Analysis of Boundary Element Method and Finite Difference Method for the Laplace Equation

A THESIS Submitted to Department of Mathematics, Adama Science and Technology University, for the Partial Fulfillment of the Requirements for the Degree of Masters in Science in Mathematics.

(Numerical Analysis)

BY:

Abener Tewodros

Advisor:

Dr. Tekle Gemechu

Adama, Ethiopia

September, 2017

Approval sheet

Name of student	signature	date
Abener Tewodros	________________	________________
Name of Advisor	signature	date
Tekle Gemechu (PhD)	________________	________________
Name of Examiner	signature	date
________________	________________	________________
Name of Dep't head (HOD)	signature	date
________________	________________	________________
Name of school dean	signature	date
________________	________________	________________
Name of SGS dean	signature	date
________________	________________	________________

ACKNOWLEDGEMENT

First of all, I would like to thank almighty God who gave me long life and his mother Virgin Mary for helping me in every moment of my life and to reach the precious time of my study.

Next, my peculiar cordial thanks go to my advisor, Dr. Tekle Gemechu for his support and guidance in this study by giving me notes, published articles and books.

Finally, I would like to thank my families and my dear friends who gave me constructive ideas.

Contents

List of figures

List of Tables

List of symbols

$\forall x$ for all x

Ω domain

Γ boundary of Ω.

$\frac{du}{dx}$ the first derivative of u with respect to the independent variable x.

$\frac{d^2u}{dx^2}$ the second derivative of u with respect to the independent variable x.

$\int f\, dx$ the integral of f with respect to dx.

$\int_A f\, dA$ integral of f on the set A.

$\mathbb{R}$ the set of all real numbers.

$\mathbb{R}^2$ plane $\{(x, y): x, y \in \mathbb{R}\}$

∇^2 the Laplacian

λ eigen value

Abstract

The main intention of this study is to examine the boundary element method and finite difference method for finding an approximate solution of two dimensional Laplace equation. Since the two dimensional Laplace equation is considered, the boundaries of the domain are one dimensional and discretization is carried on the one dimensional boundary for the boundary element method and domain discretization is carried on for that of the finite difference method. A general description of both methods is explicated in this thesis. This explanation discusses on the types of element for the Boundary element method for the two dimensional Laplace equation. And in addition there is no detail idea on how to show stability consistency and convergence. So, we analyze stability, convergence and consistency of both presented methods and for this we apply theorems with some proofs and mainly focus on boundary element method. And also we compare the approximate solution of the two dimensional Laplace equation by these proposed methods based on test problems.

Chapter One

1. Introduction

1.1 Background of the study

As a result of the progress of computational mathematics, numerical methods are the most widely used to solve problems arising in the real world. In most realistic engineering problems exact solutions are out of question and so we use approximate (numerical) solutions. Numerical analysis is a branch of mathematics that deals with the study of algorithms that use numerical approximation for the problems of mathematical analysis. Numerical methods can not truly bloom until the invention and then the wider availability of the electronic computers in the early 1960s (Alexander, Daisy, 2005). Many Engineering problems in fluid dynamics, hydrodynamics elasticity and etc are governed by partial differential equations (Hassan, Saeid, and Ahmad, 2016)

Finite difference method, abbreviated as FDM, is a numerical method for solving differential equations by approximating them with difference equations, in which finite differences approximate the derivatives. Popular difference formulas at an interior node, say x_j, for a discrete function u include:

- The Backward difference: $(D^{-}u)_j = \frac{u_j - u_{j-1}}{h}$
- The forward difference:$(D^{+}u)_j = \frac{u_{j+1} - u_j}{h}$
- The central difference: $(D^{\pm}u)_j = \frac{u_{j+1} - u_{j-1}}{2h}$
- The central second difference: $(D^{2}u)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$

The above difference formulations are used, especially the central second difference to approximate the Laplace equation at an interior node (x_i, y_j) and results in a five point stencil. Discretization is carried on the domain, and hence it is a finite domain method. The given domain is discretized into a uniform grid (Randall J. LeVeque, 2007).

The boundary element, abbreviated as BEM, is one of the recent numerical methods which is suitable for linear partial differential equations. Until the beginning of the eighties, the BEM was known as Boundary integral Equation method (BIEM). As a method for solving problems of mathematical physics, it has its origin in the work of G.Green (Green, 1828). He formulated, in 1828, the integral representation of the solution for the Dirichlet and Neumann problems of the Laplace equation by introducing the so called Green's function for these problems. In 1872, Betti presented a general method for integrating the equations of elasticity and deriving their solution in integral form (Betti, 1872). In 1885, Somagliana used Betti's reciprocal theorem to derive the integral representation of the solution for the elasticity problem, including in its expression the body forces, the boundary displacements and the tractions (Somigliana, 1885). In the BEM the concept of discretization is carried on the boundary of the domain of interest. In addition the idea of fundamental solutions is also used in the formulation of the BEM. Multiple efforts for the BEM started around 1962. A turning point that that set up a series of connected efforts, which soon developed into a movement can be traced back to 1967. In the 1970s the BEM was still a tiro-numerical technique, but saw an exponential growth. The term *boundary element* refers to the technique that uses discretization of the boundary, called *elements* and was first used by Brebbia.

The *Laplace equation* is a second order elliptic partial differential equation named after Pierre Simon Laplace who first studied its properties. The general theory of solutions to Laplace's equation is known as *potential theory*. The solutions of the Laplace equation are the Harmonic functions which are significant in many fields of electromagnetism, astronomy, and fluid dynamics. So solving the Laplace equation is a crucial problem as it may be applied to many Engineering problems. Exact (analytic or closed form) solutions for the Laplace equation are available and the numerical results can be compared.

Error, also called absolute error, in numerical analysis is defined as the difference between the exact or analytic value and the approximate value. And the relative error is defined as the absolute value of the absolute error divided by the magnitude of the exact value.

1.2 Statement of the problem

The method of solving the Potential equation by transforming it into a boundary integral equation goes back to the beginning of the 20th century. In 1900, Fredholm employed it in the potential theory to ascertain the unknown boundary quantities from the prescribed ones (Fredholm, 1903)

However, with the advent of modern computers the method came forth again and started to be used as a computational method in the beginning of the 1960's. Jawson (Jaswon, 1963) and Symm (Symm, 1963), presented a solution algorithm for the potential equation; Jawson and Ponter developed a numerical technique to solve the boundary integral equation for the classical Saint-Venant torsion problem of non circular bars (Jaswon and Ponter, 1963).

In the last 25 years, the number of publications on BEM solutions for the potential equations has increased enormously.

Hassan Ghassemi, Saied Panahi, Ahmad Reza Kohansal wrote a journal on solving the Laplace's equation by the FDM and BEM using Mixed Boundary Conditions (Hassan Ghassemi, Saeid Panahi, and Ahmad Reza Kohansal, 2016), but didn't say about stability, consistency and convergence in detail. The main intention of numerical methods is to solve real world problems, and stability, consistency and convergence are the crucial parts. One can apply a numerical method and get the desired solution for a real world problem if consistency, stability and convergence are guaranteed.

So, in the study we are going to deal with the boundary element method with constants elements for the two dimensional Laplace equation and compare the results with that of the finite difference method for a rectangular domain and address **consistency, stability and convergence**.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this study is to analyze the boundary element method and the finite difference method for the Laplace equation in two dimensions.

1.3.2 Specific objectives

The specific objectives of the present study are:

- to test the stability, consistency and convergence of the methods proposed.
- to compare Boundary element method with finite difference method.
- to analyze error (error analysis)

Chapter two

2. Review of the related literature

2.1 preliminaries

2.1.1 The Gauss-Green theorem

It is a fundamental identity that relates the integral of the derivative of a function over a domain Ω to the integral of that function on its boundary.

Suppose Ω is a two dimensional domain with a boundary Γ. Assume $f = f(x, y)$. First let's work out the derivative of f with respect to x.

$$\int_\Omega \frac{\partial f}{\partial x} d\Omega = \int_{y_1}^{y_2} (\int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx) dy = \int_{y_1}^{y_2} \{f(x_2, y) - f(x_1, y)\} dy \tag{2.1}$$

Where

$x_1 = x_1(y)$ and $x_2 = x_2(y)$

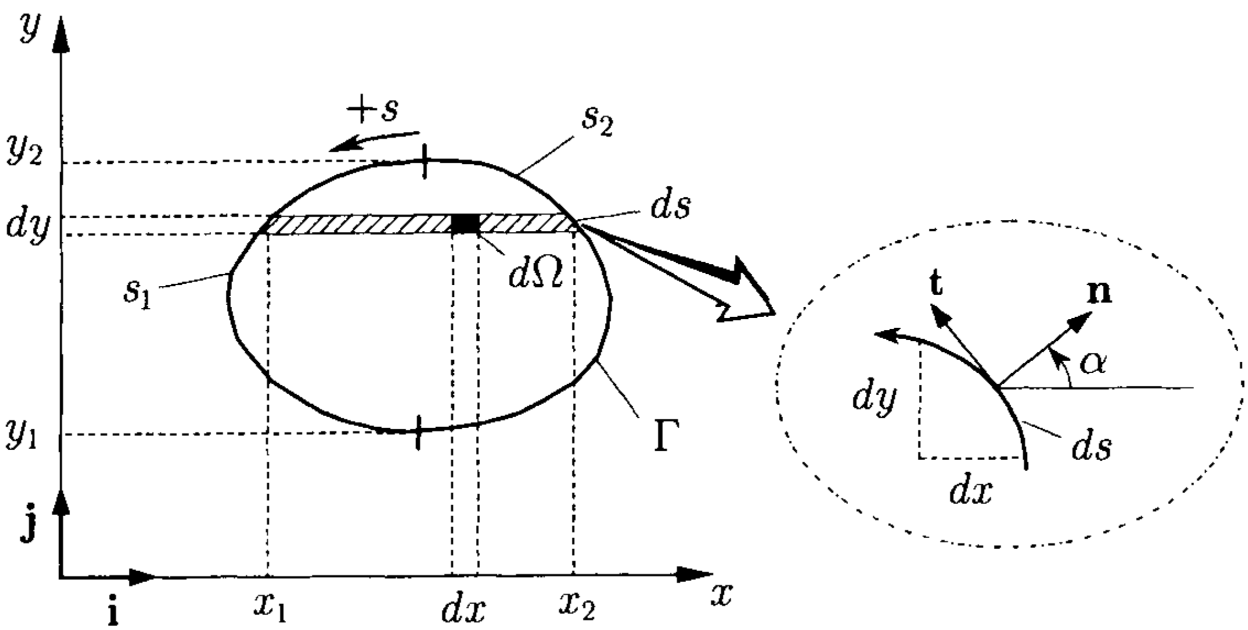


Fig 2.1 Integration over a plane domain Ω bounded by a curve Γ.

From the detail of fig 2.1, we have

$$\frac{dy}{ds} = cos\alpha = n_x \qquad \Rightarrow \quad dy = n_x ds \tag{2.2}$$

$$-\frac{dx}{ds} = sin\alpha = n_y \qquad \Rightarrow \quad dx = -n_y ds \tag{2.3}$$

Where n_x and n_y are the components of the unit vector $\boldsymbol{n}$, which is normal to the boundary Γ. the negative sign is due to the fact that the dx and the $sin\alpha$ have opposite signs when the angle α is measured in the counter clockwise sense with respect to the positive x-direction.

Consequently, eq (2.1) becomes

$$\int_{y_1}^{y_2}\{f(x_2,y)-f(x_1,y)\}dy=\int_{s_2} f(x_2,y)n_x ds-\int_{s_1} f(x_1,y)n_x ds \tag{2.4}$$

Using uniform direction for the integration over s, both terms in eq (2.4) can be combined in a single expression

$$\int_\Omega \frac{\partial f}{\partial x}d\Omega=\int_\Gamma f n_x ds \tag{2.5}$$

Interchanging x with y in eq (2.5), we obtain

$$\int_\Omega \frac{\partial f}{\partial y}d\Omega=\int_\Gamma f n_y ds \tag{2.6}$$

If g is another function of x and y, the eqs (2.5) and (2.6) result in

$$\int_\Omega \frac{\partial (fg)}{\partial x}d\Omega=\int_\Gamma fgn_x ds=\int_\Omega g\frac{\partial f}{\partial x}d\Omega+\int_\Omega f\frac{\partial g}{\partial x}d\Omega \tag{2.7}$$

$$\Rightarrow\ \int_\Omega g\frac{\partial f}{\partial x}d\Omega=-\int_\Omega f\frac{\partial g}{\partial x}d\Omega+\int_\Gamma fgn_x ds \tag{2.8}$$

$$\int_\Omega \frac{\partial (fg)}{\partial y}d\Omega=\int_\Gamma fgn_y ds=\int_\Omega g\frac{\partial f}{\partial y}d\Omega+\int_\Omega f\frac{\partial g}{\partial y}d\Omega \tag{2.9}$$

$$\Rightarrow\ \int_\Omega g\frac{\partial f}{\partial y}d\Omega=-\int_\Omega f\frac{\partial g}{\partial y}d\Omega+\int_\Gamma fgn_y ds \tag{2.10}$$

Equations (2.8) and (2.10) state the integration by parts in two dimensions and are known as the Gauss-Green theorem.

2.1.2 The Divergence theorem of Gauss

* A result of an application of the Green-Gauss theorem.

Consider the vector field $\boldsymbol{u}=u\mathbf{i}+v\mathbf{j}$, where $\mathbf{i},\mathbf{j}$ denote the unit vectors along the x and y axes and $u=u(x,y), v=v(x,y)$ its components. Applying eq (2.5) for $f=u$ and eq (2.6) for $f=v$ and adding, gives

$$\int_\Omega\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)d\Omega=\int_\Gamma(un_x+vn_y)ds \tag{2.11}$$

Equation (2.11) can be written using vector notation as

$$\int_\Omega \nabla.\boldsymbol{u}d\Omega=\int_\Gamma \boldsymbol{u}.\boldsymbol{n}\,ds \tag{2.12}$$

In which the symbolic vector ∇ (read as 'nabla vector') is defined as

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}$ and represents the differential operator that produces the gradient of a scalar field.

The quantity $\nabla.\boldsymbol{u}$ is referred to as the divergence of the vector field $\boldsymbol{u}$ at a domain point inside the domain Ω.

And the quantity $\boldsymbol{u}.\boldsymbol{n}$ is the flux of the vector field at a point on the boundary Γ.

Equation (2.12) relates the total divergence to the flux of a vector field and it is known as the divergence theorem of Gauss.

2.1.3 Green's second identity

Consider the function $u = u(x,y) and\ v = v(x,y)$ which are twice continuously differentiable in Ω and once differentiable on Γ. Applying eq (2.8) for $g = v, f = \frac{\partial u}{\partial x}$ and eq (2.10) for $g = u, f = \frac{\partial u}{\partial y}$ and again applying eq (2.8) for $g = u, f = \frac{\partial v}{\partial x}$ and eq (2.10) for $g = u, f = \frac{\partial v}{\partial y}$ and with some computations, we obtain

$$\int_{\Omega}(v\nabla^2 u - u\nabla^2 v)d\Omega = \int_{\Gamma}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})ds \tag{2.13}$$

Where ∇^2 is known as the Laplace operator and it is defined as

$$\nabla^2 \equiv \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.14}$$

And

$\frac{\partial}{\partial n} \equiv n.\nabla = n_x\frac{\partial}{\partial x} + n_y\frac{\partial}{\partial y}$ is the operator that produces the derivative of a scalar function in the direction of $\boldsymbol{n}$. Equation (2.13) is known as Green's second identity.

2.1.4 Dirac delta function

For the one dimensional case it is a generalized function or distribution on the real number line that is zero everywhere except at zero, with an integral of one over the entire real line. The delta function is sometimes thought of as a hypothetical function whose graph is infinitely high, infinitely thin spike at the origin, with total area one under the spike. Physically it represents the density of an idealized point mass or charge. It was introduced by theoretical physicist Paul Dirac. That is

$$\delta(x) = \begin{cases} +\infty & x=0 \\ 0 & x\neq 0 \end{cases} \tag{2.15}$$

And $$\int_{-\infty}^{+\infty}\delta(x)\,dx = 1 \tag{2.16}$$

Analogously, for the two dimensional case $\delta(Q)$ is defined as (for $Q \in \mathbb{R}^2$)

$$\delta(Q) = \begin{cases} +\infty & Q=(0,0) \\ 0 & Q\neq(0,0) \end{cases} \tag{2.17}$$

And $$\int_{\mathbb{R}^2} \delta(Q) = 1. \tag{2.18}$$

In conformity to the one dimensional Dirac delta function, $\delta(x - x_0)$, the two dimensional $\delta(Q - Q_0)$ is defined as

$$\delta(Q - P) = \begin{cases} +\infty & Q=P \\ 0 & Q\neq P \end{cases} \text{ for a point source at } P. \tag{2.19}$$

2.1.5 Fundamental solution

Mathematically the fundamental solution of a problem is the solution of the governing differential equation when the Dirac delta is acting as a forcing term, appears on the right hand side. The name "fundamental" came from the fact that it is the solution of the most fundamental problem in mechanics, which deals with a unit source with an infinite body.

It is the particular solution of the problem corresponding to the Dirac delta distribution. Provided that the Dirac delta possesses the singular nature, the fundamental solution is also singular.

Consider a point source placed at point P of the xy plane. Its density at Q may be expressed mathematically by the Dirac delta function as

$$f(Q) = \delta(Q - P) \tag{2.20}$$

And the potential $v = v(Q, P)$ produced at point Q satisfies the equation

$$\nabla^2 v = \delta(Q - P) \tag{2.21}$$

Now for the two dimensional Laplace equation, the fundamental solution is given by

$$v = \frac{1}{2\pi} lnr \tag{2.22}$$

The above equation can be obtained by transforming equation (2.21) into polar coordinate system.

That is

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \delta(Q - P) \tag{2.23a}$$

And the transformation

$$x = rcos\theta, \qquad y = rsin\theta \qquad r^2 = x^2 + y^2 \qquad \theta = \arctan\left(\frac{y}{x}\right) \tag{2.23b}$$

Now $$v(x, y) = v(r, \theta).$$

So $$\nabla^2 v(r, \theta) = \delta(Q - P)$$

By using (2.23a) and (2.23b) and chain rule, we arrive at

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}.\frac{\partial v}{\partial r} + \frac{1}{r^2}.\frac{\partial^2 v}{\partial \theta^2} = \delta(Q-P) \tag{2.24}$$

Since the solution is axisymmetric with respect to the source, it's independent of the polar angle θ and thus equation (2.24) becomes

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}.\frac{\partial v}{\partial r} = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = \delta(Q-P) \tag{2.25}$$

where

$$r = |Q-P| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

The right hand side of eq (2.25) vanishes at all points of the plane, except at the origin $r = 0$, where it has an infinite value. Aside from point $r = 0$, eq (2.25) is written as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0 \tag{2.26}$$

which gives after integration twice

$$v = Alnr + B \tag{2.27}$$

where A and B are arbitrary constants. Since we are looking for a particular solution, we may set $B = 0$. The other constant A, may be determined as follows.

Due to the axisymmetric nature of the problem, it is

$$\frac{\partial v}{\partial n} = \frac{\partial v}{\partial r} = Alnr \tag{2.28}$$

And $ds = rd\theta$

Application of Green's second identity for $u = 1\ and\ v = Alnr$, yields

$$-\int_\Omega \nabla^2 v d\Omega = -\int_\Gamma \frac{\partial v}{\partial n} ds \tag{2.29}$$

Where Ω is a circle with center at point P and radius r.

So the above relation can be written as

$$-\int_\Omega \delta(Q-P) d\Omega = -\int_0^{2\pi} A\frac{1}{r} r d\theta \tag{2.30}$$

Then $1 = 2\pi A$

$\Rightarrow \quad A = \frac{1}{2\pi}$

Hence, the fundamental solution becomes

$$v = \frac{1}{2\pi} lnr \tag{2.31}$$

2.1.6 Application of the fundamental solution with a singularity to the Green's second identity

Let p be a point on the boundary Γ of our given domain Ω.

The Laplace equation is given by

$$\nabla^2 u = 0,\ x \in\ \Omega \tag{3.1a}$$

Recall that the fundamental solution to the 2D Laplace equation is given by

$$v = \frac{1}{2\pi} lnr$$

Substituting $u\ and\ v$ into the Green's second identity gives

$$\int_\Omega (v\nabla^2 u - u\nabla^2 v) d\Omega = \int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma \tag{3.1b}$$

Note:

p is on the boundary and q is inside the domain. Thus, the fundamental solution v has a singularity at p, since $r = \sqrt{|q-p|}$ (the usual Euclidean distance from p to q) and $\lim_{q\to p} r = 0$ Hence v becomes undefined at $p \in \Gamma$. However if the Dirac delta function is applied

i.e. v satisfies$\nabla^2 v = -\delta(p,q)$, and since $\nabla^2 u = 0$ (eq 3.1a), the left hand side of eq (3.1b) gives

$$\int_\Omega (v\nabla^2 u - u\nabla^2 v) d\Omega = \int_\Omega (-u(q)\delta(p,q) d\Omega = -u(p) \tag{3.2}$$

Hence, we have

$$u(p) = -\int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma \tag{3.3}$$

Thus, the fundamental with a singularity is still applicable to the Green's second identity.

Equation (3.3) sets up a relation between $u\ at\ p$ in the domain with u and its normal derivative $\frac{\partial u}{\partial n}$ on the boundary. Nevertheless, only u is known in the BVP; $\frac{\partial u}{\partial n}$ is unknown. In this case, eq (3.3) cannot be used to calculate u in the domain.

We first move q onto Γ, thus using the Dirac delta function, the integral on the right hand side of eq (3.2) is given by

$$\int_{\Omega}(-u(q)\delta(p,q)d\Omega = -\frac{\alpha}{2\pi}u(p) \qquad p \in \Omega \tag{3.4a}$$

where α is the angle subtended by Ω at q in Γ. (General case)

Depending on the location of the point p.

$$\int_{\Omega} u(q)\delta(p,q)d\Omega = \begin{cases} u(p) & if\ p\ is\ inside\ the\ domain\ \Omega \\ \frac{u(p)}{2} & if\ p\ is\ on\ smooth\ boundary\ \Gamma \\ \frac{\alpha}{2\pi}u(p) & if\ p\ is\ on\ a\ non-smooth\ boundary\ \Gamma \end{cases}$$

So the integral representation of the solution for the Laplace equation (3.1a) at points $p \in \Gamma$ where the boundary is not smooth is given by

$$\frac{\alpha}{2\pi}u(p) = -\int_{\Gamma}\left[v(p,q)\frac{\partial u(q)}{\partial n_q} - u(q)\frac{\partial v(p,q)}{\partial n_q}\right]ds_q \tag{3.4b}$$

For points p, where the boundary is smooth, eq(3.4b) is given as

$$\frac{1}{2}u(p) = -\int_{\Gamma}\left[v(p,q)\frac{\partial u(q)}{\partial n_q} - u(q)\frac{\partial v(p,q)}{\partial n_q}\right]ds_q \tag{3.4c}$$

2.1.7 Description of the BEM

The BEM is a more recent numerical method used to find approximate solutions to boundary value problems. It complements the finite element method (FEM) to solve boundary value problems. The main difference between the BEM and the FEM is that the FEM is a domain method. It means that the whole region of interest (domain) has to be discretized. And the boundary element method divides only the boundary into elements. So this discretization of the boundary reduces the dimension of the problem. The 3-D problem becomes a 2-D problem; similarly the 2-D problem becomes a 1-D problem. Two approximations are made over each of these elements. One is about the geometry of the boundary, while the other has to do with the variation of the unknown quantity over the element. The usually employed boundary elements are the constant element, linear element and the parabolic element. On each element, we make out the extreme (end) points and the nodal points. The later are the points at which values of the boundary quantities are assigned.

Constant elements

In this case,

i) The boundary segment is approximated by a straight line which connects its extreme points.
ii) The node is placed at the midpoint of the straight line

iii) The boundary quantity is assumed to be constant along the element and equal to its value at the nodal point (fig 3.1)

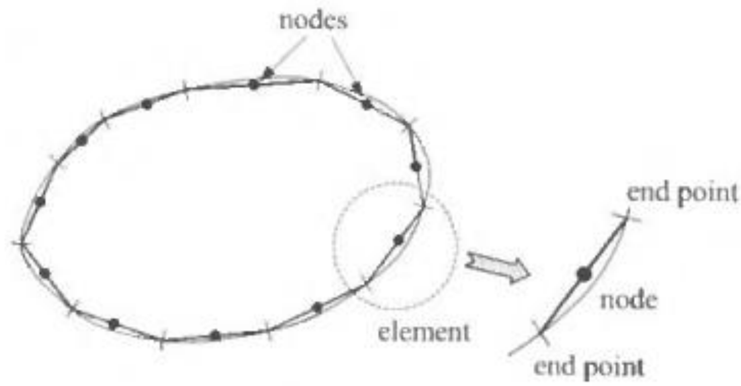


Fig 3.1 constant elements

Linear elements

In this case

i) The boundary segment is again approximated by a straight line connecting its extreme points.
ii) The element has two nodal points usually placed at the end points.
iii) The boundary quantity is assumed to vary linearly between the nodal points (fig 3.2)

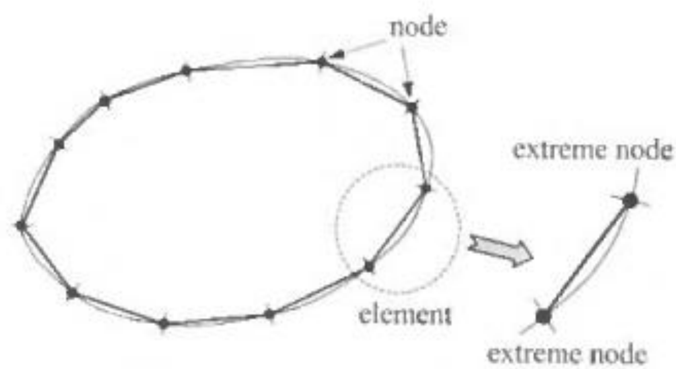


Fig 3.2 linear elements

Parabolic (quadratic) elements

In this case

i) The geometry is approximated by a parabolic arc.
ii) The element has three nodal points, two of which are placed at the end and the third somewhere in-between, usually at the midpoint (fig 3.3).

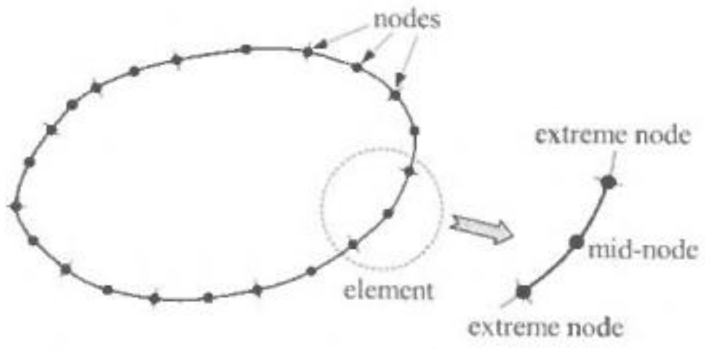


Fig 3.3 quadratic elements

The basic steps in the process of solving a BVP using BEM are

1) Determine the governing PDE with boundary conditions.
2) Find the corresponding fundamental solution of the governing equation.
3) Transform the given domain (of course for this thesis 2-D), the PDEs, and boundary conditions into a set of BIEs. The main mathematical tool for this transformation is the Green's identity which is discussed above.
4) Divide the boundary of the domain into finite number of elements. It could be constant elements, linear elements or quadratic elements.
5) Discretize the already obtained boundary integral equations for the discretized boundary.
6) Put the discretized BIE's in matrix form.
7) With the aid of the prescribed conditions solve the solution matrix, and obtain a function value at each node or the numerical approximation to the solution of our BVP.

Chapter Three

3. Research methodology

This section comprises methods and materials used to attempt the study.

3.1 Study design

The study comprises documentary review and experimental design.

3.2 Procedures of the study

To come upon the objective of the study, the following procedures are used:

1) For the Boundary Element Method
   - Defining the problem.
   - Finding fundamental solution.
   - Discretizing the boundary of the domain of interest.
   - Creating BIE (boundary integral equations) (transformation of BVP to BIE).
   - Discretizing the resulting boundary integral equation.
   - Forming system of linear equations.
   - Writing computer codes using Matlab.
   - Solving the resulting system.
   - Finding the solution for a domain point.
   - Error analysis.
   - Discuss on the Convergence, Consistency and Stability.

2) For the Finite Difference Method
   - Discretization of the given domain of interest.
   - Substitution of difference equations in the given governing differential equation.
   - Formation system of linear equations with the aid of the prescribed conditions.
   - Solving the resulting system.
   - Error analysis
   - Discuss on the Convergence, Consistency and Stability.

3.3 Instrumentation

Studying the BEM and FDM using illustrative figures, Matlab codes.

Chapter Four

4.1 The BEM with constant elements

The boundary Γ is discretized into N elements, which are numbered in the anti clock wise sense. The values of the boundary quantity u and its normal derivative $\frac{\partial u}{\partial n}$ are assumed constant over each element and equal to their value at the midpoint of the element.

The discretized form of eq (3.4c) is expressed for a given point p_i on Γ as

$$\frac{1}{2}u^i = -\sum_{j=1}^{N}\int_{\Gamma_j} v(p_i,q)\frac{\partial u(q)}{\partial n_q}ds_q + \sum_{j=1}^{N}\int_{\Gamma_j} u(q)\frac{\partial v(p_i,q)}{\partial n_q}ds_q \tag{4.1}$$

where

Γ_j is the segment (straight line) on which the $j-$th node is located and over which integration is carried out.

p_i is the nodal point of the $i-$th element. For constant elements the boundary is smooth at the nodal points, and hence $\alpha = \pi$.

Since $u \; and \; \frac{\partial u}{\partial n}$ are assumed constant, they can be moved outside the integral. So eq (4.1) can be written as

$$-\frac{1}{2}u^i + \sum_{j=1}^{N}\left(\int_{\Gamma_j}\frac{\partial v}{\partial n}ds\right)u^j = \sum_{j=1}^{N}\left(\int_{\Gamma_j} v ds\right)u_n^j \tag{4.2}$$

where $u^j = u \; at \; the \; j-th \; element$

$$u_n^j = \frac{\partial u}{\partial n} \; at \; the \; j-th \; element$$

The integrals involved in the above equation relate the node p_i, where the fundamental solution is applied, to the node $p_j (j = 1,2,\ldots,N)$ (fig 3.4). Their values express the contribution of the nodal values u^j and u_n^j to the formation of the value $\frac{1}{2}u^i$. For this reason, they are referred to as *influence coefficients*. These coefficients are denoted by $\widehat{H}_{ij}$ and G_{ij} which are defined as

$$\widehat{H}_{ij} = \int_{\Gamma_j}\frac{\partial v(p_i,q)}{\partial n_q}ds \qquad and \qquad G_{ij} = \int_{\Gamma_j} v(p_i,q)ds \tag{4.3}$$

where the point p_i remains as a reference point, while the point q varies over the $j-th$ element, which is the integration point.

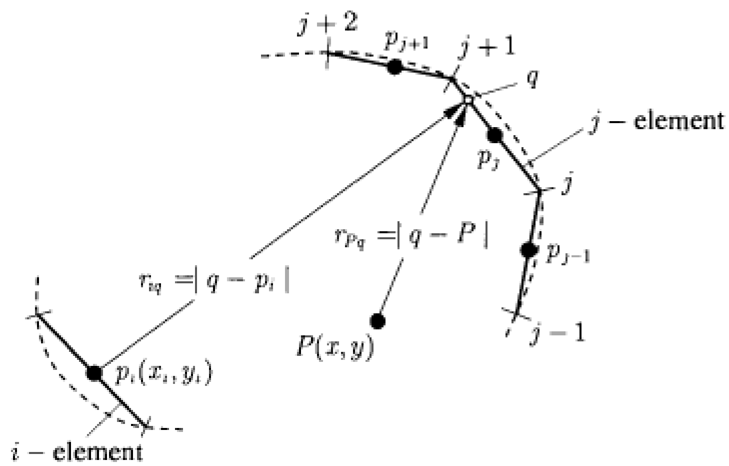


Fig 3.4 Nodal point location and relative distances for constant element discretization

Using eq (4.3) into eq (4.2) the discrete form of the solution becomes

$$-\frac{1}{2}u^i + \sum_{j=1}^{N}\left(\widehat{H}_{ij}\right)u^j = \sum_{j=1}^{N}(G_{ij})u_n^j \tag{4.4}$$

Furthermore, setting

$H_{ij} = \widehat{H}_{ij} - \frac{1}{2}\delta_{ij}$ where δ_{ij} is the Kronecker delta given as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4.5}$$

we get the following

$$\sum_{j=1}^{N}\left(H_{ij}\right)u^j = \sum_{j=1}^{N}(G_{ij})u_n^j \tag{4.6}$$

Eq (4.6) is applied consecutively for all nodes $p_i(i = 1,2, \dots, N)$ affording a system of N linear algebraic equations, which are arranged in a matrix form

$$[H]\{u\} = [G]\{u_n\} \tag{4.7}$$

where $H\ and\ G$ are $N \times N$ square matrices, and $\{u\}\ and\ \{u_n\}$ are vectors of dimension N.

If there is a need of rearrangement in the matrices, then we move the unknowns to the left hand side. This procedure is more desirable for writing computer program.

The solution of the simultaneous equations eq (4.7) yields the unknown boundary quantities. Therefore, knowing all the boundary quantities on Γ, the solution u can be computed at any point in the domain by virtue of eq (3.4a) for $\alpha = 2\pi$. Applying the same discretization as in eq (4.1), we arrive at

$$u(P) = \sum_{j=1}^{N}\left(H_{ij}\right)u^j + \sum_{j=1}^{N}(G_{ij})u_n^j \tag{4.8}$$

The line integrals G_{ij} and $\widehat{H}_{ij}$ are evaluated numerically using Gaussian quadrature. Two cases are discerned for the integrals of the influence coefficients.

(i) Off-diagonal elements

In this case the point $p_i(x_i, y_i)$ lies outside the j- element, meaning that $r = |q - p_i|$ does not vanish and, accordingly, the integral is regular.

The Gaussian integration is performed over the interval $-1 \leq \xi \leq 1$,

$$\int_{-1}^{1} f(\xi)\, d\xi = \sum_{k=1}^{n} \varpi_k f(\xi_k) \tag{4.9}$$

where n is the number of Gauss points, ϖ_k and ξ_k are the weights and abscissas of the Gaussian quadrature of order n.

ξ_k's are roots of Legendre's polynomial

$$P_n(\xi) = \frac{1}{2^n n!}\frac{d^n\left(\xi^2-1\right)^n}{d\xi^n} \text{ and } \varpi_k = \frac{2\left(1-\xi_k^2\right)}{n^2\left(P_n(\xi_k)\right)^2}$$

Table 1. Abscissas and weights in the Gauss quadrature

n (nº of Gauss points)	$\boldsymbol{\xi_k}$	$\boldsymbol{\varpi_k}$
2	± 0.57735	1.000000
3	0	0.888889
	± 0.774597	0.555556
4	± 0.339981	0.652145
	± 0.861136	0.347855
5	0	0.568889
	± 0.538469	0.478629
	± 0.90618	0.236927

Let us consider the element j over which the integration will be carried out. This element is defined by the coordinates (x_j, y_j) and (x_{j+1}, y_{j+1}) of its extreme points, which are expressed in *global system* having axes x and y, and origin at point O.

Subsequently, a local system of axes x' and y' is introduced at point p_j of the element. The local coordinates $(x', 0)$ of the point q on the j-th element are related to the global coordinates of the xy system through the expressions.

$$x = \frac{x_{j+1}+x_j}{2} + \frac{x_{j+1}-x_j}{l_j}x' \tag{4.10}$$

$$y = \frac{y_{j+1}+y_j}{2} + \frac{y_{j+1}-y_j}{l_j}x', \qquad -\frac{l_j}{2} \le x' \le \frac{l_j}{2} \tag{4.11}$$

where l_j is the length of the $j-th$ element and is given in terms of the coordinates of the end points as

$$l_j = \sqrt{\left(x_{j+1}-x_j\right)^2 + \left(y_{j+1}-y_j\right)^2} \tag{4.12}$$

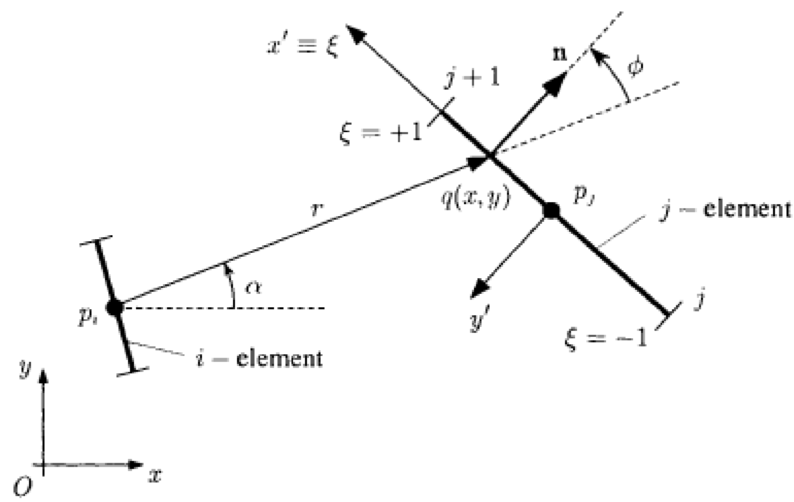


Fig 3.5 Global and local coordinate systems

Expressions that map the global coordinates onto the integration interval [-1,1] are obtained by introducing to eq (4.10) the geometric relation

$$\frac{x'}{\frac{l_j}{2}} = \xi \tag{4.13}$$

Thus, the resulting coordinate transformation becomes

$$x(\xi) = \frac{x_{j+1}+x_j}{2} + \frac{x_{j+1}-x_j}{2}\xi \tag{4.14}$$

$$y(\xi) = \frac{y_{j+1}+y_j}{2} + \frac{y_{j+1}-y_j}{2}\xi \tag{4.15}$$

Furthermore,

$$ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\frac{x_{j+1}-x_j}{2}\right)^2 + \left(\frac{y_{j+1}-y_j}{2}\right)^2}\, d\xi$$

$$= \frac{l_j}{2} d\xi \tag{4.16}$$

And then the integrals of the influence coefficients are assessed numerically in the following way.

(A) The integral of G_{ij}

$$G_{ij} = \int_{\Gamma_j} v\, ds = \int_{-1}^{1} \frac{1}{2\pi} \ln[r(\xi)]\ \frac{l_j}{2}\ d\xi = \frac{l_j}{4\pi} \sum_{k=1}^{n} \ln[r(\xi_k)]\, \varpi_k \tag{4.17}$$

where

$$r(\xi_k) = \sqrt{[x(\xi_k) - x_i]^2 + [y(\xi_k) - y_i]^2} \tag{4.18}$$

(B) The integral of $\widehat{H}_{ij}$

Notice that $\quad ds cos\phi = r d\alpha$

Then $\quad \widehat{H}_{ij} = \int_{\Gamma_j} \frac{\partial v}{\partial n} ds = \int_{\Gamma_j} \frac{1}{2\pi} \frac{cos\phi}{r} ds = \int_{\Gamma_j} \frac{1}{2\pi} da = \frac{a_{j+1} - a_j}{2\pi} \qquad (4.19)$

The angles $a_{j+1}\ and\ a_j$ are computed from

$$\tan a_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} \quad \text{and} \tag{4.20}$$

$$\tan a_j = \frac{y_j - y_i}{x_j - x_i} \tag{4.21}$$

where $x_{j+1}, y_{j+1}, x_j\ and\ y_j$ are extreme points of the j-th element.

(ii) Diagonal elements, $i = j$

In this case the node p_i coincides with node p_j, and r lies on the element. Consequently, it is $\phi = \pi/2$ or $\phi = 3\pi/2$, which yields $cos\phi = 0$. Moreover, we have

$$x_{p_j} = \frac{x_{j+1} + x_j}{2}, \qquad y_{p_j} = \frac{y_{j+1} + y_j}{2} \tag{4.22}$$

And

$$r(\xi) = \sqrt{\left[x(\xi) - x_{p_j}\right]^2 + \left[y(\xi) - y_{p_j}\right]^2} = \frac{l_j}{2}\ |\xi| \tag{4.23}$$

Hence,

$$G_{jj} = \int_{\Gamma_j} \frac{1}{2\pi} lnr ds = 2 \int_0^{l_j/2} \frac{1}{2\pi} lnr dr \tag{4.24}$$

$$= \frac{1}{\pi} [r lnr - r]_0^{l_j/2} = \frac{1}{\pi} \frac{l_j}{2} \left[\ln\left(\frac{l_j}{2}\right) - 1\right] = G_{ii} \tag{4.25}$$

and

$$\widehat{H}_{jj} = \frac{1}{2\pi}\int_{\Gamma_j}\frac{cos\phi}{r}ds = \frac{1}{2\pi}\int_{-1}^{1}\frac{cos\phi}{|\xi|}d\xi \qquad (4.26)$$

$$= \frac{2}{2\pi}[cos\phi * \ln|\xi|]_0^1 = 0 = H_{ii}$$

And therefore, the solution at any domain point can be found by applying

$u(p) = -\int_{\Gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma$ where p is a domain point.

This can be done by dividing the boundary integral into the number of elements.

i.e. $u(p) = -\left[\int_{\Gamma_1}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_1 + \int_{\Gamma_2}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_2 + \cdots + \int_{\Gamma_N}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_N\right]$
where N is the number of elements.

4.2 Finite difference scheme for the 2D Laplace equation.

In the finite difference method; finite differences are used to approximate the derivatives.

That is,

Consider a twice differentiable function say u which is a function of two variables x and y.

$u = u(x, y)$. Then

$U_x = \frac{\partial U}{\partial x} \cong \frac{U_{i+1,j} - U_{i,j}}{h}$ where $U_{i,j} = U(x_i, y_j)$ and $h = \Delta x$. (4.27)

$U_y = \frac{\partial U}{\partial y} \cong \frac{U_{i,j+1} - U_{i,j}}{k}$ where $k = \Delta y$. (4.28)

And also

$$U_{xx} = \frac{\partial^2 U}{\partial x^2} \cong \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} \qquad (4.29)$$

and

$$U_{yy} = \frac{\partial^2 U}{\partial y^2} \cong \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}. \qquad (4.30)$$

(4.27) and (4.28) are called forward differences and (4.29) and (4.30) are central differences.

Then

$$U_{xx} + U_{yy} = 0 \qquad (4.31a)$$

$$\Rightarrow \quad \frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2}+\frac{U_{i,j+1}-2U_{i,j}+U_{i,j-1}}{k^2}=0 \tag{4.31b}$$

Equation (4.31a) is the governing equation of the two dimensional Laplace equation.

4.3 Stability Analysis and Consistency.

4.3.1 Matrix eigen value analysis

Let A be a square matrix of order N. Then, A is said to be stable if all the eigen values are less than zero or all real parts are zero (if it is a complex).

4.3.2 Von Neumann analysis

To check the stability of the finite difference method for the two dimensional Laplace equation, we use the **Von Neumann analysis** condition for stability.

Von Neumann stability condition says

Assume that a numerical scheme admits a solution of the form

$$u_{ij}=a^{(i)}(\omega)e^{zj\omega\Delta x} \tag{4.38}$$

where ω is the wave number and $z=\sqrt{-1}$.

Define

$$G(\omega)=\frac{a^{(i+1)}(\omega)}{a^{(i)}(\omega)} \tag{4.39}$$

where $G(\omega)$ is an amplification factor, which governs the growth of the Fourier component $a(\omega)$.

The von Neumann stability condition is given by

$$|G(\omega)|\le 1 \text{ for } 0\le\omega\Delta x\le\pi. \tag{4.40}$$

Now for the 2D Laplace equation,

Let $u_{i,j}=a^{(i)}(\omega)e^{zj\omega\Delta x}$.

Then,

$$u_{i+1,j}=a^{(i+1)}(\omega)e^{zj\omega\Delta x}$$

$$u_{i-1,j}=a^{(i-1)}(\omega)e^{zj\omega\Delta x}$$

$$u_{i,j+1}=a^{(i)}(\omega)e^{z(j+1)\omega\Delta x}$$

$u_{i,j-1} = a^{(i)}(\omega)e^{z(j-1)\omega\Delta x}$

Substituting the above into

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = 0$$

Or

$$2(\frac{1}{h^2} + \frac{1}{k^2})U_{i,j} = \frac{U_{i+1,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} + U_{i,j-1}}{k^2}$$

Gives

$$2\left(\frac{1}{h^2} + \frac{1}{k^2}\right)\left(a^{(i)}(\omega)e^{zj\omega\Delta x}\right) = \frac{\left(a^{(i+1)}(\omega)e^{zj\omega\Delta x} + a^{(i-1)}(\omega)e^{zj\omega\Delta x}\right)}{h^2} + \frac{\left(a^{(i)}(\omega)e^{z(j+1)\omega\Delta x}\right) + a^{(i)}(\omega)e^{z(j-1)\omega\Delta x}\right)}{k^2} \quad (4.41)$$

$$G(\omega) = \frac{a^{(i+1)}(\omega)}{a^{(i)}(\omega)}$$

Divide both sides of (4.41) by $a^{(i)}(\omega)e^{zj\omega\Delta x}$.

$$\frac{2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)\left(a^{(i)}(\omega)e^{zj\omega\Delta x}\right)}{a^{(i)}(\omega)e^{zj\omega\Delta x}} = \frac{\left(a^{(i+1)}(\omega)e^{zj\omega\Delta x} + a^{(i-1)}(\omega)e^{zj\omega\Delta x}\right)}{h^2 a^{(i)}(\omega)e^{zj\omega\Delta x}} + \frac{\left(a^{(i)}(\omega)e^{z(j+1)\omega\Delta x}\right) + a^{(i)}(\omega)e^{z(j-1)\omega\Delta x}\right)}{k^2 a^{(i)}(\omega)e^{zj\omega\Delta x}} \quad (4.42)$$

$$\Rightarrow \quad 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) = \frac{a^{(i+1)}(\omega) + a^{(i-1)}(\omega)}{h^2 a^{(i)}(\omega)} + \frac{e^{z(j+1)\omega\Delta x} + e^{z(j-1)\omega\Delta x}}{k^2 e^{zj\omega\Delta x}}$$

$$\Rightarrow \quad 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) = \frac{1}{h^2}\left(G(\omega) + \frac{1}{G(\omega)}\right) + \frac{1}{k^2}\left(e^{z\omega h} + e^{-z\omega h}\right)$$

$$\Rightarrow \quad \frac{1}{h^2}\left(G(\omega) + \frac{1}{G(\omega)}\right) = 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) - \frac{1}{k^2}\left(e^{z\omega h} + e^{-z\omega h}\right)$$

$$\Rightarrow \quad \frac{1}{h^2}\left(\frac{G^2(\omega)+1}{G(\omega)}\right) = 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) - \frac{2}{k^2}\frac{\left(e^{z\omega h} + e^{-z\omega h}\right)}{2}$$

$$\Rightarrow \quad \frac{1}{h^2}\left(\frac{G^2(\omega)+1}{G(\omega)}\right) = 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) - \frac{2}{k^2}\cos(\omega h)$$

$$\Rightarrow \quad G^2(\omega) + 1 = \left[2h^2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) - \frac{2h^2}{k^2}\cos(\omega h)\right]G(\omega)$$

$$\Rightarrow \quad G^2(\omega) - \left[2h^2\left(\frac{1}{h^2} + \frac{1}{k^2}\right) - \frac{2h^2}{k^2}\cos(\omega h)\right]G(\omega) + 1 = 0$$

Now,

$$G(\omega)=\frac{\left[2h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{2h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\left[2h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{2h^2}{k^2}\cos(\omega h)\right]^2-4}}{2}$$

$$G(\omega)=\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1}$$

The Von Neumann condition then is

To show $|G(\omega)|\le 1$

Or

$$\left|\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1}\right|\le 1 \qquad (4.43)$$

Take limit as $h\to 0, k\to 0$.

$$\lim_{\substack{h\to 0\\k\to 0}}\left|\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1}\right|$$

$$=\left|\lim_{\substack{h\to 0\\k\to 0}}\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1}\right|$$

$$=\left|\left[\lim_{\substack{h\to 0\\k\to 0}}\left(h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right)\right]\pm\lim_{\substack{h\to 0\\k\to 0}}\sqrt{\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1}\right|$$

$$=\left|\left[\lim_{\substack{h\to 0\\k\to 0}}\left(\left(1+\frac{h^2}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right)\right]\pm\sqrt{\lim_{\substack{h\to 0\\k\to 0}}\left[\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-1\right]}\right|$$

$$=\left|\left[\lim_{\substack{h\to 0\\k\to 0}}1+\lim_{\substack{h\to 0\\k\to 0}}\frac{h^2}{k^2}-\lim_{\substack{h\to 0\\k\to 0}}\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\lim_{\substack{h\to 0\\k\to 0}}\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-\lim_{\substack{h\to 0\\k\to 0}}1}\right|$$ and using

$$\lim_{\substack{\alpha\to 0\\\beta\to 0}}\frac{\alpha}{\beta}=1$$

$$\Rightarrow\left|\left[\lim_{\substack{h\to 0\\k\to 0}}1+\lim_{\substack{h\to 0\\k\to 0}}\frac{h^2}{k^2}-\lim_{\substack{h\to 0\\k\to 0}}\frac{h^2}{k^2}\cos(\omega h)\right]\pm\sqrt{\lim_{\substack{h\to 0\\k\to 0}}\left[h^2\left(\frac{1}{h^2}+\frac{1}{k^2}\right)-\frac{h^2}{k^2}\cos(\omega h)\right]^2-\lim_{\substack{h\to 0\\k\to 0}}1}\right|$$

$$= \left| [1+1-1] \pm \sqrt{\left[\lim_{\substack{h\to 0\\ k\to 0}} 1 + \lim_{\substack{h\to 0\\ k\to 0}} \frac{h^2}{k^2} - \lim_{\substack{h\to 0\\ k\to 0}} \frac{h^2}{k^2}\cos(\omega h) \right]^2 - 1} \right|$$

$$= \left| 1 \pm \sqrt{[1+1-1] - 1} \right|$$

$$= \left| 1 \pm \sqrt{1-1} \right| = |1 \pm 0| = 1$$

Hence, $|G(\omega)| \leq 1$. i.e., the Von Neumann condition is satisfied.

Therefore, for small h and k the finite difference scheme for the 2D Laplace equation is stable.

4.3.3 Consistency

Consistency is related to local error. A numerical method is called consistent if the local truncation error decays sufficiently fast as the step size gets smaller and smaller. The local truncation error is the error given by dividing the difference between the exact solution and approximate solution by the step size.

4.3.4 Lax-Richtmyer Equivalence Theorem

The Lax-Richtmyer Equivalence Theorem is often called the Fundamental Theorem of Numerical Analysis. It states that a consistent numerical method is convergent iff it is stable.

4.4 Numerical test example

The two dimensional Laplace equation governs many real world problems. One is the steady state heat conduction. Assume a plate (or rectangular bar), which is considered as a two dimensional, is insulated on three of its sides and there is a constant heat distribution on the rest side of the plate. Steady state heat conduction is the form of conduction that happens when the temperature difference(s) driving the conduction are constant, so that, the spatial distribution of temperatures in the conducting bar does not change any further. Since the BCs are of Dirichlet type, the heat distribution on the boundary is given. If it were Neumann BCs type, it means the heat flux is given at the boundary. Heat flux or thermal flux is a flow of heat energy per unit of area per unit of time.

Before going to solve the 2D Laplace equation with prescribed conditions by FDM or BEM, let's see the exact solution.

4.4.1 Exact solution

The 2D Laplace equation is given by

$$u_{xx} + u_{yy} = 0 \quad \text{on } (0,1) \times (0,1) \quad \text{and} \quad u = u(x, y)$$

with Dirichlet boundary conditions

$$u(0, y) = 0 \qquad u(1, y) = 1$$

$$u(x, 0) = 0 \qquad u(x, 1) = 0$$

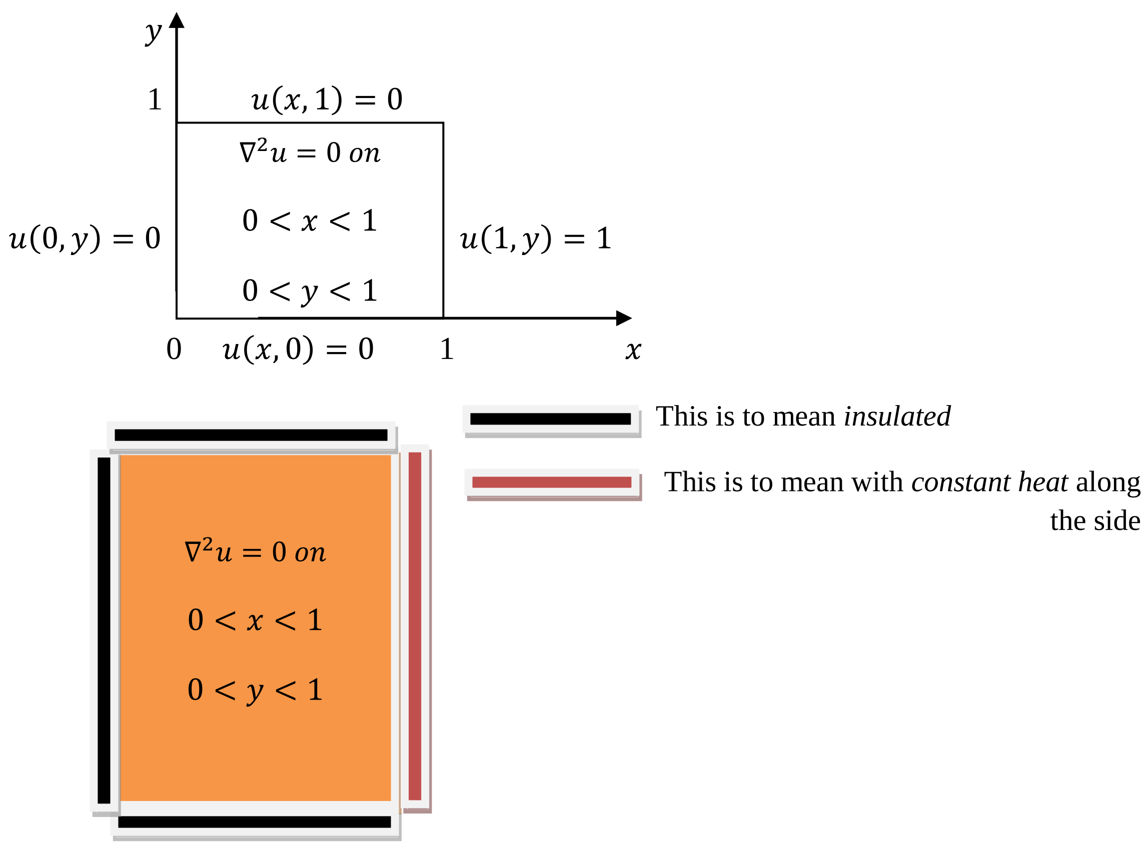


Fig. 4.1 Steady state heat conduction on a plate

Now, by using the well known method, separation of variables, and superposition principle, our exact solution is given by:-

$$u(x,y) = \sum_{k=1}^{\infty}\left(\frac{8}{(2\mathrm{k}-1)\pi\sinh\big((2k-1)\pi\big)}\right)\sinh\big((2k-1)\pi x\big)\sin\big((2k-1)\pi y\big)$$

which are Harmonic functions.

Solution graphs

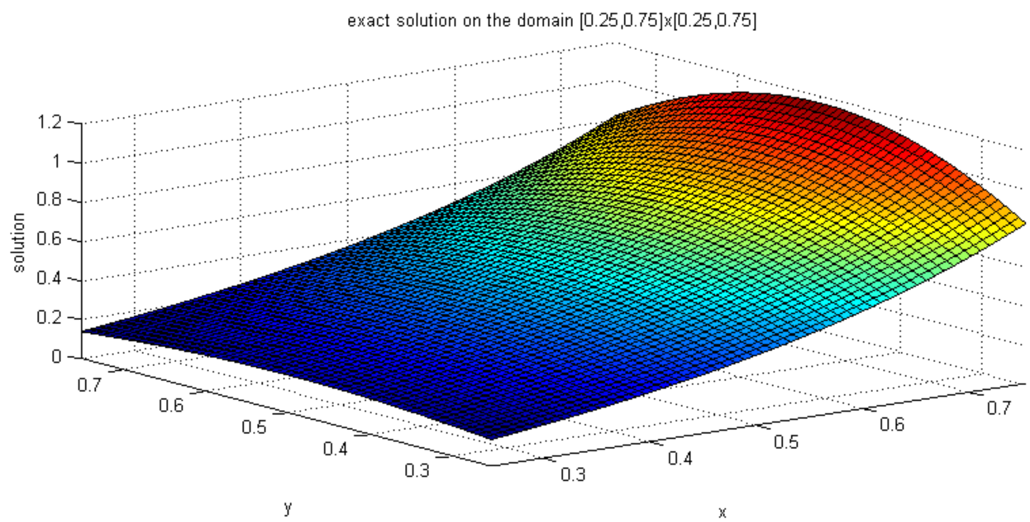


Fig 4.2 Exact solution graph on the domain [0.25,0.75]x[0.25,0.75]

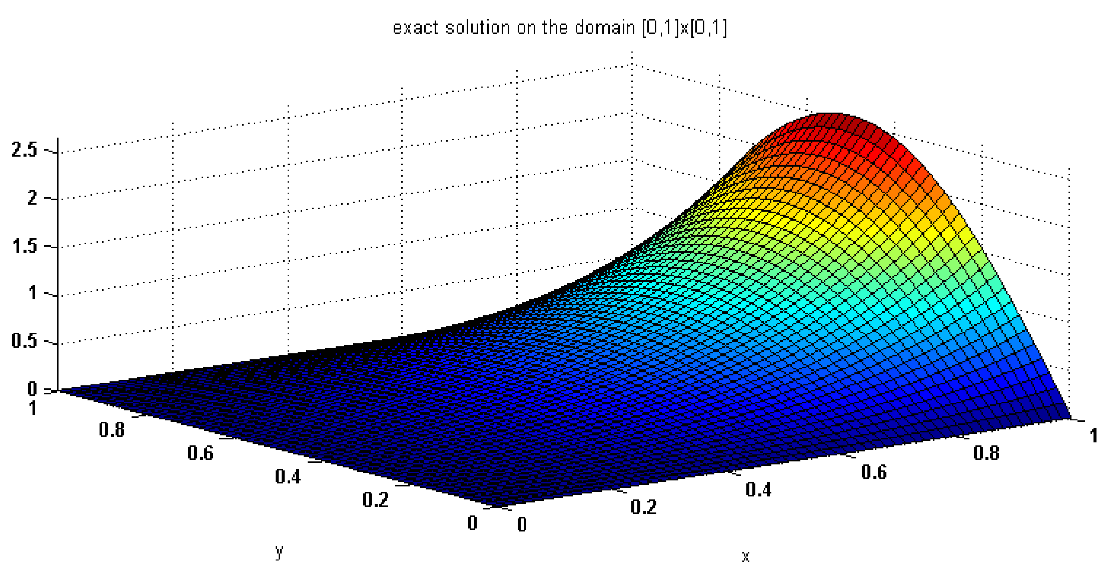


Fig 4.3 Exact solution graph on the domain [0,1]x[0,1]

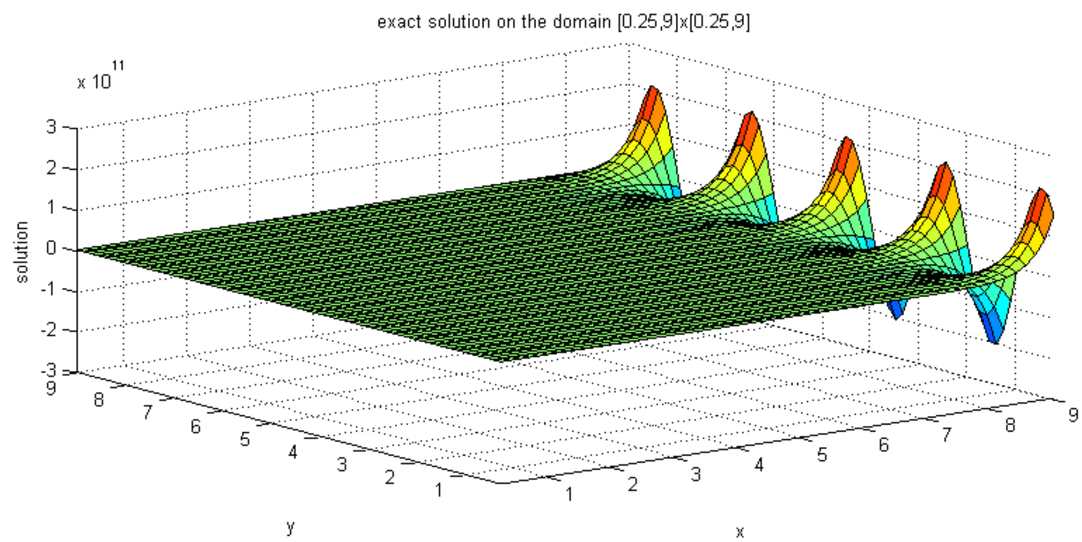


Fig 4.4 Exact solution graph on the domain [0.25,9]x[0.25,9]

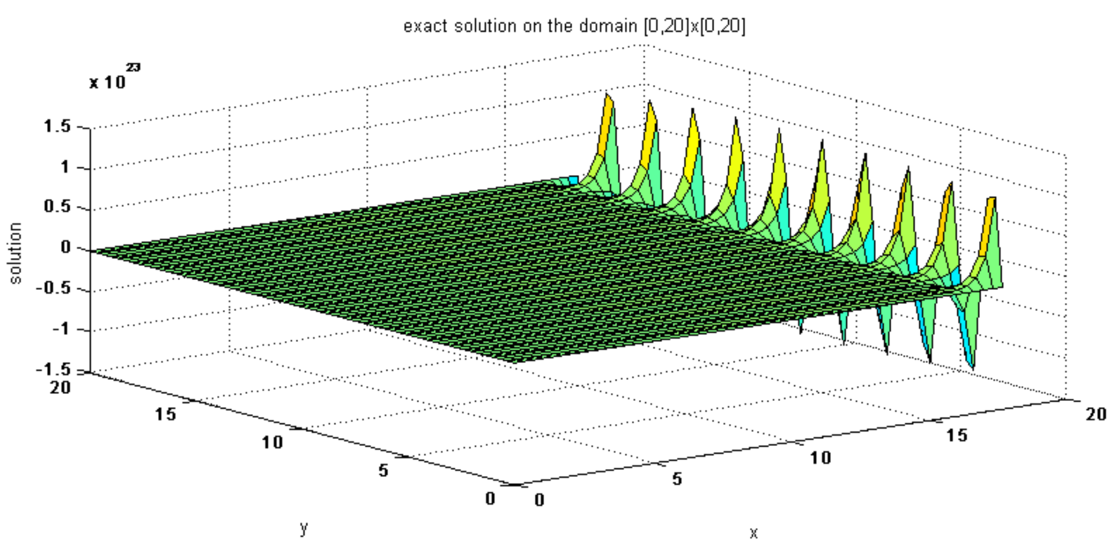


Fig 4.5 Exact solution graph on the domain [0,20]x[0,20]

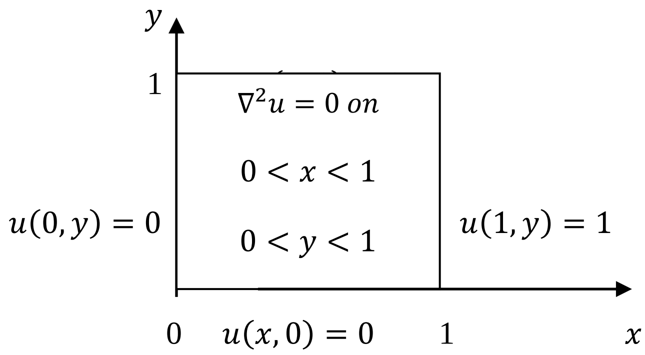
4.4.2 FDM

Now, recall that the two dimensional Laplace equation is given by

$u_{xx} + u_{yy} = 0 \ \ , u = u(x, y)$

where $x \ and \ y$ are the independent variables and u is the dependent one.

And consider the following domain with boundary conditions.



Step 1: Discretization of the given domain

Assume $h = \frac{1}{3} \ and \ k = \frac{1}{3}$.

Now

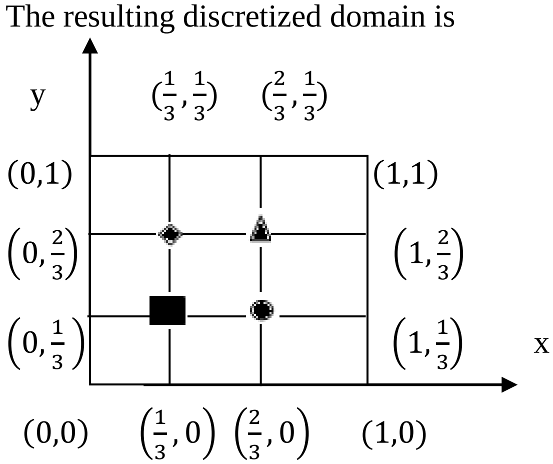
$x_0 = 0$ and $x_i = ih \ , \ for \ i = 0,1,2,3$

$y_0 = 0$ and $y_i = ik \ , \ for \ k = 0,1,2,3$

$\Rightarrow x_0 = 0, \ x_1 = \frac{1}{3}, \ x_2 = \frac{2}{3}, \ x_3 = 1$

$\Rightarrow y_0 = 0, \ y_1 = \frac{1}{3}, \ y_2 = \frac{2}{3}, \ y_3 = 1$

The resulting discretized domain is



where

■ $= (x_1, y_1) = \left(\frac{1}{3}, \frac{1}{3}\right)$ ◆ $= (x_1, y_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$

● $= (x_2, y_1) = \left(\frac{2}{3}, \frac{1}{3}\right)$ ▲ $= (x_2, y_2) = \left(\frac{2}{3}, \frac{2}{3}\right)$

Step 2: Substitution in $U_{xx} + U_{yy} = 0$

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = 0$$

Step 3: Formation of system of linear equations

Now from step 2 and the given values of $h\ and\ k$, we have the following.

Given $h = k$, for $i = 1,2,3\ and\ j = 1,2,3$

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2} = 0$$

$$\frac{1}{h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j} + U_{i,j+1} - 2U_{i,j} + U_{i,j-1}\right) = 0$$

$$U_{i+1,j} - 2U_{i,j} + U_{i-1,j} + U_{i,j+1} - 2U_{i,j} + U_{i,j-1} = 0$$

$$U_{i+1,j} - 4U_{i,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} = 0$$

$$4U_{i,j} = U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} \tag{4.32}$$

$$4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1} = 0 \tag{4.33}$$

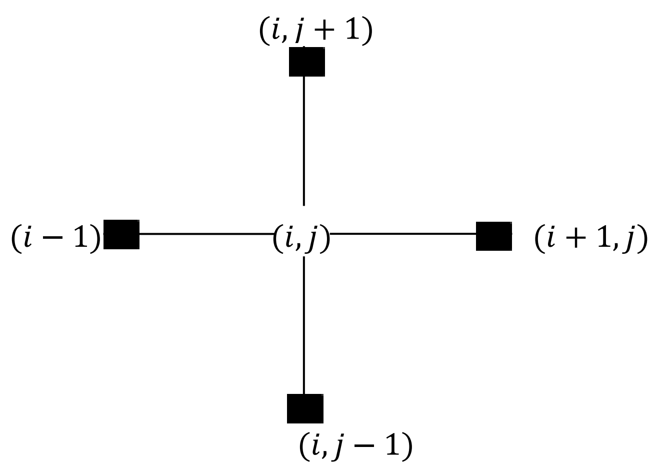


Fig 4.6 The five point stencil for the two dimensional Laplace equation

Now,

For $i = 1,\ j = 1,$

$$4U_{11} - U_{21} - U_{01} - U_{12} - U_{10} = 0 \tag{4.34}$$

For $i = 1,\ j = 2,$

$$4U_{12} - U_{22} - U_{02} - U_{13} - U_{11} = 0 \tag{4.35}$$

For $i = 2,\ j = 1,$

$$4U_{21} - U_{31} - U_{11} - U_{22} - U_{20} = 0 \tag{4.36}$$

For $i = 2,\ j = 2,$

$$4U_{22} - U_{32} - U_{12} - U_{23} - U_{21} = 0 \tag{4.37}$$

From the prescribed boundary conditions we get

$u_{00} = 0$	$u_{01} = 0$	$u_{02} = 0$	$u_{03} = 0$
$u_{10} = 0$			$u_{13} = 0$
$u_{20} = 0$			$u_{23} = 0$
$u_{30} = 0$	$u_{31} = 1$	$u_{32} = 1$	$u_{33} = 1$

Substituting the prescribed conditions into the above (4.34—4.37) gives,

$$4U_{11} - U_{21} - 0 - U_{12} - 0 = 0$$

$$\Rightarrow 4U_{11} - U_{21} - U_{12} = 0$$

$$4U_{12} - U_{22} - 0 - 0 - U_{11} = 0$$

$$\Rightarrow 4U_{12} - U_{22} - U_{11} = 0$$

$$4U_{21} - 1 - U_{11} - U_{22} - 0 = 0$$

$$\Rightarrow 4U_{21} - U_{11} - U_{22} = 1$$

$$4U_{22} - 1 - U_{12} - 0 - U_{21} = 0$$

$$\Rightarrow 4U_{22} - U_{12} - U_{21} = 1$$

Now rearranging gives the following system

$$\begin{cases} 4U_{11} - U_{12} - U_{21} + 0u_{22} = 0 \\ -U_{11} + 4U_{12} + 0u_{21} - U_{22} = 0 \\ -U_{11} + 0u_{12} + 4U_{21} - U_{22} = 1 \\ 0u_{11} - U_{12} - U_{21} + 4U_{22} = 1 \end{cases}$$

The above system can be written in the form

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Step 4: Solving the resulting system using LU decomposition.

By using LU decomposition, we get the following solution.

$$\begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{pmatrix} = \begin{pmatrix} 0.1250 \\ 0.1250 \\ 0.3750 \\ 0.3750 \end{pmatrix}$$

Case 2: assume $h = k = \frac{1}{8}$

Then

Step 1: Discretization of the given domain

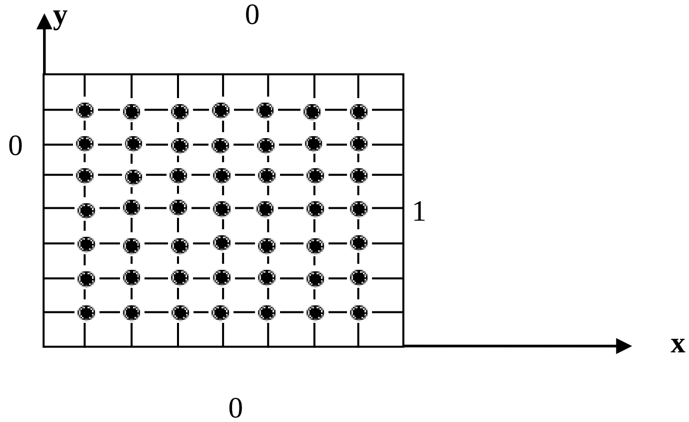
$h = \frac{1}{8}$ *and* $k = \frac{1}{8}$

Now

$x_0 = 0$ and $x_i = ih$, $for\ i = 0,1,2,\ldots,8$

$y_0 = 0$ and $y_i = ik$, $for\ k = 0,1,2,\ldots,8$

The resulting discretized domain is



Step 2: Substitution

$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = 0$$

Step 3: Formation of system of linear equations

It results a system of 49 linear equations since 49 internal nodes are considered.

$AX = B$. where A is a 49×49 matrix X *and* B are vectors of dimension 49.

A *and* B are found from the finite difference equations and boundary conditions respectively.

X is the column matrix containing the unknowns.

Then,

$$X = \begin{pmatrix} 0.0203 \\ 0.0380 \\ 0.0513 \\ 0.0592 \\ 0.0613 \\ 0.0580 \\ 0.0506 \\ 0.0431 \\ 0.0805 \\ 0.1081 \\ 0.1241 \\ 0.1281 \\ 0.1202 \\ 0.1012 \\ 0.0718 \\ 0.1326 \\ 0.1764 \\ 0.2011 \\ 0.2066 \\ 0.1936 \\ 0.1621 \\ 0.1113 \\ 0.2019 \\ 0.2639 \\ 0.2971 \\ 0.3037 \\ 0.2853 \\ 0.2424 \\ 0.1717 \\ 0.2997 \\ 0.3800 \\ 0.4199 \\ 0.4259 \\ 0.4015 \\ 0.3506 \\ 0.2758 \\ 0.4453 \\ 0.5366 \\ 0.5766 \\ 0.5783 \\ 0.5444 \\ 0.4826 \\ 0.4862 \\ 0.6689 \\ 0.7443 \\ 0.7718 \\ 0.7663 \\ 0.7150 \\ 0.5494 \end{pmatrix}$$

which is the solution at all the 49 internal nodes.

Solution graph

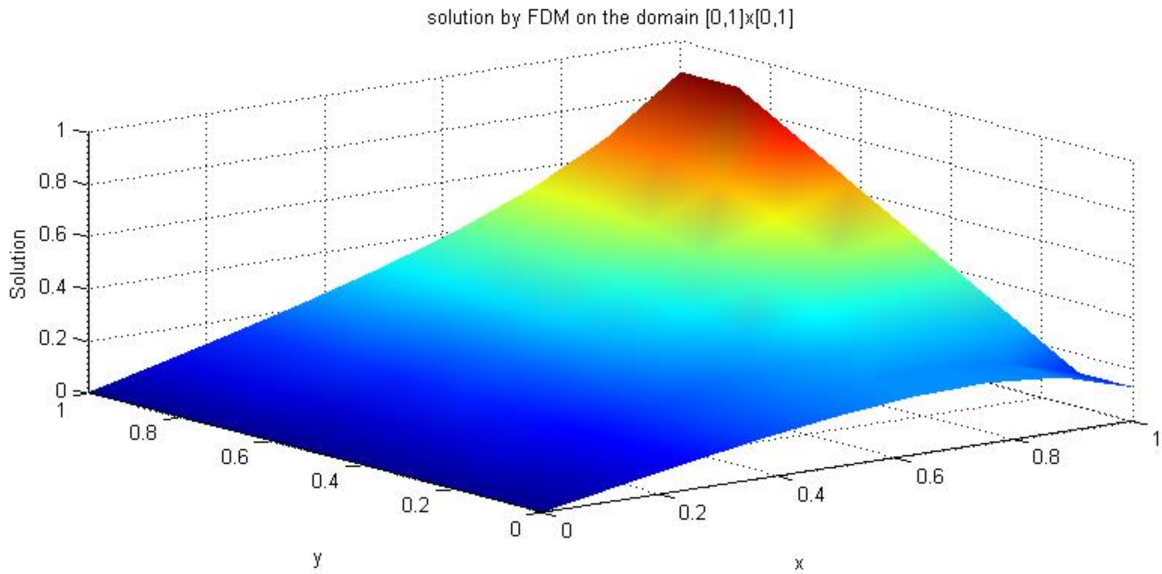


Fig 4.7 FDM solution graph on the domain [0,1]x[0,1]

Table 2 (Comparison of Finite Difference Method solution and exact solution) ($h = k = \frac{1}{8}$)

x	y	Approx. solution	Exact solution	Error	Local Truncation Error (LTE)
0.25	0.25	0.0805	0.1359	0.0554	0.0061555555556
0.25	0.5	0.1241	0.1908	0.0667	0.0074111111111
0.25	0.75	0.1936	0.1360	0.0576	0.0064
0.5	0.25	0.2019	0.3640	0.1621	0.0180111111111
0.5	0.5	0.2971	0.5000	0.2029	0.0225444444444
0.5	0.75	0.2853	0.3640	0.0787	0.0087444444444
0.75	0.25	0.4453	0.8640	0.4187	0.0465222222222
0.75	0.5	0.5766	1.0812	0.5046	0.056066666667
0.75	0.75	0.5444	0.8640	0.3196	0.035511111111

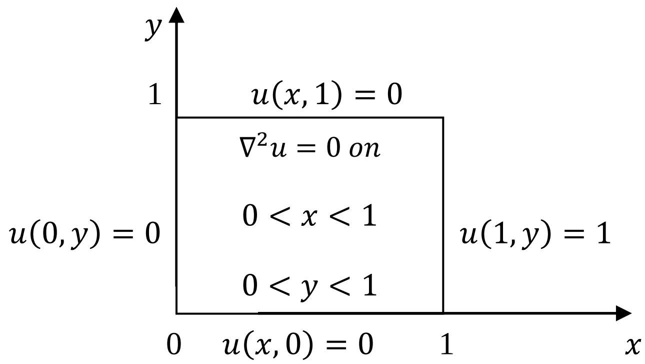
Note: $Error = |exact - approximate|$

$LTE = \frac{Error}{step\ size}$ where the step size is equal to nine.

Now, let's solve the same problem by BEM.

4.5.2 The Boundary Element Method for the 2D Laplace equation.

Consider the same domain as above.



Step 1: finding the corresponding fundamental equation.

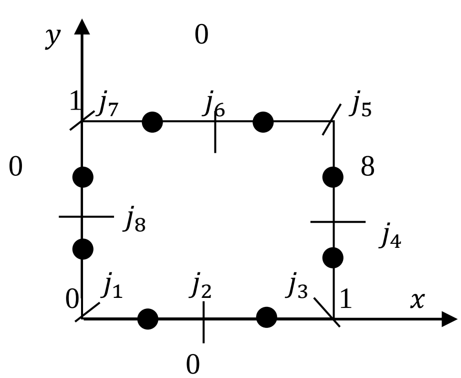
This is already done above. And the fundamental solution for the two dimensional Laplace equation is

$$v = \frac{1}{2\pi} lnr$$

Step 2: transforming the given differential equation into a set of BIEs (Boundary integral Equations) with the aid of the boundary conditions.

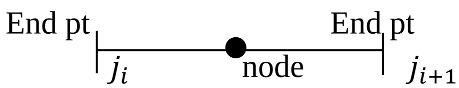
$$\frac{1}{2}u = \int_{\Gamma} \left(v * \frac{\partial u}{\partial n} - u * \frac{\partial v}{\partial n} \right) ds$$

Step 3: discretizing the boundary of the given domain.



$0 \le x \le 1 \qquad 0 \le y \le 1$

Here, we have 8 elements and 8 nodes.



This is the jth element. $j = 1,2,\ldots,8$. $(N = 8)$

For $j = 1$, the end points of the first element are $(0,0) and \left(\frac{1}{2},0\right). And\ u^1 = 0.$

For $j = 2$, the end points of the second element are $\left(\frac{1}{2},0\right) and\ (1,0). And\ u^2 = 0.$

For $j = 3$, the end points of the third element are $(1,0) and\ \left(1,\frac{1}{2}\right). And\ u^3 = 1.$

For $j = 4$, the end points of the fourth element are $\left(1,\frac{1}{2}\right)\ and\ (1,1). And\ u^4 = 1.$

For $j = 5$, the end points of the fifth element are $(1,1)\ and\ \left(\frac{1}{2},1\right). And\ u^5 = 0.$

For $j = 6$, the end points of the sixth element are $\left(\frac{1}{2},1\right)\ and\ (0,1). And\ u^6 = 0.$

For $j = 7$, the end points of the seventh element are $(0,1)\ and\ \left(0,\frac{1}{2}\right). And\ u^7 = 0.$

For $j = 8$, the end points of the eighth element are $\left(0,\frac{1}{2}\right)\ and\ (0,0). And\ u^8 = 0.$

Step 4: discretizing the already obtained BIEs for the discretized boundary.

$$\frac{1}{2}u^i = -\sum_{j=1}^{8}\int_{\Gamma_j} v(p_i,q)\frac{\partial u(q)}{\partial n_q}ds_q + \sum_{j=1}^{8}\int_{\Gamma_j} u(q)\frac{\partial v(p_i,q)}{\partial n_q}ds_q.$$

And resulting

$$\sum_{j=1}^{8}(H_{ij})u^j = \sum_{j=1}^{8}(G_{ij})u_n^j \tag{4.44}$$

Step 5: putting the discretized BIEs in matrix form.

Now,

(i) For $i \neq j$, $(i = 1,2,\ldots,8\ and\ j = 1,2,\ldots,8)$

$$G_{ij} = \int_{\Gamma_j} v\, ds = \int_{-1}^{1}\frac{1}{2\pi}\ln[r(\xi)]\ \frac{l_j}{2}\ d\xi = \frac{l_j}{4\pi}\sum_{k=1}^{n}\ln[r(\xi_k)]\,\varpi_k \tag{4.45}$$

where

$$r(\xi_k) = \sqrt{[x(\xi_k)-x_i]^2 + [y(\xi_k)-y_i]^2} \tag{4.46}$$

Notice that $dscos\phi = rd\alpha$

Then $\hat{H}_{ij} = \int_{\Gamma_j} \frac{\partial v}{\partial n} ds = \int_{\Gamma_j} \frac{1}{2\pi} \frac{cos\phi}{r} ds = \int_{\Gamma_j} \frac{1}{2\pi} da = \frac{a_{j+1} - a_j}{2\pi}$ (4.47)

The angles a_{j+1} and a_j are computed from

$$\tan a_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} \quad \text{and} \tag{4.48}$$

$$\tan a_j = \frac{y_j - y_i}{x_j - x_i} \tag{4.49}$$

where x_{j+1}, y_{j+1}, x_j and y_j are extreme points of the j-th element.

(ii) Diagonal elements, $i = j$ ($i = 1,2, \dots ,8$)

In this case the node p_i coincides with node p_j, and r lies on the element. Consequently, it is $\phi = \pi/2$ or $\phi = 3\pi/2$, which yields $cos\phi = 0$. Moreover, we have

$$x_{p_j} = \frac{x_{j+1} + x_j}{2}, \qquad y_{p_j} = \frac{y_{j+1} + y_j}{2}$$

And

$$r(\xi) = \sqrt{\left[x(\xi) - x_{p_j}\right]^2 + \left[y(\xi) - y_{p_j}\right]^2} = \frac{l_j}{2} \, |\xi| \tag{4.50}$$

Hence,

$$G_{jj} = \int_{\Gamma_j} \frac{1}{2\pi} lnr ds = 2 \int_0^{l_j/2} \frac{1}{2\pi} lnr dr$$

$$= \frac{1}{\pi} [rlnr - r]_0^{l_j/2} = \frac{1}{\pi} \frac{l_j}{2} [\ln\left(\frac{l_j}{2}\right) - 1] \tag{4.51}$$

and

$$\hat{H}_{jj} = \frac{1}{2\pi} \int_{\Gamma_j} \frac{cos\phi}{r} ds = \frac{1}{2\pi} \int_{-1}^{1} \frac{cos\phi}{|\xi|} d\xi$$

$$= \frac{2}{2\pi} [cos\phi * \ln|\xi|]_0^1 = 0 \tag{4.52}$$

We perform three-point Gaussian quadrature $(n = 3)$ to evaluate eq (4.45), where

$\xi_1 = -0.7746$ $\xi_2 = 0$ $\xi_3 = 0.7746$ $\omega_1 = 0.5556$ $\omega_2 = 0.8889$ and $\omega_2 = 0.5556$.

$$P_n(\xi) = \frac{1}{2^n n!} \frac{d^n (\xi^2 - 1)^n}{d\xi^n} \text{ and } \varpi_k = \frac{2(1 - \xi_k^2)}{n^2 (P_n(\xi_k))^2}$$

$P_3(\xi) = \frac{1}{2^3 3!}\frac{d^3(\xi^2-1)^3}{d\xi^3} = \frac{1}{48}\frac{d^2}{d\xi^2}(6\xi(\xi^2-1)^2) = \frac{1}{8}(\xi^2-1)^2 + \frac{1}{8}4\xi(\xi^2-1) = \frac{d}{d\xi}\left(\frac{1}{2}\xi(\xi^2-1) + \frac{1}{2}3\xi^2 - \frac{1}{2}\right) = \frac{1}{2}\xi^3 + \frac{1}{2}3\xi^2 - \frac{1}{2}\xi - \frac{1}{2} = \frac{1}{2}(\xi^3 + 3\xi^2 - \xi - 1)$

The matrices are formed by using Matlab.

Now,

$$G = \begin{pmatrix} -0.1899 & -0.0244 & 0.0031 & 0.0179 & 0.0179 & 0.0031 & -0.0244 & -0.1306 \\ -0.1306 & -0.1899 & -0.0447 & -0.0086 & 0.0031 & 0.0031 & -0.0086 & -0.0447 \\ -0.0244 & -0.1306 & -0.1899 & -0.0244 & 0.0031 & 0.0179 & 0.0179 & 0.0031 \\ -0.0086 & -0.0447 & -0.1306 & -0.1899 & -0.0447 & -0.0086 & 0.0031 & 0.0031 \\ 0.0179 & 0.0031 & -0.0244 & -0.1306 & -0.1899 & -0.0244 & 0.0031 & 0.0179 \\ 0.0031 & 0.0031 & -0.0086 & -0.0447 & -0.1306 & -0.1899 & -0.0447 & -0.0086 \\ 0.0031 & 0.0179 & 0.0179 & 0.0031 & -0.0244 & -0.1306 & -0.1899 & -0.0244 \\ -0.0447 & -0.0086 & 0.0031 & 0.0031 & -0.0086 & -0.0447 & -0.1306 & -0.1899 \end{pmatrix}$$

$$H = \begin{pmatrix} -0.5000 & 0 & 0.0738 & 0.1250 & 0.0512 & 0.0738 & 0 & 0 \\ 0.1250 & -0.5000 & 0.1250 & 0.0512 & 0.0738 & -0.4262 & 0.0512 & 0.0512 \\ 0 & 0.1250 & -0.5000 & 0 & -0.4262 & 0.0512 & 0.0512 & 0.0738 \\ 0.0512 & -0.3750 & 0.1250 & -0.5000 & -0.3750 & 0.0512 & 0.0738 & 0.0738 \\ 0.0512 & 0.0512 & 0 & 0 & -0.5000 & 0 & 0.0738 & 0 \\ -0.4262 & 0.0738 & 0.0512 & 0.1250 & 0.2500 & -0.5000 & 0.1250 & 0.0512 \\ 0.0738 & 0.0738 & 0.0512 & 0.0738 & 0.0738 & 0.2500 & -0.5000 & 0 \\ 0.1250 & 0.0512 & 0.0738 & 0.0738 & 0.0512 & 0.1250 & 0 & -0.5000 \end{pmatrix}$$

$U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ as given. (Dirichlet boundary condition)

Thus, $U_n = \begin{pmatrix} -0.5910 \\ -1.2698 \\ 3.5061 \\ 0.0377 \\ -0.4874 \\ -0.8252 \\ 0.2338 \\ -0.4673 \end{pmatrix}$

And therefore, the solution at any domain point can be found by applying

$u(p) = -\int_{\Gamma}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma$ where p is a domain point.

This can be done by dividing the boundary integral into the number of elements.

i.e. $u(p) = -\left[\int_{\Gamma_1}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_1 + \int_{\Gamma_2}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_2 + \cdots + \int_{\Gamma_8}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)d\Gamma_{\mathrm{N}}\right]$ where $N = 8$ is the number of elements.

E.g. To find the approximate solution at interior point say $(x, y) = (0.25,0.25)$,

$$u(0.25,0.25) = -\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2\pi}\ln(r_1) * U_{n_1} - U_1\frac{1}{2\pi r_1}\right)dx + \int_{\frac{1}{2}}^1\left(\frac{1}{2\pi}\ln(r_2) * U_{n_2} - U_2\frac{1}{2\pi r_2}\right)dx + \int_0^{\frac{1}{2}}\left(\frac{1}{2\pi}\ln(r_3) * U_{n_3} - U_3\frac{1}{2\pi r_3}\right)dx + \int_{\frac{1}{2}}^1\left(\frac{1}{2\pi}\ln(r_4) * U_{n_4} - U_4\frac{1}{2\pi r_4}\right)dx + \int_1^{\frac{1}{2}}\left(\frac{1}{2\pi}\ln(r_5) * U_{n_5} - U_5\frac{1}{2\pi r_5}\right)dx + \int_{\frac{1}{2}}^0\left(\frac{1}{2\pi}\ln(r_6) * U_{n_6} - U_6\frac{1}{2\pi r_6}\right)dx + \int_1^{\frac{1}{2}}\left(\frac{1}{2\pi}\ln(r_7) * U_{n_7} - U_7\frac{1}{2\pi r_7}\right)dx + \int_{\frac{1}{2}}^1\left(\frac{1}{2\pi}\ln(r_8) * U_{n_8} - U_8\frac{1}{2\pi r_8}\right)dx\right]$$

where

$$r_1 = d\left((0.25,0.25),\left(\frac{1}{4},0\right)\right) = \sqrt{(0.25-0.25)^2 + (0.25-0)^2}$$

$$r_2 = d\left((0.25,0.25),\left(\frac{3}{4},0\right)\right) = \sqrt{(0.25-0.75)^2 + (0.25-0)^2}$$

$$r_3 = d\left((0.25,0.25),(1,0.25)\right) = \sqrt{(0.25-1)^2 + (0.25-0.25)^2}$$

$$r_4 = d\left((0.25,0.25),\left(1,\frac{3}{4}\right)\right) = \sqrt{(0.25-1)^2 + (0.25-0.75)^2}$$

$$r_5 = d\left((0.25,0.25),\left(\frac{3}{4},1\right)\right) = \sqrt{(0.25-0.75)^2 + (0.25-1)^2}$$

$$r_6 = d\left((0.25,0.25),\left(\frac{1}{4},1\right)\right) = \sqrt{(0.25-0.25)^2 + (0.25-1)^2}$$

$$r_7 = d\left((0.25,0.25),\left(0,\frac{1}{4}\right)\right) = \sqrt{(0.25-0)^2 + (0.25-0.25)^2}$$

$$r_8 = d\left((0.25,0.25),\left(0,\frac{3}{4}\right)\right) = \sqrt{(0.25-0)^2 + (0.25-0.75)^2}$$

And

$$U=\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\\U_7\\U_8\end{pmatrix}=\begin{pmatrix}0\\0\\1\\1\\0\\0\\0\\0\end{pmatrix}\quad,\quad U_n=\begin{pmatrix}U_{n_1}\\U_{n_2}\\U_{n_3}\\U_{n_4}\\U_{n_5}\\U_{n_6}\\U_{n_7}\\U_{n_8}\end{pmatrix}=\begin{pmatrix}-0.5910\\-1.2698\\3.5061\\0.0377\\-0.4874\\-0.8252\\0.2338\\-0.4673\end{pmatrix}$$

Table 3 (Comparison of 8 nodes Boundary Element Method solution and exact solution)

x	y	**Approx. solution (8)**	**Exact solution**	**Error**	**Local truncation Error (LTE)**
0.25	0.25	0.0875	0.1359	0.0484	0.00605
0.25	0.5	0.1216908	0.1908	0.0691092	0.00863865
0.25	0.75	0.090769	0.1360	0.045231	0.005653875
0.5	0.25	0.289839	0.3640	0.074161	0.0092645125
0.5	0.5	0.204399	0.5000	0.295601	0.036950125
0.5	0.75	0.2013019	0.3640	0.1626981	0.0203372625
0.75	0.25	0.302469	0.8640	0.561531	0.070191375
0.75	0.5	0.5730164	1.0812	0.5081836	0.06352295
0.75	0.75	0.5026648	0.8640	0.3613352	0.0451669

(8) is to mean with 8 nodes.

Table 4 (Comparison of 8 nodes Boundary Element Method solution , FDM solution and exact solution)

x	y	**Approx. sol$^{\underline{n}}$ (FDM) (49)**	**Approx. sol$^{\underline{n}}$ (BEM) (8)**	**Exact Solution**
0.25	0.25	0.0805	0.0875	0.1359
0.25	0.5	0.1241	0.1216908	0.1908
0.25	0.75	0.1936	0.090769	0.1360
0.5	0.25	0.2019	0.289839	0.3640

0.5	0.5	0.2971	0.204399	0.5000
0.5	0.75	0.2853	0.2013019	0.3640
0.75	0.25	0.4453	0.302469	0.8640
0.75	0.5	0.5766	0.6030164	1.0812
0.75	0.75	0.5444	0.5026648	0.8640

(49) is to mean with 49 nodes.

Stability

The stability of the boundary element method for the given two dimensional Laplace equation can be checked by matrix eigen value analysis.

Recall the resulting system of linear equations written in matrix form results

$$G=\begin{pmatrix} -0.1899 & -0.0244 & 0.0031 & 0.0179 & 0.0179 & 0.0031 & -0.0244 & -0.1265 \\ -0.1265 & -0.1899 & -0.0447 & -0.0086 & 0.0031 & 0.0031 & -0.0086 & -0.0447 \\ -0.0244 & -0.1265 & -0.1899 & -0.0244 & 0.0031 & 0.0179 & 0.0179 & 0.0031 \\ -0.0086 & -0.0447 & -0.1265 & -0.1899 & -0.0447 & -0.0086 & 0.0031 & 0.0031 \\ 0.0179 & 0.0031 & -0.0244 & -0.1265 & -0.1899 & -0.0244 & 0.0031 & 0.0179 \\ 0.0031 & 0.0031 & -0.0086 & -0.0447 & -0.1265 & -0.1899 & -0.0447 & -0.0086 \\ 0.0031 & 0.0179 & 0.0179 & 0.0031 & -0.0244 & -0.1265 & -0.1899 & -0.0244 \\ -0.0447 & -0.0086 & 0.0031 & 0.0031 & -0.0086 & -0.0447 & -0.1265 & -0.1899 \end{pmatrix}$$

G is a square matrix of size 8×8.

Now, we use Matlab to find the eigen values of G.

The resulting eigen values are

$\lambda_1 = -0.3741 + 0.0000i$

$\lambda_2 = -0.3227 + 0.1086i$

$\lambda_3 = -0.3227 - 0.1086i$

$\lambda_4 = -0.1421 + 0.0819i$

$\lambda_5 = -0.1421 - 0.0819i$

$\lambda_6 = -0.0593 + 0.0000i$

$\lambda_7 = -0.0781 + 0.0451i$

$\lambda_8 = -0.0781 - 0.0451i$

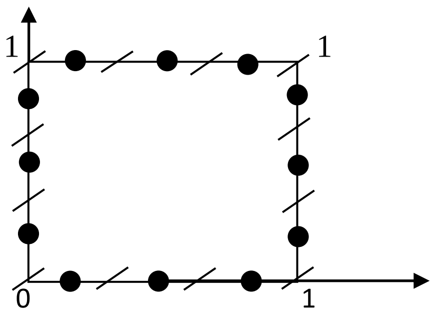
As seen above all the eigen values of G are complex numbers and the real part of all is less than zero.

$i.e. Re(\lambda_i) < 0 \; \forall \; i = 1,2, \dots ,8.$

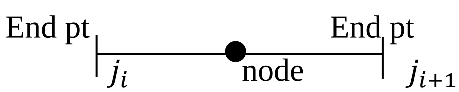
Hence by the matrix eigen value stability condition, it is stable, which implies the BEM for the given BVP is stable.

Case 2: Assume we have 12 elements and hence 12 nodes.

Now the discretization becomes



Here, we have 12 elements and 12 nodes.



. This is the jth element. $j = 1,2, \dots ,12.$ $(N = 12)$

For $j = 1$, the end points of the first element are $(0,0)$ and $\left(\frac{1}{3}, 0\right)$. $And\ u^1 = 0.$

For $j = 2$, the end points of the second element are $\left(\frac{1}{3}, 0\right)$ and $\left(\frac{2}{3}, 0\right)$. $And\ u^2 = 0.$

For $j = 3$, the end points of the third element are $\left(\frac{2}{3}, 0\right)$ and $(1,0)$. $And\ u^3 = 0.$

For $j = 4$, the end points of the fourth element are $(1,0)$ and $\left(1, \frac{1}{3}\right)$. $And\ u^4 = 1.$

For $j = 5$, the end points of the fifth element are $\left(1, \frac{1}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. $And\ u^5 = 1.$

For $j = 6$, the end points of the sixth element are $\left(1, \frac{2}{3}\right)$ and $(1,1)$. $And\ u^6 = 1.$

For $j = 7$, the end points of the seventh element are $(1,1)$ and $\left(\frac{2}{3}, 1\right)$. $And\ u^7 = 0.$

For $j = 8$, the end points of the eighth element are $\left(\frac{2}{3}, 1\right)$ and $\left(\frac{1}{3}, 1\right)$. $And\ u^8 = 0.$

For $j = 9$, the end points of the ninth element are $\left(\frac{1}{3}, 1\right)$ and $(0,1)$. $And\ u^9 = 0.$

For $j = 10$, the end points of the tenth element are $(0,1)$ and $\left(0,\frac{2}{3}\right)$. And $u^{10} = 0$.

For $j = 11$, the end points of the eleventh element are $\left(0,\frac{2}{3}\right)$ and $(0,\frac{1}{3})$. And $u^{11} = 0$.

For $j = 12$, the end points of the twelfth element are $\left(0,\frac{1}{3}\right)$ and $(0,0)$. And $u^{12} = 0$.

Now, the influence coefficient matrices are computed and are given as

$$G = \begin{pmatrix}
-0.1481 & -0.0378 & -0.0100 & 0.0010 & 0.0060 & 0.0140 & 0.0140 & 0.0060 & 0.0010 & -0.0100 & -0.0194 & -0.0096 \\
-0.1086 & -0.1481 & -0.0378 & -0.0194 & -0.0096 & 0.0034 & 0.0060 & 0.0010 & 0.0010 & -0.0060 & -0.0273 & -0.0513 \\
-0.0378 & -0.1086 & -0.1481 & -0.0513 & -0.0273 & -0.0060 & 0.0010 & 0.0010 & 0.0060 & 0.0034 & -0.0096 & -0.0194 \\
-0.0100 & -0.0378 & -0.1086 & -0.1481 & -0.0378 & -0.0100 & 0.0010 & 0.0060 & 0.0140 & 0.0140 & 0.0060 & 0.0010 \\
-0.0060 & -0.0273 & -0.0513 & -0.1086 & -0.1481 & -0.0378 & -0.0194 & -0.0096 & 0.0034 & 0.0060 & 0.0010 & 0.0010 \\
0.0034 & -0.0096 & -0.0194 & -0.0378 & -0.1086 & -0.1481 & -0.0513 & -0.0273 & -0.0060 & 0.0010 & 0.0010 & 0.0060 \\
0.0140 & 0.0060 & 0.0010 & -0.0100 & -0.0378 & -0.1086 & -0.1481 & -0.0378 & -0.0100 & 0.0010 & 0.0060 & 0.0140 \\
0.0060 & 0.0010 & 0.0010 & -0.0060 & -0.0273 & -0.0513 & -0.1086 & -0.1481 & -0.0378 & -0.0194 & -0.0096 & 0.0034 \\
0.0010 & 0.0010 & 0.0060 & 0.0034 & -0.0096 & -0.0194 & -0.0378 & -0.1086 & -0.1481 & -0.0513 & -0.0273 & -0.0060 \\
0.0010 & 0.0060 & 0.0140 & 0.0140 & 0.0060 & 0.0010 & -0.0100 & -0.0378 & -0.1086 & -0.1481 & -0.0378 & -0.0100 \\
-0.0194 & -0.0096 & 0.0034 & 0.0060 & 0.0010 & 0.0010 & -0.0060 & -0.0273 & -0.0513 & -0.1086 & -0.1481 & -0.0378 \\
-0.0513 & -0.0273 & -0.0060 & 0.0010 & 0.0010 & 0.0060 & 0.0034 & -0.0096 & -0.0194 & -0.0378 & -0.1086 & -0.1481
\end{pmatrix}$$

G is a 12×12 matrix.

And

$$H = \begin{pmatrix}
-0.5000 & 0 & 0 & 0 & 0.0198 & 0.0314 & 0.0314 & 0.0424 & 0.0512 & 0 & 0.0738 & 0.0512 \\
0.2500 & -0.5000 & 0 & 0.0738 & 0.0512 & 0.0314 & 0.0424 & 0.0512 & -0.4488 & 0.0226 & 0.0512 & 0.1250 \\
0 & 0.2500 & -0.5000 & 0.1250 & 0.0512 & 0.0226 & 0.0512 & -0.4488 & 0.0424 & 0.0314 & 0.0512 & 0.0738 \\
0 & 0 & 0.2500 & -0.5000 & 0 & 0 & -0.4488 & -0.4488 & 0.0314 & 0.0314 & 0.0424 & 0.0512 \\
0.0226 & 0.0226 & -0.3750 & 0 & -0.5000 & 0 & -0.4262 & 0 & 0.0314 & 0.0424 & 0.0512 & 0.0512 \\
0.0314 & 0.0512 & -0.4262 & 0 & 0 & -0.5000 & -0.3750 & 0.0512 & 0.0226 & 0.0512 & 0.0512 & 0.0424 \\
0.0314 & 0.0314 & -0.4488 & 0 & 0 & 0 & -0.5000 & 0 & 0 & 0.0512 & 0.0424 & 0.0314 \\
0.0424 & -0.4488 & 0.0512 & 0.0226 & 0.0512 & 0.1250 & 0.2500 & -0.5000 & 0 & 0.0738 & 0.0512 & 0.0314 \\
-0.4488 & 0.0512 & 0.0424 & 0.0314 & 0.0512 & 0.0738 & 0 & 0.2500 & -0.5000 & 0.1250 & 0.0512 & 0.0226 \\
0.0512 & 0.0424 & 0.0314 & 0.0314 & 0.0424 & 0.0512 & 0 & 0 & 0.2500 & -0.5000 & 0 & 0 \\
0.0738 & 0.0512 & 0.0314 & 0.0424 & 0.0512 & 0.0512 & 0.0226 & 0.0512 & 0.2500 & -0.5000 & 0 & 0 \\
0.1250 & 0.0512 & 0.0226 & 0.0512 & 0.0512 & 0.0424 & 0.0314 & 0.0512 & 0.2500 & -0.5000 & 0 & 0
\end{pmatrix}$$

H is also a 12×12 matrix.

Then, u_n can be calculated from $\sum_{j=1}^{12}\left(H_{ij}\right)u^j = \sum_{j=1}^{12}(G_{ij})u_n^j$

where $u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

And hence

$$u_n = \begin{pmatrix} 0.2595 \\ -1.1729 \\ -2.4578 \\ 5.0871 \\ 0.2394 \\ 3.3894 \\ -2.7879 \\ -0.6584 \\ -0.2419 \\ 0.0474 \\ -0.4069 \\ -0.2839 \end{pmatrix}$$

And therefore, the solution at any domain point can be found by applying

$u(p) = -\int_\Gamma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma$ where p is a domain point.

This can be done by dividing the boundary integral into the number of elements.

i.e. $u(p) = -\left[\int_{\Gamma_1} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma_1 + \int_{\Gamma_2} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma_2 + \cdots + \int_{\Gamma_{12}} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) d\Gamma_{12} \right]$
where 12 is the number of elements.

Table 5 (Comparison of Boundary Element Method solution for 12 nodes and exact solution)

x	y	BEM solution (12) 3pt Gauss	Exact solution	Error	Local Truncation Error (LTE)
0.25	0.25	0.2266	0.1359	0.09074	0.0075616667
0.25	0.5	0.34023	0.1908	0.14942	0.0124516667
0.25	0.75	0.3157	0.1360	0.17970	0.014975
0.5	0.25	0.3823	0.3640	0.01832	0.0015266667
0.5	0.5	0.5454	0.5000	0.0454	0.00378333333
0.5	0.75	0.6323	0.3640	0.2683	0.02235833333
0.75	0.25	0.7172	0.8640	0.1468	0.01223333333
0.75	0.5	0.8899	1.0812	0.1913	0.0159416667
0.75	0.75	1.0046	0.8640	0.1406	0.0117166667

(12) is to mean with 12 nodes.

Stability

The stability can be checked by matrix eigen value analysis.

The eigen values of G are

-0.3565 + 0.0000i

-0.3153 + 0.0685i

-0.3153 - 0.0685i

-0.1438 + 0.0576i

-0.1438 - 0.0576i

-0.1314 + 0.0000i

-0.0851 + 0.0476i

-0.0851 - 0.0476i

-0.0561 + 0.0305i

-0.0561 - 0.0305i

-0.0444 + 0.0106i

-0.0444 - 0.0106i

As seen above all the eigen values of G have a real part of all is less than zero.

$i.e. Re(\lambda_i) < 0 \;\forall\; i = 1,2,\dots,12.$

Hence by the matrix eigen value stability condition, it is stable, which implies the BEM for the given BVP is stable.

Table 6 (Comparison of Boundary Element Method solution for 12 nodes, 8 nodes and exact solution)

x	y	**Approx. solution (12)**	**Exact solution**	**Approx. solution (8)**	**Error (12)**	**Error (8)**
0.25	0.25	0.2266	0.1359	0.0875	0.09074	0.0484
0.25	0.5	0.34023	0.1908	0.1216908	0.14942	0.0691092
0.25	0.75	0.3157	0.1360	0.090769	0.17970	0.045231
0.5	0.25	0.3823	0.3640	0.289839	0.01832	0.074161
0.5	0.5	0.5454	0.5000	0.204399	0.0454	0.295601
0.5	0.75	0.6323	0.3640	0.2013019	0.2683	0.1626981
0.75	0.25	0.7172	0.8640	0.302469	0.1468	0.561531
0.75	0.5	0.8899	1.0812	0.5730164	0.1913	0.5081836
0.75	0.75	1.0046	0.8640	0.5026648	0.1406	0.3613352

And in addition let's use 4-point Gauss quadrature. And we apply the same process.

In the 4-point Gauss quadrature, the weights and the abscissas are given as

ξ_k	ϖ_k
± 0.339981	0.652145
± 0.861136	0.347855

For $\boldsymbol{k} = 1,2,3,4$.

And the solution is shown in table 7.

Table 7 (Comparison of Boundary Element Method solution for 12 nodes with 4-point Gauss quadraure and exact solution)

x	y	**BEM solution (12)**	**Exact solution**	**Error (12)**	**Local Truncation Error (LTE)**
0.25	0.25	0.1985	0.1359	0.0626	0.0052166667
0.25	0.5	0.2803	0.1908	0.0895	0.0745833333
0.25	0.75	0.2834	0.1360	0.1474	0.0122833333
0.5	0.25	0.3705	0.3640	0.0065	0.0005416667
0.5	0.5	0.5238	0.5000	0.0238	0.0019833333
0.5	0.75	0.4924	0.3640	0.1284	0.0107
0.75	0.25	0.7509	0.8640	0.1131	0.009425
0.75	0.5	0.9157	1.0812	0.1655	0.0137916667
0.75	0.75	0.7856	0.8640	0.0784	0.0065333333

Table 8 (Comparison of all solutions)

x	y	**Approx. sol$^{\underline{n}}$ (FDM) (49)**	**BEM sol$^{\underline{n}}$ (8)**	**Exact Solution**	**BEM solution (12) 4pt Gauss**	**BEM solution (12) 3pt Gauss**
0.25	0.25	0.0805	0.0875	0.1359	0.1985	0.2266
0.25	0.5	0.1241	0.1216908	0.1908	0.2803	0.34023
0.25	0.75	0.1936	0.090769	0.1360	0.2834	0.3157
0.5	0.25	0.2019	0.289839	0.3640	0.3705	0.3823
0.5	0.5	0.2971	0.204399	0.5000	0.5238	0.5454
0.5	0.75	0.2853	0.2013019	0.3640	0.4924	0.6323
0.75	0.25	0.4453	0.302469	0.8640	0.7509	0.7172
0.75	0.5	0.5766	0.6030164	1.0812	0.9157	0.8899
0.75	0.75	0.5444	0.5026648	0.8640	0.7856	1.0046

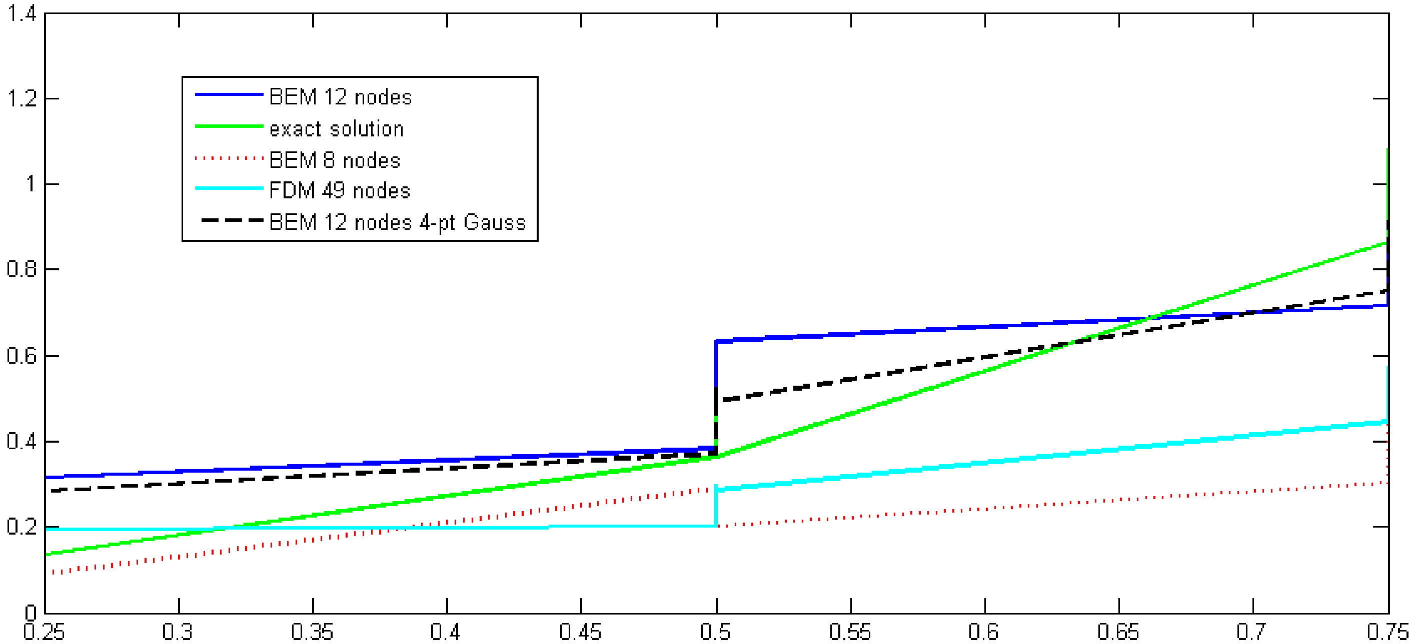


Fig 4.8 Comparison of solutions for every nodal point value of x.

4.5 Discussions

As seen from the graphs and tables, the finite difference scheme yields a maximum error of 0.3284, which is at the point (0.75,0.75), by using 49 nodes and the boundary element scheme gives a maximum error of 0.561531 at the point (0.75,0.5) just by using 8 nodes. From table 6 the BEM with 12 nodes has a maximum error of 0.2683 at the point (0.5, 0.75). And an error about 0.1655 at the point (0.75,0.5) with 12 nodes and a 4-point Gauss quadrature for the boundary element method. This shows how the boundary element method is very powerful.

As ascertained from fig 4.7 the graph of the Finite difference solution resembles to that of the graph of the exact solution. And from table 1, the local truncation error of the finite difference method approaches to zero and hence it is consistent. The stability is checked by the well known Von Neumann Analysis.

And in addition, as shown in fig 4.8, the boundary element method solution with 12 nodes and 4-point Gauss quadrature is in a good agreement with the exact solution than that of the BEM solutions with 12 nodes, 8 nodes and the finite difference method solutions with 49 nodes.

As the number of step size in the discretization process of the finite difference scheme increases the point wise error (error at the same domain point) decreases as compared to the exact solution.

As the number of elements increase in the boundary element method, the approximate solution approaches to the exact solution. In addition having a higher point Gauss quadrature will give better results.

And in addition as observed from the tables, the local truncation error is getting smaller and smaller which shows that the boundary element method is consistent.

Our given plate (or rectangular bar), which is considered as a two dimensional, is insulated on three of its sides and there is a constant heat distribution on the rest side of the plate as shown earlier. All partial derivatives of temperature *with respect to space* may either be zero or have nonzero values, but all derivatives of temperature at any point *with respect to time* are uniformly zero. Our solution is a temperature distribution inside the plate or rectangular bar (interior of the plate or the rectangular bar).The graphs and the tables show the temperature distribution inside the plate or (rectangular bar).

Chapter five

5.1 Conclusion

From observations of the results obtained by application of the finite difference method and boundary element method, to the boundary value problem with two dimensional Laplace equation as a governing equation, the boundary element method is very efficient as compared to the FDM. In contrast to the BEM, the finite difference scheme was so simplified to implement, as much work is demanded in implementation of the BEM.

The Laplace equation governs many real world problems like steady state heat conduction. And the study of these phenomena is studied in the so called potential theory. So as seen from the present work potential distribution exerted on domain can be successfully predicted through the presented methods. In addition, for linear and elliptic problems the boundary element method is superior over the finite difference method. The Laplace equation in any dimension is linear and elliptic. The FDM uses discretization of the whole domain but in the BEM the discretization is carried only on the boundary and this results in reduction of dimension which greatly minimizes the cost of computation. And to get the same level of accuracy problems the finite difference scheme needs many nodes as compared to the BEM. The BEM is used to find solutions at any domain point once the unknowns in the boundary are determined, but this cannot be done in the FDM. To accomplish this, change in the mesh grid must be done. The matrix resulting from the BEM scheme is fully populated and dense, but in that of the FDM the matrix is sparse tridiadonal and large. Nevertheless, this is not a serious disadvantage, because to obtain the same level of accuracy as the FDM solution, the BEM needs only a relatively modest number of nodes and elements. The BEM with 8 nodes 8 elements and 12 nodes 12 elements is stable for the posed problem. The stability is checked after formation of the influence coefficients matrix. And as seen in the discussion part it is consistent by the analysis of Local truncation error and thus, becomes convergent by the Lax- Richtmayer theorem. In general, we use Matrix eigen value analysis to check stability, local truncation error for consistency and the Lax- Richtmayer equivalence theorem for convergence for Boundary Element Method.

5.2 Recommendation

Analyzing the boundary element method for the three dimensional Laplace equation with different boundary conditions with real world applications and compare with other numerical methods.

References

Alexander H.-D. Cheng, Daisy T. Cheng, (2005). Heritage and early history of the boundary element method. Elsevier. pp. 268–302.

Green G. (1828). An Essay on the Application on Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham.

Hassan Ghassemi, Saeid Panahi, and Ahmad Reza Kohansal (2016). Solving the Laplace's Equation by the FDM and BEM Using Mixed Boundary Conditions. American Journal of Applied Mathematics and Statistics, vol. 4, no. 2: 37-42. doi: 10.12691/ajams-4-2-2

Jaswon M.A (1963). Integral Equation Methods in Potential Theory, I, Proceeding of the Royal Society, London, Vol.275 (A), pp.23-32.

Jaswon M.A and Ponter A.R (1963). An Integral Equation Solution of the torsion Problem, Proceeding of the Royal Society, London, Vol.275 (A), pp.237-246.

John Katsikadelis (2002). Boundary Elements: Theory and Applications: Elsevier.

Mark Davis (2010-11), Finite Difference Methods, Department of mathematics, Imperial College London, pp.14-15.

Randall J. LeVeque (2007). Finite Difference Methods for Ordinary and Partial Differential Equations, USA: Society for Industrial and Applied Mathematics, pp. 60-62.

Symm G.T (1963). Integral Equation Methods in Potential Theory, II, proceeding of the Royal Society, London, Vol.275 (A), pp.33-46.

Youssef F. Rashed (2001), The First Step to The Direct Integral Forms, Dept. of Structural Engineering, Cairo University, Giza Egypt, Boundary Element Communications.

Youssef F. Rashed (2001). An introduction to Boundary Element, Dept. of Structural Engineering, Cairo University, Giza Egypt, Boundary Element Communications.

Youssef F. Rashed (2002), Fundamental Solutions: I-Simple and Compound Operators, Dept. of Structural Engineering, Cairo University, Giza Egypt, Boundary Element Communications.