ADAMA SCINCE AND TECHNOLOGY UNIVERSTY

SCHOOL OF GRADUATE STUDY

APPLIED MATHEMATICS PROGRAM

NUMERICAL SOLUTION FOR SOLVING BURGER'S FISHER EQUATIONBY USING ADOMAIDECOMPOSITION METHOD

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER'S IN NATURAL SCIENCE (NUMERICAL ANALYSIS)

By

TEGENE LEGESSE

ADAMA, ETHIOPIA

SEPTEMBER, 2017

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LIST OF ABBREVIATIONS

ADM = Adomain decomposition method

VIM = Variational Iteration method

ODE = Ordinary Differential Equation

PDE = Partial Differential Equation

IC = Initial condition

BC = Boundary condition

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Abstract

This thesis work focuses on the Adomain decomposition method for solving the Burger's Fisher equation. In this thesis work, different one-dimensional linear and non-linear illustrative Burger's equations are solved using the Adomain decomposition method (ADM). All calculations have been done to find their approximate solutions in the form of series with easily computable Components and based on the results obtained the convergence Properties and accuracy of the Adomain decomposition method is discussed and compared with the analytical (exact) solution.

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CHAPTER ONE

1. Introduction

1.1 Back ground of the study

In many branches of science and engineering we come across equations, which contain besides the dependent and independent variables, different derivatives of the dependent variable with respect to the independent variable or variables. These equations are called differential equations.

For instance $\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} - 2y^2 = 3x$ and $\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = K\frac{\partial^2 U}{\partial x^2}$ etc.

We can classify differential equations into two parts ordinary differential equations and partial differential equations.

An ordinary differential equation (ODE) is an equations which involves only one independent variable and derivatives of the dependent variable with respect to the independent variable.

Eg. $\frac{dy}{dx} + y = \sin x$ and $(\frac{dy}{dx})^2 + xy = 1+x$

A partial differential equation is an equation which involves more than one independent variables (say x, y, z,) and dependent function u(x, y, z,), ie, a function of the independent variables and the partial derivatives of the dependent function u with respect to the independent variables. It is an equation in which there are two or more independent variables and the derivatives that occurred in it.

Eg. $U(x, y) = 6xy - 8x^2y$ and $U_{xx}+U_{yy} = 0$

The order of differential equation is the order (degree) of the highest derivative appearing in the equation. A differential equation is linear if those terms involving the unknown function contains only product of a single factor of the unknown function or one of its derivatives with other (known) constants or functions of the independent variables.

In linear differential equations, the unknown function and its derivatives do not appear raised to a power other than 1.

Eg. $y\frac{dy}{dx}$ + xy = 1+ x and $\frac{d^2U}{dt^2} + u^2$ = 0 are non- linear differential equations.

The second order linear partial differential equations with two independent variables (x and y) and one dependent variable U of the form:

$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU = G$

where A, B, C, D, E, F, G are constants is:

- Parabolic if $B^2 - 4AC = 0$ eg. The Heat equation ($U_t - c^2U_{xx} = 0$)
- Elliptic if $B^2 - 4AC < 0$ eg. The Laplace equation ($U_{xx} + U_{yy} = 0$)
- Hyperbolic if $B^2 - 4AC >$ eg. The wave equation ($U_{tt} - c^2U_{xx} = 0$)

Partial differential equation describe many types of physical phenomena in engineering and science ranging from transient heat equation through vibration of strings and plates to fluid flows and the behavior of electric and magnetic fields [1].

The Burger's equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of dynamics, traffic flow, heat conduction and non- linear acoustic waves [2]. This equation is also encountered in chemical kinetics and population dynamics which includes problems, such as non-linear evolution of a population in nuclear reaction and branching. Moreover, the same equation occurs in logistic propagation growth model, flame propagation, Neurophysiology, autocatalytic chemical reaction and branching Brownian motion process. The Burger's equation is a mixed hyperbolic parabolic partial differential equation (PDE). It is named for Johannes Martinus Burger (1895-1981). This equation is a mixed hyperbolic-parabolic partial differential equation (PDE) the convection term in non-linear and the source terms are 1st and 3rd order.

Hyperbolic equations can be used to model a wide variety of phenomena that involve wave motion or the advective transport of substances (advective transport refers to a substance being carried along with fluid motion).

These problems are generally time dependent so that their solutions are dependent on time as well as one or more spatial (space) variables.
Shock waves occur in explosions, traffic flow, glacier waves, airplanes breaking the sound barrier and so on are modeled by non-linear hyperbolic PDEs.

For instance $U_t + \alpha UU_x = 0$ and $U_t + UU_x = 0$

The solutions of differential equations are functions as opposed to standard algebraic equations whose solutions are numbers. For most partial differential equations we are not able to find their exact solutions and sometimes we do not even know whether a unique solution exists. For these reasons, in most cases the only alternative way to solve partial differential equation is to approximate their solutions numerically. Numerical methods are methods used to solve partial differential equations that constitute an invisible part of modern engineering and science due to the wide range applicability of the Burger's equation, finding its numerical solution is very crucial. A lot of works have been done in order to find the numerical solution of this equation using different numerical methods (techniques). For example an analytical study of the Fisher equation by ADM and VIM for solving the generalized Fisher equation, numerical solution for solving Burge's equation, a novel approach for solving Fisher equation using Exponential function method.
In recent years some works have been done in order to find the numerical solution of the Burger's equation by using different iterative methods.
In this research work we use the iterative numerical methods such as ADM to find the numerical solution of the Burger's equation. So the approximate solution of this equation (unknown function) is calculated in the form of series which its components are computed by applying recursive relation. The existence and uniqueness of the solution and the convergence of the proposed iterative method is proved. An illustrative example is studied to demonstrate the accuracy of the presented iterative method.

Hence, in this research work, the researcher tried to solve the numerical solution of an illustrative Burger’s Fisher equation by using Adomian decomposition method (ADM) and has the easiness and accuracy of this method in solving non-linear partial differential equations.

1.2 Statement of the problem

This research work focuses on the study of Adomain decomposition Method for solving Burger's Fisher equation. The researcher's ambition is to investigate the convergence and accuracy of this method for solving linear and non-linear Burger's equation by addressing the following basic questions.

i) Which iterative Method is used to solve the Burger's equation?

ii) Which type of PDE is solved by ADM?

iii) Is ADM an accurate iterative method to solve Burger's equation?

iv) Does the numerical solution obtained by ADM converges to the exact solution?

1.3 Objective of the study

1.3.1 General objective

The general objective of this research work is to investigate the easiness and accuracy of the Adomain decomposition for solving linear and non-linear Burger's Fisher equation.

1.3.2 Specific objectives

The specific objectives of this research work are to:-

- Describe the iterative method used to solve the Burger's equation and its procedure.
- Find out the numerical solution of a practical Burger's equation using the proposed iterative method.
- Compare the given iterative method in solving linear and non-linear Burger's equation with the exact solution by analyzing their errors.
- discuss the convergence, easiness and accuracy of the proposed iterative method in solving the Burger's equation.

1.4 Significance of the study

The significance/purpose/ of this research work is :

- ❖ To know the best iterative method used to approximate the Burger's equation that converges to the exact solution
- ❖ The findings of this research work may encourage others to do more study in this area
- ❖ To understand the easiness and accuracy of the iterative method used in solving non-linear Burger's equation.
- ❖ At the end of this study the numerical solution of the Burger's equation obtained is used as a resource (reference) for other researches for further study.

1.5 Scope of the study

This researcher work is restricted to solve the linear and non – linear Burger's equation using one of the iterative methods, ie the Adomain decomposition method.

1.6 The limitations of the study

The limitations of the study are:

- ❖ Time constraint
- ❖ Shortage of sufficient budget
- ❖ Shortage of materials
- ❖ Luck of internet access in the work place
- ❖ Shortage of network to communicate with my advisor
- ❖ Shortage of review literature

CHAPTER TWO

Related Mathematical preliminaries

2.1 Description of Burger's equation

Differential equation is an equation which contains the dependent variable, independent variables and different derivatives of the dependent variable with respect to the independent variable or variables.

Differential equation has two parts, ordinary differential equation and partial differential equation.

An ordinary differential equation (ODE) is an equations which involves only one independent variable and derivatives of a function with respective of this independent variable.

A partial differential equation is any equation involving more than one independent variable and at least one partial derivative of that function. ie, a function of the independent variables and the partial derivatives of the dependent function u with respect to the independent variables. It is an equation in which there are two or more independent variables and the derivatives that occurring in it.

The solution a PDE is some region R of the space of the independent variable is a function that has all the partial derivatives appearing in the PDE in some domain D, containing R and satisfy the PDE everywhere in R.

The Burger's Fisher equation is a fundamental partial differential equation from fluid mechanics. This equation is used to model a wide range of phenomena in physics, engineering, chemistry and biology.It occurs in various areas of applied mathematics, such as modeling of dynamics, traffic flow, heat conduction and non- linear acoustic waves [2].This equation is a mixed hyperbolic parabolic partial differential equation (PDE).

For a given field u(x, t), the mathematical model of the generalized Burger's Fisher equation is modeled by:

$$\frac{\partial U}{\partial t} + \alpha u^{\delta}\frac{\partial U}{\partial x} - \frac{\partial^2 U}{\partial x^2} = \beta u(1-u^{\delta}), 0 \le x \le L,\ 0 \le t \le T \qquad (1)$$

With initial conditions:

$U(x, 0) = U_O(x)$, $x \in \{-L, L\}$ and boundary condition:

$U(-L, t) = u(L, t) = 0$, $t \geq 0$ Where α, δ and β are constants (parameters).

If α = o and δ = β= 1 this equation can be reduce to the Huxley equation

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$ which describe nerve pulse propagation in nerve and wall motion in liquid crystals [3]. This Burger's equation is used as a model for the propagation of a mutant gene, with u denoting the density of an advantageous. This equation is encountered in chemical kinetics [4] and population dynamics which includes problems such as non-linear evolution of a population in a nuclear reaction and branching. Moreover, the same equation occurs in logistic population growth model [5], flame propagation, neurophysiology, autocatalytic chemical reaction, and branching Brownian motion process.

In this equation If δ = 1, β = 0 (absent) and α is any arbitrary constant, then the Burger's Fisher equation runs to the Burger's equation: $\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = d\frac{\partial^2 U}{\partial x^2}$ which is used to model:

- advection diffusion equation (d is diffusion coefficient or viscosity).
- The speed of Sound wave traveling in viscous medium
- One dimensional turbulence motion
- Waves in fluid filled electric tubes and magneto-hydro dynamic waves in a medium.

If α = β = 0 (absent), the Burgers' fisher equation reduced to the heat equation:

$$\frac{\partial U}{\partial t} = v\frac{\partial^2 U}{\partial x^2} .$$

If we integrate it one time wrt time t using the initial condition and we obtain These are typical examples of Burgers' equations. So now days the Burger's equation has attracted much attention Therefore, finding the numerical solution of this equation is very important. In order to obtain an approximate solution of equation (1),

$$\therefore \int_o^t \left(\frac{\partial U}{\partial t}\right) dt = \beta \int_0^t (\mathrm{u}(1-u^\delta))dt - \alpha \int_0^t (u^\delta \frac{\partial u}{\partial x})dt + \int_0^t (\frac{\partial^2 U}{\partial x^2})\mathrm{dt}$$

$$\mathrm{U}(x, t) = f(x)+\beta \int_0^t F(u(x,t))dt - \alpha \int_0^t F_1(u(x,t))dt + \int_0^t D^2(u(x,t))dt$$

Where $F(u(x, t)) = u(x, t) - u^\delta(x, t)$

$F_1(u(x, t)) = u^\delta(x, t)\frac{\partial u(x,t)}{\partial x}$

$D^2(u(x, t)) = \frac{\partial^2 U(x,t)}{2x^2}$

In equ.(2), we assume that f(x) is bounded for all x in j= [0, L].

The terms $F(u(x, t))$, $F_1(u(x, t))$ and $D^2(u(x, t))$are Lipchitz conditions with:

$|F(u) - F(u')| \leq L_1|u - u'|$

$|F1(u) - F_1(u')| \leq L_2|u - u'|$

$|D^2(u) - D^2(u')| \leq L_3|u - u'|$

α_1= T(|β| L_1+|α|L_2+L_3

β_1= 1-T (1-x_1) and

β_2= 1-TLx_1

2.2 Definition of operators [14]

2.2.1 An operator is a transformation that transforms a function in to another function.

2.2.2 An operator that obeys the distributive law L(f+g) = Lf + Lg, where f and are functions.

2.2.3 An operator is said to be linear if for every pair of functions f and g and Scalar t, L(f+g) = Lf +Lg and L(tf) = Ltf.

2.2.4 A differential operator is a transformation which transforms a differentiable function in to its derivatives.

Eg Dy = y' = $\frac{dy}{dx}$

2.2.3 The inverse operator L^{-1} is an integral operator that transforms differential function back to its anti- derivative.

ie. L_x^{-1} (.) = $\int_0^x (.)dx$, Where (.) is the given function.

2.3 The Adomian decomposition method

For some differential equation it is difficult to find their exact solution analytically. So numerical methods are computational alternative methods used to solve such problems.

The Adomain decomposition method is a semi analytical method introduced by the American mathematician G. Adomian(1923-1996).It was based on the search for a solution(un known function) in the form of a series and on decomposing the non-linear operator into a series in which the terms are calculated recursively using the Adomian polynomials.

The Adomain decomposition method (ADM) is a method for solving linear and non-linear functional equations (algebraic, differential, partial, partial differential, integral, integro-differential equations). The method was developed from the 1970s to the 1990s by George Adomian, chair of the Center for Applied Mathematics at the University of Georgia [6] .This method consist of splitting the given equation into linear and non-linear Parts, inverting the higher-order derivative operator contained in the linear operator on solution, decomposing the nonlinear function in terms of special Polynomials called

Adomian polynomials [7], and finding the successive terms of series solution by recurrent relation using Adomian polynomials. The Adomain polynomials mathematically generalize to a maclaurin series about an arbitrary external parameter: which gives the solution method more flexibility than direct Taylor series expansion [8].The ADM has been successfully applied to solve a large class of linear and nonlinear problems such that algebraic equation differential equation [9],system of differential equation[10],partial differential equation PDE [11] and system of partial differential equation [12]".By the Adomain decomposition method, the original non-linear equation is directly soluble (form a solution) and does not require linearization, perturbation and discretization of the variables. Therefore, it is not affected by errors associated to discretization.

2.4 .Existence and convergence of iterative methods

Adomain Decomposition Method (ADM): is an iterative Method used to solve linear and non- linear partial differential equations

Theorem 2.4 1 Let 0<x<1, then the Burgers fisher equation (1) has a unique solution.

Proof: Let U and U' be two different solutions of equ. (5). Then

$|u - u'| = |\beta| \int_0^t F(u(x,t))dt - \alpha \int_0^t F1\,(u(x,t))dt + \int_0^t D^2(u(x,t))dt$

$\leq \quad |\beta| \int_0^t F(u(x,t)) - F(u'(x,t)dt + |\alpha| \int_0^t F1\big(u(x,t)\big) \; -$

$F1(u'(x,t))dt \int_0^t D^2(u(x,t)) - D^2(u(x,t)) - D^2(u(x,t))dt$

$\leq T(\ |\beta|L_1+|x|\ L_2+L_3)|u - u'|$

$= \alpha_1|u - u'|$

From which we get $(1-\alpha_1)\ |u - u'| \ \leq 0$,

Since $0<\alpha_1<1$, then $|u - u'| = 0 = u = u'$

Hence $|en + 1|$= max $\forall t \in \hat{j}$ $|en| \leq \beta|en|$

2.5 Convergence property

Whenever we use a numerical method to solve a differential equation, we should be concerned about the accuracy and convergence properties of the method. In practice we must apply the method on and we wish to ensure that the numerical solution obtained is a sufficiently good approximation to the true solution. For real problems we generally do not have the true solution to compare against, and we must rely on some combination of the following techniques to gain confidence in our numerical results. Validation on test problems, the method should be tested on simpler problems for which the true solution is known, or on problems for which a highly accurate comparison solution can be computed by other means. A method for solving differential equation is said to be convergent if the approximate solutions Un approaches the exact solution for all values in the domain.

CHAPTER THREE

Research Design and Methodology

3.1 Design of the Study

In this study, we select an appropriate Burger's equation to be solved by ADM and then find the numerical solution of this equation using the proposed method. Therefore, the one dimensional non-linear and linear Burger's equations: $U_t + UU_x - dU_{xx} = 0$ and $U_t - U_{xx} = 0$ are taken as test examples and solved using the proposed method and the solutions are examined using appropriate values for the unknowns. Based on the results obtained the convergence of this method has been compared to the exact solution.

3.2 Procedure of the study

The researcher of this study has used the following procedures.

- Investigate iterative methods used to solve the Burger's equation.
- State an illustrative Burger's equation and solve it using the proposed iterative method.
- Examine the solutions of ADM and exact method using different values for the unknowns.
- Analyze e the convergence of this method by comparing the results obtained.
- Write MATLAB code for some scheme and sketch their graphs.
- Analyze the errors.

3.3 Expected outcome

The expected outcome of this research work is to assess the easiness and accuracy of the ADM for approximating the solution of Burger's Fisher equation.

Chapter four

Results and discussion

4.1 Adomain decomposition Method (ADM) for the Numerical solution of Burgers Fisher equation.

. The Burger's Fisher equation is a mixed hyperbolic-parabolic partial differential equation that can be solved by iterative method. In this chapter, we will solve linear and non- linear Burger's equation by one of the iterative methods, the Adomain decomposition method and make comparison to discuss the convergence of this method. The Adomian decomposition method is applied to the following general non-linear equation: Lu+Nu+Ru = g(x) , where:

u is the unknown function,

L is the highest order derivative linear operator which is assumed to be easily invertible

R is a linear differential operator of order less than L

N represents the non-linear operator and g is the source term.

To solve differential equations using the Adomain decomposition method (ADM) we should first define the linear and invers operator as follows.

$L_t(.) = \frac{\partial(\,.\,)}{\partial t}$

$L_t^{-1}(.) = \int_0^t (.)dt$

The next step will be writing the given differential equation in standard (operator) form: LU+ Nu + Ru = g(x) (1)

Where N is non – linear differential operator

L and R are linear differential operators

and $L(U) = U_t = \frac{\partial U}{\partial t}$

$N(u) = UU_x = U\frac{\partial U}{\partial x}$

$R(u) = U_{xx} = \frac{\partial^2 U}{\partial X^2}$

Now multiplying both sides of the eq.1 by inverse operator L_t^{-1} we have:

$L_t^{-1}LU + L_t^{-1}NU + L_t^{-1}RU = 0$ (2)

But using the initial condition the result of ：

L_t^{-1}LU = $\int_0^t U_t \; dt$

= $\int_0^t \frac{\partial U}{\partial t}$ dt

= U(x, t) $)|_{0}^{t}$

L_t^{-1}LU = U(x, t) – U (x, 0)

Now substituting this value in equ,2 , we have the following.

U(x, t) – U (x, 0) +L_t^{-1}NU+L_t^{-1}RU = 0

U(x, t) = U (x, 0) _ L_t^{-1}NU – L_t^{-1}RU

But U(x, t) = $\sum_{n=0}^{\infty} Un\,(x, t)$ and N(U) = N($\sum_{n=0}^{\infty} Un\,(x, t)$) = $\sum_{n=0}^{\infty} An$,

Where An are the Adomain Polynomials.

∴ $\sum_{n=0}^{\infty} U_n(x, t)$ = U (x, 0) _ L_t^{-1}NU – L_t^{-1}RU

= U (x, 0) _ $\sum_{n=0}^{\infty} An$ – L_t^{-1}RU (3)

This will give us the solutions of given differential equation.

According to the Adomain decomposition method, the non-linear operator Nu is decomposed as:

N (U) = UUx = ($\sum_{n=0}^{\infty} U_n\,(x, t\,)$) = $\sum_{n=0}^{\infty} An$.

Where An, n≥0 are the Adomian polynomial determined formally as follows

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=o}^{\infty} \lambda^i u_i\right)\right]\right]_{\lambda=0}$$

Then, after finding the Adomain polynomials (necessary conditions for the unknown functions), we come back toequ.3 to find Un(x, t):Where the components u_0, u_1… are usually determined recursively by:

$$\begin{cases} U_0(x, t) \quad = U(x, 0) = f(x) \\ U_{n+1}(x, t) = -\, L_t^{-1}\mathrm{NU} - L_t^{-1}\mathrm{RU} \end{cases}$$

After finding the unknown functions Uo, U1, U2,., the solution of the differential equation will be U(x, t) = $\sum_{n=0}^{\infty} Un\,(x, t\,)$ = $U_0 + U_1 + U_2 + U_3 + \ldots$

Test example1

Given the Burger's Fisher equation $U_t + UU_x - dU_{xx} = 0$

Where d is the diffusion constant and When the parameters $\alpha = \delta = 1$ an $\beta = 0$

It is a one – dimensional non – linear Parabolic equation with initial condition:

$U_0(x, t) = U(x, o) = sin\pi x$

Boundary condition: U (0, t) = U (1, t) = 0

Given the exact solution: $U(x, t) = \sin(\pi x) \exp(12dt)$

Aim to find the solution of the unknown function, U(x, t) by ADM?

To solve this equation using the Adomain decomposition method (ADM) we should define first the linear and invers operator as follows.

$L_t(\bullet) = \frac{\partial(.)}{\partial t}$

$L_t^{-1}(\bullet) = \int_0^t (.)dt$

The next step will be writing the given differential equation in standard (Operator) form: LU+ Nu – dRu = 0 (1)

Where N is non – linear differential operator

L and R are linear differential operators.

and $L(U) = U_t = \frac{\partial U}{\partial t}$

$N(u) = UU_x = U\frac{\partial U}{\partial x}$

$R(u) = Uxx = \frac{\partial^2 U}{\partial X^2}$

Now multiplying both sides of eq.1 by inverse operator weL_t^{-1} have:

$L_t^{-1}LU + L_t^{-1}NU - dL_t^{-1}RU = 0$ (2)

But using the initial condition the result of :

$$L^{-1}L_tU = \int_0^t U_t\ dt$$

$$= \int_0^t \frac{\partial U}{\partial t} dt$$

$$= U(x, t) |_0^t$$

$$= U(x, t) - U(x, 0)$$

$$= \underline{\underline{\mathbf{U(x, t) - Sin\pi x}}}$$

Now substituting this value to equ.2, we have:

U(x, t) − sinπx + L_t^{-1}Nu −dL_t^{-1}Ru = 0

U(x, t) = sinπx − L_t^{-1}NU +dL_t^{-1}Ru

and N(U) =$\sum_{n=0}^{\infty} An$, where An are the Adomain polynomials.

So $\sum_{n=0}^{\infty}$ U_n (x, t) = U (x, 0) −L_t^{-1}N (u) + dL_t^{-1}R (u)

:. U_n (x, t) = f(x) − $L_t^{-1} \sum_{n=0}^{\infty}$ An + dL_t^{-1}R (u_n) (3)

To find An, we use the domain polynomials formula as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}(N(\textstyle\sum_{i=0}^{\infty} \lambda^i Ui))\right]_{=0} \quad , \quad n \geq 0$$

For n = 0, we have:

$$A_0 = 1\left[\frac{d^0}{d\lambda^0}(N(\textstyle\sum_{i=0}^{\infty} \lambda^0 U_0))\right]_{\lambda=0}$$

$$= \left[\frac{d}{d}(N(\textstyle\sum_{n=0}^{\infty} U_0))\right]_{=0}$$

= N (U_0) b/se N(U) = UU_x

:. **A_0 = $\underline{U_0\ U_{0_x}}$**

For n = 1, we have:

$$A_1 = 1\left[(\frac{d^1}{d\lambda^1}(N(\textstyle\sum_{n=0}^{\infty} \lambda^1 U_1))\right]_{\lambda=0}$$

$$= \left[\frac{d}{d\lambda}(N(\textstyle\sum_{n=0}^{\infty} \lambda U_1))\right]_{=0}$$

$$= \left[\frac{d}{d\lambda}(N(\lambda^0 U_0 + \lambda^1 U_1))\right]_{=0}$$

$$= \left[\frac{d}{d\lambda}(N(U_0 + \lambda U1))\right]_{=0}$$

$$= \frac{d}{d\lambda}\left[(U_0 + \lambda U1)\ (U_0 + \lambda U1)_x\right]_{\lambda=0}$$

$$= \left[U_1(U_0 + \lambda U_1)_x + (U_0 + \lambda U_1)\ (U_1)_x\right]_{\lambda=0}$$

$$= U_1(U_0) + U_0\ (U_{1x})$$

:. **A_1 = $\underline{U_{0x}\ U_1 + U_0\ U_{1x}}$**

For n = 2, we have:

$$A_2 = \frac{1}{2!}\left[\left(\frac{d^2}{d\lambda^2}\left(N\left(\sum_{i=0}^{\infty} \lambda^2 U_2\right)\right)\right]_{\lambda=0}$$

$$= \frac{1}{2}\left[\frac{d^2}{d\lambda^2} N\left(U_0 + \lambda SU_1 + \lambda^2 U_2\right)x)\right]_{\lambda=0}$$

$$= \frac{1}{2}\left[\frac{d^2}{d\lambda^2}\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)\left(U_0 + \lambda U_1 + \lambda^2 U^2\right)x\right]_{\lambda=0}$$

$$= \frac{1}{2}\left[\frac{d}{d\lambda}\left(\frac{d}{d\lambda}\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)x\right]_{\lambda=0}$$

$$= \frac{1}{2}\left[\frac{d}{d\lambda}\left(U_1 + 2\lambda U_2\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)x + \left(U_0 + \lambda U_1 + \lambda^2 U_2\right)\left(U_1 + 2\lambda U_2\right)x\right]_{\lambda=0}$$

$$= \frac{1}{2}\left[\frac{d}{d\lambda}\left(U_1 + 2\lambda U_2\right)\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)x + \left(U_0 + \lambda U_1 + \lambda^2 U_2\right)\left(U_1 + 2\lambda U_2\right)x\right]_{\lambda=0}$$

$$= \frac{1}{2}\Big[2U_2\left(U_0 + \lambda U_1 + \lambda^2 U_2\right)x + \left(U_1 + 2\lambda U_2\right)\left(U_1 + 2\lambda U_2\right)x +$$

$$\left(U_1 + 2\lambda U_2\right)\left(U_1 + 2U_2\right)x + \left(U_0 + \lambda U_1 + \lambda^2 U_2\right)\left(2U_2\right)x\Big]_{\lambda=0}$$

$$= \frac{1}{2}\left(2U_2\left(U_0\right)x + U_1\left(U_1\right)x + U_1\left(U_1\right)x + U_0\left(2U_2\right)x\right.$$

$$= \frac{1}{2}\left(2U_2U_{0}x + 2U_1U_{1}x + 2U_0U_{2}x\right)$$

$$\therefore \mathbf{A_2 = \underline{U_2U_0x + U_1U_1x + U_0U_2x}}$$

For n = 3, we have:

$$A_3 = \frac{1}{3!}\left[\frac{d^3}{d\lambda^3}\left(N\left(\sum_{i=0}^{3} \lambda^3 U_3\right)\right)\right]_{\lambda=0}$$

$$= \frac{1}{6}\left[\frac{d^3}{d\lambda^3}\left(N\left(\sum_{i=0}^{3} \lambda^3 U_3\right)\right)\right]_{\lambda=0}$$

$$= \frac{1}{6}\left[\frac{d^3}{d\lambda^3}\left(N\left(\lambda^0 U_0 + \lambda^1 U_1 + \lambda^2 U_2 + \lambda^2 U_3\right)\right)\right]_{\lambda=0}$$

$$= \frac{1}{6}\left[\frac{d^3}{d\lambda^3}\left(N\left(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3\right)\right)\right]_{\lambda=0}$$

$$= \frac{1}{6}\left[\frac{d^3}{d\lambda^3}\left(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3\right)\left(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3\right)x\right)\right]_{\lambda=0}$$

$= \frac{1}{6}\Big[\frac{d^2}{d\lambda^2}\big((U_1 + 2\lambda U_2 + 3\lambda^2 U_3)(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3)x + (U_1 + 2\lambda U_2 +$

$3\lambda^2 U_3)x \quad (U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3)\big)\Big]_{\lambda=0}$

$= \frac{1}{6}\Big[\frac{d}{d\lambda}\big(2U_2 + 6\lambda U_3)(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3)x + (U_1 + 2\lambda U_2 + 3\lambda^2 U_3)x\,(U_1 +$

$2\lambda U_2 + 3\lambda^2 U_3) + (2U_2 + 6\lambda U_3)x\,(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3) + (U_1 + 2\lambda U_2 +$

$3\lambda^2 U_3)(U_1 + 2\lambda U_2 + 3\lambda^2 U_3)x\Big]_{\lambda=0}$

$= \frac{1}{6}\Big[(6U_3\,(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3)_x + (U_1 + 2\lambda U_2 + 3\lambda^2 U_3)x\,(2U_2 + 6\lambda U_3) +$

$(2U_2 + 6\lambda U_3)x\,(U_1 + 2\lambda U_2 + 3\lambda^2 U_3) + (2U_2 + 6\lambda U_3)(U_1 + \lambda U_2 + 3\lambda^2 U_3)x\Big] +$

$(6\lambda U_3)_x\,(U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3) + (U_1 + 2\lambda U_2 + 3\lambda^2 U_3)(2U_2 + 6\lambda U_3)x +$

$(2U_2 + 6\lambda U_3)(U_1 + 2\lambda U_2 + 3\lambda^2 U_3)x + (2U_2 + 6\lambda U_3)x\,(U_1 + 2\lambda U_2 + 3\lambda^2 U_3)\Big]_{\lambda=0}$

$= \frac{1}{6}\Big(6U_3\,(U_0)x + (U_1)x\,(2U_2) + (2U_2)x + (U_1) + 2U_2(U_1)x + (6U_3)_x\,(U_0) + U_1(2U_2)x$

$+ \quad 2U_2\,(U_1)x + (2U2)x\,(U_1).$

$= \frac{1}{6}\big(6U_3U_{0x} + 2U_2U_{1x} + 2U_1U_{2x} + 2U_2U_{1x} + 6U_3xU_0 + 2U_1U_2\,x + 2U_2U_{1x} +$

$2U_1U_2x.$

$= \frac{1}{6}\,(6U_3U_{0x} + 6U_2U_{1x} + 6U_1U_{2x} + 6U_0U_{3x}$

:. $\mathbf{A_3 = \underline{U_3U_{0x} + U_2U_1x + U_1U_2\,x + U_0U_3x}}$

A_4 and the other terms are done in the same way.

Now after finding some Adomain polynomials, we coming back to equ.3 to determine the recursive relationship that will be used to come to the solution:

$$\begin{cases} U_0(x\,,t) = U(x,0) = \sin\pi x = f(x) \\ U_{n+1}\,(x\,,t) = -\,L_t^{-1}N(U) + dL_t^{-1}R(U), \qquad n \geq o \\ \qquad = -\,L_t^{-1}(A_n) + dL_t^{-1}R\,(U) \end{cases}$$

By solving this, we can find the solution of the unknown Function.

$U_0(x, t) = \sin\pi x = f(x)$

For n = 0 ,we have:

$U_1(x , t) = -L_t^{-1}P(A_o) + dL_t^{-1}R(U_o)$

$= -L_t^{-1}(U_0U_{0x}) + dL_t^{-1}(U_{0xx})$

$= -\int_0^t U_oU_{ox}dt + d\int_0^t U_{oxx}dt$

$= -\int_0^t(\sin\pi x)\frac{d}{dx}(\sin\pi x)dt \ \ +d\int_0^t\frac{d^2}{dx^2}(\sin\pi x)dt$

$= -\int_0^t(\sin\pi x)(\pi\cos\pi x)dt - d\int_0^t(\pi^2\sin\pi x)dt$

$= -\int_0^t \pi\sin\pi x\cos\pi x dt - d\int_0^t \pi^2\sin\pi x\, dt$

$= -\pi\sin\pi x\cos\pi x\int_0^t dt - \pi^2 d\sin\pi x\int_0^t dt$

$= -\pi\sin\pi x\cos\pi x.t\,|_0^t - \pi^2 d\sin\pi x.t|_0^t$

$= -\pi t\sin\pi x\cos\pi x - \pi^2 td\sin\pi x$

:. $U_1(x , t) = -\pi t\sin\pi x(\cos\pi x + \pi d)$

For n =1,we have:

$U_2(x , t) = -L_t^{-1}(A_1) + dL_t^{-1}R(U_1)$

$= -L_t^{-1}(U_1U_{ox} + U_oU_{1x}) + dL_t^{-1}(U_{1xx})$

$= -\int_0^t (U_1U_{0x} + U_0U_{1x})\,dt + d\int_0^t (U_{1xx})\,dt$

But $U_{0x} = \frac{d}{dx}(\sin\pi x) = \pi\cos\pi x$

$U_{1x} = \frac{d}{dx}(-\pi t\sin\pi x\cos\pi x - \pi^2 td\sin\pi x)$

$= -\pi t\frac{d}{dx}(\sin\pi x\cos\pi x) - \pi^2 td\frac{d}{dx}(\sin\pi x)$

$= -\pi t(\pi\cos\pi x.\cos\pi x + \sin\pi x. -\pi\sin\pi x) - \pi^2 td(\pi\cos\pi x)$

$= -\pi t.\Pi(\cos^2\pi x + \sin^2\pi x) - \pi^3 td\cos\pi x$

$= -\pi^2 t(1) - \pi^3 td\cos\pi x$

:. $U_{1x} = -\pi^2 t - \pi^3 td\cos\pi x$

$U_{1xx} = \frac{d}{dx}(-\pi^2 t - \pi^3 td\cos\pi x)$

$U_1xx = 0 - \pi^3 td \frac{d}{dx}(\cos\pi x)$

$= -\pi^3 td(-\pi\sin\pi x)$

$\therefore$ $U_1xx = \pi^4 td\sin\pi x$

$U_0xU_1 = \pi\cos\pi x(-\pi t\sin\pi x\cos\pi x - \pi^2 td\sin\pi x)$

$= \pi\cos\pi x(-\frac{1}{2}\pi t\sin 2\pi x - \pi^2 td\sin\pi x)$

$\therefore$ $U_0xU1 = -\frac{1}{2}\pi^3 t\sin 2\pi x - \frac{1}{2}\pi^2 td\sin 2\pi x\cos\pi x$

$U_0U_1x = \sin\pi x(-\pi^2 t - \pi^3 td\cos\pi x)$

$= -\pi^2 t\sin\pi x - \pi^3 td\underline{\sin\pi x\cos\pi x}$

$= -\pi^2 t\sin\pi x - \pi^3 td(\frac{1}{2}\sin 2\pi x)$

$\therefore$ $U_0U_1x = -\frac{1}{2}\pi^3 td\sin 2\pi x - \pi^2 t\sin\pi x$

Note 1. $\sin 2\pi x = \sin\pi x\cos\pi\pi x + \cos\pi x\sin\pi x$

$= 2\sin\pi x\cos\pi x$

2. $\cos 2\pi x = \cos\pi x\cos\pi x - \sin\pi x\sin\pi x$

$= \cos^2\pi x - \sin^2\pi x$

3. $\sin\pi x\cos\pi x = \frac{1}{2}\sin 2\pi x$

Now let us come back to U_2

$U_2(x, t) = -\int_0^t (U_1U_{Ox} + U_oU_{1x})\,dt + d\int_0^t (U_{1xx})dt$

$= -\int_0^t (-\frac{1}{2}\pi^3 td\sin 2\pi x - \frac{1}{2}\pi^2 td\sin 2\pi x\cos\pi x - \frac{1}{2}\pi^3 td\sin 2\pi x - \pi^2 t\sin\pi x)\,dt$

$+ d\int_0^t (\pi^4 t\sin\pi x)dt$

$= -\int_0^t (-\pi^3 t\sin 2\pi x - \frac{1}{2}\pi^2 t\sin 2\pi x\cos\pi x - \pi^2 t\sin\pi x)\,dt + d\int_0^t (\pi^4 t\sin\pi x)dt$

$= \int_0^t \pi^3 td\sin 2\pi x\,dt + \int_0^t \frac{1}{2}\pi^2 t\sin 2\pi x\cos\pi x\,dt + \int_0^t \pi^2 t\sin\pi x\,dt +$

$d\int_0^t \pi^4 t\sin\pi x\,dx$

$= \pi^3 d\sin 2\pi x\int_0^t t\,dt + \frac{1}{2}\pi^2\sin 2\pi x\cos\pi x\int_0^t t dt + \pi^2\sin\pi x\int_0^t t\,dt + \pi^4 d\sin\pi x\int_0^t t\,dt$

$= \pi^3 d\sin 2\pi x \cdot \frac{t^2}{2} + \frac{1}{2}\pi 2\sin 2\pi x\cos\pi x \cdot \frac{t^2}{2} + \pi^2\sin\pi x \cdot \frac{t^2}{2} + \pi^4 d\sin\pi x \cdot \frac{t^2}{2}$

$= \frac{1}{2}\pi^3 t^2 \sin 2\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x\cos\pi x + \frac{1}{2}\pi^2 t^2 \sin\pi x + \frac{1}{2}\pi^4 t^2 \sin\pi x$

$= \frac{1}{2}\pi^4 t^2 d\sin\pi x + \frac{1}{2}\pi^3 t^2 d\sin 2\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x\cos\pi x + \frac{1}{2}\pi^2 t^2 \sin\pi x$

:. $U_2(x, t) = \frac{1}{2}\pi^4 t^2 d^2 \sin\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x(\cos\pi x + 2\pi d)$

For n = 2, we have:

$U_3(x, t) = -L_t^{-1}(A_2) + dL_t^{-1}R(U_2)$

$= -L_t^{-1}(U_2U_{0x} + U_1U_{1x} + U_oU_{2x}) + dL_t^{-1}(U_{2xx})$

$= -\int_0^t (U_2U_{0X} + U_1U_1x + U_OU_{2x})dt + d\int_0^t (U_{2xx})\, dt$

But $U_0x = \pi\cos\pi x$

$U_1x = -\pi^3 td\cos\pi x - \pi^2 t$

$U_2 = \frac{1}{2}\pi^4 t^2 d^2 \sin\pi x + \frac{1}{2}\pi^3 t^2 d\sin 2\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x\cos\pi x$

$U_2x = \frac{d}{dx}\left(\frac{1}{2}\pi^4 t^2 d^2 \mathrm{in}\pi x + \frac{1}{2}\pi^3 t^2 d\sin 2\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x\cos\pi x\right)$

$U_2x = \frac{1}{2}\pi^5 t^2 d^2 \cos\pi x + \pi^4 t^2 d\mathrm{Cos}2\pi x + \frac{1}{2}\pi^3 t^2 \mathrm{Cos}2\pi x\cos\pi x - \frac{1}{4}\pi^3 t^2 \sin 2\pi\sin\pi x$

$U_2xx = \frac{d}{dx}\left(\frac{1}{2}\pi^5 t^2 d^2 \cos\pi x + \pi^4 t^2 d\mathrm{Cos}2\pi x + \frac{1}{2}\pi^3 t^2 \cos 2\pi x\cos\pi x - \frac{1}{4}\pi^3 t^2 \sin 2\pi x\sin\pi x\right)$

$= \frac{1}{2}\pi^5 t^2 d^2 . -\pi\sin\pi x + \pi^4 t^2 d^2 . -2\pi\sin 2\pi x + \frac{1}{2}\pi^3 t^2 (-2\pi\sin 2\pi x\cos\pi x + \Pi . -\sin\pi x\cos 2\pi x) -$

$\frac{1}{4}\pi^3 t^2 (2\pi\cos 2\pi x\sin\pi x + \pi\cos\pi x\sin 2\pi x)$

$= -\frac{1}{2}\pi^6 t^2 d^2 \sin\pi x - 2\pi^5 t^2 d\sin 2\pi x - \pi^4 t^2 \sin 2\pi x\cos\pi x - \frac{1}{2}\pi^4 t^2 \sin\pi x\cos 2x$

$-\frac{1}{2}\pi^4 t^2 \sin\pi x\cos 2\pi x - \frac{1}{4}\pi^4 t^2 \sin 2\pi x\cos\pi x$

:. $U2xx = -\frac{1}{2}\pi^6 t^2 d^2 \sin\pi x - 2\pi^5 t^2 d\sin 2\pi x - \frac{1}{4}\pi^4 t^2 \sin 2\pi x\cos\pi x -$

$\pi^4 t^2 \sin\pi x\cos 2\pi x$

$U_oxU_2 = \pi\cos\pi x\left(\frac{1}{2}\pi^4 t^2 d^2 \sin\pi x + \frac{1}{2}\pi^3 t^2 d\sin 2\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x\cos\pi x\right)$

$U_oxU_2 = \frac{1}{4}\pi^5 t^2 d^2 \sin 2\pi x + \frac{1}{2}\pi^4 t^2 d\sin 2\pi x\cos\pi x + \frac{1}{4}\pi^3 t^2 \sin 2\pi x\cos^2\pi x$

$U_1U_{1x} = (-\pi t\sin\pi x\cos\pi x - \pi^2 td\mathrm{in}\pi x)(-\pi^3 td\cos\pi x - \pi^2 t)$

$= (-\frac{1}{2}\pi t\sin 2\pi x - \pi^2 td\sin\pi x)(-\pi^3 td\cos\pi x - \pi^2 t)$

$= \frac{1}{2}\pi^4t^2 d\sin 2\pi x\cos\pi x + \frac{1}{2}\pi^3t^2\sin 2\pi x + \pi^5t^2d^2\sin\pi x\cos\pi x \ \ + \pi^4t^2 d\sin\pi x$

$= \frac{1}{2}\pi^4t^2\sin 2\pi x\cos\pi x + \frac{1}{2}\pi^5t^2d^2\sin 2\pi x + \frac{1}{2}\pi^3t^2\sin 2\pi x + \pi^4t^2 d\sin\pi x$

$\mathbf{U_1U_{1x} = \frac{1}{2}\pi^5t^2d^2\sin 2\pi x + \pi^4t^2d\sin\pi x + \frac{1}{2}\pi^4t^2\sin 2\pi x\cos\pi x + \frac{1}{2}\pi^3t^2\sin 2\pi x}$

$UoU2x = \sin\pi x\,(\frac{1}{2}\pi^5t^2d^2\cos\pi x + \pi^4t^2d\cos 2\pi x + \frac{1}{2}\pi^3t^2\cos 2\pi x\cos\pi x -$

$\frac{1}{4}\pi^3t^2\sin 2\pi x\sin\pi x)$

$\mathbf{U_oU_{2x} = \frac{1}{4}\pi^5t^2d^2\sin 2\pi x + \pi^4t^2d\sin\pi x}\cos 2\pi x + \mathbf{\frac{1}{4}\pi^3t^2\sin 2\pi x\cos 2\pi x -}$

$\mathbf{\frac{1}{4}\pi^3t^2\sin 2\pi x\sin^2\pi x}$

Now let us come back to U_3

$U3(x, t) = -\int_0^t (U_{ox}U_2 + U_1U_{1x} + U_oU_{2x})\,dt + d\int_0^t (U_{2xx})\,dt$

$= -\int_0^t (\frac{1}{4}\pi^5t^2d^2\sin 2\pi x + \frac{1}{2}\pi^4t^2d\sin 2\pi x\cos\pi x + \frac{1}{4}\pi^3t^2\sin 2\pi x\cos^2\pi x +$

$\frac{1}{2}\pi^5t^2d^2\sin 2\pi x + \pi^4t^2d\sin\pi x + \frac{1}{2}\pi^4t^2\sin 2\pi x\cos\pi x + \frac{1}{2}\pi^3t^2\sin 2\pi x +$

$\frac{1}{4}\pi^5t^2d^2\sin 2\pi x + \pi^4t^2d\sin\pi x\cos 2\pi x + \frac{1}{4}\pi^3t^2\sin 2\pi x\cos 2\pi x -$

$\frac{1}{4}\pi^3t^2\sin 2\pi x\sin^2\pi x)dt + d\int_0^t (-\frac{1}{2}\pi^6t^2d^2\sin\pi x - 2\pi^5t^2d\sin 2\pi x -$

$\frac{1}{4}\pi^4t^2\sin 2\pi x\cos\pi x - \pi^4t^2\sin\pi x\cos 2\pi x)dt$

$= -\int_0^t \frac{3}{4}\pi^5t^2d^2\sin 2\pi x + \pi^4t^2d\sin 2\pi x\cos\pi x + \frac{5}{4}\pi^3t^2\sin 2\pi x\cos\pi x +$

$2\pi^4t^2\sin\pi x\cos 2\pi x + \frac{1}{4}\pi^3t^2\sin 2\pi x\cos^2\pi x - \frac{1}{4}\pi^3t^2\sin 2\pi x\sin^2\pi x +$

$d\int_0^t (-\frac{1}{2}\pi^6t^2d^2\sin\pi x - 2\pi^5t^2d\sin 2\pi x - \frac{5}{4}\pi^4t^2\sin 2\pi x\cos\pi x - \pi^4t^2\sin\pi x\cos 2\pi x)dt$

$= -\frac{3}{4}\pi^5d^2\sin 2\pi x\int_0^t t^2dt - \frac{3}{2}\pi^4d\sin 2\pi x\cos\pi x\int_0^t t^2dt - \frac{5}{4}\pi^3\sin 2\pi x\cos 2\pi x\int_0^t t^2dt$

$-2\pi^4\sin\pi x\cos 2\pi x\int_0^t t^2dt - \frac{1}{4}\pi^3\sin 2\pi x\cos^2\pi x\int_0^t t^2dt + \frac{1}{4}\pi^3\sin 2\pi x\sin^2\pi x\int_0^t t^2dt -$

$\frac{1}{2}\pi^6d^3\sin\pi x\int_0^t t^2dt - 2\pi^5d^2\sin 2\pi x\int_0^t t^2dt - \frac{5}{4}\pi^4 d\sin 2\pi x\cos\pi x\int_0^t t^2\,dt - \pi^4 d\sin\pi x$

$co2s\pi x\int_0^t t^2dt$

$= -\frac{3}{2}\pi^5 d^2 \sin 2\pi x.\frac{t^3}{3} - \frac{3}{2}\pi^4 d\sin 2\pi x\cos\pi x.\frac{t^3}{3} - \frac{5}{4}\pi^3 \sin 2\pi x \cos 2\pi x .\frac{t^3}{3} - 2\pi^4 \sin\pi x\cos 2\pi x.\frac{t^3}{3} -$

$\frac{1}{4}\pi^3 \sin 2\pi x \cos^2\pi x .\frac{t^3}{3} + \frac{1}{4}\pi^3 \sin 2\pi x \sin^2\pi x .\frac{t^3}{3} - \frac{1}{2}\pi^6 d^3 \sin\pi x .\frac{t^3}{3} - 2\pi^5 d^2 \sin 2\pi x .\frac{t^3}{3} -$

$\frac{5}{4}\pi^4 d \sin 2\pi x \cos\pi x .\frac{t^3}{3} - \pi^4 d\mathrm{in}\pi x \cos 2\pi x .\frac{t^3}{3}$

$= -\frac{3}{6}\pi^5 t^3 d^2 \sin 2\pi x - \frac{3}{6}\pi^4 t^3 d\sin 2\pi x \cos\pi x - \frac{5}{12}\pi^3 t^3 \sin 2\pi x \cos 2x - \frac{2}{3}\pi^4 t^3 \sin\pi x \cos 2x$

$\frac{1}{12}\pi^3 t^3 \sin 2\pi x\cos^2\pi x + \frac{1}{12}\pi^3 t^3 \sin 2\pi\sin^2\pi x - \frac{1}{6}\pi^6 t^3 d^3 \sin\pi x - \frac{2}{3}\pi^5 t^3 d^2 \sin 2\pi x$

$\frac{5}{12}\pi^4 t^3 d\sin 2\pi x \cos\pi x - \frac{1}{3}\pi^4 t^3 d\sin\pi x\cos 2\pi x$

$= -\frac{1}{6}\pi^6 t^3 d^3 \sin\pi x - \frac{1}{3}\pi^5 t^3 d^2 \sin 2\pi x - \frac{1}{3}\pi^4 t^3 d\sin 2\pi x \cos\pi x - \frac{1}{12}\pi^3 t^3 \sin 2\pi x\cos 2\pi x$

:. $U_3(x , t) = -\frac{1}{6}\pi^6 t^3 d^3 \sin\pi x - \frac{1}{12}\pi^3 t^3 \sin 2\pi x(\cos x+2\pi d)^2$

U_4 and the next terms are done in the same way.

Now the solution of the given test equation is a series form of :

$U(x, t) = \sum_{n=0}^{\infty} Un(x,t)$

$= U_0 + U_1 + U_2 + U_3 + \ldots\ldots$

$= \sin\pi x - \pi t\sin\pi x (\cos\pi x+\pi d) + \frac{1}{2}\pi^4 t^2 d^2 \sin\pi x + \frac{1}{4}\pi^2 t^2 \sin 2\pi x (\cos\pi x+2\pi d) -$
$\frac{1}{6}\pi^6 t^3 d^3 \sin\pi x - \frac{1}{12}\pi^3 t^3 \sin 2\pi x (\cos\pi x+2\pi d)^2+\ldots$

If the approximate solution is calculated for x = 0.2…0.9 and t = 0.00001…0.00008 using the 4^{th} term approximation we can get the following result.

X	T	ADM	Exact	Abso. Error
0.2	0.00001	0.58777	0.58779	0.00002
0.3	0.00002	0.80900	0.80903	0.00003
0.4	0.00003	0.95105	0.95109	0.00004
0.5	0.00004	1.00003	1.00004	0.00001
0.6	0.00005	0.95114	0.95111	0.00003
0.7	0.00006	0.80914	0.80907	0.00007
0.8	0.00007	0.58790	0.58783	0.00007
0.9	0.00008	0.30909	0.30904	0.00005

Table1: Comparison of the ADM solution and the exact Solution for d= 0.1

X	T	ADM	Exact	Abso. Error
0.2	0.00001	0.58777	0.58778	0.00001
0.3	0.00002	0.80898	0.80901	0.00003
0.4	0.00003	0.95103	0.95105	0.00002
0.5	0.00004	1.00000	1.00000	0.00000
0.6	0.00005	0.95110	0.95106	0.00004
0.7	0.00006	0.80911	0.80902	0.00009
0.8	0.00007	0.58789	0.58779	0.00001
0.9	0.00008	0.30909	0.30901	0.00008

Table 2: Comparison of the ADM solution and the exact Solution for d = 0.01

Graphical Comparison of the ADM solution and exact solution

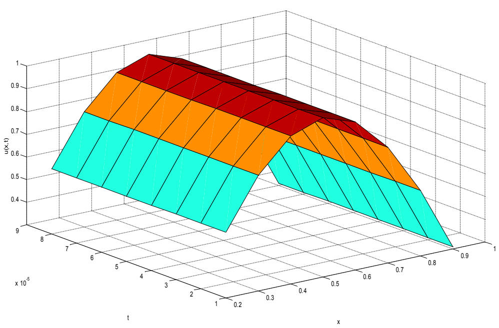

Fig.1 The graph of ADM solution

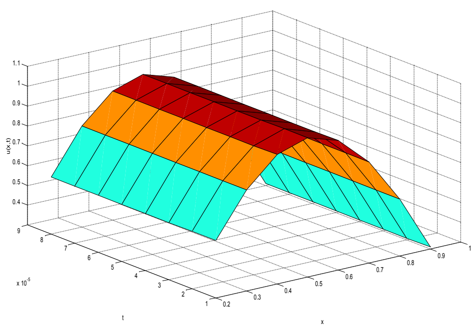

Fig.2 The graph of exact solution

.

Test example2

Given the linear partial differential equation Ut -Uxx =0 (parabolic heat or diffusion equation)

When the parameters $= \beta = 0 \; and \; \delta = 1.$

With the initial condition : U(x, 0) = sinπx **=** f(x)

and exact solution U(x, t)= = $\sin(\pi x)\, e^{-\pi^2 t}$

Aim to solve it using ADM?

Solution

To solve the given partial differential equation using ADM, we should use the following steps.

1. Defining the linear operator:
2. L_t (**.**) = $\frac{\partial(.)}{\partial t}$
3. Defining the inverse operator:

   Lt^{-1}(**.**) = $\int_0^t (.)dt$
4. Writing the given equation in an operator (standard) form.

   LU + RU = 0 (1)

   Where L and R are linear differential operators and

   LU = $\frac{\partial U}{\partial t}$

   RU= $-\frac{\partial^2 U}{\partial x^2}$

Note: since the PDE is linear there is no non-linear operator.

Now multiplying both sides of equ.1 by the inverse operator Lt^{-1} we have:

L_t^{-1}LU + L_t^{-1}RU = 0 (2)

Now by using the initial condition the result of L_t^{-1}LU = $\int_0^t LUdt$

$=\int_0^t Utdt$

=U(x, t)$|_0^t$

=U(x, t) - U(x, o)

:.L_t^{-1}LU = U(x, t) - U_o(x, t)

Substituting this value in equ.2, we have:

U(x, t) - U_O(x, t) - L_t^{-1}RU= 0

U(x, t) = U_o (x, t) + L_t^{-1} R (U)

Note: Here since the non-linear part is absent, using the Adomian polynomials is not necessary. So we have to find only the linear part of the equation as follows.

The standard decomposition technique represents the solution of U(x, t) in equ.6 as the following series:

U(x, t) = $\sum_{n=0}^{\infty} Un(x,t)$, $n \geq 0$

= U_O(x, t) $- L_t^{-1}$R(Un)

Where the components U_o,U_1,....are usually determined recursively

$$\begin{cases} U_0(x,t) = U(x,0) = \sin\pi x \\ U_{n+1}(x,t) = -L_t^{-1}R(Un) \quad n \geq 0 \end{cases}$$

By solving this we can find the solution of the given equation

U_O(x, t) = sinπx

For n = 0 we have:

U_1(x, t) = $-L_t^{-1}$R (U_O)

$= -\int_0^t R(U_0)dt$

$= -\int_0^t (-U_{0xx})dt$ b/se R (U) = - U_{XX}

= - (-U_{oxx})$\int_0^t dt$

= ($U0$)$_{XX}$$\int_0^t tdt$

= (sinπx)$_{xx}$$\int_0^t tdt$

= -π^2sinπx). $t \mid_0^t$

:. U_1(x, t) = -π^2tsinπx

For n=1, we have:

U_2(x, t) = -Lt^{-1}R (U_1)

$= -\int_0^t R(U_1)dt$

$= -\int_0^t (-U_1xx)dt$ b/se R (U) = - U_{xx}

= - $(-U_1)_{xx} \int_0^t dt$

= $(U_1)_{xx} \int_0^t dt$

= $(-\pi^2 t\sin\pi x)_{xx} \int_0^t dt$

= $-\pi^2 (\sin\pi x)_{xx} \int_0^t tdt$

= $-\pi^2 (-\pi^2 \sin\pi x) \int_0^t tdt$

= $\pi^4 (\sin\pi x).\frac{t^2}{2} |_0^t$

:. $U_2 (x, t) = \frac{1}{2}\pi^4 t^2 \sin\pi x$

For n =2, we have:

$U_3(x, t) = -Lt^{-1}R(U_2)$

= $-\int_0^t R(U_2)dt$

= $-\int_0^t (-U_{2xx})dt$ b/se R (U) = - U_{xx}

= - (-U2xx)$\int_0^t dt$

= $(U_2)xx \int_0^t dt$

= $(\frac{1}{2}\pi^4 t^2 \sin\pi x)xx \int_0^t dt$

= $\frac{1}{2}\pi^4 (\sin\pi x)xx \int_0^t t^2 dt$

= $\frac{1}{2}\pi^4 (-\pi^2 \sin\pi x) \int_0^t t^2 dt$

= $-\frac{1}{2}\pi^6 \sin\pi x \,.\frac{t^3}{3} |_0^t$

:. $U_3(x, t) = -\frac{1}{6}\pi^6 t^3 \sin\pi x$

$U_4(x, t)$ and the other terms are done in the same way.

Now the solution of the given equation can be computed as:

$U(x, t) = \sum_{n=0}^{\infty} Un(x,t)$

$= U_0(x, t) + U_1(x, t) + U_2(x, t) + U_3(x, t) + \ldots$

$= \sin\pi x - \pi^2 t \sin\pi x + \frac{1}{2}\pi^4 t^2 \sin\pi x - \frac{1}{6}\pi^6 t^3 \sin\pi x + \ldots.$

If the approximate solution is calculated for x = 0.2…0.9 and t =0.003…0.010 and approximate the solution using the 4th term approximation we get the ff

X	**T**	**ADM**	**Exact**	**Absolute error**
0.2	0.003	0.570636	0.570636	0.000000
0.3	0.004	0.777700	0.777700	0.000000
0.4	0.005	0.905262	0.905262	0.000000
0.5	0.006	0.942501	0.942501	0.000000
0.6	0.007	0.887568	0.887568	0.000000
0.7	0.008	0.537824	0.537826	0.000002
0.8	0.009	0.537824	0.537826	0.000002
0.9	0.010	0.279973	0.279974	0.000001

Table 3 : Comparison of the ADM solution and the exact Solution

Graphical Comparison of the ADM solution and exact solution

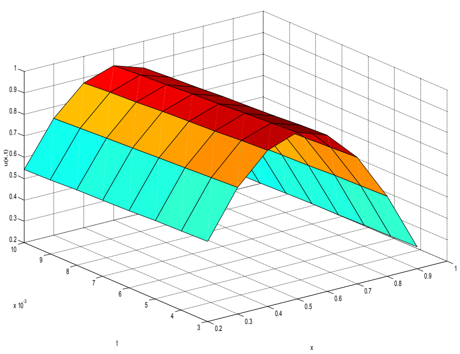

Fig .3 The graph of exact solution

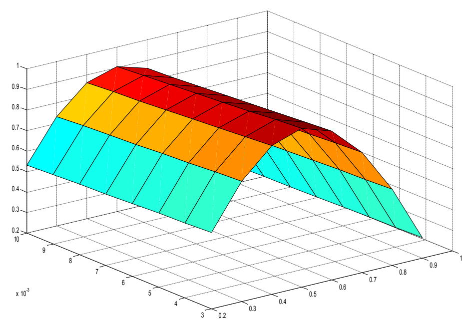

Fig.4 The graph of ADM solution

4.2 Discussion of results

In this study, to solve the given Burger's equation by Adomain Decomposition Method, first we wrote the given equation in an operator form: LU+NU-RU = 0 Next, we multiplying both sides of the above equation by the inverse operator L_t^{-1} , to have L_t^{-1}LU + L_t^{-1}NU + L_t^{-1}RU = 0. Then using the initial condition we solve for $L_t^{-1}LU$ and obtain U(x, t) - U(x, 0). Since the non- linear operator N(U)= $\sum_{n=0}^{\infty} An$, we calculate the Adomain polynomials. After finding An's (Adomain polynomials), we solve for LU+NU-RU=O recursively integrating RU twice and obtain the components of the approximate solution, U_0, U_1, U_2, U_3...

:. U(x, t) = $U_0 + U_1 + U_2 + U_3 + \ldots$ is the approximate solution of the given Burger's equation.

To examine the convergence and accuracy of the proposed method, we choose some appropriate values for the unknown variables and the calculations have been done. The results obtained are shown using tables and graphs.

Table 1and 2 displays the comparison of ADM (approximation) solution and the exact solution example 1 for different values of x, t and d. As we have seen from the tables, the maximum error for table 1 is 0.00007 and for table 2 is 0.00009 which is very small. Therefore, the ADM solution and the exact solutions are almost the same for the given differential equation.

Fig.1 and Fig.2 shows the graph of the exact solution and ADM solutions of the first test example. As seen from the two graphs, the graph of exact solution and the graph of ADM solution are .nearly the same.

Table 3 displays the comparison of ADM (approximation) solution and the exact solution of example 2 for different values of x, t and d.

As we have seen from the table, the maximum error is 0.000002 is which is very small. Therefore, the ADM solution and the exact solutions are almost the same for the given differential equation. Fig.3 shows the graph of the exact solution and ADM solutions of example 2. As seen from the two graphs, the graph of exact solution and the graph of ADM solution are almost in a perfect agreement.

Chapter Five

Conclusion and recommendation

5.1 Conclusion

The main focus of this thesis work is to investigate the easiness and accuracy of Adomain Decomposition Method in solving Burger's Fisher by comparing the results of ADM approximate solution and exact solutions.

To examine this, the researcher has solved two illustrative Burger's equations using the proposed iterative method. As we have seen from the discussion part of chapter 4, the approximate solution of ADM are calculated in the form of series with easily computable components. The approximate solution of the proposed ADM was examined by choosing some appropriate values for the unknown variables. From the results obtained, we observed that the ADM approximate solutions and the exact solutions are nearly the same and the errors are very less. Therefore, we can conclude that, the ADM is a reliable method to solve effectively, easily and accurately a large class of linear and non- linear Burger's equations with the approximate solution which converges rapidly to the exact solution for appropriate values of the unknowns.In addition to that we have seen also some short comings of the method.

- Finding the Adomain polynomials is difficult (time consuming) as $n \to \infty$
- The method needs truncation of approximate solutions.
- The method converges only for small values of numbers small region).

5.2 Recommendation

The ADM is a semi-analytical method for solving non –linear and linear Burger's Fisher equation. But since there are some short comings to apply the method, the researcher would like to comment the following points to anyone who wants to use the method.

- Use only some number of Adomain polynomials.
- Use truncation rule for the approximate solution.
- Use the method only for small region of the given domain.

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