Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method.

Chali Bulbulu Debele

A Thesis Submitted to The Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied mathematics (Differential Equations)

Office of Graduate Studies

Adama Science and Technology University

June, 2022

Adama, Ethiopia

Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method.

Chali Bulbulu Debele

Advisor : Feyissa Kebede Bushu (Ph.D.)

Co-Advisor : Tamirat Temesgen Dufera (Ph.D.)

A Thesis Submitted to the Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied mathematics (Differential Equations)

Office of Graduate Studies

Adama Science and Technology University

June, 2022

Adama, Ethiopia

Declaration

I hereby declare that this Master Thesis entitled ***"Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method"*** is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

______________________	________________	______________
Name of Student	Signature	Date

Recommendation of Advisors

We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method"*** prepared under our guidance by **Chali Bulbulu Debele**. submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics. Therefore, we recommend that the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled ***"Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method"*** and developed by **Chali Bulbulu Debele**, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by **Chali Bulbulu Debele**, have read and evaluated the thesis entitled ***"Solving Korteweg-De Vries and Forced Vander Pol-duffing Oscillator Equations Using Natural Decomposition Method"*** and examined the candidate during the open defense. This is, therefore, to certify that the thesis is accepted for partial fulfilment of the requirement of the degree of Master of Science in Applied Mathematics.

Chairperson	Signature	Date
Internal Examiner	Signature	Date
External Examiner	Signature	Date

Finally, approval and acceptance of the thesis are contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate committee (DGC) and School Graduate Committee (SGC).

Department Head	Signature	Date
School Dean	Signature	Date
Office of Postgraduate Studies, Dean	Signature	Date

Acknowledgment

First of all, I would like to thank the almighty God for caring my life, blessing all my activities and for giving me strength and perseverance to complete this study. Secondly, My special and heart full gratitude goes to my advisor Dr.Feyissa Kebede, who painstakingly guided and directed the study at the time he could barely make any time for anything beside his work. I must admit that without him this step was difficult to reach. Your guidance durring the program was excellence and it is my hope that you will continue to lend your vast experience and knowledge to me even after this program. God will pay you back and I will always pray that he bless him abundantly as he continue helping other students. I would also like to thank my Co-advisor Dr.Tamirat Temesgen, for his valuable guidance throughout my studies and all my classmates for their cooperative work through our study time and doing the thesis. Then I would like to extend my thanks to my families and all who encouraged me to complete my project and their heart full help while writing the thesis.Lastly, I would like thanks to Department of Mathematics, Adama Science and Technology University for giving the necessary materials throughout the preparation of this thesis.

Contents

List of Tables

List of Figures

List of Abbreviations

PDEs	Partial differential equations.
ODEs	Ordinary differential equations.
DEs	differential equations.
NDM	Natural Decomposition Method
NLDE	Nonlinear differential equation
KDV	Korteweg-De Vries
RKM	Runge Kutta method
N-Transform	Natural transform
FDM	Finite difference method

Abstract

The Natural decomposition method is combination of Natural transform and Adomian polynomial.This method was used to obtain the approximate analytical solutions of nonlinear differential equations. In particular, third order KdV equation and Forced Van der Pol-Duffing oscillator equation are considered. Adomian polynomial was introduce as essential tool to linearize the associated nonlinear terms in the equations since Natural transform cannot handle nonlinear terms. All the problems considered show that the Natural decomposition method is very powerful integral transform methods in solving nonlinear equations like KDV equations. In addition we obtain the numerical solution of KDV equation using finite difference method and for Forced Van der Pol-Duffing oscillator equation we use RK4 method. Then we compare the numerical solution with the approximated analytic solution of this DEs. The finite difference and RK4 method of these DEs implemented in Matlab.

Key Words: Natural transform, Natural decomposition method, Adomian polynomial, ordinary differential equations, partial differential equations.

CHAPTER 1

Introduction

1.1 Background of Study

Differential equations play a vital role in various fields of science and engineering. In most of the cases, an exact solution of nonlinear differential equations may not be obtained easily. Therefore, various types of numerical techniques have been developed by researchers to solve such nonlinear differential equations.

Investigation of approximate solutions of nonlinear differential equations plays an important role in studying nonlinear physical phenomena. Some transformations like Laplace transform, Sumudu transform, Natural transform and Alzaki transforms are unable to solve some nonlinear differential equations. Some researchers are working on combining these transformations with the Adomian polynomial in order to solve some nonlinear differential equations (Guckenheimer & Holmes, 2013).

A new integral transform method is called the Natural Decomposition Method and apply it to find analytic approximate solutions for nonlinear differential equations. The Natural decomposition method was first introduced by (Guckenheimer & Holmes, 2013), to solve linear and nonlinear differential equations that appear in many areas of science. Natural Decomposition Method is an excellent mathematical tool used for approximate solutions of linear and nonlinear differential equations. The proposed technique lead to approximate solution in the form of a rapidly convergence series and is applied directly without any unnecessary linearization, discretization, transformation or taking some restrictive assumptions.

The decomposition of a series is obtained by non-linear term which is called Adomian polynomials. By using Adomian polynomials we find a solution in the form of a series which can be determined by the recursive relationship.

There are many nonlinear differential equations with strong nonlinearity which is very difficult to solve either analytically or numerically in science and engineering. In this thesis, the forced Van der Pol-Duffing oscillator equation and KDV equation are solved by a method called the Natural Decomposition Method.

The Van der Pol-Duffing oscillator equation is a classical nonlinear oscillator, which is a very useful mathematical model for understanding different engineering problems (Guckenheimer & Holmes, 2013). The Van der Pol oscillator equation was proposed by a Dutch scientist Balthazar Van der Pol (Tsatsos, 2006), which describes triode oscillations in electrical circuits. It is a classical example of a self-oscillatory system and is now considered as a very important model to describe a variety of physical systems. The nonlinear Van der Pol-Duffing oscillator equations are difficult to solve analytically. It is governed by a second-order nonlinear ordinary differential equation.

$$\frac{d^2x}{dt^2} - \lambda(1 - x^2)\frac{dx}{dt} + \alpha x + \beta x^3 = F\cos(\omega)t,$$

with initial conditions $x(0) = a$, $x^{'}(0) = b$.

The Van der Pol equation has been studied extensively by many authors using various methods. Kumar and Varshney (2021) presented numerical simulation of Van der Pol equation using multiple scales modified lindstedt-poincare method. Kharrat and Toma (2020) used collocation method for numerical solution of Van der pol equation, (Soomro et al., 2013) compared between improved Heun's method against the classical Runge-Kutta and Mid-point methods for Van der Pol equation. Rasedee et al. (2018) used the backward difference formulation to solve Duffing- Van der Pol equation.

The well known Korteweg-De-Vries equation had been derived in 1885 by the two scientists Korteweg and De Vries to describe long wave propagation on shallow water. The dynamics of solitary waves is modeled by the KDV equation. The General Korteweg De Vries equation is of the form:

$$u_t(x,t) + \gamma u(x,t)^p u_x(x,t) + \mu u_{xxx} = 0,$$

where $p = 1, 2, 3, ...$ is a positive integer and γ, μ are positive parameters.
The Korteweg De Vries equation developed by blow is similar to the GKDV with $p = 1$

and is of the form:

$$u_t + \gamma u u_x + \mu u_{xxx} = 0,$$

u describes the elongation of the wave at place and at time. The non-linear term introduces the possibility of shock waves into the solution. The μu_{xxx} term produces dispersive broadening that can exactly compensate the narrowing caused by the non-linear term under proper conditions (Landau et al., 2007).

1.2 Statement of the problem

In this work, we consider two initial value problems.

$$\begin{cases} \dfrac{d^2x}{dt^2} - \lambda(1-x^2)\dfrac{dx}{dt} + \alpha x + \beta x^3 = F\cos(\omega)t \\ x(0) = a \\ x^{'}(0) = b, \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t + \gamma u u_x + u_{xxx} = 0 \\ u(x,0) = f_1(x). \end{cases} \tag{1.2}$$

The aim of this thesis, is to apply Natural decomposition method to find approximate analytic solution of above two problems.
This study is intended to answer the following basic questions

- How to use NDM to solve nonlinear differential equation?
- Is this NDM method is efficient to solve nonlinear DE?

1.3 Objectives of the study

1.3.1 General objective

General objective of this study is to solve a nonlinear DEs using Natural decomposition method.

1.3.2 Specific objectives

The specific objectives of this study are to:

- apply Natural decomposition method to solve nonlinear differential equation
- approximate analytical solution of Forced vander pol-duffing oscillator and Korteweg-De Vries equations
- compare the result of Natural decomposition method with existing solution.

1.4 Significance of the study

The result of this study will help

- scientific community for further investigate similar problems.
- some individuals to understand how this method work.
- the researcher as a reference for those who needs to conduct research in this area.

CHAPTER 2

Literature Review

To start with, we know that nonlinear phenomena play an important role in the field of applied mathematics and engineering. Several mathematical methods have been applied by various researchers in various fields of science and engineering to obtain the analytical solutions of nonlinear differential equations, which appeared in so many research literature. We discuss some of these methods in this chapter.

Different researchers introduced integral transforms and explored their application in solving both ordinary and partial differential equation problems arising from different discipline. Agarwal et al. (2019) used the N–Transform to solve the Maxwell's equation, Bessel's differential and linear and nonlinear Klein Gordon equations and more.

Adomian decomposition method which was developed by George Adomian between 1970s and 1990s. This method is categorized as an iterative method which provides approximate analytical solutions or semi analytical solution which is in the form of an infinite power series for nonlinear equations. The important aspect of this method is that it makes use of Adomian polynomials which give room for the convergence of the solution of nonlinear part of the differential equations and high level of accuracy of numerical solutions was generated by this method. Later, in order to improve the convergence of series solution of nonlinear differential equations this method is modified and used by (Anjum & He, 2020).

Akter and Chowdhury (2019) proposed the homotopy perturbation method. This method can be use to solve the given equations in order to get the solutions which converge to analytical solution. This homotopy analysis employs the homotopy technique from topology in order to produce a convergent series solution for nonlinear differential equations.

Dattu (2018) introduced a new integral transform called the Sumudu transform in early 90s and used this transform to obtain the solution of ordinary differential equations in control engineering problems. The most important thing about this transform is that it has several properties which make it easier to use.

Dattu (2018) developed the new integral transform called the Elzaki transform, this transform was derived from the classical Fourier integral. It was introduced purposely for solving ordinary and partial differential equations in the time domain. Due to the mathematical simplicity of the Elzaki transform and its fundamental properties, Elzaki transform was applied to solve linear system of partial differential equations with constant coefficients (Dosunmu et al., 2022). In addition, (Ige et al., 2019) explained some relationship that exists between the new integral transform called Elzaki transform and Laplace transform. Further solved the first and second order ordinary differential equations with constant and non-constant coefficients by using both Elzaki and Laplace transforms and established the fact that Elzaki transform and Laplace transform are related. Ige et al. (2019) went further to use Elzaki transform to solve ordinary differential equations with variable coefficients without necessarily leading to a new frequency.

Ige et al. (2019) applied Elzaki transform to solve some nonlinear differential equations, since this transform can not handle nonlinear terms, differential transform method was used to decompose the nonlinear term into linear term so that iteration procedure was used to obtained the solution, applied Elzaki transform on this differential equation to obtain the analytic solution. The essence of using this technique is that this solution which is expressed as an infinite series will converges fast to exact solutions.

Ige et al. (2019) considered the Klein-Gordon nonlinear hyperbolic equation with initial conditions and obtained the analytical series solution by using ADM and able to establish the decomposition method for solving the Klein-Gordon equation, likewise proved the convergence of Adomian decomposition method for nonlinear Klein-Gordon equation. Obtained the numerical solutions of the Klein-Gordon equation in the form of convergent power series and concluded that the method used executes to a specified standard in terms of efficiency, exactness(accuracy), simplicity, reliability and stability.

Various types of numerical perturbation techniques such as Euler, Runge-Kutta, homotopy perturbation, linearization, and variational iteration methods have been used to solve the nonlinear equation. A linearization method has been employed by (Guckenheimer & Holmes, 2013) for solving the Duffing and Van der Pol equations. Soomro et al. (2013) ,solved Van derpol, Rayleigh, and Duffing equations using an algebraic method (Tsatsos, 2006).

The Korteweg-de Vries equation equation has been arisen in the study of shallow water waves, (Landau et al., 2007). Particularly, it is utilized to characterize the diffusion of plasma waves in a dispersive medium and furthermore describes long waves traveling in canals too. The KDV equation has been widely studied and has gained alot of interest. Many analytical and numerical techniques were used to study the solitary waves which outcome of such equation. Kruskal and Zabusky have researched in the interaction of repetition of the initial cases and solitary waves. They detected the solitary waves undergo non-linear interaction subsequent KDV equation. It is worth mentioning, waves emanate of this interaction retaining its amplitude and shape, also solitary waves maintain its character and identities resembles such as particle conduct, they stimulated to recall solitary wave solitons.

From the above literature survey it is very relevant to point out that no attempt has been made to study approximate analytic solution of third order KDV equation and Forced Van der Pol-Duffing oscillator equation. Hence this thesis focus approximate analytic solution of third order KDV equation and Forced Van der Pol-Duffing oscillator equation by Natural decomposition method.

CHAPTER 3

Mathematical Preliminaries

In this chapter we gather some general mathematical preliminaries that we uses throughout this thesis. We shall not go into details but cite references in which the interested reader can find further details.

3.1 Differential Equation

Definition 3.1.1 An equation involving one dependent variable and its derivative concerning one or more independent variables is called a differential equation. It may be noted that most of the fundamental laws of nature can be formulated as DE's namely, the growth of a population, conduction of heat in a rod, vibration of structures, etc (Maz'ya et al., 1979).

3.1.1 Ordinary Differential Equation

Definition 3.1.2 The ordinary differential equation (ODE) is a differential equation that contains only one independent variable so that all the derivatives occurring in it are ordinary derivatives. In symbols we can express an *nth* order ordinary differential equation in one dependent variable by the general form:

$$F(x, y, y^{'}, ..., y^{n}) = 0, \tag{3.1}$$

where F is a real valued function of $n+1$ variables $x, y, y^{'}, ..., y^{n}$ (Maz'ya et al., 1979).

3.1.2 Partial Differential Equation

Definition 3.1.3 Let $u = u(x_1, ..., x_n)$ be a function of n independent variables $x_1, ..., x_n$. Then a PDE is an equation that contains the independent variables $x_1, ..., x_n$, the dependent variables or the unknown function u and its partial derivatives up to some order. It has the form:

$$F(x_1, ...x_n, u, u_{x1}, ..., u_{xn}, u_{x1x1}, ..., u_{xixj}, ...) = 0 \tag{3.2}$$

where, $u_{xj} = \dfrac{\partial u}{\partial x_j}$, $u_{xixj} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$
for, $i, j = 1, 2, ..., n$ are partial derivatives of u. (Maz'ya et al., 1979).

3.1.3 Linear and Nonlinear Differential Equations

Definition 3.1.4 Linear differential equations are those in which (i) the dependent variable and its derivatives appear only in the first degree, and (ii) products of derivatives and the dependent variable do not occur.
Definition 3.1.5 Nonlinear differential equations are differential equations that are not linear. That is the products of the unknown function and derivatives are allowed and degree of dependent variable is greater than one.
The following are examples of nonlinear DE's.

- The Van der Pol-Duffing oscillator equation:

$$\frac{d^2x}{dt^2} - \lambda(1 - x^2)\frac{dx}{dt} + \alpha x + \beta x^3 = F\cos(\omega),$$

is the second order nonlinear ordinary differential equation. It was a model for an electrical circuit with a triode valve, and was later extensively studied as a host of a rich class of dynamical behavior.

- Korteweg-De Vries equation:

$$u_t + \gamma u u_x + u_{xxx} = 0,$$

is third order nonlinear partial differential equation. This equation is considered as a mathematical model of waves on shallow water surfaces.

3.2 Initial Value Problem's (IVP's)

Many problems in science and engineering can be modeled by ODE's or PDE's, along with one or more supplementary conditions.
The n^{th}-order IVP with n initial conditions be written as

$$\frac{d^n y}{dx^n} = f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ..., \frac{d^{n-1}y}{dx^{n-1}}) \tag{3.3}$$

with initial conditions

$$y^{(n-1)}(x_0) = y_{n-1}$$

3.3 Taylor series expansion

The main purpose of series is to write a given complicated quantity as an infinite sum of simple terms and since the terms get smaller and smaller, we can approximate the original quantity by taking only the first few terms of the series. In this section, we finally develop the tool that lets us do this in most cases a way to write any reasonable function as an explicit power series. This will allow us to compute outputs of the function by plugging into the series

Our functions must behave decently near the center point of the desired power series. We say $f(x)$ is analytic at $x = a$ if it is possible to write $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some coefficients c_n, with positive radius of convergence.

Definition 3.1.6: (Maz'ya et al., 1979) Let $f(x)$ be a function which is analytic at $x = a$. Then we can write $f(x)$ as the following power series, called the Taylor series of $f(x)$ at $x = a$:

$$f(x) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + ... + \frac{f^{(n)}(a)}{n!}h^n + R_n(x)$$

where $h = x-a$, $n!$ denote the factorial of n and $R_n(x)$ is the remainder term, denoting the difference between the Taylor polynomial of degree n and the original function.
If we write the nth derivative of $f(x)$ as $f^n(x)$, this becomes

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

with coefficients

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Suppose f(x,y) and its partial derivatives through order $n+1$ are continuous throughout an open rectangular region R centered at the point (0,0), then throughout R:

$$f(x,y) = f(0,0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy}) + ... + \frac{1}{n!}(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1} y \frac{\partial^n f}{\partial x^{n-1}\partial y} + ... + y^n \frac{\partial^n f}{\partial y^n}) + \frac{1}{(n+1)!}(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + ... + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}})$$

3.4 The Natural Transform Method

The Natural transform is derived from the Fourier Integral. Landau et al. (2007), established some characteristics of natural transform which is known as N-transform. The natural transform of the function $f(t) \geq 0$ for $t \in (-\infty, \infty)$ be piecewise continuous and exponential order then on the set B the natural transform is defined by:

$$B = \{f(t) : \exists M, \lambda_1, \lambda_2 > 0, f(t) < Me^{\frac{|t|}{\lambda_j}}, t \in (-1)^j \times [0, \infty], j \in Z^+\}.$$

The Natural transform of the function $f(t)$ for $t \in (-\infty, \infty)$ is defined by

$$N[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(tu)dt. \tag{3.4}$$

Where $s, u \in (-\infty, \infty)$, $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables.
Note that equation (3.4) can be written in the form:

$$\begin{aligned} N[f(t)] &= [\int_{-\infty}^{0} e^{-st} f(ut)dt, s, u \in (-\infty, 0)] + [\int_{0}^{\infty} e^{-st} f(ut)dt, s, u \in (0, \infty)] \\ &= N^-[f(t)] + N^+[f(t)] \\ &= N[f(t)H(-t)] + N[f(t)H(t)] \\ &= R^-(s,u) + R^+(s,t) \end{aligned}$$

where $H(.)$ is the Heaviside function. It should be mentioned here, if the function $f(t)H(t)$ is defined on the positive real axis, with $t \in (0, \infty)$, then we define the Natural transform (N–Transform) for the function $f(t)$ as:

$$N^+[f(t)] = \int_{0}^{\infty} e^{-st} f(ut)dt \tag{3.5}$$

$$= R^+(s,u), s, u \in (0, \infty) \tag{3.6}$$

where $R^{+}(s,u)$ is the Natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables and the inverse of natural transform is defined by:

$$f(t) = N^{-1}[R^{+}(s,u)] \tag{3.7}$$

Example:

$$N^{+}[t^n] = \frac{n!u^n}{s^{n+1}}$$

for general form we are going to prove by induction:
Given

$$N^{+}[t^n] = \frac{n!u^n}{s^{n+1}}, \tag{3.8}$$

when $n = 1$, we have:

$$\begin{aligned} N^{+}[t] &= \int_0^\infty e^{-st}(ut)dt \\ &= u\int_0^\infty e^{-st}tdt \end{aligned}$$

By integration by part,

$$\begin{aligned} \int_0^\infty e^{-st}tdt &= \frac{-t}{s}e^{-st}|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\ &= \frac{1}{s}\int_0^\infty e^{-st}dt \\ &= \frac{-1}{s^2}e^{-st}|_0^\infty \\ &= \frac{1}{s^2} \\ N^{+}[t] &= u\int_0^\infty e^{-st}tdt \\ &= \frac{u}{s^2} \end{aligned}$$

when $n = 2$, we have:

$$\begin{aligned} N^{+}[t^2] &= \int_0^\infty e^{-st}(ut)^2dt \\ &= u^2\int_0^\infty e^{-st}t^2dt \end{aligned}$$

By integration by part,

$$\begin{aligned} \int_0^\infty e^{-st}t^2dt &= \frac{-t^2}{s}e^{-st}|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}tdt \\ &= \frac{2}{s}\int_0^\infty e^{-st}tdt \end{aligned}$$

From the above we know that $\int_0^\infty e^{-st}tdt = \dfrac{1}{s^2}$

$$\begin{aligned}\int_0^\infty e^{-st}t^2dt &= \frac{2}{s}[\frac{1}{s^2}] \\ &= \frac{2}{s^3} \\ N^+[t^2] &= u^2\int_0^\infty e^{-st}t^2dt \\ &= \frac{2u^2}{s^3}\end{aligned}$$

Therefore we conclude that

$$N^+[t^n] = \frac{n!u^n}{s^{n+1}}$$

3.4.1 Properties of N–Transform

(i) Linearity property

If a and b are any constants and $f(t)$ and $g(t)$ are functions, then

$$N^+[af(t) + bg(t)] = aN^+[f(t)] + bN^+[g(t)]$$

Proof.

$$N^+[f(t)] = \int_0^\infty e^{-st}f(ut)dt$$

and

$$N^+[g(t)] = \int_0^\infty e^{-st}g(ut)dt$$

Then if a and b are any constants.

$$\begin{aligned}N^+[af(t) + bg(t)] &= \int_0^\infty e^{-st}[af(ut) + bg(ut)]dt \\ &= a\int_0^\infty e^{-st}f(ut)dt + b\int_0^\infty e^{-st}g(ut)dt \\ &= aN^+[f(t)] + bN^+[g(t)]\end{aligned}$$

(ii) Change of scale property

If $N^+[f(t)] = R^+(s, u)$, then $N^+[f(at)] = \dfrac{1}{a}R^+(s, u)$

Proof.

$$N^{+}[f(at)] = \int_0^\infty e^{-st} f(aut)dt$$

let $p = at$ then $dp = adt \Rightarrow dt = \dfrac{dp}{a}$

$$\begin{aligned}\int_0^\infty e^{\frac{-sp}{a}} f(up)\frac{dp}{a} &= \frac{1}{a}\int_0^\infty e^{\frac{-sp}{a}} f(up)dp \\ &= \frac{1}{a}R(s,u)\end{aligned}$$

(iii)Natural transform of derivative

Let $R^{+}(s,u)$ be the Natural transform of $f(t)$ i.e, $N^{+}[f(t)]$, then

$$N^{+}[f^{n}(t)] = \frac{s^n}{u^n}R^{+}(s,u) - \sum_{j=0}^{n-1}\frac{s^{n-1-j}}{u^{n-j}}f^{j}_{(0)}$$

Proof.

When $n = 1$, from equation (3.8) $N^{+}[f'(t)]$ is given by

$$N^{+}[f'(t)] = \int_0^\infty e^{-st} f'(tu)dt$$

By integrating by part we have

$$\begin{aligned}\int_0^\infty e^{-st} f'(tu)dt &= \frac{1}{u}f(ut)e^{-st}|_0^\infty + \frac{s}{u}\int_0^\infty e^{-st} f(tu)dt \\ &= \frac{-1}{u}f(0) + \frac{s}{u}\int_0^\infty e^{-st} f(tu)dt\end{aligned}$$

Which yields to

$$\int_0^\infty e^{-st} f'(ut)dt = \frac{s}{u}R(s,u) - \frac{1}{u}f(0) \tag{3.9}$$

$n = 2$, using equation (3.8) again we have

$$N^{+}[f''(t)] = \int_0^\infty e^{-st} f''(tu)dt$$

By integrating by part we have

$$\begin{aligned}\int_0^\infty e^{-st} f''(tu)dt &= \frac{1}{u}f'(ut)e^{-st}|_0^\infty + \frac{s}{u}\int_0^\infty e^{-st} f'(tu)dt \\ &= \frac{-1}{u}f'(0) + \frac{s}{u}\int_0^\infty e^{-st} f'(tu)dt,\end{aligned}$$

since from equation (3.9) we have

$$\int_0^\infty e^{-st} f'(ut)dt = \frac{s}{u}R^+(s,u) - \frac{1}{u}f(0)$$

Then we have

$$\begin{aligned}\frac{-1}{u}f'(0) + \frac{s}{u}\int_0^\infty e^{-st} f'(tu)dt &= \frac{-1}{u}f'(0) + \frac{s}{u}[\frac{s}{u}R(s,u) - \frac{1}{u}f(0)] \\ &= \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{1}{u}f'(0)\end{aligned}$$

Therefore

$$N^+[f''(t)] = \frac{s^2}{u^2}R^+(s,u) - \frac{s}{u^2}f(0) - \frac{1}{u}f'(0)$$

In general form above two proof we conclude that:

$$N^+[f^n(t)] = \frac{s^n}{u^n}R^+(s,u) - \sum_{j=0}^{n-1}\frac{s^{n-1-j}}{u^{n-j}}f^j_{(0)}$$

for all $n \geq 0$

3.5 Natural decomposition method for nonlinear differential equations

In this section, we extend the proposed method to solve differential equations(PDE and ODE)

3.5.1 NDM for nonlinear ordinary differential equations:

Consider the general nonlinear non-homogenous ordinary differential equation:

$$\frac{d^n y(t)}{dt^n} + Ry(t) + Ny(t) = g(t) \tag{3.10}$$

where $n = 1, 2, 3, ...,$ with the initial conditions:

$$\frac{d^{n-1}y(t)}{dt^{n-1}}|_{t=0} = f_{n-1}(0), \tag{3.11}$$

where $\frac{d^n y(t)}{dt^n}$ is the linear nth order derivative which is assumed to be invertible operator, R is a linear differential operator of order less than $\frac{d^n y(t)}{dt^n}$, $g(t)$ is the source term and N is the nonlinear term.
First apply the natural transform on both sides of (3.10) and use properties of natural transform, we get:

$$\frac{s^n}{u^n}R^+(s,u) - \sum_{k=0}^{n-1} y^k(0)\frac{s^{n-k-1}}{u^{n-k}} + N^+[Ry(t)] + N^+[Ny(t)] = N^+[g(t)] \tag{3.12}$$

$$R^+(s,u) = \frac{u^n}{s^n}[\sum_{k=0}^{n-1} y^k(0)\frac{s^{n-k-1}}{u^{n-k}} + N^+[g(t)]] - \frac{u^n}{s^n}N^+[Ry(t)] + N^+[Ny(t)] \tag{3.13}$$

second applying the inverse N-transform on (3.13), we get:

$$N^{-1}[R^+(s,u)] = N^{-1}[\frac{u^n}{s^n}[\sum_{k=0}^{n-1} y^k(0)\frac{s^{n-k-1}}{u^{n-k}} + N^+[g(t)]]] - N^{-1}[\frac{u^n}{s^n}N^+[Ry(t)] + N^+[Ny(t)]] \tag{3.14}$$

$$y(t) = G(t) - N^{-1}\frac{u^n}{s^n}N^+[Ry(t)] + N^+[Ny(t)] \tag{3.15}$$

where $G(t)$ represents the term arising from the source term and the prescribed initial conditions.
We now assume an infinite series solution of the unknown function $y(t)$ of the form:

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{3.16}$$

Substituting (3.16) in to (3.15), we have:

$$\sum_{n=0}^{\infty} y_n(t) = G(t) - N^{-1}[\frac{u^n}{s^n}N^+[R\sum_{n=0}^{\infty} y_n(t) + \sum_{n=0}^{\infty} A_n]], \tag{3.17}$$

where A_n are the Adomian polnomials (Ige et al., 2018) of y_0, y_1, y_2, ... and calculate by the formula given below.

$$A_n = \frac{d^n}{n!d\lambda^n}[N(\sum_{i=0}^{n} \lambda^i y_i)]|_{\lambda=0}$$

Comparing both sides of (3.17), we can easily build the recursive relation as follows:

$$
\begin{aligned}
y_0(t) &= G(t) \\
y_1(t) &= -N^{-1}[\frac{u^n}{s^n}N^+[Ry_0 + A_0]] \\
y_2(t) &= -N^{-1}[\frac{u^n}{s^n}N^+[Ry_1 + A_1]] \\
y_3(t) &= -N^{-1}[\frac{u^n}{s^n}N^+[Ry_2 + A_2]] \\
&\vdots
\end{aligned}
$$

we have the general recursive relation as follows:

$$y_{n+1}(t) = -N^{-1}[\frac{u^n}{s^n}N^+[Ry_n + A_n]] \tag{3.18}$$

where $n = 1, 2, 3, ...$ and $n \geq 0$
Finally, we approximate the analytical solution $y(t)$ by truncated series:

$$y(t) = \lim_{N\to\infty} \sum_{n=0}^{N} y_n(t) \tag{3.19}$$

.

3.5.2 NDM for nonlinear partial differential equations:

Our focus in this work is to solve the nonlinear partial differential equations which is third order KdV equation, so we need to consider how the Natural decomposition method can be use to decompose the general nonlinear partial differential equation. According to (Landau et al., 2007), consider;

$$\frac{\partial^m u(x,t)}{\partial t^m} + Ru(x,t) + Nu(x,t) = g(x,t) \tag{3.20}$$

where $m = 1, 2, 3, ...,$ with the initial conditions:

$$\frac{\partial^{m-1} u(x,t)}{\partial t^{m-1}}|_{t=0} = f_{m-1}(x), \tag{3.21}$$

where $\frac{\partial^m u(x,t)}{\partial t^m}$ is the linear mth order derivative with respect to t, R represents the linear differential operator, $g(x,t)$ is the source term and N is the nonlinear term. First apply the natural transform on both sides of (3.20) and use properties of natural transform, we get:

$$N^{+}[\frac{\partial^m u(x,t)}{\partial t^m}] + N^{+}[Ru(x,t) + Nu(x,t)] = N^{+}[g(x,t)] \tag{3.22}$$

Note that;

$$N^{+}[\frac{\partial^m u(x,t)}{\partial t^m}] = \frac{s^m}{u^m}R(s,u) - \sum_{k=0}^{m-1}\frac{s^{m-1-k}}{u^{m-k}}\frac{\partial^k u(x,0)}{\partial t^k} \tag{3.23}$$

Now let substitute equation (3.23) into equation (3.22), we have;

$$\frac{s^m}{u^m}R^{+}(s,u) - \sum_{k=0}^{m-1}\frac{s^{m-1-k}}{u^{m-k}}\frac{\partial^k u(x,0)}{\partial t^k} + N^{+}[Ru(x,t)] + N^{+}[Nu(x,t)] = N^{+}[g(x,t)] \tag{3.24}$$

$$R^{+}(s,u) = \frac{u^m}{s^m}[\sum_{k=0}^{m-1}\frac{s^{m-1-k}}{u^{m-k}}\frac{\partial^k u(x,0)}{\partial t^k} + N^{+}[g(x,t)]] - \frac{s^m}{u^m}N^{+}[Ru(x,t)] + N^{+}[Nu(x,t)] \tag{3.25}$$

Second operating the inverse Natural transform on both side of (3.25), we get:

$$N^{-1}[R^{+}(s,u)] = N^{-1}[\frac{u^m}{s^m}[\sum_{k=0}^{m-1}\frac{s^{m-1-k}}{u^{m-k}}\frac{\partial^k u(x,0)}{\partial t^k} + N^{+}[g(x,t)]]] - N^{-1}[\frac{s^m}{u^m}N^{+}[Ru(x,t)] + N^{+}[Nu(x,t)]]$$

$$u(x,t) = G(x,t) - N^{-1}\frac{s^m}{u^m}N^{+}[Ru(x,t)] + N^{+}[Nu(x,t)] \tag{3.26}$$

where $G(x,t)$ denotes the expression that arises from the given initial conditions and the source terms after simplification.

We now assume an infinite series solution of the unknown function $u(x,t)$ of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.27}$$

Substituting (3.27) in to (3.26), we have:

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - N^{-1}[\frac{u^m}{s^m}N^{+}[R\sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n]], \tag{3.28}$$

where A_n are the Adomian polnomial (Ige et al., 2018) of $u_0, u_1, u_2, ...$ represent non-linear term and calculate by the formula given below.

$$A_n = \frac{d^n}{n!d\lambda^n}[N(\sum_{i=0}^{n}\lambda^i u_i)]|_{\lambda=0}$$

Comparing both sides of (3.28), we can easily build the recursive relation as follows:

$$\begin{aligned}
u_0(x,t) &= G(x,t)\\
u_1(x,t) &= -N^{-1}[\frac{u^m}{s^m}N^+[Ru_0 + A_0]]\\
u_2(x,t) &= -N^{-1}[\frac{u^m}{s^m}N^+[Ru_1 + A_1]]\\
u_3(x,t) &= -N^{-1}[\frac{u^m}{s^m}N^+[Ru_2 + A_2]]\\
&\vdots
\end{aligned}$$

The general recursive relation as follows:

$$u_{n+1}(x,t) = -N^{-1}[\frac{u^m}{s^m}N^+[Ru_n + A_n]], \tag{3.29}$$

where $m = 1, 2, 3, ...$ and $n \geq 0$
The analytical solution $u(x,t)$ approximated by truncate series:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t) \tag{3.30}$$

3.6 Previous work on Apllication of NDM to solve nonlinear differential equations

1, Consider the first order nonlinear ordinary differential equation of the form :

$$\frac{dv(t)}{dt} - 1 = v^2(t) \tag{3.31}$$

subject to the initial condition

$$v(0) = 0 \tag{3.32}$$

Taking the Natural transform to both sides of equation (3.31), we obtain:

$$\frac{s}{u}V(s,u) - \frac{1}{u}v(0) = N^+[v^2(t)] \tag{3.33}$$

Substituting equation(3.32) , we obtain:

$$V(s,u) = \frac{u}{s^2} + \frac{u}{s}[N^+[v^2(t)]] \tag{3.34}$$

Taking the inverse Natural transform of equation(3.34) , we obtain:

$$v(t) = t + N^{-1}[[N^{+}[v^{2}(t)]]] \tag{3.35}$$

We now assume an infinite solution of the unknown function $v(t)$ of the form:

$$v(t) = \sum_{n=0}^{\infty} v_n(t) \tag{3.36}$$

Using equation(3.36) , we can re-write *equation*(3.35) in the form

$$\sum_{n=0}^{\infty} v_n(t) = t + N^{-1}[[N^{+}[\sum_{n=0}^{\infty} A_n(t)]]], \tag{3.37}$$

where $A_n(t)$ is the Adomian polynomial representing the nonlinear term $v^2(t)$. Then from *equation*(3.37) , we can generate the recursive relation as follows:

$$\begin{aligned}
v_0(t) &= t \\
v_1(t) &= N^{-1}\frac{u}{s}N^{+}[A_0(t)] \\
v_2(t) &= N^{-1}\frac{u}{s}N^{+}[A_1(t)] \\
v_3(t) &= N^{-1}\frac{u}{s}N^{+}[A_2(t)]
\end{aligned}$$

Thus, the general recursive relation is given by

$$v_{n+1}(t) = N^{-1}[\frac{u}{s}[N^{+}[A_n(t)]]], n \geq 0 \tag{3.38}$$

Using equation(3.38),we can easily compute the remaining components of the unknown function v(t) as follows:

$$\begin{aligned}
v_1(t) &= N^{-1}[\frac{u}{s}[N^{+}[A_0(t)]]] \\
&= N^{-1}[\frac{u}{s}[N^{+}[v_0^2(t)]]] \\
&= N^{-1}[\frac{u}{s}[N^{+}[t^2]]] \\
&= N^{-1}[\frac{2u^3}{s^4}] \\
&= \frac{1}{3}t^3
\end{aligned}$$

$$
\begin{aligned}
v_2(t) &= N^{-1}[\frac{u}{s}[N^+[A_1(t)]]] \\
&= N^{-1}[\frac{u}{s}[N^+[2v_0(t)v_1(t)]]] \\
&= N^{-1}[\frac{2t^4}{3}] \\
&= N^{-1}[\frac{48u^5}{3s^6}] = \frac{2t^5}{15} \\
v_3(t) &= N^{-1}[\frac{u}{s}[N^+[A_2(t)]]] \\
&= N^{-1}[\frac{u}{s}[N^+[2v_0(t)v_2(t) + v_1^2(t)]]] \\
&= N^{-1}[\frac{17t^6}{45}] \\
&= N^{-1}[\frac{12240u^7}{45s^8}] = \frac{17t^7}{315}
\end{aligned}
$$

Then the approximate solution of the unknown function $v(t)$ is given by:

$$
\begin{aligned}
v(t) &= \sum_{n=0}^{\infty} v_n(t) \\
&= v_0(t) + v_1(t) + v_2(t) + v_3(t) + \cdots \\
&= t + \frac{1}{3}t^3 + \frac{2t^5}{15} + \frac{17t^7}{315} + \cdots
\end{aligned}
$$

Note that from the Taylor's series expansion of trigonometric function we have:

$$\tan(t) = t + \frac{1}{3}t^3 + \frac{2t^5}{15} + \frac{17t^7}{315} + \cdots$$

Therefore;

$$v(t) = \tan(t)$$

Table 3.1: Comparison between Numerical and approximated solutions for $equation$(3.31)

t	v_{RK4}	v_{NDM}	$Error = \|v_{RK4} - v_{NDM}\|$
0	0	0	0
0.005	0.00500004	0.00500001	3.12504×10^{-8}
0.01	0.0100003	0.0100001	2.50013×10^{-7}
0.015	0.0150011	0.01500034	8.43845×10^{-7}
0.02	0.02	0.0199987	2.00008×10^{-6}

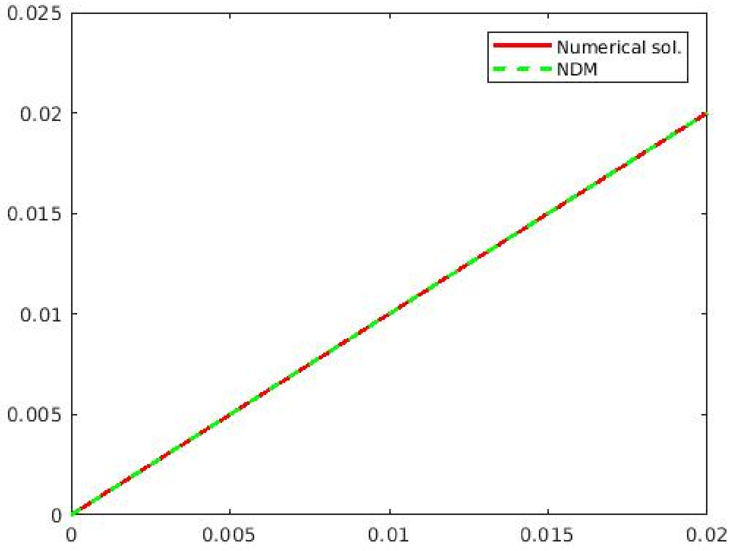

Figure 3.1: Numerical and approximated solution

2,Consider the homogenous Klein Gordon equation (Ige et al., 2019):

$$u_{tt} - u_{xx} - u = 0 \tag{3.39}$$

with the initial conditions

$$u(x,0) = 1 + \sin(x), u_t(x,0) = 0 \tag{3.40}$$

By taking the N-transform on both side of (3.39) and use the properties of N-transform, we have:

$$\frac{s^2}{u^2}R^+(s,u) - \frac{s}{u^2}u(x,0) - \frac{1}{u}u_t(x,0) = N^+[u_{xx} + u] \tag{3.41}$$

$$R^+(s,u) = \frac{1}{s}u(x,0) + \frac{u}{s^2}u_t(x,0) + \frac{u^2}{s^2}N^+[u_{xx} + u] \tag{3.42}$$

by substituting the initial condition, we have:

$$R^+(s,u) = \frac{1}{s} + \frac{\sin(x)}{s} + \frac{u^2}{s^2}N^+[u_{xx} + u] \tag{3.43}$$

By applying inverse N-transform on both side of (3.43) , we get:

$$u(x,t) = 1 + \sin(x) + N^{-1}\frac{u^2}{s^2}N^+[u_{xx} + u] \tag{3.44}$$

Now to deal with the nonlinear term, we represent the solution in an infinite series form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.45}$$

and

$$u_{xx} = \sum_{n=0}^{\infty} A_n \tag{3.46}$$

where A_n is the Adomian polnomials of $u_0, u_1, u_2, ...$
Substituting (3.45) and (3.46) in (3.44), we have:

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + \sin(x) + N^{-1}\frac{u^2}{s^2}N^{+}[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n(x,t)] \tag{3.47}$$

Comparing both sides of eq. (3.47), we can easily build the recursive relation as follows:

$$\begin{aligned}
u_0(x,t) &= 1 + \sin(x) \\
u_1(x,t) &= N^{-1}\frac{u^2}{s^2}N^{+}[A_0 + u_0(x,t)] \\
u_2(x,t) &= N^{-1}\frac{u^2}{s^2}N^{+}[A_1 + u_1(x,t)] \\
u_3(x,t) &= N^{-1}\frac{u^2}{s^2}N^{+}[A_2 + u_2(x,t)] \\
&\vdots
\end{aligned}$$

Eventually, we have the general recursive relation as follows:

$$u_{n+1}(x,t) = N^{-1}\frac{u^2}{s^2}N^{+}[A_n + u_n(x,t)] \tag{3.48}$$

$$\begin{aligned}
u_0(x,t) &= 1 + \sin(x) \\
u_1(x,t) &= N^{-1}\frac{u^2}{s^2}N^{+}[A_0 + u_0(x,t)]
\end{aligned}$$

where

$$
\begin{aligned}
A_0 &= \frac{d^0}{0!d\lambda^0}[N(\sum_{i=0}^{0}\lambda^i u_i)]l_{\lambda=0} \\
&= N[u_0] \\
&= u_{0(xx)} = -\sin(x) \\
u_1(x,t) &= N^{-1}\frac{u^2}{s^2}N^+[A_0 + u_0(x,t)] \\
&= N^{-1}\frac{u^2}{s^2}N^+[-\sin(x) + 1 + \sin(x)] \\
&= N^{-1}\frac{u^2}{s^2}N^+[1] \\
&= N^{-1}\frac{u^2}{s^3} = \frac{t^2}{2!}
\end{aligned}
$$

The approximate series solution is:

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + +... \\
&\quad = 1 + \sin(x) + \frac{t^2}{2!} + ...
\end{aligned}
$$

We can rewrite this as;

$$u(x,t) = \sin(x) + 1 + \frac{t^2}{2!} + ...$$

Note that from the Taylor's series expansion of trigonometric function we have:

$$\cosh(t) = 1 + \frac{t^2}{2!} + ...$$

Therefore;

$$u(x,t) = \sin(x) + \cosh(t)$$

Table 3.2: Comparison between numerical and approximated solutions for $equation$(3.39)

t	v_{FDM}	v_{NDM}	$Error$
0	1	1	0
0.01	1.0100	1.0100	0.0001
0.02	1.0200	1.0202	0.0002
0.03	1.0300	1.0304	0.0005
0.04	1.0400	1.0408	0.0008
0.05	1.0500	1.0512	0.0012
0.06	1.0600	1.0618	0.0018
0.07	1.0699	1.0724	0.0025
0.08	1.0799	1.0831	0.0032
0.09	1.0899	1.0839	0.0041
0.1	1.0998	1.1048	0.0050

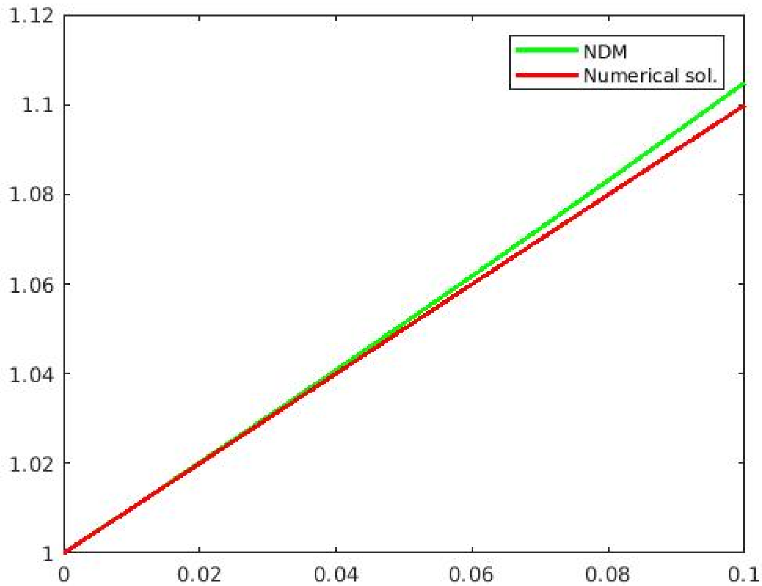

Figure 3.2: Numerical and approximated solution

CHAPTER 4

Result and Discussion

In this chapter we use the Natural decomposition method to solve both third-order nonlinear KdV and second order nonlinear Van Der Pol forced oscillator equations and to generalized the method of solving any kind of these equations by using Natural decomposition method.

4.1 Application

Problem 1

Consider the Van Der Pol forced oscillator problem (Tsatsos, 2006)

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^2\frac{dx}{dt} = \frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t), \\ x(0) = 0, \\ x^{'}(0) = 1, \end{cases} \tag{4.1}$$

where $x = x(t)$.

We can rewrite equation (4.12) as:

$$\frac{d^2x}{dt^2} = -[\frac{dx}{dt} + x + x^2\frac{dx}{dt}] + \frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t). \tag{4.2}$$

Applying the N-transform on both side of equation (4.2), we get

$$N^+[\frac{d^2x}{dt^2}] = -N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}] + N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]. \tag{4.3}$$

Note that

$$N^+[\frac{d^2x}{dt^2}] = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}x(0) - \frac{1}{u}\dot{x}(0).$$

Then equation (4.3) becomes

$$\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}x(0) - \frac{1}{u}\dot{x} = -N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}] + N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]$$
$$\frac{s^2}{u^2}R(s,u) = \frac{1}{u} - N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}] + N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]$$

$$R(s,u) = \frac{u}{s^2} - \frac{u^2}{s^2}N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}] + \frac{u^2}{s^2}N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]. \tag{4.4}$$

Applying the inverse Natural transform to equation (4.4), we have

$$N^{-1}R(s,u) = N^{-1}[\frac{u}{s^2}] - N^{-1}[\frac{u^2}{s^2}N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}]] + N^{-1}[\frac{u^2}{s^2}N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]]$$

$$x(t) = t - N^{-1}[\frac{u^2}{s^2}N^+[\frac{dx}{dt} + x + x^2\frac{dx}{dt}]] + N^{-1}[\frac{u^2}{s^2}N^+[\frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t)]] \tag{4.5}$$

Now we assume an infinite series solution of the unknown function $x(t)$ of the form

$$x(t) = \sum_{n=0}^{\infty} x_n(t)$$

Then equation (4.14) becomes

$$\sum_{n=0}^{\infty} x_n(t) = t - N^{-1}[\frac{u^2}{s^2}N^+[\sum_{n=0}^{\infty}[\dot{x}_n(t) + x_n(t) + A_n - f_n(t)]]]. \tag{4.6}$$

By comparing both side of equation (4.6), we get the following recursive relation.

$$\begin{aligned}
x_0(t) &= t \\
x_1(t) &= -N^{-1}[\frac{u^2}{s^2}N^+[\dot{x}_0(t) + x_0(t) + A_0 - f_0(t)]] \\
x_2(t) &= -N^{-1}[\frac{u^2}{s^2}N^+[\dot{x}_1(t) + x_1(t) + A_1 - f_1(t)]] \\
x_3(t) &= -N^{-1}[\frac{u^2}{s^2}N^+[\dot{x}_2(t) + x_2(t) + A_2 - f_2(t)]] \\
&\vdots
\end{aligned}$$

$$x_{n+1}(t) = -N^{-1}\frac{u^2}{s^2}N^+[\dot{x}_n(t) + x_n(t) + A_n - f_n(t)], \tag{4.7}$$

were A_n is the Adomian polynomial.

$$f(t) = \frac{5}{4}\cos(t) - \frac{1}{4}\cos(3t),$$

we know that

$$cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!}$$

$$\begin{aligned}
\Rightarrow \sum_{n=0}^{\infty} f_n(t) &= \frac{5}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (t)^{2n}}{(2n)!} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n (3t)^{2n}}{(2n)!} \\
&= 1 + \frac{1}{2!}t^2 - \frac{19}{4!}t^4 + \dots \\
x_1(t) &= -N^{-1}[\frac{u^2}{s^2}N^{+}[\dot{x}_0 + x_0 + A_0 - f_0(t)]] \\
A_0 &= \frac{d^0}{0!d\lambda^0}[N(\sum_{i=0}^{0}\lambda^i x_i)]|_{\lambda=0} = N(x_0) \\
&= x_0^2\dot{x}_0 = t^2 \\
f_0(t) &= 1 \\
x_1(t) &= -N^{-1}[\frac{u^2}{s^2}N^{+}[1 + t + t^2 - 1]] \\
&= -N^{-1}[\frac{u^2}{s^2}N^{+}[t + t^2]] \\
&= -N^{-1}[\frac{u^2}{s^2}[\frac{u}{s^2} + \frac{2u^2}{s^3}]] \\
&= -N^{-1}[\frac{u^3}{s^4} + \frac{2u^4}{s^5}] \\
&= -[\frac{t^3}{3!} + \frac{2t^4}{4!}] \\
x_2(t) &= -N^{-1}[\frac{u^2}{s^2}N^{+}[\dot{x}_1 + x_1 + A_1 - f_1(t)]] \\
A_1 &= \frac{d}{1!d\lambda}[N(\sum_{i=0}^{1}\lambda^i x_i)]|_{\lambda=0} \\
&= \frac{d}{d\lambda}[N(x_0 + \lambda x_1)]|_{\lambda=0} \\
&= \frac{d}{d\lambda}[(x_0 + \lambda x_1)^2(x_{0t} + \lambda x_{1t})]|_{\lambda=0} \\
&= 2(x_0 + \lambda x_1)(x_1)(x_{0t} + \lambda x_{1t}) + (x_0 + \lambda x_1)^2(x_{1t})|_{\lambda=0} \\
&= 2x_0x_1\dot{x}_0 + x_0^2\dot{x}_1 \\
f_1(t) &= \frac{t^2}{2} \\
x_2(t) &= -N^{-1}[\frac{u^2}{s^2}N^{+}[\dot{x}_1 + x_1 + A_1 - f_1(t)]] \\
&= -N^{-1}[\frac{u^2}{s^2}N^{+}[-t^2 - \frac{1}{2}t^3 - \frac{11}{12}t^4 - \frac{1}{2}t^5]]
\end{aligned}$$

$$
\begin{aligned}
&= N^{-1}\frac{u^2}{s^2}[\frac{2u^2}{s^3}+\frac{3u^3}{s^4}+\frac{22u^4}{s^5}+\frac{60u^5}{s^6}] \\
&= N^{-1}[\frac{2u^4}{s^5}+\frac{3u^5}{s^6}+\frac{22u^6}{s^7}+\frac{60u^7}{s^8}] \\
&= \frac{2t^4}{4!}+\frac{3t^5}{5!}+\frac{22t^6}{6!}+\frac{60t^7}{7!}.
\end{aligned}
$$

The series solution of $x(t)$ is

$$
\begin{aligned}
x(t) &= x_0 + x_1 + x_2 + ... \qquad (4.8)\\
&= t - \frac{1}{3!}t^3 + \frac{3}{5!}t^5 + \frac{22}{6!}t^6 + \frac{60}{7!}t^7 + ... \qquad (4.9)
\end{aligned}
$$

The analytical solution $x(t)$ approximated by truncated series:

$$
\begin{aligned}
x(t) &= \lim_{N\to\infty}\sum_{n=0}^{N} x_n(t) \\
&= \sum_{n=0}^{2} x_n(t) \\
&= x_0 + x_1 + x_2 \\
&= t - \frac{1}{3!}t^3 + \frac{3}{5!}t^5 + \frac{22}{6!}t^6 + \frac{60}{7!}t^7
\end{aligned}
$$

Table 4.1: Comparison between numerical fourth-order Runge–Kutta method and Natural decomposition method for problem 1

t	$x(t)_{RK4}$	$x(t)_{NDM}$	$Error = \|x(t)_{RK4} - x(t)_{NDM}\|$
0	0	0	0
0.1	0.0998	0.0998	0
0.2	0.1987	0.1987	0
0.3	0.2956	0.2956	0
0.4	0.3896	0.3897	1×10^{-4}
0.5	0.4796	0.4805	9×10^{-4}
0.6	0.5648	0.5677	2.9×10^{-3}
0.7	0.6444	0.6516	7.2×10^{-3}
0.8	0.7174	0.7334	1.6×10^{-2}
0.9	0.7832	0.8152	3.2×10^{-2}
1	0.841	0.9008	5.98×10^{-2}

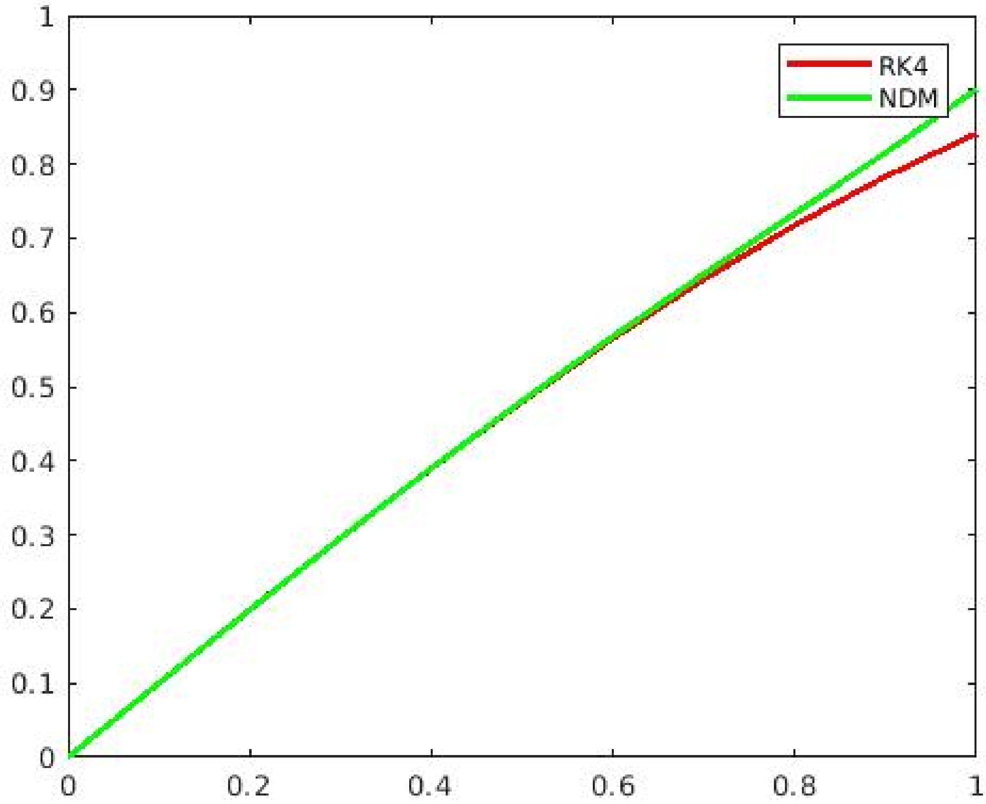

Figure 4.1: Comparing Numerical and approximate analytic solutions of problem 1

Problem 2

Consider the homogeneous KdV equation (Ige et al., 2019)

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0 \\ u(x,0) = 6x, \end{cases} \tag{4.10}$$

where $u = u(x,t)$.

Let us rewrite equation (4.10) as:

$$u_t = 6uu_x - u_{xxx} \tag{4.11}$$

Applying N-Transform to both side of equation (4.11)

$$N^+[u_t] = N^+[6uu_x - u_{xxx}] \tag{4.12}$$

Recall that

$$N^+[u_t] = \frac{s}{u}R(s,u) - \frac{1}{u}u(0)$$

Then equation (4.12) be comes:

$$\frac{s}{u}R(s,u) - \frac{1}{u}u(0) = N^{+}[6uu_x - u_{xxx}]$$

$$R(s,u) = \frac{1}{s}6x + \frac{u}{s}N^{+}[6uu_x - u_{xxx}] \tag{4.13}$$

Applying the inverse N-Transform on both side of equation (4.13),we obtain

$$N^{-1}R(s,u) = N^{-1}\frac{1}{s}6x + N^{-1}[\frac{u}{s}N^{+}[6uu_x - u_{xxx}]] \tag{4.14}$$

$$u(x,t) = 6x + N^{-1}[\frac{u}{s}N^{+}[6uu_x - u_{xxx}]] \tag{4.15}$$

We now assume an infinite series solution of the unknown function $u(x,t)$ of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Then equation (4.15) becomes:

$$\sum_{n=0}^{\infty} u_n(x,t) = 6x + N^{-1}[\frac{u}{s}N^{+}[6\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} u_{nxxx}]], \tag{4.16}$$

where A_n is an Adomian polynomial which represent the nonlinear term.
By comparing both side of equation (4.16), we can easily build the recursive relation as follows:

$$\begin{aligned}
u_0(x,t) &= 6x \\
u_1(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_0 - u_{0xxx}]] \\
u_2(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_1 - u_{1xxx}]] \\
u_3(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_2 - u_2(xxx)]] \\
&\vdots
\end{aligned}$$

We have the general recursive relation as follows

$$\begin{aligned}
u_{n+1}(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_n - u_{nxx}]] \\
u_1(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_0 - u_{0xxx}]],
\end{aligned}$$

where

$$\begin{aligned}
A_0 &= \frac{d^0}{0!d\lambda^0}[N(\sum_{i=0}^{0}\lambda^i u_i)]|_{\lambda=0} \\
&= N(u_0) = u_0\frac{\partial u_0}{\partial x} = 36x \\
u_1(x,t) &= N^{-1}[\frac{u}{s}N^{+}[216x - \frac{\partial^3 u_0}{\partial x^3}]] \\
&= 216xN^{-1}[\frac{u}{s}N^{+}[1]] \\
&= 216xN^{-1}[\frac{u}{s^2}] \\
&= 216xt \\
u_2(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_1 - u_{1xx}]]
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \frac{d}{1!d\lambda}[N(\sum_{i=0}^{1}\lambda^i u_i)]|_{\lambda=0} \\
&= \frac{d}{d\lambda}N(u_0 + \lambda u_1)|_{\lambda=0} \\
&= \frac{d}{d\lambda}(u_0 + \lambda u_1)(u_{0x} + \lambda u_{1x})|_{\lambda=0} \\
&= u_1(u_{0x} + \lambda u_{1x}) + u_{1x}(u_0 + \lambda u_1)|_{\lambda=0} \\
&= u_1 u_{0x} + u_0 u_{1x} \\
&= 216xt(6) + 6x(216t) = 2,592xt \\
u_2(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6(2592xt) - \frac{\partial^3 u_1}{\partial x^3}]] \\
&= N^{-1}[\frac{u}{s}N^{+}[6(2592xt) - 0] \\
&= 15552xN^{-1}[\frac{u}{s}N^{+}[t]] \\
&= 15552xN^{-1}[\frac{u}{s}[\frac{u}{s^2}]] \\
&= 15552xN^{-1}[\frac{u^2}{s^3}] \\
&= 15552x[\frac{t^2}{2!}] = 7,776xt^2 \\
u_3(x,t) &= N^{-1}[\frac{u}{s}N^{+}[6A_2 - u_2(xxx)]]
\end{aligned}$$

where

$$\begin{aligned}
A_2 &= \frac{d^2}{2!d\lambda^2}[N(\sum_{i=0}^{2}\lambda^i u_i)]|_{\lambda=0} \\
&= \frac{d^2}{2d\lambda^2}N(u_0+\lambda u_1+\lambda^2 u_2) \\
&= \frac{d^2}{2d\lambda^2}[(u_0+\lambda u_1+(\lambda)^2 u_2)(u_{0x}+\lambda u_{1x}+(\lambda)^2 u_{2x})]|_{\lambda=0} \\
&= u_2\frac{\partial u_0}{\partial x}+u_1\frac{\partial u_1}{\partial x}+u_0\frac{\partial u_2}{\partial x} \\
&= 46,656xt^2(1)+46,656xt^2+46,656xt^2 = 139,968xt^2 \\
u_3(x,t) &= N^{-1}\frac{u}{s}N^{+}[6(139,968xt^2)-\frac{\partial^3 u_2}{\partial x^3}] \\
&= 6(139,968x)N^{-1}[\frac{u}{s}N^{+}[t^2-0]] \\
&= 6(139,968x)N^{-1}[\frac{u}{s}N^{+}[t^2]] \\
&= 6(139,968x)N^{-1}[\frac{u}{s}[2\frac{u^2}{s^3}]] \\
&= 12(139,968x)N^{-1}[\frac{u^3}{s^4}] \\
&= 12(139,968x)[\frac{t^3}{3!}] = 279,936xt^3
\end{aligned}$$

The series solution of $u(x,t)$ is

$$u(x,t) = u_0+u_1+u_2+u_3+... \tag{4.17}$$
$$= 6x+216xt+7776xt^2+279,936xt^3+... \tag{4.18}$$

This can be written as:

$$u(x,t) = 6x[1+36t+(36t)^2+(36t)^3+...] \tag{4.19}$$

We know that the power series representation of $f(x)$ is

$$f(x) = \frac{1}{1-x} = 1+x+x+...$$

The function of power series representation of equation (4.19) is

$$u(x,t) = \frac{6x}{1-36t} \tag{4.20}$$

The analytical solution $u(x,t)$ approximated by truncated series:

$$\begin{aligned} u(x,t) &= \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t) \\ &= \sum_{n=0}^{3} u_n(x,t) \\ &= u_0 + u_1 + u_2 + u_3 \\ &= 6x[1 + 36t + (36t)^2 + (36t)^3] \end{aligned}$$

Table 4.2: Comparison between numerical and approximated solution of problem 2

t	u_{FDM}	u_{NDM}	$Error = \lvert u_{FDM} - u_{NDM}\rvert$
0	6	6	0
0.001	6.2240664	6.2240559	1.05×10^{-5}
0.002	6.4655172414	6.465343488	1.74×10^{-4}
0.003	6.726457399	6.725542272	9.15×10^{-4}
0.004	7.009345794	7.006331904	3.014×10^{-3}
0.005	7.317073171	7.309392	7.68×10^{-3}
0.006	7.65306122	7.636402176	1.67×10^{-2}
0.007	8.021390374	7.989042048	3.23×10^{-2}
0.008	8.426966292	8.368991232	5.8×10^{-2}
0.009	8.875739645	8.777929344	9.78×10^{-2}
0.01	9.375	9.217536	0.1575

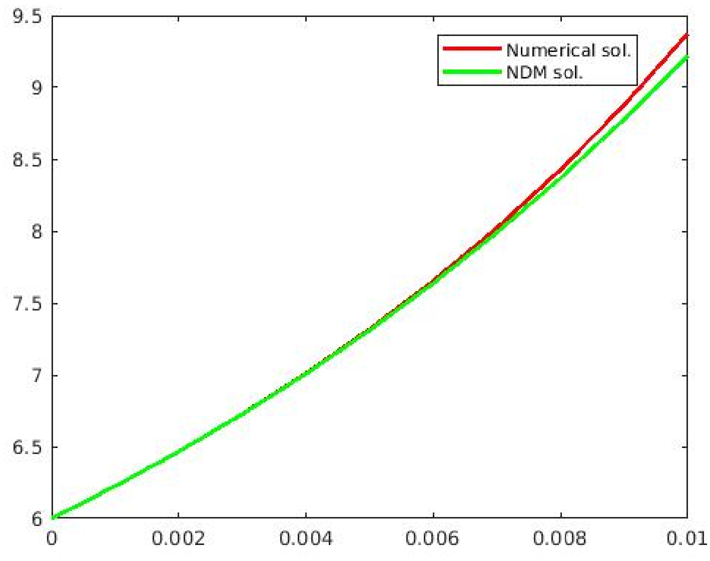

Figure 4.2: Comparing the numerical and analytical approximate solution of problem2 2.

4.2 Discussion

Natural decomposition method was used to obtain the approximate analytical solution of the third order Korteweg-de Vries (KdV) and second order Van Der Pol forced oscillator equations considered in this work. For illustration purpose, examples were selected to show how to apply Natural decomposition method to find approximate analytic solution. The approximate solution and numerical solution are compared and shows small error.

CHAPTER 5

Conclusion and Recommendation

Conclusion

In this thesis, the approximate analytical solutions of third order KdV equation and second order Van Der Pol forced oscillator equation are obtained using the Natural decompositiom method. We successfully found an approximate analytic solution of two problems and compare the result with numerical method FDM and RK4.Thus comparison show that NDM is a very powerful integral transform method for solving nonlinear differential equations. Our interests in future work is to apply NDM to other nonlinear PDEs, which arises in the fields of other disciplines.

Recommendation

This thesis specifically focused on application of Natural decomposition method to get approximate analytic solution of second order nonlinear ODE and third order nonlinear PDE. Natural decomposition method is very interesting and easy to find analytic approximate solution of differential equation.There are other equations which are more complex and more coupled than those we have considered and solved. We extend this work to multiple dimension problems in partial differential equations, which arise other discipline. We will consider these in our future work.

Research Fund Acknowledgment

This research project is funded by Adama Science and Technology University under the grant number ASTU/SM-R/418/22, Adama, Ethiopia.

References

Agarwal, R., Yadav, M. P., & Agarwal, R. P. (2019). Analytic solution of time fractional boussinesq equation for groundwater flow in unconfined aquifer. *Discontinuity, Nonlinearity, and Complexity*, *8*(3), 341–352.

Akter, M. T., & Chowdhury, M. (2019). Homotopy perturbation method for solving highly nonlinear reaction-diffusion-convection problem. *American Journal of Mathematics and Statistics*.

Anjum, N., & He, J.-H. (2020). Two modifications of the homotopy perturbation method for nonlinear oscillators. *Journal of Applied and Computational Mechanics*, *6*(Special Issue), 1420–1425.

Bildik, N., & Deniz, S. (2018). Comparative study between optimal homotopy asymptotic method and perturbation-iteration technique for different types of nonlinear equations. *Iranian Journal of Science and Technology, Transactions A: Science*, *42*(2), 647–654.

Dattu, M. K. U. (2018). New integral transform: Fundamental properties, investigations and applications. *IAETSD Journal for Advanced Research in Applied Sciences*, *5*(4), 534–539.

Dosunmu, D., Edeki, S., Achudume, C., & Udjor, V. (2022). Elzaki adomian decomposition method applied to logistic differential model. In *Journal of physics: Conference series* (Vol. 2199, p. 012019).

Guckenheimer, J., & Holmes, P. (2013). *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* (Vol. 42). Springer Science & Business Media.

Ige, O. E., Oderinu, R. A., & Elzaki, T. M. (2019). Adomian polynomial and elzaki transform method of solving fifth order korteweg-de vries equation. *Caspian Journal of Mathematical Sciences (CJMS) peer*, *8*(2), 103–119.

Ige, O. E., et al. (2018). Approximate analytical solution of nonlinear differential equations using elzaki transform: Case study: Kortewed-de vries (kdv) equations.

Kharrat, B. N., & Toma, G. (2020). Numerical solution of van der pol oscillator problem using a new hybrid method. *World Applied Sciences Journal*, *38*(4), 360–364.

Kumar, M., & Varshney, P. (2021). Numerical simulation of van der pol equation using multiple scales modified lindstedt–poincare method. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, *91*(1), 55–65.

Landau, R. H., Páez, M. J., & Bordeianu, C. C. (2007). Computational physics. *Computer Languages*, *7*, 2–3.

Maz'ya, V., Nazarov, S., & Plamenevskii, B. (1979). References to maz'ya's publications made in volume ii. *Partial Differential*, *24*, 381.

Othman, S. G., & Hasan, Y. Q. (n.d.). Fractional van der pol equation by adomian decomposition method.

Rasedee, A. F. N., Sathar, M. H. A., Ijam, H. M., Othman, K. I., Ishak, N., & Hamzah, S. R. (2018). A numerical solution for duffing-van der pol oscillators using a backward difference formulation. In *Aip conference proceedings* (Vol. 2016, p. 020120).

Soomro, A. S., Tularam, G. A., & Shaikh, M. M. (2013). A comparison of numerical methods for solving the unforced van der pol's equation. *Mathematical Theory and Modeling*, *3*(2), 66–77.

Tsatsos, M. (2006). Theoretical and numerical study of the van der pol equation. *Doctoral desertation, Aristotle University of Thessaloniki*, *4*(6).