ROBUST BIPARTITE ENTANGLEMENT IN AN OPTOMECHANICAL SYSTEM ASSISTED BY THREE LEVEL LASER.

Trhas Mehari

A Thesis Submitted to Department of Applied Physics School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Physics (Quantum Optics)

Office of Graduate Studies

Adama Science and Technology University

December 19, 2022
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Declaration

I hereby declare that this Master Thesis entitled "Robust Bipartite Entanglement in an Optomechanical System Assisted by Three Level Laser"is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

______________________________- ______________- ____________-

Name of the Student Signature Date

Recommendation

I, the advisor of this thesis, hereby certify that I have read the revised version of the thesis entitled “Robust Bipartite Entanglement in an Optomechanical System Assisted by Three Level Laser ”prepared under my guidance by Trhas Mehari submitted in Partial Fulfillment of the Requirement for the Degree of Master’s in Applied Physics (Quantum Optics). Therefore, I recommend the submission of revised version of the thesis to the department following the applicable procedures.

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Name of Principal Advisor	Signature	Date

Approval Page of M.Sc. Thesis

I, the advisor of the thesis entitled "Robust Bipartite Entanglement in an Optomechanical System Assisted by Three Level Laser "and developed by Trhas Mehari, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

________________________________- _______________- _____________-

Advisor Signature Date

We, the undersigned, members of the Board of Examiners of the thesis by Trhas Mehari have read and evaluated the thesis entitled "Robust Bipartite Entanglement in an Optomechanical System Assisted by Three Level Laser"and examined the candidate during open defense. This is, therefore, to certify that the thesis is a Master's in Applied Physics (Quantum Optics).

Chairperson Signature Date

Internal Examiner Signature Date

External Examiner Signature Date

Finally, approval and acceptance of the thesis is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

Department Head	Signature	Date
School Dean	Signature	Date
Office of Postgraduate Studies, Dean	Signature	Date

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Thank you.

List of Abbreviations

CM Correlation Matrix.

CVS Continuous Variable System.

OC Optical Cavity.

PPT Positive Partial Transpose.

QC Quantum Correlation Measurement .

QIP Quantum Information Processes .

QLE Quantum Langevin Equation.

List of Figures

Abstract

In this thesis, we theoretically analyzed the bipartite entanglement of an optomechanical system for single mode light produced by three level lasers. To this aim, we constructed the model and Hamiltonian of system. While, the system consists of one fixed mirror and one movable mirror and a degenerate three level atoms placed between them. Thus, the cavity mode is driven by a laser field and the interaction of three level atoms and a laser field is analyzed. Accordingly, the dynamics of the system can be obtained by using the nonlinear quantum Langevin equations and linearization approximation. Under the linearization approximation, the bipartite entanglement is quantified through logarithmic negativity. Accordingly, our results showed that the introduction of three level atoms makes the entanglement more robust. Interestingly, we showed that the effects of atom-field coupling, the internal energy of atoms and atomic detuning have great contribution for the enhancement of the stationary continuous variable entanglement. Thus, when three level atoms are added inside a cavity, the optomechanical coupling and the bipartite entanglement is enhanced. Such entanglements results may have spectacular contribution for constructing long-distance quantum communication network and technologies.

Contents

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INTRODUCTION

1.1 Background of the Study

One of the fastest growing areas of quantum optics is quantum optomechanics. Optomechanics is the sub-field of physics involving the study of the interaction of electromagnetic radiation(photons) with mechanical systems via radiation pressure (Luan et al., 2014). Optomechanical system is a hybrid system where mechanical and optical degrees of freedom are mutually coupled (Restrepo et al., 2017). It is a new platform for studying quantum optics (Aspelmeyer et al., 2010). A simple example, a Fabry-Perot cavity consist of one mirror is fixed while the other one is free to move in a harmonic potential. The cavity modes of frequency is driven with a laser and the light inside the cavity will push the mirror due to radiation pressure (Tesfahannes, 2021). The quantum properties of light are largely determined by the quantum state of the light mode. Such as squeezing is one of the non-classical feature of light which is generated as a result of quantum optical process such as subharmonic generation and second harmonic generation. It has also been studied (Tesfa, 2006) that three-level cascade lasers can be considered as a source of squeezed light under certain conditions.

In such lasers, the squeezing is due to the atomic coherence which can be introduced either by initially preparing the three-level atoms in a coherent superposition of the top and bottom level or coupling these levels by external coherent light. When a three-level atom makes a transition from the top to bottom level through the intermediate level, two highly correlated photons are generated (Wei et al., 1998). If these photons are identical, the laser is referred to as a degenerate three-level laser, otherwise it is a non-degenerate three-level laser. Here in this thesis we use degenerate three-level laser (Aspelmeyer et al., 2014).

The most important quantum states of light which has the minimum uncertainty , plays great role in entanglement quantization between Optomechanics and subharmonic generating is Squeezed states of light (Schnabel, 2017). Squeezed light can be generated from light in a coherent state by means of certain nonlinear optical interactions or three-level cascade laser. Some of the quantum optical processes which can generate squeezed light is three-level cascade laser.

If they are correlated, then how to characterize the correlation, both qualitatively and quantitatively, becomes an essential task. Quantum Information Theory is the study of how such tasks can be accomplished using quantum mechanical systems (Groisman et al., 2005). Any bipartite quantum state may be used as a communication channel with some degree of success, and so it is of interest to determine how to separate the correlations it contains into a classical and an entangled part (Altintas and Eryigit, 2013). Furthermore, entanglement is a property of correlations between two or more quantum systems. These correlations defy classical description and are associated with intrinsically quantum phenomena.

For this reason, entanglement has played an important role in the development and testing of quantum theory. It is also a central element in quantum information (Barnett, 2009). Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently instead, a quantum state may be given for the system as a whole. Measurements of physical properties such as position, momentum, spin, polarization, etc. performed on entangled particles are found to be appropriately correlated. Thus, classical information and quantum information based on entangled quantum systems can be used for quantum communication purposes such as teleporting quantum states (Alber et al., 2003). The efficiency of quantum information processing highly depends on the degree of entanglement. Hence, it is desirable to generate strongly entangled continuous variable state. The idea of a nonlocal effect due to entanglement was pointed out by

Einstein, Podolsky, and Rosen;hence, the name EPR state (Einstein et al., 1935).

Notably, the first demonstration of entanglement between spatially distinct mechanical oscillators has been achieved using trapped ions which reported the observation of non-classical correlations between the motion of two separated pairs of ions (Häffner et al., 2005). One route to study motional entanglement for more massive systems is via cavity quantum optomechanics (Ockeloen-Korppi et al., 2018) , which utilizes radiation-pressure and other optical forces, such as electrostriction, to generate and study non-classical motional states of mechanical oscillators from the zeptogram to kilogram scale. The entanglement quantization of this system can be analyzed (measured) by so called logarithmic negativity.

The quantum Langevin equations (Ford et al., 1988) is used to describe the full dynamics of the system including the fluctuation and the dissipation term affecting the optomechanical system interact to the environment.

1.2 Statement of the Problem

Both the theoretical and experimental studies of entanglement have been achieved for the past decade (Dowling et al., 2006), (Friis et al., 2019). One route to study entanglement for more massive systems is via cavity quantum optomechanics, which utilizes radiation-pressure and other optical forces. The simple optomechanics consists of two mirrors, where one is fixed and the other is movable under harmonic oscillator. In previous researches, several authors have studied the quantum correlation property generated by a degenerate three-level cascade laser alone using different methods. In this thesis, we will study on how to generate the bipartite entanglement of quantum optomechanical system generated by a degenerate three-level cascade laser using optomechanics.Thus, we quantify their entanglement by using a method of logarithmic negativity.

1.3 Objectives of the Study

1.3.1 General Objective of the Study

The general objective of this study is to quantify the bipartite entanglement of an optomechanical system with single-mode light produced by degenerate three level laser.

1.3.2 Specific Objectives of the Study

The specific objectives of this thesis are:

- to formulate the model for the system
- to determine the quantum Langevin equations of the system
- to extract a drift matrix, i.e all information.
- to quantify the entanglement between subsystems by using a method of logarithmic negativity.

1.4 Significance of the Study

The main aim of quantum optomechanical system is to achieve quantum control of micromechanical and nanomechanical resonators using the electromagnetic field,which have a very advanced applications such as in teleportation and quantum computation. This can be achieved if we could transfer state from one location to another by a means of quantum entanglement. So,the results of this research will contribute for these applications that are used for technological advancements. In addition to that the results of this research will be used as input or guide line for other researchers or these will be served as platform for other scientific researchers.

1.5 Outline of the Thesis

In general, this thesis consists of five chapters. Each chapter will discuss on different issues related to the thesis. The first chapter describes an overview of the thesis background, statement of the problem, objective of the thesis and the significance of the study. Chapter two focuses on the review from previous studies that was made helpful to gain knowledge and to understand the thesis better and discusses about the optomechanics, a degenerate three-level cascade laser, quantum Langevin equations and logarithmic negativity. While chapter three gives detail description about the model Hamiltonian and dynamics of the system. Chapter four presents result and discussion. Finally, chapter five provides conclusion and recommendations of the thesis.

2

LITERATURE REVIEW

2.1 Overview of the Study

In this chapter, we describe a cavity optomechanical system, a degenerate three-level cascade laser, Hamiltonian formulation of optomechanical system, the Heisenberg quantum Langevin equation to determine the system dynamics and logarithmic negativity which is used as a tool to analyze the bipartite entanglement of an optomechanical system for single mode light produced by three level laser.

2.2 Cavity Optomechanical System

In the past few decades, the mechanical oscillator has been investigated with different science and technologies (Aspelmeyer et al., 2014). Its applications range from atomic force microscope cantilever for micro size to nano-mechanical oscillator mirror interferometer for the detection of gravitational waves, displacement etc. (Meystre, 2013; Kippenberg and Vahala, 2008). The quantum control of the mechanical motion is state of the art, which spans the size range from hundreds of nanometers in the case of nano-electro-mechanical or nano-opto-mechanical systems (NEMS/NOMS) to tens of centimeters in the case of gravitational wave antennae (Gebremariam et al., 2019).

The pioneering theoretical work of cavity optomechanical system was performed by Braginsky (Caves, 1979), Caves and others. The field has undergone rapid development over the last decade with the separate development of new methods for fabricating small bulk mechanical resonators of various forms; nano-scale beams coupled to microwave cavities (Milburn, 2011), photonic-phononic crystals (Merklein et al., 2017), toroidal optical microresonators (Heylman et al., 2016), doubly clamped beams with integrated mirrors (Pruessner et al., 2008) and drumhead capacitors in superconducting microwave resonators Milburn (2011).

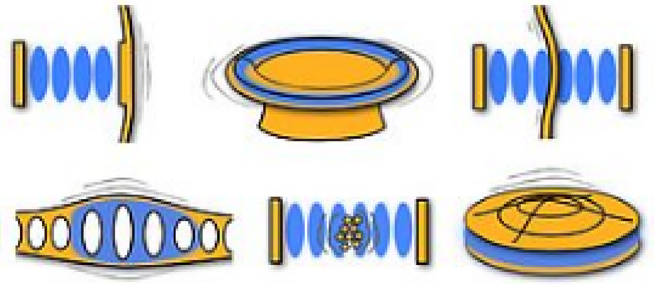

Fig. 2.1: Optomechanical Setups (schematic), from top left: Fabry-Perot cavity with a moveable end-mirror, microtoroid, membrane or nanotube inside a cavity, photonic crystal, atoms trapped inside a cavity, vibrating capacitor plates in a microwave LC circuit Marquardt and Girvin (2009).

In optomechanical systems, an optical field is used for the measurement and control of the mechanical resonator. The optical field is usually confined within a cavity, providing resonant enhancement of the field strength and sensitivity to mechanical displacements. The study of mechanical systems near the quantum limit encompasses systems having a wide range of sizes and frequencies. In order to study these mechanical resonators, some auxiliary system is required to measure and control them (Gröblacher, 2012). This auxiliary system may take the form of an electrical circuit (quantum electromechanics), an optical cavity (quantum optomechanics), or even an atomic system.

As mentioned above, optomechanics is one of the emerging fields exploring the interaction between light and mechanical motion. Such interaction originates from the mechanical effect of light, i.e., optical force. Radiation pressure force (or scattering force) and optical gradient force (or dipole force) are two typical categories of optical forces. The radiation pressure force which originates from the fact that light carries momentum plays an astonishing role in optomechanics (Ulbricht, 2021).

2.3 Degenerate three-level cascade laser

Quantum optics deals mainly with quantum properties of the light generated by various quantum optical system, such as; Laser and maser. A three-level cascade laser is a quantum optical system in which three level atoms in a cascade configuration. In this system the top, intermediate and bottom levels of a three level cascade atom can be denoted by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively.

A three-level laser is a source of coherent or chaotic light emitted by three-level atoms inside a cavity coupled to a vacuum reservoir. In the quantum theory of a laser, one usually takes into consideration the interaction of the atoms inside the cavity with the vacuum reservoir outside the cavity (Kassahun, 2008). Three-level lasers in which the crucial role is played by the coherent superpositions of top and bottom levels of the injected atoms have been studied by several authors. A three-level laser, in which the top and bottom levels of three-level atoms injected into a cavity and coupled by a strong coherent light can also generate light in a squeezed state. It appears to be quite difficult to prepare the atoms in a coherent superposition of the top and bottom levels before they are injected into the laser cavity. Moreover, it is certainly hard to find out that the atoms have decayed spontaneously before they are removed from the cavity.

Nanomechanical oscillators have recently become important candidates for exploring quantum features in macroscopics systems. In recent years, various schemes to generate bipartite entanglements have been proposed. Some researchers have proposed to entangle two mirrors in a cavity. The optomechanical entanglement between a cavity field and a vibrating end mirror has been investigated. The preparation and manipulation of quantum entanglement of mechanical oscillators are realized using optomechanical coupling via radiation pressure. Since the radiation pressure is proportional to the photon number, enhancing the photon number to increase the radiation pressure is necessary to obtain robust entanglement between a movable mirror and a cavity field. The atomic medium has recently been shown to play an important role in a cavity with a mechanical oscillator to enhance the entanglement.

2.4 Basic Tools and Techniques of Optomechanics

2.4.1 Hamiltonian Formulation of Standard Optomechanical System

The Hamiltonian of the quantum system plays a decisive role in the property-ascription rule that selects the definite-valued observables whose values become actual (Kong et al., 2017). It is the sum of the kinetic energies of all particles, plus the potential energy of the particles associated with the system.

The Hamiltonian takes different forms and can be simplified in some cases by taking into account the concrete characteristics of the system under analysis, such as single or several particles in the system, the interaction between particles, time-varying potential or time-independent one.

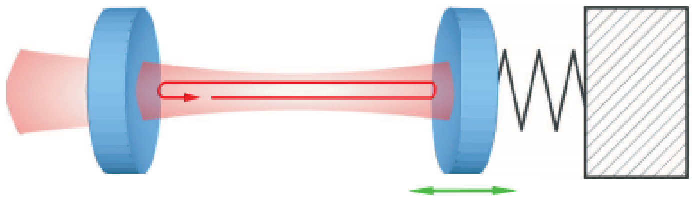

Fig. 2.2: Typical shape of cavity optomechanical system consisting of a Fabry Perot cavity with a harmonically bound end mirror (Kippenberg and Vahala, 2007).

The starting point of all our subsequent discussions will be the Hamiltonian describing the coupled system of a radiation mode interacting with a vibrational mode Fig. (2.2). The system with one movable mirror provides an additional mechanical degree of freedom. This system can be mathematically described by a single optical cavity mode coupled to a single mechanical mode. The coupling originates from the radiation pressure of the light field that eventually moves the mirror, which changes the cavity length and resonance frequency. The optical mode is driven by an external laser.

Starting from the derivation of quantum harmonic oscillator (Deniz, 2019) we can get standard Hamiltonian of optomechanical system as starting point for our system. Then, the free Hamiltonian (the sum of the kinetic energies of all particles, plus the potential energy of the particles associated with the system) is

$$\hat{H} = \hat{P}_E + \hat{K}_E \tag{2.1}$$

where $\hat{P}_E = \frac{1}{2}kx^2$ is potential energy and $\hat{K}_E = \frac{1}{2}mv^2$ is kinetic energy.
Here $\omega = \sqrt{\frac{k}{m}}$ then, $k = \omega^2 m$ and from $\hat{p} = m\hat{v}$, $\hat{v} = \frac{\hat{p}}{m}$, where k is force constant, v is speed of a particle, $\hat{x}$ is the position (displacement) operator of particle and $\hat{p}$ is the momentum operator of the particle. Then, the Hamiltonian of Eq. (2.1) becomes

$$\hat{H} = \frac{1}{2m}\left[m^2\omega^2\hat{x}^2 + \hat{p}^2\right]. \tag{2.2}$$

Now, let by trick

$$\begin{aligned}
(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) &= m^2\omega^2\hat{x}^2 + im\omega\hat{x}\hat{p} - im\omega\hat{p}\hat{x} + \hat{p}^2 \\
&= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw(\hat{x}\hat{p} - \hat{p}\hat{x}) \\
&= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw[\hat{x}, \hat{p}] \\
&= m^2\omega^2\hat{x}^2 + \hat{p}^2 + imw(i\hbar) \\
&= m^2\omega^2\hat{x}^2 + \hat{p}^2 - mw\hbar
\end{aligned}$$

$$\Rightarrow m^2\omega^2\hat{x}^2 + \hat{p}^2 = (mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) + m\omega\hbar. \tag{2.3}$$

Inserting Eq. (2.3) into Eq. (2.2) we get

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\left[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p}) + m\omega\hbar\right] \\
&= \frac{1}{2m}\left[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\right] + \frac{\omega\hbar}{2} \\
&= \frac{\omega\hbar}{\omega\hbar}\frac{1}{2m}\left[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\right] + \frac{\omega\hbar}{2} \\
&= \frac{\omega\hbar}{2m\omega\hbar}\left[(mw\hat{x} - i\hat{p})(mw\hat{x} + i\hat{p})\right] + \frac{\omega\hbar}{2} \\
&= \omega\hbar\left[\left(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}}\right)\left(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}}\right)\right] + \frac{\omega\hbar}{2}
\end{aligned}$$

Therefore,

$$\hat{H} = \omega\hbar\Big[(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}})(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}}) + \frac{1}{2}\Big]. \tag{2.4}$$

From Eq. (2.4) the terms

$$(\frac{mw\hat{x} - i\hat{p}}{\sqrt{2m\omega\hbar}}) = \hat{a}^\dagger,$$

which is called creation operator and

$$(\frac{mw\hat{x} + i\hat{p}}{\sqrt{2m\omega\hbar}}) = \hat{a}$$

is known as annihilation operator. Then, the harmonic Hamiltonian is written as

$$\hat{H} = \Big[\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big]\omega\hbar. \tag{2.5}$$

The corresponding commutation relations are $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$. The offset term, in Eq. (2.5) corresponding to the zero-point energy, makes no contribution to the Hamiltonian dynamics, and hence is often omitted. We have neglected the $\frac{1}{2}\omega\hbar$, since this will not affect our results. Thus, the Hamiltonian of optical field reads as (Law, 1995)

$$\hat{H}_{om} = \hbar\omega_c(x)\hat{a}^\dagger\hat{a} \tag{2.6}$$

where ω_c is the frequency of the optical mode and x is the position of the mechanical resonator.

In the same manner we can determine the Hamiltonian of mechanical resonator as

$$\hat{H}_{mech} = \hbar\omega_m\hat{b}^\dagger\hat{b}, \tag{2.7}$$

where $\hat{b}$ and $\hat{b}^\dagger$ are annihilation and creation operator of mechanical resonators respectively and ω_m is mechanical mode frequency. They are defined as

$$\hat{b} = (\frac{m\omega_m\hat{x} + i\hat{p}}{\sqrt{2m\omega_m\hbar}}) \tag{2.8}$$

and

$$\hat{b}^\dagger = (\frac{m\omega_m\hat{x} - i\hat{p}}{\sqrt{2m\omega_m\hbar}}) \tag{2.9}$$

and the corresponding commutation relations are $[\hat{b}, \hat{b}^{\dagger}] = 1$ and $[\hat{b}, \hat{b}] = [\hat{b}^{\dagger}, \hat{b}^{\dagger}] = 0$.

Meanwhile, we need to see how the two systems communicate to one another. Because the optical cavity and mechanical oscillator are coupled by radiation pressure and by the mechanical oscillator physically changing the length of the optical cavity. We note that $\omega_c = \frac{2\pi nc}{L}$ where $n = 1, 2, 3, ...$ and when the mirror moves by a distance x, the length of the optical cavity increases by $L + x$. Then,

$$\omega_c(x) = \frac{2\pi nc}{L + x}. \tag{2.10}$$

By Taylor expansion Eq. (2.10) becomes

$$\begin{aligned}\omega_c(x) &= 2\pi nc\left[\frac{1}{L(1 + \frac{x}{L})}\right] \\ &= \frac{2\pi nc}{L}\left[1 - \frac{x}{L} + \frac{x^2}{L^2} - \frac{x^3}{L^3} + ...\right].\end{aligned}$$

Since $x \ll L$, by taking the first order and rejecting the other terms we get

$$\omega_c(x) = \frac{2\pi nc}{L}\left[1 - \frac{x}{L}\right] \tag{2.11}$$

$$\omega_c(x) = \omega_c - \frac{\omega_c x}{L}. \tag{2.12}$$

After solving Eq. (2.8) and Eq. (2.9) simultaneously we get

$$\hat{x} = \frac{\hat{b} + \hat{b}^{\dagger}}{\sqrt{2\hbar/m\omega_m}}. \tag{2.13}$$

Then, by replacing x by operator $\hat{x}$ and inserting Eq. (2.13) into Eq. (2.12) we get

$$\omega_c(x) = \omega_c - \frac{\omega_c}{\sqrt{2}L}x_{zpt}(\hat{b} + \hat{b}^{\dagger}) \tag{2.14}$$

where $x_{zpt} = \sqrt{\frac{\hbar}{m\omega_m}}$ is the size of mechanical zero point fluctuation or width of the mechanical ground state wave function.

From Eq. (2.14) let's define the term $g = \frac{\omega_c}{\sqrt{2}L} x_{zpt}$ which is called single photon optomechanical coupling interaction (strength). It determines the amount of cavity resonance frequency shift if the mechanical oscillator is displaced by zero point uncertainty. Then, Eq. (2.14) becomes

$$\omega_c(x) = \omega_c - g(\hat{b} + \hat{b}^\dagger). \tag{2.15}$$

Finally, inserting Eq. (2.15) into Eq. (2.6) we can extract the optomechanical interaction Hamiltonian as follows,

$$\hat{H}_{opt} = \hbar \left[\omega_c - g(\hat{b} + \hat{b}^\dagger) \right] \hat{a}^\dagger \hat{a} \tag{2.16}$$

$$\hat{H}_{opt} = \hbar\omega_c \hat{a}^\dagger \hat{a} - \hbar g(\hat{b} + \hat{b}^\dagger)\hat{a}^\dagger \hat{a}. \tag{2.17}$$

Then, from Eq. (2.7) and Eq. (2.17) we get

$$\hat{H} = \hbar\omega_c \hat{a}^\dagger \hat{a} + \hbar\omega_m \hat{b}^\dagger \hat{b} - \hbar g(\hat{b} + \hat{b}^\dagger)\hat{a}^\dagger \hat{a}. \tag{2.18}$$

Equation (2.18) serve as standard optomechanical Hamiltonian of the optical and mechanical mode with the optomechanical interaction between the optical mode and the mechanical mode with out driving field for any optomechanical systems.

2.4.2 Quantum Langevin Equations

A Langevin equation is a stochastic differential equation describing the time evolution of a subset of degrees of freedom. These degrees of freedom typically are collective (macroscopic) variables changing only in comparison to the other (microscopic) variables of the system. The fast (microscopic) variables are responsible for the stochastic nature of the Langevin equation (Ford and Kac, 1987).

The quantum Langevin equations are used to describe the full dynamics of the system including the fluctuation and the dissipation term affecting the optomechanical system interact with the environment. It interacts in the external environment have corresponding decay or damping rate β and input noise $\hat{z}_{in}$ respectively. The general form of quantum Langevin equation in the optomechanics for any operator $\hat{z}$ can be written as

$$\frac{d\hat{z}}{dt} = \frac{-i}{\hbar}[\hat{z}, \hat{H}] - \beta\hat{z} + \sqrt{2\beta}\hat{z}_{in} \tag{2.19}$$

where $\hat{H}$ is the total Hamiltonian of the system.

2.4.3 The Linearization Approximation Approach

The quantum dynamics of the full non-linear system are difficult to analyze. So we linearize the dynamics of the quantum fluctuations around the semi-classical fixed points where the cavity is intensely driven under a strong field. We expand each Heisenberg operators as sums of its semi-classical steady-state value plus quantum fluctuation for any relevant operators (Tesfahannes and Alemu, 2021). The general form of linearization approach for any operator $\hat{o}$ given by Aspelmeyer et al. (2014)

$$\hat{o} = \hat{O}_s + \delta\hat{o}, \tag{2.20}$$

where $\hat{o}$ is an operator $\hat{O}_s$ is a real numbers denoting the optical and mechanical amplitudes and $\delta\hat{o}$ is the quantum fluctuation operator.

2.4.4 Logarithmic Negativity

The logarithmic negativity is an entanglement measure which is easily computable and an upper bound to the distillable entanglement (Plenio, 2005).
It is defined as

$$E_N = max[0, -ln2\bar{\eta}], \tag{2.21}$$

where $\bar{\eta}$ is the minimum symplectic eigenvalue of partial transposed cavity mode corresponding to the bipartite system of interest which is given by

$$\bar{\eta} = \sqrt{\frac{[\sum(V) - \sqrt{\sum(V)^2 - 4detV}]}{2}} \tag{2.22}$$

where F can be expressed in a form of block matrix as

$$V = \begin{pmatrix} F_A & F_C \\ F_C^T & F_B \end{pmatrix} \tag{2.23}$$

and $\sum(V) = detF_A + detF_B - 2detF_C$ where F_A, F_C and F_B are 2×2 are matrices extracted from the correlation matrix of the system. Thus, E_N is entangled if $detF_C < 0$ and $\bar{\eta} \leq 1$. Based on those frame work we can analyze the bipartite entanglement of an optomechanical system for single mode light produced by three level laser.

3

METHODOLOGY

3.1 Overview of the Study

In this chapter, we investigate theoretically bipartite entanglement of an optomechanical system for single mode light produced by three level laser. The dynamics of the system in terms of quantum Langevin equation for relevant operators are studied and using logarithmic negativity to quantify the generated and enhanced entanglement of an optomechanical system with three level laser.

3.2 Theoretical Model and Hamiltonian Formulation

We consider mechanical resonator can be described by a harmonic oscillator of frequency ω_m and decay rate γ which is one side coupled to the optical cavity with resonant frequency ω_m decay rate κ via radiation pressure. The atomic medium with three-level atoms is injected into the cavity and interacts with the cavity field. Such model of optomechanical system via three level atoms shown schematically in Fig. (3.1). One cavity mirror is fixed and the other one is movable, which is modeled as a quantum mechanical harmonic oscillator with effective mass m and frequency ω_m.

The total Hamiltonian of the whole system is written as (Huang and Agarwal, 2009)

$$\begin{aligned}\hat{H} =&\omega_c\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger e^{-i\omega_L t} - \hat{a}e^{i\omega_L t})\\ &+ \sum_{i=e,g,l}\hbar E_i\sigma_{ii} + \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + H.c)\end{aligned} \tag{3.1}$$

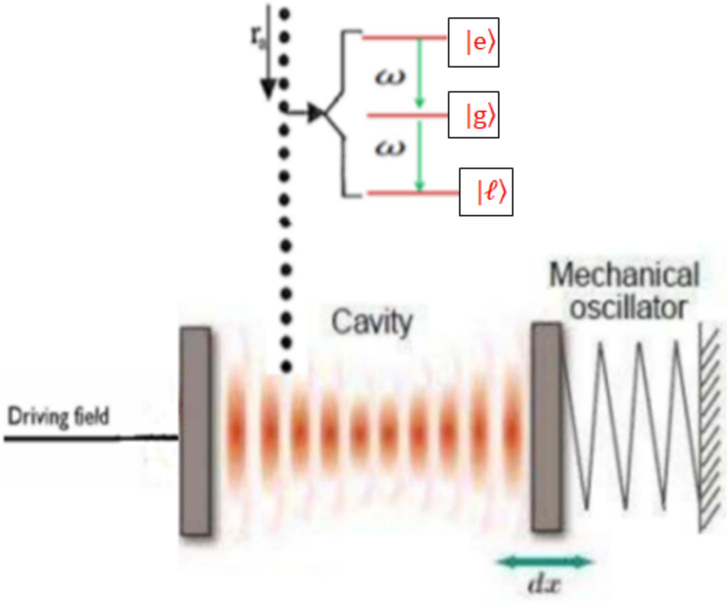

Fig. 3.1: Schematic diagram of an optomechanical system with degenerate three-level atoms injected into optomechanical cavity and interacts with cavity field.

The first term describes the energy of the optical cavity. The second term denotes the energy of the mechanical resonator interns of dimensionless position and momentum operators p and q, respectively. The third term gives the radiation pressure coupling with the coupling rate of Ω, the fourth term describes the input laser, the fifth and six terms present the energies of the atoms with the energy $E_i(i = e, g, l)$ in the level i. The pauli matrix $\sigma_{ii} = |i\rangle\langle i|$ denotes the internal energy of the atom, and the last term refers to the interaction between the atom and the cavity field with the coupling constant G (Hohenester, 2020). While $\sigma_{ge} = |g\rangle\langle e|$, $\sigma_{eg} = |e\rangle\langle g|$, $\sigma_{lg} = |l\rangle\langle g|$ and $\sigma_{gl} = |g\rangle\langle l|$ denote the flip-up and flip-down operators of the atoms.

3.3 Dynamics of the System

The dynamics of the optomechanical system assisted by three-level atoms can be described by a set of nonlinear quantum langevin equations in which the dissipation and fluctuation terms are added to the Heisenberg equations of motion as shown in Eq. (2.7). To obtain the nonlinear QLE for our system operators $\hat{q}$, $\hat{p}$, $\hat{a}$, σ_{ge} and σ_{lg} we have to apply a unitary transformation and rotating wave approximation to Eq. (3.1).

Then, the effective Hamiltonian in the interaction picture is

$$\begin{aligned}\hat{H} =&\Delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) + \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})\\ &+ \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}\end{aligned} \tag{3.2}$$

where, $\Delta = \omega_c - \omega_L$ is the optical detuning, $\beta_1 = E_e - E_g - \omega_L$, $\beta_2 = E_g - E_l - \omega_L$, thus, $\beta_1 - \Delta = \omega_c - \omega_L$ and $\beta_2 - \Delta = \omega_c - \omega_L$ represents the detuning between the classical driving fields and the cavity field. While, Ω is the single optomechanical coupling strength between the cavity field and mechanical mirror, ε is amplitude of the field driving laser and the cavity field (input field intensity related to the input power $\varepsilon = \sqrt{\frac{2\kappa P}{\hbar\omega_L}}$, where $\kappa = \frac{\pi c}{2FL}$ is optical decay rate with F is the cavity finesse and L is cavity length).

The nonlinear QLE for this system can be calculated for $\dot{\hat{q}}$, $\dot{\hat{p}}$, $\dot{\hat{a}}$ $\dot{\sigma_{ge}}$ and $\dot{\sigma_{lg}}$ by inserting Eq. (3.2) into Eq. (2.7).

For $\dot{\hat{q}}$,

$$\frac{d\hat{q}}{dt} = \dot{\hat{q}} = \frac{-i}{\hbar}[\hat{q}, \hat{H}] - \beta\hat{q} + \sqrt{2\beta}\hat{z}_{in}. \tag{3.3}$$

Then, the commutation relation for Eq. (3.2) becomes

$$\begin{aligned}[\hat{q}, \hat{H}] =&[\hat{q}, \Delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})\\ &+ \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl}) + \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}]\end{aligned} \tag{3.4}$$

$$\begin{aligned}[\hat{q}, \hat{H}] =&[\hat{q}, \Delta\hat{a}^\dagger\hat{a}] + [\hat{q}, \frac{\hbar\omega_m}{2}(\hat{p}^2+\hat{q}^2) + [\hat{q}, \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q}] + [\hat{q}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})]\\ &+ [\hat{q}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] + [\hat{q}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}].\end{aligned} \tag{3.5}$$

Now, lets solve Eq.(3.5) term by term by utilizing the following commutation relation

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C} \tag{3.6}$$

and

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}. \tag{3.7}$$

Then,

$$[\hat{q}, \hat{a}^\dagger\hat{a}] = 0, \tag{3.8}$$

$$[\hat{q}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = i\hbar^2\omega_m\hat{p}, \tag{3.9}$$

$$[\hat{q}, \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q}] = 0, \tag{3.10}$$

$$[\hat{q}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})] = 0, \tag{3.11}$$

$$\hat{q}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] = 0 \tag{3.12}$$

and

$$[\hat{q}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}] = 0 \tag{3.13}$$

Finally, by inserting Eq. (3.8), Eq. (3.9), Eq. (3.10), Eq. (3.11),Eq. (3.12) and Eq. (3.13) into Eq. (3.11), then Eq. (3.3) becomes

$$\frac{d\hat{q}}{dt} = \dot{\hat{q}} = \hbar\omega_m\hat{p}. \tag{3.14}$$

For $\hat{p}$,

$$\frac{d\hat{p}}{dt} = \dot{\hat{p}} = \frac{-i}{\hbar}[\hat{p}, \hat{H}] - \beta\hat{p} + \sqrt{2\beta}\hat{z}_{in}. \tag{3.15}$$

Then,

$$\begin{aligned}[\hat{p}, \hat{H}] =& [\hat{p}, \Delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) \\ &+ \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl}) + \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}],\end{aligned} \tag{3.16}$$

Then, in the same manner

$$[\hat{p}, \hat{a}^\dagger\hat{a}] = 0, \tag{3.17}$$

$$[\hat{p}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = i\hbar^2\omega_m\hat{q}, \tag{3.18}$$

$$[\hat{p}, \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q}] = -i\hbar^2\Omega\hat{a}^\dagger\hat{a}, \tag{3.19}$$

$$[\hat{p}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})] = 0, \tag{3.20}$$

$$[\hat{p}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] = 0 \tag{3.21}$$

and

$$[\hat{p}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}] = 0 \tag{3.22}$$

Finally, by inserting Eq. (3.17), Eq. (3.18), Eq. (3.19), Eq. (3.20),Eq. (3.21) and Eq. (3.22) into Eq. (3.16), then Eq. (3.15) becomes

$$\frac{d\hat{p}}{dt} = \dot{\hat{p}} = -\hbar\omega_m\hat{q} + \hbar g\hat{a}^\dagger\hat{a} - \gamma_m\hat{p} + \xi. \tag{3.23}$$

For $\hat{a}$,

$$\frac{d\hat{a}}{dt} = \dot{\hat{a}} = \frac{-i}{\hbar}[\hat{a}, \hat{H}] - \kappa\hat{a} + \sqrt{2\kappa}\hat{a}_{in} \tag{3.24}$$

$$\begin{aligned}[\hat{a}, \hat{H}] =& [\hat{a}, \Delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) \\ &+ \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl}) + \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}],\end{aligned} \tag{3.25}$$

Then, in the same manner

$$[\hat{a}, \hat{a}^\dagger\hat{a}] = \hbar\Delta\hat{a}, \tag{3.26}$$

$$[\hat{a}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = 0, \tag{3.27}$$

$$[\hat{a}, \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q}] = \hbar\Omega\hat{a}\hat{q}, \tag{3.28}$$

$$[\hat{a}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})] = i\hbar\varepsilon, \tag{3.29}$$

$$[\hat{a}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] = \hbar G(\sigma_{ge} + \sigma_{lg}) \tag{3.30}$$

and

$$[\hat{a}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}] = 0 \tag{3.31}$$

Finally, by inserting Eq. (3.26), Eq. (3.27), Eq. (3.28), Eq. (3.29),Eq. (3.30) and Eq. (3.31) into Eq. (3.25), then Eq. (3.24) becomes

$$\frac{d\hat{a}}{dt} = \dot{\hat{a}} = -(i\Delta + \kappa)\hat{a} - i\Omega\hat{a}\hat{q} + \varepsilon - iG(\sigma_{ge} + \sigma_{lg}) + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.32}$$

For $\hat{\sigma}_{ge}$,

$$\frac{d\hat{\sigma}_{ge}}{dt} = \dot{\hat{\sigma_{ge}}} = \frac{-i}{\hbar}[\hat{\sigma}_{ge}, \hat{H}] - \gamma\hat{\sigma}_{ge} + \sqrt{2\kappa}\hat{\sigma_{ge}}_{in} \tag{3.33}$$

$$\begin{aligned}[\hat{\sigma}_{ge}, \hat{H}] =&[\hat{a}, \Delta\hat{a}^\dagger\hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) + \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) \\ &+ \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl}) + \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}],\end{aligned} \tag{3.34}$$

Then, in the same manner

$$[\hat{\sigma}_{ge}, \hat{a}^\dagger\hat{a}] = 0, \tag{3.35}$$

$$[\hat{\sigma}_{ge}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = 0, \tag{3.36}$$

$$[\hat{\sigma}_{ge}, \hbar\Omega\hat{a}^\dagger\hat{a}\hat{q}] = 0, \tag{3.37}$$

$$[\hat{\sigma}_{ge}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})] = 0, \tag{3.38}$$

$$[\hat{\sigma}_{ge}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] = -\hbar G\hat{a}^\dagger\sigma_{le} + \hbar G\hat{a}(\sigma_{gg} - \sigma_{ee}) \tag{3.39}$$

and

$$[\hat{\sigma}_{ge}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}] = \hbar\beta_1\sigma_{ge} \tag{3.40}$$

Finally, by inserting Eq. (3.35), Eq. (3.36), Eq. (3.37), Eq. (3.38),Eq. (3.39) and Eq. (3.40) into Eq. (3.34), then Eq. (3.33) becomes

$$\frac{d\hat{\sigma}_{ge}}{dt} = \dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)\sigma_{ge} - iG\hat{a}(\sigma_{gg} - \sigma_{ee}) + iG\hat{a}^\dagger\sigma_{le} + \sqrt{2\gamma}\hat{\sigma_{ge}}_{in} \tag{3.41}$$

For $\hat{\sigma}_{lg}$,

$$\frac{d\hat{\sigma}_{lg}}{dt} = \dot{\hat{\sigma_{lg}}} = \frac{-i}{\hbar}[\hat{\sigma}_{lg}, \hat{H}] - \gamma\hat{\sigma}_{lg} + \sqrt{2\kappa}\hat{\sigma}_{lg_{in}} \tag{3.42}$$

$$
\begin{aligned}
[\hat{\sigma}_{lg}, \hat{H}] =& [\hat{a}, \Delta \hat{a}^\dagger \hat{a} + \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) + \hbar\Omega \hat{a}^\dagger \hat{a}\hat{q} + i\hbar\varepsilon(\hat{a}^\dagger - \hat{a}) \\
& + \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl}) + \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}],
\end{aligned} \tag{3.43}
$$

Then, in the same manner

$$[\hat{\sigma}_{lg}, \hat{a}^\dagger \hat{a}] = 0, \tag{3.44}$$

$$[\hat{\sigma}_{lg}, \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)] = 0, \tag{3.45}$$

$$[\hat{\sigma}_{lg}, \hbar\Omega \hat{a}^\dagger \hat{a}\hat{q}] = 0, \tag{3.46}$$

$$[\hat{\sigma}_{lg}, i\hbar\varepsilon(\hat{a}^\dagger - \hat{a})] = 0, \tag{3.47}$$

$$[\hat{\sigma}_{lg}, \hbar G(\sigma_{ge}\hat{a}^\dagger + \sigma_{lg}\hat{a}^\dagger + \hat{a}\sigma_{eg} + \hat{a}\sigma_{gl})] = \hbar G\hat{a}^\dagger\sigma_{le} + \hbar G\hat{a}(\sigma_{ll} - \sigma_{gg}) \tag{3.48}$$

and

$$[\hat{\sigma}_{lg}, \hbar\beta_1\sigma_{ee} + \hbar\beta_2\sigma_{ll}] = -\hbar\beta_2\sigma_{lg} \tag{3.49}$$

Finally, by inserting Eq. (3.44), Eq. (3.45), Eq. (3.46), Eq. (3.47),Eq. (3.48) and Eq. (3.49) into Eq. (3.43), then Eq. (3.42) becomes

$$\frac{d\hat{\sigma}_{lg}}{dt} = \dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)\sigma_{lg} - iG\hat{a}(\sigma_{ll} - \sigma_{gg}) - iG\hat{a}^\dagger\sigma_{le} + \sqrt{2\gamma}\hat{\sigma_{lg}}_{in}. \tag{3.50}$$

Therefore, the following are nonlinear QLEs of our model

$$\dot{\hat{q}} = \hbar\omega_m\hat{p}, \tag{3.51}$$

$$\dot{\hat{p}} = -\hbar\omega_m\hat{q} - \hbar\Omega\hat{a}^\dagger\hat{a} - \gamma_m\hat{p} + \xi, \tag{3.52}$$

$$\dot{\hat{a}} = -(i\Delta + \kappa)\hat{a} - i\Omega\hat{a}\hat{q} + \varepsilon - iG(\sigma_{ge} + \sigma_{lg}) + \sqrt{2\kappa}\hat{a}_{in}, \tag{3.53}$$

$$\dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)\sigma_{ge} - iG\hat{a}(\sigma_{gg} - \sigma_{ee}) + iG\hat{a}^\dagger\sigma_{le} + \sqrt{2\gamma}\hat{\sigma_{ge}}_{in}, \tag{3.54}$$

and

$$\dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)\sigma_{lg} - iG\hat{a}(\sigma_{ll} - \sigma_{gg}) - iG\hat{a}^\dagger\sigma_{le} + \sqrt{2\gamma}\hat{\sigma_{lg}}_{in}. \tag{3.55}$$

The optical cavity amplitude decay at a rate κ is affected by the radiation input noise operator $\hat{a}_{in}$ with zero mean value; its correlation function (Gardiner and Zoeller, 1991) are as follows; where $\hat{a}_{in}$ is the input vacuum noise operator

$$\langle\delta\hat{a}_{in}(t)\delta\hat{a}_{in}^\dagger(t')\rangle = \delta(t - t') \tag{3.56}$$

and

$$\langle\delta\hat{a}_{in}(t)\delta\hat{a}_{in}(t')\rangle = \langle\delta\hat{a}_{in}^\dagger(t)\delta\hat{a}_{in}(t')\rangle = 0. \tag{3.57}$$

The mechanical resonator affected by the quantum Brownian noise $\xi(t)$ leading to the damping of its oscillation with a rate γ_m possessing the correlation function

$$\langle\xi(t)\xi(t')\rangle \approx \gamma_m(2\bar{n} + 1)\delta(t - t'), \tag{3.58}$$

where $\bar{n} = [exp(\hbar\omega_m/k_BT) - 1]^{-1}$ the mean thermal phonon number, which is assumed to stay at the same environmental temperature T. This means that the evolution of the mechanical resonator is a Markovian process (Pontin et al., 2014).

When the zeroth-order equations of motion for σ_{ge} and σ_{lg} are calculated, the c-number steady-state values of Eq. (3.51) to Eq. (3.55) can be obtained. In order to obtain the steady-state solution of the system, we use the linear approximation theory to solve the last two equations, Eq. (3.54) and Eq. (3.55). The approximation holds as long as the field operator changes slowly during the average lifetime of the atoms so that they can reach the steady state in a short time. This is because the states of the atoms and the cavity modes can be generally uncorrelated whether the atoms reside in or leave the cavity.

Assuming that the initial state of a three-level atom to be $\rho_e = \rho_{ee}^0|e\rangle\langle e\rangle + \rho_{le}^0|l\rangle\langle l\rangle + \rho_{le}^0(|l\rangle\langle e\rangle + |e\rangle\langle l\rangle)$ and the atoms are injected into the cavity at a normalized injection rate r_a. Consequently, the last two equations of Eq. (3.54) and Eq. (3.55)can be written as

$$\dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)\sigma_{ge} + iGr_a\rho^0_{ee}\hat{a} + iGr_a\hat{a}^\dagger\rho^0_{le} + \sqrt{2\gamma}\hat{\sigma_{ge}}_{in}, \tag{3.59}$$

and

$$\dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)\sigma_{lg} - iGr_a\rho^0_{ll}\hat{a} - iGr_a\rho^0_{le}\hat{a}^\dagger + \sqrt{2\gamma}\hat{\sigma_{lg}}_{in}. \tag{3.60}$$

3.4 Steady State Solutions of the Equations of Motion

The quantum dynamics of the full nonlinear system is difficult to analyze. So we can proceed with linearization of the system operators around the steady state under strong intense laser driving intra-cavity field amplitude Genes et al. (2008). So we write each operator of the system as the sum of its steady state mean value and a small fluctuation with zero mean value as follows

$$\hat{q} = q_s + \delta\hat{q}, \tag{3.61}$$

$$\hat{p} = P_s + \delta\hat{p}, \tag{3.62}$$

$$\hat{a} = \alpha_s + \delta\hat{a}, \tag{3.63}$$

$$\hat{a}^\dagger = \alpha_s^* + \delta\hat{a}^\dagger, \tag{3.64}$$

$$\hat{\sigma}_{ge} = \sigma_{ges} + \delta\sigma_{ge} \tag{3.65}$$

and

$$\hat{\sigma}_{lg} = \sigma_{lgs} + \delta\sigma_{lg}, \tag{3.66}$$

where P_s, q_s , α_s, σ_{ges} and σ_{lgs} represent the amplitudes of the system, whereas $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$, $\delta\sigma_{ge}$ and $\delta\sigma_{lg}$ are the fluctuations with zero-mean values. The corresponding average values are obtained by setting the fluctuations with zero-mean values and applying derivatives to zero.

Eq. (3.51) becomes and it's mean value of momentum is zero.

$$\delta\dot{\hat{q}} = \hbar\omega_m(P_s + \delta\hat{p}) = 0 \tag{3.67}$$

The momentum P_s in the steady-state is equal to zero and the mechanical damping does not affect the steady-state of the system. Then,

$$P_s = 0. \tag{3.68}$$

To find q_s, lets insert Eq. (3.62) , Eq. (3.63), Eq. (3.64) and Eq. (3.65) into Eq. (3.52)

$$\delta\dot{\hat{p}} = -\hbar\omega_m(q_s + \delta\hat{q}) - \hbar\Omega((\alpha_s^* + \delta\hat{a}^\dagger)(\alpha_s + \delta\hat{a})) - \gamma_m(P_s + \delta\hat{p}) + \xi = 0. \tag{3.69}$$

Then,

$$-\hbar\omega_m q_s - \hbar\omega_m\delta\hat{q} - \hbar\Omega(\alpha_s^*\alpha_s + \alpha_s^*\delta\hat{a} + \alpha_s\delta\hat{a}^\dagger + \delta\hat{a}\hat{a}^\dagger) - \gamma_m P_s - \gamma_m\delta\hat{p} + \xi = 0. \tag{3.70}$$

Since the mean values of fluctuation terms $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$ and $\delta\hat{a}^\dagger$ are zero and due to the momentum in the steady-state, P_s , being equal to zero, the mechanical damping, ξ, does not affect the steady-state of the system and neglecting the small terms of $\delta\hat{a}\delta\hat{a}^\dagger$. Thus, q_s can be written as

$$q_s = -\frac{\Omega|\alpha_s|^2}{\omega_m}. \tag{3.71}$$

Since α_s, α_s^* are hermitian operators

$$\alpha_s = \alpha_s^*, \tag{3.72}$$

$\delta\hat{a}\hat{a}^\dagger$ and P_s are approximately zero for linearization.

Then, Eq. (3.70) becomes

$$\delta\dot{\hat{p}} = -\hbar\omega_m q_s - \hbar\omega_m\delta\hat{q} - \hbar\Omega(|\alpha_s|^2 + \alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger)) - \gamma_m\delta\hat{p} + \xi. \tag{3.73}$$

Now after inserting Eq. (3.71) into Eq. (3.73) we obtain

$$\delta\dot{\hat{p}} = -\hbar\omega_m\frac{g|\alpha_s|^2}{\omega_m} - \hbar\omega_m\delta\hat{q} - \hbar\Omega(|\alpha_s|^2 + \alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger)) - \gamma_m\delta\hat{p} + \xi. \tag{3.74}$$

Equation (3.74) can be simplified as

$$\delta\dot{\hat{p}} = -\hbar\Omega|\alpha_s|^2 + \hbar\Omega|\alpha_s|^2 - \hbar\omega_m\delta\hat{q} - \hbar\Omega\alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger) - \gamma_m\delta\hat{p} + \xi. \tag{3.75}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} - \hbar\Omega\alpha_s(\delta\hat{a} + \delta\hat{a}^\dagger) - \gamma_m\delta\hat{p} + \xi. \tag{3.76}$$

Again after inserting Eq. (3.61), Eq. (3.63) , Eq. (3.64) and Eq. (3.65) into Eq. (3.53) we get

$$\dot{\hat{a}} = -(i\Delta+\kappa)(\alpha_s+\delta\hat{a}) - i\Omega(\alpha_s+\delta\hat{a})(q_s+\delta\hat{q}) + \varepsilon - iG((\sigma_{ges}+\delta\sigma_{ge}) + (\sigma_{lgs}+\delta\sigma_{lg})) + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.77}$$

Similarly, the mean values of fluctuation terms $\delta\hat{q}$, $\delta\hat{p}$, $\delta\hat{a}$, $\delta\sigma_{ge}$ and $\delta\sigma_{lg}$ are zero and cavity input noise $\hat{a}_{in}$, being equal to zero,it does not affect the steady-state of the system and neglecting the small terms of $\delta\hat{a}\delta\hat{q}$ and then, Eq. (3.77) becomes

$$-\alpha_s(i\Delta + \kappa + i\Omega q_s) - iG(\sigma_{ges} + (\sigma_{lgs}) = -\varepsilon \tag{3.78}$$

$$\alpha_s = \frac{-iG(\sigma_{ges} + \sigma_{lgs}) + \varepsilon}{i\tilde{\Delta} + \kappa} \tag{3.79}$$

where, $\tilde{\Delta}$, is called the effective cavity detuning, including the radiation pressure effects defined as $\tilde{\Delta} = \Delta + \Omega q_s$.

In the same manner Eq. (3.77) can be written as

$$\begin{aligned} \delta\dot{\hat{a}} =& -\alpha_s(i\Delta + \kappa + i\Omega q_s) - iG(\sigma_{ges} + \sigma_{lgs}) + \varepsilon - (i\tilde{\Delta} + \kappa)\delta\hat{a} - i\Omega\alpha_s\delta\hat{q} \\ & - \hbar G(\delta\sigma_{ge} + \delta\sigma_{lg}) - i\Omega\delta\hat{a}\delta\hat{q} + \sqrt{2\kappa}\hat{a}_{in}. \end{aligned} \tag{3.80}$$

By neglecting the small terms of $\delta\hat{a}\delta\hat{q}$ and comparing Eq. (3.80) with Eq. (3.78) we obtain

$$\delta\dot{\hat{a}} = -(i\tilde{\Delta} + \kappa)\delta\hat{a} - i\Omega\alpha_s\delta\hat{q} - iG(\delta\sigma_{ge} + \delta\sigma_{lg}) + \sqrt{2\kappa}\hat{a}_{in}. \tag{3.81}$$

Again after inserting Eq. (3.63), Eq. (3.64) , Eq. and Eq. (3.65) into Eq. (3.59) we get

$$\dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)(\sigma_{ges} + \delta\sigma_{ge}) + iGr_a\rho^0_{ee}(\alpha_s + \delta\hat{a}) + iGr_a(\alpha^*_s + \delta\hat{a}^\dagger)\rho^0_{le} + \sqrt{2\gamma}\hat{\sigma_{ge}}_{in} \tag{3.82}$$

Similarly, the mean values of fluctuation terms $\delta\hat{a}$, $\delta\hat{a}^\dagger$ and $\delta\sigma_{ge}$ are zero and atomic input noise $\hat{\sigma_{ge_{in}}}$, being equal to zero,it does not affect the steady-state of the system and then, Eq. (3.82) becomes

$$-(\gamma + i\beta_1)\sigma_{ges} + iGr_a(\rho^0_{ee} + le^0)\alpha_s = 0 \tag{3.83}$$

Then,

$$\sigma_{ges} = \frac{iGr_a(\rho^0_{ee} + le^0)}{(\gamma + i\beta_1)} \tag{3.84}$$

In the same manner Eq. (3.82) can be written as

$$\dot{\hat{\sigma_{ge}}} = -(\gamma+i\beta_1)\sigma_{ges}+iGr_a(\rho^0_{ee}+le^0)\alpha_s-(\gamma+i\beta_1)\delta\sigma_{ge}+iGr_a\rho^0_{ee}\delta\hat{a}+iGr_a\rho^0_{le}\delta\hat{a}^\dagger)+\sqrt{2\gamma}\hat{\sigma_{ge_{in}}} \tag{3.85}$$

Now comparing Eq. (3.85) with Eq. (3.83) we obtain

$$\dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)\delta\sigma_{ge} + iGr_a\rho^0_{ee}\delta\hat{a} + iGr_a\rho^0_{le}\delta\hat{a}^\dagger + \sqrt{2\gamma}\hat{\sigma_{ge_{in}}} \tag{3.86}$$

Again after inserting Eq. (3.63), Eq. (3.64) , Eq. and Eq. (3.66) into Eq. (3.60) we get

$$\dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)(\sigma_{lgs} + \delta\sigma_{lg}) - iGr_a\rho^0_{ll}(\alpha_s + \delta\hat{a}) - iGr_a\rho^0_{le}(\alpha^*_s + \delta\hat{a}^\dagger) + \sqrt{2\gamma}\hat{\sigma_{lg_{in}}}. \tag{3.87}$$

Similarly, the mean values of fluctuation terms $\delta\hat{a}$, $\delta\hat{a}^\dagger$ and $\delta\sigma_{gl}$ are zero and atomic input noise $\hat{\sigma_{gl_{in}}}$, being equal to zero,it does not affect the steady-state of the system and then, Eq. (3.87) becomes

$$-(\gamma - i\beta_2)\sigma_{gls} - iGr_a(\rho^0_{ll} - le^0)\alpha_s = 0 \tag{3.88}$$

Then,

$$\sigma_{gls} = \frac{-iGr_a(\rho^0_{ll} - \rho^0_{le})}{(\gamma - i\beta_2)}\alpha_s \tag{3.89}$$

In the same manner Eq. (3.87) can be written as

$$\dot{\hat{\sigma_{lg}}} = -(\gamma-i\beta_2)\sigma_{gls}-iGr_a(\rho^0_{ll}-\rho_S le^0)\alpha_s-(\gamma-i\beta_2)\delta\sigma_{lg}-iGr_a\rho^0_{ll}\delta\hat{a}-iGr_a\rho^0_{le}\delta\hat{a}^\dagger+\sqrt{2\gamma}\hat{\sigma_{lg_{in}}}. \tag{3.90}$$

Now comparing Eq. (3.90) with Eq. (3.88) we obtain

$$\dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)\delta\sigma_{lg} - iGr_a\rho^0_{ll}\delta\hat{a} - iGr_a\rho^0_{le}\delta\hat{a}^\dagger + \sqrt{2\gamma}\hat{\sigma}_{lg_{in}}. \tag{3.91}$$

Therefore, from Eq. (3.51),Eq. (3.76) and Eq. (3.81) and Eq. (3.91) the linearized quantum Langevin equation for the fluctuation operators can be written as

$$\delta\dot{\hat{q}} = \hbar\omega_m\delta\hat{p}, \tag{3.92}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} - \hbar\Omega\alpha_s(\delta\hat{a}^\dagger + \delta\hat{a}) - \gamma_m\delta\hat{p} + \xi, \tag{3.93}$$

$$\delta\dot{\hat{a}} = -(i\tilde{\Delta} + \kappa)\delta\hat{a} - i\Omega\alpha_s\delta\hat{q} - iG(\delta\sigma_{ge} + \delta\sigma_{lg}) + \sqrt{2\kappa}\hat{a}_{in}, \tag{3.94}$$

$$\delta\dot{\hat{a}}^\dagger = -(-i\tilde{\Delta} + \kappa)\delta\hat{a}^\dagger + i\Omega\alpha_s\delta\hat{q} + iG(\delta\sigma_{eg} + \delta\sigma_{gl}) + \sqrt{2\kappa}\delta\hat{a}^\dagger_{in}, \tag{3.95}$$

$$\dot{\hat{\sigma_{ge}}} = -(\gamma + i\beta_1)\delta\sigma_{ge} + iGr_a\rho^0_{ee}\delta\hat{a} + iGr_a\rho^0_{le}\delta\hat{a}^\dagger + \sqrt{2\gamma}\hat{\sigma}_{ge_{in}}, \tag{3.96}$$

$$\dot{\hat{\sigma_{eg}}} = -(\gamma - i\beta_1)\delta\sigma_{eg} - iGr_a\rho^0_{ee}\delta\hat{a}^\dagger - iGr_a\rho^0_{le}\delta\hat{a} + \sqrt{2\gamma}\hat{\sigma}_{eg_{in}}, \tag{3.97}$$

$$\dot{\hat{\sigma_{lg}}} = -(\gamma - i\beta_2)\delta\sigma_{lg} - iGr_a\rho^0_{ll}\delta\hat{a} - iGr_a\rho^0_{le}\delta\hat{a}^\dagger + \sqrt{2\gamma}\hat{\sigma}_{lg_{in}}, \tag{3.98}$$

and

$$\dot{\hat{\sigma_{gl}}} = -(\gamma + i\beta_2)\delta\sigma_{gl} + iGr_a\rho^0_{ll}\delta\hat{a}^\dagger + iGr_a\rho^0_{le}\delta\hat{a} + \sqrt{2\gamma}\hat{\sigma}_{gl_{in}}. \tag{3.99}$$

3.5 Quantum Fluctuation

The squeezed lighted generated by three level laser and optical cavity with the mechanical resonator is able to enhance bipartite robust CV entanglement in system

(Vitali et al., 2007b) that is the quantum correlations between appropriate quadratures of the two intra-cavity fields and the position and momentum of the resonator. Introducing the quadratures for the operators as

$$\delta\hat{x} = (\frac{\delta\hat{a}^{\dagger} + \delta\hat{a}}{\sqrt{2}}) \tag{3.100}$$

and

$$\delta\hat{y} = i(\frac{\delta\hat{a}^{\dagger} - \delta\hat{a}}{\sqrt{2}}). \tag{3.101}$$

$$\delta\hat{U}_1 = (\frac{\delta\sigma_{ge} + \delta\sigma_{eg}}{\sqrt{2}}) \tag{3.102}$$

$$\delta\hat{V}_1 = (\frac{\delta\sigma_{lg} + \delta\sigma_{gl}}{\sqrt{2}}) \tag{3.103}$$

$$\delta\hat{U}_2 = i(\frac{\delta\sigma_{eg} - \delta\sigma_{ge}}{\sqrt{2}}). \tag{3.104}$$

$$\delta\hat{V}_2 = i(\frac{\delta\sigma_{gl} - \delta\sigma_{lg}}{\sqrt{2}}). \tag{3.105}$$

and the input noise quadratures

$$\hat{x}_{in} = (\frac{\hat{a}_{in}^{\dagger} + \hat{a}_{in}}{\sqrt{2}}) \tag{3.106}$$

$$\hat{y}_{in} = i(\frac{\hat{a}_{in}^{\dagger} - \hat{a}_{in}}{\sqrt{2}}). \tag{3.107}$$

$$\delta\hat{U}_{1_{in}} = (\frac{\delta\sigma_{ge_{in}} + \delta\sigma_{eg_{in}}}{\sqrt{2}}). \tag{3.108}$$

$$\delta\hat{V}_{1_{in}} = (\frac{\delta\sigma_{lg_{in}} + \delta\sigma_{gl_{in}}}{\sqrt{2}}). \tag{3.109}$$

$$\delta\hat{U}_{2_{in}} = i(\frac{\delta\sigma_{gl_{in}} - \delta\sigma_{lg_{in}}}{\sqrt{2}}). \tag{3.110}$$

After inserting Eq. (3.100) into Eq. (3.93) we get

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} - \hbar\Omega\alpha_s(\sqrt{2}\delta\hat{x}) - \gamma_m\delta\hat{p} + \xi, \tag{3.111}$$

But, let's define the term $G_0 = \sqrt{2}\Omega\alpha_s$ which is the effective coupling of the optical cavity fluctuation with mechanical resonator. Then, Eq. (3.111) becomes

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q} - \hbar G_0\delta\hat{x} - \gamma_m\delta\hat{p} + \xi. \tag{3.112}$$

Now, let's derivate Eq. (3.100) on both sides with respect to time

$$\delta\dot{\hat{x}} = (\frac{\delta\dot{\hat{a}}^\dagger + \delta\dot{\hat{a}}}{\sqrt{2}}). \tag{3.113}$$

After inserting Eq. (3.94) and Eq. (3.95) into Eq. (3.113) we get

$$\begin{aligned}\delta\dot{\hat{x}} =& (\frac{-(-i\tilde{\Delta} + \kappa)\delta\hat{a}^\dagger + i\Omega\alpha_s\delta\hat{q} + iG(\delta\sigma_{eg} + \delta\sigma_{gl}) + \sqrt{2\kappa}\delta\hat{a}_{in}^\dagger - (i\tilde{\Delta} + \kappa)\delta\hat{a} - i\Omega\alpha_s\delta\hat{q}}{\sqrt{2}}) \\ &+ (\frac{-iG(\delta\sigma_{ge} + \delta\sigma_{lg}) + \sqrt{2\kappa}\hat{a}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.114}$$

Now after inserting Eq. (3.100), Eq. (3.101), Eq. (3.104), Eq. (3.105) and Eq. (3.106) into Eq. (3.114) and rearranging we get

$$\delta\dot{\hat{x}} = -\kappa\delta\hat{x} - \tilde{\Delta}\delta\hat{y} - G\delta\hat{U}_2 + G\delta\hat{V}_2 + \sqrt{2\kappa}\delta\hat{x}_{in}. \tag{3.115}$$

Again let's derivate Eq. (3.101) on both sides with respect to time

$$\delta\dot{\hat{y}} = i(\frac{\delta\dot{\hat{a}}^\dagger - \delta\dot{\hat{a}}}{\sqrt{2}}). \tag{3.116}$$

After inserting Eq. (3.95) and Eq. (3.94) into Eq. (3.116) we get

$$\begin{aligned}\delta\dot{\hat{y}} =& i[(\frac{-(-i\tilde{\Delta} + \kappa)\delta\hat{a}^\dagger + i\Omega\alpha_s\delta\hat{q} + iG(\delta\sigma_{eg} + \delta\sigma_{gl}) + \sqrt{2\kappa}\delta\hat{a}_{in}^\dagger + (i\tilde{\Delta} + \kappa)\delta\hat{a} + i\Omega\alpha_s\delta\hat{q}}{\sqrt{2}}) \\ &+ (\frac{iG(\delta\sigma_{ge} + \delta\sigma_{lg}) - \sqrt{2\kappa}\hat{a}_{in}}{\sqrt{2}})].\end{aligned} \tag{3.117}$$

Now after inserting Eq. (3.100), Eq. (3.101), Eq. (3.102), Eq. (3.103) and Eq. (3.107) into Eq. (3.117) and rearranging we get

$$\delta\dot{\hat{y}} = -G_0\delta\hat{q} - \tilde{\Delta}\delta\hat{x} - \kappa\delta\hat{y} - G\delta\hat{U}_1 - G\delta\hat{V}_1 - \sqrt{2\kappa}\delta\hat{y}_{in}. \tag{3.118}$$

Again let's derivate Eq. (3.102) on both sides with respect to time

$$\delta\dot{\hat{U}}_1 = (\frac{\delta\dot{\sigma}_{ge} + \delta\dot{\sigma}_{ge}}{\sqrt{2}}). \tag{3.119}$$

After inserting Eq. (3.96) and Eq. (3.97) into Eq. (3.119) we get

$$\begin{aligned}\delta\dot{\hat{U}}_1 =&(\frac{-(\gamma+i\beta_1)\delta\sigma_{ge}+iGr_a\rho^0_{ee}\delta\hat{a}+iGr_a\rho^0_{le}\delta\hat{a}^\dagger+\sqrt{2\gamma}\hat{\sigma_{ge}}_{in}-(\gamma-i\beta_1)\delta\sigma_{eg}-iGr_a\rho^0_{ee}\delta\hat{a}^\dagger}{\sqrt{2}})\\&+(\frac{-iGr_a\rho^0_{le}\delta\hat{a}+\sqrt{2\gamma}\hat{\sigma_{eg}}_{in}}{\sqrt{2}}).\end{aligned} \tag{3.120}$$

Now after inserting Eq. (3.101), Eq. (3.102), Eq. (3.104) and Eq. (3.108) into Eq. (3.120) and rearranging we get

$$\delta\dot{\hat{U}}_1 = Gr_a(\rho^0_{le}-\rho^0_{ee})\delta\hat{y}-\gamma\delta\hat{U}_1+\beta_1\delta\hat{U}_2+\sqrt{2\gamma}\delta\hat{U}_{1_{in}}. \tag{3.121}$$

In similar manner the rest fluctuation are as follows

$$\delta\dot{\hat{U}}_2 = Gr_a(\rho^0_{le}+\rho^0_{ee})\delta\hat{x}-\beta_1\delta\hat{U}_1-\gamma\delta\hat{U}_2-\sqrt{2\gamma}\delta\hat{U}_{1_{in}} \tag{3.122}$$

$$\delta\dot{\hat{V}}_1 = Gr_a(\rho^0_{ll}-\rho^0_{ee})\delta\hat{y}-\gamma\delta\hat{V}_1-\beta_2\delta\hat{V}_2+\sqrt{2\gamma}\delta\hat{V}_{1_{in}} \tag{3.123}$$

and

$$\delta\dot{\hat{V}}_2 = -Gr_a(\rho^0_{ll}+\rho^0_{ee})\delta\hat{x}+\beta_2\delta\hat{V}_1-\gamma\delta\hat{V}_2-\sqrt{2\gamma}\delta\hat{V}_{2_{in}} \tag{3.124}$$

Finally, the linearized QLE with fluctuation and dissipation terms of our system are written as

$$\delta\dot{\hat{q}} = \hbar\omega_m\delta\hat{p}, \tag{3.125}$$

$$\delta\dot{\hat{p}} = -\hbar\omega_m\delta\hat{q}-\hbar G_0\delta\hat{x}-\gamma_m\delta\hat{p}+\xi, \tag{3.126}$$

$$\delta\dot{\hat{x}} = -\kappa\delta\hat{x}-\tilde{\Delta}\delta\hat{y}-G\delta\hat{U}_2+G\delta\hat{V}_2+\sqrt{2\kappa}\delta\hat{x}_{in}, \tag{3.127}$$

$$\delta\dot{\hat{y}} = -G_0\delta\hat{q}-\tilde{\Delta}\delta\hat{x}-\kappa\delta\hat{y}-G\delta\hat{U}_1-G\delta\hat{V}_1-\sqrt{2\kappa}\delta\hat{y}_{in}, \tag{3.128}$$

$$\delta\dot{\hat{U}}_1 = Gr_a(\rho^0_{le}-\rho^0_{ee})\delta\hat{y}-\gamma\delta\hat{U}_1+\beta_1\delta\hat{U}_2+\sqrt{2\gamma}\delta\hat{U}_{1_{in}}, \tag{3.129}$$

$$\delta\dot{\hat{U}}_2 = Gr_a(\rho^0_{le}+\rho^0_{ee})\delta\hat{x}-\beta_1\delta\hat{U}_1-\gamma\delta\hat{U}_2-\sqrt{2\gamma}\delta\hat{U}_{2_{in}}, \tag{3.130}$$

$$\delta\dot{\hat{V}}_1 = Gr_a(\rho^0_{ll} - \rho^0_{ee})\delta\hat{y} - \gamma\delta\hat{V}_1 - \beta_2\delta\hat{V}_2 + \sqrt{2\gamma}\delta\hat{V}_{1_{in}}, \tag{3.131}$$

and

$$\delta\dot{\hat{V}}_2 = -Gr_a(\rho^0_{ll} + \rho^0_{ee})\delta\hat{x} + \beta_2\delta\hat{V}_1 - \gamma\delta\hat{V}_2 - \sqrt{2\gamma}\delta\hat{V}_{2_{in}}. \tag{3.132}$$

The above Eq. (3.125)up to Eq. (3.132) can be written in matrix form as:

$$\dot{u}(t) = Au(t) + n(t) \tag{3.133}$$

where $u(t) = (\delta\hat{q}, \delta\hat{p}, \delta\hat{x}, \delta\hat{y}, \delta\hat{U}_1, \delta\hat{U}_2, \delta\hat{V}_1, \delta\hat{V}_2)^T$ is the vector of quadrature fluctuation operators, A is the so-called drift matrix (contains information about the system), and $n(t) = (0, \xi(t), \sqrt{2\kappa}\delta\hat{x}_{in}(t), -\sqrt{2\kappa}\delta\hat{y}_{in}(t), \sqrt{2\gamma}\delta\hat{U}_{1_{in}}(t),$ $-\sqrt{2\gamma}\delta\hat{U}_{2_{in}}(t), \sqrt{2\gamma}\delta\hat{V}_{1_{in}}(t), -\sqrt{2\gamma}\delta\hat{V}_{2_{in}}(t))^T$ is the vector of noise quadrature operators associated with the noise terms and the exponent T represent the transpose. Here we assumed that $\hbar = 1$.

$$\begin{pmatrix} \delta\dot{\hat{q}} \\ \delta\dot{\hat{p}} \\ \delta\dot{\hat{x}} \\ \delta\dot{\hat{y}} \\ \delta\dot{\hat{U}}_1 \\ \delta\dot{\hat{U}}_2 \\ \delta\dot{\hat{V}}_1 \\ \delta\dot{\hat{v}}_2 \end{pmatrix} = \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} \delta\hat{q} \\ \delta\hat{p} \\ \delta\hat{x} \\ \delta\hat{y} \\ \delta\hat{U}_1 \\ \delta\hat{U}_2 \\ \delta\hat{V}_1 \\ \delta\hat{V}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \xi \\ \sqrt{2\kappa}\delta\hat{x}_{in} \\ -\sqrt{2\kappa}\delta\hat{y}_{in} \\ \sqrt{2\gamma}\delta\hat{U}_{1_{in}} \\ \sqrt{2\gamma}\delta\hat{U}_{2_{in}} \\ \sqrt{2\gamma}\delta\hat{V}_{1_{in}} \\ \sqrt{2\gamma}\delta\hat{V}_{2_{in}} \end{pmatrix} \tag{3.134}$$

and

$$A = \begin{pmatrix} 0 & \omega_m & 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega_m & -\gamma_m & G_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\kappa & \tilde{\Delta} & 0 & -G & 0 & G \\ -G_0 & 0 & \tilde{\Delta} & -\kappa & -G & 0 & G & 0 \\ 0 & 0 & 0 & Gr_a(\rho^0_{le} - \rho^0_{ee}) & -\gamma & \beta_1 & 0 & 0 \\ 0 & 0 & Gr_a(\rho^0_{le} + \rho^0_{ee}) & 0 & -\beta_1 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & Gr_a(\rho^0_{ll} - \rho^0_{ee}) & 0 & 0 & -\gamma & -\beta_2 \\ 0 & 0 & -Gr_a(\rho^0_{ll} + \rho^0_{ee}) & 0 & 0 & 0 & -\beta_2 & -\gamma \end{pmatrix}. \tag{3.135}$$

Equation (3.133) can be written in compact form as $\dot{u}(t) = Au(t) + n(t)$, whose solution is $u(t) = M(t)u(0) + \int_0^\infty M(s)n(t-s)ds$, where $M(t) = exp(At)$. The system is stable and reaches its steady state when all of the values of A have negative real parts so that $M(\infty) = 0$. The stability refers to a time independent state of motion while dynamic stability refers to a time dependent state of motion.

The stability condition can be formulated by using the Routh-Hurwitz criteria (Schumacher, 2013; 202, 2021). In spite of this fact, the explicit inequalities are quite cumbersome. Here, we check the stability of the steady-state solutions with numerical calculation to ensure that the parameters we used always satisfy the stability conditions. Due to the Gaussian nature of the quantum noise terms in Eq.(3.133) and the linearized dynamics, the steady-state quantum fluctuations of the system is a bipartite Gaussian state , fully characterized by the 8×8 covariance matrix V with its entries defined as

$$V_{ij} = \Big(\langle v_i(\infty)v_j(\infty) + v_j(\infty)v_i(\infty) \rangle \Big)/2. \tag{3.136}$$

In the stable system such a CM is given by

$$V = \int_0^\infty M(s)DM^T(s)ds. \tag{3.137}$$

If the system satisfied stability conditions $M(\infty) = 0$, since Eq.(3.137) is equivalent to the following Lyapunov equation Anderson et al. (1986) for the steady-state CM

$$AV + VA^T = -D, \tag{3.138}$$

where V is the system CM contains all information about the steady state and D is the diffusion matrix stemming from the noise correlations, which is defined as

$$\frac{1}{2}\langle n_i(t)n_j(s) + n_j(s) \rangle = D_{ij}\delta(t-s). \tag{3.139}$$

The diffusion matrix (noise) is a diagonal matrix, that is $D = diag[0, \gamma_m(2\bar{n}+1), \kappa, \kappa, \gamma, \gamma, \gamma, \gamma]$. Here we have assumed the mechanical resonator is of high quality factor $Q = \omega_m/\gamma_m \gg 1$, which is typically satisfied under the current experiment conditions (Vitali et al., 2007a).

Thus, the bipartite entanglement of our system can be quantified numerically using the correlation matrix 8×8 of the two bipartite entanglement

$$V = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & v_{26} & v_{27} & v_{28} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & v_{36} & v_{37} & v_{38} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} & v_{46} & v_{47} & v_{48} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} & v_{56} & v_{57} & v_{58} \\ v_{61} & v_{62} & v_{63} & v_{64} & v_{65} & v_{66} & v_{67} & v_{68} \\ v_{71} & v_{72} & v_{73} & v_{74} & v_{75} & v_{76} & v_{77} & v_{78} \\ v_{81} & v_{82} & v_{83} & v_{84} & v_{85} & v_{86} & v_{87} & v_{88} \end{pmatrix}. \tag{3.140}$$

Finally, we employ on logarithmic negativity E_N a quantity which has been already proposed as a measure of entanglement in the continuous variable case as defined in Eq. (2.9) Eq. (2.10) and Eq. (2.11). Based on those framework we can quantify the bipartite entanglement dynamics behavior of an optomechanical system with three level atom.

The CM allows to calculate the stationary entanglement. Indeed, to calculate the pairwise entanglement, we reduce the (8×8) covariance matrix V to a (4×4) sub-matrix V_R. There are four such cases of the sub-matrix V_R: (i) if the indices i and j for the element V_{ij} are confined to the set $1, 2, 3, 4$, the submatrix $V_R = [V_{ij}]$ is produced by the first four rows and columns of V and correspond to the covariance between the first intracavity photon-phonon coupling. Similarly, (ii) if the indices run over $3, 4, 5, 6$, V_R is the covariance matrix of the second intracavity optical cavity and three level atom interaction. (iii) If the indices run over $1, 2, 5, 6$, V_R designates the covariance between the two mechanical resonator mode and three level atom.

4

RESULT AND DISCUSSION

In this section, we discusses the numerical results of the analysis of bipartite entanglement of system for single mode light produced by three level laser. The entanglement properties can be verified by experimentally measuring the corresponding CM through logarithmic negativity. To measure E_N at the steady state, one has to measure all the determined independent entries of the correlation matrix V. This has been experimentally realized (Li et al., 2020), (Zhou et al., 2011) for the case of steady-state entanglement via three-level atoms in a hybrid levitated optomechanical system.

The parameters used from (Li et al., 2020), (Zhou et al., 2011) are optical cavity length $L = 3\times10^{-3}m$, the speed of light $c = 3\times10^{8}m/s$, Plancks constant $h = 6.626\times10^{-34}J.s$, Boltzmanns constant $K_B = 1.38\times10^{-23}J/K$, wavelength of driven laser $\lambda = 810\times10^{-9}m$, the power of driven laser $P = 50\times10^{-6}W$, frequency of mechanical oscillator $\omega_m = 2\pi MHz$ and damping for the mechanical rate $\gamma_m = 200\pi Hz$. Accordingly, we study effects of atom injected for entanglement of optical cavity with mechanical resonator, entanglement of atom with optical cavity, and entanglement of atom with mechanical resonator are investigated by plotting graphs as follows.

Fig.(4.1) shows E_N as a function of the normalized detuning $\tilde{\Delta}/\omega_m$ between optical cavity with mechanical resonator; here the internal energy of the atoms are assumed to be equal as $\rho_{ee} = \rho_{ll} = \rho_{le} = 0.5$ (Li et al., 2020) and atomic detunings also assumed equal $\beta_1 = \beta_2 = \pi MHz$. In this case power $P = 50\times10^{-6}W$, the mechanical oscillator has a frequency $\omega_m = 10\pi MHz$, a damping rate $\omega_m = 2\pi MHz$, its temperature is $T = 42\mu K$, the internal energy of the atoms $\rho_{ee} = \rho_{ll} = \rho_{le} = 0.5$, atomic decay rate of $\gamma = 2\pi MHz$,atomic detunings $\beta_1 = \beta_2 = \pi MHz$ and decay rate of optical cavity $\kappa = 0.2MHz$. The red line refers to the coupling of the atom and the cavity field of $G = 2\pi \times 2 \times 10^6 Hz$, the blue line refers to $G = 2\pi \times 1.7 \times 10^6 Hz$ and the green line refers to $G = 2\pi \times 1.5 \times 10^5 Hz$. The dependence of bipartite entanglement on the coupling of the atom and the cavity field G, as it's value increases or decreases causes the enhancement of entanglement.

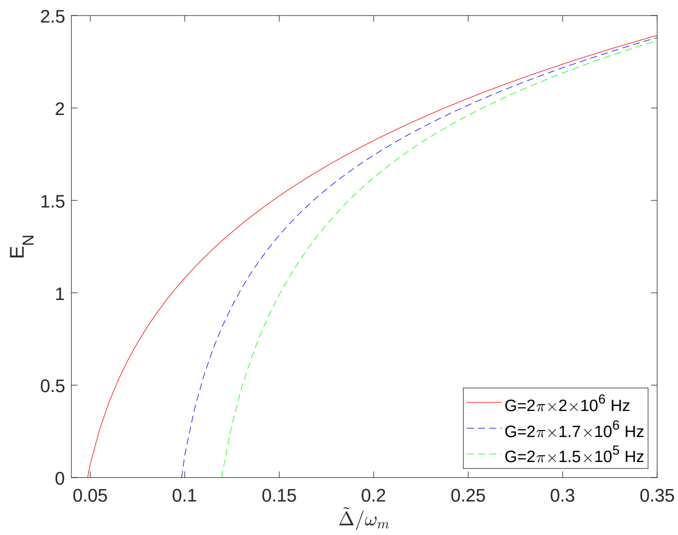

Fig. 4.1: Plot of the logarithmic negativity E_N as a function of the normalized detuning $\tilde{\Delta}/\omega_m$ between optical cavity with mechanical resonator for internal energy of the atoms are assumed to be equal.

As coupling of the atom and the cavity field G increases the optomechanical entanglement also increases. In addition to this the lower value of coupling of the atom and the cavity field G causes decreases in entanglement between optical cavity and mechanical resonators.

Fig.(4.2) illustrates that the logarithmic negativity E_N as a function of the normalized

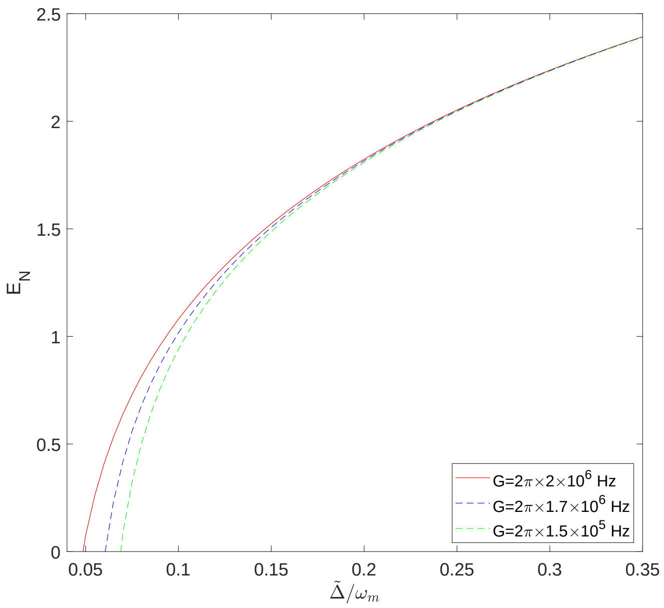

Fig. 4.2: Plot of the logarithmic negativity E_N as a function of the normalized detuning $\tilde{\Delta}/\omega_m$ between optical cavity with mechanical resonator for internal energy of the atoms are assumed to be different.

detuning $\tilde{\Delta}/\omega_m$ between optical cavity with mechanical resonator. Unlike description of Fig.(4.1) the internal energy of the atoms are assumed to be taken different values as $\rho_{ee} = 0.2$, $\rho_{ll} = 0.8$, $\rho_{le} = 0.4$ and atomic detunings assumed equal. In this case the internal energy of the atoms $\rho_{ee} = 0.2$, $\rho_{ll} = 0.8$, $\rho_{le} = 0.4$. The red line refers to the coupling of the atom and the cavity field of $G = 2\pi \times 2 \times 10^6 Hz$, the blue line refers to $G = 2\pi \times 1.7 \times 10^6 Hz$ and the green line refers to $G = 2\pi \times 1.5 \times 10^5 Hz$. Other parameters are the same as in Fig.(4.1). The dependence of bipartite entanglement on the coupling of the atom and the cavity field G, as it's value increases or decreases causes less enhancement of entanglement. After certain increment of normalized detuning, coupling of the atom and the cavity field G increases equally causes less increment of the optomechanical entanglement. Thus, in order to get robust bipartite entanglement the internal energy of the atoms must be equal with the increment of the coupling of the atom and the cavity field G.

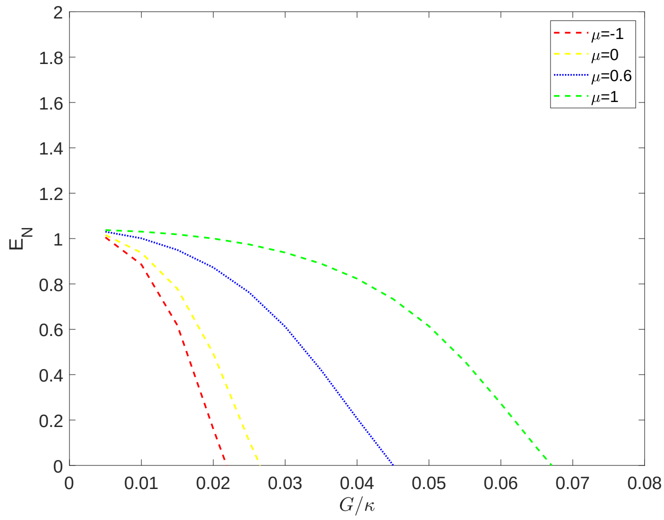

Fig. 4.3: Plot of the entanglement E_N versus effective atom-field coupling G of different coherent superposition parameter η: in this case $\tilde{\Delta} = 3\kappa$ and the driving power $P = 50\mu W$. The red line refers to $\mu = -1$, the yellow line refers to $\mu = 0$, the blue line refers to $\mu = 0.6$ and the green line refers to $\mu = 1$. Other parameters are the same as in Fig.(4.1)

Fig.(4.3) indicates that the entanglement E_N versus effective atom-field coupling G of different coherent superposition parameter η (Sete and Eleuch, 2015). In order to investigate the impact of the atomic coherence on the steady-state optomechanical entanglement, we introduced a new coherent superposition parameter $\eta \in [-1, 1]$ to characterize the initial atomic medium, i.e., $\rho_{ee}^0 = (1 - \eta)/2$ and $\rho_{ll}^0 = (1 + \eta)/2$, and the corresponding atomic coherence, i.e., $\rho_{le}^0 = \sqrt{1 - \eta^2}/2$ (Sete and Eleuch, 2015). It is easy to find that ρ_{ee}^0 (ρ_{ll}^0) decreases (increases) with the increase of η. In contrast, the atomic coherence ρ_{le}^0 increases first and then decreases to 0 with the increase of η. The optimal coherence of the atomic initial state can be attained when $\eta = 0$. The initial population of the atomic states affects significantly the effective optomechanical coupling. Furthermore, when the atomic population is changed from the upper level to the lower level, the bipartite entanglements E_N increase monotonically. The entanglement maximum appears at $\eta = 1$ and $\rho_{le} = 0$. Therefore, the large initial population of lower level $\langle l|$ can enhance the optomechanical entanglement. In addition, the bipartite entanglements E_N decrease with the increase of G if all atoms are initially in the upper level, i.e., $(\eta = 1)$. In contrast, when the atoms are initially in the lower atomic level, i.e., $(\eta = 1)$, the bipartite entanglements E_N increase with the increase of G. Consequently, the injected atomic medium should be prepared initially in the lower level at a fixed atomic coherence ρ_{le} to attain a large entanglement.

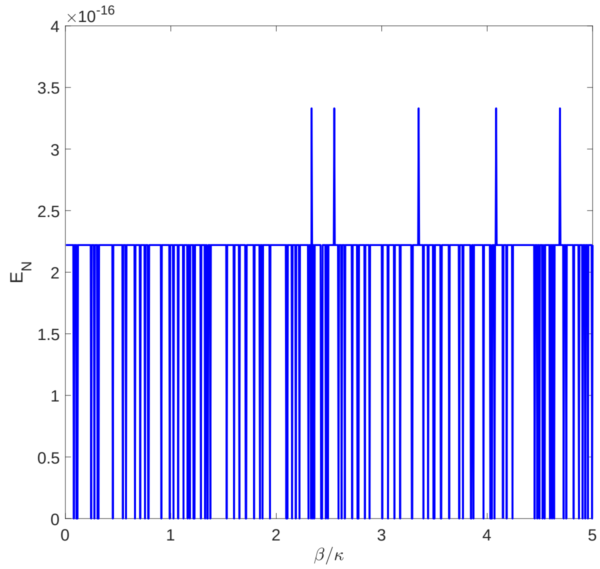

Fig. 4.4: Plot of the entanglement E_N as functions of the atomic detuning β: in the case the optimal cavity detuning $\tilde{\Delta} = 3\kappa$, $\beta_2 = \tilde{\Delta}$, $G = 1.6\pi KHz$ and $P = 50\mu W$ .Other parameters are the same as in Fig.(4.1)

In Fig.(4.4) we investigate the dependence of the optomechanical entanglements, E_N, upon the atomic detunings d1 and d2 near the optimal effective cavity field detuning β with $\tilde{\Delta} = 3\kappa$. The entanglements are not a monotonic function of the atomic detunings β_1 and β_2. Further, for a given atomic detuning β, the maximum entanglement appears in the region of $\beta_2 = \tilde{\Delta}$. In contrast, when the atomic detuning β_2 is fixed, the bipartite entanglement decreases first and then increases with the increase of the atomic detuning β_2 and the minimum entanglement is attained at $\beta_2 = \tilde{\Delta}$. The results suggest that one should carefully choose the atomic detunings, which impact significantly the optomechanical entanglement and its maximum.

5

CONCLUSION AND RECOMMENDATIONS

5.1 CONCLUSION

In conclusion, we have analyzed the bipartite entanglement of an optomechanical system for single mode light produced by three level lasers to generates the stationary continuous-variable entanglement. Particularly, we have studied the dependence of bipartite entanglement on the coupling of the atom and the cavity field , as its value increases or decreases causes the enhancement of entanglement. In addition to this the lower value of coupling of the atom and the cavity field causes decreases in entanglement between optical cavity and mechanical resonators. After certain increment of normalized detuning, coupling of the atom and the cavity field G increases equally causes less increment of the optomechanical entanglement.

Thus, in order to get robust bipartite entanglement the internal energy of the atoms must be equal with the increment of the coupling of the atom and the cavity field. Accordingly, our results showed that the introduction of three level atoms makes the entanglement more robust. Interestingly, we showed that the effects of atom-field coupling, the internal energy of atoms and atomic detuning have great contribution for the enhancement of the stationary continuous variable entanglement. Thus, when three level atoms are added inside a cavity, the optomechanical coupling and the bipartite entanglement is improved.

5.2 RECOMMENDATIONS

In this work, we have studied robust bipartite entanglement in an optomechanical system assisted by three level laser. Further studies will be conducted on the enhancement of entanglement at several mechanical oscillators and cavities with few atoms injected among them. This might be at cold and warm temperatures

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Bibliography

(2021). Using Eigenvalues and Eigenvectors to Find Stability and Solve ODEs. [Online; accessed 2021-07-12].

Alber, G., Beth, T., Horodecki, M., Horodecki, P., Horodecki, R., Rötteler, M., Weinfurter, H., Werner, R., and Zeilinger, A. (2003). *Quantum information: An introduction to basic theoretical concepts and experiments*, volume 173. Springer.

Altintas, F. and Eryigit, R. (2013). Control and manipulation of entanglement between two coupled qubits by fast pulses. *Quantum information processing*, 12(6):2251–2268.

Anderson, B., Agathoklis, P., Jury, E., and Mansour, M. (1986). Stability and the matrix lyapunov equation for discrete 2-dimensional systems. *IEEE Transactions on Circuits and Systems*, 33(3):261–267.

Aspelmeyer, M., Gröblacher, S., Hammerer, K., and Kiesel, N. (2010). Quantum optomechanicsthrowing a glance. *JOSA B*, 27(6):A189–A197.

Aspelmeyer, M., Kippenberg, T. J., and Marquardt, F. (2014). Cavity optomechanics. *Reviews of Modern Physics*, 86(4):1391.

Barnett, S. (2009). *Quantum information*, volume 16. Oxford University Press.

Caves, C. M. (1979). Microwave cavity gravitational radiation detectors. *Physics Letters B*, 80(3):323–326.

Deniz, C. (2019). Quantum harmonic oscillator. In *Oscillators-Recent Developments*. IntechOpen.

Dowling, M. R., Doherty, A. C., and Wiseman, H. M. (2006). Entanglement of indistinguishable particles in condensed-matter physics. *Physical Review A*, 73(5):052323.

Einstein, A., Podolsky, B., and Rosen, N. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical review*, 47(10):777.

Ford, G. and Kac, M. (1987). On the quantum langevin equation. *Journal of statistical physics*, 46(5-6):803–810.

Ford, G. W., Lewis, J. T., and Oconnell, R. (1988). Quantum langevin equation. *Physical Review A*, 37(11):4419.

Friis, N., Vitagliano, G., Malik, M., and Huber, M. (2019). Entanglement certification from theory to experiment. *Nature Reviews Physics*, 1(1):72–87.

Gardiner, C. and Zoeller, P. (1991). Quantum noise springer-verlag. *Berlin, Heidelberg, New York*.

Gebremariam, T., Mazaheri, M., Zeng, Y., and Li, C. (2019). Dynamical quantum steering in a pulsed hybrid opto-electro-mechanical system. *JOSA B*, 36(2):168–177.

Genes, C., Vitali, D., and Tombesi, P. (2008). Emergence of atom-light-mirror entanglement inside an optical cavity. *Physical Review A*, 77(5):050307.

Gröblacher, S. (2012). *Quantum opto-mechanics with micromirrors: combining nano-mechanics with quantum optics*. Springer Science & Business Media.

Groisman, B., Popescu, S., and Winter, A. (2005). Quantum, classical, and total amount of correlations in a quantum state. *Physical Review A*, 72(3):032317.

Häffner, H., Hänsel, W., Roos, C., Benhelm, J., Chek-al Kar, D., Chwalla, M., Körber, T., Rapol, U., Riebe, M., Schmidt, P., et al. (2005). Scalable multiparticle entanglement of trapped ions. *Nature*, 438(7068):643–646.

Heylman, K. D., Thakkar, N., Horak, E. H., Quillin, S. C., Cherqui, C., Knapper, K. A., Masiello, D. J., and Goldsmith, R. H. (2016). Optical microresonators as single-particle absorption spectrometers. *Nature Photonics*, 10(12):788–795.

Hohenester, U. (2020). *Nano and Quantum Optics.* Springer.

Huang, S. and Agarwal, G. (2009). Enhancement of cavity cooling of a micromechanical mirror using parametric interactions. *Physical Review A*, 79(1):013821.

Kassahun, F. (2008). *Fundamentals of Quantum Optics.* Lulu. com.

Kippenberg, T. J. and Vahala, K. J. (2007). Cavity opto-mechanics. *Optics express*, 15(25):17172–17205.

Kippenberg, T. J. and Vahala, K. J. (2008). Cavity optomechanics: back-action at the mesoscale. *science*, 321(5893):1172–1176.

Kong, C., Xiong, H., and Wu, Y. (2017). Coulomb-interaction-dependent effect of high-order sideband generation in an optomechanical system. *Physical Review A*, 95(3):033820.

Law, C. (1995). Interaction between a moving mirror and radiation pressure: A hamiltonian formulation. *Physical Review A*, 51(3):2537.

Li, G., Nie, W., Wu, Y., Liao, Q., Chen, A., and Lan, Y. (2020). Manipulating the steady-state entanglement via three-level atoms in a hybrid levitated optomechanical system. *Physical Review A*, 102(6):063501.

Luan, X., Huang, Y., Li, Y., McMillan, J. F., Zheng, J., Huang, S.-W., Hsieh, P.-C., Gu, T., Wang, D., Hati, A., et al. (2014). An integrated low phase noise radiation-pressure-driven optomechanical oscillator chipset. *Scientific reports*, 4(1):1–7.

Marquardt, F. and Girvin, S. M. (2009). Optomechanics (a brief review). *arXiv preprint arXiv:0905.0566.*

Merklein, M., Stiller, B., Vu, K., Madden, S. J., and Eggleton, B. J. (2017). A chip-integrated coherent photonic-phononic memory. *nature Communications*, 8(1):1–7.

Meystre, P. (2013). A short walk through quantum optomechanics. *Annalen der Physik*, 525(3):215–233.

Milburn, G. (2011). 5. an introduction to quantum optomechanics. *acta physica slovaca*, 61(1).

Ockeloen-Korppi, C., Damskägg, E., Pirkkalainen, J.-M., Asjad, M., Clerk, A., Massel, F., Woolley, M., and Sillanpää, M. (2018). Stabilized entanglement of massive mechanical oscillators. *Nature*, 556(7702):478–482.

Plenio, M. B. (2005). Logarithmic negativity: a full entanglement monotone that is not convex. *Physical review letters*, 95(9):090503.

Pontin, A., Bonaldi, M., Borrielli, A., Cataliotti, F., Marino, F., Prodi, G., Serra, E., and Marin, F. (2014). Detection of weak stochastic forces in a parametrically stabilized micro-optomechanical system. *Physical Review A*, 89(2):023848.

Pruessner, M., Stievater, T., and Rabinovich, W. (2008). Reconfigurable filters using mems resonators and integrated optical microcavities. In *2008 IEEE 21st International Conference on Micro Electro Mechanical Systems*, pages 766–769. IEEE.

Restrepo, J., Favero, I., and Ciuti, C. (2017). Fully coupled hybrid cavity optomechanics: Quantum interferences and correlations. *Physical Review A*, 95(2):023832.

Schnabel, R. (2017). Squeezed states of light and their applications in laser interferometers. *Physics Reports*, 684:1–51.

Schumacher, M. (2013). Stability and entanglement in an optomechanical system.

Sete, E. A. and Eleuch, H. (2015). Anomalous optical bistability and robust entanglement of mechanical oscillators using two-photon coherence. *JOSA B*, 32(5):971–982.

Tesfa, S. (2006). Entanglement amplification in a nondegenerate three-level cascade laser. *Physical Review A*, 74(4):043816.

Tesfahannes, T. G. (2021). Enhanced optomechanically induced transparency via atomic ensemble in optomechanical system. *Quantum Information Processing*, 20(3):1–12.

Tesfahannes, T. G. and Alemu, D. (2021). Stationary entanglement dynamics in a hybrid opto-electro-mechanical system.

Ulbricht, H. (2021). Testing fundamental physics by using levitated mechanical systems. In *Molecular Beams in Physics and Chemistry*, pages 303–332. Springer.

Vitali, D., Gigan, S., Ferreira, A., Böhm, H., Tombesi, P., Guerreiro, A., Vedral, V., Zeilinger, A., and Aspelmeyer, M. (2007a). Optomechanical entanglement between a movable mirror and a cavity field. *Physical review letters*, 98(3):030405.

Vitali, D., Tombesi, P., Woolley, M., Doherty, A., and Milburn, G. (2007b). Entangling a nanomechanical resonator and a superconducting microwave cavity. *Physical Review A*, 76(4):042336.

Wei, C., Suter, D., Windsor, A. S., and Manson, N. B. (1998). ac stark effect in a doubly driven three-level atom. *Physical Review A*, 58(3):2310.

Zhou, L., Han, Y., Jing, J., and Zhang, W. (2011). Entanglement of nanomechanical oscillators and two-mode fields induced by atomic coherence. *Physical Review A*, 83(5):052117.