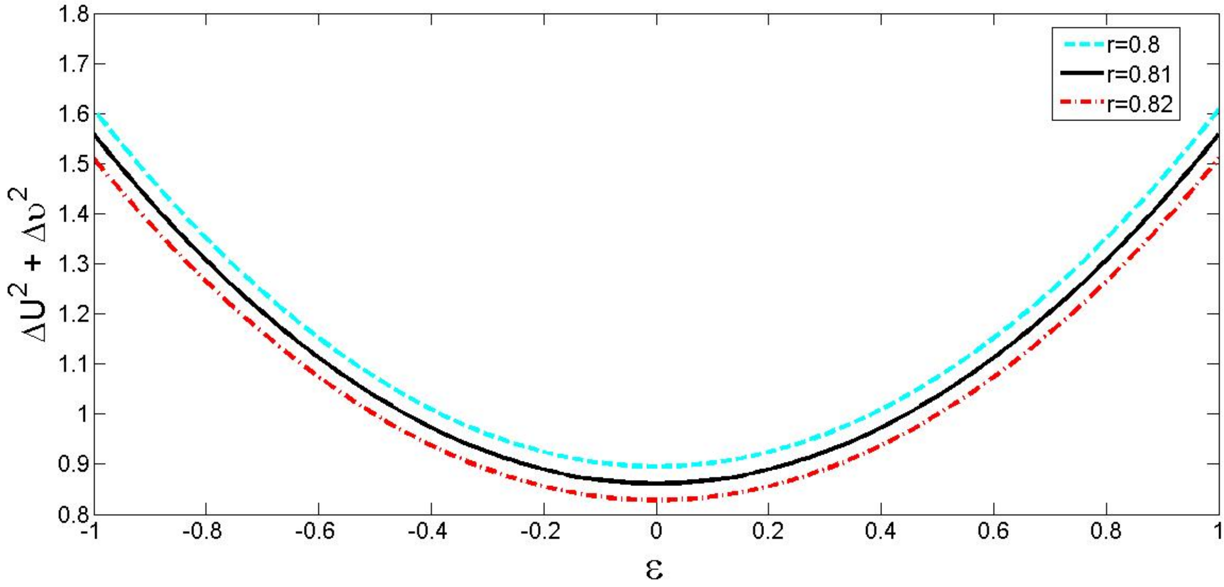


*Fig. 4.6:* Plots of variance of the sum of EPR-operators [Eq.(4.111)] versus epsilon for $\theta = 0, \kappa = 0.1, A = 2$, and different values of squeeze parameter,for r = 0.8(dashed curve), r = 0.81(solid curve) and r = 0.82(dotted curve).

As we see from the figure the entanglement of the cavity increases with squeeze parameter(r).

consider the case in which the cavity mode is coupled to vacuum reservoir, upon setting $r = 0$ in Eq.(4.111) we get,

$$\Delta U^2 + \Delta v^2 = \frac{2\kappa}{\kappa - A}. \tag{4.112}$$

Eq.(4.112), represents the variance of the sum of the EPR-operators of the cavity coupled to vacuum reservoir.

# 5

# Conclusions

In this thesis, we have studied quantum dynamics of light produced by two-mode v laser coupled to squeezed vacuum reservoir. We first obtained the master equation for the system under consideration. Using this master equation, we obtained c-number Langevin equation associated with the normal ordering and the correlation properties of the noise forces. Applying this equations, we have calculated the quadrature variance, the mean photon number, the variance of photon number and the entanglement of the cavity modes. Our results showed that the light produced by the system under consideration is in squeezed state and the degree of squeezing increases with squeeze parameter(r) and decreases with linear gain coefficient. Furthermore, we have also obtained that the mean and variance of photon number increases with linear gain coefficient. On the other hand, we have calculated the variance of the sum of EPR-operators with the aid of the solutions of c-number Langevin equation and Duan et al. criteria. The results showed that the entanglement of the cavity mode increases with squeeze parameter(r) and decreases with linear gain coefficient.

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