EXPONENTIALLY FITTED TENSION SPLINE METHOD FOR SINGULARLY PERTURBED TIME DELAY REACTION DIFFUSION PROBLEMS

Ermias Argago Megiso

A Thesis Submitted to the Department of Applied Mathematics
School of Applied Natural Science

In Partial Fulfillment of the Requirement for the Degree of
Master's in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2022

Adama, Ethiopia

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Ermias Argago Megiso

Advisor: Mesfin Mekuria Woldaregay (PhD)

Co-Advisor: Tekle Gemechu Dinka (PhD)

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DECLARATION

I hereby declare that this Master Thesis entitled "Exponentially Fitted Tension Spline Method for Singularly Perturbed Time Delay Reaction Diffusion" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled “Exponentially Fitted Tension Spline Method for Singularly Perturbed Time Delay Reaction Diffusion” prepared under my guidance by Ermias Argago Megiso. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

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Co-advisor	Signature	Date

APPROVAL PAGE

We, the advisors of the thesis entitled "Exponentially Fitted Tension Spline Method for Singularly Perturbed Time Delay Reaction Diffusion" and developed by Ermias Argago Megiso, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by Ermias Argago Megiso have read and evaluated the thesis entitled "Exponentially Fitted Tension Spline Method for Singularly Perturbed Time Delay Reaction Diffusion" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics.

Chairperson	Signature	Date
Tamirat Temesgen Dufera (Ph.D.)		23-06-2022
Internal Examiner	Signature	Date
Prof. Gemechis File Duressa		23-06-2022
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CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ACRONYMS

PDEs	Partial Differential Equations
BVP	Boundary Value Problem
SPP	Singularly Perturbed Problems
DDEs	Delay Differential Equations
SPDDEs	Singularly Perturbed Delay Differential Equations
ROC	Rate of Convergence

ABSTRACT

In this study, exponentially fitted tension spline method is presented for solving singularly perturbed time delay reaction-diffusion problems. Crank-Nicolson method is used for time discretization and exponentially fitted tension spline method is applied for spatial discretization. Using the comparison principle and solution bound, the scheme's stability is analyzed. The scheme's uniform convergence is proved. The suggested approach is proven to be uniformly convergent in space and time, with virtually second order in both direction. The scheme's theoretical analysis is supported by numerical test examples for various values of ε *and mesh size.*

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Singularly perturbed differential equations (SPDEs) are differential equations in which its highest order derivative is multiplied by a small positive perturbation parameter (ε). Such types of differential equation arise in various areas of science and engineering. For example in the modeling of semiconductor devices, hydrodynamics Miller et al. (1996), chemical reactor theory, aerodynamics, reaction-diffusion process, quantum mechanics, optimal control (Kumar and Rao, 2010).

Numerical solution of a singularly perturbed differential equation exhibits a multiscale character (Russell, 2016). That is there are (is) a thin layer(s) of the domain where the solution changes rapidly or jumps suddenly forming a boundary layer(s), while a way from the layer(s) the solution behaves regularly or changes slowly in the outer region. As a result such types of problems are called boundary layer problems (File and Reddy, 2013).

Because of the multiscale character of the solution, standard numerical methods which are effective in solving most mathematical problems perform badly when applied to singular perturbation problems. Finite Element Method (FEM), Finite Difference Method (FDM), Spline Approximation (SPA) and Finite Volume Methods (FVM) are some of the standard numerical methods used to solve mathematical problems numerically. However, these standard numerical methods are not parameter-uniform for singular perturbation problems (Miller et al., 2012).

Early numerical solutions of singularly perturbed differential equations were obtained by using a standard finite difference operator on a uniform mesh. In this approach, as the singular perturbation parameter decreases in magnitude, the mesh is refined sufficiently to capture the boundary or interior layers. Even for problems in one dimension, such methods are inefficient and inaccurate (Roos, 2012). Due to these problems, it is necessary to construct suitable numerical methods that are uniformly convergent with respect to perturbation parameter (ε). In order to solve such type of the problems, there are two general approaches, such as fitted operator method which reflects the singularly perturbed nature of the differential operator and fitted mesh methods (Miller et al., 1996).

Differential difference equations or delay differential equations (DDEs) are differential equations where the evolution of the system does not only depend on the present state of the system but also depends on the past history. We can classify DDEs as retarded and neural type.

We consider DDEs as retarded type if the delay argument does not exist in the highest order derivative term but if it exist, we consider it as neural type (Woldaregay et al., 2021).

Mathematical modeling with delay differential equations is widely used for analysis and predictions in various areas of life sciences, for example, population dynamics, epidemiology, immunology, physiology, and neural networks. The time delays or time lags, in these models, can be related to the duration of certain hidden processes like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period Rihan et al. (2018), robotics, control theory, bio sciences, particularly in micro scale heat transfer (Angasu et al., 2021).

A reaction-diffusion equation comprises a reaction term and a diffusion term. The typical form is as follows:

$$\frac{\partial u}{\partial t} = D\Delta u + f(u), \tag{1.1}$$

where $u = u(x,t)$, is a state variable and describes density or concentration of a substance, a population at position $x \in \Omega \subset R$ at time t is (Ω a open set). Δ denotes the Laplace operator so the first hand side describes the "diffusion" including D as diffusion coefficient. The second term, $f(u)$ is a smooth function $f : R \to R$ and describes process with really 'change' the present u i.e something happens to it (birth, death, chemical reaction) not just diffuse in the space. It is also possible, that the reaction term depends not on u but also on the first derivative of u and or explicitly on x where $u(x,t) \in R^m$. Instead of a scalar equation one can also introduce system of reaction diffusion equation which are of the form in (Kuttler, 2011)

$$\frac{\partial u}{\partial t} = D\Delta u + f(x,u,\Delta u). \tag{1.2}$$

Singularly perturbed differential difference equations or singularly perturbed delay differential equations (SPDDEs) are differential equations in which its highest order derivative is multiplied by a small perturbation parameter and having delay parameters on the terms different from the highest derivative Woldaregay et al. (2021). Singularly perturbed parabolic delay differential equation of the form (Wang, 1963)

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} = vg(u(x,t-\tau)) + c[f(u(x,t-\tau) - u(x,t)]$$

represents a simplified mathematical model for the control system of furnace used to produce metal sheets, where u is the temperature distribution in a metal sheet moving at an instantaneous material strip velocity v and heated by a distributed temperature source given by the function f; both v and f are dynamically changed by a controller monitoring the current temperature distribution. The controller's finite speed, on the other hand, causes a fixed delay of length. When $\tau = 0$ this problem becomes a thermal problem without time delay.

In SPDDEs model process, the evaluation not only depends on the current state of the system but also depends on the past history. A number of model problems in science and engineering take the forms of SPDDEs. To list a few of them: neuron variability model in computational neuroscience, optimal control theory problems, and model describing the motion of sunflower etc in (Woldaregay and Duressa, 2021a).

In general, when the perturbation parameter ε tends to zero in such case the smoothness of the solution of the SPDDEs descend and it forms a boundary layers Govindarao et al. (2020). However most of the problems have no known analytical or exact solution so, developing or formulating numerical methods or techniques are critical things to solve the problems numerically (Woldaregay and Duressa, 2021a).

1.2 Statement of the Problem

For singularly perturbed time delay reaction diffusion problems, most numerical methods are not ε-uniform. When ε is very small, standard numerical methods such as FDM, FEM, FVM and collocation method lead to oscillations or divergence in the computed solution Woldaregay and Duressa, (2021a). To avoid this oscillation or divergence, a large number of mesh size is required. This is not practical because of the limited size of computer memory and round-off error. Therefore, to overcome or fix this drawback associated with classical or standard methods we have to use the fitted mesh technique which have a fine mesh in the boundary layer region or fitted operator techniques (Miller et al., 2012).

In some cases the cubic spline does not provide an accurate picture of the shape of the graph. In those cases we often use a tension spline. A tension spline is a cubic spline which has had tension factor applied to it, to stretch the graph closer to the given points. A tension factor τ is a number that we use to generate different conditions for our spline. When our tension factor is 0 we generate a normal cubic spline.

Developing a numerical method whose convergence does not depend on the perturbation parameter has a great importance to the scientific research area. Owning this, this study deals with formulating a uniform numerical method using exponentially fitted tension spline in spatial direction and Crank-Nicolson in temporal direction to find a numerical solution of a singularly perturbed time delay reaction-diffusion problems.

1.3 Objectives

1.3.1 General Objective

The general objective of this study is to formulate uniform numerical method for solving singularly perturbed time delay reaction-diffusion problems.

1.3.2 Specific Objectives

The specific objective of this study are:

- to construct exponentially fitted tension spline method.
- to show stability of the scheme
- to show the uniform convergence of the scheme.

1.4 Significance of the Study

The result obtained from this study

- used as reference material for scholars who works on this research area.
- offer a numerical method for solving singularly perturbed time dependent reaction-diffusion problems.

1.5 Delimitation of the Study

The study is delimited to singularly perturbed delay reaction-diffusion problems of the form:

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + a(x,t)u(x,t) = b(x,t)u(x,t-\tau) + f(x,t), \tag{1.3}$$

with interval condition

$$u(x,t) = \psi_b(x,t),\ (x,t) \in [0,1] \times [-\tau, 0],$$

and boundary conditions

$$u(0,t) = \psi_l(t), \qquad t \in [0,T],$$

$$u(1,t) = \psi_r(t), \qquad t \in [0,T],$$

where $\varepsilon \in (0,1]$ is the singular perturbation parameter, and τ is a delay parameter. The functions $a(x,t), b(x,t), \psi_l(x,t), \psi_r(x,t), \psi_b(x,t)$, and $f(x,t)$ are assumed to be sufficiently smooth and bounded for the existence of unique solution.

CHAPTER TWO

LITERATURE REVIEW

The solution methods of SPDDEs have received high attention in recent year because of their wide applications in different research areas. It is of theoretical and practical interest to consider numerical methods for such problems. Here we have some published papers that shows the numerical treatment of singularly perturbed delay differential equations.

Chakravarthy and Kumar,(2019) proposed an adaptive mesh method for time dependent singularly perturbed differential difference equations. Their method converges uniformly with respect to the perturbation parameter.

Kumar and Kadalbajoo, (2012) used Taylor's series approximation for the delay terms and converted the problem into equivalent BVPs. The authors computed the numerical solution using a B-spline collocation method on Shishkin mesh.

Kanth and Kumar, (2020) used a hybrid scheme for a class of linear and non linear singularly perturbed convection delay problems on piece-wise uniform. Their scheme comprises with tension spline scheme with in the boundary layer region and mid point approximation in the outer region on piece-wise uniform mesh. The scheme showed to be ε-uniform convergence.

File and Reddy, (2013) used a domain decomposition method for solving singularly perturbed differential difference equations with the layer behavior.

Kanth and Kumar, (2017) used tension spline numerical treatment for singularly perturbed convection delayed dominated diffusion equation.

Adilaxmi et al. (2019) used an initial value technique using exponentially fitted non standard or classical finite difference method. Swamy et al. (2016) used a fitted non standard finite difference method for singularly perturbed differential difference equation with mixed shifts. The contribution of this thesis is developing uniformly convergent scheme using an exponentially fitted tension spline method and formulating the stability and uniform convergence analysis of the scheme.

Woldaregay et al. (2021) developed exponentially fitted numerical scheme for singularly perturbed convection-diffusion reaction problems involving delay terms on the convection and reaction terms. The problem exhibits an exponential boundary layer on the left or right side of the domain. They treated terms with delay by using Taylor's series approximation and the resulting singularly perturbed boundary value problem is solved using a specially designed exponentially fitted finite difference method. The stability of the scheme is analyzed and investigated using a comparison principle and solution bound. The formulated scheme converges uniformly with linear order of convergences.

Govindarao et al. (2020) developed a numerical scheme for a singularly perturbed delay parabolic reaction-diffusion initial-boundary value problem. For the discretization of the time derivative, they used the implicit Euler scheme on the uniform mesh and for spatial discretization used the central difference scheme on the Shiskin mesh, which provides a second-order convergent rate. To improve the order of convergence they applied Richardson extrapolation technique and proved that the method converges uniformly with respect to the perturbation parameter.

Fitted operator mid point upwind finite difference method developed for singularly perturbed differential equation having small delay on the convection and reaction terms Woldaregay and Duressa, (2021b). The developed scheme converges ε- uniformly and they showed the stability.

Kumar and Rao, (2010) developed robust finite difference method for singularly perturbed reaction-diffusion problems. They discretized the problem using a suitable combination of fourth order compact difference scheme and central difference scheme on generalized Shishkin mesh and the convergence analysis is given and proved to be almost fourth order uniformly convergent in maximum norm with respect to singular perturbation parameter(ε).

Turuna et al. (2020) solved singularly perturbed one dimensional parabolic convection-diffusion problems. They developed suitable numerical methods for the provided problem. They used non standard finite difference method for spatial derivatives discretization and implicit Runge-kutta method for the semi-discrete scheme. The convergence of the method is analyzed, and it is shown to be first order convergent.

Gowrisankar and Natesan,(2017) proposed ε-uniformly convergent numerical scheme for singularly perturbed parabolic convection-diffusion problems with a delay. They divided the domain using a piece-wise uniform adaptive mesh in the spatial direction and a uniform mesh in the temporal direction. Further they discretized the time derivative by the backward-Euler scheme and the spatial derivatives by the upwind finite difference scheme. They work out the stability analysis. Then proved that the proposed scheme is ε-uniform convergence of first-order in time and up to a logarithmic factor in space. Finally the numerical result showed the ε-uniform first-order convergence.

Woldaregay and Duressa,(2021a) developed exponentially fitted tension spline method for singularly perturbed differential-difference equations having delay and advance in the reaction terms.They used Taylor series approximation to approximate the terms with the shifts. The resulting singularly perturbed boundary value problem is treated using an exponentially fitted tension spline method. The stability and uniform convergence of the scheme are discussed and proved. The developed scheme gives an accurate result with linear order uniform convergence.

Chakravarthy et al. (2015) constructed exponentially fitted finite difference scheme for second order singularly perturbed delay differential equations with large delay on a uniform by the method based on cubic spline in compression. They verified ε-uniform convergence of the scheme by taking a variety of examples and the proposed method provided significant advantage for the considered problems.

Chakravarthy and Rao, (2012) developed modified fourth order Numerov method for singularly perturbed differential-difference equation of mixed type which comprises both terms having delay and advance shift. They proved that, layer behavior is maintained when the shift parameters are smaller than perturbation parameter. It is also observed that the layer behavior of the solution is no longer maintained and the solution exhibits oscillatory behavior when shift parameters are larger than perturbation parameter. Finally they concluded that as the grid size h decreases, the maximum absolute errors decrease, which shows the convergence to the computed solution.

Rao and Chakravarthy, (2014a) treated singularly perturbed boundary value problem. They treated the considered problems using a fitted numerical scheme for second order singularly perturbed delay differential equations via cubic spline in compression and used Taylor's approximation to tackle the term containing the small shift. Also, they introduced a fitting factor in a tridiagonal finite difference method to handle the perturbation parameter and used Thomas algorithm to solve the tridiagonal system. The method shown to have second order parameter uniform convergence and provided significant advantage.

Ramesh and Kadalbajoo, (2008) time dependent singularly perturbed convection-diffusion type is treated numerically. They first used Taylor series approximation to approximate the retarded term and the resulting time-dependent singularly perturbed differential equation is approximated using parameters uniform numerical methods based on Euler implicit, upwind and mid point upwind finite difference schemes. The schemes L^h_{up} and L^h_{mp} are proved to be parameter uniform convergent.

Erdogan and Amiraliyev,(2012) singularly perturbed initial value problem for linear second-order delay differential equation is studied. In order to solve the considered problem they used exponentially fitted difference scheme in a uniform mesh. Finally, the proposed scheme showed the uniform convergence with respect to perturbation parameter and mesh parameter.

As mentioned above all studies where developed difference scheme for singularly perturbed delay differential equations. To the best of our knowledge, the exponentially fitted tension spline method has not been developed for treating singularly perturbed time delay differential equations. As a result in this thesis we developed an accurate and uniformly convergent numerical scheme using exponentially fitted tension spline to spatial discretization and Crank-Nicolson to temporal discretization.

CHAPTER THREE

METHODOLOGY

3.1 Description of the Analytical Analysis

3.1.1 Continuous Problem

On the domain $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ and $\varphi = \varphi_l \cup \varphi_b \cup \varphi_r$ on smooth boundary, consider a class of second order singularly perturbed parabolic reaction-diffusion equations having a term with a time delay is given by.

$$L_\varepsilon u_\varepsilon(x,t) \equiv \left(\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} + au\right)(x,t) = -b(x,t)u(x,t-\tau) + f(x,t), \tag{3.1}$$

with interval condition

$$u(x,t) = \psi_b(x,t)\ ,\ (x,t) \in [0,1] \times [-\tau, 0], \tag{3.2}$$

and boundary conditions

$$\begin{aligned} u(0,t) &= \psi_l(0,t), \qquad t \in [0,T], \\ u(1,t) &= \psi_r(1,t), \qquad t \in [0,T], \end{aligned} \tag{3.3}$$

where $\varepsilon \in (0,1]$ is the singular perturbation parameter, and τ is a delay parameter. The functions $a(x,t),\ b(x,t),\ \psi_l(x,t),\ \psi_r(x,t),\ \psi_b(x,t)$, and $f(x,t)$ are assumed to be sufficently smooth and bounded functions that satisy $a(x,t) \geq \alpha > 0,\ b(x,t) \geq \beta > 0\ \ (x,t) \in \bar{D}$.

3.1.2 Bounds on the Solution and Its Derivatives

To guarantee the existence and uniqueness of the solution of singularly perturbed parabolic reaction diffusion problems with retarded parameter $(3.1) - (3.3)$, it is presumed that given data $a(x,t), b(x,t)$ and $f(x,t)$ is holder's continuous and the compatibility conditions at $(0,0), (1,0), (0,-\tau)$ and $(1,-\tau)$ corner points are fulfilled. Then we can define the compatibility condition's as:

$$\begin{cases} \psi_b(0,0) = \psi_l(0), \\ \psi_b(1,0) = \psi_r(0), \\ \frac{\partial \psi_l(0)}{\partial t} - \varepsilon\frac{\partial^2 \psi_b(0,0)}{\partial x^2} + a(0,0)\psi_b(0,0) = f(0,0) - b(0,0)\psi_b(0,-\tau), \\ \frac{\partial \psi_r(0)}{\partial t} - \varepsilon\frac{\partial^2 \psi_b(1,0)}{\partial x^2} + a(1,0)\psi_b(1,0) = f(1,0) - b(1,0)\psi_b(1,-\tau). \end{cases} \quad (Khari\ andKumar, 2022) \tag{3.4}$$

The differential operator L_ε in(3.1) satisfies the following Maximum principle.

Lemma 3.1 *Maximum principle. Assume that a,* $b \in C^0(\bar{D})$ *and let* $\phi \in C^2(D) \cap C^0(\bar{D})$*. Suppose*

that $\phi \geq 0$ on φ. Then $L_\varepsilon \phi \geq 0$ in D implies that $\phi \geq 0$ in $\bar{D}$.
The stability of L_ε and an ε-uniform bound for the solution of (3.1) in the maximum norm are established by the following theorem. (Khari and Kumar,2022)

Theorem 3.1 *Let υ be any function in the domain of definition of the differential operator L_ε in* (3.1). *Then $\|\upsilon\| \leq (1+\alpha T)\max\{\|L_\varepsilon \upsilon\|, \|\upsilon\|\varphi\}$, and any solution u_ε of* (3.1) *has the ε-uniform upper bound. $\|u_\varepsilon\| \leq (1+\alpha T)\max\{\|f\|,\ \|\psi\|\varphi\}$, where the constant $\alpha = \max_{\bar{D}}\{0, 1-\gamma\} \leq 1$. (Khari and Kumar,2022)*

Theorem 3.2 *Let a,b,f $\in C^{\alpha}(\bar{D}), \psi_l \in C^{1+\alpha/2}([0,T]), \psi_b \in C^{(2+\alpha,1+\alpha/2)}(\varphi)$,*
$\psi_r \in C^{1+\alpha/2}([0,T])$, and assume that the compatibility conditions (3.4) *and* (3.1) *are fulfilled. Then* (3.1) *has a unique solution u_ε and $u_\varepsilon \in C^{(2+\alpha,1+\alpha/2)}(\bar{D})$. Further, the derivative of the solution u_ε satisfies the following:*

$$\left\| \frac{\partial^{i+j} u_\varepsilon}{\partial x^i \partial t^j} \right\| \leq C\varepsilon^{-i/2}. \tag{3.5}$$

The proof exist in (Ansari et al., 2007).
The bounds in the above theorem do not depend explicitly on the boundary layer. So we decompose u_ε in to its singular and smooth component to obtain the stronger estimate on its derivatives and of its partial derivative. We decompose u_ε the solution of (3.1) as:

$$u_\varepsilon = \upsilon_\varepsilon + \omega_\varepsilon. \tag{3.6}$$

Where ω and υ are singular and regular component. We further decompose the smooth component as:

$$\upsilon_\varepsilon = \upsilon_0 + \varepsilon \upsilon_1. \tag{3.7}$$

Further we define υ_0 and υ_1 as

$$\begin{aligned} \frac{\partial \upsilon_0}{\partial t} + a\upsilon &= b\upsilon(x,t-\tau) + f,\ (x,t) \in D, \\ \upsilon_0(x,t) &= \psi_b(x,t), \qquad 0n\ \varphi_b. \end{aligned} \tag{3.8}$$

$$\begin{aligned} L_\varepsilon \upsilon_1 = -b\upsilon_1(x,t-\tau) + \frac{\partial^2 \upsilon_0}{\partial x^2}, &\qquad (x,t) \in D, \\ \upsilon_1(x,t) = 0, &\qquad (x,t) \in \varphi. \end{aligned} \tag{3.9}$$

The function υ_0 is the solution of the reduced problem. Furthermore υ_ε satisfies

$$\begin{aligned} L_\varepsilon \upsilon_\varepsilon = -b\upsilon_\varepsilon(x,t-\tau) + f,\ (x,t) \in D,\ \upsilon_\varepsilon = \psi_b(x,t),\ (x,t) \in \varphi_b, \\ \upsilon_\varepsilon(0,t) = \upsilon_0(0,t),\ (x,t) \in \varphi_l,\ \upsilon_\varepsilon(1,t) = \upsilon_0(1,t),\ (x,t) \in \varphi_r. \end{aligned} \tag{3.10}$$

With υ_ε thus defined, it follows that ω_ε is determined and that it satisfies

$$\begin{aligned} L_\varepsilon \omega_\varepsilon &= -b\omega_\varepsilon(x,t-\tau),\ (x,t)\in D,\ \omega_\varepsilon = 0\ (x,t)\in\varphi_b, \\ \omega_\varepsilon(0,t) &= \psi_l(t)-\upsilon_0(0,t)\ (x,t)\in\varphi_l,\ \omega_\varepsilon(1,t)=\psi_r(t)-\upsilon_0(1,t),\ (x,t)\in\varphi_r. \end{aligned} \tag{3.11}$$

It is also convenient to write

$$\omega_\varepsilon = \omega_l + \omega_r, \tag{3.12}$$

where ω_l and ω_r are defined by

$$\begin{aligned} L_\varepsilon \omega_l &= -b\omega_l(x,t-\tau),(x,t)\in D, \\ \omega_l(0,t) &= \psi_l(t)-\varphi_0(0,t),\ (x,t)\in\varphi_l,\ \omega_l=0,\ (x,t)\in\varphi_b\cup\varphi_r, \end{aligned} \tag{3.13}$$

$$\begin{aligned} L_\varepsilon \omega_r &= -b\omega_r(x,t-\tau),(x,t)\in D, \\ \omega_r(1,t) &= \psi_r(t)-\upsilon_0(1,t),\ (x,t)\in\varphi_r,\ \omega_r=0,\ (x,t)\in\varphi_l\cup\varphi_b. \end{aligned} \tag{3.14}$$

The subsequent theorem provides the explicit bounds for the singular component ω_ε, the smooth component υ_ε, and all its partial derivatives.

Theorem 3.3 *Suppose the data* $a,\ b,\ f\in C^{(2+\alpha,1+\alpha/2)}(\bar{D})$, $\psi_l\in C^{(2+\alpha/2)}([0,T])$, $\psi_b\in C^{(4+\alpha,2+\alpha/2)}(\varphi)$, $\psi_r\in C^{(2+\alpha/2)}([0,T]),\alpha\in(0,1)$ *and compatibility conditions* (3.4) *of high order at corner points are satisfied. Then the integer i , j such that* $0\le i+2j\le 4$ *we have the following estimate:*

$$\left\|\frac{\partial^{i+j}\upsilon}{\partial x^i\partial t^j}\right\|_{\bar{D}} \le C(1+\varepsilon^{1-i/2}), \tag{3.15}$$

$$\left\|\frac{\partial^{i+j}\omega_l}{\partial x^i\partial t^j}\right\| \le C\varepsilon^{-i/2}e^{\frac{-x}{\sqrt{\varepsilon}}}, \tag{3.16}$$

$$\left\|\frac{\partial^{i+j}\omega_r}{\partial x^i\partial t^j}\right\| \le C(1+\varepsilon^{-i/2})e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}. \tag{3.17}$$

The proof exist in (Ansari et al., 2007)

Theorem 3.4 *The partial derivative of ω (x,t) satisfies:*

$$\left\|\frac{\partial^{i+j}\omega}{\partial x^i\partial t^j}\right\| \le C\varepsilon^{-i/2}\left(e^{\frac{-x}{\sqrt{\varepsilon}}}+e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}\right),\ (x,t)\in(\bar{D}), for\ integer\ i,\ j \text{ such that } 0\le i+2j\le 4.$$

Using estimations of Theorem 3.3 and decomposition (3.12) proof of the above theorem is completed.

3.2 Formulation of Numerical Scheme

3.2.1 Temporal Semi-discretization

The time domain [0,T] is discretized using a uniform mesh with time step Δt as $\Omega_t^M = \{t_j = j\Delta t,\ j = 0,1,2,3,..M,\ t_M = T,\ \Delta t = T/M\}$ and $\Omega_t^m = \{t_j = j\Delta t,\ j = 0,1,2,3,..m,\ t_m = \tau,\ \Delta t = \tau/m\}$ where M is the number of mesh points in time direction in the interval [0,T] and m is the number of mesh points in $[-\tau,0]$. Note that $T = k\tau$ for some positive integer k. For temporal derivative term of (3.1)-(3.3) approximation we use averaged Crank-Nicolson method, which gives a system of boundary value problems

$$\left(1+\frac{\Delta t}{2}L^{\Delta t}\right)U^{j+1}(x) = \begin{cases} \left(1-\frac{\Delta t}{2}L^{\Delta t}\right)U^{j}(x) + \Delta t b(x,t_{j+1/2})\psi_b(x) + \Delta t f(x,t_{j+1/2}), \\ \text{for } j = 0,1,2,3,4\ \ldots,m, x\in\Omega_x, \\ \left(1-\frac{\Delta t}{2}L^{\Delta t}\right)U^{j}(x) + \Delta t b(x,t_{j+1/2})U_{j+1/2-m}(x) + \Delta t f(x,t_{j+1/2}), \\ \text{for } j = m+1, m+2,\ldots,M-1,\ x\in\Omega_x, \end{cases} \tag{3.18}$$

with the discretized boundary conditions

$$U^{j+1}(0) = \psi(t_{j+1}),\ U^{j+1}(1) = \psi(t_{j+1}), \tag{3.19}$$
$$\text{for } j = 1,2,3,4\ldots,M,$$

where

$$L^{\Delta t}U^{j+1}(x) = -\varepsilon U_{xx}^{j+1}(x) + a(x,t_{j+1/2})U^{j+1}(x),$$

Here, $U_{j+1}(x)$ is denoted for the approximation of $U(x,t_{j+1})$ at the $(j+1)$th time level and $a(x,t_{j+1/2}) = \frac{a(x,t_j+1)+a(x,t_j)}{2}$, similarly for $b(x,t_{j+1/2})$ and $f(x,t_{j+1/2})$.

The Local Truncation Error in the Temporal Direction

Lemma 3.2 *Suppose that*

$$\left|\frac{\partial^k u(x,t)}{\partial t^k}\right| \le C, (x,t)\in\bar{D},\ k = 0,1,2,. \tag{3.20}$$

The local truncation error in the temporal direction is given by

$$\|e_{j+1}\| \le C_1(\Delta t)^3. \tag{3.21}$$

Proof. Using Taylor's series approximation for $u(x,t_j)$ and $u(x,t_{j+1})$ centering at $t_{j+1/2}$, we obtain

$$
\begin{aligned}
u(x,t_j) &= u(x,t_{j+1/2}) - \frac{\Delta t}{2}u_t(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{tt}(x,t_{j+1/2}) + O((\Delta t)^3),\\
u(x,t_{j+1}) &= u(x,t_{j+1/2}) + \frac{\Delta t}{2}u_t(x,t_{j+1/2}) + \frac{(\Delta t)^2}{8}u_{tt}(x,t_{j+1/2}) + O((\Delta t)^3).
\end{aligned} \tag{3.22}
$$

From the above approximation in (3.22), we obtain

$$
\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} = u_t(x,t_{j+1/2}) + (O(\Delta t)^2). \tag{3.23}
$$

Substituting (3.23) in to (3.1) we obtain.

$$
\begin{aligned}
\frac{u(x,t_{j+1}) - u(x,t_j)}{\Delta t} &= \varepsilon u_{xx}(x,t_{j+1/2}) - a(x,t_{j+1/2})u(x,t_{j+1/2}) + f(x,t_{j+1/2}) + O((\Delta t)^2)\}\\
&\quad + \begin{cases} b(x,t_{j+1/2})\alpha(x), \\ \text{for } j = 0,1,\ldots,m \\ b(x,t_{j+1/2-m})u(x,t_{j+1/2-m}) \\ \text{for } j = m+1,\ldots,M-1, \end{cases}
\end{aligned} \tag{3.24}
$$

where

$$
u(x,t_{j+1/2}) = \frac{u(x,t_{j+1}) + u(x,t_j)}{2} + O((\Delta t)^2),\; f(x,t_{j+1/2}) = \frac{f(x,t_{j+1}) + f(x,t_j)}{2} + O((\Delta t)^2). \tag{3.25}
$$

Since the error $e_{j+1}(x) = u(x,t_{j+1}) - U_{j+1}(x)$ satisfies the semidiscrete differential operator

$$
\left(1 + \frac{\Delta t}{2}L^{\Delta t}\right) e_{j+1}(x) = O((\Delta t)^3), e_{j+1}(0) = 0 = e_{j+1}(1). \tag{3.26}
$$

Hence by applying the maximum principle, we obtain

$$
\|e_{j+1}\| \le C_1(\Delta t)^3. \tag{3.27}
$$

Next, we need to show the bound for the global error of the temporal discretization. Let us denote TE_{j+1} as the global error up to the $(j+1)^{th}$ time step.

Global Error of the Temporal Discretization

Lemma 3.3 *The global error at t_{j+1} is given by*

$$
\|TE\| \le C_1(\Delta t)^2, j = 1,2,\ldots,M-1. \tag{3.28}
$$

Proof. Using the local error up to the $(j+1)^{th}$ time step given in the above, we obtain the global error at the $(j+1)^t h$ time step as

$$
\begin{aligned}
\|TE\| &= \left\| \sum_{l=1}^{j+1} el \right\| \leq \|e_1\| + \|e_2\| + \|e_3\| + \cdots + \|e_{j+1}\| \\
&\leq C_1 T(\Delta t)^2 \text{ since } (j+1)\Delta t \leq T \\
&= C(\Delta t)^2,\ C_1 T = C, \qquad (3.29)
\end{aligned}
$$

where C is a constant independent of ε and Δt.

3.3 Spatial Discretization

The spatial domain is discretized into N equal number of subintervals, each of length h. Let $0 = x_0,\ x_N = 1$, and $x_i = ih,\ i = 0, 1, 2, \ldots, N$, be the mesh points. For spatial discretization, we apply an exponentially fitted tension spline method which helps us to control the influence of the singular perturbation parameter.

3.3.1 Exponentially Fitted Tension Spline Method

Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be a uniform partition for $[0,1]$ such that $x_i = ih, \quad i = 0, 1, 2, 3, \ldots N$. A function $S(x,\tau) = S(x)$ is a class of $C^2(\bar{\Omega})$, which interpolates $u(x)$ at the mesh points x_i depending on the parameter τ, reduces to cubic spline in $\bar{\Omega}$ as $\tau \to 0$, and is termed as a parameteric cubic spline function Khan and Aziz, (2005), Adivi Sri Venkata and Palli, (2017) . In $[x_i, x_{i+1}]$, the spline function $S(x)$ satisfies the differential equation

$$
S''(x) - \tau S(x) = [S''(x_i) - \tau S(x_i)]\frac{x_{i+1} - x}{h} + [S''(x_{i+1}) - \tau S(x_{i+1})]\frac{x - x_i}{h}, \qquad (3.30)
$$

where $S(x_i) = u(x_i)$ and $\tau > 0$ is termed as a cubic spline in compression. Solving the linear second-order differential equation in (3.30) and determining the arbitrary constants from the interpolation conditions $S(x_{i+1}) = u(x_{i+1}),\ S(x_i) = u(x_i)$, we get

$$
\begin{aligned}
S(x) &= \frac{h^2}{\lambda^2 \sinh h\lambda}[M_{i+1}\sinh\frac{\lambda(x - x_i)}{h}) + M_i \sinh(\frac{\lambda(x_{i+1} - x)}{h})] \qquad (3.31) \\
&- \frac{h^2}{\lambda^2}[(M_{i+1} - \frac{\lambda^2}{h^2}u(x_{i+1}))(\frac{x - x_i}{h}) + (M_i - \frac{\lambda^2}{h^2}u(x_i)(\frac{x_{i+1} - x}{h})],
\end{aligned}
$$

where $\lambda = h\tau^1/2$ and $M_k = u''(x_K)$ for $k = i \pm 1, i$.
Now, differentiating (3.31) and letting $x \to x_i$, on the interval $[x_i, x_i + 1]$, we obtain

$$
S'(x_i^+) = \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{\lambda^2}[M_{i+1}(1 - \frac{\lambda}{\sinh\lambda}) + M_i(\lambda\coth\lambda - 1)], \qquad (3.32)
$$

and on the interval $[x_{i-1}, x_i]$, we obtain

$$
S'(x_i^-) = \frac{u(x_i) - u(x_{i-1})}{h} + \frac{h}{\lambda^2}[M_i(\lambda\coth\lambda - 1) + M_{i-1}(1 - \frac{\lambda}{\sinh\lambda})], \qquad (3.33)
$$

Equating the left and right hand derivatives at x_i gives

$$\frac{u(x_i)-u(x_{i-1})}{h}+\frac{h}{\lambda^2}[M_i(\lambda\coth\lambda-1)+M_{i-1}(1-\frac{\lambda}{\sinh\lambda})] \tag{3.34}$$
$$=\frac{u(x_{i+1})-u(x_i)}{h}-\frac{h}{\lambda^2}[M_{i+1}(1-\frac{\lambda}{\sinh\lambda})+M_i(\lambda\coth\lambda-1)],$$

Rearranging, we obtain the tridiagonal system

$$\lambda_1 M_{i-1}+2\lambda_2 M_i+\lambda_1 M_{i+1}=\frac{u(x_{i-1})-2u(x_i)+u(x_{i+1})}{h^2},\ i=1,2,3,\ldots,N-1, \tag{3.35}$$

where $\lambda_1=\frac{1}{\lambda^2}(\frac{\lambda}{\sinh\lambda}-1)$ and $\lambda_2=\frac{1}{\lambda_2}(1-\lambda\coth\lambda)$ The condition of continuity given in (3.35) ensures the continuity of the first order derivatives of the spline $S(x)$ at interior nodes.

Now substituting

$$-\varepsilon M_k+a(x_k,t_{j+1/2})U^{j+1}(x_k)=\begin{cases}\varepsilon M_k+a(x_k,t_{j+1/2})U^j(x_k)+\Delta t b(x_k,t_{j+1/2})\alpha(x_k)+\\ \Delta t f(x_k,t_j+1/2) \text{ for } j=0,1,2,3,4...,m,\ x\in\Omega_x,\\ \varepsilon M_k+a(x_k,t_{j+1/2})U^j(x_k)+\Delta t b(x_k,t_j+1/2)U(x_k,t_{j+1/2-m})+\\ \Delta t f(x_k,t_{j+1/2}) \text{ for } j=m+1\ldots,M-1\ x\in\Omega_x,\end{cases} \tag{3.36}$$

for $k=i-1,i$ and $i+1$, $M_k=u''(x_k)$ we obtain
$L^{\Delta t,h}\equiv\frac{-\varepsilon}{h^2}[\,U_{i-1,j+1}-2U_{i,j+1}+U_{i+1,j+1}]\ +\lambda_1 a(x_{i-1,j+1/2})U_{i-1,j+1}+2\lambda_2 a(x_{i,j+1/2})U_{i,j+1}$
$+\lambda_1 a(x_{i+1,j+1/2})U_{i+1,j+1}=\frac{\varepsilon}{h^2}[\,U_{i-1,j}-2U_{i,j}+U_{i+1,j}]\ +\lambda_1 a(x_{i-1,j+1/2})U_{i-1,j}$
$+2\lambda_2 a(x_{i,j+1/2})U_{i,j}+\lambda_1 a(x_{i+1,j+1/2})U_{i+1,j}$

$$+\begin{cases}\lambda_1\Delta t b(x_{i-1},t_{j+1/2})\alpha(x_{i-1})+2\lambda_2\Delta t b(x_i,t_{j+1/2})\alpha(x_i)+\lambda_1\Delta t b(x_{i+1},t_{j+1/2})\alpha(x_{i+1})+\\ \lambda_1\Delta t f(x_{i-1},t_{j+1/2})+2\lambda_2\Delta t f(x_i,t_{j+1/2})+\lambda_1\Delta t f(x_{i+1},t_{j+1/2}),\\ \text{for } j=0,1,2,\ldots,m,\ x\in\Omega_x,\\ \lambda_1\Delta t b(x_{i-1},t_{j+1/2})U_{i-1,j+1/2-m}+2\lambda_2\Delta t b(x_i,t_{j+1/2})U_{i,j+1/2-m}+\lambda_1\Delta t b(x_{i+1},t_{j+1/2})U_{i+1,j+1/2-m}\\ +\lambda_1\Delta t f(x_{i-1},t_{j+1/2})+2\lambda_2\Delta t f(x_i,t_{j+1/2})+\lambda_1\Delta t f(x_{i+1},t_{j+1/2}),\\ \text{for } j=m+1,\ldots,M-1,\ x\in\Omega_x.\end{cases} \tag{3.37}$$

3.3.2 Deriving the Exponential Fitting Factor

Here we consider the general reaction-diffusion problems given by

$$-\varepsilon U''(x)+a(x)U(x)=f(x), \tag{3.38}$$

boundaryconditions

$$U(0)=\psi_1,\ U(1)=\psi_2, \tag{3.39}$$

where $a(x)$ and $f(x)$ are sufficiently smooth functions and moreover the coefficient functions is assumed to satisfy $a(x) \geq \gamma > 0$ for all $x \in (0,1)$.

The problems in (3.38) exhibits twin layers, the domain is divided in to three sub-domains, and the boundary layer problem is converted to a regular problem by appropriate transformations using stretching variables we consider, the asymptotic expansion solution of (3.38) and (3.39).

$$U(x,\varepsilon) = \sum_{i=0}^{\infty}[U_i(x) + v_i(\mu) + w_i(\omega)]\varepsilon^i \tag{3.40}$$

where $\mu = x/\sqrt{\varepsilon}$ and $\omega = (1-x)/\sqrt{\varepsilon}$.

Then the zeros order of the asymptotic expansion is given by

$$U(x) = U_0(x) + v_0(\mu) + w_0(\omega) + O(\varepsilon) \tag{3.41}$$

$$\text{where } U_0(x) = f(x)/a(x) \tag{3.42}$$

is the solution of the reduced problem (3.38) and (3.39) which does not satisfy the boundary conditions, v_0 is the left boundary layer correction and w_0 is the right boundary layer correction. The solution v_0, w_0 satisfies the differential equations,

$$-\frac{d^2 v_0(\mu)}{d\mu^2} + a(0)v_0(\mu) = 0;\ \mu \in (0,\infty) \tag{3.43}$$

$$-\frac{d^2 w_0(\omega)}{d\omega^2} + a(1)w_0(\omega) = 0;\ \omega \in (0,\infty) \tag{3.44}$$

$$\text{with } v_0(\mu = 0) + w_0(\omega = 1/\sqrt{\varepsilon}) = \psi_1 - U_0(0), \tag{3.45}$$

$$v_0(\mu = 1/\sqrt{\varepsilon}) + w_0(\omega = 1) = \psi_2 - U_0(0), \tag{3.46}$$

$$v_0(\mu = \infty) = w_0(\omega = \infty) = 0.$$

The solution of (3.43) and (3.44) will be

$$v_0(\mu) = Ae^{-\sqrt{a(0)}\mu}, \tag{3.47}$$

$$w_0(\omega) = Be^{-\sqrt{a(1)}\omega}. \tag{3.48}$$

Therefore, zeros order solution of (3.41) and (3.42) becomes

$$U(x) = U_0(x) + Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x} + Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-x)} + O(\varepsilon) \tag{3.49}$$

where A and B are determined as

$$A = \frac{(\psi_1 - U_0(0)) - (\psi_2 - U_0(1))e^{-\sqrt{\frac{a(1)}{\varepsilon}}}}{1 - e^{-\left(\sqrt{\frac{a(0)}{\varepsilon}} + \sqrt{\frac{a(1)}{\varepsilon}}\right)}}, \tag{3.50}$$

$$B = \frac{(\psi_2 - U_0(1)) - (\psi_1 - U_0(0))e^{-\sqrt{\frac{a()}{\varepsilon}}}}{1 - e^{-\left(\sqrt{\frac{a(0)}{\varepsilon}} + \sqrt{\frac{a(1)}{\varepsilon}}\right)}}. \tag{3.51}$$

Now we divided the interval $[0,1]$ in to N equal parts with constant mesh length $h,$ let $0 = x_0, x_1, \ldots, x_N = 1$ be the mesh points. Then we have $x_i = ih, i = 0,1,\ldots,N$. We choose m_1 and m_2 such that $x_{m1} = \sqrt{\varepsilon}$ and $x_{m2} = 1 - \sqrt{\varepsilon}$. Then in the interval $[0,\sqrt{\varepsilon}]$ the boundary layer will be in the left hand side $(i.e.$ at $x = 0)$ and in the interval $[1-\sqrt{\varepsilon},1]$ the boundary layer will be in the right hand side $(i.e.$ at $x = 1)$.
At $x = x_i$ equation(3.38) can be written as

$$-\varepsilon U''(x_i) + a(x_i)U(x_i) = f(x_i)$$

for convenience, we take $a(x_i) = a_i,\ f(x_i) = f_i,\ U(x_i) = U_i$.
Now here we consider the finite difference for U''

$$U'' = \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + a_iU_i = f_i,\ i = 1,2,\ldots,N-1$$

and substituting in Eq(3.38) it becomes

$$-\varepsilon\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + a_iU_i = f_i, \tag{3.52}$$

Boundary layer problems

Here, we introduce a fitting factor σ in the scheme (3.52) to hinder the infulence of perturbation as.

$$-\varepsilon\sigma\left(\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2}\right) + a_iU_i = f_i, \tag{3.53}$$

Now to determine the fitting factor σ, we use boundary layer asymptotic solution with outer solution.

$$U_0(x_i) + v_0(x_i) = U_i = U_0(x_i) + Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}, \tag{3.54}$$

where A is given above in Eq. (3.50). The solution converges uniformly to the solution of (3.38).
As h$\rightarrow$ 0 Eq. (3.53) becomes

$$-\lim_{h\to 0}\frac{\sigma}{\rho}\left(U_{i-1} - 2U_i + U_{i+1}\right) = \lim_{h\to 0}\left(f_i - b(ih)U_i\right), \tag{3.55}$$

where $\rho = \frac{h}{\sqrt{\varepsilon}}$, using (3.54) we get

$$U_{i-1} = U_0(x_{i-1}) + Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}e^{\sqrt{a(x(i))}\rho} \text{ and} \tag{3.56}$$

$$U_{i+1} = U_0(x_{i+1}) + Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}e^{-\sqrt{a(x(i))}\rho} \tag{3.57}$$

Then we get
$U_{i-1}-2U_i+U_{i+1}=(U_0(x_{i-1}-2U_0(x_i)+U_0(x_{i+1})+Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right).$
Now substituting this in to (3.55) we get

$$-\lim_{h\to 0}\frac{\sigma}{\rho^2}\left(U_0(x_{i-1}-2U_0(x_i)+U_0(x_{i+1})+Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)\right)$$
$$=\lim_{h\to 0}\left(f_i-a(ih)U_0(x_i)-a(ih)Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}\right)$$

$$-\lim_{h\to 0}\frac{\sigma}{\rho^2}\left(U_0(x_{i-1}-2U_0(x_i)+U_0(x_{i+1})+Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}x_i}\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)\right)$$
$$=-\lim_{h\to 0}\left(b(ih)Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_i}\right)$$

$$-\lim_{h\to 0}\frac{\sigma}{\rho^2}\left(U_0(x_{i-1}-2U_0(x_i)+U_0(x_{i+1})+Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_{ih}}\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)\right)$$
$$=-\lim_{h\to 0}\left(a(ih)Ae^{-\sqrt{\frac{a(x(i))}{\varepsilon}}x_{ih}}\right),\ \text{since } x_i=ih.$$

Simplifying the above we obtain

$$\frac{\sigma}{\rho^2}\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)=a(x(i))$$

$$\sigma=\frac{\rho^2 a(x(i))}{\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)}$$

$$\sigma=\frac{\rho^2 a(x(i))}{\left(e^{\sqrt{a(x(i))}\rho}-2+e^{-\sqrt{a(x(i))}\rho}\right)}$$

$$\sigma=\frac{\rho^2 a(x(i))}{\left(e^{\sqrt{a(x(i))}\rho/2}-e^{-\sqrt{a(x(i))}\rho/2}\right)^2}.$$

Finally the fitting factor for the boundary layer problems will be

$$\sigma=\frac{(\frac{\rho}{2})^2 a(x(i))}{\left(sinh(\sqrt{a(x(i))}\rho/2)\right)^2} \tag{3.58}$$

Substituting (3.58) in to (3.37) we obtain the required scheme as.
$(\frac{-\Delta t\varepsilon\sigma}{2h^2}+\lambda_1+\frac{\Delta t}{2}\lambda_1 a(x_{i-1},t_{j+1/2}))U_{i-1,j+1}+(\frac{-\Delta t\varepsilon\sigma}{h^2}+2\lambda_2+\Delta t\lambda_2 a(x_i,t_{j+1/2}))U_{i,j+1}+(\frac{-\Delta t\varepsilon\sigma}{2h^2}+\lambda_1+\frac{\Delta t}{2}\lambda_1 a(x_{i+1},t_{j+1/2}))U_{i+1,j+1}=(\frac{\Delta t\varepsilon\sigma}{2h^2}+\lambda_1-\frac{\Delta t}{2}\lambda_1 a(x_{i-1},t_{j+1/2}))U_{i-1,j}$

$$(-\tfrac{\Delta t\varepsilon\sigma}{h^2}+2\lambda_2-\Delta t\lambda_2 a(x_i,t_{j+1/2}))U_{i,j}+(\tfrac{\Delta t\varepsilon\sigma}{2h^2}+\lambda_1-\tfrac{\Delta t}{2}\lambda_1 a(x_{i+1},t_{j+1/2}))U_{i+1,j}$$

$$+\begin{cases}\lambda_1\Delta t b(x_{i-1},t_{j+1/2})\alpha(x_{i-1})+2\lambda_2\Delta t b(x_i,t_{j+1/2})\alpha(x_i)+\lambda_1\Delta t b(x_{i+1},t_{j+1/2})\alpha(x_{i+1})+\\ \lambda_1\Delta t f(x_{i-1},t_{j+1/2})+\lambda_2\Delta t f(x_i,t_{j+1/2})+\lambda_1\Delta t f(x_{i+1},t_{j+1/2}),\\ \text{for } j=0,1,2,\dots,m,\ x\in\Omega_x,\\ \lambda_1\Delta t b(x_{i-1},t_{j+1/2})U_{i-1,j+1/2-m}+2\lambda_2\Delta t b(x_i,t_{j+1/2})U_{i,j+1/2-m}+\lambda_1\Delta t b(x_{i+1},t_{j+1/2})U_{i+1,j+1/2-m}\\ +\lambda_1\Delta t f(x_{i-1},t_{j+1/2})+2\lambda_2\Delta t f(x_i,t_{j+1/2})+\lambda_1\Delta t f(x_{i+1},t_{j+1/2}),\\ \text{for } j=m+1,\dots,M-1,\ x\in\Omega_x.\end{cases} \tag{3.59}$$

Here we have matrix representation for the above scheme

$$\begin{pmatrix} v^- & v^c & v^+ & 0 & \cdot & \cdot & 0\\ 0 & v^- & v^c & v^+ & \cdot & \cdot & \cdot\\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & v^- & v^c & v^+ & 0\\ 0 & \cdot & \cdot & 0 & v^- & v^c & v^+ \end{pmatrix}\begin{pmatrix} U_{0,j+1}\\ U_{1,j+1}\\ \cdot\\ \cdot\\ \cdot\\ U_{N-1,j+1}\\ U_{N,j+1}\end{pmatrix}$$

$$=\begin{pmatrix} z^- & z^c & z^+ & 0 & \cdot & \cdot & 0\\ 0 & z^- & z^c & z^+ & \cdot & \cdot & \cdot\\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & z^- & z^c & z^+ & 0\\ 0 & \cdot & \cdot & 0 & z^- & z^c & z^+ \end{pmatrix}\begin{pmatrix} U_{0,j}\\ U_{1,j}\\ \cdot\\ \cdot\\ \cdot\\ U_{N-1,j}\\ U_{N,j}\end{pmatrix}+K(x_i,t_j)$$

where

$$K=\begin{cases}\lambda_1\Delta t[b(x_{i-1},t_{j+1/2})\alpha(x_{i-1})+f(x_{i-1},t_{j+1/2})]+2\lambda_2\Delta t[b(x_i,t_{j+1/2})\alpha(x_i)+f(x_i,t_{j+1/2})]\\ +\lambda_1\Delta t[b(x_{i+1},t_{j+1/2})\alpha(x_{i+1})+f(x_{i+1},t_{j+1/2})],\ \text{for } j=0,1,2,\dots,m\\ \lambda_1\Delta t[b(x_{i-1},t_{j+1/2})U_{i-1,j+1/2-m}+f(x_{i-1},t_{j+1/2})]+2\lambda_2\Delta t[b(x_i,t_{j+1/2})U_{i,j+1/2-m}\\ +f(x_i,t_{j+1/2})]+\lambda_1\Delta t[b(x_{i+1},t_{j+1/2})U_{i+1,j+1/2-m}+f(x_{i+1},t_{j+1/2})],\ \text{for } j=m+1,\dots,M-1\end{cases}$$

$$\begin{aligned} v_i^- &= \frac{-\Delta t}{2}(\frac{\varepsilon\sigma}{h^2}-\lambda_1 a(x_{i-1},t_{j+1/2}))+\lambda_1, & z_i^- &= \frac{\Delta t}{2}(\frac{\varepsilon\sigma}{h^2}-\lambda_1 a(x_{i-1},t_{j+1/2}))+\lambda_1,\\ v_i^c &= \Delta t(\frac{\varepsilon\sigma}{h^2}+\lambda_2 a(x_i,t_{j+1/2}))+2\lambda_2, & z_i^c &= -\Delta t(\frac{\varepsilon\sigma}{h^2}+\lambda_2 a(x_i,t_{j+1/2}))+2\lambda_2,\\ v_i^+ &= \frac{-\Delta t}{2}(\frac{\varepsilon\sigma}{h^2}-\lambda_1 a(x_{i+1},t_{j+1/2}))+\lambda_1, & z_i^+ &= \frac{\Delta t}{2}(\frac{\varepsilon\sigma}{h^2}-\lambda_1 a(x_{i+1},t_{j+1/2}))+\lambda_1.\end{aligned}$$

3.4 Stability and Uniform Convergence Analysis

The next lemma gives the bounds of the solution of the semi discrete problem (3.18) as well as those of its derivatives.

Lemma 3.4 *Let U_{j+1} (x) be the solution of problem* (3.18) *at the time level* $j+1$. *Then, we have*

$$\left|\frac{d^kU_{j+1}(x)}{dx^k}\right| \leq C\left[1+\varepsilon^{\frac{-k}{2}}\left(\exp\left(-\sqrt{\frac{\beta}{\varepsilon}}\,x\right)+\exp\left(-\sqrt{\frac{\beta}{\varepsilon}}\,(1-x)\right)\right)\right] \tag{3.60}$$

where $0 \leq k \leq 4$, $a \geq \beta > 0$.

Proof See (Clavero & Gracia, 2010).

Lemma 3.5 . *For a fixed number of mesh N, for all positive integers of k and for* $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp -Cx_i/\sqrt{\varepsilon}}{\varepsilon^{k/2}} = 0. \tag{3.61}$$

$$\lim_{\varepsilon\to 0}\max_{1\leq i\leq N-1}\frac{\exp -C(1-x_i)/\sqrt{\varepsilon}}{\varepsilon^{k/2}} = 0. \tag{3.62}$$

Proof See Munyakazi and Patidar,(2010) and (Patidar, 2007)

Lemma 3.6 *Semidiscrete Maximum Principle*

Let $Q_{j+1}(x)$ be a sufficently smooth function on $\bar{\Omega}_x$. If $Q_{j+1}(0) \geq 0$, $Q_{j+1}(1) \geq 0$ and $(1+(\Delta t/2)L^{\Delta t})Q_{j+1}(x) \geq 0$, $\forall x \in \Omega_x$, then $Q_{j+1}(x) \geq 0\ \forall x \in \bar{\Omega}_x$.

Proof. Assume that there exist $x^* \in \bar{\Omega}_x = [0,1]$suchthat$Q_{j+1}(x^*) = min_{x\in\bar{\Omega}}Q_{j+1}(x) < 0$. From the above assumption it is obvious that $x^* \notin \{0,1\}$ implies that $x^* \in (0,1)$ since $Q_{j+1}(x^*)$= $\min_{x\in\bar{\Omega}_x} Q_{j+1}(x)$, using the property in calculus, we have $(d/dx)Q_{j+1}(x^*) = 0$ and $(d^2/dx^2)Q_{j+1}(x^*) \geq 0$ then we obtain $(1+(\Delta t/2)L^{\Delta t})Q_{j+1}(x^*) \leq 0$ which is in contradiction to $(1+(\Delta t/2)L^{\Delta t})Q_{j+1}(x^*) \geq 0\ \forall x \in \bar{\Omega}_x$. Therefore, we conclude that $Q_{j+1} \geq 0\ \forall x \in \bar{\Omega}_x$.

Lemma 3.7 *Discrete Comparison Principle Let there exist a comparision function* $Q_{i,j+1}$ *such that* $(1+(\Delta t/2)L^{\Delta t,h})U_{i,j+1} \leq (1+(\Delta t/2)L^{\Delta t,h})Q_{i,j+1}$, $i = 1,2,\ldots,N-1$ *and if* $U_{0,j+1} \leq Q_{0,j+1}$ *and* $U_{N,j+1} \leq Q_{N,j+1}$,*then* $U_{i,j+1} \leq Q_{i,j+1}$ $i = 0,1,2,\ldots,N$.

Proof. The matrix of $(1+(\Delta t/2)L^{\Delta t,h})U_{i,j+1}$ of size $(N+1)\times(N+1)$ with its entries for $i =$

$1,2,\ldots,N-1$ are

$$D_{i,i-1} = \frac{-\Delta t}{2}\left(\frac{a(0)}{4(\sinh(\sqrt{\frac{a(0)}{\varepsilon}}\frac{h}{2}))^2} - \lambda_1 a(x_{i-1,t_{j+1/2}})\right) + \lambda_1 < 0,$$

$$D_{i,i} = \Delta t\left(\frac{a(0)}{4(\sinh(\sqrt{\frac{a(0)}{\varepsilon}}\frac{h}{2}))^2} + \lambda_2 a(x_{i-1,t_{j+1/2}}) + 2\lambda_2\right) > 0,$$

$$D_{i,i+1} = \frac{-\Delta t}{2}\left(\frac{a(0)}{4(\sinh(\sqrt{\frac{a(0)}{\varepsilon}}\frac{h}{2}))^2} - \lambda_1 a(x_{i+1,t_{j+1/2}})\right) + \lambda_1 < 0. \tag{3.63}$$

As a result, the coefficient matrix satisfies the M matrix property. So there is an inverse matrix that is nonnegative. This proves the discrete solution's existence and uniqueness

Lemma 3.8 *Uniform Stability Estimate The solution $U_{i,j+1}$ of the discrete scheme in (3.60) satisfies the bound*

$$|U_{i,j+1}| \leq \frac{\|(1+(\Delta t/2)L^{\Delta t,h})U_{i,j+1}\|}{1+\Delta t\gamma/2} + k\max\{|\psi_1(t_{j+1})|, |\psi_2(t_{t+1})|\}, \tag{3.64}$$

where $a(x_i, t_{j+1/2}) \geq \gamma > 0$.

proof. Let us consider a barrier function $\mu^{\pm}_{i,j+1}$ as $\mu^{\pm}_{i,j+1} = \frac{\|(1+(\Delta t/2)L^{\Delta t,h})U_{i,j+1}\|}{1+\Delta t\gamma/2} + k\max\{|\psi_1(t_{j+1})|, |\psi_2(t_{t+1})|\} \pm U_{i,j+1}$. Now we can show $\mu^{\pm}_{0,j+1} \geq 0$ and $\mu^{\pm}_{N,j+1} > 0$.

$$\begin{aligned}
&\left(1+\frac{\Delta t}{2}L^{\Delta t,h}\right)\mu^{\pm}_{i,j+1} = \mu^{\pm}_{i,j+1} + \frac{\Delta t}{2}\left(-\varepsilon\sigma(\rho)D^+D^-\mu^{\pm}_{i,j+1} + a(x_i,t_{j+1/2})\mu^{\pm}_{i,j+1}\right) \\
&= \left((1+(\frac{\Delta t}{2})L^{\Delta t,h})a(x_i,t_{j+1})\right)\left(\frac{\|(1+(\frac{\Delta t}{2})L^{\Delta t,h})U_{i,j+1}\|}{1+\frac{\Delta t\gamma}{2}} + k\max\{|\psi_1(t_{j+1})|, |\psi_2(t_{t+1})|\}\right) \\
&\pm U_{i,j+1} \pm \frac{\Delta t}{2}\left(-\varepsilon\sigma(\rho)D^+D^-U_{i,j+1} + a(x_i,t_{j+1/2})U_{i,j+1}\right) = \left((1+(\frac{\Delta t}{2})L^{\Delta t,h})a(x_i,t_{j+1})\right) \\
&\left(\frac{\|(1+(\frac{\Delta t}{2})L^{\Delta t,h})U_{i,j+1}\|}{1+\frac{\Delta t\gamma}{2}} + k\max\{|\psi_1(t_{j+1})|, |\psi_2(t_{t+1})|\}\right) \pm \left(1+\frac{\Delta t}{2}a(x_i,t_{j+1/2})\right)U_{i,j+1} \geq 0.
\end{aligned} \tag{3.65}$$

Using the discrete comparison principle, we obtain $\mu^{\pm}_{i,j+1} \geq 0,\ i = 0,1,2\ldots,N.$

Lemma 3.9 *If $Q_{i,j+1}$ be any mesh function such that $Q_{0,j+1} = Z_{N,j+1} = 0$. Then,*

$$|Q_{i,j+1}| \leq \frac{1}{\gamma}\max_{1\leq k\leq N-1}|L^{\Delta t,h}Q_{k,j+1}|. \tag{3.66}$$

Proof. Consider two barrier functions of the form $\mu^{\pm}_{i,j+1} = S \pm Q_{i,j+1}$, where $S = (\frac{1}{\gamma}) \max_{1\le k\le N-1} |L^{\Delta t,h} Q_{k,j+1}|$. It is easily shown that $\mu^{\pm}_{0,j+1} \geq 0$, $\mu^{\pm}_{N,j+1} \geq 0$. Next, we show that $L^{\Delta t,h}\mu^{\pm}_{i,j+1} \geq 0$.

$$L^{\Delta t,h}\mu^{\pm}_{i,j+1} = L^{\Delta t,h}(S \pm Q_{i,j+1}) = \pm L^{\Delta t,h} Q_{i,j+1} + \frac{b(x,t_{j+1/2})}{\gamma} \max_{1\le k\le N-1} |L^{\Delta t,h} Q_{k,j+1}| \geq 0. \tag{3.67}$$

Hence, using the discrete comparison principles gives

$$|Q_{i,j+1}| \leq \frac{1}{\gamma} \max_{1\le k\le N-1} |L^{\Delta t,h} Q_{k,j+1}|. \tag{3.68}$$

The truncation error of the scheme (3.63) is given by:

$$\left(U^{j+1}(x_i) + \frac{\Delta t}{2} a(x,t_{j+1}) U^{j+1}(x) - \frac{\varepsilon \Delta t}{2} U^{j+1}_{xx}(x)\right) - (\frac{\varepsilon \Delta t \sigma}{2h^2}(U^{j+1}_{i+1} - 2U^{j+1}_i + U^{j+1}_{i-1}) + U^{j+1}(x_i) + \frac{\Delta t}{2} a(x_i,t_{j+1}) + \alpha U^{j+1}_{i-1} + \alpha U^{j+1}_{i+1} + 2\lambda_2 + \Delta t \lambda_2 a(x_i,t_{j+1})) U^{j+1}_i)$$

where $\alpha = \lambda_1 + \frac{\Delta t}{2}\lambda_1 a(x_i,t_{j+1})$ and $2\lambda_2 = 1 - 2\lambda_1$.
Taylor series expansion of the terms U^{j+1}_{i+1} and U^{j+1}_{i-1}

$$\begin{aligned} U^{j+1}_{i+1} &= U^{j+1}(x_i) + h(U^{j+1}_x(x_i)) + \frac{h^2}{2}(U^{j+1}_{xx}(x_i)) + \frac{h^3}{6}(U^{j+1}_{xxx}(x_i)) \\ &\quad + \frac{h^4}{24}(U^{j+1}xxxx(x_i)) + \frac{h^5}{120}(U^{j+1}_{xxxxx}(x_i)) + \dots \\ U^{j+1}_{i-1} &= U^{j+1}(x_i) - h(U^{j+1}_x(x_i)) + \frac{h^2}{2}(U^{j+1}_{xx}(x_i)) - \frac{h^3}{6}(U^{j+1}_{xxx}(x_i)) \\ &\quad + \frac{h^4}{24}(U^{j+1}_{xxxx}(x_i)) - \frac{h^5}{120}(U^{j+1}_{xxxxx}(x_i)) + \dots \end{aligned} \tag{3.69}$$

Using the truncated Taylor series expansions of the terms U^{j+1}_{i+1} and U^{j+1}_{i-1} yields.

$$\begin{aligned} \left(L^{\Delta t} U_{j+1}(x) - L^{\Delta t,h} U_{i,j+1}\right) &= (U^{j+1}(x_i) + \frac{\Delta t}{2} a(x_i,t_{j+1}) - \frac{\varepsilon \Delta t}{2}(U^{j+1}_{xx}(x_i)) \\ &\quad -(-\frac{\varepsilon \Delta t \sigma}{2h^2}\left(h^2(U^{j+1}_{xx}(x)) + \frac{h^2}{12}(U^{j+1}_{xxxx}(x))\right) + U^{j+1}(x_i) \\ &\quad + \frac{\Delta t}{2} a(x_i,t_{j+1}) + \alpha h^2 U^{j+1}_{xx}(x_i) + \alpha \frac{h^4}{12} U^{j+1}_{xxxx}(x_i)). \end{aligned} \tag{3.70}$$

Next we use a truncated Taylor series expansion of a fitting factor (σ) of order five.

$$\sigma = 1 - \frac{\zeta_i^2 h^2}{12} + \frac{\zeta_i^4 h^4}{240}. \tag{3.71}$$

Now, substituting (3.71) into (3.70), we obtain

$$\left(L^{\Delta t}U_{j+1}(x)-L^{\Delta t,h}U_{i,j+1}\right)=\left(\frac{\varepsilon}{12}(((u_{xx}(t))_i\zeta_i^2+(u_{xxxx}(t))_i)\right)h^2+$$
$$\varepsilon\left(\frac{\zeta_i^4}{240}(u_{xx}(t))_i-\frac{\zeta_i^4}{144}u_{xx}(t))_i\right)h^4+\left(\frac{\varepsilon\zeta_i^4}{2880}(u_{xxxx}(t))_i\right)h^6+\alpha h^2U_{xx}^{j+1}(x)+\frac{\alpha h^4}{12}U_{xxxx}^{j+1}(x). \tag{3.72}$$

Simplifying (3.72) we obtain

$$\left(L^{\Delta t}U_{j+1}(x)-L^{\Delta t,h}U_{i,j+1}\right)=\left(\frac{\varepsilon}{12}(((U_{xx}^{j+1}(x))\zeta_i^2+(U_{xxxx}^{j+1}(x)))+\alpha U_{xx}^{j+1}(x)\right)h^2+$$
$$\left(\varepsilon(\frac{\zeta_i^4}{240}(U_{xx}^{j+1}(x))_i-\frac{\zeta_i^4}{144}(U_{xx}^{j+1}(x)))+\frac{\alpha}{12}U_{xxxx}^{j+1}(x)\right)h^4+\left(\frac{\varepsilon\zeta_i^4}{2880}(u_{xxxx}(t))_i\right)h^6 \tag{3.73}$$

Using lemma (3.4), (3.5) and the relation $N^{-2}>N^{-4}>N^{-6}\dots$ The discrete problem satisfies the following bound:

$$\|U_{j+1}(x)-U_{i,j+1}\|\leq CN^{-2} \tag{3.74}$$

Theorem 3.5 *Let u and U be the solution of (3.1)-(3.3) and (3.63) respectively. Finally the uniform error bound holds:*

$$\|u-U\|\leq C(h^2+(\Delta t)^2) \tag{3.75}$$

Proof. *The combination of temporal and spatial error bounds gives the required result*

CHAPTER FOUR

NUMERICAL RESULTS AND DISCUSSION

Example 1 *Consider the following example of SPPRD problem with retarded argument. (Rao and Chakravarthy,2014b)*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{(1+x^2)}{2} u(x,t) + u(x,t-\tau) = t^3, & \\ u_\varepsilon(x,t) = 0, & (x,t) \in [0,1] \times [-1,0] \\ u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, & t \in [0,2]. \end{cases} \tag{4.1}$$

Example 2 *Consider the following example of SPPRD problem with retarded argument. (Khari and Kumar,2022)*

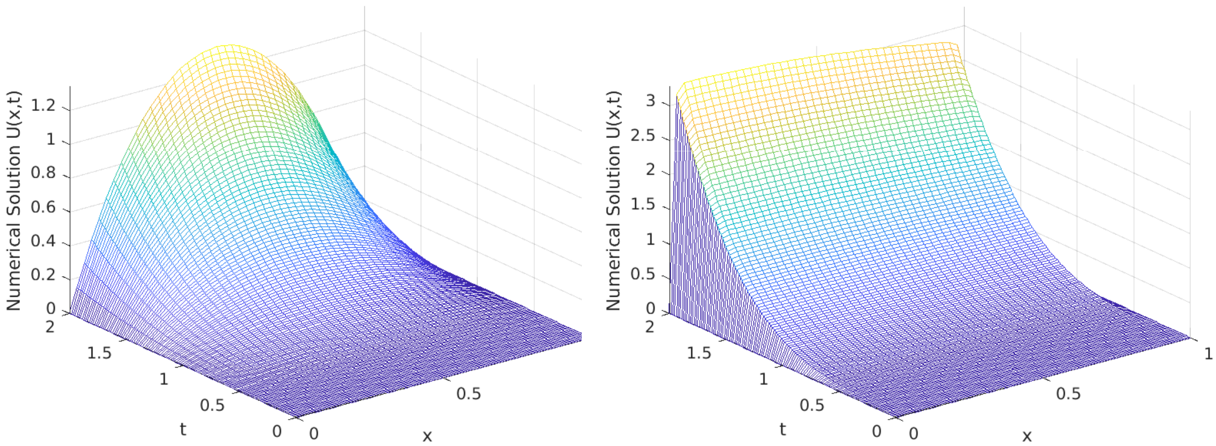
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + x^2 u(x,t) = t^3 - u(x,t-\tau), & \\ u_\varepsilon(x,t) = 0, & (x,t) \in [0,1] \times [-1,0] \\ u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, & t \in [0,2]. \end{cases} \tag{4.2}$$

Example 3 *Consider the following example of SPPRD problem with retarded argument. (Singh, Kumar, and Kumar,2018)*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (1.1+x^2) u(x,t) = t^3 - u(x,t-\tau), & \\ u_\varepsilon(x,t) = 0, & (x,t) \in [0,1] \times [-1,0] \\ u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, & t \in [0,2]. \end{cases} \tag{4.3}$$

Table 4.1: Example 1, Maximum Absolute Error of the Scheme and Rate of Convergence of the Scheme

$\varepsilon \downarrow$	$M=64$	128	256	512	1024
	$N=20$	40	80	160	320
2^{0}	1.0209e-04	7.0347e-05	4.0148e-05	2.1341e-05	1.1004e-05
	0.53728	0.80916	0.91170	0.95560	1.1153
2^{-1}	2.3379e-04	1.6992e-04	9.8649e-05	5.2802e-05	2.7253e-05
	0.4604	0.7849	0.9017	0.95418	0.9493
2^{-2}	3.8218e-04	3.30384e-04	1.8106e-04	9.7936e-05	5.0845e-05
	0.33094	0.7468	0.8866	0.9457	0.9709
2^{-4}	6.0202e-04	5.0106e-04	3.0467e-04	1.6607e-04	8.6501e-05
	0.2648	0.7177	0.87545	0.94100	0.9712
2^{-6}	4.2439e-03	8.8457e-04	3.5017e-04	1.8423e-04	9.4623e-05
	2.2623	1.3369	0.92655	0.96125	0.9789
2^{-8}	2.2925e-02	4.6993e-03	1.0750e-03	2.2823e-04	9.6683e-05
	2.2864	2.1281	2.2358	1.2392	0.98228
2^{-10}	8.9288e-02	2.4218e-02	4.9511e-03	1.1946e-03	2.7467e-04
	1.8824	2.2903	2.0512	2.1208	2.2335
$E^{N,M}$	8.9288e-02	2.4218e-02	4.9511e-03	1.1946e-03	2.7467e-04
$r^{N,M}$	1.8824	2.2903	2.0512	2.1208	2.2335

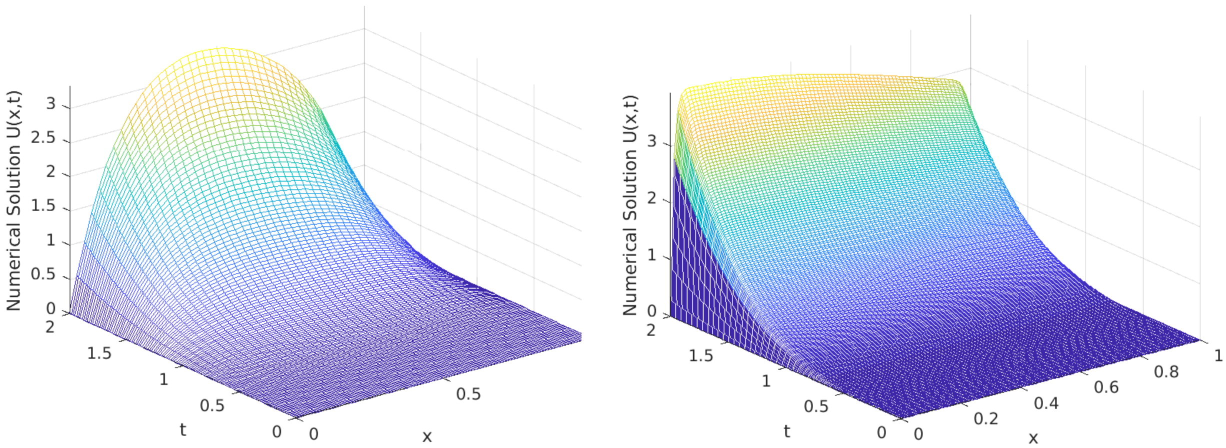


$\varepsilon = 2^{-1}$ $\qquad$ $\varepsilon = 2^{-10}$

Figure 4.1: Solution of Example 1, with Boundary Layer Formation

Table 4.2: Example 2, Maximum Absolute Error of the Scheme and Rate of Convergence of the Scheme

$\varepsilon \downarrow$	$M = 64$ $N = 20$	128 40	256 80	512 160	1024 320
2^{0}	1.4410e-04	8.3638e-05	4.4794e-05	2.3156e-05	1.1847e-05
	0.78484	0.90085	0.9519	0.95560	1.0225
2^{-1}	3.5638e-04	2.1086e-04	1.1388e-04	5.9094e-05	3.0094e-05
	0.7571	0.8889	0.9017	0.9464	1.0038
2^{-2}	6.4213e-04	3.9344e-04	2.1573e-04	1.1270e-04	5.7568e-05
	0.70672	0.86692	0.93674	0.96915	0.98450
2^{-4}	1.0071e-03	6.6229e-04	3.7209e-04	1.9656e-04	1.0095e-04
	0.6047	0.83181	0.92068	0.9613	0.9808
2^{-6}	3.9903e-03	8.1168e-04	4.1671e-04	2.2002e-04	1.1291e-04
	2.2975	0.96187	0.9214	0.96246	0.98154
2^{-8}	2.2316e-02	4.56683e-03	1.0299e-03	2.2688e-04	1.1671e-04
	2.288	2.1487	2.1825	0.95900	0.9803
2^{-10}	8.8696e-02	2.3903e-02	4.8842e-03	1.1725e-03	2.6934e-04
	1.8917	2.2910	1.4269	2.1221	2.1699
2^{-12}	2.1312e-01	9.2178e-02	2.4753e-02	5.0491e-03	1.2449e-03
	1.2092	1.8968	2.2935	2.0200	2.0749
$E^{N,M}$	2.1312e-01	9.2178e-02	2.4753e-02	5.0491e-03	1.2449e-03
$r^{N,M}$	1.2092	1.8968	2.2935	2.0200	2.0749

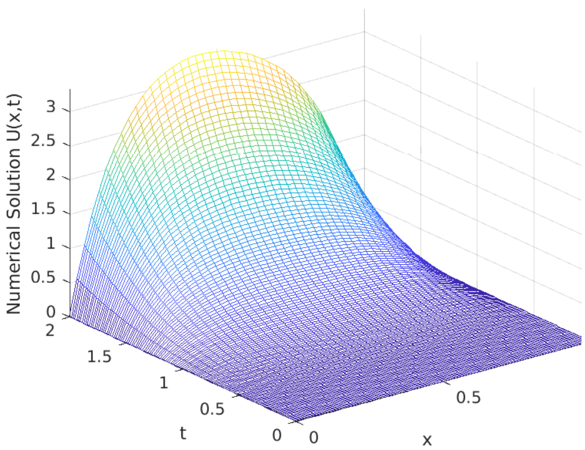


$\varepsilon = 2^{-6}$ $\qquad$ $\varepsilon = 2^{-12}$

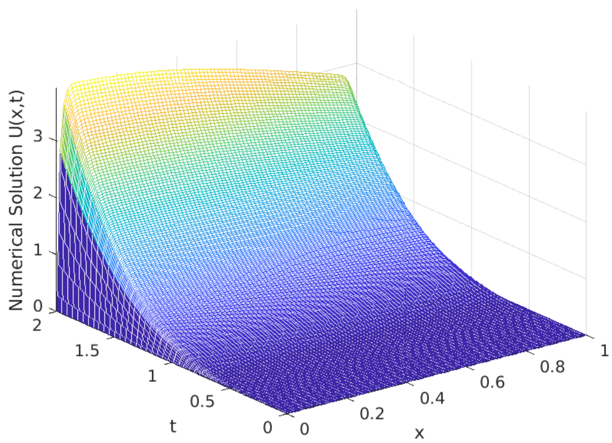
Figure 4.2: Solution of Example 2, with Boundary Layer Formation

Table 4.3: Example 3, Maximum Absolute Error of the Scheme and Rate of Convergence of the Scheme

$\varepsilon \downarrow$	$M = 64$ $N = 20$	128 40	256 80	512 160	1024 320
2^0	2.9930e-05	4.7101e-05	3.1928e-05	1.8101e-05	9.5076e-06
	-0.6.5417	0.56094	0.81875	0.92892	1.1112
2^{-1}	4.8144e-05	1.0535e-04	7.3860e-05	4.2333e-05	2.2518e-05
	-1.1298	0.5123	0.8030	0.91070	9.7498
2^{-2}	1.6908e-04	1.7639e-04	1.2913e-04	7.5015e-05	4.0153e-05
	-0.061063	0.4499	0.783574	0.90167	0.95351
2^{-4}	1.3694e-03	3.2000e-04	2.1906e-04	1.2466e-04	6.6178e-05
	2.0974	0.54675	0.81333	0.91358	0.95823
2^{-6}	7.2293e-03	1.6486e-03	3.5961e-04	1.4320e-04	7.3755e-05
	2.1326	2.1967	1.3284	0.95722	0.97366
2^{-8}	3.3115e-02	7.7596e-03	1.8118e-03	4.2553e-04	9.3382e-05
	2.0934	2.0986	2.090	2.1880	1.2673
2^{-10}	1.0114e-01	3.4597e-02	8.0418e-03	1.8959e-03	4.6360e-04
	1.5476	2.1051	2.0846	2.0319	2.0927e
2^{-12}	2.0856e-01	1.0410e-01	3.5372e-02	8.1868e-03	1.9433e-03
	1.0025	1.5573	2.1112	2.0748	2.0097
$E^{N,M}$	2.0856e-01	1.0410e-01	3.5372e-02	8.1868e-03	1.9433e-03
$r^{N,M}$	1.0025	1.5573	2.1112	2.0748	2.0097



$\varepsilon = 2^{-4}$



$\varepsilon = 2^{-12}$

Figure 4.3: Solution of Example 3, with Boundary Layer Formation

Table 4.4: Example (1) ε-Uniform Error and ε-Uniform Rate of Convergence of the Proposed Scheme and Results in (Rao and Chakravarthy, 2014b).

Scheme↓	$M = 64$	128	256	512	1024
	$N = 20$	40	80	160	320
	Proposed method				
$E^{N,M}$	8.9288e-02	2.4218e-02	4.9511e-03	1.1946e-03	2.7467e-04
$r^{N,M}$	1.8824	2.2903	2.0512	2.1208	2.2335
	Rao and Chakravarthy,2014b				
$E^{N,M}$	5.3429e-02	2.7108e-02	1.3653e-02	6.8512e-03	3.4318e-03
$r^{N,M}$	0.9789	0.9895	0.9947	0.9974	0.9987

DISCUSSION

Test examples 2 and 3, the exact solution is not given. The double mesh approach is applied to analyze the numerical solution $U^{N,M}$ to the numerical solution $U^{2N,2M}$, which is computed on a mesh that is two times finer in space and two times finer in time, in order to assess the accuracy and parameter uniform convergence of the proposed numerical scheme. Tables 4.2 and 4.3 respectively show the maximum point-wise error $E_{\varepsilon}^{N,M}$ and the ROC O(N) for examples 2 and 3. In each column of tables of 4.1, 4.2 and 4.3 (or for each N and M) as $\varepsilon \to 0$ the maximum absolute error becomes uniform this shows that the convergence of the scheme is independent of perturbation parameter. One observes in these tables that the scheme converges with order of convergence two. The result given in table 4.4 shows the proposed scheme is more accurate than the scheme in (Rao and Chakravarthy,2014b). Based on the results given in tables (4.1-4.3), the proposed scheme is accurate, stable and ε-uniformly convergent with rate of convergence two.

CONCLUSION and RECOMMENDATION

CONCLUSION

In this thesis exponentially fitted tension spline method is developed for solving singularly perturbed time delay reaction-diffusion problems. The solution of considered problem exhibits two exponential boundary layers of thickness O(ε) on the left and right spatial domains. We employed the crank-Nicolson approach in temporal discretization and an exponentially fitted tension spline in spatial discretization to set up the numerical scheme. The analytical solution is explored in terms of its bounds and attributes. An exponential fitting parameter is established based on the derivation from the asymptotic solution to stabilize the influence of the singular perturbation on the discrete solution. The proposed scheme's stability and convergence analysis are examined. The performance of this scheme is illustrated through a numerical examples. The approach is examined for stability and convergence, and it is shown to that second order in space and second order in time.

RECOMMENDATION

In this thesis, we have developed more accurate, higher order and uniformly convergent numerical scheme using exponentially fitted operator tension spline method on uniform mesh for solving singularly perturbed time delay reaction diffusion problems. But the fitted mesh technique is not addressed in this thesis. This will be addressed in the future.

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