SOLVING BOUSSINESQ-LIKE B(M,N) EQUATIONS USING HOMOTOPY ANALYSIS METHOD

Abdi Hailu Tesema

A Thesis Submitted to the Department of Applied Mathematics,

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics (Differential Equations)

Office of Graduate Studies

Adama Science and Technology University

June, 2024

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Declaration

I hereby declare that this Master Thesis entitled " ***Solving Boussinesq-like B(m,n) equations using Homotopy Analysis Method***" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Solving Boussinesq-like B(m,n) equations using Homotopy Analysis Method "*** prepared under our guidance by **Abdi Hailu Tesema** submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics. Therefore, we recommend that the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled ***"Solving Boussinesq-like B(m,n) equations using Homotopy Analysis Method"*** and developed by **Abdi Hailu Tesema**, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

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Co-advisor	Signature	Date

Approval of Board of Examiners

We, the undersigned, members of the Board of Examiners of the thesis by **Abdi Hailu Tesema**, have read and evaluated the thesis entitled ***"Solving Boussinesq-like B(m,n) equations using Homotopy Analysis Method"*** and examined the candidate during the open defense. This is, therefore, to certify that the thesis is accepted for partial fulfilment of the requirement of the degree of Master of Science in Applied Mathematics.

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List of Abbreviations

ADM	Adomain Decomposition Method
DEs	Differential Equations
HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method
KdV	Korteweg-De Vries
PDEs	Partial Differential Equations
PT	Perturbation Technique
SEIRS	Susceptible Exposed Infectious Recovered Susceptible
VIM	Variational Iteration Method

Abstract

In this thesis, we have used the homotopy analysis method (HAM) to obtain approximate analytical solutions of Boussinesq-like $B(m, n)$ *equations. The Homotopy Analysis Method is a mathematical technique used to solve nonlinear differential equations. This method employs the concept of the homotopy from topology to generate a convergent series solution for nonlinear equations. To assess the efficiency of the HAM, we conducted a comparative analysis between its results and the results obtained by the well-known numerical techniques(FDM). Two problems involving Boussinesq-like B(m,n) equations were considered to demonstrate the efficiency and effectiveness of the homotopy analysis method. The results were visualized through plotted graphs using matlab code.*

Keywords: *Homotopy Analysis Method, Boussinesq-like* $B(m, n)$ *equations.*

CHAPTER ONE
INTRODUCTION

1.1 Background of the study

Most problems are essentially nonlinear and expressed as nonlinear differential equations. In particular, nonlinear DEs appear frequently in physics, engineering, and several other fields. Unlike linear differential equations, solving nonlinear ones is generally more challenging. Due to the complexity of nature, practically many processes and phenomena in science and engineering are intrinsically nonlinear, and they are represented by nonlinear PDEs. For this reason, nonlinear PDEs have piqued the interest of many mathematicians and applied scientists(Loyinmi & Idowu, 2023). Nonlinear PDEs are frequently used to describe a wide range of processes and real-world phenomena, such as genetic configurations, mutation, and variations in Fisher's equation, magnetic flux, intensity, and quantum field theory in sine Gordon equation, shallow water waves and patterns in Korteweg-De Vries (KdV) equation, advection-diffusion mechanisms and dynamics in Burgers-equation, Huxley's and so on (Morenikeji et al., 2021).

Due to the difficulty in obtaining the exact solutions of nonlinear differential equations, semi-analytic solutions and numerical solutions are provided (Loyinmi & Idowu, 2023). However, it is crucial to carefully assess the consistency and accuracy of these methods to ensure reliable results. Solving nonlinear problems, especially analytically, can pose challenges in many cases. Traditionally, perturbation techniques have been widely used to tackle such problems. These techniques rely on the presence of small or large parameters, known as perturbation quantities, to simplify the analysis. However, there are numerous nonlinear problems in science and engineering that lack these perturbation quantities, making the application of perturbative techniques difficult. To address this limitation, non-perturbative techniques have been developed. These techniques do not rely on the assumption of small parameters and provide alternative approaches to solving nonlinear problems. By considering the problem as a whole, rather than breaking it down into perturbation quantities, non-perturbative techniques offer a broader scope of applicability.

Despite the advantages of both perturbative and many non-perturbative techniques, they still face challenges when it comes to adjusting or controlling the convergence region. Convergence refers to the process of reaching a stable and accurate solution. In the context of solving nonlinear problems, it is important to have methods that allow for easy adjustment or control of the convergence region.

To overcome such problems, the Homotopy Analysis Method (HAM) is developed and introduced first by (S.-J. Liao, 1992). The HAM is a general approximate analytic approach used to obtain series solutions of nonlinear equations of various types, including algebraic equations, ordinary differential equations, partial differential equations and differential–integral equations (Słota et al., 2019). This method is valid no matter whether a nonlinear problem contains small/large physical parameters or not, which is essentially required in perturbation techniques(S. Liao, 2012). More importantly, unlike all perturbation and traditional non-perturbation methods, the homotopy analysis method provides us with both the freedom to choose proper base functions for approximating a nonlinear problem and a simple way to ensure the convergence of the solution series (Abbas, 2014)

Perturbation technique (PT) is widely used to investigate physical systems that can be exactly solved but contain small perturbation parameters(Roozi et al., 2011). When applying PT to such a system, expansion around the perturbation parameter is involved, and approximations are expressed as power series of these parameters. However, non-perturbative techniques have been established to explore physical problems that do not have a small physical parameter that can be used as the perturbation parameter(S. Liao, 2005).

Boussinesq-like $B(m,n)$ equations are a class of nonlinear partial differential equations that are derived from the Boussinesq equation(Boussinesq, 1877), which is a well-known equation in fluid dynamics. In this thesis we have proposed Boussinesq-like $B(m,n)$ equations of the form:

$$u_{tt} + (u^m)_{xx} - (u^n)_{xxxx} = 0, \qquad m, n > 1, \tag{1.1}$$

where $u = u(x,t)$.
In this equations $u(x,t)$ represents the unknown function that describes the displacement of the fluid particles in the horizontal direction at position x and time t. The parameters m and n represent the exponents in the nonlinear terms of the equations.

Boussinesq-like $B(m, n)$ equations have wide-ranging applications in the study of wave propagation, coastal engineering, tsunami modeling, nonlinear optics, and fluid dynamics. These equations provide valuable insights into the behavior of waves and help in understanding and predicting complex wave phenomena in different physical systems. Given the significance and practical relevance of this problem, many researchers have applied various techniques to find solutions for this model. Some of the methods used to obtain solutions for the Boussinesq-like $B(m, n)$ equations include the ADM, VIM, HPM and others. These techniques have been employed to explore and analyze the behavior of the Boussinesq-like $B(m, n)$ equations in different scenarios.

The motivation of present study is the application of HAM for nonlinear partial differential equations. The HAM can be used to solve some complicated highly nonlinear PDEs so as to enrich and deepen our understandings about these interesting nonlinear problems. For example, By means of the HAM, an explicit analytic approximation of the optimal exercise boundary of American put option was gained, which is often valid for a couple of decades prior to expiry, whereas the asymptotic and perturbation formulas are valid only for a couple of days or weeks in general. The two-dimensional $2nd\_order$ Gelfand equation was solved by means of the HAM in a rather easy way by transferring it into an infinite number of linear PDEs governed by a very simple $4th$-order linear operator(S. Liao & Tan, 2007). The wave-resonance criterion for an arbitrary number of traveling waves with large amplitudes was first established using the HAM (S. Liao, 2012). All of these applications show the originality, validity and generality of the HAM for nonlinear problems.

1.2 Statement of the problem

Many mathematical methods have been employed by different authors to solve numerous physical phenomena. One such phenomenon is the Boussinesq-like $B(m,n)$ equations, a nonlinear partial differential equation that describes the behavior of certain physical systems, particularly in the field of fluid dynamics. Although extensive research has been conducted on this equation using different methods, there is a noticeable gap in the literature when it comes to systematically applying the HAM to solve the Boussinesq-like $B(m,n)$ equations . The existing literature on the Boussinesq-like $B(m,n)$ equations primarily focuses on analytical and numerical techniques, such as perturbation methods, numerical simulations and exact solutions for specific cases.

While the HAM has been used to solve various differential equations, to the best of our knowledge it has not yet been applied to solve Boussinesq-like $B(m,n)$ equations. By addressing this gap, this thesis aims to investigate and develop a systematic framework for solving the Boussinesq-like $B(m,n)$ equations using the HAM. The HAM offers the potential to provide accurate and efficient analytical approximate solutions, which can enhance our understanding of the dynamics and behavior of the Boussinesq-like $B(m,n)$ equations. Therefore, this thesis aims to fill the gap in the literature by employing the HAM to obtain analytical approximate solutions for the Boussinesq-like $B(m,n)$ equations, thereby contributing to the broader field of nonlinear dynamics and partial differential equations.

1.3 Objectives

1.3.1 General Objective

The general objective of this study is to employ the Homotopy Analysis Method for solving Boussinesq-like $B(m, n)$ equations.

1.3.2 Specific Objectives

The specific objectives of this study are to:

- find approximate analytical solutions of Boussinesq-like $B(m, n)$ equations by taking different cases,
- compare the results obtained through numerical method with the approximate analytical solutions,
- investigate the effectiveness of the homotopy analysis method.

1.4 Significance of the Study

The significance of this study are to:

- serve the scholars who works on this area as a reference material.
- help the researchers as they motivate in mathematical physics for further study.
- gain an understanding of how the HAM is utilized to solve nonlinear problems.

1.5 Scope of the study

This thesis is mainly to provide a valuable computational tool for obtaining approximate analytical solutions to these equations, contributing to the understanding and analysis of fluid dynamics.

CHAPTER TWO
LITERATURE REVIEW

The homotopy is a fundamental concept in topology, which can be traced back to Jules Henri Poincar's (1854–1912), a French mathematician. Based on the homotopy, two methods have been developed(Abbasbandy & Zakaria, 2008). (Iba et al., 1991) in their article showed that the homotopy continuation method dating back to 1930s, which is a global convergent numerical method mainly for nonlinear algebraic equations. The other is homotopy analysis method proposed in 1992 by Shi- jun Liao, which is an analytic approximation method with guarantee of convergence, mainly for nonlinear differential equations(S.-J. Liao & Cheung, 2003)

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (Alomari et al., 2009). This method has been successfully applied to solve many types of non-linear problems. The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer power. The HAM is better since it does not involve discretization of the variables hence it is free from rounding off errors and does not require large computer memory or time. Unlike the homotopy continuation method that is a global convergent numerical method mainly for nonlinear algebraic equations, the HAM is a kind of analytic approximation method mainly for nonlinear differential equations.

Furthermore, the HAM provides larger freedom to choose the auxiliary linear operator L. Most importantly, the so-called convergence-control parameter is introduced for the first time in the homotopy equation so that the HAM provides us a convenient way to guarantee the convergence of series(Al-Hayani & Fahad, 2019). (S.-J. Liao, 1992) used HAM to solve a two-dimensional non linear progressive gravity wave and he obtained a better analytic approximation of the fourth order than those given by Perturbation methods.

(Abbasbandy, 2006) applied homotopy analysis method to non linear equation arising in heat transfer. In his work, the basic idea of the HAM was introduced and then applied to obtain the solution of nonlinear equations arising in heat transfer. He explained further that the nonlinear equations of the heat radiation and conduction equations of a fin in the steady state and in the free space are solved through HAM. Comparisons were made between HAM, perturbation method and HPM. The HAM provided a convenient way to control the convergence of approximation series which is a fundamental qualitative difference in analysis between HAM and other methods. Also it has been shown that the HPM and perturbation method are valid only for small parameter, but in HAM we can choose control-convergence parameter h in an appropriate way. This shows the validity and great potential of HAM for non linear problem in science and engineering.

(Domairry et al., 2009) employed HAM to solve nonlinear differential equation governing Jeffery–Hamel flow. In their work they showed that the Jeffery–Hamel flow has been studied and its nonlinear ordinary differential equation has been solved through HAM. The obtained solution in comparison with the numerical ones represents a remarkable accuracy. The results also indicate that HAM can provide us with a convenient way to control and adjust the convergence region.

(Jafari et al., 2010), in their article showed that higher dimensional initial boundary value problems of variable coefficients can be solved by means of an analytic technique, namely the HAM. There also comparisons were made between the ADM, the exact solution and the HAM. They discovered that all the examples solved shows that the results obtained from HAM was in perfect agreement with those obtained by ADM.

(Yıldırım, 2009) applied HPM to obtain exact special solutions with solitary patterns for Boussinesq-like $B(m, n)$ equations with fully nonlinear dispersion. The homotopy perturbation method is generally used to find physical solutions satisfying boundary/initial conditions.

(Gupta & Gupta, 2012), applied homotopy analysis method to solve Nonlinear Cauchy problem. They showed by of means HAM, that the solutions of some linear Cauchy problems of parabolic-hyperbolic type are exactly obtained in the form of convergent Taylor series. The HAM contains the auxiliary parameter h, that provide a convenient way of controlling the convergent region of series solutions. The results reveal that the proposed method is very effective.

(Duan et al., 2012) in their article showed that higher dimensional initial boundary value problems of variable coefficients can be solved by means of an analytic technique, namely the homotopy analysis method .There also comparisons were made between the ADM, the exact solution and the HAM. They discovered that all the examples solved shows that the results obtained from HAM was in perfect agreement with those obtained by ADM(Babolian et al., 2004). It was apparently seen that HAM is a very powerful and efficient technique in finding analytical solutions for wider classes of linear and non linear problems.

(Chioma et al., 2019) applied homotopy analysis method to epidemic model to defend the propagation of an infectious disease. Firstly, they formulated an SEIRS mathematical model and showed that the population classes are non-negative and then they determined the basic reproduction number of the disease by using the next generation matrix method. from their numerical solution, they demonstrated the ability of HAM to converge very fast, they saw that the HAM converges in just few iterations. For the accuracy of the method, it has been established by (S.-J. Liao, 1992). Hence, they conclude that HAM is a very efficient and accurate method in solving nonlinear mathematical Model.

Based on the literature survey conducted, it appears that there have been no studies on the approximate analytical solution of the Boussinesq-like $B(m, n)$ equations using the HAM. Therefore, this thesis aims to fill this research gap by focusing on obtaining an approximate analytical solution of the Boussinesq-like $B(m, n)$ equations using HAM.

CHAPTER THREE
PRELIMINARIES

In this chapter, the homotopy analysis method (HAM) is systematically described in details as a whole. Mathematical theorems related to the so-called homotopy-derivative operator and deformation equations are proved, which are helpful to gain high-order approximations.

3.1 Partial Differential Equation

A partial differential equation is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function (Greiner et al., 1997)

$$F(x_1, ..., x_n; u, \frac{du}{dx_1}, ..., \frac{du}{dx_n}; \frac{d^2u}{dx_1x_2}, ..., \frac{d^2u}{dx_1x_n}; ...) = 0. \tag{3.1}$$

3.1.1 Nonlinear Partial Differential Equation

A non-linear partial differential equation is a partial differential equation that is not a linear equation in the unknown function and its derivatives. The Boussinesq-like $B(m, n)$ equations is an example of nonlinear PDEs.

3.2 Taylor series

The essential tool in the development of approximate analytical solution is Taylor's series. Taylor's series will enable us to approximate a function with a polynomial, and polynomials are easy to compute.

Definition 3.1 *(Moreno et al., 1996) Let $f(x)$ be a function which is analytic at $x = x_0$. Then we can write $f(x)$ as the power series, called the Taylor series of $f(x)$ at $x = x_0$.*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x - x_0)}{3!}(x - x_0)^3 + \dots, \tag{3.2}$$

If we write the n^{th} derivative of $f'(x)$ as $f^n(x)$, this becomes:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n. \tag{3.3}$$

3.3 Concept of homotopy

The homotopy analysis method proposed by (S.-J. Liao, 1992) is based on the concept of the homotopy, a fundamental concept in topology and differential geometrySen & Sterman (1983). The concept of the homotopy can be traced back to Jules Henri Poincaré (1854 — 1912), a French mathematician. Shortly speaking, a homotopy describes a kind of continuous variation or deformation in mathematics. For example, a circle can be continuously deformed into a square or an ellipse, the shape of a coffee cup can deform continuously into the shape of a doughnut. However, the shape of a coffee cup can not be distorted continuously into the shape of a football. In particular, we consider the following definition.

Definition 3.2 *(S. Liao, 2012) A homotopy between two continuous functions $f(x)$ and $g(x)$ from a topological space X to a topological space Y is formally defined to be a continuous function $H : X \times [0,1] \longrightarrow Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then $H(x;0) = f(x)$ and $H(x;1) = g(x)$.*

Let $C[a,b]$ denote the set of all continuous real functions on the interval $[a,b]$. If a continuous function $f \in C[a,b]$ can be deformed continuously into another continuous function $g \in C[a,b]$, one can construct a homotopy

$H : f(x) \sim g(x)$, in the way

$$\begin{aligned} H(x;q) &= (1-q)f(x) + qg(x), \quad x \in [a,b], \quad q \in [0,1] && (3.4) \\ &= f(x) + q[g(x) - f(x)], \quad x \in [a,b], \quad q \in [0,1]. && (3.5) \end{aligned}$$

Definition 3.3 *(Al-Hayani & Fahad, 2019) The embedding parameter $q \in [0,1]$ in a homotopy of functions or equations is called homotopy-parameter.*

Example: (S. Liao, 2012) Let X and Y be any topological spaces, $f(x) = sin(\pi x)$ and $g(x) = 8x(x-1)$ in the interval $x \in [0,1]$. Since f and g are continuous in interval $[0,1]$ then $H : f(x) \sim g(x)$.
Define $H : X \times [0,1] \longrightarrow Y$ by:

$$H(x;q) = (1-q)f(x) + qg(x) \tag{3.6}$$
$$= (1-q)sin(\pi x) + q8x(x-1), \quad \forall x \in X, \quad \forall q \in [0,1]. \tag{3.7}$$

Especially, when $q = 0$, we have

$$H(x;0) = sin(\pi x), \qquad x \in [0,1], \tag{3.8}$$

and when $q = 1$, it holds

$$H(x;1) = 8x(x-1), \qquad x \in [0,1], \tag{3.9}$$

respectively. So, as the embedding parameter $q \in [0,1]$ increases from 0 to 1, the real function $H(x;q)$ varies continuously from a trigonometric function $sin(\pi x)$ to a polynomial $8x(x-1)$, as shown in Figure 3.1.

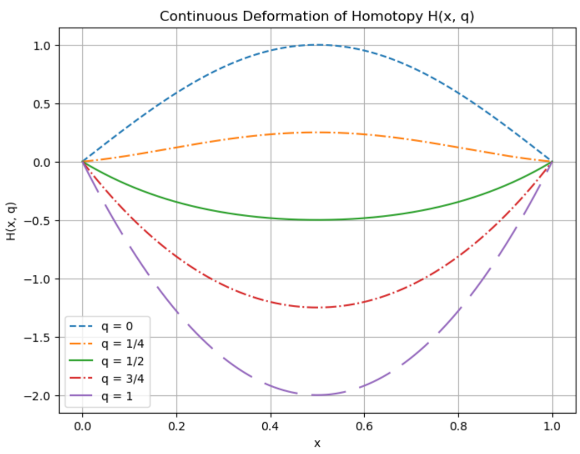

Figure 3.1: Continuous deformation of the homotopy $H(x;q) : sin(\pi x) \sim 8x(x-1)$ when $q = 0$, $q = 1/4$, $q = 1/2$, $q = 3/4$, and $q = 1$.

3.4 Derivation of Newton-iterative method using homotopy

Following the book of Liao (S. Liao, 2012), we can use the idea of homotopy to deduce the Newton's iterative method for solving nonlinear algebraic equations.

Consider a nonlinear algebraic equation

$$f(x) = 0, \tag{3.10}$$

where $f(x) \in C^{\infty}[a, b]$ is a continuous real function. Assume that the above equation has at least one solution in the region $x \in [a, b]$.

Let $x_0 \in [a, b]$ denote an initial guess of the unknown solution x. Obviously, $f(x) - f(x_0) \in C^{\infty}[a, b]$ can be deformed continuously to $f(x) \in C^{\infty}[a, b]$, i.e. they are homotopic. Thus, we can construct such a homotopy of function

$$H(x; q) = (1 - q)[f(x) - f(x_0)] + qf(x),$$

where $q \in [0, 1]$ is the homotopy-parameter. When $q = 0$ and $q = 1$, we have

$$H(x, 0) = f(x) - f(x_0), \quad H(x, 1) = f(x),$$

respectively. Thus, as q increases from 0 to 1, $H(x; q)$ varies continuously from $f(x) - f(x_0)$ to $f(x)$. Such kind of continuous variation is called deformation in topology (Sen & Sterman, 1983).
Now, enforcing $H(x; q) = 0$, i.e

$$(1 - q)[f(x) - f(x_0)] + qf(x) = 0, \qquad q \in [0, 1],$$

we have now a parameter-family of algebraic equations. The solution of the above parameter-family of algebraic equations is dependent upon the homotopy-parameter q. Replacing x by $\widetilde{x}(q)$, this family of equations can be rewritten more precisely in the form

$$(1 - q)\left\{f[\widetilde{x}(q)] - f(x_0)\right\} + qf[\widetilde{x}(q)] = 0. \tag{3.11}$$

When $q = 0$, we have

$$f[\widetilde{x}(0)] - f(x_0) = 0,$$

whose solution is

$$\widetilde{x}(0) = x_0.$$

When $q = 1$, we have

$$f[\widetilde{x}(1)] = 0,$$

which is exactly the same as the original algebraic equation $f(x) = 0$. So, it holds

$$\widetilde{x}(1) = x.$$

Therefore, as the homotopy-parameter q increases from 0 to 1, $\widetilde{x}(q)$ deforms from the initial guess x_0 to the solution x of the original equation $f(x) = 0$. So, Equation (3.11) defines a homotopy of function $\widetilde{x}(q)$: $x_0 \sim x$. For simplicity, the parameter- family of equations (3.11) is called the zeroth-order deformation equation, because it defines a continuous deformation from the initial guess x_0 to the solution x of the original equation $f(x) = 0$.

Note that $\widetilde{x}(q)$ is a function of the homotopy-parameter q. Assume that $\widetilde{x}(q)$ is analytic at $q = 0$ so that it can be expanded into a Maclaurin series with respect to the homotopy-parameter q, i.e.

$$\widetilde{x}(q) \sim x_0 + \sum_{k=1}^{+\infty} x_k q^k, \tag{3.12}$$

where $\widetilde{x}(0) = x_0$ is employed, and

$$x_k = \frac{1}{k!} \frac{d^k \widetilde{x}(q)}{dq^k}|_{q=0} = D_k[\widetilde{x}(q)]. \tag{3.13}$$

Here, the series (3.12) is called homotopy-Maclaurin series, D_k is called the homotopy-derivative operator, and $D_k[\widetilde{x}(q)]$ is called the k^{th}-order homotopy- derivative of $\widetilde{x}(q)$.

Note that $D_1[\widetilde{x}(q)] = x_1$.

Assuming that $\widetilde{x}(q)$ is analytic for $q \in [0,1]$, then

$$\widetilde{x}(1) = x = x_0 + \sum_{k=1}^{\infty} x_k.$$

and the M^{th} -order approximation of x is given by

$$x \approx x_0 + \sum_{k=1}^{M} x_k. \tag{3.14}$$

According to the fundamental theorem in calculus about Taylor series, the coefficient x_k of the homotopy-Maclaurin series (3.12) is unique. Therefore, the governing equation of x_k is unique, too, and can be deduced directly from the zeroth-order deformation equation (3.11). First, differentiating the zeroth-order deformation equation (3.12) with respect to the homotopy-parameter q, we have

$$f^{'}[\widetilde{x}(q)]\frac{d\widetilde{x}(q)}{dq} + f(x_0) = 0. \tag{3.15}$$

Then, setting $q = 0$ in the above equation and using the relationship $\widetilde{x}(0) = x_0$, we have

$$f^{'}[\widetilde{x}(0)]\frac{d\widetilde{x}(q)}{dq}\bigg|_{q=0} + f(x_0) = 0.$$

According to the equation (3.19), we have

$$\frac{d\widetilde{x}(q)}{dq}\bigg|_{q=0} = x_1.$$

Thus, we obtain the so-called 1st-order deformation equation

$$x_1 f^{'}(x_0) + f(x_0) = 0,$$

whose solution is

$$x_1 = -\frac{f(x_0)}{f'(x_0)}.$$

Using (3.14), we have the First-order homotopy-approximation

$$x \approx x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The method employed here is the widely recognized iterative approach known as the Newton-Raphson scheme. In fact, one can give a family of iteration formulas in a similar way. This shows the potential of the homotopy approach.

3.5 Properties of homotopy derivative

This section is devoted to the homotopy derivative and its properties.

Definition 3.4 *(S. Liao, 2012) Let ϕ be a function of the homotopy parameter q, then*

$$D_m(\phi) = \frac{1}{m!}\frac{d^m\phi}{dq^m}|_{q=0} \tag{3.16}$$

is called the m^{th} order homotopy derivative of ϕ, where $m\geq 0$ is an integer, and D_m is called the operator of m^{th} order homotopy derivative.

Theorem 3.1 *(S. Liao, 2012) For two arbitrary homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \qquad \Psi = \sum_{k=0}^{+\infty} w_k q^k$$

where ϕ and Ψ are analytic in $q \in [0, a)$, it holds

$$i. \quad D_m(\phi) = u_m \tag{3.17}$$

$$ii. \quad D_m(q^k\phi) = D_{m-k}(\phi) = \begin{cases} u_{m-k}, & 0 \leq k \leq m \\ 0, & otherwise \end{cases} \tag{3.18}$$

$$iii. \quad D_m(\Psi\phi) = \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\Psi) = \sum_{k=0}^{m} u_k w_{m-k} \tag{3.19}$$

$$= \sum_{k=0}^{m} D_{m-k}(\phi)D_k(\Psi) = \sum_{k=0}^{m} u_{m-k} w_k \tag{3.20}$$

$$iv. \quad D_m(\phi^{n+1}) = \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\phi^n) = \sum_{k=0}^{m} D_{m-k}(\phi)D_k(\phi^n) \tag{3.21}$$

where $m \geq 0$, $n \geq 1$, and $0 \leq k \leq m$ are integers.

***Proof*.**(i). According to Taylor theorem (Moreno et al., 1996), the unique coefficient u_m of the Maclaurin series

$$\phi = \sum_{m=0}^{+\infty} u_m q^m$$

with respect to the homotopy-parameter q is given by

$$D_m(\phi) = \frac{1}{m!}\frac{d^m\phi}{dq^m}|_{q=0}$$

which gives theorem (3.1) by means of the definition (3.4) of $D_m(\phi)$.

(ii). Let

$$\phi = \sum_{j=0}^{\infty} u_j q^j,$$

then

$$\begin{aligned} q^k \phi &= q^k \sum_{j=0}^{\infty} u_j q^j \\ &= \sum_{j=0}^{\infty} u_j q^{j+k} \end{aligned}$$

Let $k + j = m \longrightarrow j = m - k$ when $j = 0 \longrightarrow m = k$, then

$$q^k \phi = \sum_{m=k}^{\infty} u_{m-k} q^m,$$

when $j \longrightarrow \infty$ then $m \longrightarrow \infty$. By Theorem(3.1)(i)

$$D_m(q^k \phi) = u_{m-k} = D_{m-k}(\phi), \ \ when \ \ 0 \leq k \leq m$$

and

$$D_m(q^k \phi) = 0, \ \ when \ \ k \geq m$$

(iii). According to Leibnitz's rule for derivatives of product, it holds

$$\frac{\partial^m(\phi\Psi)}{\partial q^m} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{\partial^k \phi \partial^{m-k} \Psi}{\partial q^k \partial q^{m-k}} = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{\partial^k \Psi \partial^{m-k} \phi}{\partial q^k \partial q^{m-k}}$$

which gives according to definition (3.4) and theorem (3.1) that

$$\begin{aligned} D_m(\phi\Psi) = \frac{1}{m!} \frac{\partial^m(\phi\Psi)}{\partial q^m}|_{q=0} &= \sum_{k=0}^{m} \left(\frac{1}{k!} \frac{\partial^k(\Psi)}{\partial q^k}|_{q=0} \right) \left[\frac{1}{(m-k)!} \frac{\partial^{m-k}\phi}{\partial q^{m-k}}|_{q=0} \right] \\ &= \sum_{k=0}^{m} D_k(\phi) D_{m-k}(\Psi) = \sum_{k=0}^{m} u_k w_{m-k} \end{aligned}$$

(iv). write $\Psi = \phi^n$. By Theorem(3.1)(iii)

$$\begin{aligned} D_m(\phi^{n+1}) = D_m(\phi\Psi) &= \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\Psi) \\ &= \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\phi^n) \end{aligned}$$

Theorem 3.2 *(S. Liao, 2012) If*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k \quad and \quad \Psi = \sum_{k=0}^{+\infty} w_k q^k$$

are two homotopy Maclaurin series, where ϕ and Ψ analytic in $q \in [0, a)$, f and g are independent of the homotopy-parameter $q \in [0, 1]$, then it holds

$$D_m(f\phi + g\Psi) = fD_m(\phi) + gD_m(\Psi) = fu_m + gw_m \tag{3.22}$$

Proof. Since D_m by definition (3.4) is a linear operator, and moreover f and g are independent of q, then

$$\begin{aligned} D_m(f\phi + g\Psi) =& D_m(f\phi) + D_m(g\Psi) \\ =& fD_m(\phi) + gD_m(\Psi \end{aligned}$$

By Theorem (i), $D_m(\phi) = u_m$ and $D_m(\Psi) = w_m$, then

$$D_m(f\phi + g\Psi) = fu_m + gw_m$$

Theorem 3.3 *(S. Liao, 2012) Let L denote a linear operator independent of the homotopy- parameter $q \in [0, 1]$. For two homotopy-Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k, \quad \Psi = \sum_{k=0}^{+\infty} w_k q^k,$$

where ϕ and Ψ are analytic in $q \in [0, a)$, it holds

$$D_m(L\phi) = L\{D_m(\phi)\} = L(u_m) \tag{3.23}$$

and

$$D_m(\Psi L(\phi)) = \sum_{k=0}^{m} D_{m-k}(\Psi) L\left((D_k(\phi))\right) \tag{3.24}$$

$$= \sum_{k=0}^{m} w_{m-k} L(u_k) \tag{3.25}$$

where $m \geq 0$ is an integer.

Proof. Since L is linear and independent of q, using theorem (3.1), we have

$$L(\phi) = L\left(\sum_{k=0}^{+\infty} u_k q^k\right) = \sum_{k=0}^{+\infty} L(u_k) q^k$$

Using the theorem (3.1), one has $D_m[L(\phi)]= L(u_m)$ on the other side, according to theorem (3.1), it holds $L[D_m(\phi)] = L(u_m)$. Thus,

$$D_m(L\phi) = L\{D_m(\phi)\} = L(u_m)$$

holds. Then, according to theorem (3.1), we have

$$\begin{aligned} D_m(\Psi L(\phi)) &= \sum_{n=0}^{m} D_{m-n}(\Psi) D_n[L(\phi)] \\ &= \sum_{n=0}^{m} D_{m-n}(\Psi) L[D_n(\phi)] \\ &= \sum_{n=0}^{m} w_{m-n} L(u_n) \end{aligned}$$

Theorem 3.4 *(S. Liao, 2009) For an arbitrary homotopy Maclaurin series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

then

$$i.\ D_m(\phi^2) = \sum_{n=0}^{m} u_{m-n} u_n, \tag{3.26}$$

$$ii.\ D_m(\phi^3) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} u_k \tag{3.27}$$

$$iii.\ D_m(\phi^\delta) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_\delta-1=0}^{r_\delta-2} u_{r_\delta-2-r_\delta-1} u_{r_\delta-1}, \tag{3.28}$$

where $m \geq 0$ and $\delta \geq 2$ are positive integer.

Proof.(i). By Theorem (3.1)(i) and Theorem (3.1)(iii)

$$D_m(\phi^2) = \sum_{n=0}^{m} D_{m-n}(\phi) D_n(\phi) = \sum_{n=0}^{m} u_{m-n} u_n$$

(ii). By Theorem (3.4) (i), we have

$$D_m(\phi^2) = \sum_{k=0}^{m} D_{m-k}(\phi) D_k(\phi) = \sum_{k=0}^{m} u_{m-k} u_k$$

Then by Theorem (3.1)(iii)

$$D_m(\phi^3) = \sum_{n=0}^{m} D_{m-n}(\phi) D_n(\phi^2) = \sum_{n=0}^{m} u_{m-n} \sum_{k=0}^{n} u_{n-k} u_k$$

(iii).This statement can be proved by the method of mathematical induction.
1. When $\delta = 2$ it's true by Theorem (3.4)(i)
2. Suppose the statement true when $\delta = k$, i.e.

$$D_m(\phi^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_k-1=0}^{r_k-2} u_{r_k-2-r_k-1} u_{r_k-1}$$

where $m \geq 0$ and $k \geq 2$ are positive integer

Replacing r_j by r'_{j+1} and m by r'_1 , the above expression reads

$$D_{r'_1}(\phi^k) = \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k} u'_{r_k}$$

Using the above expression by Theorem (3.1)(i) and by Theorem (3.1)(iv)

$$\begin{aligned} D_m(\phi^{k+1}) &= \sum_{r'_1}^{m} D_{m-r'_1}(\phi) D_{r'_1}(\phi^k) \\ &= \sum_{r'_1}^{m} u_{m-r'_1} \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2} \sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3} \sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4} \cdots \sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k} u'_{r_k} \end{aligned}$$

Thus, it is true for $\delta = k + 1$.
Hence, it is true for any integer $\delta \geq 2$.

3.6 Generalized zeroth-order deformation equation

The starting point of the use of the HAM is to construct the so-called zeroth- order deformation equation, which builds a connection (i.e. a continuous mapping/deformation) between a given nonlinear problem and a relatively much simpler linear ones. So, the zeroth-order deformation equation is a base of the HAM.

Definition 3.5 *For a given nonlinear differential equation $N[u(x,t)] = 0$, where N is a nonlinear differential operator, $u(x,t)$ is an unknown function, x denotes a vector of spatial independent variables, t denotes the temporal independent variable, $h \neq 0$ is an auxiliary parameter, $H(x,t)$ denotes a non-zero auxiliary function. Then the generalized zeroth-order deformation equation is given by(S. Liao, 1997)*

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)], \qquad q \in [0,1],$$

such that, the corresponding homotopy-series is dependent upon not only the physical independent-variables x and t but also the non-zero auxiliary parameter h.

3.7 High-order deformation equation

In this section, we apply the properties of homotopy derivative to obtain a formula for high order deformation.

Lemma 3.1 *(S.-J. Liao & Cheung, 2003) let*

$$\phi = \sum_{m=0}^{\infty} u_m q^m,$$

denote a homotopy-Maclaurin series, where $q \in [0, 1]$ is the homotopy-parameter, and L denotes an auxiliary linear operator which has the property $L[0] = 0$ and is independent of q. It holds

$$D_m \{(1-q)L(\phi - u_0)\} = L(u_m - \chi_m u_{m-1}),$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \ > 1. \end{cases}$$

Proof. Since L is a linear operator independent of q, then

$$\begin{aligned} (1-q)L\{\phi - u_0\} &= L\{(1-q)(\phi - u_0)\} \\ &= L\{\phi - q\phi + u_0 q - u_0\} \\ \implies (1-q)L\{\phi - u_0\} &= L\{\phi - q\phi + u_0 q - u_0\} \end{aligned}$$

Take D_m to both sides

$$\begin{aligned} D_m\left((1-q)L\{\phi - u_0\}\right) =& D_m\left(L\{\phi - q\phi + u_0 q - u_0\}\right) \\ =& L\left(D_m\{\phi - q\phi + u_0 q - u_0\}\right) \\ =& L\left(D_m(\phi) - D_m(q\phi) + D_m(u_0 q) - D_m(u_0)\right) \\ =& L\left(u_m - u_{m-1} + u_0 D_m(q) - 0\right) \\ =& L\left(u_m - u_{m-1} + u_0 D_m(q)\right) \\ \implies D_m\left((1-q)L\{\phi - u_0\}\right) =& L\left(u_m - u_{m-1} + u_0 D_m(q)\right) \end{aligned}$$

When $m = 1$

$$\begin{aligned} D_1\left((1-q)L\{\phi - u_0\}\right) =& L\left(u_1 - u_0 + u_0 D_1(q)\right) \\ =& L\left(u_1 - u_0 + u_0.1\right) \\ =& L(u_1) \end{aligned}$$

When $m = 2$

$$\begin{aligned} D_2\left((1-q)L\{\phi - u_0\}\right) =& L\left(u_2 - u_1 + u_0 D_2(q)\right) \\ =& L\left(u_2 - u_1 + u_0.0\right) \\ =& L(u_2 - u_1) \end{aligned}$$

When $m = 3$

$$\begin{aligned} D_3\left((1-q)L\left\{\phi - u_0\right\}\right) =& L\left(u_3 - u_2 + u_0 D_3(q)\right) \\ =& L\left(u_3 - u_2 + u_0.0\right) \\ =& L(u_3 - u_2) \end{aligned}$$

and so on, then

$$D_m\left\{(1-q)L(\phi - u_0)\right\} = L(u_m - \chi_m u_{m-1})$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

3.8 Homotopy Analysis Method(HAM)

In order to show the basic idea of HAM, consider the differential equation(Chidozie, 2015)

$$N[u(x,t)] = 0, \tag{3.29}$$

where N is a nonlinear operator, x and t denote the independent variables and u is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. Now, the deformation equation of zero-order is constructed as:

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)], \tag{3.30}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $H(x,t)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, equation (3.30) becomes

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t) \tag{3.31}$$

respectively. Thus as q increases from 0 to 1, the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)q^m, \tag{3.32}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{3.33}$$

The convergence of the series (3.32) depends upon the auxiliary parameter h. If it is convergent at $q=1$, one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t), \tag{3.34}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao(S. Liao, 1997). Define the vectors

$$\vec{u_n} = (u_0(x,t), u_1(x,t), ..., u_n(x,t)). \tag{3.35}$$

Differentiating the zeroth-order deformation equation (3.30) m-times with respect to q and then dividing them by $m!$ and finally setting $q=0$, we get the following mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\Re_m(\vec{u}_{m-1}), \tag{3.36}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{3.37}$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1, \\ 1, & \text{if } m \ > 1. \end{cases} \tag{3.38}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (3.36) with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation softwares such as Maple, Mathematica and Matlab.

CHAPTER FOUR
RESULT AND DISCUSSION

4.1 Application

In this section, we give some computational results of numerical experiments with methods for solving Boussinesq-like $B(m,n)$ equations.

4.1.1 Problems:

Problem 1: Consider Boussinesq-like $B(m,n)$ equations when $m = n = 2$

$$\begin{cases} u_{tt} + (u^2)_{xx} - (u^2)_{xxxx} = 0 \\ u(x,0) = \frac{-2}{3} + e^{\frac{1}{2}x} \\ u_t(x,0) = \frac{1}{2}e^{\frac{1}{2}x}, \end{cases} \tag{4.1}$$

where $u = u(x,t)$.

To solve equation (4.1) by means of the homotopy analysis method let us consider the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2},$$

with the property that

$$L(c + c_1 t) = 0,$$

which implies that

$$L^{-1}(.) = \int_0^t \int_0^t (.)dtdt$$

We now define the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} + \frac{\partial^2 (\phi(x,t;q))^2}{\partial x^2} - \frac{\partial^4 (\phi(x,t;q))^2}{\partial x^4} \tag{4.2}$$

Construct the zeroth-order deformation equation by

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\phi(x,t;q)], \tag{4.3}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $H(x,t)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (4.3) becomes

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t) \tag{4.4}$$

We choose auxiliary function $H(x,t) = 1$, then we obtain the m^{th} order deformation equation

$$\begin{aligned} L[u_m(x,t) - \chi_m u_{m-1}(x,t)] &= h\Re_m(\vec{u}_{m-1}), \\ u_m(x,t) &= \chi_m u_{m-1}(x,t) + hL^{-1}\Re_m(\vec{u}_{m-1}), \end{aligned}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^2}{\partial t^2}(u_{m-1}(x,t)) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right) - \frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right),$$

then

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) + hL^{-1}\left(\frac{\partial^2}{\partial t^2}u_{m-1}(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right)\right. \\ &\quad \left. - \frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right)\right) \\ &= \chi_m u_{m-1}(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_{m-1}(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right)\right. \\ &\quad \left. - \frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i}\right)\right) dt\,dt \end{aligned}$$

since

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1, \\ 1, & \text{if } m > 1. \end{cases}$$

and $m = 0, 1, 2, 3, ...$

For solving equation (4.1) by using HAM, we choose the initial guess from initial condition

$$u_0(x,t) = \frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x}$$

$$
\begin{aligned}
u_1(x,t) &= \chi_1 u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{1} u_i u_{0-i}\right)\right.\\
&\left.-\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{1} u_i u_{0-i}\right)\right) dtdt\\
&= \chi_1 u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}((u_0(x,t))^2) - \frac{\partial^4}{\partial x^4}((u_0(x,t))^2)\right) dtdt\\
&= 0.u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}((u_0(x,t))^2) - \frac{\partial^4}{\partial x^4}((u_0(x,t))^2)\right) dtdt\\
&= 0.\left(-\frac{2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x}\right) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}\left(u_0(x,t)^2\right)\right.\\
&\left.-\frac{\partial^4}{\partial x^4}\left(u_0(x,t)^2\right)\right) dt\, dt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}((u_0(x,t))^2) - \frac{\partial^4}{\partial x^4}((u_0(x,t))^2)\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}(\frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x}) + \frac{\partial^2}{\partial x^2}((\frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x})^2)\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}((\frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x})^2)\right) dtdt\\
&= h\int_0^t\int_0^t \left(0 + \frac{\partial^2}{\partial x^2}((\frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x})^2) - \frac{\partial^4}{\partial x^4}((\frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x})^2)\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}[\frac{4}{9} - \frac{4}{3}e^{\frac{1}{2}x} - \frac{2}{3}te^{\frac{1}{2}x} + e^x(1+t+\frac{t^2}{4})]\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}[\frac{4}{9} - \frac{4}{3}e^{\frac{1}{2}x} - \frac{2}{3}te^{\frac{1}{2}x} + e^x(1+t+\frac{t^2}{4})]\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{-1}{3}e^{\frac{1}{2}x} - \frac{1}{6}te^{\frac{1}{2}x} + e^x(1+t+\frac{t^2}{4})\right)\\
&- \left[\frac{-1}{12}e^{\frac{1}{2}x} - \frac{1}{24}te^{\frac{1}{2}x} + e^x(1+t+\frac{t^2}{4})\right] dtdt\\
&= h\int_0^t\int_0^t \left(\frac{-1}{4}e^{\frac{1}{2}x} - \frac{1}{8}te^{\frac{1}{2}x}\right) dtdt = h\int_0^t \left(\frac{-1}{4}te^{\frac{1}{2}x} - \frac{1}{16}t^2e^{\frac{1}{2}x}\right) dt\\
&= h\left(\frac{-1}{8}t^2e^{\frac{1}{2}x} - \frac{1}{48}t^3e^{\frac{1}{2}x}\right)\\
u_1(x,t) &= \frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) &= \chi_2 u_1(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_1(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{1} u_i u_{1-i}\right)\right. \\
&\left. -\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{1} u_i u_{1-i}\right)\right) dtdt \\
&= \chi_2 u_1(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_1(x,t) + \frac{\partial^2}{\partial x^2}(2u_0u_1) - \frac{\partial^4}{\partial x^4}(2u_0u_1)\right) dtdt \\
&= 1.(\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}(\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(2u_0u_1) - \frac{\partial^4}{\partial x^4}(2u_0u_1)\right) dtdt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}(\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(2u_0u_1) - \frac{\partial^4}{\partial x^4}(2u_0u_1)\right) dtdt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left((\frac{-1}{4}he^{\frac{1}{2}x} - \frac{1}{8}hte^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(\frac{1}{6}ht^2e^{\frac{1}{2}x} + \frac{1}{36}ht^3e^{\frac{1}{2}x} - (\frac{1}{4}ht^2 + \frac{1}{8}ht^3 + \frac{1}{48}ht^4)e^x)\right) dtdt \\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}(\frac{1}{6}ht^2e^{\frac{1}{2}x} + \frac{1}{36}ht^3e^{\frac{1}{2}x} - (\frac{1}{4}ht^2 + \frac{1}{8}ht^3 + \frac{1}{48}ht^4)e^x)\right) dtdt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left((\frac{-1}{4}he^{\frac{1}{2}x} - \frac{1}{8}hte^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{1}{24}ht^2e^{\frac{1}{2}x} + \frac{1}{144}ht^3e^{\frac{1}{2}x} - (\frac{1}{4}ht^2 + \frac{1}{8}ht^3 + \frac{1}{48}ht^4)e^x\right) dtdt \\
&- h\int_0^t\int_0^t \left(\frac{1}{96}ht^2e^{\frac{1}{2}x} + \frac{1}{576}ht^3e^{\frac{1}{2}x} - (\frac{1}{4}ht^2 + \frac{1}{8}ht^3 + \frac{1}{48}ht^4)e^x\right) dtdt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left((\frac{-1}{4}he^{\frac{1}{2}x} - \frac{1}{8}hte^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{1}{32}ht^2e^{\frac{1}{2}x} + \frac{1}{192}ht^3e^{\frac{1}{2}x}\right) dtdt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\int_0^t \left((\frac{-1}{4}hte^{\frac{1}{2}x} - \frac{1}{16}ht^2e^{\frac{1}{2}x})\right) dt \\
&+ h\int_0^t \left(\frac{1}{96}ht^3e^{\frac{1}{2}x} + \frac{1}{768}ht^4e^{\frac{1}{2}x}\right) dt \\
&= (\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x}) + h\left((\frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x})\right) \\
&+ h\left(\frac{1}{384}ht^4e^{\frac{1}{2}x} + \frac{1}{3840}ht^5e^{\frac{1}{2}x}\right) \\
&= \frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x} + \frac{-1}{8}h^2t^2e^{\frac{1}{2}x} - \frac{1}{48}h^2t^3e^{\frac{1}{2}x} \\
&+ \frac{1}{384}h^2t^4e^{\frac{1}{2}x} + \frac{1}{3840}h^2t^5e^{\frac{1}{2}x}
\end{aligned}
$$

$$
\begin{aligned}
u_2(x,t) = & \frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x} - \frac{1}{8}h^2t^2e^{\frac{1}{2}x} - \frac{1}{48}h^2t^3e^{\frac{1}{2}x} + \frac{1}{384}h^2t^4e^{\frac{1}{2}x} \\
& + \frac{1}{3840}h^2t^5e^{\frac{1}{2}x}
\end{aligned}
$$

At $h=-1$, then we have:

$$
\begin{aligned}
u_1(x,t) &= \frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x} \\
&= \frac{1}{8}t^2e^{\frac{1}{2}x} + \frac{1}{48}t^3e^{\frac{1}{2}x} \\
u_2(x,t) &= \frac{-1}{8}ht^2e^{\frac{1}{2}x} - \frac{1}{48}ht^3e^{\frac{1}{2}x} - \frac{1}{8}h^2t^2e^{\frac{1}{2}x} - \frac{1}{48}h^2t^3e^{\frac{1}{2}x} + \frac{1}{384}h^2t^4e^{\frac{1}{2}x} \\
&+ \frac{1}{3840}h^2t^5e^{\frac{1}{2}x} \\
&= \frac{1}{8}t^2e^{\frac{1}{2}x} + \frac{1}{48}t^3e^{\frac{1}{2}x} - \frac{1}{8}t^2e^{\frac{1}{2}x} - \frac{1}{48}t^3e^{\frac{1}{2}x} + \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
&= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
u_3(x,t) &= \chi_3 u_2(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_2(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{2} u_i u_{2-i}\right)\right) dtdt \\
&- h\int_0^t\int_0^t \frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{2} u_i u_{2-i}\right) dtdt \\
&= \chi_3 u_2(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_2(x,t) + \frac{\partial^2}{\partial x^2}\left(2u_0u_2 + u_1^2\right)\right) dtdt \\
&- h\int_0^t\int_0^t \frac{\partial^4}{\partial x^4}\left(2u_0u_2 + u_1^2\right) dtdt \\
&= 1.(\frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x}) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}(\frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x})\right) dtdt \\
&+ h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(\frac{-1}{288}t^4e^{\frac{1}{2}x} - \frac{1}{2880}t^5e^{\frac{1}{2}x} + (\frac{1}{48}t^4 + \frac{1}{120}t^5 + \frac{1}{1440}t^6)e^x)\right) dtdt \\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}(\frac{-1}{288}t^4e^{\frac{1}{2}x} - \frac{1}{2880}t^5e^{\frac{1}{2}x} + (\frac{1}{48}t^4 + \frac{1}{120}t^5 + \frac{1}{1440}t^6)e^x)\right) dtdt \\
u_3(x,t) &= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} + h\int_0^t\int_0^t \left(\frac{1}{32}t^2e^{\frac{1}{2}x} + \frac{1}{192}t^3e^{\frac{1}{2}x}\right) dtdt \\
&+ h\int_0^t\int_0^t \left(-\frac{1}{1152}t^4e^{\frac{1}{2}x} - \frac{1}{11520}t^5e^{\frac{1}{2}x} + (\frac{1}{48}t^4 + \frac{1}{120}t^5 + \frac{1}{1440}t^6)e^x\right) dtdt \\
&- h\int_0^t\int_0^t \left(-\frac{1}{4608}t^4e^{\frac{1}{2}x} - \frac{1}{46080}t^5e^{\frac{1}{2}x} + (\frac{1}{48}t^4 + \frac{1}{120}t^5 + \frac{1}{1440}t^6)e^x\right) dtdt
\end{aligned}
$$

$$
\begin{aligned}
u_3(x,t) &= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} + h\int_0^t\int_0^t\left(\frac{1}{32}t^2e^{\frac{1}{2}x} + \frac{1}{192}t^3e^{\frac{1}{2}x}\right)dtdt \\
&+ h\int_0^t\int_0^t\left(-\frac{1}{1536}t^4e^{\frac{1}{2}x} + \frac{1}{4608}t^4e^{\frac{1}{2}x} - \frac{1}{15360}t^5e^{\frac{1}{2}x} + \frac{1}{46080}t^5e^{\frac{1}{2}x}\right)dtdt \\
&= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} + h\int_0^t\int_0^t\left(\frac{1}{32}t^2e^{\frac{1}{2}x} + \frac{1}{192}t^3e^{\frac{1}{2}x}\right)dtdt \\
&+ h\int_0^t\int_0^t\left(-\frac{1}{1152}t^4e^{\frac{1}{2}x} + \frac{1}{4608}t^4e^{\frac{1}{2}x} - \frac{1}{11520}t^5e^{\frac{1}{2}x} + \frac{1}{46080}t^5e^{\frac{1}{2}x}\right)dtdt \\
&= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} + h\int_0^t\int_0^t\left(\frac{1}{32}t^2e^{\frac{1}{2}x} + \frac{1}{192}t^3e^{\frac{1}{2}x}\right)dtdt \\
&+ h\int_0^t\int_0^t\left(-\frac{1}{1536}t^4e^{\frac{1}{2}x} - \frac{1}{15360}t^5e^{\frac{1}{2}x}\right)dtdt \\
u_3(x,t) &= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
&+ h\int_0^t\int_0^t\left(\frac{1}{32}t^2e^{\frac{1}{2}x} + \frac{1}{192}t^3e^{\frac{1}{2}x} - \frac{1}{1536}t^4e^{\frac{1}{2}x} - \frac{1}{15360}t^5e^{\frac{1}{2}x}\right)dtdt \\
&= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
&+ h\int_0^t\left(\frac{1}{96}t^3e^{\frac{1}{2}x} + \frac{1}{768}t^4e^{\frac{1}{2}x} - \frac{1}{7680}t^5e^{\frac{1}{2}x} - \frac{1}{92160}t^6e^{\frac{1}{2}x}\right)dt \\
&= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
&+ h\left(\frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} - \frac{1}{46080}t^6e^{\frac{1}{2}x} - \frac{1}{645120}t^7e^{\frac{1}{2}x}\right)
\end{aligned}
$$

At $h=-1$, we have

$$u_3(x,t) = \frac{1}{46080}t^6e^{\frac{1}{2}x} + \frac{1}{645120}t^7e^{\frac{1}{2}x}$$

the components are:

$$
\begin{aligned}
u_0(x,t) &= \frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x} \\
u_1(x,t) &= \frac{1}{8}t^2e^{\frac{1}{2}x} + \frac{1}{48}t^3e^{\frac{1}{2}x} \\
u_2(x,t) &= \frac{1}{384}t^4e^{\frac{1}{2}x} + \frac{1}{3840}t^5e^{\frac{1}{2}x} \\
u_3(x,t) &= \frac{1}{46080}t^6e^{\frac{1}{2}x} + \frac{1}{645120}t^7e^{\frac{1}{2}x} \qquad .
\end{aligned}
$$

$$\vdots$$

From homotopy series, the approximate analytical solutions of given nonlinear equation is given as

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)$$

Thus, at $h = -1$, the $3rd$ iteration HAM approximate analytical solution is given by

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{3} u_m(x,t)$$

$$\begin{aligned} u(x,t) = \frac{-2}{3} + e^{\frac{1}{2}x} + \frac{1}{2}te^{\frac{1}{2}x} + \frac{1}{8}t^2e^{\frac{1}{2}x} + \frac{1}{48}t^3e^{\frac{1}{2}x} + \frac{1}{384}t^4e^{\frac{1}{2}x} \\ + \frac{1}{3,840}t^5e^{\frac{1}{2}x} + \frac{1}{46,080}t^6e^{\frac{1}{2}x} + \frac{1}{645,120}t^7e^{\frac{1}{2}x} \end{aligned}$$

$$u(x,t) = \frac{-2}{3} + [1 + \frac{1}{2}t + \frac{1}{8}t^2 + \frac{1}{48}t^3 + \frac{1}{384}t^4 + \frac{1}{3,840}t^5 + \frac{1}{46,080}t^6 + \frac{1}{645,120}t^7]e^{\frac{1}{2}x}$$

The exact solution of this problem obtained by (Zaidan et al., n.d.) is:

$$u(x,t) = \frac{-2}{3} + e^{\frac{1}{2}x}\left(1 + \sum_{n=0}^{\infty} \frac{t^{n+1}}{2^{n+1} * (n+1)!}\right)$$

Table 4.1: Approximate solution(HAM) and Exact solution of Boussinesq-like $B(m,n)$ equation when $m = n = 2$ at $t = 0.01$ and in the interval $x \in [0,1]$

t	x	Exact solution	HAM solution	Absolute Error
0.01	0.00	0.3383458541927343	0.3383458541927344	5.5511151231257827e-17
0.01	0.10	0.3898739480088277	0.3898739480088277	5.5511151231257827e-17
0.01	0.20	0.4440439436890385	0.4440439436890387	1.6653345369377348e-16
0.01	0.30	0.5009912944384584	0.5009912944384584	0.0000000000000000e+00
0.01	0.40	0.5608583982965109	0.5608583982965111	2.2204460492503131e-16
0.01	0.50	0.6237949542062230	0.6237949542062232	2.2204460492503131e-16
0.01	0.60	0.6899583363395573	0.6899583363395575	2.2204460492503131e-16
0.01	0.70	0.7595139876148133	0.7595139876148134	1.1102230246251565e-16
0.01	0.80	0.8326358333901002	0.8326358333901002	0.0000000000000000e+00
0.01	0.90	0.9095067163673244	0.9095067163673246	2.2204460492503131e-16
0.01	1.00	0.9903188537941839	0.9903188537941842	2.2204460492503131e-16

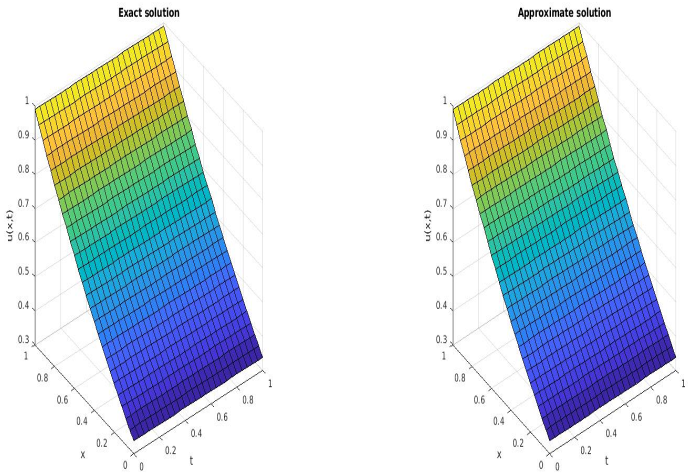

Figure 4.1: The exact solution and 3^{rd} order approximate solution of $B(2,2)$ at $t = 0.01$, $h = -1$, and in the interval $x \in [0,1]$.

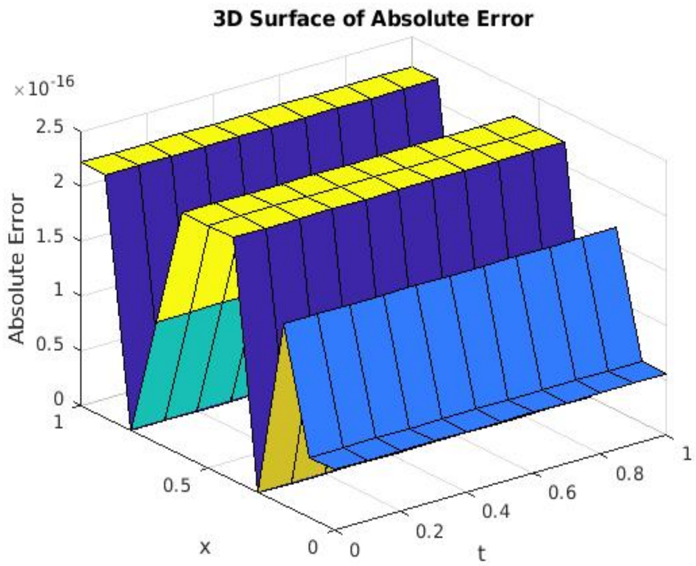

Figure 4.2: The Absolute error between exact and HAM solution of $B(2,2)$ at $t = 0.01$ and $h = -1$.

Computational Simulations

For the given equation 4.1, by using Finite Difference Method(FDM), we can discretize this differential in the following form:

$$(u_{tt})_i^j = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2}$$
$$(u^2)_{xx} = \frac{u_{i+1,j}^2 - 2u_{i,j}^2 + u_{i-1,j}^2}{(\Delta x)^2}$$
$$(u^2)_{xxxx} = \frac{u_{i+2,j}^2 - 4u_{i+1,j}^2 + 6u_{i,j}^2 - 4u_{i-1,j}^2 + u_{i-2,j}^2}{(\Delta x)^4}$$

The discretized equation using the finite difference method is as follows:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} + \frac{u_{i+1,j}^2 - 2u_{i,j}^2 + u_{i-1,j}^2}{(\Delta x)^2} - \frac{u_{i+2,j}^2 - 4u_{i+1,j}^2 + 6u_{i,j}^2 - 4u_{i-1,j}^2 + u_{i-2,j}^2}{(\Delta x)^4} = 0$$

We can rearrange this equation to solve for u_i^{j+1}, the unknown value at the next time step and after solving for the unknown, we get the scheme to be coded in Python and matlab.

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} - \frac{(\Delta t)^2}{(\Delta x)^2}(u_{i+1,j}^2 - 2u_{i,j}^2 + u_{i-1,j}^2) + \frac{(\Delta t)^2}{(\Delta x)^4}(u_{i+2,j}^2 - 4u_{i+1,j}^2 + 6u_{i,j}^2 - 4u_{i-1,j}^2 + u_{i-2,j}^2)$$

Table 4.2: Numerical solution(FDM) and Exact solution of Boussinesq-like $B(m,n)$ equation when $m=n=2$ at $t=0.01$, and in the interval $x \in [0,1]$

t	x	Exact solution	FDM solution	Absolute Error
0.01	0.00	0.338346	0.333333	5.013000×10^{-3}
0.01	0.10	0.389874	0.384604	5.270000×10^{-3}
0.01	0.20	0.444044	0.438504	5.540000×10^{-3}
0.01	0.30	0.500991	0.495168	5.824000×10^{-3}
0.01	0.40	0.560858	0.554736	6.122000×10^{-3}
0.01	0.50	0.623795	0.617359	6.436000×10^{-3}
0.01	0.60	0.689958	0.683192	6.766000×10^{-3}
0.01	0.70	0.759514	0.752401	7.113000×10^{-3}
0.01	0.80	0.832636	0.825158	7.478000×10^{-3}
0.01	0.90	0.909507	0.901646	7.861000×10^{-3}
0.01	1.00	0.990319	0.982055	8.264000×10^{-3}

Figure 4.3: The Exact solution and Numerical solution(FDM) of $B(2,2)$ at $t=0.01$ and in the interval $x \in [0,1]$

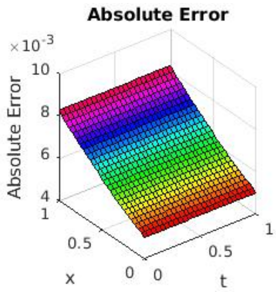

Figure 4.4: The Absolute error between exact and Numerical solution(FDM) of $B(2,2)$ at $t=0.01$ and in the interval $x \in [0,1]$

Problem 2: Consider Boussinesq-like $B(m,n)$ equations when $m=n=3$

$$\begin{cases} u_{tt}+(u^3)_{xx}-(u^3)_{xxxx}=0 \\ u(x,0)=\frac{\sqrt{6}}{2}sinh(\frac{x}{3}) \\ u_t(x,0)=\frac{-\sqrt{6}}{6}cosh(\frac{x}{3}), \end{cases} \tag{4.5}$$

where $u=u(x,t)$.

To solve equation (4.5) by means of the homotopy analysis method let us consider the linear operator

$$L[\phi(x,t;q)]=\frac{\partial^2\phi(x,t;q)}{\partial t^2},$$

with the property that

$$L(c+c_1t)=0,$$

which implies that

$$L^{-1}(.)=\int_0^t\int_0^t(.)dtdt$$

We now define the nonlinear operator as

$$N[\phi(x,t;q)]=\frac{\partial^2\phi(x,t;q)}{\partial t^2}+\frac{\partial^2(\phi(x,t;q))^3}{\partial x^2}-\frac{\partial^4(\phi(x,t;q))^3}{\partial x^4} \tag{4.6}$$

Construct the zeroth-order deformation equation by

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)N[\phi(x,t;q)], \tag{4.7}$$

where $q\in[0,1]$ is the embedding parameter, $h\neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $H(x,t)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (4.3) becomes

$$\phi(x,t;0)=u_0(x,t), \quad \phi(x,t;1)=u(x,t) \tag{4.8}$$

We choose auxiliary function $H(x,t)=1$, then we obtain the m^{th} order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\Re_m(\vec{u}_{m-1}),$$
$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hL^{-1}\Re_m(\vec{u}_{m-1}),$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{\partial^2}{\partial t^2}(u_{m-1}(x,t)) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right) - \frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)$$

then

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hL^{-1}\left(\frac{\partial^2}{\partial t^2}(u_{m-1}(x,t)) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right) - hL^{-1}\left(\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right)$$

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_{m-1}(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right) - h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right) dtdt$$

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_{m-1}(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right) - h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{m-1} u_i\left(\sum_{k=0}^{m-1-i} u_k u_{m-1-i-k}\right)\right)\right) dtdt$$

since

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1, \\ 1, & \text{if } m \ > 1. \end{cases}$$

and $m = 0, 1, 2, 3, ...$

For solving equation (4.5) by using HAM, we choose the initial guess from initial condition

$$u_0(x,t) = \frac{\sqrt{6}}{2}sinh(\frac{x}{3}) - \frac{\sqrt{6}}{6}tcosh(\frac{x}{3})$$

$$
\begin{aligned}
u_1(x,t) &= \chi_1 u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{0} u_i\left(\sum_{k=0}^{0-i} u_k u_{0-i-k}\right)\right)\right)\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}\left(\sum_{i=0}^{0} u_i\left(\sum_{k=0}^{0-i} u_k u_{0-i-k}\right)\right)\right) dtdt\\
u_1(x,t) &= \chi_1 u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}(u_0(x,t))^3 - \frac{\partial^4}{\partial x^4}(u_0(x,t))^3\right) dtdt\\
&= 0.u_0(x,t) + h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}(u_0(x,t))^3 - \frac{\partial^4}{\partial x^4}(u_0(x,t))^3\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}u_0(x,t) + \frac{\partial^2}{\partial x^2}(u_0(x,t))^3 - \frac{\partial^4}{\partial x^4}(u_0(x,t))^3\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial t^2}(\frac{\sqrt{6}}{2}sinh(\frac{x}{3}) - \frac{\sqrt{6}}{6}tcosh(\frac{x}{3})) + \frac{\partial^2}{\partial x^2}(u_0(x,t))^3\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}(u_0(x,t))^3\right) dtdt\\
&= h\int_0^t\int_0^t \left((0) + \frac{\partial^2}{\partial x^2}(u_0(x,t))^3 - \frac{\partial^4}{\partial x^4}(u_0(x,t))^3\right) dtdt\\
&= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(\frac{\sqrt{6}}{2}sinh(\frac{x}{3}) - \frac{\sqrt{6}}{6}tcosh(\frac{x}{3}))^3 - \frac{\partial^4}{\partial x^4}(\frac{\sqrt{6}}{2}sinh(\frac{x}{3})\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{6}tcosh(\frac{x}{3}))^3\right) dtdt\\
u_1(x,t) &= h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(\frac{3}{4}\sqrt{6}sinh^3(\frac{x}{3}) - \frac{3}{4}\sqrt{6}tcosh(\frac{x}{3})sinh^2(\frac{x}{3}))\right) dtdt\\
&+ h\int_0^t\int_0^t \left(\frac{\partial^2}{\partial x^2}(\frac{1}{4}\sqrt{6}t^2cosh^2(\frac{x}{3})sinh(\frac{x}{3}) - \frac{1}{36}\sqrt{6}t^3cosh^3(\frac{x}{3}))\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}(\frac{3}{4}\sqrt{6}sinh^3(\frac{x}{3}) - \frac{3}{4}\sqrt{6}tcosh(\frac{x}{3})sinh^2(\frac{x}{3}))\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{\partial^4}{\partial x^4}(\frac{1}{4}\sqrt{6}t^2cosh^2(\frac{x}{3})sinh(\frac{x}{3}) - \frac{1}{36}\sqrt{6}t^3cosh^3(\frac{x}{3}))\right) dtdt\\
u_1(x,t) &= h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{2}sinh(\frac{x}{3})cosh^2(\frac{x}{3}) + \frac{\sqrt{6}}{4}sinh^3(\frac{x}{3})\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{7}{12}\sqrt{6}tcosh(\frac{x}{3})sinh^2(\frac{x}{3})\right) dtdt\\
&+ h\int_0^t\int_0^t \left(\frac{-\sqrt{6}}{6}tcosh^3(\frac{x}{3}) + \frac{\sqrt{6}}{18}t^2sinh^3(\frac{x}{3}) + \frac{7}{36}\sqrt{6}t^2sinh(\frac{x}{3})cosh^2(\frac{x}{3})\right) dtdt\\
&+ h\int_0^t\int_0^t \left(\frac{-\sqrt{6}}{54}t^3cosh(\frac{x}{3})sinh^2(\frac{x}{3}) - \frac{\sqrt{6}}{108}t^3cosh^3(\frac{x}{3})\right) dtdt\\
&- h\int_0^t\int_0^t \left(\frac{10}{81}\sqrt{6}sinh(\frac{x}{3})cosh^2(\frac{x}{3})\right) dtdt\\
&+ h\int_0^t\int_0^t \left(\frac{-7}{36}\sqrt{6}sinh^3(\frac{x}{3}) + \frac{5}{27}\sqrt{6}tcosh^3(\frac{x}{3}) + \frac{61}{108}\sqrt{6}tcosh(\frac{x}{3})sinh^2(\frac{x}{3})\right) dtdt
\end{aligned}
$$

$$+ h\int_0^t\int_0^t \left(\frac{-61}{324}\sqrt{6}t^2 sinh(\frac{x}{3})cosh^2(\frac{x}{3}) - \frac{5}{81}\sqrt{6}t^2 sinh^3(\frac{x}{3})\right) dtdt$$
$$+ h\int_0^t\int_0^t \left(\frac{10}{486}\sqrt{6}t^3 cosh(\frac{x}{3})sinh^2(\frac{x}{3})\right) dtdt$$
$$+ h\int_0^t\int_0^t \left(\frac{7}{972}\sqrt{6}t^3 cosh^3(\frac{x}{3})\right) dtdt$$

$$u_1(x,t) = h\int_0^t\int_0^t \left(\frac{-\sqrt{6}}{18} sinh(\frac{x}{3})cosh^2(\frac{x}{3}) + \frac{\sqrt{6}}{18} sinh^3(\frac{x}{3})\right) dtdt$$
$$- h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{54} tcosh(\frac{x}{3})sinh^2(\frac{x}{3})\right) dtdt$$
$$+ h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{54} tcosh^3(\frac{x}{3}) - \frac{\sqrt{6}}{162}t^2 sinh^3(\frac{x}{3}) + \frac{\sqrt{6}}{162}t^2 sinh(\frac{x}{3})cosh^2(\frac{x}{3})\right) dtdt$$
$$+ h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{486}t^3 cosh(\frac{x}{3})sinh^2(\frac{x}{3}) - \frac{\sqrt{6}}{486}t^3 cosh^3(\frac{x}{3})\right) dtdt$$
$$u_1(x,t) = h\int_0^t\int_0^t \left(\frac{-\sqrt{6}}{18} sinh(\frac{x}{3}) + \frac{\sqrt{6}}{54} tcosh(\frac{x}{3}) + \frac{\sqrt{6}}{162}t^2 sinh(\frac{x}{3})\right) dtdt$$
$$- h\int_0^t\int_0^t \left(\frac{\sqrt{6}}{486}t^3 cosh^3(\frac{x}{3})\right) dtdt$$
$$= h\int_0^t \left(\frac{-\sqrt{6}}{18} tsinh(\frac{x}{3}) + \frac{\sqrt{6}}{108}t^2 cosh(\frac{x}{3}) + \frac{\sqrt{6}}{486}t^3 sinh(\frac{x}{3}) - \frac{\sqrt{6}}{1944}t^4 cosh^3(\frac{x}{3})\right) dt$$
$$= h\left(\frac{-\sqrt{6}}{36}t^2 sinh(\frac{x}{3}) + \frac{\sqrt{6}}{324}t^3 cosh(\frac{x}{3}) + \frac{\sqrt{6}}{1944}t^4 sinh(\frac{x}{3}) - \frac{\sqrt{6}}{9720}t^5 cosh^3(\frac{x}{3})\right)$$

At $h = -1$, we have

$$u_1(x,t) = \frac{\sqrt{6}}{36}t^2 sinh(\frac{x}{3}) - \frac{\sqrt{6}}{324}t^3 cosh(\frac{x}{3}) - \frac{\sqrt{6}}{1944}t^4 sinh(\frac{x}{3}) + \frac{\sqrt{6}}{9720}t^5 cosh^3(\frac{x}{3})$$

From homotopy series, the approximate analytical solutions of given nonlinear equation is given as

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)$$

Thus, at $h = -1$, the 2^{nd} iteration HAM approximate analytical solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t)$$

Thus,

$$u(x,t) = \frac{\sqrt{6}}{2} sinh(\frac{x}{3}) - \frac{\sqrt{6}}{6} tcosh(\frac{x}{3}) + \frac{\sqrt{6}}{36} t^2 sinh(\frac{x}{3}) - \frac{\sqrt{6}}{324} t^3 cosh(\frac{x}{3})$$
$$- \frac{\sqrt{6}}{1944} t^4 sinh(\frac{x}{3}) + \frac{\sqrt{6}}{9720} t^5 cosh^3(\frac{x}{3})$$

The exact solution of this problem obtained by (Yıldırım, 2009) is:

$$u(x,t) = \frac{\sqrt{6}}{2} sinh\left(\frac{x-t}{3}\right)$$

Table 4.3: Approximate solution(HAM) and Exact solution of Boussinesq-like $B(m,n)$ equation when $m = n = 3$ at $t = 0.01$, and in the interval $x \in [0,1]$

t	x	Exact solution	HAM solution	Absolute Error
0.01	0.00	-0.004082490465	-0.004082490465	2.940096083259e-14
0.01	0.10	0.036747857742	0.036747857741	6.006861674734e-13
0.01	0.20	0.077619040682	0.077619040681	1.231376112187e-12
0.01	0.30	0.118576474987	0.118576474985	1.863370568955e-12
0.01	0.40	0.159665673130	0.159665673127	2.497418938319e-12
0.01	0.50	0.200932294004	0.200932294000	3.134159598517e-12
0.01	0.60	0.242422193655	0.242422193651	3.774258683364e-12
0.01	0.70	0.284181476240	0.284181476236	4.418465593403e-12
0.01	0.80	0.326256545259	0.326256545254	5.067724018204e-12
0.01	0.90	0.368694155118	0.368694155112	5.722478046977e-12
0.01	1.00	0.411541463081	0.411541463075	6.383338302385e-12

Figure 4.5: The exact solution and Approximate solution (HAM) of $B(3,3)$ at $t = 0.01$ and $h = -1$.

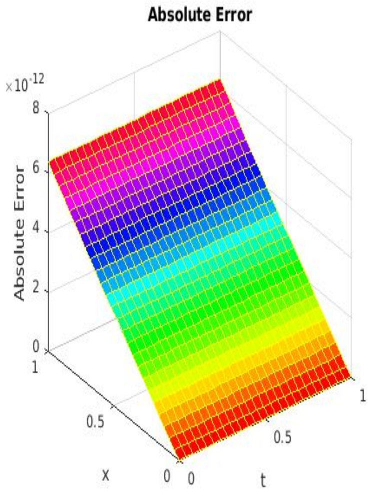

Figure 4.6: The Absolute error of $B(3,3)$ at $t = 0.01$ and $h = -1$.

Computational Simulations

For the given equation 4.5, by using Finite Difference Method(FDM), we can discretize this differential in the following form:

$$(u_{tt})_i^j = \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2}$$
$$(u^2)_{xx} = \frac{u_{i+1,j}^3 - 2u_{i,j}^3 + u_{i-1,j}^3}{(\Delta x)^2}$$
$$(u^2)_{xxxx} = \frac{u_{i+2,j}^3 - 4u_{i+1,j}^3 + 6u_{i,j}^3 - 4u_{i-1,j}^3 + u_{i-2,j}^3}{(\Delta x)^4}$$

The discretized equation using the finite difference method is as follows:

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} + \frac{u_{i+1,j}^3 - 2u_{i,j}^3 + u_{i-1,j}^3}{(\Delta x)^2} - \frac{u_{i+2,j}^3 - 4u_{i+1,j}^3 + 6u_{i,j}^3 - 4u_{i-1,j}^3 + u_{i-2,j}^3}{(\Delta x)^4} = 0$$

We can rearrange this equation to solve for u_i^{j+1}, the unknown value at the next time step and after solving for the unknown, we get the scheme to be coded in Python and matlab.

$$u_i^{j+1} = 2u_i^j - u_i^{j-1} - \frac{(\Delta t)^2}{(\Delta x)^2}(u_{i+1,j}^3 - 2u_{i,j}^3 + u_{i-1,j}^3) + \frac{(\Delta t)^2}{(\Delta x)^4}(u_{i+2,j}^3 - 4u_{i+1,j}^3 + 6u_{i,j}^3 - 4u_{i-1,j}^3 + u_{i-2,j}^3)$$

Table 4.4: Numerical solution(FDM) and Exact solution of Boussinesq-like $B(m, n)$ equation when $m = n = 3$ at $t = 0.01$ in the interval $x \in [0, 1]$

t	x	Exact solution	FDM solution	Absolute Error
0.01	0.00	-0.004083	-0.004083	0.000000
0.01	0.10	0.036748	0.040832	0.004085
0.01	0.20	0.077619	0.081710	0.004091
0.01	0.30	0.118576	0.122679	0.004102
0.01	0.40	0.159666	0.163784	0.004118
0.01	0.50	0.200932	0.205070	0.004138
0.01	0.60	0.242422	0.246585	0.004163
0.01	0.70	0.284181	0.288374	0.004193
0.01	0.80	0.326257	0.330483	0.004227
0.01	0.90	0.368694	0.372960	0.004266
0.01	1.00	0.411541	0.415851	0.004309

Figure 4.7: The exact solution and Numerical solution of $B(3, 3)$ at $t = 0.01$

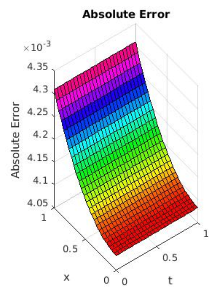

Figure 4.8: The Absolute error of $B(3, 3)$ at $t = 0.01$.

4.2 Discussion

In this thesis, we employed the homotopy analysis method to solve Boussinesq-like $B(m, n)$ equations. Initially, we selected an initial approximation (u_0), a linear operator (L), an auxiliary function $(H(x, t) \neq 0)$, and a convergence control parameter $(h \neq 0)$ to derive an approximate analytical solution using the HAM. Subsequently, we graphically obtained the approximate analytical solution using the HAM. A comparison was then made between the HAM solution and FDM solutions for both $B(2, 2)$ and $B(3, 3)$. The comparison demonstrated a small absolute error between the HAM solution and the exact solutions. The small absolute error indicates the efficiency of these methods. Consequently, the HAM outperforms the FDM as it yields a minimal absolute error. Hence, based on these findings, we can confidently assert that the HAM is an effective and powerful method, at least for this specific problem.

CONCLUSION AND RECOMMENDATION

Conclusion

This thesis successfully applies the homotopy analysis method to obtain approximate analytical solution of the Boussinesq-like $B(m, n)$ equations. The homotopy analysis method offers a systematic and efficient approach to obtain approximate analytical solutions for these complex equations. Through this study, it has been demonstrated that the homotopy analysis method can effectively handle Boussinesq-like $B(m, n)$ equations, generating infinite power series solutions for appropriate initial conditions. The method's ability to control the convergence of approximation series is a significant advantage, distinguishing it from other solution techniques. After comparing the results obtained from the HAM and the FDM, it has been observed that the HAM offers highly accurate approximations that closely resemble the exact values of the given problems. These results demonstrate the effectiveness of the homotopy analysis method as a powerful mathematical tool for solving Boussinesq-like $B(m, n)$ equations. It is worth noting that the auxiliary parameter in this method enables fast convergence of the solutions. The computations in this paper were performed using matlab and Python.

Recommendation

This thesis specifically focuses on the application of the homotopy analysis method to solve the approximate analytic solution of the Boussinesq-like $B(m, n)$ equations. The homotopy analysis method is an interesting approach that allows for finding approximate analytical solutions to nonlinear partial differential equations. It is recommended that future researchers compare the homotopy analysis method with other methods to assess its efficiency and validity. Additionally, the homotopy analysis method can be extended to solve nonlinear problems related to Boussinesq-like $B(m, n)$ equations for various values of $m > 3$ and $n > 3$ to know the behaviour of the system and other related equations.

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