BOUNDARY ELEMENT METHOD FOR SOLVING THE HEAT FLOW IN RESISTIVE SWITCHING DEVICES WITH DIFFERENT THERMAL CONDUCTIVITY

By
Abduljabar Ibrahim Abdo

A Thesis Submitted to Department of Applied Mathematics
School of Applied Natural Science

Office of Graduate Studies

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Approval Page

This is certify that the thesis prepared by **Abduljabar Ibrahim** entitled " **Boundary Element Method for Solving the Heat Flow in Resistive Switching Devices with Variable Thermal Conductivity** " submitted in partial fulfillment of the requirement for the degree of Master of Science in Applied Mathematics with the regulation of the University and meets the accepted standards with respect to originality.

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Abstract

In this thesis we solved the heat equation in resistive switching devices using boundary element method. In this method the heat equation with the boundary condition converted into the boundary integral equation by applying the Green's second identity and using fundamental solution. Then, the boundary of the domain was divided into finite small boundary elements. By adding up the integral over each element, the BIE is discretized into system of linear equation. Then we solved this system of equation to obtain, both the temperature and the normal derivative of the temperature(flux) distribution on the boundary of the device. Finally, we used this boundary values in boundary integral equation to determine the temperature distribution at any point inside the domain.

Keywords: Fundamental solution, Green's formula, Boundary integral representation, Boundary integral equation, Boundary element method, Resistive switching device.

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Notations and Abbreviations

Ω	The domain (open bounded set).
$\Gamma := \partial\Omega$	The boundary of Ω.
$\overline{\Omega}$	Closure of domain Ω.
$C(\Omega)$	The space of continuous function on Ω.
$C^m(\Omega)$	The set of all continuous functions and m-times differentiable on Ω.
κ	Thermal conductivity of the device.
q	Normal derivative of the temperature.
∇	The gradient operator.
Δ	The Laplace operator.
δ	The Dirac delta function.
BEM	Boundary Element Method.
BIE	Boundary Integral Equation.
DEs	Differential Equations.
PDEs	Partial differential equations.
BVPs	Boundary value problems.
CMOS	Complementary metal-oxide-semiconductor
RRAM	Resistive Random Access Memory

Chapter 1

Introduction

1.1 Background of the Study

Resistive switching is a physical phenomena where the dielectric suddenly changes its resistance under the action of a strong electric field or current. The change of resistance is a non-volatile and reversible process, which can be applied to create a memory device. An interesting application of resistive switching is the fabrication novel non-volatile resistive random access memory (RRAM), which is a metal-insulator-metal layered capacitor like device[3]. In this devices there is an electrode which is an ordinary metal such as Au, Pt, Al, Ag, etc. It is an electrical conductor used to make contact with a nonmetallic part of a circuit. There is also dielectric in the devices that are transition metal oxide (TMO) like TiO_2, NiO, SrO_3, etc [1].

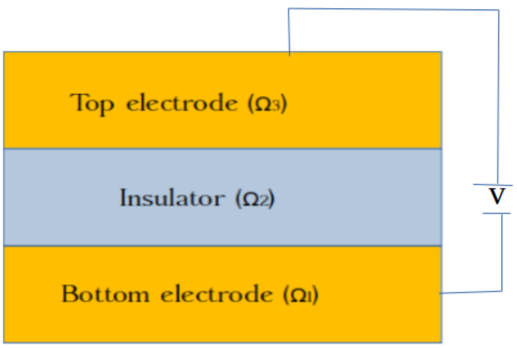

Figure 1.1: Schematic diagram of a typical resistive switching device[5]

In fact, transitional metallic oxide in typical systems Figure(1.1) usually exhibit high dielectric constants, which is considered a desirable feature towards dense electronic integration. Resistive random-access memory is a type of non-volatile random access (RAM) computer memory that works by changing the resistance across a dielectric solid state material[3].

When we consider the insulating film material, the change in resistance can be forced very rapidly at reasonable right energy[3, 5]. Because of such unique properties, resistive switching devices are well suited for applications in digital memories, programmable logic, etc.

The Joule heating effects also play an important role in switching device[5, 7]. Any model of switching behavior in these device must include the heat equation,

$$-\nabla \cdot (\kappa \nabla T) = S \tag{1.1}$$

This is the Fourier heat law, where S is the heat source and κ is the thermal conductivity. This model represents an excellent approximation for a wide range of heat source profiles commonly encountered in the modeling of resistive switching devices [6].

In this study, we apply the boundary element method for solving the heat equation in a metal-insulator-metal structure, with heat sources within the insulator. The BEM is a classical tool that solves boundary value problems. It is numerical method, in essence a numerical integral equation representations of governing PDEs. There is an intermediate step, whereby the integral equation is reformulated so that actual BVPs can be solved. A process that results is known as BIE formulations[7]. It is derived by applying Green's identity for any point in the surface. Then, this boundary integral equation is descritized to a number of elements.

The BEM requires discretizating only the boundary of the domain. The approximate solution of the boundary value problem obtained by BEM is parametrized by a finite set of parameters living on the boundary[17].

BEM reduce the problem dimension by one. This property is advantageous as it reduces the size of the solution system leading to improved computational efficiency. To obtain this reduction of dimension, it is necessary to formulate the governing equation as a boundary integral equation. Also to achieve a boundary integral formulation it is necessary to to determine an appropriate fundamental solution[27].

Another advantage of the BEM is that it allows an explicit expression for the solution at an internal point. The domain integration also can be simply and accurately performed in the BEM.

Therefore, we apply the BEM to solve the heat flow in resistive switching devices with variable thermal conductivity, which is a composite domain. The temperature distribution in this domain can be solved by applying the BEM separately to each of its subdomains. The reason is that the fundamental solution is valid only for homogeneous domains. Since a region is only piecewise homogeneous, the numerical procedures described can be applied to each homogeneous subregion as they were separated from the others [27]. Then combining the separated form as a unique system, which generates a formulation involving both the dependent variable T and the flux q. This allows the temperature and its flux at the boundary to be applied directly in the computations of the temperature at an internal point. Generally, the BEM is used to determine the the heat flow in the resistive switching device at any point in the multi domain.

1.2 Statement of the Problem

We consider a layered device of dimensions l_{x_1} and l_{x_2} , where it is illustrated schematically in the Figure (1.2). Let the metal electrodes have heights of t_1 and t_3 and the insulator has height h. The bottom and top electrodes have thermal conductivities k_1 and k_3 respectively, whereas the insulator has a thermal conductivity of k_2.

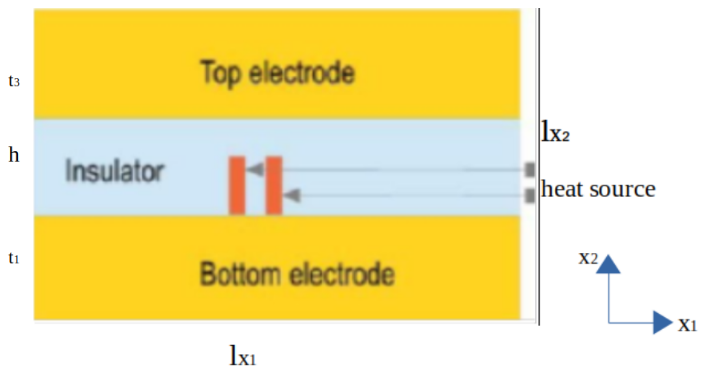

Figure 1.2: Schematic profile view

Assume the device occupies a domain $\Omega \subset \mathbb{R}^2$ and $T(x)$ be the temperature distribution of the device at a point $x = (x_1, x_2) \in \Omega$ and $\kappa(x)$ be its thermal conductivity. The heat equation (1.1) is written as;

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\kappa(x)\frac{\partial T(x)}{\partial x_i}\right) = S(x) \tag{1.2}$$

where $S(x)$ is given heat source at a position x. We assume that there is no heat source in the two electrodes, say $S = s_1 = 0$ and $S = s_3 = 0$, while the heat source in the insulator known to be $S = s_2$. We also impose zero temperature boundary condition at the outer edges of each layer, i.e., $T(x) = 0$. Let, $\Omega = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where,

$$\begin{aligned}
\overline{\Omega_1} &= \{x:\ 0 \leq x_1 \leq l_{x_1},\ 0 \leq t_1\} \\
\overline{\Omega_2} &= \{x:\ 0\ \leq x_1 \leq l_{x_1},\ t_1\ \leq t_1 + h\} \\
\overline{\Omega_3} &= \{x:\ 0 \leq x_1 \leq l_{x_1},\ t_1 + h \leq l_{x_2}\}
\end{aligned}$$

Therefore, we state the boundary value problem as follows: Find the temperature distribution $T(x)$ of the resistive switching device at a position $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ satisfying;

$$\begin{aligned}
-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\kappa(x)\frac{\partial T(x)}{\partial x_i}\right) &= S(x) \text{ in } \Omega \\
T(x) &= 0 \quad \text{on } \partial\Omega
\end{aligned} \tag{1.3}$$

Where $S(x)$ the heat source is defined as,

$$S(x) = \begin{cases} 0, & \text{for the top electrode,} \\ s_2, & \text{for the insulator,} \\ 0, & \text{for the bottom electrode.} \end{cases}$$

and since, the device is made up of different materials, we will have different thermal conductivities κ depending on subdomain of Ω. Hence, we describe $\kappa(x)$ as follows.

$$\kappa(x) = \begin{cases} k_1, & \text{for the top electrode,} \\ k_2, & \text{for the insulator,} \\ k_3, & \text{for the bottom electrode.} \end{cases}$$

We will compute and simulate the heat distribution of the resisitive switching device using boundary element method. We will do so in such a way that temperature and the heat flux are continuous at the contact points of each layers, because of the continuity of the temperature and the flux at the interface of the subdomain.

This study is intended to answer the following basic questions:

1. How can we solve temperature distribution in resistive switching devices with variable thermal conductivity using boundary element method?

2. What is the type of algebric system of equations that resulted from the discretization of the BIEs?

3. What is the solution of heat equation in a metal-insulator-metal structure?

1.3 Objective of the Study

1.3.1 General Objective of the Study

The general objective of this study is to apply the boundary element method to compute the heat distribution of a resistive switching device with a variable thermal conductivity.

1.3.2 Specific Objectives of the Study

The specific objectives of this study are to:

1. formulate Boundary integral equation to heat equation in each layer of a metal-insulator-metal structure by appling Green's second identity.

2. descritize the formulated boundary integral equation and obtain the system of equation for the descritized problem.

3. determine the solutions of heat equation in a metal-insulator-metal structure by BEM.

4. obtain the numerical solution of the problem and graphically show the result.

1.4 Significant of the Study

This study has the following importance;

- The process of this research great contribution for the future engineering research work.

- This study serves the researchers who have an interest to conduct further study on the topic or related topic.

Chapter 2

Literature Review

In this section, we review different study about boundary element method and the related methods that are directly related to the objectives of this study. Until the beginning of the eighth, the BEM was known as boundary integral equation method.

In different times, many researchers developed different BIEs for different potential problem. The BIE was originated by the work of G.Green [24]for solving problems of mathematical physics. He formulated the integral representation of the solution for the Dirichlet and Neumann problems of the Laplace equation by introducing the so called Green's function for these problems. Betti [25] presented a general method for integrating the equations of elasticity and deriving their solution in integral form. Katsikadelis, J. T.[10] used boundary integral equation method to analyze clamped plates on elastic foundation. In 1885, Somagliana [26] used Betti's reciprocal theorem to derive the integral representation of the solution for the elasticity problem, including in its expression the body forces, the boundary displacements and the tractions. David L. Clements, M. Haselgrove [18] used the boundary integral equation method for the solution of problems involving elastic slabs.

In the mid eighteenth century, physical problems such as the conduction patterns of heat and the study of vibrations led to the study of boundary integral equation. The theory behind the boundary integral equations was developed by Fredholm [12]. He established solution to potential problems on the basis of limiting descritization procedure and identified the Fredhlom integral equations. While applications of BIE in potential theory was developed by Mazya V.G.[13]. In the BEM the concept of discretization is carried on the boundary of the domain of interest. In addition, the idea of fundamental solutions is also used in the formulation of the BEM. Multiple efforts for the BEM started around 1962. A turning point that set up a series of connected efforts, which soon developed into a movement can be traced back to 1967.

The BEM was originated with C.A.Brebbia et al. [8] at Southampton University in England and has been used across the entire spectrum of engineering science (such as heat flows, wave propagation, electromagnetic theory, fluid mechanics and etc) that can be ascertained by perusing through books on BEM that have appeared in recent years. It is a technique that uses discretization of the boundary called elements.

The boundary element method used to solve a problem formulated in terms of integral equations relating boundary values numerically. If values of interior domain are needed, they can be calculated from the data of the boundary. Only in certain cases (example, non linearity, inhomogeneity certain types of body forces) does the BEM require discretization of the interior domain. There are two basic categories BEM, namely indirect and direct. In the former case which is also known as source method, the boundary integral formulation is accomplished in terms of fictitious source density from which the physical quantities of the problem can be recovered. In the latter case, Manolis,G.D.[13] sate the boundary integral formulation is a accomplished in terms of physical quantities such as potential, fluxes, etc., which are computed directly without use of intermediate steps.

Jawson and Ponter[33], presented a solution algorithm for the potential equation and developed a numerical technique to solve the boundary integral equation for the classical Saint-Venant torsion problem of non circular bars. In the last 25 years, the number of publications on BEM solutions for the potential equations has increased enormously. Hassan Ghassemi, Saied Panahi, Ahmad Reza Kohansal[34] wrote a journal on solving the Laplace's equation by the BEM using Mixed Boundary Conditions.

Ashok T. Ramu and Dmitri B. Strokov [3] apply the Fourier series solution of the heat in a composite device. They compute the temperature distribution in the devices using Fourier series solution method.

From the above literature survey it is very relevant to point out that no attempt has been made to study the boundary element method on the domain of multi subdomains. This thesis shows a scheme that find the solution of heat conduction in multi subdomains device using the BEM. The BEM solves multi subdomains the device problem, which is difficult to solve with analytical method.This study contribute to improve our understanding how to transform the BVP to the BIE and showing equivalence of BVP with BIE. At the present, the use of partial differential equations and integral equations are commonly encountered in the fields of science and engineering. This issue has addressed some efficient computational tools, recent trends and developments regarding the numerical methods such as BEM for the solutions of partial differential equations and integral equations arising in physical models.

Chapter 3

Mathematical Preliminaries

In this chapter, we pick out some mathematical concepts, definitions, notations and useful formulas that are required for the development and understanding of the BIE. Also these relations could placed here to show the reader their important role in the theoretical foundation and development of the BEM. They will be used for the transformation of the differential equations, which govern the response of physical systems within a domain, into integral equations on the boundary. The concepts are found in references [8, 10, 17, 20, 27, 29]. Since we work in two dimension, we emphasize the theories in $\mathbb{R}^2$.

3.1 Partial Differential Equation

A PDE is an equation which includes derivatives of an unknown function with respect to two or more independent variables. PDE's describe the behavior of many engineering phenomena such as: Heat flow and distribution, fluid flow (air or liquid), vibration etc.

Definition 3.1.1. *[13] Let $u = u(x_1 \cdots x_n)$ be a function of n independent variables $x_1 \cdots x_n$. Then a PDE is an equation that contains the independent variables $x_1 \cdots x_n$, the dependent variables or the unknown function u and its partial derivatives up to some order. It has the form:*

$$F\left(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_1}, \cdots, u_{x_ix_j}, \cdots\right) = 0$$

where,$u_{x_j} = \dfrac{\partial u}{\partial x_j}, u_{x_ix_j} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$ *for,* $i, j = 1, 2, 3 \cdots, n$ *are partial derivatives of u.*

3.1.1 Boundary value problem

In the field of differential equations, the boundary value problem is a differential equation together with a set of additional constraints called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Definition 3.1.2. ***(Boundary value problem)[14]*** *It is a problem, in which the differential equation has values assigned on the physical boundary of the domain in which the problem is specified. For example,*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla^2 u = f \quad \text{in } \Omega \\ u(0,t) = u_1 \quad \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial t}(0,t) = u_2 \quad \text{on } \partial\Omega \end{cases}$$

3.2 Dirac Delta Function

The Dirac delta function is important in determining the fundamental solution. The fundamental solution is the solution of the governing differential equation due to a concentrated excitation such as a point force in solid and fluid mechanics, a point charge in electrostatics. The mathematical representation of this concentrated force can be achieved with the aid of the Dirac delta function $\delta(x)$.

Definition 3.2.1. *[27] The symbol $\delta(x)$ is the Dirac delta function of x in 2D for $x \in \Omega \subset \mathbb{R}^2$ is defined by:*

$$\delta(x) = \begin{cases} \infty, & for\, x = 0 \\ 0, & for\, x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\, dx = \begin{cases} 1, & for\, x \in \Omega \\ 0, & for\, x \notin \Omega \end{cases}$$

The Dirac delta function $\delta(x,y)$ in two dimensions has the following shifting properties.

$$\begin{aligned} \int_{\Omega} f(y)\delta(x,y)d\Omega(y) &= \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases} \\ \int_{\Omega} f(y)\frac{\partial}{\partial x_i}\delta(x,y)d\Omega(y) &= \begin{cases} -\dfrac{\partial}{\partial x_i} f(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases} \end{aligned} \tag{3.1}$$

in which x and y represent two points in space and $f(x) = f(x_i)$ is a differentiable function.

3.3 Gauss Theorem

The Gauss theorem is important formula we need in the development of BIE formulations.

Definition 3.3.1. *[20] For a closed domain Ω in two dimensions with boundary Γ,the Gauss theorem states that:*

$$\int_{\Omega} \nabla \overrightarrow{F}\, d\Omega = \int_{\Gamma} \overrightarrow{F}.n_y\, d\Gamma$$

For any differentiable function $F(x_i)$, where n_i is the component of the outward normal.

Definition 3.3.2. Divergence[20]. *The divergence of a function $F(x_1,x_2)$ is given by:*

$$divF = \Delta.F = \frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2}$$

3.4 Green's identities

Using the Gauss theorem, we can establish the following Green's identities [23]. Consider the function u and w which are twice continuously differentiable in a domain Ω and once differentiable on the boundary Γ. Then,

Theorem 3.4.1. ***Green's first identity***
Let $\Omega \in \mathbb{R}^2$ be a domain bounded by the closed smooth contour Γ and let n_y denote the unit normal vector to the boundary Γ directed into the exterior of Γ and corresponding to a point y

$\in \Gamma$. *Then, for every function u which is once continuously differentiable in the closed domain* $\overline{\Omega} = \Omega \cup \Gamma$, $u \in C^1(\Omega)$ *and every function w which is twice continuously differentiable in* $\overline{\Omega}$, $w \in C^2(\Omega)$, *then*

$$\int_\Omega u\Delta w \, d\Omega = -\int_\Omega \nabla u.\nabla w \, d\Omega + \int_\Gamma u\frac{\partial w}{\partial n_y} \, d\Gamma_y \tag{3.2}$$

Proof. By applying Gauss theorem,

$$\int_\Omega \nabla \overrightarrow{F} \, d\Omega = \int_\Gamma \overrightarrow{F}.n_y \, d\Gamma_y$$

to the vector function $\overrightarrow{F} = (F_1, F_2) \in C^1(\overline{\Omega})$ (with the components) F_1, F_2 being once continuously differentiable in the $\overline{\Omega} = \Omega \cup \Gamma$ closed domain , $F_1, F_2 \in C^2(\overline{\Omega})$ defined by

$$\overrightarrow{F} = u\nabla w = u\left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\right) = (F_1, F_2)$$

$$\text{the components are,} \quad F_1 = \frac{\partial w}{\partial x_1}, F_2 = \frac{\partial w}{\partial x_2}$$

Then,

$$\begin{aligned}
\nabla\left(\overrightarrow{F}\right) = \nabla(u\nabla w) &= \frac{\partial}{\partial x_1}\left(u\frac{\partial w}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(u\frac{\partial w}{\partial x_2}\right) \\
&= \frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} + u\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_2} + u\frac{\partial^2 w}{\partial x_2^2} \\
&= \frac{\partial u}{\partial x_1}\frac{\partial w}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial w}{\partial x_2} + u\left(u\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2}\right) \\
&= \nabla u \nabla w + u\Delta w
\end{aligned} \tag{3.3}$$

On the other hand, according to the definition of the normal derivative,

$$n_y.\overrightarrow{F} = n_y.(u\nabla w) = u(n_y.\nabla w) = u\frac{\partial w}{\partial n_y} \tag{3.4}$$

so that finally,

$$\int_\Omega \nabla \overrightarrow{F} d\Omega = \int_\Omega \nabla(u\nabla w + u\Delta w) d\Omega$$

and from equation (3.3) and using equation (3.4), we have,

$$\int_\Gamma u\frac{\partial w}{\partial n_y} d\Gamma_y = \int_\Gamma \overrightarrow{F}.n_y \, d\Gamma_y$$

which proves,

$$\int_\Omega u\Delta w \, d\Omega = -\int_\Omega \nabla u.\nabla w \, d\Omega + \int_\Gamma u\frac{\partial w}{\partial n_y} \, d\Gamma_y$$

□

Theorem 3.4.2. ***Green's second identity,***
Consider the function $u \in C^2(\overline{\Omega})$ *and* $w \in C^2(\overline{\Omega})$ *and once differentiable on* Γ. *Then,*

$$\int_\Omega (u\Delta w - w\Delta u) \, d\Omega = \int_\Gamma \left(u\frac{\partial w}{\partial n_y} - w\frac{\partial u}{\partial n_y}\right) d\Gamma_y \tag{3.5}$$

Proof. By interchanging the role of u and w in the Green's first identity (3.2),

$$\begin{aligned}\int_\Omega u\Delta w\, d\Omega &= -\int_\Omega \nabla u.\nabla w\, d\Omega + \int_\Gamma u\frac{\partial w}{\partial n_y}\, d\Gamma_y \quad ....(i)\\ \int_\Omega w\Delta u\, d\Omega &= -\int_\Omega \nabla w.\nabla u\, d\Omega + \int_\Gamma w\frac{\partial u}{\partial n_y}\, d\Gamma_y \quad ....(ii)\end{aligned}$$

Then by subtracting (ii) from (i) we obtain,

$$\int_\Omega (u\Delta w - w\Delta u)\, d\Omega = \int_\Gamma \left(u\frac{\partial w}{\partial n_y} - w\frac{\partial u}{\partial n_y}\right) d\Gamma_y$$

□

3.5 Fundamental solution

The fundamental solution for the problem below,

$$-\nabla^2 u(x,y) = f(x,y), \quad \forall\, x,y \in \mathbb{R}^2 \tag{3.6}$$

is the solution when the Dirac delta is acting as a forcing term. It is the particular solution of the problem corresponding to the Dirac delta distribution.

Definition 3.5.1. *The fundamental solution $G(x,y)$ for problems satisfies*

$$-\nabla^2 G(x,y) = \delta(x,y), \quad \forall\, x,\, y \in \mathbb{R}^2\ [27] \tag{3.7}$$

Where the $\delta(x,y)$ denotes a unit source concentrated at the point x.

No function has this property in fact it is a distribution rather than a function, but it can be thought of as a limit of function whose integrals over space are unity, and whose support (the region where the function is nonzero) shrinks to a point. Now, the fundamental solution of the equation(3.6), for $f = 0$ is given by,

$$G(x,y) = \frac{1}{2\pi}\ln\left[\frac{1}{r(x,y)}\right] \tag{3.8}$$

Where $r(x,y)$ is the distance between x and y, given by

$$r(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

To derive this fundamental solution G(x,y) above we see for the "radial" solution in $\mathbb{R}^2$ such that $u(x) = G(|x|)$, $|x| \ = \ r$=$\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For $i = 1,2,...,n$
Then,

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}\ (x \neq\ 0)$$

We thus have,

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= G'(r)\frac{x_i}{r} \\ \frac{\partial^2 u}{\partial x_i x_i} &= G''(r)\frac{x_i^2}{r^2} + G'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \end{aligned} \tag{3.9}$$

Then, take sum over equation (3.9),

$$\sum_1^n \frac{\partial^2 u}{\partial x_i x_i} = \sum_1^n G''(r)\frac{x_i^2}{r^2} + \sum_1^n G'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

We obtain,

$$\nabla^2 u = G''(r) + \frac{n-1}{r}G'(r)$$

Hence, $\nabla^2 u$=0 if and only if

$$G''(r) + \frac{n-1}{r}G'(r) = 0$$

If $G' \neq 0$, we deduce

$$\begin{aligned} \frac{G''}{G'} &= \frac{1-n}{r} \\ \ln(G') &= (1-n)\ln r \end{aligned}$$

For n=2(two dimension),

$$G = A\ln r + B$$

where A and B are arbitrary constants. Since we are looking for a particular solution, we may set $B = 0$. The other constant A, may be determined as follows. Due to the axisymmetric nature of the problem, it is

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r} = A\ln r \quad \text{and} \quad d\Gamma = r d\theta$$

Application of Green's second identity for $u = 1$ and $v = A\ln r$, yields

$$\int_\Omega \Delta v \, d\Omega = \int_\Gamma u\frac{\partial v}{\partial \nu} \, d\Gamma$$

Where Ω is a circle with center at point x and radius r. So the above relation can be written as,

$$\begin{aligned} -\int_\Omega \Delta\delta(x,y) \, d\Omega &= \int_0^{2\pi} A \, r\frac{1}{r}d\theta \\ \text{Then,} \quad -1 &= 2\pi A \\ \implies A &= -\frac{1}{2\pi} \end{aligned}$$

Hence, the fundamental solution becomes,

$$G = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$$

3.6 Representation Formula

To derive the BIE corresponding to the equation (3.10), we apply the second Green's identity. In this equation f is an arbitrary source defined in Ω.

$$-\nabla^2 u = f \quad \text{in } \Omega \tag{3.10}$$

The function u is determined as a particular solution of equation (3.10). Using the fundamental solution $G(x,y)$ of equation (3.7) and Green's second identity we have the following,

$$\int_\Omega G(x,y)\nabla^2 u(y) d\Omega - \int_\Omega u(y)\nabla^2 G(x,y)\, d\Omega \;=\; \int_\Gamma \left[G(x,y)\frac{\partial u(y)}{dn} - u(y)\frac{\partial G(x,y)}{dn}\right] d\Gamma \tag{3.11}$$

Then using the definition of the fundamental solution in equation(3.1) and rearranging the equation(3.11) we have,

$$\int_\Omega u(y)\delta(x,y)d\Omega \;=\; \int_\Gamma \left[G(x,y)\frac{\partial u(y)}{dn} - u(y)\frac{\partial G(x,y)}{dn}\right] d\Gamma + \int_\Omega G(x,y)f(y)d\Omega \tag{3.12}$$

Since, $\int_\Omega u(y)\delta(x,y)\, d\Omega \;=\; u(x)$ from the shifting property of the Dirac delta function, we have the following representation formula;

$$u(x) = \int_\Gamma \left[G(x,y)\frac{\partial u(y)}{dn} - u(y)\frac{\partial G(x,y)}{dn}\right] d\Gamma + \int_\Omega G(x,y)f(y)d\Omega,\ \forall x \in \Omega. \tag{3.13}$$

Now represent, $q \;=\; \dfrac{\partial u}{\partial n}$ and $F(x,y) = \dfrac{\partial G(x,y)}{\partial n}$ and substitute into equation (3.13) then,

$$u(x) = \int_\Gamma G(x,y)q(y)\, d\Gamma - \int_\Gamma u(y)F(x,y)\, d\Gamma + \int_\Omega G(x,y)f(y)d\Omega,\ \forall x \in \Omega. \tag{3.14}$$

Equation (3.14) is the representation integral for equation (3.10). Once the boundary values of both u and q are known on Γ, equation (3.14) can be applied to calculate u everywhere in Ω, if needed.

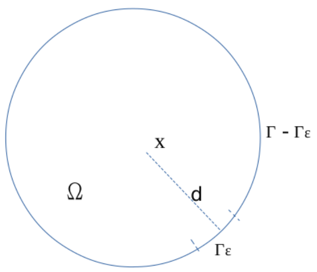

Figure 3.1: Limits as x approaches boundary Γ.

To solve the boundary values of u and q on Γ, we let x tend to Γ to obtain a BIE from equation(3.14). To do this, we consider the following limit:

$$\lim_{x\to\Gamma} u(x) = \lim_{x\to\Gamma}\left\{\int_\Gamma G(x,y)\; q(y)d\Gamma - \int_\Gamma u(y)F(x,y)\, d\Gamma + \int_\Omega G(x,y)f(y)d\Omega\right\}, \quad \forall x \in \Omega. \tag{3.15}$$

Consideration of the limit process is necessary for each integral on the right hand side of equation(3.15). We first divide the boundary Γ into two parts: $\Gamma - \Gamma_\varepsilon$ and Γ_ε , where Γ_ε is a small segment with length 2ε centered around the point to which x will approach (figure 3.1). The first integral on the righthand side of Eq(3.15) is evaluated as:

$$\lim_{x\to\Gamma}\int_\Gamma G(x,y)q(y)\,d\Gamma(y) = \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon} G(x,y)q(y)\,d\Gamma(y) + \lim_{\substack{d\to 0\\ \varepsilon\to 0}}\int_{\Gamma_\varepsilon} G(x,y)q(y)\,d\Gamma(y_\varepsilon)$$

where y_ε is a point on Γ_ε . When ε is small, Γ_ε can be regarded as a straight line segment (assuming Γ is smooth). The integration on the line segment Γ_ε shown in figure below from point 1 to point 2 can be expressed as,

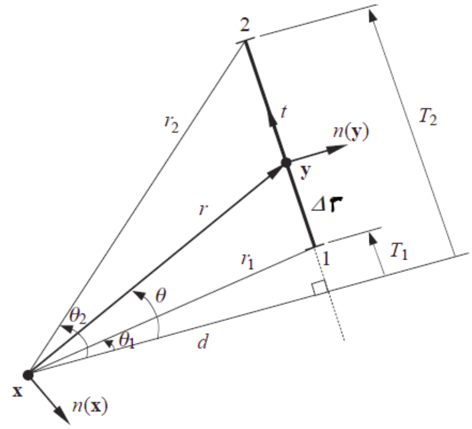

Figure 3.2: Analytical integration on an arbitrary line segment.[27]

It can be evaluated analytically as follows. Note that on Γ_ε, $r = \frac{d}{\cos\theta}$, $d\Gamma_\varepsilon = \frac{r\,d\theta}{\cos\theta}$

$$\int_{\Gamma_\varepsilon} G(x,y)d\Gamma = \frac{1}{2\pi}\left[-(\theta_2-\theta_1)\,d + 2R - T_2\log r_2 + T_1\log r_1\right] \tag{3.16}$$

$$\int_{\Gamma_\varepsilon} F(x,y)d\Gamma = \frac{1}{2\pi}\left[-(\theta_2-\theta_1)\right] \tag{3.17}$$

These results can be used to evaluate directly the coefficients of the BIE for 2D potential problems using constant elements. If the source point x is on the element of integration (at the midpoint), we have:

$$\theta_2-\theta_1 = \pi,\ d = 0,\ r_1 = r_2 = R,\ T_1 = -T_2 = -R$$

From equation (3.16), we obtain,

$$\lim_{\substack{d\to 0\\ \varepsilon\to 0}}\int_{\Gamma_\varepsilon} G(x,y)q(y)\,d\Gamma(y_\varepsilon) = 0$$

Therefore,

$$\lim_{x\to\Gamma}\int_\Gamma G(x,y)q(y)\,d\Gamma(y) = \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon} G(x,y)q(y)\,d\Gamma(y) = \int_\Gamma G(x,y)q(y)\,d\Gamma(y) \tag{3.18}$$

Similarly, the second integral on the right hand side of equation (3.15) is evaluated as:

$$\lim_{x\to\Gamma}\int_\Gamma F(x,y)u(y)\,d\Gamma(y) = \lim_{\varepsilon\to 0}\int_{\Gamma-\Gamma_\varepsilon} F(x,y)u(y)\,d\Gamma(y) + \lim_{\substack{d\to 0\\ \varepsilon\to 0}}\int_{\Gamma_\varepsilon} F(x,y)u(y)\,d\Gamma(y_\varepsilon)$$

Applying the equation (3.17), we obtain:

$$\lim_{\substack{d\to 0\\ \varepsilon\to 0}} \int_{\Gamma_\varepsilon} F(x,y)u(y)\, d\Gamma(y_\varepsilon) = -\frac{1}{2}u(x), \quad x \in \Gamma.$$

Therefore,

$$\begin{aligned}\lim_{x\to\Gamma} \int_{\Gamma} F(x,y)u(y)\, d\Gamma(y) &= \lim_{\varepsilon\to 0} \int_{\Gamma-\Gamma_\varepsilon} F(x,y)u(y)\, d\Gamma(y) - \frac{1}{2}u(x) \\ &= \int_{\Gamma} F(x,y)u(y)\, d\Gamma(y) - \frac{1}{2}u(x), \quad x\in\Gamma\end{aligned}$$

Again the third integral on the right hand side of equation (3.15) is:

$$\lim_{x\to\Gamma} \int_{\Omega} G(x,y)f(y)d\Omega = \int_{\Omega} G(x,y)f(y)d\Omega, \quad \forall x \in \Omega. \tag{3.19}$$

Substituting equations (3.18)-(3.19) into equation (3.15) and combining the free terms we obtain the following BIE,

$$c(x)u(x) = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)u(y)d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega \tag{3.20}$$

where,

$$\begin{aligned} G(x,y) &= \frac{1}{2\pi}\ln\frac{1}{r(x,y)} \\ c(x)(\text{free term coefficient}) &= \begin{cases} 1 & \text{if } x \in \Omega \\ \frac{1}{2} & \text{if } x \in \partial\,\Omega \text{ and } \partial\Omega \text{ is smooth at x.} \\ 0 & \text{if } x \notin \Omega \end{cases}\end{aligned}$$

The equation (3.20) is called boundary integral equation of the PDE in equation (3.10).

3.7 Gaussian quadrature

The Gaussian integration is performed over the interval $-1 \le \xi \le 1$,

$$\int_{-1}^{1} f(\xi) = \sum_{k=1}^{n} \varpi_k f(\xi_k)$$

where n is the number of Gauss points, and ξ_k and ϖ_k $(k = 1,2,...,n)$ are the abscissas and weights of the Gaussian quadrature of order n respectively. For n Gauss points are chosen, so that its values and the corresponding weights can be obtained using Legendre's orthogonal polynomials. It is approximating a non-polynomial function $f(\xi)$ by a polynomial of degree $2n-1$. The Legendre's polynomials and weights are defined as,

$$p_n(\xi) = \frac{d^n\left(\xi^2-1\right)}{d\xi^n\, 2^n n!} \quad \text{and} \quad \varpi_k = \frac{2\left(1-\xi_k\right)^2}{n^2\left[p_{n-1}(\xi_k)\right]^2}$$

such that the coordinates ξ_k of the integration points are the zeros of these polynomials. Example, for two gauss points $(n = 2)$;

$$f(\xi) = p_2(\xi) = \frac{1}{2}\left(3\xi^2-1\right) = 0$$

This implies $\xi_1 = -0.577$ and $\xi_2 = 0.577$ and the corresponding weights $\varpi_1 = \varpi_2 = 1$. Similarly, we can determine the value of the point of integration and weights of the Gaussian quadrature of order n[20].

3.8 Boundary Element method

We apply the boundary elements to discretize the BIEs into a finite number of elements, those are not necessarily equal. It is used to solve the the BIEs numerically for the unknown boundary variables. Two approximations are made over each of these elements. One is about the geometry (line segment or parabolic arc) of the boundary, while the other has to do with the variation of the unknown boundary quantity (changed or varies) over the element[10]. In the BEM, the equation (3.10) is written in a BIE form in equation (3.20) as;

$$c(x)\;u(x) = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)u(y)d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega. \tag{3.21}$$

where,

$$c(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ \frac{1}{2} & \text{if } x \in \partial\,\Omega \text{ and } \partial\Omega \text{ is smooth at x.} \\ 0 & \text{if } x \notin \Omega \end{cases}$$

$G(x,y)$ is a fundamental solution, u is the potential and $q(y)$is its derivative in the normal direction n to the boundary surface denoted by Γ. Now, the BIE in equation(3.21) for $x \in \Omega$, becomes

$$u(x) = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)u(y)d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega. \tag{3.22}$$

If we move the interior point x to the boundary point ($x \in \Gamma$), we have,

$$\frac{u(x)}{2} = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)u(y)d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega. \tag{3.23}$$

3.8.1 Discretization of the Boundary Integral Equations

The usually employed boundary elements to discretize the BIE are the constant element, linear element and the quadratic element[27]. This elements are discussed as follows,

- Constant elements
  In the constant elements, we divide the boundary Γ into line segments (elements) and place one node at the middle of each element (figure 3.3). In this case the total number of elements and nodes are equal. The potential at boundary is assumed to be constant along the element and equal to its value at the nodal point (figure 3.3). The discretization of the BIEs using constant elements is straightforward[10].

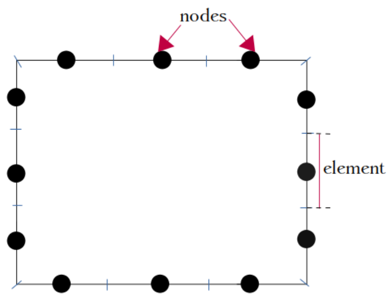

Figure 3.3: constant element.

- Linear elements
  In this case, The boundary segment is again approximated by a straight line connecting its extreme points. It has two nodal points usually placed at the end points. The potential at the boundary is assumed to vary linearly between the nodal points (fig 3.4). In this boundary element the analytical integrations of the coefficients are no longer available[8].

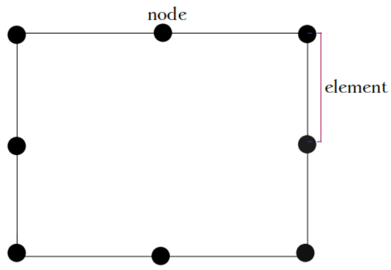

Figure 3.4: linear element.

- Quadratic element
  In the quadratic element, the geometry is approximated by a parabolic arc. The element in this case has three nodal points, two of which are placed at the end and the third some where in between, usually at the midpoint (fig 3.5). The quadratic elements used for curved boundaries, because of the more accurate representation of the geometry. However, the use of quadratic elements also presents some challenges. Analytical integrations of the coefficients are no longer available.

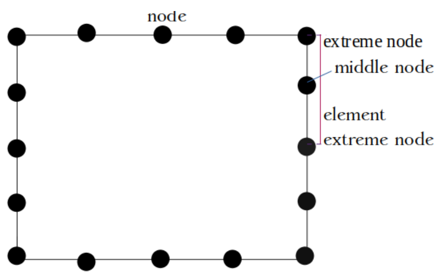

Figure 3.5: quadratic element

Now, we use the constant element to discretize the boundary of the given domain of the problem. Thus, the discretized form of Equation (3.23) is expressed for a given point x on Γ is given as;

$$\frac{u_i}{2} = \sum_{j=1}^{N}\int_{\Gamma_j} Gq\, d\Gamma - \sum_{j=1}^{N}\int_{\Gamma_j} Fu\, d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega$$

Since, the values of the boundary quantity u_j and its normal derivative q_j are constant on a constant element, we have the following equation,

$$\frac{u_i}{2} = \sum_{j=1}^{N} q_j \int_{\Gamma_j} G\, d\Gamma - \sum_{j=1}^{N} u_j \int_{\Gamma_j} F\, d\Gamma + \int_{\Omega} G(x,y)f(y)d\Omega \tag{3.24}$$

Where,

$$G = -\frac{1}{2\pi}|\eta-\xi| \quad \text{and} \quad F = -\frac{(\eta-\xi).n}{2\pi|\eta-\xi|^2} \quad [30]$$

The integrals give the influence of some element j on the fundamental solution that is centered at node i on the boundary in equation (3.24) are need to be approximated as,

$$\hat{F}_{ij} = \int_{\Gamma_j} F\, d\Gamma \text{ and } G_{ij} = \int_{\Gamma_j} G\, d\Gamma \tag{3.25}$$

These integrals give the influence of some element j on the fundamental solution that is centered at node i on the boundary. For a single point i on the boundary,

$$\frac{u_i}{2} + \sum_{j=1}^{N} u_j \hat{F}_{ij} = \sum_{j=1}^{N} q_j G_{ij} + \int_{\Omega} G(x,y) f(y) d\Omega$$

This is done by moving the singular point i around the boundary and evaluating at every node $i = 1,\ldots,N$. Then, we have N equations,

$$\sum_{j=1}^{N} q_j G_{ij} + \int_{\Omega} G(x,y) f(y) d\Omega = \sum_{j=1}^{N} u_j F_{ij} \tag{3.26}$$

where,

$$F_{ij} = \begin{cases} \hat{F}_{ij} & \text{for, } i \neq j \\ \hat{F}_{ij} + \frac{1}{2} & \text{for, } i = j \end{cases} \tag{3.27}$$

The domain integral involving function f in equation (3.26) has the form,

$$\int_{\Omega} f\, G\, d\Omega \tag{3.28}$$

This integral can be performed by subdividing the domain Ω into a series of cells as over which a numerical integration formula can be applied. Here we discretizing the boundary with elements and the domain with cells. Let the domain is discretized into a certain number E' of integration cells $\Omega^{(e')}$ as(fig. 3.6) and subsequently use numerical integration in these cells[30].

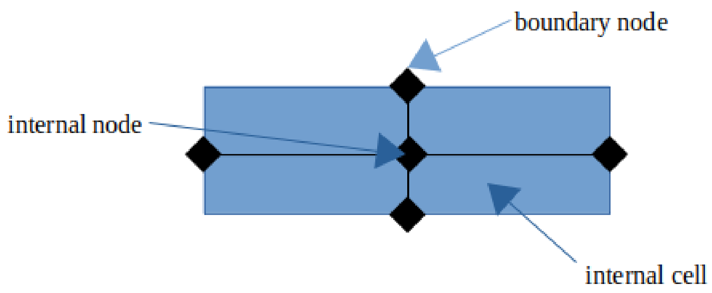

Figure 3.6: Boundary elements and internal cells

The cells of integration can now be employed to compute expression (3.28) as,

$$D_i = \sum_{e'=1}^{E'} \int_{\Omega^{(e')}} f \left(-\frac{1}{2\pi} \ln|\eta_i - \xi_i| \right) d\Omega \tag{3.29}$$

Where i is one of the boundary nodes. Thus, equation (3.26 written in matrix form as follows,)

$$GQ + D_i = Fu$$

Where F and G are $N \times N$ matrices, whose elements are given by equation (3.25) and equation (3.27) and u and Q are vectors of length N. i.e, $u = (u_1, u_2, ...u_n)$, $Q = (q_1, q_2, ..., q_n)$

$$\begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \ddots & G_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1N} \\ F_{21} & F_{22} & \ddots & F_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ F_{N1} & F_{N2} & \cdots & F_{NN} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \tag{3.30}$$

In the BEM approach, we form a standard linear system of equations as follows by using the boundary condition at each node and switching the columns in the two matrices in equation(3.30).

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \ddots & c_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix}, \quad \text{or} \quad \mathbf{C}\lambda = \mathbf{d} \tag{3.31}$$

where C is the coefficient matrix, λ is the unknown vector (with unknown u or q at each node), and d is the known right hand side vector.

By solving equation(3.31), we obtain all the unknown boundary variables on each element. After the surface distributions of u and q are known on boundary in equation(3.23), the value of u and q at any point x inside the domain can be found by using equation(3.22).

But, if the potential problems we consider is in composite domains, we can solve it by applying the BEM separately to each of its subdomains. After we treat them separately, the subdomains may then be combined in a single matrix equation by using the continuity conditions at the interfaces. These conditions express,

- Continuity of the potential. The values of the potential on each side of the interface (assuming perfect bonding) separating two subdomains are equal,

$$\begin{aligned} {u_{In}}^1 &= {u_{In}}^2 \equiv {u^I}^1 \\ {u_{In}}^3 &= {u_{In}}^4 \equiv {u^I}^2 \end{aligned}$$

- Continuity of the flux. The flux q is a quantity related to the physical problem described by the potential problem. The outcoming flux from one subdomain is equal to the incoming flux in the adjacent subdomain. Thus q the flux along the normal to the interface, its continuity across the interface requires:

$$\begin{aligned} {q_{In}}^1 &= -{q_{In}}^2 \equiv {q^I}^1 \\ {q_{In}}^3 &= -{q_{In}}^4 \equiv {q^I}^2 \end{aligned} \tag{3.32}$$

The minus sign in the equation(3.32) is explained by the fact that the positive direction for the flux coincides with the outward normal vector n. This means that the two flux vectors at the common interface of adjacent subdomains are of opposite directions, because their corresponding normal vectors are opposite too[20].

Thus, solving this single matrix equation solves the value of the problem in a subdomains at the boundary of the whole domain. Now, after we obtain the value of the problem on the boundary of each sub domains, we can solve the internal value of the problem of the the subdomains of the device again separately.

Evaluation of Influence Coefficients

The integrals G_{ij} and $\hat{F}_{ij}$ defined in equation (3.24) are evaluated numerically using a standard Gaussian quadrature or analytical integration. Two cases are distinguished for the integrals of the influence coefficients.

Off-diagonal elements, $i \neq j \quad (i = 1, 2, ..., M)$
In this case to determine the value of influence coefficient we consider the element j over which the integration will be carried out in figure (3.7). This element is defined by the coordinates (x_{1j}, x_{2j}) and (x_{1j+1}, x_{2j+1}) of its extreme points. The integrals of the influence coefficients are evaluated numerically in the following way:

- The integral of G_{ij} is evaluated using Gaussian quadrature integration method.

$$G_{ij} = \int_{\Gamma_j} G \, d\Gamma = \int_{-1}^{1} \frac{1}{2\pi} \left[\ln\left(\frac{1}{r(\xi)} \right) \right] d\Gamma$$

  Since,

$$d\Gamma = \sqrt{dx_1{}^2 + dx_2{}^2} = \sqrt{\left(\frac{x_{1j+1} - x_{1j}}{2} \right)^2 + \left(\frac{x_{2j+1} - x_{2j}}{2} \right)^2} \, d\xi = \frac{l_j}{2} d\xi$$

  Now,

$$G_{ij} = \int_{-1}^{1} \frac{1}{2\pi} \left[\ln\left(\frac{1}{r(\xi)} \right) \right] \frac{l_j}{2} d\xi = \frac{l_j}{4\pi} \sum_{k=1}^{2} \left[\ln\left(\frac{1}{r(\xi_k)} \right) \right] \varpi_k$$

  Therefore,

$$G_{ij} = \frac{l_j}{4\pi} \left[\ln\left(\frac{1}{r(\xi_1)} \right) \varpi_1 + \ln\left(\frac{1}{r(\xi_2)} \right) \varpi_2 \right]$$

  Where,

$$l_j = \sqrt{(x_{1j+1} - x_{1j})^2 + (x_{2j+1} - x_{2j})^2}$$

$$r(\xi_k) = \sqrt{[x_1(\xi_k) - x_{1i}]^2 + [x_2(\xi_k) - x_{2i}]^2}$$

  Which obtained by coordinate transformations as,

$$\begin{aligned} x_1(\xi_k) &= \frac{x_{1j+1} + x_{1j}}{2} + \frac{x_{1j+1} - x_{1j}}{2} \xi_k \\ x_2(\xi_k) &= \frac{x_{2j+1} + x_{2j}}{2} + \frac{x_{2j+1} - x_{2j}}{2} \xi_k \end{aligned}$$

- The integral of $\hat{F}_{ij}$ can also be evaluated analytically from Fig (3.7)

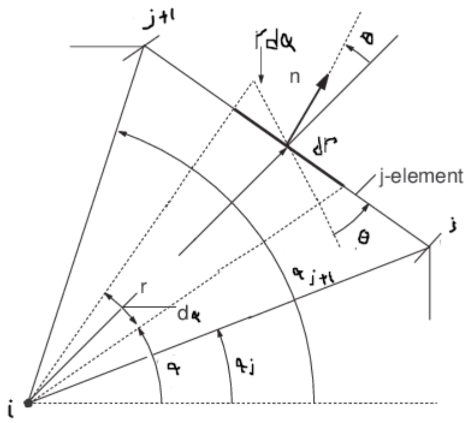

Figure 3.7: Definition of angles involved in the numerical integration over constant elements.

From figure(3.7), we have

$$d\Gamma\cos\theta = r\,d\alpha \tag{3.33}$$

This can be used to derive the expression,

$$\hat{F_{ij}} = \int_{\Gamma_j}\frac{\partial G}{\partial n}\,d\Gamma = \int_{\Gamma_j}\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)\,d\Gamma = -\int_{\Gamma_j}\frac{1}{2\pi}\frac{\partial}{\partial n}\ln(r)\;d\Gamma = -\int_{\Gamma_j}\frac{1}{2\pi r}\frac{\partial}{\partial n}r\,d\Gamma \tag{3.34}$$

Now we introduce the relation which facilitate the differentiation in equation (3.34) of the integral equations. Let $x(x_1,x_2)$ points inside the domain Ω, while $y(\xi,\eta)$ is point on the boundary. The angle between the x_1-axis and the vector $\mathbf{r}$ is α and the angle between the x_1-axis and the unit vector $\mathbf{n}$ normal to the boundary at point y by β. Using these two angles, we also define the angle θ as

$$\theta = \text{angle}(\mathbf{r},\mathbf{n}) = \beta - \alpha$$

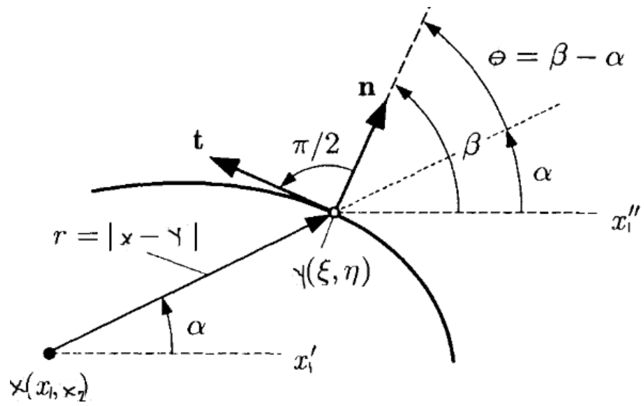

Figure 3.8: Geometric definitions of relative position of a field point x and a boundary point y.[10]

From figure (3.8)[10],

$$\cos\alpha = \frac{\xi - x_1}{r}$$

$$\sin\alpha = \frac{\eta - x_2}{r}$$

where

$$r=\sqrt{(\xi-x_1)^2+(\eta-x_2)^2} \tag{3.35}$$

Differentiation of equation(3.35) yields,

$$\frac{\partial r}{\partial x_1}=-\frac{\partial r}{\partial \xi}=-\frac{x_1-\xi}{r}=-\cos\alpha \tag{3.36}$$

$$\frac{\partial r}{\partial x_2}=-\frac{\partial r}{\partial \eta}=-\frac{x_2-\eta}{r}=-\sin\alpha \tag{3.37}$$

Noting that,

$$\cos\beta=\frac{\partial n}{\partial x_1}=n_{x_1} \quad \text{and} \quad \cos\beta=\frac{\partial n}{\partial x_2}=n_{x_2} \tag{3.38}$$

We can derive the following expression for the derivative of r.

$$\begin{aligned}
\frac{\partial r}{\partial n} &= n_{x_1}\frac{\partial r}{\partial \xi}+n_{x_2}\frac{\partial r}{\partial \eta} \\
&= \cos\beta\frac{\partial r}{\partial \xi}+\sin\beta\frac{\partial r}{\partial \eta}, \quad \text{from eq.(3.38).} \\
&= \cos\beta\cos\alpha+\sin\beta\sin\alpha, \quad \text{from eqs.(3.36) and (3.37).} \\
&= \cos(\beta-\alpha) \\
&= cos\theta
\end{aligned} \tag{3.39}$$

Now, by substituting equations (3.33) and (3.39) into equation (3.34) we obtain,

$$\hat{F_{ij}}=\int_{\Gamma_j}\frac{\partial G}{\partial n}\,d\Gamma=-\int_{\Gamma_j}\frac{1}{2\pi r}\frac{\partial}{\partial n}r\,d\Gamma=-\int_{\Gamma_j}\frac{1}{2\pi}\underbrace{\frac{\cos\theta}{r}\,d\Gamma}=-\int_{\Gamma_j}\frac{1}{2\pi}\,d\alpha=-\left(\frac{\alpha_{j+1}-\alpha_j}{2\pi}\right)$$

Where, the angle α_{j+1} and α_j are computed from

$$\tan\alpha_{j+1}=\frac{x_{2\,j+1}-x_{2i}}{x_{1\,j+1}-x_{1i}} \text{ and } \tan\alpha_j=\frac{x_{2\,j}-x_{2i}}{x_{1\,j}-x_{1i}}$$

where $x_{1\,j+1}$, $x_{2\,j+1}$, $x_{1\,j}$ and $x_{2\,j}$ are extreme points of the j-th element.

Diagonal elements, $i = j \quad (i=1,2,...,M)$
In this case the node i coincides with element j, and r lies on the element. Consequently, it is $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, which yields $\cos\theta=0$.

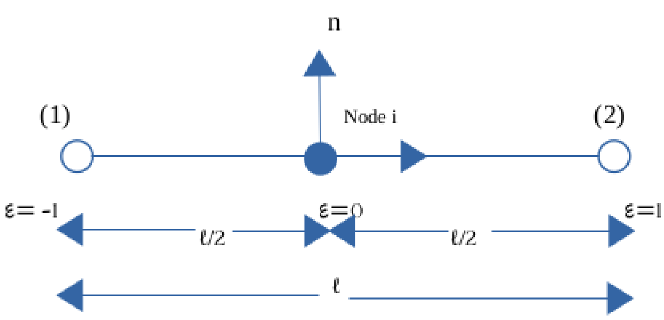

Figure 3.9: Element coordinate system

For the constant elements the $\hat{F}_{jj}$ and G_{jj} integrals can be easily computed analytically. The $\hat{F}_{ii}$ terms is identically zero due to the orthogonality between the normal n and the surface of the element[20] as (fig. (3.9)), i.e.,

$$\hat{F}_{ii} = \int_{\Gamma_j} F d\Gamma = \int_{\Gamma_j} \frac{\partial G}{\partial r}\frac{\partial r}{\partial n} d\Gamma \equiv 0$$

The integrals in G_{ii} requires special handling.

$$G_{jj} = \int_{\Gamma_j} G d\Gamma = \int_{\Gamma_j} \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) \, d\Gamma \tag{3.40}$$

In order to integrate easily equation (3.40) we change the coordinates to the homogeneous coordinate ξ over a segment (fig. (3.9)) such that, $r = \xi|\frac{\ell}{2}|$, where ℓ is the element length.

$$\begin{aligned} G_{jj} &= \int_{-1}^{1} \frac{1}{2\pi} \ln\left(\frac{1}{r}\right) d\Gamma = \int_{0}^{1} \frac{1}{\pi} \ln\left(\frac{1}{r}\right) \, dr \\ &= \frac{1}{2\pi} \int_{0}^{1} \ln\left(\frac{1}{r}\right) \, dr = \frac{1}{2\pi} \int_{0}^{1} \ln\left(\frac{2}{\xi \, \ell}\right) \, d\xi \\ &= \frac{l_j}{2\pi} \left[\ln\left(\frac{2}{\ell}\right) + \int_{0}^{1} \ln\left(\frac{1}{\xi}\right) \, d\xi \right] = G_{ii} \end{aligned} \tag{3.41}$$

The last integral in equation (3.41)is equal to 1,i.e,

$$G_{jj} = \frac{l_j}{2\pi} \left[\ln\left(\frac{2}{l_j}\right) + 1 \right] \tag{3.42}$$

Chapter 4

Analysis of Heat Flow in Resistive Switching Devices with Different Thermal Conductivities

Now, we can solve the temperature distribution of a body having different coefficients of thermal conductivity as shown in figure (4.1) using BEM. The temperature in this domain can be solved by applying the BEM separately to each of its subdomains. The reason is that the fundamental solution is valid only for homogeneous domains. Since the problem defined over a region which is only piecewise homogeneous, the numerical procedures described can be applied to each homogeneous subregion as they were separated from the others [27]. Discretizing the boundaries of these subdomains, and discretization on either side of each interface parts of the boundaries between subdomains.

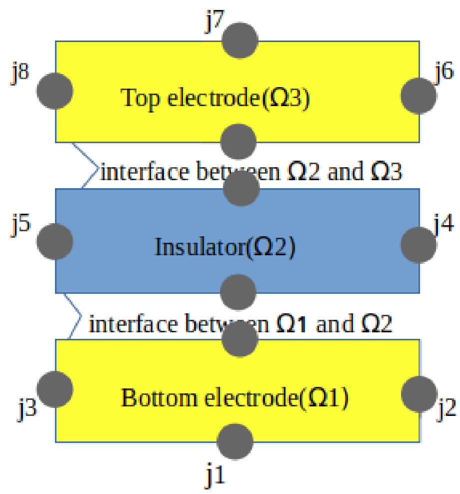

Figure 4.1: Boundary element discretization of composite domain.

Generally, the problem can be treated separately first by the BEM and then assembled together with the interface conditions as follows,
For region (Ω_1), since there is no heat source for the bottom electrode the equation(1.3) be-

comes,

$$\begin{aligned} \Delta T(x) &= 0, \quad x \text{ in } \Omega \\ T(x) &= \overline{T}, \quad x \text{ on } \Gamma \end{aligned} \tag{4.1}$$

The BIE of equation(4.1) from equation(3.22) for $x \in \Omega_1$ is given as;

$$T(x) = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)T(y)d\Gamma \quad \text{as } f = 0 \tag{4.2}$$

The BIE of equation(4.1) from equation(3.23) for $x \in \Gamma$ is also given as;

$$\frac{T(x)}{2} = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)T(y)d\Gamma \tag{4.3}$$

The discretized BIE of equation (4.3) is as follows,

$$\frac{T_i}{2} = \sum_{j=1}^{N} \int_{\Gamma_j} Gqd\Gamma - \sum_{j=1}^{N} \int_{\Gamma_j} FT\, d\Gamma \tag{4.4}$$

Since T_j and q_j are constant on each element, then equation(4.4) becomes,

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j \int_{\Gamma_j} G\, d\Gamma - \sum_{j=1}^{N} T_j \int_{\Gamma_j} F\, d\Gamma \tag{4.5}$$

The integrals in equation (4.5) are need to be approximated as,

$$\hat{F_{ij}} = \int_{\Gamma_j} F\, d\Gamma \text{ and } G_{ij} = \int_{\Gamma_j} G\, d\Gamma$$

Now for a single point i on the boundary;

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j G_{ij} - \sum_{j=1}^{N} T_j \hat{F_{ij}}$$

Now, we have N equations,

$$\sum_{j=1}^{N} T_j F_{ij} = \sum_{j=1}^{N} q_j G_{ij}$$

Where,

$$F_{ij} = \begin{cases} \hat{F_{ij}} & \text{for, } i \neq j \\ \hat{F_{ij}} + \frac{1}{2} & \text{for, } i = j \end{cases}$$

Hence, the matrix equation of the problem given as:

$$\begin{pmatrix} F_{11}^1 & F_{12}^1 & F_{13}^{I^1} & F_{14}^1 \\ F_{21}^1 & F_{22}^1 & F_{23}^{I^1} & F_{24}^1 \\ F_{31}^1 & F_{32}^1 & F_{33}^{I^1} & F_{34}^1 \\ F_{41}^1 & F_{42}^1 & F_{43}^{I^1} & F_{44}^1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_{I^1} \\ T_3 \end{pmatrix} = \begin{pmatrix} G_{11}^1 & G_{12}^1 & G_{13}^{I^1} & G_{14}^1 \\ G_{21}^1 & G_{22}^1 & G_{23}^{I^1} & G_{24}^1 \\ G_{31}^1 & G_{32}^1 & G_{33}^{I^1} & G_{34}^1 \\ G_{41}^1 & G_{42}^1 & G_{43}^{I^1} & G_{44}^1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ {q_I}^1 \\ q_3 \end{pmatrix} \tag{4.6}$$

Where

$T_1,\ T_2,\ T_3,\ q_1,\ q_2,\ q_3$ nodal values of the external boundary of bottom electrode.

${T_I}^1,\ {q_I}^1$ nodal values on the interface of bottom electrode and insulator.

For region (Ω_2), since there is heat source (s_2) for the insulator, the equation(1.3) becomes,

$$\begin{aligned} -k_2 \Delta T(x) &= s_2, \;\; x \text{ in } \Omega \\ T(x) &= \overline{T}, \;\; x \text{ on } \Gamma \end{aligned} \tag{4.7}$$

The BIE of equation(4.7) from equation(3.22) for $x \in \Omega_2$ is given as;

$$T(x) = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma + \int_\Omega G(x,y)f(y)d\Omega. \quad \text{as } f = s_2$$

The BIE of equation(4.7) from equation(3.23) for $x \in \Gamma$ is also given as ;

$$\frac{T(x)}{2} = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma + \int_\Omega G(x,y)f(y)d\Omega \tag{4.8}$$

The discretized form of equation (4.8) is expressed for a given point x on Γ as;

$$\frac{T_i}{2} = \sum_{j=1}^{N} \int_{\Gamma_j} Gq \, d\Gamma - \sum_{j=1}^{N} \int_{\Gamma_j} FT \, d\Gamma + \frac{1}{k_2} \int_{\Omega_2} s_2 \, G \, d\Omega$$

Since T_j and q_j are constant on constant elements, we have the following,

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j \int_{\Gamma_j} G \, d\Gamma - \sum_{j=1}^{N} T_j \int_{\Gamma_j} F \, d\Gamma + \frac{1}{k_2} \int_{\Omega_2} s_2 \, G \, d\Omega \tag{4.9}$$

These two integrals in equation (4.9) are need to be approximated as,

$$\hat{F_{ij}} = \int_{\Gamma_j} F \, d\Gamma \text{ and } G_{ij} = \int_{\Gamma_j} G \, d\Gamma$$

For a single point i on the boundary

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j G_{ij} - \sum_{j=1}^{N} T_j \hat{F_{ij}} + \frac{1}{k_2} \int_{\Omega_2} s_2 \, G \, d\Omega$$

This is also done by moving point i around the boundary and evaluating at every node $i = 1, \ldots, N$. Now we have N equations,

$$\sum_{j=1}^{N} T_j F_{ij} = \sum_{j=1}^{N} q_j G_{ij} + \frac{1}{k_2} \int_{\Omega_2} s_2 \, G \, d\Omega \tag{4.10}$$

Where,

$$F_{ij} = \begin{cases} \hat{F_{ij}} & \text{for, } i \neq j \\ \hat{F_{ij}} + \frac{1}{2} & \text{for, } i = j \end{cases}$$

In this subdomain, a domain integral involving function s_2 in equation(4.10) is evaluated using equation (3.29);

$$D_i = \sum_{e'=1}^{E'} \frac{1}{k_2} \int_{\Omega_2^{(e')}} s_2 \, G \, d\Omega \tag{4.11}$$

D_i is the result different for each position, where i is one of the boundary nodes. Thus, equation (4.10) written in matrix form as follows,

$$FT = GQ + D_i$$

Now, we can write the problem in a matrix form by applying the interface condition in equation(3.32) as;

$$\begin{pmatrix} F_{11}^{I^1} & F_{12}^{2} & F_{13}^{I^2} & F_{14}^{2} \\ F_{21}^{I^1} & F_{22}^{2} & F_{23}^{I^2} & F_{24}^{2} \\ F_{31}^{I^1} & F_{32}^{2} & F_{33}^{I^2} & F_{34}^{2} \\ F_{41}^{I^1} & F_{42}^{2} & F_{43}^{I^2} & F_{44}^{2} \end{pmatrix} \begin{pmatrix} T_I^1 \\ T_4 \\ T_{I^2} \\ T_5 \end{pmatrix} = \begin{pmatrix} -G_{11}^{I^1} & G_{12}^{2} & G_{13}^{I^2} & G_{14}^{2} \\ -G_{21}^{I^1} & G_{22}^{2} & G_{23}^{I^2} & G_{24}^{2} \\ -G_{31}^{I^1} & G_{32}^{2} & G_{33}^{I^2} & G_{34} \\ -G_{41}^{I^1} & G_{42}^{2} & G_{43}^{I^2} & G_{44}^{2} \end{pmatrix} \begin{pmatrix} {q_I}^1 \\ q_4 \\ {q_I}^2 \\ q_5 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} \tag{4.12}$$

Where

$T_4,\ T_5,\ q_4,\ q_5$ nodal values of the external boundary of insulator.

${T_I}^1,\ {T_I}^2,\ {q_I}^1,\ {q_I}^2$ nodal values on the interface of bottom electrode and insulator, and insulator and top electrode.

For region (Ω_3), since there is no heat source for the top electrode, the equation(1.3) becomes,

$$\begin{aligned} \Delta T(x) &= 0, \ \ x \text{ in } \Omega \\ T(x) &= \overline{T}, \ \ x \text{ on } \Gamma \end{aligned} \tag{4.13}$$

The BIE of equation(4.13) from equation (3.22)for $x \in \Omega_3$ is given as;

$$T(x) = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma \quad \text{as } f = 0 \tag{4.14}$$

The BIE of equation(4.13) from equation(3.23) for $x \in \Gamma$ is also given as;

$$\frac{T(x)}{2} = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma \tag{4.15}$$

The discretized BIE of equation (4.15) is as follows,

$$\frac{T_i}{2} \quad = \quad \sum_{j=1}^{N} \int_{\Gamma_j} Gqd\Gamma - \sum_{j=1}^{N} \int_{\Gamma_j} FT\, d\Gamma \tag{4.16}$$

since T_j and q_j are constant on each element, then equation (4.16) becomes,

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j \int_{\Gamma_j} G\, d\Gamma - \sum_{j=1}^{N} T_j \int_{\Gamma_j} F\, d\Gamma \tag{4.17}$$

These two integrals in equation (4.17) are need to be approximated as,

$$\hat{F_{ij}} = \int_{\Gamma_j} F\, d\Gamma \text{ and } G_{ij} = \int_{\Gamma_j} G\, d\Gamma$$

Now for a single point i on the boundary;

$$\frac{T_i}{2} = \sum_{j=1}^{N} q_j G_{ij} - \sum_{j=1}^{N} T_j \hat{F_{ij}}$$

For simplicity define

$$F_{ij} = \begin{cases} \hat{F_{ij}} & \text{for, } i \neq j \\ \hat{F_{ij}} + \frac{1}{2} & \text{for, } i = j \end{cases}$$

Now we have N equations,

$$\sum_{j=1}^{N} T_j F_{ij} = \sum_{j=1}^{N} q_j G_{ij}$$

then, its matrix equation,

$$FT = GQ$$

We obtain the following BEM equation by apply the interface condition.

$$\begin{pmatrix} F^{I^2}_{11} & F^3_{12} & F^3_{13} & F^3_{14} \\ F^{I^2}_{21} & F^3_{22} & F^3_{23} & F^3_{24} \\ F^{I^2}_{31} & F^3_{32} & F^3_{33} & F^3_{34} \\ F^{I^1}_{41} & F^3_{42} & F^3_{43} & F^3_{44} \end{pmatrix} \begin{pmatrix} T^2_I \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} = \begin{pmatrix} -G^{I^2}_{11} & G^3_{12} & G^3_{13} & G^3_{14} \\ -G^{I^2}_{21} & G^3_{22} & G^3_{23} & G^3_{24} \\ -G^{I^2}_{31} & G^3_{32} & G^3_{33} & G^3_{34} \\ -G^{I^2}_{41} & G^3_{42} & G^3_{43} & G^3_{44} \end{pmatrix} \begin{pmatrix} {q_I}^2 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} \tag{4.18}$$

Where

$T_6,\ T_7,\ T_8,\ q_6,\ q_7,\ q_8$ nodal values of the external boundary of top electrode.

${T_I}^2,\ {q_I}^2$ nodal values on the interface of insulator and top electrode.

The final system of equations for the whole region is obtained by adding the set of equations (4.6), (4.12) and (4.18) for each subregion together with compatibility and equilibrium conditions between their interfaces is,

$$\begin{pmatrix} F^1_{11} & F^1_{12} & F^1_{14} & F^{I^1}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{21} & F^1_{22} & F^1_{24} & F^{I^1}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{31} & F^1_{32} & F^1_{34} & F^{I^1}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{41} & F^1_{42} & F^1_{44} & F^{I^1}_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{11} & F^2_{12} & F^2_{14} & F^{I^2}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{21} & F^2_{22} & F^2_{24} & F^{I^2}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{31} & F^2_{32} & F^2_{34} & F^{I^2}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{43} & F^2_{42} & F^2_{44} & F^{I^2}_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{11} & F^3_{12} & F^3_{13} & F^3_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{21} & F^3_{22} & F^3_{23} & F^3_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{31} & F^3_{32} & F^3_{33} & F^3_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{41} & F^3_{42} & F^3_{43} & F^3_{44} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_{I}1 \\ T_4 \\ T_5 \\ T_{I}2 \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} =$$

$$\begin{pmatrix} G^1_{11} & G^1_{12} & G^{I^1}_{13} & G^1_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{21} & G^1_{22} & G^{I^1}_{23} & G^1_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{31} & G^1_{32} & G^{I^1}_{33} & G^1_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{41} & G^1_{42} & G^{I^1}_{43} & G^1_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{11} & G^2_{12} & G^2_{14} & G13^{I^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{21} & G^2_{22} & G^2_{24} & G^{I^2}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{31} & G^2_{32} & G^2_{34} & G^{I^2}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{41} & G^2_{42} & G^2_{44} & G^{I^2}_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{11} & G^3_{12} & G^3_{13} & G^3_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{21} & G^3_{22} & G^3_{23} & G^3_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{31} & G^3_{32} & G^3_{33} & G^3_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{41} & G^3_{412} & G^3_{43} & G^3_{44} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{I}1 \\ q_4 \\ q_5 \\ q_{I}2 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.19}$$

Since, the temperature of the device is zero on the boundary and the above equation becomes,

$$\begin{pmatrix} G^1_{11} & G^1_{12} & G^{I^1}_{13} & G^1_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{21} & G^1_{22} & G^{I^1}_{23} & G^1_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{31} & G^1_{32} & G^{I^1}_{33} & G^1_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ G^1_{41} & G^1_{42} & G^{I^1}_{43} & G^1_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{11} & G^2_{12} & G^2_{14} & G13^{I^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{21} & G^2_{22} & G^2_{24} & G^{I^2}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{31} & G^2_{32} & G^2_{34} & G^{I^2}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^{I^1}_{41} & G^2_{42} & G^2_{44} & G^{I^2}_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{11} & G^3_{12} & G^3_{13} & G^3_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{21} & G^3_{22} & G^3_{23} & G^3_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{31} & G^3_{32} & G^3_{33} & G^3_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{41} & G^3_{412} & G^3_{43} & G^3_{44} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{I}1 \\ q_4 \\ q_5 \\ q_{I}2 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} F^1_{11} & F^1_{12} & F^1_{14} & F^{I^1}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{21} & F^1_{22} & F^1_{24} & F^{I^1}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{31} & F^1_{32} & F^1_{34} & F^{I^1}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ F^1_{41} & F^1_{42} & F^1_{44} & F^{I^1}_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{11} & F^2_{12} & F^2_{14} & F^{I^2}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{21} & F^2_{22} & F^2_{24} & F^{I^2}_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{31} & F^2_{32} & F^2_{34} & F^{I^2}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{I^1}_{43} & F^2_{42} & F^2_{44} & F^{I^2}_{43} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{11} & F^3_{12} & F^3_{13} & F^3_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{21} & F^3_{22} & F^3_{23} & F^3_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{31} & F^3_{32} & F^3_{33} & F^3_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & F^{I^2}_{41} & F^3_{42} & F^3_{43} & F^3_{44} \end{pmatrix} \begin{pmatrix} T_1 = 0 \\ T_2 = 0 \\ T_3 = 0 \\ T_{I^1} \\ T_4 = 0 \\ T_5 = 0 \\ T_{I^2} \\ T_6 = 0 \\ T_7 = 0 \\ T_8 = 0 \end{pmatrix}$$

Moving the unknown terms to the one side, we obtain the following final equation.

$$\begin{pmatrix} -\hat{F}^{1^I}_{13} & 0 & G^1_{11} & G^1_{12} & G^{I^1}_{13} & G^1_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{F}^{1^I}_{23} & 0 & G^1_{21} & G^1_{22} & G^{I^1}_{23} & G^1_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{F}^{1^I}_{33} - 05 & 0 & G^1_{31} & G^1_{32} & G^{I^1}_{33} & G^1_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{F}^{1^I}_{43} & 0 & G^1_{41} & G^1_{42} & G^{I^1}_{43} & G^1_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\hat{F}^{I^1}_{11} - 0.5 & -\hat{F}^{I^2}_{13} & 0 & 0 & 0 & -G^{I^1}_{11} & G^2_{12} & G^2_{14} & G13^{I^2} & 0 & 0 & 0 \\ -\hat{F}^{I^1}_{21} & -\hat{F}^{I^2}_{23} & 0 & 0 & 0 & -G^{I^1}_{21} & G^2_{22} & G^2_{24} & G^{I^2}_{23} & 0 & 0 & 0 \\ -\hat{F}^{I^1}_{31} & -\hat{F}^{I^2}_{33} - 0.5 & 0 & 0 & 0 & -G^{I^1}_{31} & G^2_{32} & G^2_{34} & G^{I^2}_{33} & 0 & 0 & 0 \\ -\hat{F}^{I^1}_{41} & -\hat{F}^{I^2}_{43} & 0 & 0 & 0 & -G^{I^1}_{41} & G^2_{42} & G^2_{44} & G^{I^2}_{43} & 0 & 0 & 0 \\ 0 & -\hat{F}^{I^2}_{11} - 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{11} & G^3_{12} & G^3_{13} & G^3_{14} \\ 0 & -\hat{F}^{I^2}_{21} & 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{21} & G^3_{22} & G^3_{23} & G^3_{24} \\ 0 & -\hat{F}^{I^2}_{31} & 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{31} & G^3_{32} & G^3_{33} & G^3_{34} \\ 0 & -\hat{F}^{I^2}_{41} & 0 & 0 & 0 & 0 & 0 & 0 & -G^{I^2}_{41} & G^3_{412} & G^3_{43} & G^3_{44} \end{pmatrix} \begin{pmatrix} T_{I^1} \\ T_{I^2} \\ q_1 \\ q_2 \\ q_3 \\ q_{I^1} \\ q_4 \\ q_5 \\ q_{I^2} \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -D_1 \\ -D_2 \\ -D_3 \\ -D_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.20}$$

Solving equation(4.20), we obtain the value of the problem on the boundary of each sub domains. And therefore, the solution at any point in the subdomains can be found by applying equation (3.22) separately for each domains.

Chapter 5

Result and Discussion

5.1 Numerical Experiment

If we consider a layered device of dimensions $l_{x_1} = 2$ units and $l_{x_2} = 2$ units , where it is illustrated in the figure (5.1).

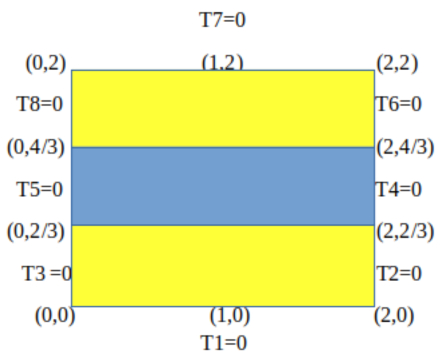

Figure 5.1: Discretization of the of the domain.

Now, from the figure (5.1),we have the following values,
For $j^1 = 1$, the end points of the first element of Ω_1 are $(0,0)$ and $(2,0)$.
For $j^1 = 2$, the end points of the second element of Ω_1 are $(2,0)$ and $(2,\frac{2}{3})$.
For $j^1 = I^1$, the end points of the I^1 element of the interface of bottom electrode and insulator are $(2,\frac{2}{3})$ and $(0,\frac{2}{3})$.
For $j^1 = 3$, the end points of the third element of Ω_1 are $(0,\frac{2}{3})$ and $(0,0)$.
For $j^2 = I^1$, the end points of the I^1 element of the interface of bottom electrode and insulator are and $(0,\frac{2}{3})$ and $(2,\frac{2}{3})$.
For $j^2 = 1$, the end points of the first element of Ω_2 are $(2,\frac{2}{3})$ and $(2,\frac{4}{3})$.
For $j^2 = I^2$, the end points of the I^2 element of the interface of insulator and top electrode are $(2,\frac{4}{3})$ and $(0,\frac{4}{3})$.
For $j^2 = 2$, the end points of the second element of Ω_2 are $(0,\frac{4}{3})$ and $(0,\frac{2}{3})$.
For $j^3 = I^2$, the end points of the I^2 element of the interface of insulator and top electrode are $(0,\frac{4}{3})$ and $(2,\frac{4}{3})$.
For $j^3 = 2$, the end points of the second element of Ω_3 are $(2,\frac{4}{3})$ and $(2,2)$.
For $j^3 = 3$, the end points of the third element of Ω_3 are $(2,2)$ and $(0,2)$.

For $j^3 = 4$, the end points of the fourth element of Ω_3 are $(0,2)$ and $(0,\frac{4}{3})$.

We obtain the temperature and flux on the boundary of the the layered device by using equation (4.20). Now, the computation of influence coefficients F_{ij} and G_{ij} which is the entry of the matrix in equation(4.20), are as follows:

Matrix Components G^1_{12} and F^1_{12}

we calculate the matrix components G^1_{12} and F^1_{12}, i.e., we collocate the load point on node 1 and carry out the integration along the element Γ_2, as shown in Figure (5.2).

$$
\begin{aligned}
\xi &= e_{x_1} \\
n &= e_{x_1} \\
\eta &= 2e_{x_1} + x_2 e_{x_2} \\
d\Gamma &= dx_2 \\
\eta - \xi &= e_{x_1} + x_2 e_{x_2} \\
|\eta - \xi| &= \sqrt{1 + x_2^2} \\
(\eta - \xi).n &= 1 + x_2.0 = 1
\end{aligned}
$$

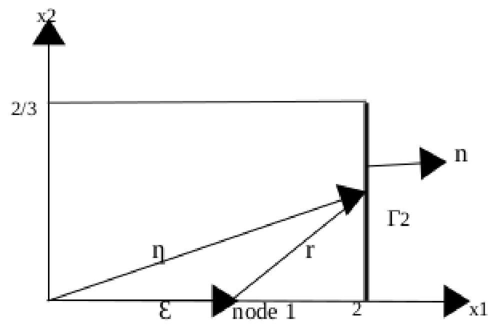

Figure 5.2: Calculation of matrix elements G^1_{12} and F^1_{12} with $r = \eta - \xi$: integration over Γ_2

Then substitution of equations given in above into (3.25) leads to,

$$
\begin{aligned}
G^1_{12} &= -\frac{1}{2\pi}\int_0^{\frac{2}{3}} \ln\sqrt{1+x_2^2}\, dx_2 \\
&= -\frac{1}{2\pi}\left[x_2 \ln\left(x_2^2+1\right) - 2x_2 + 2\arctan\left(x_2\right)\right]\Big|_0^{\frac{2}{3}} \\
&= -0.0139772
\end{aligned}
$$

and

$$
\begin{aligned}
F^1_{12} &= -\frac{1}{2\pi}\int_0^{\frac{2}{3}} \frac{1}{(1+x_2^2)} dx_2 = -\frac{1}{2\pi}\int_0^{\frac{2}{3}} \frac{1}{(x_2^2+1)} dx_2 \\
&= \frac{-1}{2\pi}\left[\arctan\left(x_2\right)\right]\Big|_0^{\frac{2}{3}} \\
&= -0.0935835
\end{aligned}
$$

Now, it follows from the symmetry of the problem that,

$$G^1_{14} = G^1_{34} = G^1_{32} = G^1_{12} \quad \text{and} \quad F^1_{14} = F^1_{34} = F^1_{32} = F^1_{12}$$

Matrix Components G^1_{23} and F^1_{23}

To calculate the components G^1_{23} and F^1_{23} , we place the load point on node 2 and integrate along the element Γ_3, as shown in Figure (5.3).

$$
\begin{aligned}
\xi &= 2e_{x_1} + \frac{1}{3}e_{x_2} \\
n &= e_{x_2} \\
\eta &= x_1 e_{x_1} + \frac{2}{3}e_{x_2} \\
d\Gamma &= -dx_1 \\
\eta - \xi &= (x_1 - 2)\, ex_1 + \frac{1}{3}e_{x_2} \\
|\eta - \xi| &= \sqrt{(x_1-2)^2 + \left(\frac{1}{3}\right)^2} \\
(\eta - \xi).n &= (x_1 - 2).0 + \frac{1}{3}.1 = \frac{1}{3}
\end{aligned}
$$

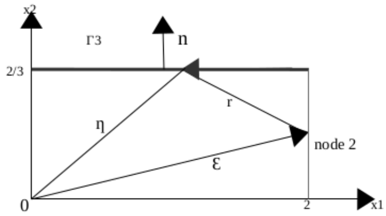

Figure 5.3: Calculation of matrix elements G^1_{23} and F^1_{23} with $r = \eta - \xi$: integration over Γ_3

By substitution of equations given in above into (3.25) leads to,

$$
\begin{aligned}
G^1_{23} &= \frac{1}{2\pi}\int_2^0 \ln\sqrt{(x_1-2)^2 + \left(\frac{1}{3}\right)^2}\, dx_1 \\
&= \frac{1}{2\pi}\left[(x_1-2)\ln\left((x_1-2)^2 + \left(\frac{1}{3}\right)^2\right) - 2x_1 + \frac{2}{3}\arctan(3x_1 - 6)\right]\Bigg|_2^0 \\
&= 0.03748336
\end{aligned}
$$

and

$$
\begin{aligned}
F^1_{23} &= \frac{1}{2\pi}\int_2^0 \frac{1}{3}\frac{1}{\left((x_1-2)^2 + \left(\frac{1}{3}\right)^2\right)} dx_1 \\
&= \frac{1}{2\pi}\left[\arctan(3x_2 - 6)\right]\Bigg|_2^0 \\
&= -0.2237157
\end{aligned}
$$

Now, it follows from the symmetry of the problem that,

$$
G^1_{21} = G^1_{41} = G^1_{43} = G^1_{23} \quad \text{and} \quad F^1_{21} = F^1_{41} = F^1_{43} = F^1_{23}
$$

Matrix Components G^1_{24} and F^1_{24}

To calculate the components G_{24} and F_{24} , we place the load point on node 2 and integrate along the element Γ_4, as shown in Figure (5.4).

$$
\begin{aligned}
\xi &= 2e_{x_1} + \frac{1}{3}e_{x_2} \\
n &= -e_{x_1} \\
\eta &= x_2 e_{x_2} \\
d\Gamma &= -dx_2 \\
\eta - \xi &= \left(x_2 - \frac{1}{3}\right)e_{x_2} - 2e_{x_1} \\
|\eta - \xi| &= \sqrt{\left(x_2 - \frac{1}{3}\right)^2 + 4} \\
(\eta - \xi).n &= \left(x_2 - \frac{1}{3}\right).0 + 2(1) = 2
\end{aligned}
$$

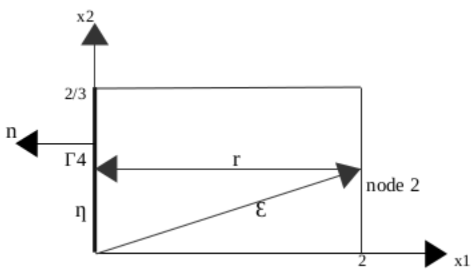

Figure 5.4: Calculation of matrix elements G^1_{24} and F^1_{24} with $r = \eta - \xi$: integration over Γ_4

$$\begin{aligned}
G_{24}^{1} &= \frac{1}{2\pi}\int_{\frac{2}{3}}^{0}\ln\sqrt{\left(x_2-\frac{1}{3}\right)^2+4}\,dx_2\\
&= \frac{1}{2\pi}\left[\left(x_2-\frac{1}{3}\right)\ln\left(\left(x_2-\frac{1}{3}\right)^2+2^2\right)-2x_2+4\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{2}{3}}^{0}\\
&= -0.14806
\end{aligned}$$

and

$$\begin{aligned}
F_{24}^{1} &= \frac{1}{2\pi}\int_{\frac{2}{3}}^{0}2\frac{1}{\left(x_2-\frac{1}{3}\right)^2+4}dx_2\\
&= \frac{1}{4\pi}\left[\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{2}{3}}^{0}\\
&= -1.505975
\end{aligned}$$

The remaining components are obtained from

$$G_{24}^{1}=G_{42}^{1}\quad\text{and}\quad F_{24}^{1}=F_{42}^{1}$$

Matrix Components G_{13}^{1} and F_{13}^{1}

To calculate the components G_{13} and F_{13} , we place the load point on node 1 and integrate along the element Γ_3, as shown in Figure (5.5).

$$\begin{aligned}
\xi &= e_{x_1}\\
n &= e_{x_2}\\
\eta &= x_1e_{x_1}+\frac{2}{3}e_{x_2}\\
d\Gamma &= -dx_1\\
\eta-\xi &= (x_1-1)\,ex_1+\frac{2}{3}e_{x_2}\\
|\eta-\xi| &= \sqrt{(x_2-1)^2+\left(\frac{2}{3}\right)^2}\\
(\eta-\xi)\,.n &= (x_1-1)\,.0+\frac{2}{3}.1=\frac{2}{3}
\end{aligned}$$

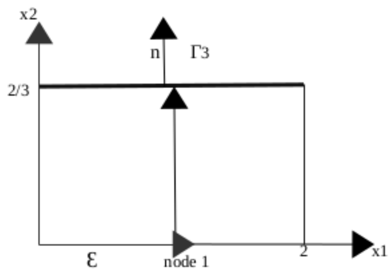

Figure 5.5: Calculation of matrix elements G_{13}^{1} and F_{13}^{1} with $r=\eta-\xi$: integration over Γ_3

$$\begin{aligned}
G_{13}^{1} &= \frac{1}{2\pi}\int_{2}^{0}\ln\sqrt{(x_1-1)^2+\left(\frac{2}{3}\right)^2}\,dx_1\\
&= \frac{1}{2\pi}\left[(x_1-1)\ln\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)-2x_1+\frac{4}{3}\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_{2}^{0}\\
&= 0.51956933
\end{aligned}$$

and

$$\begin{aligned} F_{13}^1 &= \frac{1}{2\pi}\int_2^0 \frac{2}{3}\frac{1}{\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)}dx_1 \\ &= \frac{1}{2\pi}\left[\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_2^0 \\ &= -0.1564165 \end{aligned}$$

and, by virtue of the symmetry of the problem,

$$G_{13}^1 = G_{31}^1 \quad \text{and} \quad F_{13}^1 = F_{31}^1$$

Diagonal Terms

If the load point and the field point are located on the same element, the vector $r = (\eta - \xi)$ is perpendicular to n and therefore $(\eta - \xi).n = 0$. Thus,the main diagonal of the matrix F vanishes, i.e.,

$$F_{11}^1 = F_{22}^1 = F_{33}^1 = F_{44}^1 = 0 \tag{5.1}$$

For G_{11}^1 we obtain

$$\begin{aligned} \xi &= e_{x_1} \\ \eta &= x_1 e_{x_1} \\ d\Gamma &= dx_1 \\ \eta - \xi &= (x_1 - 1)\, e_{x_1} \\ |\eta - \xi| &= |x_1 - 1| \end{aligned}$$

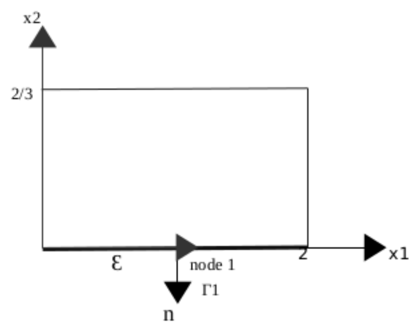

Figure 5.6: Calculation of matrix elements G_{11}^1 and F_{11}^1 with $r = \eta - \xi$: integration over Γ_1

$$\begin{aligned} G_{11}^1 &= -\frac{1}{2\pi}\int_0^2 \ln|x_1-1|\,dx_1 \\ &= -\frac{1}{2\pi}\left(\int_0^1 \ln(1-x_1)\,dx_1 + \int_1^2 \ln(x_1-1)\right)dx_1 \\ &= -\frac{1}{2\pi}\left[-(1-x_1)\ln(1-x_1)-(1-x_1)\right]\Bigg|_0^1 - \frac{1}{2\pi}\left[(x_1-1)\ln(x_1-1)-(x_1-1)\right]\Bigg|_1^2 \\ &= 0.3183099 \end{aligned}$$

Again, symmetry considerations lead to

$$G_{33}^1 = G_{11}^1 \tag{5.2}$$

Similarly, we can calculate

$$\begin{aligned}
\xi &= \frac{1}{3}e_{x_2} \\
\eta &= x_2 e_{x_2} \\
d\Gamma &= dx_2 \\
\eta - \xi &= \left(x_2 - \frac{1}{3}\right) e_{x_2} \\
|\eta - \xi| &= |x_2 - \frac{1}{3}|
\end{aligned}$$

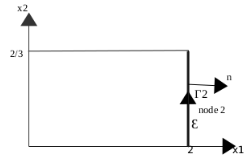

Figure 5.7: Calculation of matrix elements G^1_{22} and F^1_{22} with $r = \eta - \xi$: integration over Γ_2

$$\begin{aligned}
G^1_{22} &= -\frac{1}{2\pi}\int_0^{\frac{2}{3}} \ln|x_2 - \frac{1}{3}|\, dx_2 \\
&= -\frac{1}{2\pi}\left(\int_0^{\frac{1}{3}} \ln\left(\frac{1}{3} - x_2\right) + \int_{\frac{1}{3}}^{\frac{2}{3}} \ln\left(x_2 - \frac{1}{3}\right)\right) dx_2 \\
&= -\frac{1}{2\pi}\left[-\left(\frac{1}{3} - x_2\right)\ln\left(\frac{1}{3} - x_2\right) + \left(\frac{1}{3} - x_2\right)\right]\Bigg|_0^{\frac{1}{3}} - \frac{1}{2\pi}\left[\left(x_2 - \frac{1}{3}\right)\ln\left(x_2 - \frac{1}{3}\right) - \left(x_2 - \frac{1}{3}\right)\right]\Bigg|_{\frac{1}{3}}^{\frac{2}{3}} \\
&= \frac{1}{\pi}\left(\frac{1}{3}\ln 3 + \frac{1}{3}\right) \\
&= 0.22266968
\end{aligned}$$

and from symmetry,

$$G^1_{22} = G^1_{44}$$

We calculate similarly for the in the subdomain 2 (insulator).

$$\begin{aligned}
G^2_{12} &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}} \ln\sqrt{1 + x_2^2}\, dx_2 \\
&= -\frac{1}{2\pi}\left[x_2 \ln\left(x_2^2 + 1\right) - 2x_2 + 2\arctan(x_2)\right]\Bigg|_{\frac{2}{3}}^{\frac{4}{3}} \\
&= -0.0735779
\end{aligned}$$

and

$$\begin{aligned}
F^2_{12} &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}} \frac{1}{(1 + x_2^2)} dx_2 = -\frac{1}{2\pi}\int_0^{\frac{2}{3}} \frac{1}{(x_2^2 + 1)} dx_2 \\
&= \frac{-1}{2\pi}\left[\arctan(x_2)\right]\Bigg|_{\frac{2}{3}}^{\frac{4}{3}} \\
&= -0.054000
\end{aligned}$$

It follows from the symmetry of the problem that,

$$G^2_{14} = G^2_{34} = G^2_{32} = G^2_{12} \quad \text{and} \quad F^2_{14} = F^2_{34} = F^2_{32} = F^2_{12}$$

$$\begin{aligned}
G^2_{23} &= \frac{1}{2\pi}\int_2^0 \ln\sqrt{(x_1 - 2)^2 + \left(\frac{1}{3}\right)^2}\, dx_1 \\
&= \frac{1}{2\pi}\left[(x_1 - 2)\ln\left((x_1 - 2)^2 + \left(\frac{1}{3}\right)^2\right) - 2x_1 + \frac{2}{3}\arctan(3x_1 - 6)\right]\Bigg|_2^0 \\
&= 0.03748336
\end{aligned}$$

and

$$\begin{aligned} F_{23}^2 &= \frac{1}{2\pi}\int_2^0 \frac{1}{3}\frac{1}{\left((x_1-2)^2+\left(\frac{1}{3}\right)^2\right)}dx_1 \\ &= \frac{1}{2\pi}\left[\arctan(3x_2-6)\right]\Big|_2^0 \\ &= -0.2237157 \end{aligned}$$

Now, it follows from the symmetry of the problem that,

$$G_{21}^2 = G_{41}^2 = G_{43}^2 = G_{23}^2 \quad \text{and} \quad F_{21}^2 = F_{41}^2 = F_{43}^2 = F_{23}^2$$

$$\begin{aligned} G_{24}^2 &= \frac{1}{2\pi}\int_{\frac{4}{3}}^{\frac{2}{3}} \ln\sqrt{\left(x_2-\frac{1}{3}\right)^2+4}\,dx_2 \\ &= \frac{1}{2\pi}\left[\left(x_2-\frac{1}{3}\right)\ln\left(\left(x_2-\frac{1}{3}\right)^2+2^2\right)-2x_2+4\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{4}{3}}^{\frac{2}{3}} \\ &= -0.3692488 \end{aligned}$$

and

$$\begin{aligned} F_{24}^2 &= \frac{1}{2\pi}\int_{\frac{4}{3}}^{\frac{2}{3}} 2\frac{1}{\left(x_2-\frac{1}{3}\right)^2+4}dx_2 \\ &= \frac{1}{4\pi}\left[\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{4}{3}}^{\frac{2}{3}} \\ &= -0.0368959 \end{aligned}$$

The remaining components are obtained from symmetry of the domain,

$$G_{24}^2 = G_{42}^2 \quad \text{and} \quad F_{24}^2 = F_{42}^2$$

$$\begin{aligned} G_{13}^2 &= \frac{1}{2\pi}\int_2^0 \ln\sqrt{(x_1-1)^2+\left(\frac{2}{3}\right)^2}\,dx_1 \\ &= \frac{1}{2\pi}\left[(x_1-1)\ln\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)-2x_1+\frac{4}{3}\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_2^0 \\ &= 0.51956933 \end{aligned}$$

and

$$\begin{aligned} F_{13}^2 &= \frac{1}{2\pi}\int_2^0 \frac{2}{3}\frac{1}{\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)}dx_1 \\ &= \frac{1}{2\pi}\left[\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_2^0 \\ &= -0.1564165 \end{aligned}$$

and, by virtue of the symmetry of the problem,

$$G_{13}^2 = G_{31}^2 \quad \text{and} \quad F_{13}^2 = F_{31}^2$$

Since, the main diagonal of the matrix F vanishes, i.e.,

$$F_{11}^2 = F_{22}^2 = F_{33}^2 = F_{44}^2 = 0$$

$$\begin{aligned}
G_{11}^2 &= -\frac{1}{2\pi}\int_0^2 \ln|x_1-1|\,dx_1\\
&= -\frac{1}{2\pi}\left(\int_0^1 \ln(1-x_1)\,dx_1+\int_1^2 \ln(x_1-1)\right)dx_1\\
&= -\frac{1}{2\pi}\left[-(1-x_1)\ln(1-x_1)-(1-x_1)\right]\Big|_0^1-\frac{1}{2\pi}\left[(x_1-1)\ln(x_1-1)-(x_1-1)\right]\Big|_1^2\\
&= 0.3183099
\end{aligned}$$

Again, symmetry considerations lead to

$$G_{33}^2 = G_{11}^2$$

$$\begin{aligned}
G_{22}^2 &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}} \ln|x_2-\frac{1}{3}|\,dx_2\\
&= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}} \ln\left(x_2-\frac{1}{3}\right)dx_2\\
&= -\frac{1}{2\pi}\left[\left(x_2-\frac{1}{3}\right)\ln\left(x_2-\frac{1}{3}\right)-\left(x_2-\frac{1}{3}\right)\right]\Big|_{\frac{2}{3}}^{\frac{4}{3}}\\
&= -0.0478201
\end{aligned}$$

and by symmetry consideration,

$$G_{22}^2 = G_{22}^2$$

We calculate similarly for the in the subdomain 3 (top electrode).

$$\begin{aligned}
G_{12}^3 &= -\frac{1}{2\pi}\int_{\frac{4}{3}}^{2} \ln\sqrt{1+x_2^2}\,dx_2\\
&= -\frac{1}{2\pi}\left[x_2\ln(x_2^2+1)-2x_2+2\arctan(x_2)\right]\Big|_{\frac{4}{3}}^{2}\\
&= -0.1405414
\end{aligned}$$

and

$$\begin{aligned}
F_{12}^3 &= -\frac{1}{2\pi}\int_{\frac{4}{3}}^{2}\frac{1}{(1+x_2^2)}dx_2 = -\frac{1}{2\pi}\int_0^{\frac{2}{3}}\frac{1}{(x_2^2+1)}dx_2\\
&= \frac{-1}{2\pi}\left[\arctan(x_2)\right]\Big|_{\frac{4}{3}}^{2}\\
&= -0.02862457
\end{aligned}$$

Now, it follows from the symmetry of the problem that,

$$G_{14}^3 = G_{34}^3 = G_{32}^3 = G_{12}^3 \quad \text{and} \quad F_{14}^3 = F_{34}^3 = F_{32}^3 = F_{12}^3$$

$$\begin{aligned}
G_{23}^3 &= \frac{1}{2\pi}\int_2^0 \ln\sqrt{(x_1-2)^2+\left(\frac{1}{3}\right)^2}\,dx_1\\
&= \frac{1}{2\pi}\left[(x_1-2)\ln\left((x_1-2)^2+\left(\frac{1}{3}\right)^2\right)-2x_1+\frac{2}{3}\arctan(3x_1-6)\right]\Big|_2^0\\
&= 0.03748336
\end{aligned}$$

and

$$\begin{aligned}
F_{23}^3 &= \frac{1}{2\pi}\int_2^0 \frac{1}{3}\frac{1}{\left((x_1-2)^2+\left(\frac{1}{3}\right)^2\right)}dx_1\\
&= \frac{1}{2\pi}\left[\arctan(3x_2-6)\right]\Big|_2^0\\
&= -0.2237157
\end{aligned}$$

Now, it follows from the symmetry of the problem that,

$$G_{21}^3 = G_{41}^3 = G_{43}^3 = G_{23}^3 \quad \text{and} \quad F_{21}^3 = F_{41}^3 = F_{43}^3 = F_{23}^3$$

$$\begin{aligned}
G_{24}^3 &= \frac{1}{2\pi}\int_{\frac{2}{3}}^{0} \ln\sqrt{\left(x_2-\frac{1}{3}\right)^2+4}\,dx_2 \\
&= \frac{1}{2\pi}\left[\left(x_2-\frac{1}{3}\right)\ln\left(\left(x_2-\frac{1}{3}\right)^2+2^2\right)-2x_2+4\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{2}{3}}^{0} \\
&= 0.062209
\end{aligned}$$

and

$$\begin{aligned}
F_{24}^3 &= \frac{1}{2\pi}\int_{\frac{2}{3}}^{0} 2\frac{1}{\left(x_2-\frac{1}{3}\right)^2+4}dx_2 \\
&= \frac{1}{4\pi}\left[\arctan\left(\frac{x_2}{2}-\frac{1}{6}\right)\right]\Bigg|_{\frac{2}{3}}^{0} \\
&= -0.02628422
\end{aligned}$$

The remaining components are also obtained from symmetry.

$$G_{24}^3 = G_{42}^3 \quad \text{and} \quad F_{24}^3 = F_{42}^3$$

$$\begin{aligned}
G_{13}^3 &= \frac{1}{2\pi}\int_{2}^{0} \ln\sqrt{(x_1-1)^2+\left(\frac{2}{3}\right)^2}\,dx_1 \\
&= \frac{1}{2\pi}\left[(x_1-1)\ln\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)-2x_1+\frac{4}{3}\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_{2}^{0} \\
&= 0.51956933
\end{aligned}$$

and

$$\begin{aligned}
F_{13}^3 &= \frac{1}{2\pi}\int_{2}^{0} \frac{2}{3}\frac{1}{\left((x_1-1)^2+\left(\frac{2}{3}\right)^2\right)}dx_1 \\
&= \frac{1}{2\pi}\left[\arctan\left(\frac{3x_1-3}{2}\right)\right]\Bigg|_{2}^{0} \\
&= -0.1564165
\end{aligned}$$

and, by virtue of the symmetry of the problem,

$$G_{13}^3 = G_{31}^3 \quad \text{and} \quad F_{13}^3 = F_{31}^3$$

Since,the main diagonal of the matrix F vanishes, i.e.,

$$F_{11}^3 = F_{22}^3 = F_{33}^3 = F_{44}^3 = 0$$

$$\begin{aligned}
G_{11}^3 &= -\frac{1}{2\pi}\int_{0}^{2} \ln|x_1-1|\,dx_1 \\
&= -\frac{1}{2\pi}\left(\int_{0}^{1}\ln(1-x_1)\,dx_1+\int_{1}^{2}\ln(x_1-1)\right)dx_1 \\
&= -\frac{1}{2\pi}\left[-(1-x_1)\ln(1-x_1)+(1-x_1)\right]\Big|_{0}^{1}-\frac{1}{2\pi}\left[(x_1-1)\ln(x_1-1)-(x_1-1)\right]\Big|_{1}^{2} \\
&= 0.3183099
\end{aligned}$$

Again, symmetry considerations lead to

$$G^3_{33} = G^3_{11}$$

$$\begin{aligned} G^3_{22} &= -\frac{1}{2\pi}\int_{\frac{4}{3}}^{2} \ln|x_2 - \frac{1}{3}|\, dx_2 \\ &= -\frac{1}{2\pi}\int_{\frac{4}{3}}^{2} \ln\left(x_2 - \frac{1}{3}\right) dx_2 \\ &= -\frac{1}{2\pi}\left[\left(x_2-\frac{1}{3}\right)\ln\left(x_2-\frac{1}{3}\right) - \left(x_2 - \frac{1}{3}\right)\right]\Bigg|_{\frac{4}{3}}^{2} \\ &= 0.02939741 \end{aligned}$$

and from symmetry,

$$G^3_{44} = G^3_{22}$$

Now, to determine the domain integral, we use equation(3.29). In this case we use the complete domain as one integration cell and the boundary is discretised with four constant elements. i.e,

$$D_i = -\frac{1}{2\pi}\int_{\Omega} \ln|\xi_i - \eta_i^1| \tag{5.3}$$

with

$$\begin{aligned} \xi &= e_{x_1} \\ \eta &= x_1 e x_1 + x_2 e x_2 \\ \eta - \xi &= (x_1 - 1) e x_1 + x_2 e x_2 \\ |\eta - \xi| &= \sqrt{(x_1-1)^2 + x_2^2} \end{aligned}$$

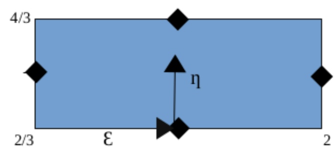

Figure 5.8: Domain Integral by Cell Integration with element 1

Then, substitute the above equations into equation(5.3) and integrate it analytically as,

$$\begin{aligned} D_1 &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\int_0^2 \ln\sqrt{(x_1-1)^2 + x_2^2}\, dx_1 dx_2 \\ &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\left[(x_1-1)\ln\left((x_1-1)^2 + x_2^2\right) - 2x_1 + 2x_2 \arctan\left(\frac{(x_1-1)}{x_2}\right)\right]\Bigg|_0^2 dx_2 \\ &= -\frac{1}{\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\left[\ln\left(1+x_2^2\right) - 2 + 2x_2\arctan\left(\frac{1}{x_2}\right)\right] dx_2 \\ &= -\frac{1}{2\pi}\left[x_2\ln\left(1+x_2^2\right) + 2\left(1+x_2^2\right)\arctan\left(\frac{1}{x_2}\right) + \arctan\left(x_2\right) - 4x_2\right]\Bigg|_{\frac{2}{3}}^{\frac{4}{3}} \\ &= 0.09720469 \end{aligned}$$

From symmetry, it follows that

$$D_1 = D_3$$

To calculate D_4 , we need

$$\begin{aligned} \xi &= \frac{1}{3}e_{x_2} \\ \eta &= x_1ex_1 + x_2ex_2 \\ \eta - \xi &= x_1ex_1 + \frac{2}{3}x_2ex_2 \end{aligned}$$

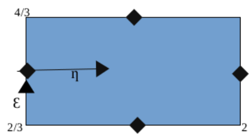

Figure 5.9: Domain Integral by Cell Integration with element 4

The integration can be carried out analytically, and we obtain

$$\begin{aligned}
D_4 &= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\int_0^2 \ln\sqrt{x_1^2+(\frac{2}{3}x_2)^2}dx_1dx_2 \\
&= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\left[x_1\ln\left(x_1^2+(\frac{2}{3}x_2)^2\right)-2x_1+\frac{4}{3}x_2\arctan\left(\frac{3x_1}{2x_2}\right)\Big|_0^2\right]dx_2 \\
&= -\frac{1}{2\pi}\int_{\frac{2}{3}}^{\frac{4}{3}}\left[2\ln\left(4+(\frac{2}{3}x_2)^2\right)-4+\frac{4}{3}x_2\arctan\left(\frac{3}{x_2}\right)\right]dx_2 \\
&= -\frac{3}{4\pi}\left[\frac{2}{3}x_2\ln\left(4+(\frac{2}{3}x_2)^2\right)+\left(\left(\frac{2}{3}x_2\right)^2+4\right)\arctan\left(\frac{3}{x_2}\right)+2\arctan\left(\frac{2}{3}x_2\right)-2x_2\right]\Big|_{\frac{2}{3}}^{\frac{4}{3}} \\
&= -\frac{3}{4\pi}\left[\frac{8}{9}\ln\left(4+\left(\frac{8}{9}\right)^2\right)+\left(\left(\frac{8}{9}\right)^2+4\right)\arctan\left(\frac{9}{4}\right)+2\arctan\left(\frac{8}{9}\right)-\frac{8}{3}-\left(\frac{4}{9}\ln\left(4+\left(\frac{4}{9}\right)^2\right)+\left(\left(\frac{4}{9}\right)^2+4\right)\arctan\left(\frac{9}{2}\right)+2\arctan\left(\frac{8}{9}\right)-\frac{4}{3}\right)\right] \\
&= -0.544847266
\end{aligned}$$

Again, using the property of symmetry, we obtain

$$D_2 = D_4$$

Then by substituting the above values in equation(4.19), we obtain the following

$$\begin{pmatrix}
0.5 & -0.09358 & -0.09358 & -0.15642 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.22372 & 0.5 & 0 & -0.22372 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.15642 & -0.09358 & -0.09358 & 0.5 & & 0 & 0 & 0 & 0 & 0 \\
-0.22372 & 0 & 0.5 & -0.22372 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & -0.05400 & -0.05400 & -0.15642 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.22372 & 0.5 & -0.036896 & -0.22372 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.15642 & -0.05400 & -0.05400 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.22372 & -0.036896 & 0 & -0.22372 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.028624 & -0.15642 & -0.028624 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.22372 & 0.5 & -0.22372 & 0.0622 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.15642 & -0.02862 & 05 & -0.02862 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.22372 & 0.062209 & -0.22372 & 0.5
\end{pmatrix}
\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_{I1} \\ T_4 \\ T_5 \\ T_{I2} \\ T_6 \\ T_7 \\ T_8 \end{pmatrix} =$$

$$\begin{pmatrix}
0.31831 & -0.01398 & 0.519569 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.03748 & 0.2226697 & 0.03748 & -0.14806 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.519569 & -0.01398 & 0.31831 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.03748 & -0.14806 & 0.03748 & 0.2226697 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.31831 & -0.07358 & -0.07358 & -01564 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.03748 & -0.04782 & -0.36925 & 0.03748 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.156416 & -0.07358 & -0.07358 & 0.31831 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.03748 & 0.03748 & -0.04782 & 0.03748 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.31831 & -0.14054 & 0.519569 & -0.14054 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.029397 & 0.03748 & 0.06221 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.519569 & -0.14054 & 0.31831 & -0.14054 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.06221 & 0.03748 & 0.029397
\end{pmatrix}
\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{I1} \\ q_4 \\ q_5 \\ q_{I2} \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} +
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.014064 \\ -0.07883 \\ 0.014064 \\ -0.07883 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.4}$$

Now, by substituting the boundary value we obtain the following equation,

$$\begin{pmatrix}
0.5 & -0.09358 & -0.09358 & -0.15642 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.22372 & 0.5 & 0 & -0.22372 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.15642 & -0.09358 & -0.09358 & 0.5 & & 0 & 0 & 0 & 0 & 0 \\
-0.22372 & 0 & 0.5 & -0.22372 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & -0.05400 & -0.05400 & -0.15642 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.22372 & 0.5 & -0.036896 & -0.22372 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.15642 & -0.05400 & -0.05400 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.22372 & -0.036896 & 0 & -0.22372 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.5 & -0.028624 & -0.15642 & -0.028624 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.22372 & 0.5 & -0.22372 & 0.0622 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.15642 & -0.02862 & 05 & -0.02862 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.22372 & 0.062209 & -0.22372 & 0.5
\end{pmatrix}
\begin{pmatrix} T_1=0 \\ T_2=0 \\ T_3=0 \\ T_{I1} \\ T_4=0 \\ T_5=0 \\ T_{I2} \\ T_6=0 \\ T_7=0 \\ T_8=0 \end{pmatrix} =$$

$$\begin{pmatrix} 0.31831 & -0.01398 & 0.519569 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03748 & 0.2226697 & 0.03748 & -0.14806 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.519569 & -0.01398 & 0.31831 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03748 & -0.14806 & 0.03748 & 0.2226697 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.31831 & -0.07358 & -0.07358 & -01564 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.03748 & -0.04782 & -0.36925 & 0.03748 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.156416 & -0.07358 & -0.07358 & 0.31831 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.03748 & 0.03748 & -0.04782 & 0.03748 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.31831 & -0.14054 & 0.519569 & -0.14054 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.029397 & 0.03748 & 0.06221 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.519569 & -0.14054 & 0.31831 & -0.14054 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.06221 & 0.03748 & 0.029397 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{I1} \\ q_4 \\ q_5 \\ q_{I2} \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.014064 \\ -0.07883 \\ 0.014064 \\ -0.07883 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.5}$$

Then, the equation (5.5) becomes,

$$A = \begin{pmatrix} 0.15642 & 0 & 0.31831 & -0.01398 & 0.519569 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.22372 & 0 & 0.03748 & 0.2226697 & 0.03748 & -0.14806 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0.519569 & -0.01398 & 0.31831 & -0.01398 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.22372 & 0 & 0.03748 & -0.14806 & 0.03748 & 0.2226697 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 0.15642 & -0.5 & 0 & 0 & -0.31831 & -0.07358 & -0.07358 & -01564 & 0 & 0 & 0 \\ 0.22372 & 0.22372 & 0 & 0 & 0 & -0.03748 & -0.04782 & -0.36925 & 0.03748 & 0 & 0 & 0 \\ 0.15642 & -0.5 & 0 & 0 & 0 & 0.156416 & -0.07358 & -0.07358 & 0.31831 & 0 & 0 & 0 \\ 0.22372 & 0.223726 & 0 & 0 & 0 & -0.03748 & 0.03748 & -0.04782 & 0.03748 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & -0.31831 & -0.14054 & 0.519569 & -0.14054 \\ 0 & 0.22372 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.029397 & 0.03748 & 0.06221 \\ 0 & 0.15642 & 0 & 0 & 0 & 0 & 0 & 0 & -0.519569 & -0.14054 & 0.31831 & -0.14054 \\ 0 & 0.22372 & 0 & 0 & 0 & 0 & 0 & 0 & -0.03748 & 0.06221 & 0.03748 & 0.029397 \end{pmatrix} \tag{5.6}$$

$$x = \begin{pmatrix} T_{I^1} \\ T_{I^2} \\ q_1 \\ q_2 \\ q_3 \\ q_{I^1} \\ q_4 \\ q_5 \\ q_{I^2} \\ q_6 \\ q_7 \\ q_8 \end{pmatrix}, \qquad b \quad = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.014064 \\ 0.07883 \\ -0.014064 \\ 0.07883 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Which is has form $Ax = b$,
Then solve for x, we obtain the temperature at the interface of the device,

$$\begin{pmatrix} T_{I^1} \\ T_{I^2} \end{pmatrix} = \begin{pmatrix} -0.058486 \\ -0.137645 \end{pmatrix}$$

The flux at the interface of the device,

$$\begin{pmatrix} q_{I^1} \\ q_{I^2} \end{pmatrix} = \begin{pmatrix} 0.18134 \\ -0.290806 \end{pmatrix}$$

The flux at the boundary of the device,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{pmatrix} = \begin{pmatrix} -0.101318 \\ 0.18134 \\ 0.089438 \\ -0.34396 \\ 0.0912814 \\ 0.281871 \\ -0.158132 \\ 0.281871 \end{pmatrix}$$

Next, we determine the value of temperature (T) at any point interior of the domain separately.
For the point $x \in \Omega_1$, the problem given as follows;

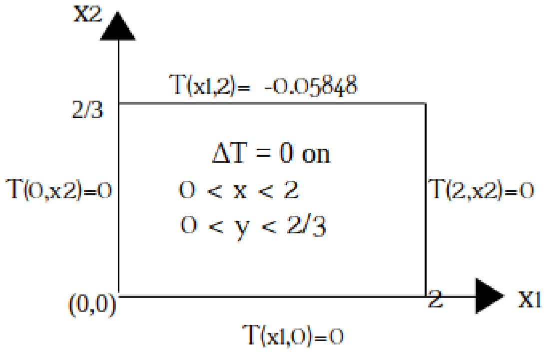

Figure 5.10: Bottom electrode

By using the separation of variables method, and superposition principle, our exact solution of the problem in Ω_1 is given by:

$$T(x_1,x_2) = \frac{1}{\sinh\left(\frac{n}{6}\pi\right)}\left(\int_0^2 -0.05848\sin\left(\frac{n\pi}{2}x_1\right)dx_1\right)\sin\left(\frac{n\pi}{2}x_1\right)\sinh\left(\frac{n\pi}{2}x_2\right)$$

Now, we use the obtained boundary values, to determine the values at any point in the subdomain Ω_1 using the following equation from equation(4.2).

$$T(x) = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma$$

This can be done by dividing the boundary integral into the number of elements.

$$T(x) = \int_{\Gamma_1} G(x,y)q(y)d\Gamma_1 + \int_{\Gamma_2} G(x,y)q(y)d\Gamma_2 + \cdots - \int_{\Gamma_1} F(x,y)T(y)d\Gamma_1 - \int_{\Gamma_2} F(x,y)T(y)d\Gamma_2 - \cdots$$

i.e,

$$\begin{aligned} T(x) &= \int_{\Gamma_1}\frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right)q_1\,dx_1 + \int_{\Gamma_2}\frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right)q_2\,dx_2 + \int_{\Gamma_{I^1}}\frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right)q_{I^1}\,dx_1 + \int_{\Gamma_3}\frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right)q_3\,dx_2 + \\ &\quad \int_{\Gamma_1}\frac{1}{2\pi}\left(\frac{1}{r_1}\right)T_1\,dx_1 + \int_{\Gamma_2}\frac{1}{2\pi}\left(\frac{1}{r_2}\right)T_2\,dx_2 + \int_{\Gamma_{I^1}}\frac{1}{2\pi}\left(\frac{1}{r_3}\right)T_{I^1}\,dx_1 + \int_{\Gamma_3}\frac{1}{2\pi}\left(\frac{1}{r_4}\right)T_3\,dx_2 \end{aligned}$$

E.g. To find the approximate temperature at interior point say $(x_1,x_2) = (0.25,0.25)$,

$$\begin{aligned} T(0.25,0.25) &= \int_0^2\frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right)*q_1\,dx_1 + \int_0^{\frac{2}{3}}\frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right)*q_2\,dx_2 + \int_2^0\frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right)*q_{I^1}\,dx_1 + \int_{\frac{2}{3}}^0\frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right)*q_4\,dx_2 + \\ &\quad \int_2^0\frac{1}{2\pi}(\frac{1}{r_1})*T_1\,dx_1 + \int_0^{\frac{2}{3}}\frac{1}{2\pi}\left(\frac{1}{r_2}\right)*T_2\,dx_2 + \int_2^0\frac{1}{2\pi}\left(\frac{1}{r_3}\right)*T_{I^1}\,dx_1 + \int_{\frac{2}{3}}^0\frac{1}{2\pi}\left(\frac{1}{r_4}\right)*T_4\,dx_2 \end{aligned}$$

Where

$$\begin{aligned} r_1 &= d\left[(0.25,0.25),(1,0)\right] = \sqrt{(0.25-1)^2+(0.25-0)^2}, \quad r_2 = d\left[(0.25,0.25),(2,\frac{1}{3})\right] = \sqrt{(0.25-2)^2+(0.25-\frac{1}{3})^2} \\ r_3 &= d\left[(0.25,0.25),(\frac{1}{3},1)\right] = \sqrt{(0.25-1)^2+(0.25-\frac{1}{3})^2}, \quad r_4 = d\left[(0.25,0.25),(0,\frac{1}{3})\right] = \sqrt{(0.25-0)^2+(0.25-\frac{1}{3})^2} \end{aligned}$$

Table 5.1: The temperature inside the bottom electrode of the device

Internal point	Coordinates		BEM solution
Points	x_1	x_2	$T(x_1,x_2)$
1	0.25	0.25	-0.0432
2	0.5	0.0	0.0002
3	0.50	0.25	-0.0020
4	0.75	0.25	0.0309
5	0.75	0.5	0.0210
6	1	0.25	0.0507
7	1	0.50	0.0148
8	1.25	0.25	0.0397
9	1.25	0.50	0.0296
10	1.50	0.0	0.0164
11	1.5	0.25	0.0167
12	1.50	0.25	0.0155
13	1.75	0.25	-0.0105

Graphically we can show the solution as follows,

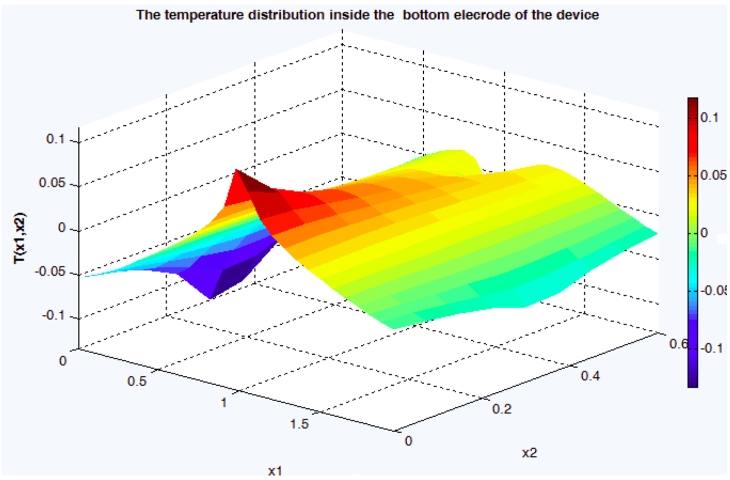

Figure 5.11: BEM solution graph on the domain $[0,2]\times[0,\frac{2}{3}]$

For the point $x \in \Omega_2$, the problem given as follows;

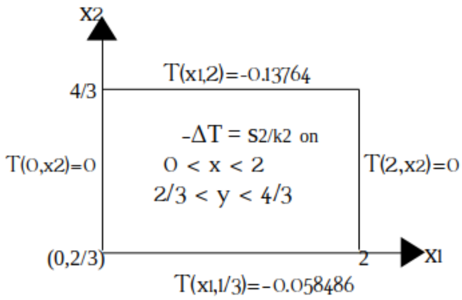

Figure 5.12: Insulator

Now, we use the obtained boundary values, to determine the values at any point in the subdomain Ω_2 using the following equation from equation(4.8).

$$T(x) = \int_{\Gamma} G(x,y)q(y)d\Gamma - \int_{\Gamma} F(x,y)T(y)d\Gamma + \frac{s_2}{k_2}\int_{\Omega} G(x,y)d\Omega$$

This can be done by dividing the boundary integral into the number of elements.

$$T(x) = \int_{\Gamma_1} G(x,y)q(y)d\Gamma_1 + \int_{\Gamma_2} G(x,y)q(y)d\Gamma_2 + \cdots - \int_{\Gamma_1} F(x,y)T(y)d\Gamma_1 - \int_{\Gamma_2} F(x,y)T(y)d\Gamma_2 - \cdots + \frac{s_2}{k_2}\int_{\Omega} G(x,y)d\Omega$$

i.e,

$$\begin{aligned} T(x) \quad = \quad & \int_{\Gamma_1} \frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right) q_1\, dx_1 + \int_{\Gamma_2} \frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right) q_2\, dx_2 + \int_{\Gamma_{I1}} \frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right) q_{I1}\, dx_1 + \int_{\Gamma_3} \frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right) q_3\, dx_2 + \\ & \int_{\Gamma_1} \frac{1}{2\pi}\left(\frac{1}{r_1}\right) T_1\, dx_1 + \int_{\Gamma_2} \frac{1}{2\pi}\left(\frac{1}{r_2}\right) T_2\, dx_2 + \int_{\Gamma_{I1}} \frac{1}{2\pi}\left(\frac{1}{r_3}\right) T_{I1}\, dx_1 + \int_{\Gamma_3} \frac{1}{2\pi}\left(\frac{1}{r_4}\right) T_3\, dx_2 \end{aligned}$$

E.g. To find the approximate temperature at interior point say $(x_1, x_2) = (0.25, 0.25)$,

$$\begin{aligned} T(0.25, 0.75) \quad = \quad & \int_0^2 \frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right) * q_{I_1}\, dx_1 + \int_{\frac{2}{3}}^{\frac{4}{3}} \frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right) * q_4\, dx_2 + \int_2^0 \frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right) * q_{I_2}\, dx_1 + \int_{\frac{4}{3}}^{\frac{2}{3}} \frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right) * q_5\, dx_2 + \\ & \int_0^2 \frac{1}{2\pi}\ln(r_1) * T_{I1}\, dx_1 + \int_{\frac{2}{3}}^{\frac{4}{3}} \frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right) * T_4\, dx_2 + \int_2^0 \frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right) * T_{I_2}\, dx_1 + \int_{\frac{4}{3}}^{\frac{2}{3}} \frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right) * T_5 dx_2 + \\ & \frac{s_2}{k_2}\int_0^2 \int_{\frac{2}{3}}^{\frac{4}{3}} \frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right) dx_2\, dx_1 \end{aligned}$$

Where

$$\begin{aligned} r_1 \quad &= \quad d\left[(0.25, 0.75), (1, \frac{2}{3})\right] = \sqrt{(0.25-1)^2 + (0.75 - \frac{2}{3})^2}, \quad r_2 = d\left[(0.25, 0.75), (2, 1)\right] = \sqrt{(0.25-2)^2 + (0.75-1)^2} \\ r_3 \quad &= \quad d\left[(0.25, 0.75), (1, \frac{4}{3})\right] = \sqrt{(0.25-1)^2 + (0.75 - \frac{4}{3})^2}, \quad r_4 = d\left[(0.25, 0.75), (0, 1)\right] = \sqrt{(0.25-0)^2 + (0.75-1)^2} \end{aligned}$$

Table 5.2: The temperature inside the insulator of the device

Internal point	Coordinates		BEM solution
Points	x_1	x_2	$T(x_1,x_2)$
1	0.0	0.6667	0.2679
2	0.0	0.9167	0.2765
3	0.25	0.6667	0.2581
4	0.25	0.9167	0.2699
5	0.25	1.1667	0.2682
6	0.50	0.6667	0.2040
7	0.50	0.9167	0.2212
8	0.75	0.6667	0.1731
9	0.75	0.9167	0.1889
10	0.75	1.1667	0.2210
11	1	0.9167	0.1431
12	1	1.1667	0.2788
13	1.25	0.9167	0.0300
13	1.5	1.1667	-0.1603
15	1.75	1.1667	-0.2910

Graphically we can show the solution as follows,

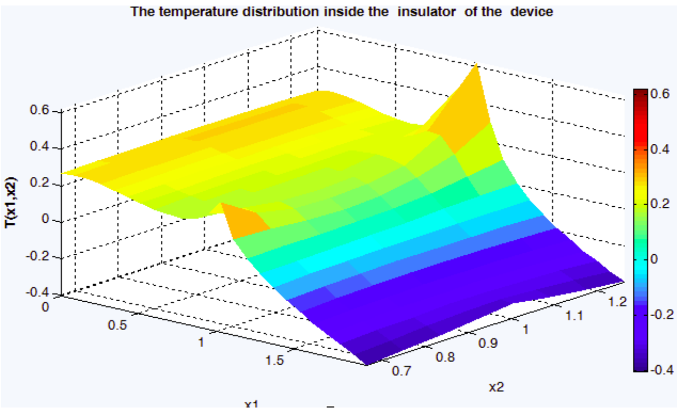

Figure 5.13: BEM solution graph on the domain $[0,2]\times[\frac{2}{3},\frac{4}{3}]$

For the point $x \in \Omega_3$, the problem given as follows.

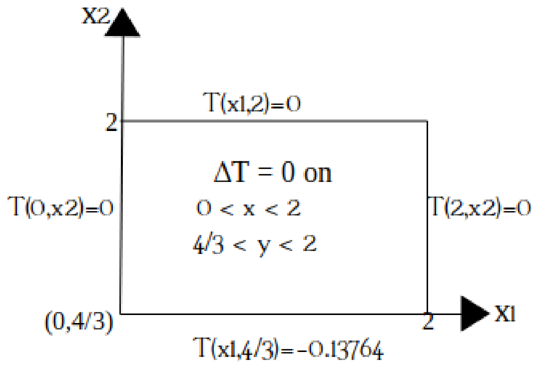

Figure 5.14: Top electrode

By using the separation of variables method, and superposition principle, our exact solution of the problem in Ω_3 is given by:

$$T(x_1,x_2) = \frac{1}{\sinh\left(\frac{n}{6}\pi\right)}\left[\int_0^2 -0.13764\sin\left(\frac{n\pi}{2}x_1\right)dx_1\right]\sin\left(\frac{n\pi}{2}x_1\right)\sinh\left(\frac{n\pi}{2}x_2\right)$$

Now, we use the obtained boundary values, to determine the values at any point in the subdomain Ω_3 using the following equation from equation(4.14).

$$T(x) = \int_\Gamma G(x,y)q(y)d\Gamma - \int_\Gamma F(x,y)T(y)d\Gamma$$

This can be done by dividing the boundary integral into the number of elements.

$$T(x) = \int_{\Gamma_1} G(x,y)q(y)d\Gamma_1 + \int_{\Gamma_2} G(x,y)q(y)d\Gamma_2 + \cdots - \int_{\Gamma_1} F(x,y)T(y)d\Gamma_1 - \int_{\Gamma_2} F(x,y)T(y)d\Gamma_2 - \cdots \quad (5.7)$$

i.e,

$$\begin{aligned} T(x) \quad = \quad & \int_{\Gamma_I^1} \frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right)q_{I_2}\,dx_1 + \int_{\Gamma_6}\frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right)q_6\,dx_2 + \int_{\Gamma_7}\frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right)q_7\,dx_1 + \int_{\Gamma_8}\frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right)q_8\,dx_2 + \\ & \int_{\Gamma_{I_2}}\frac{1}{2\pi}\left(\frac{1}{r_1}\right)T_{I_2}\,dx_1 + \int_{\Gamma_6}\frac{1}{2\pi}\left(\frac{1}{r_2}\right)T_6\,dx_2 + \int_{\Gamma_7}\frac{1}{2\pi}\left(\frac{1}{r_3}\right)T_7\,dx_1 + \int_{\Gamma_8}\frac{1}{2\pi}\left(\frac{1}{r_4}\right)T_8\,dx_2 \end{aligned}$$

E.g. To find the approximate temperature at interior point say $(x_1,x_2) = (0.25, 1.5)$,

$$\begin{aligned} T(0.25,1.5) \quad = \quad & \int_0^2 \frac{1}{2\pi}\ln\left(\frac{1}{r_1}\right)*q_{I_1}\,dx_1 + \int_{\frac{2}{3}}^{\frac{4}{3}}\frac{1}{2\pi}\ln\left(\frac{1}{r_2}\right)*q_6\,dx_2 + \int_2^0\frac{1}{2\pi}\ln\left(\frac{1}{r_3}\right)*q_7\,dx_1 + \int_{\frac{4}{3}}^{\frac{2}{3}}\frac{1}{2\pi}\ln\left(\frac{1}{r_4}\right)*q_8\,dx_2 + \\ & \int_2^0\frac{1}{2\pi}(\frac{1}{r_1})*T_{I_1}\,dx_1 + \int_{\frac{2}{3}}^{\frac{4}{3}}\frac{1}{2\pi}\left(\frac{1}{r_2}\right)*T_6\,dx_2 + \int_2^0\frac{1}{2\pi}\left(\frac{1}{r_3}\right)*T_7\,dx_1 + \int_{\frac{4}{3}}^{\frac{2}{3}}\frac{1}{2\pi}\left(\frac{1}{r_4}\right)*T_8\,dx_2 \end{aligned}$$

Where

$$\begin{aligned} r_1 \quad &= \quad d\left[(0.25,1.5),(1,\frac{4}{3})\right] = \sqrt{(0.25-1)^2+(1.5-\frac{4}{3})^2}, \quad r_2 = d\left[(0.25,1.5),(2,\frac{5}{3})\right] = \sqrt{(0.25-2)^2+(1.5-\frac{5}{3})^2} \\ r_3 \quad &= \quad d\left[(0.25,1.5),(1,2\right] = \sqrt{(0.25-1)^2+(1.5-2)^2}, \quad r_4 = d\left[(0.25,1.5),(0,\frac{5}{3})\right] = \sqrt{(0.25-0)^2+(1.5-\frac{5}{3})^2} \end{aligned}$$

Table 5.3: The temperature inside the top electrode of the device

Internal point	Coordinates		BEM solution
Points	x_1	x_2	$T(x_1,x_2)$
1	0.0	1.3333	0.1071
2	0.0	1.5833	0.1388
3	0.0	1.3333	0.1040
4	0.25	1.3333	0.1287
5	0.25	1.5833	0.1261
6	0.5	1.3333	0.1707
7	0.5	1.5833	0.1431
8	0.5	1.8333	0.0933
9	0.75	1.3333	0.3000
10	0.75	1.5833	0.1990
11	0.75	1.8333	0.0865
12	1.0	1.8333	0.0616
13	1.25	1.5833	0.1686
14	1.25	1.8333	0.0569
15	1.50	1.3333	0.1145
15	1.50	1.5833	0.0781
16	1.50	1.8333	0.0303
17	1.75	1.3333	0.0417
17	1.75	1.5833	0.0128

Graphically we can show the solution as follows,

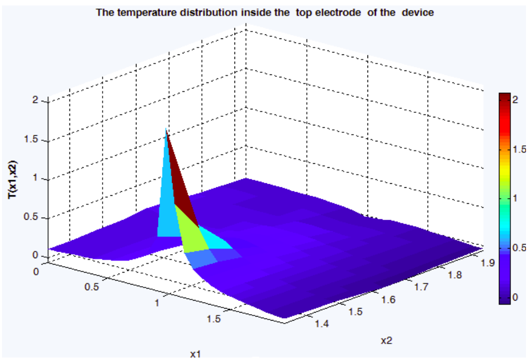

Figure 5.15: BEM solution graph on the domain $[0,2]\times[\frac{4}{3},2]$

The temperature distribution at a discrete point in the subdomains.

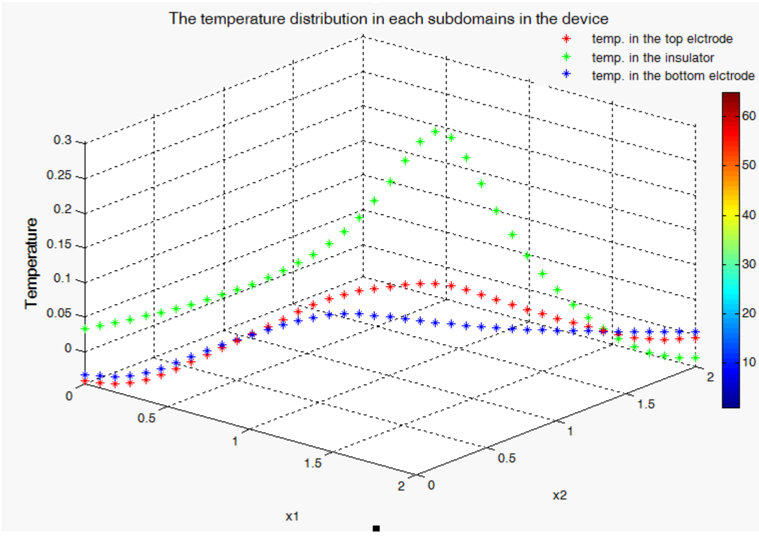

Figure 5.16: Comparison of temperatures in each subdomains.

5.2 Discussions

The given metal-insulator-metal structure, which is considered as a two dimensional, is insulated on the outer edge of the layer and there is temperature s_2 at the insulator as shown earlier. The temperature on boundary of this domain is zero but, its normal derivative (flux) have nonzero values on the boundary of the each subdomains. Our solution is a temperature distribution of the of resistive switching device with variable thermal conductivity. The graphs and the tables show the temperature distribution inside the domain. As seen from the graphs and tables, the temperature distribution in the device decrease from center of the domain to its edge due to there no temperature on the boundary of the domain.

Chapter 6

Conclusion

In this thesis, we have given brief discussion on the the BEM for the heat flow in the device with variable thermal conductivity. We used the BEM for computing the temperature distribution in a metal-insulator-metal structure with heat sources within the insulator. To do this we have formulated the BIE for each layer. Then we discretize the BIE and the given domain into a number of elements for each layers. After that combined the equation of each layers using the continuity conditions at the interfaces, we obtained a large linear system. By solving this linear system we solved the temperature and the flux at the boundary of the device. Finally, using the temperature and flux obtained at the boundary, we computed the distribution of the temperature inside the resistive device separately foreach sub domains.

6.1 Recommendation

We will now suggest some ideas on how this thesis might be modified and extended, including some ideas for future work, theoretically, it is straightforward to extend the BEM to three-dimensional problems of heat conduction, medium using the fundamental solution for the three-dimensional problems.
This study can be considered the following recommendation:

- More work on the numerical analysis, convergence and stability.
- The researcher may compare the value obtained by BEM with other methods.
- Considering the problem for more complicated material property.

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Bibliography

[1] Larentis, S., Nardi, F., Balatti, S., Gilmer, D. C., & Ielmini, D. (2012). Resistive switching by voltage-driven ion migration in bipolar RRAM—Part II: Modeling. *IEEE Transactions on Electron Devices,* 59(9), 2468-2475.

[2] Hickmott, T. W. (1964). Impurity conduction and negative resistance in thin oxide films. *Journal of Applied Physics*, 35(7), 2118-2122.

[3] Ramu, A. T., & Strukov, D. B. (2013). Thermal modeling of resistive switching devices. *IEEE Transactions on Electron Devices* 60(6), 1938-1943.

[4] Alexandrov, A. S., Bratkovsky, A. M., Bridle, B., Savel'Ev, S. E., Strukov, D. B., & Stanley Williams, R. (2011). Current-controlled negative differential resistance due to Joule heating in TiO2. *Applied Physics Letters,* 99(20), 202104.

[5] Kang, J., Huang, P., Gao, B., Li, H., Chen, Z., Zhao, Y., ... & Liu, X. (2016). Design and application of oxide-based resistive switching devices for novel computing architectures. *IEEE Journal of the Electron Devices Society,* 4(5), 307-313.

[6] Lee, M. J., Lee, C. B., Lee, D., Lee, S. R., Chang, M., Hur, J. H., ... & Chung, U. I. (2011). A fast, high-endurance and scalable non-volatile memory device made from asymmetric Ta 2 O 5- x/TaO 2- x bilayer structures. *Nature materials,* 10(8), 625.

[7] Pickett, M. D., Strukov, D. B., Borghetti, J. L., Yang, J. J., Snider, G. S., Stewart, D. R., & Williams, R. S. (2009). Switching dynamics in titanium dioxide memristive devices. *Journal of Applied Physics,* 106(7), 074508.

[8] Brebbia, C. A., & Telles, J. T. (1984). E, Wrobel, LC: Boundary element techniques. Bedim Springer.

[9] Antes, M. Y., Karinski, Y. S., & Yankelevsky, D. Z. (2008). The modified Shanks transform for the solution of elastic problems by boundary integral equation (BIE) method. *Communications in Numerical Methods in Engineering,* 24(1), 65-82.

[10] Katsikadelis, J. T. (2002). Boundary elements: *theory and applications.* Elsevier.

[11] Austin, T. M., Trew, M. L., & Pullan, A. J. (2006). Solving the cardiac bidomain equations for discontinuous conductivities. *IEEE Transactions on Biomedical Engineering,* 53(7), 1265-1272.

[12] Aliabadi, M. H. (2002). The boundary element method, *applications in solids and structures* (Vol. 2). John Wiley & Sons.

[13] Maz'ya, V. G., Nazarov, S. A., & Plamenevskii, B. A. (1979). References to Maz'ya's Publications Made in Volume II. *Partial Differential*, 24, 381.

[14] Zou, F. (2015). *A boundary element method for modelling piezoelectric transducer based structural health monitoring* (Doctoral dissertation, Imperial College London).

[15] Yu, C., Li, G., Kumar, S., Yang, K., & Jin, R. (2014). Phase transformation synthesis of novel Ag2O/Ag2CO3 heterostructures with high visible light efficiency in photocatalytic degradation of pollutants. *Advanced materials,* 26(6), 892-898.

[16] Gaul, L., Kögl, M., & Wagner, M. (2003). Boundary Element Method for Elastic Continua. In *Boundary Element Methods for Engineers and Scientists* (pp. 141-173). Springer, Berlin, Heidelberg.

[17] Trevelyan, J. (1994). *Boundary elements for engineers: theory and applications* (Vol. 1). Computational Mechanics.

[18] Clements, D. L., & Haselgrove, M. (1985). The boundary integral equation method for the solution of problems involving elastic slabs. *International journal for numerical methods in engineering,* 21(4), 663-670.

[19] Orlowska, S., Beroual, A., & Fleszynski, J. (2004). Boundary integral equation method in the prediction of dielectric characteristics of humidified composite insulator glass–epoxy core. *Journal of Physics D: Applied Physics,* 37(5), 758.

[20] Banerjee, P. K., & Butterfield, R. (1981). *Boundary element methods in engineering science* (Vol. 17, p. 578). London: McGraw-Hill.

[21] Harbrecht, H., & Kalmykov, I. (2019). Sparse grid approximation of the Riccati operator for closed loop parabolic control problems with Dirichlet boundary control.

[22] J.Chen, J. T., Wu, C. S., Chen, K. H., & Lee, Y. T. (2006). Degenerate scale for the analysis of circular thin plate using the boundary integral equation method and boundary element methods. *Computational Mechanics,* 38(1), 33-49.

[23] Theotokoglou, E. E. (2004). Boundary integral equation method to solve embedded planar crack problems under shear loading. *Computational Mechanics,* 33(5), 327-333.

[24] Green G. (1828). An Essay on the Application on Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham.

[25] Kanwal, R. P. (1969). Integral equations formulation of classical elasticity. *Quarterly of Applied Mathematics,* 27(1), 57-65.

[26] Rizzo, F. J. (1967). An integral equation approach to boundary value problems of classical elastostatics. *Quarterly of applied mathematics,* 25(1), 83-95

[27] Liu, Y. (2009). Fast multipole boundary element method: theory and applications in engineering. Cambridge university press.

[28] Cheng, A.H.D.,&Cheng,D. (2005) Heritage and early history of the boundary element method.Engineering Analysis with the boundary elements,29(3) 268-302

[29] Evans,L C. (2010) Partial differential equations (v.19).American Mathematical soc..

[30] Gaul, L., Kögl, M., & Wagner, M. (2013). Boundary element methods for engineers and scientists: an introductory course with advanced topics. Springer Science & Business Media.

[31] Becker, A. A. (1992). The boundary element method in engineering: a complete course. McGraw-Hill Companies.

[32] Mushtaq, M., Shah, N. A., & Muhammad, G. (2010). Advantages and disadvantages of boundary element methods for compressible fluid flow problems. Journal of American Science, 6(1), 162-165.

[33] Jaswon, M. A., & Ponter, A. R. (1963). An integral equation solution of the torsion problem. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 273(1353), 237-246.

[34] Ghassemi, H., Panahi, S., & Kohnsal, A. R. (2016). Solving the Laplace's equation by FDM and BEM using mixed boundary conditions. American Journal of Applied Mathematics and Statistics, 4(2), 37-42.