SOLVING 2-D LAPLACE EQUATION BY BOUNDARY ELEMENT METHOD WITH H-MATRIX

BY

HANA ENDRIS

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By

Hana Endris

Advisor

Lemi Guta(PhD)

Co-Advisor

Mr.Yadeta Chimdesa

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Approval Sheet

Name of Student	Signature	Date
Advisor	Signature	Date
External Examiner	Signature	Date
Internal Examiner	Signature	Date
Chair person	Signature	Date
HoD	Signature	Date
School Dean	Signature	Date
SGS Dean	Signature	Date

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Abstract

Hierarchical matrices uses a data-sparse representation of fully populated matrices. In this study we solve 2-D Laplace equation using boundary element method. The difference equation resulting from boundary element approximation of a problem was a full system of linear equation. H-matrices used to transform the coefficient matrix resulting from boundary element method of 2-D Laplace equation in to sparse matrix to improved efficiency of the solution. Compare the computational times and memory usage of the original matrix and the sparse matrix. Finally, numerical results assess the efficiency of the method on 2-D Laplace equation.

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Notations and Abbreviations

PDEs	Partial differential equations.
FDM	Finite difference method.
FEM	Finite element method.
BEM	Boundary element method.
BC	Boundary condition.
H-Matrix	Hierarchical matrix.
BIEM	Boundary integral equation method.
BIM	Boundary integral method.
∇	Laplace operator.
BVP	Boundary value problem.
2-D	Two dimensional.
H-tree(T_I)	Cluster tree.
$T_{I\times J}$	Block cluster tree.
R(k) -Matrix	Low-rank matrix.
CPU	Central processing unit.
O (n)	Order of n.
O (n^2)	Order of n^2.
$u(x, y)$	The exact solution of the problems.

Chapter 1

Introduction

1.1 Background of the Study

Numerical Analysis is a branch of mathematics that deals with the development and use of numerical methods for solving problems. One concern is the development of appropriate numerical models for applied problems. Another is the construction and analysis of efficient algorithms for various mathematical problems such as those of numerical integration and differentiation, and combinatorial and constrained optimization problems.

The boundary element method (BEM) is a numerical method, in essence a numerical integration of integral equation representations of the governing differential equations (DEs). There is an intermediate step, however, whereby the integral equation is reformulated so that actual BVPs can be solved, a process that results is known as Boundary Integral Equation (BIE) formulations. In many cases, the BEM requires a boundary only discretization of the problem, in contrast to both FEM and FDM that require full domain discretizations (Manolis,G.D. 2008).

Until the beginning of the eighties, the BEM was known as boundary integral equation method(BIEM). G.Green formulated in 1828 the integral representation of the solution of the Dirichlet and Neuman problems of the Laplace equation by introducing the so-called Green's function for the problems (Wang, C. T., & Brodsky, R. F.,1950).
The boundary integral equation(BIE) is derived using Green's theorem by applying Green's identity for any point in the surface.

Equation involving one or more partial derivatives of a function of two or more independent variables is called partial differential equations. The solution of partial differential equations is the most important topic in many areas of science and engineering (e.g. in Heat transfer, fluid dynamics, quantum mechanics, etc.).

The vast majority of PDEs model cannot be solved analytically. So, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically.

Laplace′s equation is widely used in physics because its solutions u(known as harmonic functions) occur in problems of electrical, hydrodynamics, magnetic, and gravitational potentials of steady-state temperatures. The equation was discovered by the French mathematician and astronomer Pierre-Simon Laplace (1749-1827) (Zachmanoglou, E. C.,& Thoe, D. W. 1986).
Laplace′s equation states that the sum of the second-order partial derivatives of u, the unknown function, with respect to the Cartesian coordinates, equals zero.

i.e;

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.1.1}$$

A BVP is a problem, in which some data on the boundary is given to solve for the rest of the boundary and the domain Ω. Due to various complexities, BVPs do not always have closed form solutions or analytic solutions. It is often inevitable to resort to some kinds of approximations and numerical methods.

Hierarchical matrices (or short H-matrices) are efficient data-sparse representations of certain densely populated matrices. The basic idea is to split a given matrix into a hierarchy of rectangular blocks and approximate each of the blocks by a low-rank matrix. The low-rank matrices can be represented as a product of two rectangular matrices. let $A^{'} \subset A$ be a subblock of size $n^{'} \times n^{'}$ and $\text{rank}(A^{'}) = k$, where $k << n^{'}$. Then there exist two matrices $B, C \in R^{n^{'} \times k}$, such that $A^{'} = BC^T$. Having this representation for $A^{'}$, then storage for its $(n^{'})^2$ entries reduce to $2kn^{'}$ entries which are the total entries of B and C. Therefore, this research work focused on the elapsed time and memory usage of BEM solution for Laplace equation with solution of H-matrix.

To do this, the researcher discussed on some formulae which is used to transform the BVPs into an equivalent BIE problem. The use of the fundamental solution (Green's function) and the Green's integral theorems are useful tools for this purpose to numerically approximate Laplace equation by using BEM and compare the computational time and memory usage of the approximate solution of H-matrix and BEM.

1.2 Statement of the Problem

Systems of linear equations may arise from discretization of Laplace equation and other problems. To solve these large systems of linear equations

$$Ax = b \tag{1.2.1}$$

we can apply iterative methods. Even though these methods are theoretically founded, they may suffer from slow convergence. To speed up the convergence of iterative methods, hierarchical-matrix (H-matrix) preconditioning is a key technique to solve the system (1.1.2) (Yang,F. 2008).

This study uses the boundary element discrtization of 2-D Laplace equation and the result becomes full matrix. Then apply H-matrix technique in order to avoid the order of n^2 ($O(n^2)$) complexity. Finally, compare the cpu time and memory utilized by this method and the usual BEM.

This study intended to answer the following basic questions:

- How to descrtize the 2-D Laplace equation using BEM?
- How to solve the resulting system with H-matrix ?
- How much cpu time and memory is needed ? When using H-matrix ?

1.3 Objective of the Study

1.3.1 General Objective of the Study

The main objective of this study is to solve the 2-D Laplace equation by boundary element method(BEM) with H-matrix.

1.3.2 Specific Objective of the Study

The specific objectives of this study is :
(1) To Descritiz the Laplace equation by BEM
(2) To approximate the dense matrix resulting from the 2-D Laplace equation to sparse matrix using H-matrix.
(3) To investigate the efficiency of the method by analyzing the utilization of computational resource.

1.4 Benefit and Beneficiaries

The solutions of Laplace equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials (Zachmanoglou, E. C.,& Thoe, D. W. 1986). The other important aspect in which the BEM distinguishes itself from other numerical method is the fact that only the boundary of the domain needs to be discretized. In many other numerical methods, such as the FEM, FDM or FVM, in addition to the boundary the interior of the domain also needs to be discretized.
Therefore, solution of this equation subjected to different boundary condition(BC) highly important in these fields. In addition, it help other researchers on this area for further work.

1.5 Delimitation of the Study

Among the Numerical methods available to solve PDE of this type like finite element method(FEM), boundary element method(BEM) and finite difference method(FDM), this study focused on boundary element method to discrtize the 2-D Laplace equation and uses H-matrix to solve the resulting system of linear equation.

Chapter 2

Literature Review

The theory behind integral equations was developed by Fredholm (1903), while applications followed later in potential theory and elastostatics (Mazya, V. G. 1991). A comprehensive treatise on the theory of singular integral equations is that of Mikhlin,S.G (2014).

Numerous articles have been appeared on the use of numerical methods based on BIE formulations in all fields of engineering science such as potential theory, potential fluid flow, acoustics, torsion, electric and magnetic field theories, electrostatics, electrodynamics, plates and shells, heat conduction, viscoelasticity, thermo elasticity, fracture mechanics, plasticity, water waves, viscous flow, ground water flow, corrosion, contact problems and coupled field problems which can be surmised by looking through a number of text books that have appeared in the last twenty years (Bonnet,M. 1999).

2.1 Boundary Element Method(BEM)

The BEM was originated with C. A. Brebbia et al.(1970)at Southampton University in England and has been used across the entire spectrum of engineering science (like; structural mechanics, fluid mechanics, wave propagation, electromagnetic theory, acoustics, fracture mechanics and corrosion), that can be ascertained by perusing through books on the BEM that have appeared in recent years (Brebbia, C. A. et al.,2012)

Using BEM, a problem is formulated in terms of integral equations relating boundary values and solution is then found numerically. If values in the interior domain are needed, they can be calculated from the boundary data. Thus, the dimension of the problem at hand is reduced by one, while the resulting system of algebraic equations is invariably smaller in size compared to similar systems obtained from use of the FEM or the FDM. Only in certain cases (example, nonlinearities, inhomogeneities, certain types of body forces) does the BEM require discretization of the interior domain.There are two basic categories of BEM, namely indirect and direct. In the former case, which is also known as the source method, the boundary integral formulation is accomplished in terms

of fictitious source densities from which the physical quantities of the problem can be recovered. In the latter case, the boundary integral formulation is accomplished in terms of physical quantities such as potentials, fluxes, displacements, etc., which are computed directly without use of intermediate steps (Manolis, G.D. 2008).

Fredholm (1903) established the existence of integral equation solutions to potential problems on the basis of a limiting discretization procedure and identified the Fredholm integral equations of the first, second and third type (Fredholm, I.1903). The first application of a direct BEM was in fluid mechanics (specifically for the ax symmetric jet problem) by Trefftz(1926), who used the method of successive approximations to satisfy his integral equation (Trefftz, E. 1926). Prager examined doubly symmetric potential flow past an elliptic cylinder using a direct-type boundary integral formulation and subsequently divided the surface of the problem into elements, thus reducing the integral equations into a system of algebraic equations (Prager, W. 1928).

2.2 Hierarchical Matrix(H-matrix)

The general framework of H -matrices goes back to Hackbusch, W.(1999) who introduces H -matrices and the basic algebraic operations. He showed that matrix-vector and matrix-matrix operations can be performed with almost linear complexity for a certain class of matrices and his book gives a comprehensive overview of the current state of the art in the field of hierarchical matrices (Hackbusch, W 2015). An extensive introduction to H -matrices is given by Bebendor, M. (2008).

From an algorithmic point of view, H-matrices have a relatively simple structure, the matrix is split into a hierarchy of submatrices, and each submatrix is approximated by low rank. Since all submatrices are independent, simple recursive algorithms can be used to perform various operations (Bebendorf, M., & Venn, R. 2012).

A major advantage of hierarchical matrix methods compared to multigrid techniques is that they can handle jumping and anisotropic coefficients without the need of specially adapted algorithms (Borm, S., & Gordes, J. 2013).

Most arithmetic algorithms for H-matrices are based on a sequence of low-rank updates that are applied to submatrices of a given H-matrix. This study introduces an algorithm that performs similar updates on H-matrices in linear time. Numerical experiments demonstrate that the new approach leads to preconditioners for FEM and BEM problems that require O(n log n) operations for setup and, at least for two-dimensional problems, O(n) units of storage (Borm, S.,& Reimer, K. 2013).

Chapter 3

Methodology of the Study

3.1 Study Design

The study involves numerical implementation with C and matlab, so it is an experimental research. For the study boundary of the domain is discretized in to boundary elements. Then, we add up the integrals over each element, such that the BIE is discrtized into system of linear equation whose coefficient matrix is fully populated and obtain a function value at each node or the numerical approximation to the solution of our BVP.
Finally, the resulting linear system is solved using BEM and H-matrix .

3.2 Study Procedures

To obtain the objective of this study we will follow the following steps.

✓ Change the BVPs(PDEs) in to BIEs formulation using green second identity and fundamental solution of the problem,

✓ Discretize the boundary integral, in to constant elements,

✓ Add up the integral over each element, such that the BIE is discretized in to a system of linear equation

✓ solve the resulting system by using BEM with matlab and H-matrix with a C programming.

3.3 Source of Information

The applicable source of information for this study are: books, published journals , articles and related studies from the internet.

Chapter 4

Mathematical Preliminaries

4.1 Classification of Partial Differential Equation

Partial differential equations (PDEs) are used to formulate problems related to many phenomena that is distributed in space and time. Consider the second order linear partial differential equation in two independent variable x and y.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \tag{4.1.1}$$

where the coefficients A,B,C,D,E,F and G are constants or are functions of the independent variables x and y.

The classification of PDEs are based on the sign of the discriminant $B^2 - 4AC$ as follows:

1) If $B^2 - 4AC > 0$ the PDE is called hyperbolic. For instance the wave equation described by $u_{tt} = c^2 u_{xx}$ is a hyperbolic PDE.

2) If $B^2 - 4AC = 0$ the PDEs is called parabolic. For instance the heat equation described by $u_t = u_{xx}$ is a parabolic PDE.

3) If $B^2 - 4AC < 0$ the PDE is called Elliptic. For instance the Poisson equation described by $u_{xx} + u_{yy} = f$ and the Laplace equation described by $u_{xx} + u_{yy} = 0$ are elliptic PDEs.

4.1.1 2-D Laplace Equation

Laplace Equation is a second order linear partial differential equation (PDE) that appears in many areas of science and engineering such as: electricity, fluid flow and steady heat conduction. Solution of this equation in a domain requires the specification of certain

conditions that the unknown function must satisfy at the boundary of the domain.
The Laplace equation is an equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.1.2}$$

The solution $u(x, y)$ satisfies the boundary condition on $\partial\Omega$, where $\partial\Omega$ the boundary of domain Ω.

Boundary conditions is a condition, that is required to be satisfied at all or part of the boundary of a region in which a set of differential equation is to be solved.

4.2 Boundary Element Method(BEM)

This is a numerical method for solving partial differential equations encountered in mathematical physics and engineering. In Boundary element method, governing differential equations changes to integral equations that are applied on the boundary. These integrals are numerically integrated over the boundary. BEM is simply and geometrically applied for any complex shape (Ghassemi, H. et al., 2016).

In this way due to the complexity of integrating functions, the analytical integration is not recommended to calculate integrals and numerical integration and Gaussian integration method is used instead of it. The total integral is calculated by adding all the integrals on all elements our approach to solve this equation is that the use of fixed element according to the geometry and boundary conditions (Ghassemi, H. et al., 2016).

4.2.1 Fundamental Function

Before one applies the BEM to a particular problem, it is necessary to obtain a fundamental solution for the problem.
The fundamental solution to the Laplace equation is,

$$\lambda = \frac{1}{2\pi} ln|Q - P| = \frac{1}{2\pi} lnr \tag{4.2.1}$$

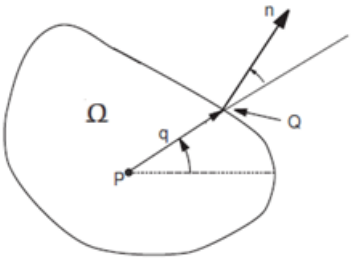

Figure 4.1: Domain with source point P

Where,

$|Q - P| = \sqrt{(n-x)^2 + (m-y)^2} = r$
And $Q(n, m)$ is any point in the domain and $P(x, y)$ is a point source.

The fundamental solution of Laplace equation is also known as the free space Greens function for Laplace equation. The fundamental solution of a particular equation is the weighting function that is used in the boundary element formulation of that equation.

4.2.2 Green's Second Identity

Consider the function $u = u(x, y)$ and $v = v(x, y)$ which are twice continuously differentiable in a domain Ω and once on a boundary Γ.
Applying Gauss-Green theorem
Theorem 1 (Gauss-Green Theorem)
The Gauss-Green theorem is a fundamental identity, which relates the integral of the derivative of a function over a domain Ω to the integral of that function on its the boundary Γ. The domain may be two or three-dimensional. For simplicity of presentation, this relationship is derived for the two-dimensional case. Consider the plane domain Ω bounded by the curve Γ.
i.e Gauss-Green theorem means $(\int_\Omega g\frac{\partial f}{\partial x}d\Omega = -\int_\Omega f\frac{\partial g}{\partial x}d\Omega + \int_\Gamma fgnxds)$.

for $g = v, f = \frac{\partial u}{\partial x}$ and for $g = v, f = \frac{\partial u}{\partial y}$ then adding the resulting equation we arrive at the following

$$\int_\Omega v(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})d\Omega = -\int_\Omega (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y})d\Omega + \int_\Gamma v(\frac{\partial u}{\partial x}nx + \frac{\partial u}{\partial y}ny)ds \qquad (4.2.2)$$

Similarly applying Gauss-Green theorem for $g = u, f = \frac{\partial v}{\partial x}$ and for $g = u, f = \frac{\partial v}{\partial y}$ then adding the resulting equations, we obtain

$$\int_\Omega u(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})d\Omega = -\int_\Omega (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y})d\Omega + \int_\Gamma u(\frac{\partial v}{\partial x}nx+\frac{\partial v}{\partial y}ny)ds \qquad (4.2.3)$$

Subtracting Eq.(4.2.3) from Eq.(4.2.2) yields

$$\int_\Omega (v\nabla^2 u - u\nabla^2 v)d\Omega = \int_\Gamma v(\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n})ds \qquad (4.2.4)$$

Where ∇^2 is known as the Laplace operator or harmonic operator and
$\frac{\partial}{\partial n} = n.\nabla = (n_x i + n_y j).(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}) = n_x\frac{\partial}{\partial x}+n_y\frac{\partial}{\partial y}$
is the operator that produces the derivative of scalar function in the direction of n.
Equation(4.2.4) is known as Green's second identity for the Laplace operator or Green's reciprocal identity.

4.2.3 Boundary Element Method for Laplace Equation

The Integral Formulation

Boundary value problem (BVP) can be solved by the boundary element method (BEM). This method transforms the BVP in to equivalent integral equation, which after discretization of the boundary, result into a set of linear algebraic equations.
The boundary value problem, governed by Laplace equation in to two dimensions can be written as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \qquad (4.2.5)$$

Where

- $\nabla^2 = \nabla.\nabla$ = Laplacian operator
- u= potential function(the function describe the property)
- x, y = Cartesian coordinate axis

Subject to the boundary conditions
$u = f$ on Γ_1 and $\frac{\partial u}{\partial n} = g$ on Γ_2, where f and g represent the given boundary value.
Using the fundamental solution for the Laplace operator and Greens second identity , the BVP transforms in the following boundary integral equation. The integral relation for the potential $u(p)$ for p at different location of the domain have been abundantly derived in preliminary and are readily available in various books on boundary element method (Katiskadelis, J. T. 2002) and given by

$$k(p)u(p) = -\int_\Gamma (\lambda(p,q)\frac{\partial u(q)}{\partial n_q}-u(q)\frac{\partial \lambda(p,q)}{\partial n_q})ds_q \qquad (4.2.6)$$

When the coefficients $k(p)$ is given by

$$k(p) = \begin{cases} 1, & for \quad p \in \Omega_i \\ \frac{1}{2}, & for \quad p \quad on \quad \Gamma \\ 0, & for \quad p \in \Omega_e \end{cases} \tag{4.2.7}$$

Eq.(4.2.6) is the integral representation of the solution for Laplace eq.(1.1) at point $p \in \Gamma$ and since the boundary is smooth for the constant element Eq.(4.2.6) becomes

$$\frac{1}{2}u(p) = -\int_{\Gamma}(\lambda(p,q)\frac{\partial u(q)}{\partial n_q} - u(q)\frac{\partial \lambda(p,q)}{\partial n_q})ds_q \tag{4.2.8}$$

Equation (4.2.8) constitutes a consistent relation between the boundary values of u and $\frac{\partial u}{\partial n}$, meaning that only one of the quantities u and $\frac{\partial u}{\partial n}$ can be prescribed at each point of the boundary. Again we take equation (4.2.8) as an integral equation on the boundary Γ, since a BIE with unknown quantity is not prescribed by the BC. As a result, for constant element the Dirichlet problem $u = f$ on boundary Γ, so that equation (4.2.8) is written as,

$$\frac{1}{2}f = \int_{\Gamma}(f\frac{\partial \lambda}{\partial n} - \lambda\frac{\partial u}{\partial n})ds \tag{4.2.9}$$

In which the only unknown is the function $\frac{\partial u}{\partial n}$ on Γ. For the Neumann problem ($\frac{\partial u}{\partial n} = g$), with only unknown function u on boundary Γ, equation (4.2.8) becomes;

$$\frac{1}{2}u = \int_{\Gamma}(u\frac{\partial \lambda}{\partial n} - \lambda g)ds \tag{4.2.10}$$

For the problems with mixed boundary conditions, equation (4.2.8) is treated as two separate equations,

$$\frac{1}{2}f = \int_{\Gamma}(f\frac{\partial \lambda}{\partial n} - \lambda\frac{\partial u}{\partial n})ds \quad on\Gamma_1 \tag{4.2.11}$$

$$\frac{1}{2}u = \int_{\Gamma}(u\frac{\partial \lambda}{\partial n} - \lambda g)ds \quad on\Gamma_2 \tag{4.2.12}$$

4.2.4 Discrtization of Laplace Equation by Bondary element method

The objective of the BEM is to discretize the boundary into a finite number of elements, not necessarily equal, which are called a BEs. Two approximations are made over each of these elements. One is about the geometry of the boundary, while the other has to do with the variation of the unknown boundary quantity over the element. The usually employed boundary elements are the constant element, the linear element and the parabolic or quadratic element (Katiskadelis, J. T. 2002). On each element, we distinguish the extreme points or end points and the nodes or nodal points. The latter are the points at which values of the boundary quantities are assigned. In case of constant elements the boundary segment is approximated by a straight line, which connects its extreme points. The node is placed at the mid-point of the straight line and the boundary quantity is assumed to be constant along the element and equal to its value at the nodal point. Type of elements using the BEM is shown as fig(4.2) (Ghassemi, H. et al. (2016).

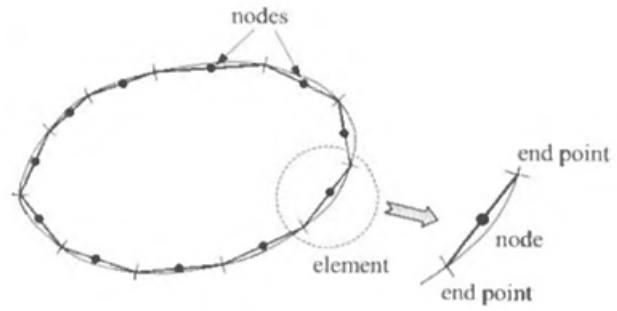

Figure 4.2: Constant element

For constant element, the boundary Γ is discretized into N constant elements, which are numbered in the counter clockwise sense. The value of the boundary quantity u and its normal derivative $\frac{\partial u}{\partial n} = q$ are assumed constant over each element and equal to their value at the mid- point of the element.
The element containing the j^{th} node is denoted by Γ_j. The discretized form of equation (4.2.8) is expressed for a given point p_i on Γ as,

$$\frac{1}{2}u_i = -\sum_{j=1}^{N}\int_{\Gamma_j}\lambda(p_i,q)\frac{\partial u(q)}{\partial n_q}ds_q + \sum_{j=1}^{N}\int_{\Gamma_j}u(q)\frac{\partial \lambda(p_{i,q})}{\partial n_q}ds_q \tag{4.2.13}$$

Where Γ_j is the segment (straight line) on which the j^{th} node is located and over which integration is carried out, and p_i is the nodal point of the i^{th} element and u_i is the value

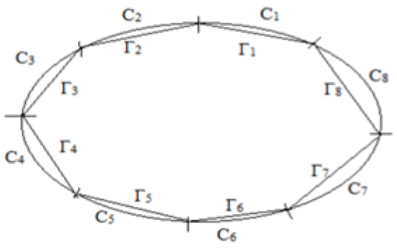

Figure 4.3: A discretization of a boundary into N=8 elements

of the function u at the point p_i.

A discretization of a $2D$ domain up to N boundary elements leads to the BEM system of algebraic equations. The physical boundary Γ is partitioned in to N parts C_j, $j = 1, 2, ...N$. Each physical partition C_j is represented by a numerical partition Γ_j. The union $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ is called a numerical boundary.

The element contaning the j^{th} node is denoted by Γ_j. The BIE is then applied at the i^{th} node and integration over Γ is estimated by the sum of integration on all over the N elements.

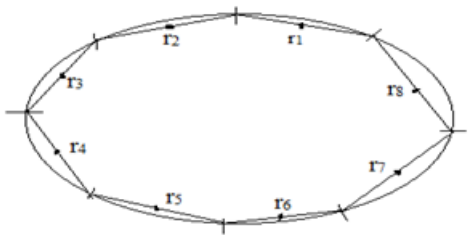

Figure 4.4: Constant element node $\Gamma_1, ..., \Gamma_8$ corresponding to the discretization in Figure 4.3

The boundary is always smooth for constant elements since we are always evaluating the function at the mid- point of the line connecting the two edges of the element. As a consequence $k(p) = \frac{1}{2}$. Moreover, the value of u and $\frac{\partial u}{\partial n} = q$ are constants on each element so that they can be taken out of the integral sign. Denoting by u_j and q_j, the values of u and q respectively on the j^{th} element equation (4.2.13) may be written as,

$$\frac{-1}{2}u_i + \sum_{j=1}^{N} u_j(\int_{\Gamma_j} z ds) = \sum_{j=1}^{N} q_j(\int_{\Gamma_j} \lambda(p_i, q) ds) \tag{4.2.14}$$

Where, $\frac{\partial \lambda(p_i,q)}{\partial n_q} = z$

Now there are two types of integrals to be carried out over the elements. These are,

$$H_{ij} = \int_{\Gamma_j} z ds \quad and \quad G_{ij} = \int_{\Gamma_j} \lambda(p_i, q) ds \tag{4.2.15}$$

Where point p_i remains constant (reference point), while the point q varies over the j^{th} element (integration point). These are called influence coefficients since their values express the contribution of the nodal values u_j and q_j to the formation of $c_i u_i$.

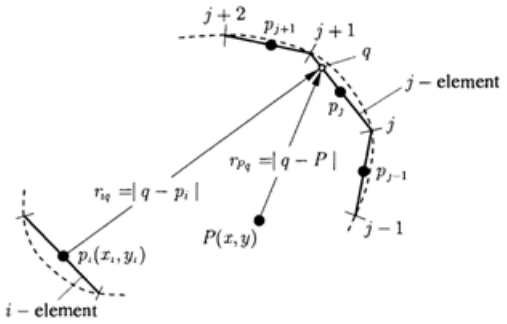

Figure 4.5: Nodal point location and relative distance for constant element discretization

Then, inserting the notation of equation (4.2.15) in to equation (4.2,14) it becomes,

$$\frac{-1}{2} u_i + \sum_{j=1}^{N} u_j H_{ij} = \sum_{j=1}^{N} q_j G_{ij} \tag{4.2.16}$$

Now, it is assumed that the position of i can also vary from 1 to N, since the fundamental solution is applied at each node successively. Since half of all of the u_i and q_i are specified by the BCs, we need N equations like eq. (4.2.16). This is done by moving the singular point i around the boundary and evaluating at every node p_i (i =1,2, ,N). For Simplicity define,

$$H_{ij} = \begin{cases} H_{ij}, & for \quad i \neq j \\ H_{ij} + 0.5, & for \quad i = j \end{cases} \tag{4.2.17}$$

Now, we have N equations,

$$\sum_{j=1}^{N} u_j H_{ij} = \sum_{j=1}^{N} q_j G_{ij} \tag{4.2.18}$$

Then, it becomes a matrix equation,

$$HU = GQ \tag{4.2.19}$$

Where H and G are $N \times N$ matrices, whose elements are given by equation (4.2.15) and equation (4.2.17), and U and Q are vectors of length N.
i.e $U = (u_1, u_2, ...u_n)^T, Q = (q_1, q_2, ..., q_n)^T$

$$\begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} \tag{4.2.20}$$

Again, let us assume mixed boundary conditions. Here from BCs, N_1 values of u and N_2 values of q are known on Γ_1 and Γ_2 respectively. That is $(\Gamma_1 + \Gamma_2 = \Gamma)$, hence there are only N unknowns in the system of equation (4.2.20). To introduce these BCs in to equation (4.2.20) one has to rearrange the system by moving columns of H and G from one side to other. Once all unknown are passed to the left side one can write,

$$AX = F \tag{4.2.21}$$

Where, X is a vector of unknown u's and q's boundary values and F is found by multiplying the corresponding columns by the known values u's and q's . It is interesting to point out that the unknowns are now a mixture of the potential and its derivative.

The solution of the system (4.2.21) gives a BEM approximation of the unknowns in X in the grid nodes Γ_j. Once this is done it is possible to calculate any internal value of u and its derivative.
The value of u's are calculated at any internal point i using formula which can be written as,

$$u(p) = \sum_{j=1}^{N} u_j H_{ij} - \sum_{j=1}^{N} q_j G_{ij} \tag{4.2.22}$$

The coefficients H_{ij} and G^{ij} defined in equation (4.2.15) are evaluated numerically using a standard Gaussian quadrature.

Then the linear system of equation(4.2.21) is full matrix and we solve this matrix using GEM and H-matrix.

4.3 Hierarchical Matrix(H-matrix)

H-matrix technique uses a data-sparse representations of a dense matrix which means that these matrices can be described by only few data. The basic idea is to split a given matrix into a hierarchy of a rectangular blocks and approximate each of the blocks by low-rank matrix. The low rank matrices can be represented as a product of the rectangular matrices.

The H-matrix format relies on a hierarchical tree structure called block cluster tree,which obtained by hierarchical partitioning of an index set in to sublocks which is called cluster tree. During the construction of the cluster tree (H-tree) at each level of partitioning these subblocks have to satisfy the admissibility condition, that determines whether they are a leaf or should be further partitioned and applying this admissibility condition to the Cartesian product of index set gives us an admissible partitioning in the sense the most blocks can be approximated by low-rank matrix (R(k)-matrices) successfully (Yang, F. 2008).

To use H-matrix techniques, one critical step is to represent a problem in H-matrix format and the main advantage of H-matrix is the reduction of storage requirement.The storage complexity of H-matrices is in $O(kn \log n)$ and $O(n)$.

The approaches to construct H-matrices can be divided into two categories,geometric approaches and algebraic approaches
For geometric H-matrix construction approaches,an index $i \in I$ represents an index set of finite elements or boundary element basis functions $(\phi_i)_{i \in I}$, for algebraic H-matrix construction approaches, I represents a row or column index set of a matrix (Hackbusch, W. 2015).

The geometric H-matrix construction approaches use the geometric information underlying a problem such as the domain information and the basis functions used for discretization. In some applications, system matrices are given but geometric information may not be available. In these cases, the geometry of H-matrix construction approaches can not be applied, but the algebraic H-matrix construction approaches which only relay on matrix information can be used (Grasedyck, L. et al.,2009)

The definition of H-matrix is mainly based on the definition of a tree:

Tree

Assume that V is a nonempty set and $E \subseteq V \times V$ a binary relation on V. A pair $T = (V, E)$ where the set $V = V(T)$ is the vertex set of T and the set $E = E(T)$ is

the edge set of T is called a tree if the following conditions hold

(1) The unique vertex $v \in V$ s called a root of the tree and is denoted by root(T) iff for all $w \neq v$ it holds that $(w, v) \notin V$

(2) For each vertex $v \in V$or root(T),there is a unique simple path from the root(T)to v namely one can find a sequence of vertices $(v_i)_{i=0}^{l}$ with v_0=root(T),$v_1 = v$ such that $(v_{i-1}, v_i) \in E$ for all $1 \leq i \leq l$ and we have $v_i \neq v_j$ if $i \neq j$ for all $0 \leq i, j \leq 1$.

Notation(Sons, Leaves and Level)

Let $T = (V, E)$ be a tree then we have the following term
(1) For a vertex $v \in V$ we define the set of all its sons by
$S(v) = \{w \in V | (v, w) \in E\}$
(2) The set of all leaves of the tree T is defined by
$L(T) = \{v \in V | S(v) = \emptyset\}$.
(3) The level of a tree T is defined recursively by
$T^{(0)} = \{root(T)\}, T^{(l)} = \{v \in V | \exists w \in T_{(l-1)} : (w, v) \in E\}$,
for $l \in N$ and we write level(v)=l if $v \in T^{(l)}$. The leaves of T on level l are also denoted by $l(T, l) = l(T) \bigcap T^{(l)}$.

The process to construction H-matrix involves four steps,
(1) The first step is to construct an index cluster tree T_I,
(2) The second step is to construct the block cluster tree $T_I \times_J$ using the index cluster tree T_I andT_J,
(3) The third step is to define an admissible condition
(4) The fourth step is to approximate the block cluster tree by Rk-matrices (Low-rank matrices) depend on admissible condition.

4.3.1 Cluster Tree T_I

The concept of a cluster tree is directly based on the concept of a tree. The tree T_I) is called a cluster tree over an index set I with root $T_I = I$ if the following condition are hold.
(1) $I \in V$ is the root of T_I and $\forall v \in V, v \neq \emptyset \Rightarrow v \subseteq I$

(2) If $v \in V$ is not a leaf i.e $S(v) \neq \emptyset$, then it is equal to the disjoint union of its sons,that is $v = \bigcup_{w \in (S(v))} w$.

Then $v \in V$ is called cluster. Note also that we identify V and T(I), namely instead of$v \in V$ we have $v \in T_I$.
To illustrate the concept of the cluster tree, we consider the following example

Example

Let $I = I_0^{(0)} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the root of the cluster tree T_I and $n = 2^l$ (here l=3).As shown figure 4.6 starting from the root with only one cluster,T(I) is iteratively constructed by splitting each index set $T_i^{(j)}$ in to two successors $I_{2i}^{(j+1)}$ and $I_{2i+1}^{(j+1)}$ for $0 \leq i, j \leq l$. Finally at the level l all cluster (vertices) are leaves of the cluster tree i.e $L(T) = \{T_i^{(3)}\}_{i=0}^{7}$.

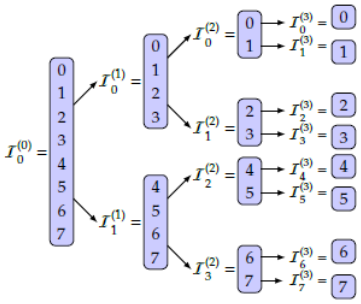

Figure 4.6: Cluster tree T(I)

4.3.2 Block Cluster Tree $(T_{I\times J})$

A block cluster tree $(T_{I\times J})$ describes an hierarchical partitioning over the Cartesian product of an index set I.Note that in general,for non-square matrices belonging to$R^{I\times J}$ we need two different cluster trees T_I and T_J .Henceforth we also consider another cluster tree T_J based on the index set J with $|J| = m$. To construct a block cluster tree $T_{I\times J}$ satisfies the following conditions .
(1) $T^{(0)}I \times J = I \times J$
(2) If $v \in T^l$ then there exist $s \in T^{(l)}(I)$ and $t \in T^{(l)}(j)$ such that $v = s \times t$.
(3) For sons of $v = s \times t, s \in T(I)$ and $t \in T(j)$ we have,

$S(v) = \{\emptyset$ if $S(s) = \emptyset$ or $S(t) = \emptyset$.
$S(v) = \{s^{'} \times t^{'} : s^{'} \in S(s)$ and $t^{'} \in S(t)$ otherwise$\}$

4.3.3 Admissibility Condition

Admissibility conditions are used to determine whether to approximate a matrix block $s \times t$ by an Rk-matrix during the construction of H-matrices. If a block $s \times t$ is admissible, then it will be approximated by an Rk-matrix.
The admissibility conditions vary for different H-matrix construction approaches. In geometric H-matrix construction approaches, admissibility conditions are defined using the geometric information underlying a problem such as the domain information and the supports of index clusters.

Given T_I, T_J , and s and $t \in T_{I \times J}$, the general form of an admissibility condition in geometric H-matrix construction approaches can be defined as:

$$s \times tisadmissible \Longleftrightarrow min\{diam(\Omega_s), diam(\Omega_t) \leq \eta dist(\Omega_s, \Omega_t), \eta > 1 \tag{4.3.1}$$

where $\eta \in R$ is a parameter to control the number of R-kmatrix blocks. Ω_s and Ω_t are the supports of the cluster s and t respectively.
i.e

$\Omega_s = u_{i \in s} supp(\phi_i)$ and $\Omega_t = u_{i \in t} supp(\phi_i)$ with ϕ_i the basis function related to index i.
The diameter of cluster $diam(s)$ and the distance between two cluster $dist(s,t)$ are defined using euclidean norms as follows:

$diam(s) = max\{\|x_i - x_j\| : x_i, x_j \in \Omega_s\}$
$dist(s,t) = min\{\|x_i - x_j\| : x_i \in \Omega_s \text{ and } x_j \in \Omega_t\}$
Equation (4.3.1) is called the standard admissibility condition for geometric H-matrix construction approaches. If minimum in (4.3.1) is replaced by maximum, we get a strong admissibility condition (Borm,S. et al.,2003).

The admissibility condition for algebraic H-matrix construction approaches is simply defined using the information contained in a matrix graph.
$s \times t$ is admissible $\Longleftrightarrow$ s and t are not connected in the matrix graph.
i.e

$$s \times tisadmissible \Longleftrightarrow s \cap t = \emptyset \tag{4.3.2}$$

Equation (4.3.2) is called the weak admissibility condition(Yang, F. 2008).

Admissible Partitioning

The purpose of both admissible condition is to construct an appropriate partitioning P of a block cluster tree $T_I \times_J$ that ensures an error bound for a matrix A and its approximation A_H in the H-matrix format, this partitioning is called admissible partitioning.

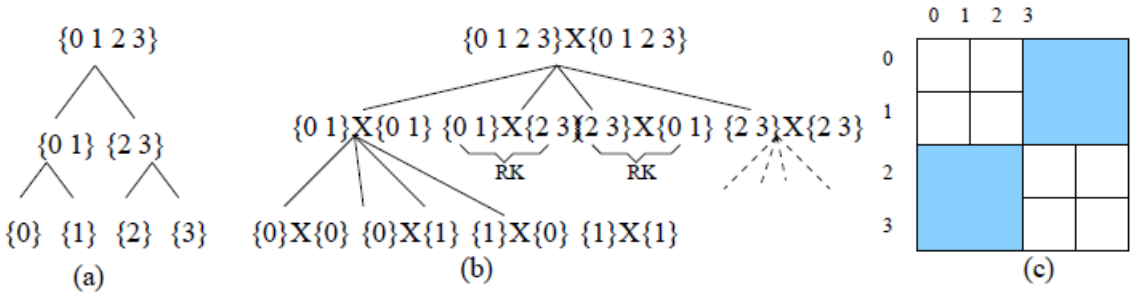

Figure 4.7: (a) is T_I . (b) is $T_{I\times I}$. (c) is the corresponding H-matrix. In (b) RK denotes a block satisfying the given admissibility condition. In (c) the dark color blocks are Rk-matrices and the white color blocks are full matrix blocks.

Example of an index cluster tree T_I, block cluster tree $T_{I\times I}$ and H-matrix in the above (fig 4.7).

4.3.4 Rk-matrix (Low Rank Matrices)

Rk-matrices (low rank matrices) are the basic building blocks of H-matrices. The low-rank matrices can be represents as a product of two rectangular matrices. let $A^{'} \subset A$ be a subblock of size $n^{'} \times n^{'}$ and rank$(A^{'}) = k$ where $k << n^{'}$. Then there exist two matrices $B, C \in R^{n^{'}\times k}$ such that $A^{'} = BC^{T}$. Having this representation for $A^{'}$, then storage for its $(n^{'})^2$ entries reduce to $2kn^{'}$ entries which are the total entries of B and C.

Chapter 5

Result and Discussion

5.1 Numerical Experiment

Example: Consider the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{5.1.1}$$

with boundary conditions $u(0, y) = 100, u(1, y) = 200 \quad and \quad u_n(x, 0) = u_n(x, 1) = 0$ as shown below in a region $0 \leq x \leq 1, 0 \leq y \leq 1$ with boundary condition indicated in Fig (5.1).
The exact solution of this problem is:

$$u(x, y) = 100(1 + x) \tag{5.1.2}$$

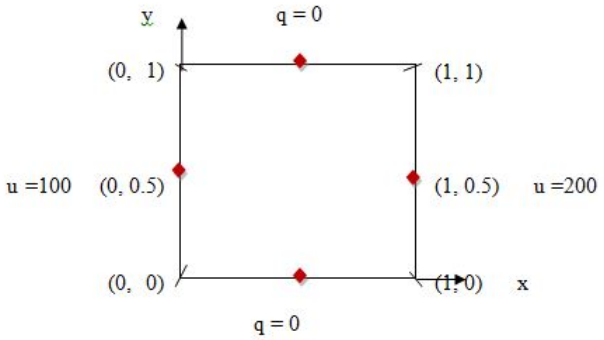

Figure 5.1: Boundary element discretization and BCs for problem

We partition the boundary in to 4 elements, then using equ.(4.2.18) it becomes,

$$\sum_{j=1}^{4} u_j H_{ij} = \sum_{j=1}^{4} q_j G_{ij} \tag{5.1.3}$$

i.e
$i = 1, u_1(H^{'}{}_{11} + \frac{1}{2}) + u_2 H^{'}_{12} + u_3 H^{'}_{13} + u_4 H^{'}_{14} = q_1 G_{11} + q_2 G_{12} + q_3 G_{13} + q_4 G_{14}$
$i = 2, u_1 H^{'}{}_{21} + u_2(H^{'}_{22} + \frac{1}{2}) + u_3 H^{'}_{23} + u_4 H^{'}_{24} = q_1 G_{21} + q_2 G_{22} + q_3 G_{23} + q_4 G_{24}$
$i = 3, u_1 H^{'}{}_{31} + u_2 H^{'}_{32} + u_3(H^{'}_{33} + \frac{1}{2}) + u_4 H^{'}_{34} = q_1 G_{31} + q_2 G_{32} + q_3 G_{33} + q_4 G_{34}$
$i = 4, u_1 H^{'}{}_{41} + u_2 H^{'}_{42} + u_3 H^{'}_{43} + u_4(H^{'}_{44} + \frac{1}{2}) = q_1 G_{41} + q_2 G_{42} + q_3 G_{43} + q_4 G_{44}$

Then, by substituting the given boundary values,
$u_1(H^{'}{}_{11} + \frac{1}{2}) + 200 H^{'}_{12} + u_3 H^{'}_{13} + 100 H^{'}_{14} = q_2 G_{12} + q_4 G_{14}$
$u_1 H^{'}{}_{21} + 200(H^{'}_{22} + \frac{1}{2}) + u_3 H^{'}_{23} + 100 H^{'}_{24} = q_2 G_{22} + q_4 G_{24}$
$u_1 H^{'}{}_{31} + 200 H^{'}_{32} + u_3(H^{'}_{33} + \frac{1}{2}) + 100 H^{'}_{34} = q_2 G_{32} + q_4 G_{34}$
$u_1 H^{'}{}_{41} + 200 H^{'}_{42} + u_3 H^{'}_{43} + 100(H^{'}_{44} + \frac{1}{2}) = q_2 G_{42} + q_4 G_{44}$

Rearranging the system by moving columns of H and G from one side to the other like passing unknowns to the left hand side and others to the right hand side, system of equation becomes,
$u_1(H^{'}{}_{11} + \frac{1}{2}) + u_3 H^{'}_{13} - q_2 G_{12} - q_4 G_{14} = -200 H^{'}_{12} - 100 H^{'}_{14}$
$u_1 H^{'}{}_{21} + u_3 H^{'}_{23} - q_2 G_{22} - q_4 G_{24} = -200(H^{'}_{22} + \frac{1}{2}) - 100 H^{'}_{24}$
$u_1 H^{'}{}_{31} + u_3(H^{'}_{33} + \frac{1}{2}) - q_2 G_{32} - q_4 G_{34} = -200 H^{'}_{32} - 100 H^{'}_{34}$
$u_1 H^{'}{}_{41} + u_3 H^{'}_{43} - q_2 G_{42} - q_4 G_{44} = -200 H^{'}_{42} - 100(H^{'}_{44} + \frac{1}{2})$
Then, in matrix form it can be written as,

$$\begin{bmatrix} H^{'}_{11} + \frac{1}{2} & H_{13} & -G_{12} & -G_{14} \\ H^{'}_{21} & H^{'}_{23} & -G_{22} & -G_{24} \\ H^{'}_{31} & H^{'}_{33} & -G_{32} & -G_{34} \\ H^{'}_{41} & H^{'}_{41} & -G_{42} & -G_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ q_2 \\ q_4 \end{bmatrix} = \begin{bmatrix} -200 H^{'}_{12} - 100 H^{'}_{14} \\ -200(H^{'}_{22} + \frac{1}{2}) - 100 H^{'}_{24} \\ -200 H^{'}_{32} - 100 H^{'}_{34} \\ -200 H^{'}_{42} - 100(H^{'}_{44} + \frac{1}{2}) \end{bmatrix} \tag{5.1.4}$$

For the integrals of the influence coefficients H_{ij} and G_{ij} for which $i = j$ and $i \neq j$, the computation is as follows:
1)The integral of G_{ij}

$$G_{ij} = \int_{\Gamma_j} \lambda ds = \int_{-1}^{1} \frac{1}{2\pi} ln[r(\varepsilon)] \frac{l_j}{2} d\varepsilon = \frac{l_j}{4\pi} \sum ln[r(\varepsilon_k)] w_k \tag{5.1.5}$$

where,
$r(_k) = \sqrt{[x(\varepsilon_k) - x_i]^2 + [y(\varepsilon_k) - y_i]^2}$
2) The integral of $H^{'}_{ij}$

This integral can also be evaluated analytically from Fig(5.2)
$dscos\phi = rd\alpha$

This can be used to derive the expression,

$$H'_{ij} = \int_{\Gamma j} \frac{\partial \lambda}{\partial n} ds = \int_{\Gamma j} \frac{cos\phi}{2\pi r} ds = \int_{\Gamma j} \frac{1}{2\pi} d\alpha = \frac{\alpha_{j+1} - \alpha_j}{2\pi} \tag{5.1.6}$$

The angle α_{j+1} and α_j are computed from

$$tan\alpha_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} \quad and \quad tan\alpha_j = \frac{y_j - y_i}{x_j - x_i} \tag{5.1.7}$$

Where, x_{j+1}, y_{j+1} and x_j, y_j are the coordinate of the extreme points of the j^{th} element.

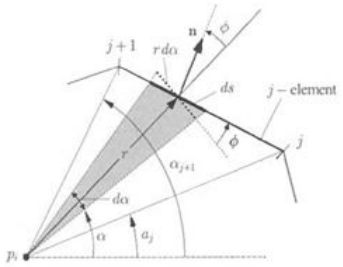

Figure 5.2: Definition of angles involved in the numerical integration over constant elements

Diagonal elements $i = j$. In this case,the node p_i coincide with p_j and r lies on the element. Consequently,$\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, which gives $cos\phi = 0$. Moreover, we have;

$$x_{pj} = \frac{x_{j+1} + x_j}{2}, y_{pj} = \frac{y_{j+1} + y_j}{2} \tag{5.1.8}$$

$$r(_\varepsilon) = \sqrt{[x(\varepsilon) - x_{pj}]^2 + [y(\varepsilon) - y_{pj}]^2} \tag{5.1.9}$$

$$G_{jj} = \int_{-1}^{1} \frac{1}{2\pi} lnr = 2\int_0^{\frac{l_j}{2}} \frac{1}{2\pi} lnr(\varepsilon) dr = \frac{1}{\pi}[rlnr - r]_0^{\frac{l_j}{2}} = \frac{l_j}{2\pi}[ln[\frac{l_j}{2}] - 1] \tag{5.1.10}$$

$$H'_{jj} = \int_{\Gamma j} \frac{cos\phi}{2\pi r} ds = \frac{2}{2\pi}[cos\phi ln(\varepsilon)]_0^1 = 0 \tag{5.1.11}$$

Where,

$$H_{ij} = \begin{cases} H'_{ij}, & for \quad i \neq j \\ H'_{ij} + 0.5 & for \quad i = j \end{cases} \tag{5.1.12}$$

Using equation (5.1.9), $G_{jj} = -0.2695$ since length of the i^{th} elements $l_j = 1$, for $j = 1, 2, 3, 4$. And, using equation (5.1.10), $H_{jj} = 0, for j = 1, 2, 3, 4$.

a) The integral of H_{ij} is evaluated analytically using equation (5.1.6)
$H_{12} = \frac{\alpha_{j+1} - \alpha_j}{2\pi}$

$tan\alpha_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} = \frac{1-0}{1-0.5} = \alpha_{j+1} = tan^{-1}2 = 63.4349 \quad and$
$tan\alpha_j = \frac{y_j - y_i}{x_j - x_i} = \frac{0-0}{1-0.5} = \alpha_j = tan^{-1}0 = 0$

This implies

$H_{12} = \frac{63.4349 - 0}{2\pi} = 10.095779$
$H_{13} = \frac{\alpha_{j+1} - \alpha_j}{2\pi}$
$tan\alpha_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i} = \frac{1-0}{0-0.5} = \alpha_{j+1} = tan^{-1}-2 = -63.4349 \quad and$
$tan\alpha_j = \frac{y_j - y_i}{x_j - x_i} = \frac{1-0}{1-0.5} = \alpha_j = tan^{-1}2 = 63.4349$

This implies
$H_{13} = \frac{-63.4349 - 63.4349}{2\pi} = -20.1919$

Similarly, $H_{14} = 10.0959, H_{21} = 10.09599, H_{23} = -18.55199, H_{24} = 8.45591, H_{31} = -20.1919, H_{32} = 10.0959, H_{34} = 10.0959, H_{41} = -18.55189, H_{42} = 8.4559, and H_{43} = 10.095985$

b) The integral of G_{ij} is computed using equation (5.1.5). This equation is evaluated using Gauss-Legendre integration method. For this, two Gauss points $n = 2$ are chosen so that its values and the corresponding weights can be obtained using Legendres orthogonal polynomials approximating a non-polynomial function $f(\varepsilon) = ln[r(\varepsilon)]$ by a polynomial of degree $2n - 1$.
Therefore,

$G_{ij} = \frac{l_j}{4\pi} \int_{-1}^{1} ln[r(\varepsilon)] d\varepsilon = \frac{l_j}{4\pi} \sum_{k=1}^{2} ln[r(\varepsilon_k)] w_k$
That is,
$\int_{-1}^{1} f(\varepsilon) d\varepsilon = \frac{l_j}{4\pi} [f(\varepsilon_1) w_1 + f(\varepsilon_2) w_2]$

Also, Legendre's polynomials and weights are defined as,

$p_n(\varepsilon) = \frac{d^n(\varepsilon^2-1)^n}{2^n n!} d\varepsilon^n$
such that the coordinates ε_k of the integration points are the zeros of these polynomials.
$w_k = \frac{2(1-\varepsilon_k)^2}{n^2[p_{n-1}(\varepsilon_k)]^2}$

Therefore, for two gauss points $n = 2$;
$f(\varepsilon) = p_{2(\varepsilon)} = \frac{1}{2}3\varepsilon^2 - 1) = 0$

This implies $\varepsilon_1 = -0.57$ and $\varepsilon_2 = 0.57$ and the corresponding weights $w_1 = w_2 = 1$.
Therefore $G_{12} = \frac{l_2}{4\pi}[r(\varepsilon_1)]w_1 + ln[r(\varepsilon_2)]w_2\, x(\varepsilon_1) = 1, y(\varepsilon_1) = 0.211, r(\varepsilon_1) = 0.54, ln[r(\varepsilon_1)] = -0.61$ and $ln[r(\varepsilon_1)] = -0.06, G_{12} = -0.054$

Similarly, $G_{13} = 0.0063, G_{14} = -0.05, G_{21} = -0.054, G_{23} = -0.054, G_{24} = 6.369, G_{32} = -0.054, G_{34} = -0.054, and G_{42} = 6.369$
Then,by substitution,

$$\begin{bmatrix} 0.5 & -20.19 & 0.054 & 0.054 \\ 10.09 & -18.55 & 0.26 & -6.36 \\ -20.19 & 0.5 & 0.05 & 0.05 \\ -18.55 & 10.09 & -6.36 & 0.26 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ q_2 \\ q_4 \end{bmatrix} = \begin{bmatrix} -3028.7458 \\ -945.591 \\ -3028 \\ -1741.18 \end{bmatrix} \tag{5.1.13}$$

Using Gaussian elimination method the results of the boundary values can be presented like the followings;

Table 5.1: Shows the numerical and the analyticaly solution of the problem for $N = 4$

node	x	y	Numerical value	Analyticaly value	error
1	0.5	0	150	150	0
2	1	0.5	200	200	0
3	0.5	1	150	150	0
4	0	0.5	100	100	0

Similarly, the Laplace equation (5.1.1) is solved with the help of the internal points $IN = 9$ and $N = 16$ boundary of the region using BEM. A large linear system $Ax = b$ is obtained, where A is a full matrix.
Using Gaussian elimination method we solve the above system.

$$\begin{bmatrix}
-0.5 & 0 & 0 & 0 & 0.01 & 0.002 & 0.03 & -0.09 & 0.03 & 0.03 & 0.04 & 0.04 & 0.01 & 0.02 & 0.04 \\
0 & -0.5 & 0 & 0 & 0.02 & 0.01 & 0.01 & -0.03 & 0.03 & 0.04 & 0.04 & 0.04 & 0.002 & 0.012 & 0.02 \\
0 & 0 & -0.5 & 0 & 0.03 & 0.02 & 0.01 & 0.01 & 0.04 & 0.04 & 0.04 & 0.04 & 0.03 & -0.01 & 0.05 \\
0 & 0 & 0 & -0.5 & 0.07 & 0.04 & 0.02 & 0.01 & 0.04 & 0.04 & 0.03 & 0.03 & -0.01 & -0.01 & 0.02 \\
0.07 & 0.01 & 0.04 & 0.2 & 0.01 & 0.06 & 0.03 & 0.01 & 0.04 & 0.04 & 0.03 & 0.02 & -0.01 & 0.01 & -1.03 \\
0.08 & 0.03 & 0.05 & 0.1 & 0.06 & 0.1 & 0.05 & 0.01 & 0.06 & 0.1 & 0.3 & 0.1 & -0.1 & -0.1 & \\
0.02 & 0.03 & 0.05 & 0.06 & 0.03 & 0.06 & 0.1 & 0.03 & 0.1 & 0.1 & 0.1 & 0.1 & -0.01 & -1.03 & \\
0.02 & 0.03 & 0.04 & 0.04 & 0.02 & 0.03 & 0.01 & 0.06 & 0.2 & 0.04 & 0.01 & 0.02 & -1.03 & -0.01 & \\
0.03 & 0.03 & 0.04 & 0.04 & 0.01 & 0.02 & 0.02 & 0.04 & -0.5 & 0 & 0 & 0 & 0.1 & 0.01 & \\
0.03 & 0.04 & 0.03 & 0.04 & 0.01 & 0.01 & 0.01 & 0.01 & 0 & -0.5 & 0 & 0 & 0.02 & 0.01 & \\
0.04 & 0.04 & 0.03 & 0.03 & -0.03 & 0.01 & 0.02 & 0.01 & 0 & 0 & -0.5 & 0 & 0.04 & 0.02 & \\
0.04 & 0.04 & 0.03 & 0.03 & -0.01 & -0.01 & -0.01 & 0.01 & 0 & 0 & 0 & -0.5 & 0.07 & 0 & \\
0.04 & 0.05 & 0.03 & 0.02 & -0.01 & -0.01 & -1.03 & -1.03 & 0.01 & 0.01 & 0.04 & 0.2 & 0.1 & 0.06 & \\
0.06 & 0.05 & 0.03 & 0.02 & -0.01 & -0.01 & -0.01 & -0.01 & 0.02 & 0.02 & 0.05 & 0.1 & 0.06 & 0.1 & \\
0.09 & 0.05 & 0.02 & 0.02 & -0.02 & -1.03 & 0.02 & -0.01 & 0.02 & 0.03 & 0.05 & 0.1 & 0.03 & 0.1 &
\end{bmatrix}$$

Table 5.2: Shows the numerical and the analyticaly solution of the problem for $N = 16$

node	x	y	Numerical value	Analyticaly value	error
1	0.250	0	124.8004	125	0.1996
2	0.375	0	137.3227	137.5	0.1773
3	0.625	0	162.6773	162.5	0.1773
4	0.875	0	188.1232	187.5	0.6232
5	1	0.250	200	200	0
6	1	0.375	200	200	0
7	1	0.625	200	200	0
8	1	0.875	200	200	0
9	0.875	1	188.1232	187.5	0.6232
10	0.625	1	162.6773	162.5	0.1773
11	0.375	1	137.3227	137.5	0.1773
12	0.250	1	111.8768	112.5	0.6232
13	0	0.875	100	100	0
14	0	0.625	100	100	0
15	0	0.375	100	100	0
16	0	0.250	100	100	0

The obtained systems are solved by H-matrix discussed in chapter 4 section 4.3 and Gaussian elimination method. The exact solution and the approximate solution of the given problem are shown in Fig(5.1). As we see in the blow graph the exact solution and the approximate solution of 2-D Laplace's equation with mixd boundary condition. From the blow two graphs, we can understand that the approximate solution of the given problem is almost resembles to the exact solution of 2-D Laplace equation, the only difference occur due to the error presents. The error decreases when the number of boundary element increases. So that the numerical solution converges to the analytical solution.

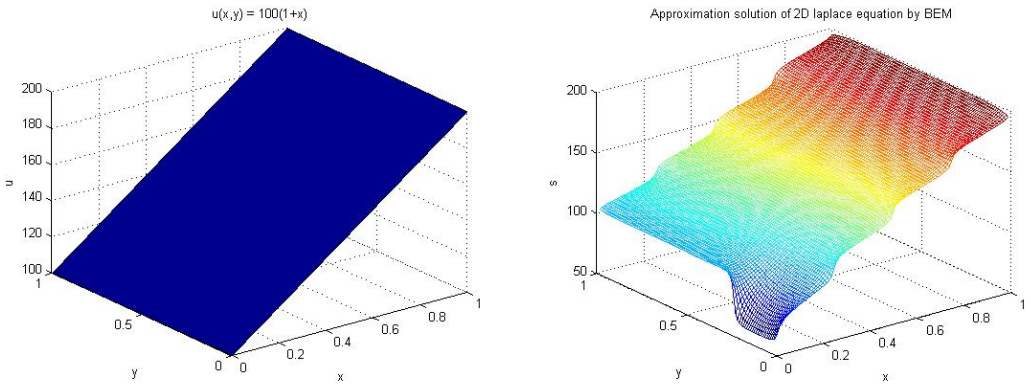

Figure 5.3: Exact and Approximation solution of 2-D Laplace equation respectively

In the previous chapter, we have seen the two numerical methods for solving the Laplace equation. Then, we get the solution of the given problem using the two numerical method. Now we also compare the CPU time and memory usage of the two methods.

Table 5.3: shows the time taken and memory usage of GEM and H-matrix

Method	Time taken	Memory usage
GEM	188.69 sec	961.45 Mb
H-matrix	81.91 sec	0.063 Mb

Table (5.3) indicated the time required and memory usage for solving the full linear system of 2-D Laplace equation using Gaussian elimination method. From this table we understand the time taken and memory usage to solve the problem by using Gaussian elimination method are 188.692704 sec and 961.458 MB respectively and the elapsed time and memory usage to solve the problem using H-matrix method are 81.9 sec and 0.063 MB respectively.
The computation time and memory usage of H-matrix approach is compared with the computation time and memory usage of Gaussian elimination method.

Generally,from the above table, we observed that CPU time and memory usage of H-Matrix technique is less than the CPU time and memory usage of Gaussian elimination method. Therefor, H-matrix methods better interms of CPU time and memory usage for solving Laplace equation in two dimension.

Chapter 6

Conclusion

In this study, we solved 2-D Laplace equation with mixed boundary condition using boundary element method . A large full linear system has been obtained after the spatial discretization of the domain using boundary element method. Using Gaussian elimination method and H-matrix, we have solved the required linear system. Finally, we show that accessing matrix using H-matrix techniques, to solve a large sparse linear system and reduces the computation time and memory usage.
As general, for this study, we showed that H-matrix is best method rather than Gaussian elimination method to solve the Laplace equation in 2 dimensional with less computational time and memory usage.

6.1 Recommendation

The results obtained in this study, solving the 2-D Laplace problem using H-matrix method and Gaussian elimination method, then comparison of H-matrix method and Gaussian elimination method interms of computational time and memory usage.
This study can be considered the following recommendation:
(1) Apply H-matrix method for other method, for instant FDM, FEM and FVM.
(2) The researcher may compare H-matrix with other methods interms of stability, consistency and convergence.

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