Finite Element Method For Solving the Wave Equation in 3D with Unbounded Domain

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Notations and Abbreviations

ABC	Absorbing Boundary Condition
BC	Boundary Conditions
PDE	Partial Differential Equation
PEC	Perfect Electeric circuit
FEM	Finite Element Method

Abstract

This thesis presents an overview of the acoustic scattering of 3D wave equation and the finite element method for the solution strategies in unbounded domain. First, the mathematical formulation of Exterior Helmholtz equation will be presented and we discretize the solution domain for exterior Helmholtez problem.Then by using finite element methods (FEM) we solve the exterior Helmholtez problem by taking weak formulation and we will compare first and second order absorbing boundary condition.

Key words: *Finite Element method Numerical solution for exterior helmoholtz equation*

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Chapter 1

Introduction

1.1 Background of the study

Finite element methods (FEM) for time-harmonic acoustics governed by the reduced wave equation (Helmholtz equation), have been an active research area for nearly 40 years. Initial applications of finite element methods for time-harmonic acoustics focused on interior problems with complex geometries including direct and modal coupling of structural acoustic systems for forced vibration analysis, frequency response of acoustic enclosures, and waveguides (A. Craggs,1972 and G.M.L. Gladwell,1966). In recent years, tremendous progress in the development of improved finite element methods for time-harmonic acoustics including exterior problems in unbounded domains, which incorporate knowledge of wave behavior into the algorithm, combined with parallel sparse iterative and domain decomposition solvers, are moving the application of FEM into the higher frequency (wave number) regimes.

The exterior acoustics problem in unbounded domains presents a special challenge for finite element methods. In order to use the FEM for exterior problems, the unbounded domain is usually truncated by an artificial boundary Γ_a yielding a bounded computational domain. Reducing the size of the bounded domain reduces the computation cost, but must be balanced by the ability to minimize any spurious wave reflection with a computationally efficient and geometrically flexible truncation boundary treatment. The first complete finite element approach for modeling acoustic radiation and scattering in unbounded domains appears in the impedance matching technique presented (Hunt et al.,1975). Recent numerical treatments including infinite elements, absorbing layers,local absorbing boundary conditions, and exact nonlocal boundary conditions have proven to be effective in handling acoustic scattering problems in unbounded domains, especially for large-scale problems requiring iterative and parallel solution methods and for modeling inhomogeneities and acoustic-structure interaction. The method of choice depends on the shape and complexity of the scattering object, inhomogeneities, frequency range, and resolution requirements, among other parameters. A natural way of modeling the acoustic region exterior to a scattering/radiating object is to introduce a boundary element discretization of the surface Γ based on an integral representation of the exact solution in the exterior (A. Burton and G. Miller,1971). Using the free-space Greens function (fundamental solution), the boundary element method (BEM) only requires surface discretization on Γ, reducing the d-dimensional problem to a (d-1)-dimensional one; and automatically satisfies the required Sommerfeld radiation condition at infinity (P. Ortiz and E. Sanchez.,2001).

In this thesis, recent developments in finite element methods for time-harmonic acoustics, including treatments of unbounded domains are reviewed. Topics include mathematical formulation of exterior helmoholtz equation, local absorbing (non reflecting) boundary conditions for exterior problems; discretization methods which reduce numerical dispersion error arising in the approximation of short unresolved waves

1.2 Statement of the problem

A number of researchers have addressed the aproximation solution of 3D wave equation with bounded domain. The work of Lord Rayleigh (1870) and W. Ritz (1909) on variational methods and the weighted-residual approach taken by B. G. Galerkin (1915) and others form the theoretical framework to the finite element method. With a bit of a stretch, one may even claim that Schellbachs approximate solution to Plateaus problem (find a surface of minimum area enclosed by a given closed curve in three dimensions). All these works extended the well established the approximation solution of 3D wave equation in different forms and do not work on the solution of wave equation with unbounded domain. In this research work, we will continue to carryout numerical approximation of 3D wave equation with unbounded domain using finite element method. Although finite element methods have been around for a considerable time, there is still much work to be done on their application to acoustic scattering problems. Firstly, we remark that the finite element method was originally developed for the numerical solution of problems on bounded domains. However,often in acoustic scattering applications the computational domain may be unbounded domain. It is difficult to discretize. A secondly, when the wave number k becomes large, the accuracy of the standard finite element method deteriorates. So, this study is intended to answer the following basic questions:

- What is the mathematical formulation of exterior Helmholtz problem for unbounded domain ?
- What is the solution of exterior boundary-value problem of acoustics scattering ?
- Which absorbing boundary condition (1st or 2nd order) is better ?

1.3 Objective of the study

1.3.1 General objective

- The general objective of this study is to solve three-dimensional wave equations with unbounded domain using finite element method.

1.3.2 Specific objectives

Specific objectives of the study are:

- To solve exterior boundary-value problem of acoustics scattering.
- To derive The exterior Helmholtz Equation for the domain is unbounded.
- To compare 1st order and 2nd order absorbing boundary condition.

1.4 Benefits of the study

The formulation of problems in many diverse fields of studies in engineering and physical sciences such as thermostatic, magnetostatic problem, electrostatics, ideal fluid flow and flow in porous media are described through wave equation. Solving wave equation is very crucial area in the real life application. The benefites of the study are:

- To determine the FEM solution of three- dimensional wave equations with unbounded domain.
- To calculate the radar cross section.
- Moreover it provides several guides to all natural science students,teachers and anyone who works on this area.

1.5 Limitation of the Study

It was difficult to assess the over all aspects of the FEM technique for solving three-dimensional wave equation with unbounded domain. On the progress of this thesis we face some problems such as shortage of time, lack of reference books, and inaccessibility of internet.

Chapter 2

Literature Review

The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science. Historically, all major practical advances of the method have taken place since the early 1950s in conjunction with the development of digital computers. However, interest in approximate solutions of field equations dates as far backs in time as the development of the classical field theories (e.g.elasticity, electro-magnetism) themselves. The work of (Lord Rayleigh , 1870) and (W. Ritz , 1909) on variational methods and the weighted-residual approach taken by (B. G. Galerkin , 1915) and others form the theoretical framework to the finite element method. With a bit of a stretch, one may even claim that Schellbachs approximate solution to Plateaus problem (find a surface of minimum area enclosed by a given closed curve in three dimensions), which dates back to 1851 is a rudimentary application of the finite element method (Turn Most researchers agree that the era of the finite element method begins with a lecture presented by (R. Courant, 1943) to the American Association for the Advancement of Science. In his work, Courant used the Ritz method and introduced the pivotal concept of spatial discretization for the solution of the classical torsion problem. Courant did not pursue his idea further, since computers were still largely unavailable for his research. More than a decade later Ray W. Clough, Jr. of the University of California, Berkeley, and his colleagues reinvented the FEM as a natural extension of matrix struc- tural analysis and published their first work in 1956. In few years following its introduction to the engineering community, the FEM has attracted the attention of applied mathematicians, particularly those interested in numerical solution of PDEs. (G. Strang and G.J., 1973), Fix authored the first conclusive treatise on mathematical aspects of the method, focusing exclusively on its application to the solution of problems emanating from standard variational theorems. The finite element has been subject to intense research, both at the mathematical and technical level, and thousands of scientific articles and hundreds of books about it have been authored. By the beginning of the 1990s, the method clearly dominated the numerical solution of problems in the field of structural analysis, structural mechanics and solid mechanics. Moreover, the finite element method clearly competes in popularity with the finite difference method in the areas of heat transfer and fluid mechanics. The first part of the project involved familiarization with the Finite Element Method (FEM), through the reading of the textbooks by (John Wiley and Sons,1993),(Polycarpou,2006) and (Volakis et al,2001), to understand how Maxwells equations are discretized into linear matrix equations. For meshing, before setting up our own triangular meshing codes, the documentation of DistMesh (P. Persson and G. Strang,2004), an open source MATLAB-based meshing library. Next, a study of the two dimensional boundary value problem described by (Polycarpou,2006) and (John Wiley and Sons,1993) and the three major boundary conditions (Dirichlet, Neumann and Mixed Boundary Conditions) was undertaken. The weak form was developed using these ideas, and the mathematical foundations underlying them, as outlined by (Volakis et al) and (Kesavan). The Galerkin formulation of (John Wiley and Sons,1993) was identified to be used for subsequent computational work, and a Helmoltz equation was set up for a general two dimensional scattering problem.

2.1 Mathematical preliminaries

- **The Helmholtz Equation**

  This derivation starts from the scalar wave equation

$$u_{tt} = c^2 \vartriangle u + f(t, x) \tag{2.1.1}$$

  Where c is the local speed of propagation for waves and $f(t, x)$ is a source that injects waves into the solution. Suppose we look for solution with a single angular time frequency ,

$$u(t, x) = \varphi(x)e^{-i\omega t}, f(t, x) = g(x)e^{-i\omega t} \tag{2.1.2}$$

  Entering this into equation(2.1.1) we obtain

$$-\omega^2\varphi(x)e^{-i\omega t} = c^2(\Delta\varphi(x)e^{-i\omega t}) + g(x)e^{-i\omega t} \tag{2.1.3}$$

  Hence, after dividing by $e^{-i\omega t}$ and reordering the terms,

$$\Delta\varphi(x) + k^2\varphi(x) = -\frac{g(x)}{c^2} \tag{2.1.4}$$

  With $f(t, x) = \dfrac{g(x)}{c^2}$.
  Where ω is the frequency of an eigenmode and $k = \dfrac{\omega}{c}$. Equation (2.1.4) is called Helmholtz equation or sometimes called reduced wave equation. Mathematically, the problem is about the Eigen values of the laplacian operator. The solution φ for the corresponding wave numbers are the standing waves inherent to the geometry of the domain. The Helmholtz equation is the basis for a large number of numerical methods for computational acoustics; they are called spectral methods since they do not simulate time-dependent pressure fields but the responses of the environment to different frequencies instead.

- **Boundary conditions (BC.):** expressed by constraints on the values taken by the field and its normal derivative at a surface, will reflect the properties of the surface. For perfectly reflective surfaces we have two possible cases:

- **Neumann condition:** when the normal derivative of the potential field is given at the boundary, i.e., if n is the unit normal pointing outward from the surface of artificial boundary:

$$\frac{\partial u(x)}{\partial n} = 0, x \in \Omega$$

  In acoustics this condition holds for complex amplitude of pressure,when the surface material has much higher acoustic impedance than the acoustic impedance of the scatterer.In this case the surface is called sound hard.

- **Dirichlet condition:** when the value of the potential field is given at the boundary:

$$u(x) = 0, x \in \Omega$$

  This condition appears inside the domain, e.g. for complex amplitude pressure in acoustics, when the material of the surface has very low acoustic impedance compared to the acoustic impedance of the carrier medium. In this case the surface is called sound soft.. In most real cases the surface is not perfectly reflecting and both the potential and its normal derivative are different from zero at the boundary.

- **Sommerfeld Radiation Condition:** The problems, which are usually considered in relation with the wave equation in three dimensional unbounded domains are scattering problems. In this case the wave function is specified as

$$u(x,t) = u(x,t)_{int} + u(x,t)_{scat}$$

Where both functions $u(x,t)_{inc}$ and $u(x,t)_{scat}$ satisfy the wave equation. Function $u(x,t)_{int}$ is the potential of the incident field, while $u(x,t)_{scat}$ is the potential of the scattered field, which arises due to the presence of one or several scatterers. In the absence of scatterers $u(x,t) = u(x,t)_{inc}$ is some given function. To understand the scattered field we may turn our attention to the Huygens principle, which represents wave propagation as emission of secondary wave from the points located on the current wave front. When the primary wave described $u(x,t)_{inc}$ reaches the scatterer boundary the secondary waves are generated from the boundary points located at the intersection of the boundary and the wave front. Due to finite speed of wave propagation spatial points far from the boundary do not know about these secondary waves, so these waves can be thought as waves outgoing from the boundary points. where $f \equiv 0$ and, therefore, the total scattered field,$u(x,t)_{scat}$, can be seen now as a superposition of outgoing waves, corresponding potential in the frequency domain should satisfy condition

$$lim_{r \longrightarrow \infty} r \left(\frac{\partial u_{scat}}{\partial r} - iku_{scat} \right) = 0$$

This condition is called Sommerfeld radiation condition or radiation condition. It states that the scattered field consists of outgoing waves only. Solutions of the Helmholtz equation which satisfies the radiation condition are called radiating solutions or radiating functions.

2.2 Local Absorbing Boundary Condition

Absorbing boundary conditions should annihilate (absorb) any spurious reflections at the artificial boundary (which are incoming). For local absorbing boundary conditions defined on a sphere, the development is based on the idea of annihilating (absorbing) radial terms in the Atkinson-Wilcox radial expansion in powers of $\frac{1}{kr}$

$$u(r,\theta,\varphi,k) = \frac{e^{ikr}}{ikr}\sum_{l=0}^{\infty}\frac{f_l(\theta,\varphi,k)}{(kr)^l} \tag{2.2.1}$$

This expansion is valid for radius $r > r_0$, where r_0 is the radius of a spherical (or spheroidal)surface circumscribing the target/radiator, labeled Γ, and any inhomogeneities of the domain Ω outside r_0, and in particular the radius R of the truncation surface, the exterior domain must be homogeneous and may not contain any objects/obstacles. Bayliss et al(1982). showed that a sequence of local differential operators can be used to annihilate terms in this expansion. The first two local operators acting on the expansion for u, with their corresponding remainders are

$$G_1 u = (\frac{\partial}{\partial r} - B_1)u = o(\frac{1}{kr})^3 \tag{2.2.2}$$

$$G_2 u = (\frac{\partial}{\partial r} + \frac{2}{r} - B_1)(\frac{\partial}{\partial r} - B_1)u = o(\frac{1}{kr})^5 \tag{2.2.3}$$

where

$$B_1 = ik - \frac{1}{r} \tag{2.2.4}$$

In the case of the second-order operator G_2, the second-order radial derivative is replaced by second-order angular derivatives using the Helmholtz equation expressed in spherical coordinates. Setting the remainders to zero results on approximate local radiation boundary conditions which are easily implemented in standard finite element methods. The corresponding local BGT(Bayliss-Gunzburger-Turkel) boundary conditions are defined by relating the radial (normal) derivative to Dirichlet data in the form of the differential map,

$$\frac{\partial u}{\partial r} = B_j u \tag{2.2.5}$$

where for $j = 1$, the first-order operator B_1 is defined in (2.2.4) and for $j = 2$, the second-order BGT operator is

$$B_2 = B_1 + \frac{1}{2B_1}\,\Delta_\Gamma \tag{2.2.6}$$

We consider the spherical coordinate system (r, θ, φ), where $r \geq 0$ is the radial coordinate, $0 \leq \theta \leq 2\Pi$ is the longitude or azimuth, and $0 \leq \varphi \leq \Pi$ is the polar angle or latitude. The three-dimensional BGT conditions may be expressed as Robin boundary conditions on a sphere of radius R (using the Helmholtz equation to eliminate the second radial derivative from BGT-2). The second-order angular derivatives appearing in (2.2.6) are defined by

$$\triangle_{\Gamma}\, u = \nabla_{\Gamma_a} \nabla_{\Gamma_a} u$$
$$= \frac{1}{r^2 sin\theta} \frac{\partial}{\partial\theta}(sin\theta \frac{\partial u}{\partial\theta}) + \frac{1}{r^2 sin^2\theta} \frac{\partial^2 u}{\partial\varphi^2}$$

2.3 Variational formulation of wave problem

Finite element approximation methods are most naturally defined interms of weak formulations; we briefly indicate how hyperbolic problems can be cast in weak form. Moreover, the weak formulation provides a relatively simple way to develop the existence and uniqueness of so called weak solutions.

Let $\Omega \subset R^3$ be a bounded domain with Dirichlet boundary condition and $f \in C^2(\overline{\Omega})$. Consider the wave equation for the unknown function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$

$$u_{tt} - \nabla u - f = 0 \tag{2.3.1}$$

$$u = 0 \quad on \quad \partial\Omega \tag{2.3.2}$$

Now let us replace the classical representation (2.3.1) to (2.3.2) by a weak or variational formulation. The idea is to multiply the equation (2.3.1) with a so called test function $w \in H_0^1(\Omega)$ and then integrate both sides of the new equation over the domain Ω.

$$\int_{\Omega} w \triangle u d\mathbf{x} = -\int_{\Omega} \nabla w . \nabla u d\mathbf{x} + \int_{\partial\Omega} wu.nd\sigma \tag{2.3.3}$$

Using the known rules of integration by parts and then employing the Dirichlet boundary condition (2.3.2), equivalently it holds

$$\int_{\Omega}(u_{tt}w + \nabla uw - fw)d\mathbf{x} - \int_{\partial\Omega} w\nabla u.nd\sigma = 0, \forall w \in V$$

This called **weak form** of the wave problem.

Chapter 3

Research Methodology

In this thesis, we focused on solving the forward problem to compute the scattering of acoustic wave motion with unbounded domain and the mathematical details of the Finite Element Method and the comparison of Absorbing Boundary conditions (1st and 2nd orders). To solve wave motion in spherical scatter placed in air. first introduce the artificial boundary to divide the exterior domain into bounded domain.Then Mesh truncation schemes used were the second order ABC. These problems were solved by writing our own computer programs in MATLAB from scratch. For meshing spherical, we generated our own codes. DistMesh can be used for meshing. DistMesh mesh generators in this class of problems. The mesh connectivity matrices generated by these libraries have to be suitably reordered for use in an FEM code. Therefore, interface code for DistMesh had to be written in MATLAB.

3.1 Study prosedure

To attain the objective of the study, the following procedures are undertaken:

- introduce the artificial boundary to divide the exterior domain into the unbounded domain
- we can obtain an analytical representation of the solution of problem by separation

of variables. Using this analytical representation, we obtain an analytical representation of the DtN operator.

- Imposing a boundary condition using the DtN operator, called the DtN boundary condition, we reduce the original exterior problem.
- Deriving the weak formulation corresponding to the reduced equations for a typical element,
- Calculate the stiffness matrix and load vector for each element in the domain.
- Solving the system of equations obtained.
- Using Matlab language, the problem algorithm is implemented to solve the numerical problems and obtain a function value at each node, analyzing using and interpreting the results.

3.2 Study Area and period

The study site of this research in Adama Science and Technology University (ASTU) and the period is from January, 2017 to July , 2017.

Chapter 4

Result and Discussion

4.1 Mathematical Formulation of Exterior Helmholtz Problem

For exterior problem the most common example is the scattering problem and we focused on unbounded domain. Then the domain of the solution is outside a bounded open set $\Omega \subset R^3$,describing the scatterer.

Example: In the scattering problem the objective is to find the wave that is scattered off Ω from in an incident plane wave $u(x)_{inc} = e^{i\omega x}$ coming in from infinity.

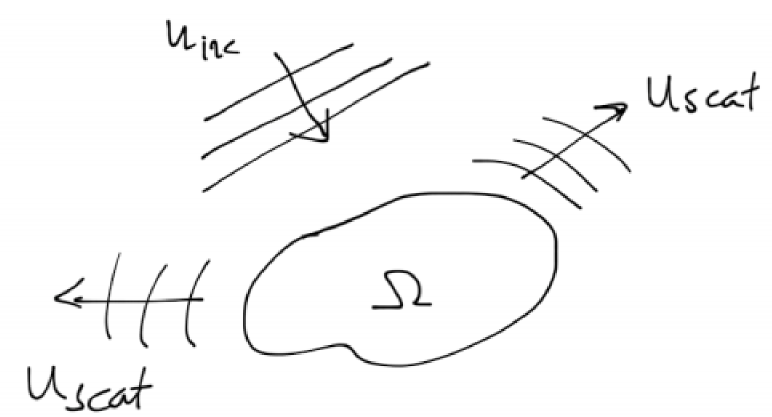

Figure 4.1: Scattering of wave on unbounded domain

The equation is

$$\triangle u(x) + \omega^2 \frac{1}{c^2} u(x) = f(x), x \notin \Omega \tag{4.1.1}$$

with inhomogenous boundary conditions on Dirichlet or Neumann type:

$$\begin{cases} u(x) = g(x), x \in \partial\Omega, (Dirchlet) \\ \dfrac{\partial u(x)}{\partial n} = h(x), x \in \partial\Omega, (Neumann) \end{cases} \tag{4.1.2}$$

Note that c^2 is constant in (4.1.1). In another version of the scattering problem (4.1.1) is set in all of R^3 but c^2 is allowed to vary inside a compact set. Also note that there is no source, $f(x) = 0$, and the waves are instead generated by the in-homogeneous boundary conditions $g(x)$ or $h(x)$.

We let u_{tot} be the sum of the known incident wave u_{inc} and the unknown scattered wave u_{scat}. The total field $u_{tot} = u_{inc} + u_{scat}$ satisfies (4.1.1) with homogeneous boundary conditions on $\partial\Omega$, either Dirichlet with $g = 0$ or Neumann with $h = 0$ depending on the physics of the waves considered. Since u_{inc} clearly also satisfies (4.1.1), so does u_{scat} and on the boundary $u(x)_{inc} + u(x)_{scat} = 0$. Consequently, we have

$$\triangle\, u_{scat} + \omega^2 u_{scat} = f(x), x \notin \Omega \tag{4.1.3}$$

and one of

$$u(x)_{scat} = -u(x)_{inc}, x \in \partial\Omega, (Dirchlet) \tag{4.1.4}$$

$$\frac{\partial u(x)_{scat}}{\partial n} = -\frac{\partial u(x)_{inc}}{\partial n}, x \in \partial\Omega, (Neumann) \tag{4.1.5}$$

The problems (4.1.1, 4.1.2) and, equivalently, (4.1.3, 4.1.4) are well-posed if additional boundary conditions are given at infinity, namely

$$lim_{r \longrightarrow \infty} r \left(\frac{\partial u(x)_{scat}}{\partial n} - iku(x)_{scat} \right) = 0 \tag{4.1.6}$$

where $\dfrac{\partial}{\partial n}$ is differentiation in the radial direction and r is the radius. This is called the Sommerfeld radiation condition or simply the outgoing condition. Without this condition the solution is not uniquely determined.

$$\begin{cases} \triangle\, u - k^2 u = 0, in, \Omega^+ \\ \partial_n u + \beta u = g, on \Gamma := \partial\Omega \\ lim_{r \longrightarrow \infty} r \left(\dfrac{\partial u(x)_{scat}}{\partial n} - iku(x)_{scat} \right) = 0 \end{cases} \tag{4.1.7}$$

Therefor this equation called Exterior Helmholtz equation.

4.2 Discretization Methods for Exterior Helmholtz Problems

The finite element method (FEM) is a well-established method for the numerical solution of boundary value problems on bounded domains. The application of this method on unbounded exterior domains usually involves a decomposition of the exterior. In scattering problems, the obstacle is enclosed by an artificial boundary. The FEM is used for discretization of the bounded domain between the obstacle and the artificial boundary. On the artificial boundary, one can prescribe absorbing boundary conditions(ABC) that incorporate (exactly or approximately) the far-field behavior into the finite element model.

4.2.1 Introduction of an Artificial Boundary

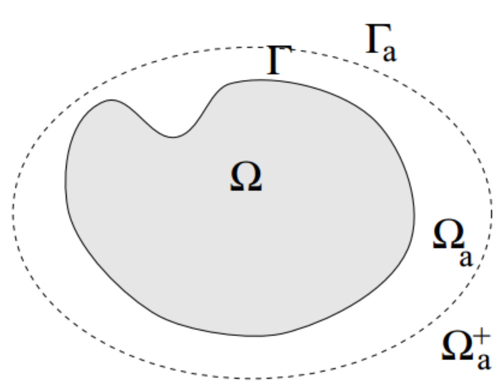

Figure 4.2: Scatterer and artificial boundary

As before, let $\Omega \subset R^3$ be a sufficiently regular domain and denote by $\Omega^+ := R^3 \setminus \overline{\Omega}$ the exterior.

Consider the Neumann boundary value problem

$$\begin{cases} \triangle\, u - k^2 u = 0, in\ \Omega^+ \\ \partial_n u + \beta u = g, on\ \Gamma := \partial\Omega \\ lim_{r \longrightarrow \infty} r \left(\dfrac{\partial u(x)_{scat}}{\partial n} - iku(x)_{scat} \right) = 0 \end{cases} \tag{4.2.1}$$

Here, $u(x)$ represents the spatial part of the acoustic pressure or velocity potential, with wavenumber $k \in C, Im(k) \geq 0$. The sign convention for the phase is $e^{i\omega t}$, where $i = \sqrt{-1}$ and ω is the natural frequency. The normal derivative $(\frac{\partial u}{\partial n}) = \nabla u.\mathbf{n}$ defines the gradient in the direction of the unit outward vector normal to $\partial\Omega$. In the above, $r = \|\mathbf{x}\|$ is a radius centered near the origin of the sound source. The Sommerfeld radiation condition allows only outgoing waves proportional to $exp(ikr)$ at infinity. The radiation condition requires that energy flux at infinity be positive, thus ensuring unique solutions. Finite element methods typically introduce an artificial boundary Γ_a, which divides the original unbounded domain into two regions: a bounded computational domain Ω_a discretized with the finite element method and an infinite residual region $\Omega_a^+ = R \setminus \Omega_a$. Reducing the size of the bounded computational domain decreases the computational cost and memory storage. Methods for modeling the exterior complement $\Omega_a^+ = R \setminus \Omega_a$, i.e., the infinite region exterior to the artificial boundary , can be divided into three main categories: local or non-local absorbing (non-reflecting) boundary conditions, infinite elements, and absorbing layers. Infinite element methods represent the exterior complement by assuming a radial approximation with outgoing wave behavior. Matched absorbing layers attempt to decay outgoing waves in a relatively thin layer exterior to Γ_a. For the non-reflecting (absorbing) boundary conditions, the outgoing wave solution in Ω_a^+ is represented by a relation of the unknown solution and its derivative on the artificial truncation boundary Γ_a. Options include matching exact analytical series solutions as used in the non-local

Dirichlet-to - Neumann (DtN) map , and various local approximations. The artificial boundary Γ_a is usually taken to be a surface defined in separable coordinates for efficiency, e.g., a sphere or spheroid. Formulations on non-separable boundaries have also been developed; in this case the formulations are usually applied directly to the surface of the scatterer, thus completely avoiding discretization Ω_a. Unbounded domain treatments may also be derived for acoustic waveguide problems. For absorbing boundary conditions, the originally unbounded exterior problem is replaced by an equivalent reduced problem defined on the bounded domain : Find $u(x) \in R^3$, such that

$$\begin{cases} \triangle\, u - k^2 u = 0, in\ \Omega^+ \\ \partial_n u + \beta u = g, on\ \Gamma := \partial\Omega \\ \dfrac{\partial u}{\partial n} = -Bu, in\ \Gamma_a \end{cases} \tag{4.2.2}$$

where B is a linear operator called the Dirichlet-to-Neumann (DtN) map relating Dirichlet data to the outward normal derivative of the solution on Γ_a.

4.2.2 Finite Element Method For Exterior Helmholtz Problem

The corresponding weak (variational) form of the boundary value problem with the B_2 BGT operator is: Find the trial solution u, such that, for all test functions w and integrating we get,

$$\int_{\Omega_a} \triangle\, uwdx + \int_{\Omega_a} k^2 uwdx = \int_\Gamma fwd\Gamma, x \in \Omega_a \tag{4.2.3}$$

Then use Greens theorem to integrate Eq. (4.2.3) by parts to obtain:

$$-\int_{\Omega_a} \nabla w \nabla ud\Omega_a + k^2 + \int_{\Omega_a} w.ud\Omega_a + \int_{\Gamma_a} w\frac{\partial u}{\partial n} d\Gamma_a + \int_\Gamma fwd\Gamma \tag{4.2.4}$$

Adding the boundary condition to Eq. (4.2.17):

$$\int_{\Omega_a} \nabla w.\nabla ud\Omega_a - k^2 \int_{\Omega_a} w.ud\Omega_a + \int_{\Gamma_a} w.Bud\Gamma_a = -\beta \int_\Gamma wud\Gamma + \int_\Gamma fwd\Gamma \tag{4.2.5}$$

Adding the second order BGT operator(2.2.6) to Eq.(4.2.5) and we can write in this form.

$$B(w,u) - B_{\Gamma_a}(w,u) = F(w) \tag{4.2.6}$$

where

$B(w,u) = \int_{\Omega_a}(\nabla w.\nabla u - k^2 wu)dx + \int_{\Gamma} \beta wud\Gamma$'

$B_{\Gamma_a}(w,u) = \int_{\Gamma_a} B_1 wud\Gamma_a - \int_{\Gamma_a} \frac{1}{2B_1}\nabla_{\Gamma_a} w.\nabla_{\Gamma_a} ud\Gamma_a,$

$F(w) = \int_{\Gamma}(wg)d\Gamma$

with differential surface area $d\Gamma_a = R^2 sin\theta d\theta d\varphi$ on a spherical truncation boundary of radius R.

Conventional finite element methods partition the computational domain Ω into non-overlapping tetrahedral elements (sub-domain)Ω_e for R^3. Each element is defined by four nodes that are interconnected by six edges with continuous piecewise polynomials.

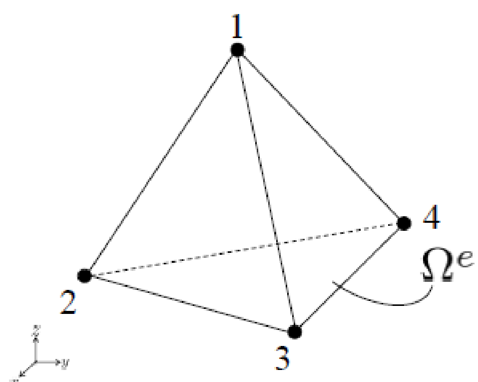

Figure 4.3: tetrahedral element

Edg No. i	node i_1	node i_2
1	1	2
2	1	3
3	1	4
4	2	3
5	4	2
6	3	4

Table 4.1: Edge numbering for a tetrahedral element

The vector field inside a single tetrahedron can be written as

$$u^e(x) = \sum_{i=1}^{6} u_i^e N_i^e(x) \tag{4.2.7}$$

where $u_i^e is$ the six unknown parameters associated with each edges i. The total field is obtained by evaluating equation (4.2.7)and $N_i^e(x)$ is the vector basis function given by

$$N_i^e(x) = l_i^e(\lambda_{i_2}^e(x)\nabla\lambda_{i_1}^e(x) - \lambda_{i_2}^e(x)\nabla\lambda_{i_1}^e(x)) \tag{4.2.8}$$

where the edge number i is defined as lying between nodes i_1 and i_2 as specified in Table. 4.2, and l_i^e is the length of edge i,. For a local node l belonging to tetrahedron e the nodal linear basis function given by

$$\lambda_l^e(x) = \frac{1}{6V^e}(a_l^e + b_l^e x + c_l^e y + d_l^e z) \tag{4.2.9}$$

node No. l	node i	node j	node k
1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Table 4.2: local indices to calculate nodal basis function

where V^e is the volume of the tetrahedron and a_l^e, b_l^e, c_l^e and d_l^e are coefficients dependent on the tetrahedron geometry. The volume of a tetrahedron e is calculated as

$$V = det \begin{pmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{pmatrix} \tag{4.2.10}$$

where $x_{1,2,3,4}^e, y_{1,2,3,4}^e$ and $z_{1,2,3,4}^e$ are $X-Y-Z$ coordinates of nodes $\{1,2,3,4\}$. For a node l the basis function coefficients calculated as follows:

$$a_l^e = (-1)^{l-1}(x_iy_jz_k - x_iy_kz_j - x_jy_iz_k + x_jy_kz_i + x_ky_iz_j - x_ky_jz_i)$$

$$b_l^e = (-1)^l(y_iz_j - y_jz_i - y_iz_k + y_kz_i + y_jz_k - y_jz_k)$$

$$c_l^e = (-1)^{l-1}(x_iz_j - x_jz_i - x_iz_k + x_kz_i + x_jz_k - x_kz_j)$$

$$d_l^e = (-1)^l(x_iy_j - x_jy_i - x_iy_k + x_ky_i + x_jy_k - x_ky_j)$$

where the nodal local indices $\{i, j, k\}$ associated with a node l are defined in a cyclic manner as shown in Table 4.2. This cyclic scheme assumes specific ordering of the local nodes around the tetrahedron, which is shown in Figure 4.4.

Now $\lambda_{i_1}^e, \lambda_{i_2}^e$ are the three-dimensional linear nodal basis function and ∇ special gradient operator. From equations (4.2.8) and (4.2.9), the edge elements are given as

$$N_i^e(x) = \frac{l_m^e}{36V^2}[(A_{xm}+B_{xm}y+C_{xm}z)X+(A_{ym}+B_{ym}x+C_{ym}z)Y+(A_{zm}+B_{zm}x+C_{zm}y)Z] \tag{4.2.11}$$

where

$A_{xm} = a_{i_1}b_{i_2} - a_{i_2}b_{i_1}$

$B_{xm} = c_{i_2}b_{i_2} - c_{i_2}b_{i_1}$

$C_{xm} = d_{i_1}b_{i_2} - d_{i_2}b_{i_1}$

$A_{ym} = a_{i_1}c_{i_2} - a_{i_2}c_{i_1}$

$B_{ym} = b_{i_1}c_{i_2} - b_{i_2}c_{i_1} = -B_{xm}$

$C_{ym} = d_{i_1}c_{i_2} - d_{i_2}c_{i_1}$

$A_{zm} = a_{i_1} d_{i_2} - a_{i_2} d_{i_1}$

$B_{zm} = b_{i_1} d_{i_2} - b_{i_2} d_{i_1} = -C_{xm}$

$C_{zm} = c_{i_1} d_{i_2} - c_{i_2} d_{i_1} = -C_{ym}$

Also N_i can be shown to satisfy the condition,

$$N_i = \begin{cases} 1, (edge = m) \\ 0, (otheredge) \end{cases} \tag{4.2.12}$$

The basis (shape) functions $N_i(x)$, associated with element nodes are C^0continuous interpolation functions . The continuous approximation is written as the linear combination,

$$u^h(x) = \sum_{i=1}^{N_{dof}} N_i(x) u_i = \mathbf{N}^T(x)\mathbf{u} \tag{4.2.13}$$

where $N \in R^{N_{dof}}$is a column vector of standard C^0 basis functions, and $\mathbf{u} \in C^{N_{dof}}$ is a column vector containing the N^{dof} (degree of freedom)unknown nodal values $u_i = u^h(xi)$, where $u^h(xi)$ is the approximation of the solution u at node x_i. Using a standard Galerkin finite element approximation, test (weighting) functions w^h are expressed as a linear span of the same basis functions. Substituting equation(4.2.13) into equation (4.2.6), integrating over the solution domain, and interchanging the summation and integration leads to the sparse, complex-symmetric (non-Hermitian) linear algebraic system,

$$\mathbf{Zd{=}f}, \tag{4.2.14}$$

where

$$Z = S - k^2 M - K_{\Gamma_a}$$

with matrix coefficients

$$(S)_{i,j} = \int_{\Omega_a} \nabla N_i . \nabla N_j dx + \int_{\Gamma} \beta N_i N_j d\Gamma,$$
$$(M)_{i,j} = \int_{\Omega_a} N_i N_j dx,$$
$$(K_{\Gamma_a})_{i,j} = \int_{\Gamma_a} B_1 N_i N_j d\Gamma_a - \int_{\Gamma} \frac{1}{2B_1} \nabla_{\Gamma} N_j d\Gamma_a$$

The excitation vector

$$(f)_j = \int_{\Gamma_a} N_i g d\Gamma_a$$

Example: consider the pde

$$\begin{cases} \triangle\, u + k^2 u = 0, in, \Omega^+ \\ u = 0, on, \Gamma \\ \dfrac{\partial u}{\partial n} = (ik + \dfrac{1}{r_0})u, in, \Gamma_a \end{cases} \tag{4.2.15}$$

The first object to be considered has a radius $\frac{\lambda}{2}$ where λ is the free space propagation wavelength. The first order ABC is imposed at $\frac{3\lambda}{2}$, a wavelength away from the object. The scatter field has the form $u(x)_0 = e^{ik_0 x}$ for $u(x)_0 = \frac{1V}{m}$, $k_0 = \frac{2\Pi}{\lambda}$. So without loss of generality, $\lambda = 1$. The weak form of differential equation

$$-\int_{\Omega_a} \triangle\, uwdx - \int_{\Omega_a} k^2 uwdx = 0, x \in \Omega_a \tag{4.2.16}$$

Then use Greens theorem to integrate Eq. (4.2.16) by parts to obtain:

$$-\int_{\Omega_a} \nabla w \nabla ud\Omega_a - k^2 \int_{\Omega_a} w.ud\Omega_a + \int_{\Gamma_a} w\frac{\partial u}{\partial n} d\Gamma_a = 0 \tag{4.2.17}$$

The tetrahedral shape functions are:

$N_i^e(x) = \frac{l_m^e}{36V^2}[(A_{xm}+B_{xm}y+C_{xm}z)X+(A_{ym}+B_{ym}x+C_{ym}z)Y+(A_{zm}+B_{zm}x+C_{zm}y)Z]$

The primary unknown quantity u is interpolated within an element e as $u^e(x) = \sum_{i=1}^{6} u_i^e N_i^e(x)$

Here u_i^e denotes the value of the unknown quantity at local node j of element e.

$\int_{\Omega_a} \left[\frac{\partial N_i}{\partial x}(\sum_{j=1}^{6} u_j^e \frac{\partial N_i}{\partial x}) + \frac{\partial N_i}{\partial y}(\sum_{j=1}^{6} u_j^e \frac{\partial N_i}{\partial y}) + \frac{\partial N_i}{\partial z}(\sum_{j=1}^{6} u_j^e \frac{\partial N_i}{\partial z})\right] d\Omega_a$

$+ \int_{\Omega_a} \beta N_i(\sum_{j=1}^{6} u_j^e d\Omega_a + \int_{\Gamma_a} N_i(\frac{\partial u}{\partial z} n_z + \frac{\partial u}{\partial y} n_y + \frac{\partial u}{\partial z} n_z) d\Gamma_a$

This equation can be represented in a matrix form given by

$$(S)_{i,j} = \int_{\Omega_a} \nabla N_i.\nabla N_j dx + \int_{\Gamma} \beta N_i N_j d\Gamma,$$

$$(M)_{i,j} = \int_{\Omega_a} N_i N_j dx,$$

$$(K_{\Gamma_a})_{i,j} = \int_{\Gamma_a} B_1 N_i N_j d\Gamma_a - \int_{\Gamma} \frac{1}{2B_1} \nabla_{\Gamma} N_j d\Gamma_a$$

For this particular problem, the concentric sphere based meshing algorithm described above was used. A plot of the total electric field as a function of the polar angle at a radius $\rho = \lambda$, is shown below on the left, for various values of the mesh discretization parameter h (expressed in units of λ).

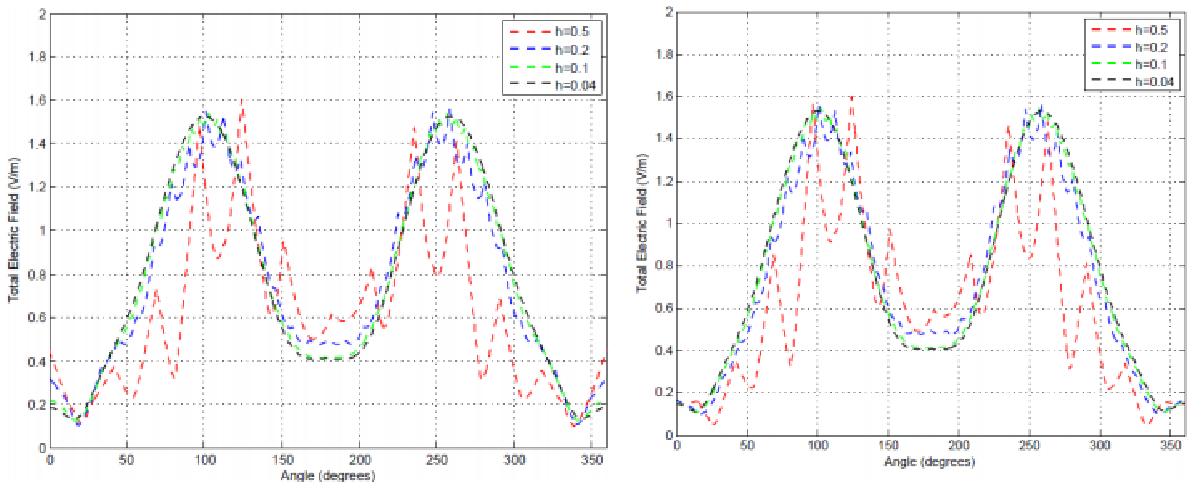

Figure 4.4: First and Second Order solutions for PEC spherical Scattering for different discretization sizes

The same experiment was then repeated but with a second order ABC imposed on a mesh truncation boundary, rather than the first order condition. The corresponding plot of the total electric field is shown in the figure on the right above. This problem has an analytical solution given by

$$E(\rho,\theta) = E_0 \sum_{n=-\infty}^{\infty} j^{-n}[J_n(k_0\rho) - \frac{J_n(k_0a)}{H_n^1(k_0a)} H_n^1(k_0\rho)]e^{jn\theta} \tag{4.2.18}$$

where E the total electric field, a is the radius of the PEC sphere, ρ and θ are the polar coordinates. Here, J_n denotes the Bessel Function of the First Kind of order n and $H_n^{(1)}$ n denotes the Hankel Function of the first Kind of order n. The absolute values of the exact, first order and second order solutions are plotted in the figure below, for a discretization size of $\frac{\lambda}{25}$, on the boundary .

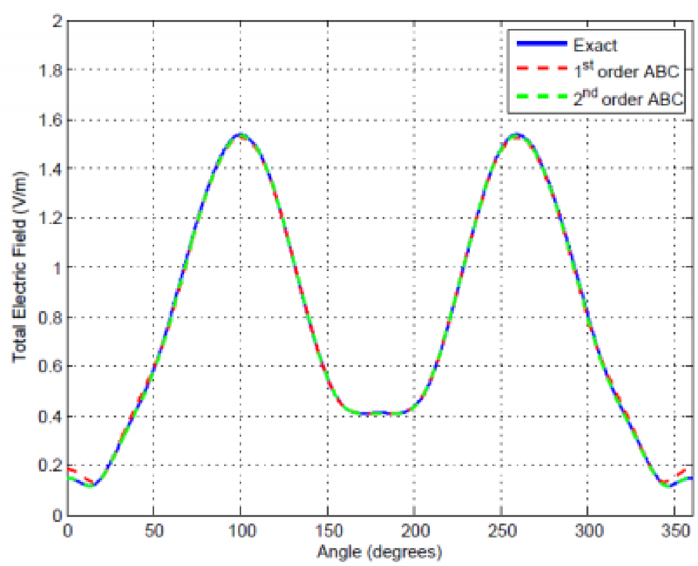

Figure 4.5: Analytical and FEM Solutions for the PEC spherical Scatterer, at discretization $\frac{\lambda}{25}$, evaluated at $\rho = \lambda$

From the plots, we can infer that

1. Both 1st and 2nd order ABC solutions become smoother as the discretization size is lowered.

2. For a discretization size of $\frac{\lambda}{25}$, the 2nd order ABC-based solution almost coincides with the analytical solution, whereas the 1st order ABC-based solution deviates from the analytical solution more significantly at very small angles ($\theta = 0^0$), moderately large angles (100^0 and 260^0) and again at around 360^0.

Scattering from a PEC sphere in Air using the second Order ABC (Effect of varying Discretization)

For a PEC sphere of radius $r = \dfrac{\lambda}{2}$, and an ABC boundary of radius $R = \dfrac{3\lambda}{2}$, the analytical and FEM solutions are compared at $\rho = \lambda$, for different discretizations, to illustrate the effect of meshing. The mesh pattern, a contour plot of the total electric field in the region, and a graph of the FEM and exact solution have been plotted for three discretization sizes, and the results are shown below.

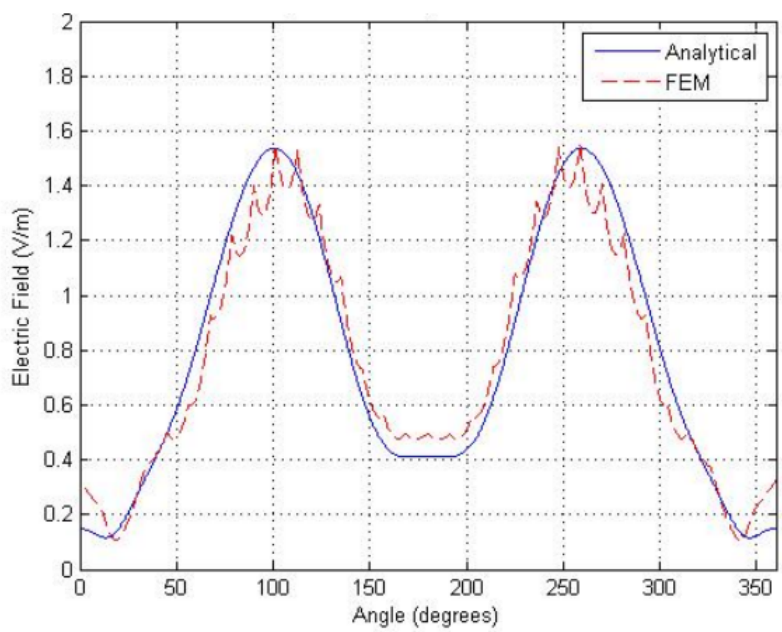

Figure 4.6: Discretization Size= 0.2

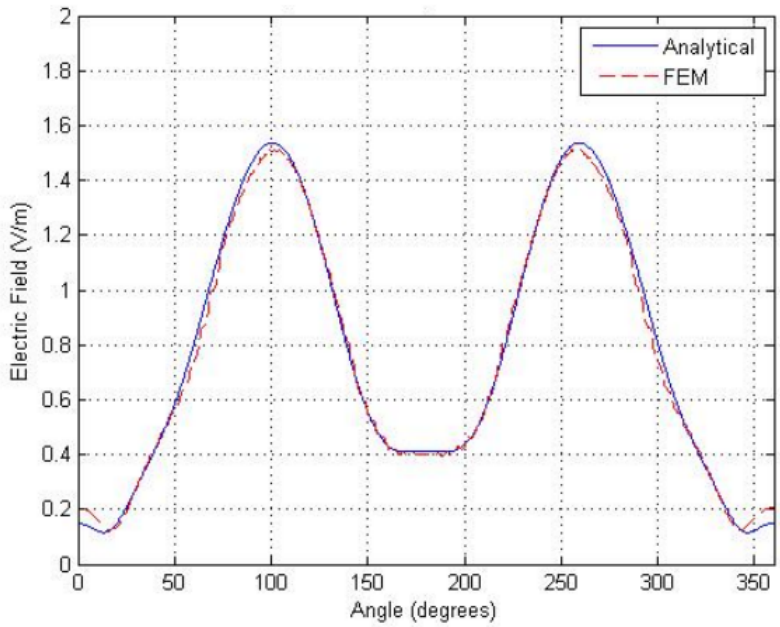

Figure 4.7: Discretization Size= 0.09

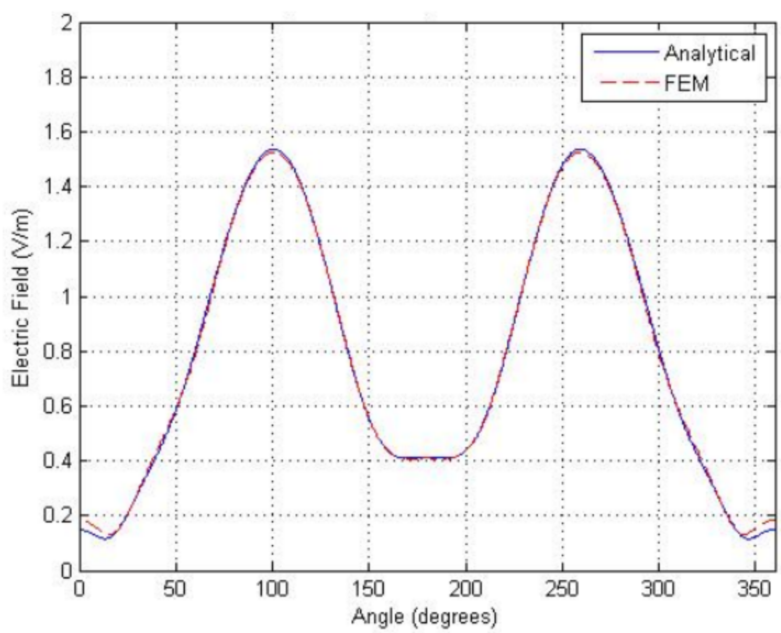

Figure 4.8: Discretization Size= 0.04

4.3 Conclusion

A framework for solving the wave equations in three dimensions has been constructed from scratch in MATLAB, comprising of meshing, preprocessing and FEM routines, with a feature to impose Mixed Boundary Conditions, in particular, the 1st and 2nd order ABCs. As a part of this thesis, a system for meshing of arbitrary shapes in three dimensions has been put in place by matlab codes with DistMesh for meshing. Completely general FEM codes have been written to compute the total and scattered field distributions in the near region of one scatterers in three dimensional domains. Results from several electromagnetic scattering situations have been mentioned and discussed in this report,comparing the 1st and 2nd order ABCs.

- The 2nd order ABC is better than the 1st order ABC as for mesh truncation, as it introduces fewer artifacts in the computational solution, by capturing the curvature of the field.

- The 2nd order ABC can be imposed at a smaller distance from the scatterer, than the 1st order ABC, without compromising fidelity of the solution.

- It is seen that for $r = 0.5\lambda R = 1.5\lambda$, the FEM solution is very nearly equal to the exact solution for the discretization size 0.04.

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