# NONSTANDARD FINITE DIFFERENCE SCHEME FOR SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION PROBLEMS

Sultan Godana Wako

A thesis Submitted to the Department of Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the
Degree of Master of Science in Mathematics
(Numerical Analysis)

Office of Graduate Studies
Adama Science and Technology University

June, 2022
Adama, Ethiopia

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Eshetu Belete Derzie (Ph.D. scholar)

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# Declaration

I hereby declare that this Master Thesis entitled **Nonstandard finite difference scheme for singularly perturbed parabolic convection-diffusion problems** is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

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# Recommendation of advisors

We, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled **Nonstandard finite difference scheme for singularly perturbed parabolic convection-diffusion problems** prepared under my/our guidance by **Sultan Godana Wako**. submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

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# Approval page of M.Sc thesis

We, the advisors of the thesis entitled **Nonstandard finite difference scheme for singularly perturbed parabolic convection-diffusion problems** and developed by **Sultan Godana Wako**, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by **Sultan Godana Wako** have read and evaluated the thesis entitled
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# Abstract

*In this thesis, a numerical study is made for solving singularly perturbed time-dependent parabolic convection-diffusion problems. To approximate the solution of singularly perturbed parabolic convection-diffusion problems, we use nonstandard finite difference method for spatial discretization and Runge kutta Fehlberg method of* $4^{th}$ *and* $5^{th}$ *order for time discretization. The method is shown to be accurate of order one. A convergence analysis has been carried out to show* $\varepsilon$*- uniform convergence of the proposed scheme.*

**Keywords**: Singular perturbation,Nanstandard finite difference,Boundary layersand Runge Kutta Fehlberg.

# CHAPTER 1

# INTRODUCTION

Many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems are described by differential equations involving large or small parameters (Farrell et al., 2000). Singular perturbation problems in which the term containing the highest order derivative is multiplied by a small parameter $\varepsilon$, occur in a number of areas of applied mathematics, science and engineering among them fluid mechanics (boundary layer problems) elasticity ( edge effort in shells ) and quantum mechanics (Gie et al., 2018).

Singularly perturbed differential equations are differential equations in which it's highest order derivative term multiplied by small parameter. Let $P_\varepsilon$ denote the original problem and $u_\varepsilon$ be its solution. Let $P_o$ denote the reduced problem of $P_\varepsilon$ (setting $\varepsilon = 0$) in $P_0$ let $u_o$ be its solution. Then the problem $P_\varepsilon$ is called a Singularly Perturbed Problem (SPP) if and only if $u_\varepsilon$ does not converge uniformly to $u_o$ in the entire domain of the definition of the problem, otherwise the problem is called Regularly Perturbed Problem (RPP) (Hofmanova et al., 2020).

The convection-diffusion equation consists of two processes, the first process is convection and is due to the movement of materials from one region to another. The second process is diffusion and is due to the movement of materials from a region of high concentration to a region of low concentration. Convection-diffusion problems are governed by typical mathematical models, which are common in fluid and gas dynamics , the equations provide the basis for describing heat and mass transfer phenomena as well as processes of continuum mechanics. A singular perturbation problem is said to be of convection-diffusion type, if the order of the differential equation is reduced by one when the perturbation parameter is set equal to zero. If the order reduces by two, it is known as a reaction-diffusion type problem (Gracia & Lisbona, 2007).

The nonstandard finite difference (NSFD) schemes developed by Mickens (2000) were proposed to compensate for the weaknesses of methods such as the SFD methods; numerical instabilities being a prime example.

As commented on Liao and Ding (2013), with regard to the positivity, boundedness, and monotonicity of solutions, NSFD schemes have performed better than SFD schemes, because it is more flexible in its construction, a NSFD scheme can more easily preserve certain properties and structures obeyed by the original equations and can have better dynamical consistency for dynamical problems. Nonstandard finite difference methods provide numerical solutions to differential equations by constructing discrete models. The method preserve the significant properties of continuous analogues and consequently give reliable numerical results.

The idea of non-standard discrete modeling method is based on the development of exact finite difference method. These are some of the shortcoming that lead to the idea of NSFD method. The SFD method is associated with some degree of stability, and its stability depends on the time step size. It is only conditionally stable.

In numerical analysis, stability is a desirable property of numerical method. Therefore, a numerical method is said to be stable if the errors at any stage of the computation does not grow in the course of the computational processes . Stability of a discretization refers to a quantitative measure of the well-posedness of the discrete problem.
A numerical scheme is consistent if the method approximating differential equation become exact as the time step size tend to zero. For a method to be consistent, the truncation error should vanish as the step size approaches zero.
A numerical method is said to be convergence if the solution of the numerical method approaches exact solution of the original differential equation in the course of the computation progresses. A stable and consistent process of discretization leads to convergence of the solution. A reaplication of this property in the semi-discrete form will be used to analyse the numerical method easly (Smith et al., 1985).

The appropriate use of varying step size produce computationally efficient integral approximating methods. In itself, this might not be sufficient to favor these methods due to the increased complication of applying them (Mondal et al., 2016).

However,they have another feature that makes them worthwhile. They incorporate in the step-size procedure an estimate of the truncation error that does not require the approximation of the higher derivatives of the function. These methods are called adaptive because they adapt the number and position of the nodes used in the approximation to ensure that the truncation error is kept within a specified bound (Mondal et al., 2016).

There is a close connection between the problem of approximating the value of a definite integral and that of approximating the solution to an initial-value problem. It is not surprising, then, that there are adaptive methods for approximating the solutions to initial-value problems and that these methods are not only efficient, but also incorporate the control of error (Gerald, 2004).

One way to guarantee accuracy in the solution of an I.V.P. is to solve the problem twice using step sizes $h$ and $h/2$ and compare answers at the mesh points corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough. The Runge-Kutta-Fehlberg method (denoted RKF45) is one way to try to resolve this problem. It has a procedure to determine if the proper step size $h$ is being used. At each step, two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the two answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size is increased.

RKF45 is an embedded ERK method, which means that it uses embedded error estimation. The method uses a fourth- and fifth-order method in the embedded error estimation (Djebali et al., 2021).

## 1.1 Statement of the problems

Classical numerical methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter $\varepsilon$ is very small. The accuracy,stablity and convergence of the methods need considaration, because the numerical treatment of singular perturbation problems is difficult, and the solution depends on perturbation parameter and mesh size (Turuna et al., 2020).

This recommends that numerical scheme that treat singularly perturbed parabolic convection-diffusion equations should be improved. The presence of the singular pertur bation parameter $\varepsilon$, leads to occurrences of oscillations or divergence in the computed solutions while using classical numerical methods. To avoid these oscillations or divergence, a large number of mesh points are required when $\varepsilon$ is very small. This is difficult and sometimes impossible to handle (Woldaregay et al., 2021).

Therefore, we use non-standard finite difference method together with Fehlberg Runge Kutta fourth and fifth order to treat the problem without creating an oscillation, to minimize drawback associated with classical numerical methods.

## 1.2 Objectives

### 1.2.1 General objective

The main objective of this study is to develop more accurate, stable, convergent and non oscillating numerical Scheme for solving singularly perturbed parabolic convection-diffusion problems.

### 1.2.2 Specific objectives

**The spacific objectives of this study are:**

1. To formulate accurate, stable and uniform convergent numerical method

2. To analyses the parameter uniform convergence of developed method.

3. To establish the stability of the method.

## 1.3 Significance of the study

The results of this study are:

- Provide a numerical scheme to solve a singularly perturbed parablic convection-diffusion real life problems.
- Further it can be serve as a reference as scholar working in the area.

## 1.4 Delimitation of the study

The study is delimited to 1-D time dependent singularly perturbed parabolic convection-diffusion equation(SPPCDE) of the form :

$$L_u(x,t) \equiv u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u_x(x,t) + b(x,t)u(x,t) = f(x,t), \tag{1.1}$$

$$(x,t) \in Q \equiv \Omega \times (0,T], \Omega = (0,1).$$

Subject to the boundary and the initial condition

$$u(0,t) = u_o, u(1,t) = u_1, u(x,0) = u(x), x \in (\bar{\Omega}), t \in [0,T].$$

where $\varepsilon$ is the perturbation parameter such that $0 < \varepsilon << 1$, the coefficient functions a(x,t), b(x,t) and the source function f (x, t) are assumed sufficiently smooth. This condition guarantees the existence of a unique solution for the problem in Eq.(1.2).
In this thesis, we assume the case $a(x,t) \geq \alpha > 0$ and $b(x,t) \geq \beta \geq 0$ which enables the existence of the boundary layer on the right side of the domain.

# CHAPTER 2

# REVIEW OF RELATED LITERATURE

## 2.1 Singular perturbation theory

Lundwing Prandtl was the first to introduce the concept of boundary layer in 1904 at the third international congruence of mathematics in Heidelberg Germany. His hypothesis was in the setting of fluid dynamics, fluid adjacent to the boundary sticks to the edge in a thin boundary layer due to friction but this friction has no effect to the flow of the fluid on the interior.

The term singular perturbation has been first coined by Frendricks and Wasow in 1946 in their paper "Singular Perturbation of Nonlinear Oscillation" Friedrichs and Wasow (1946) Wasow continued to contribute to the area of asymptotic methods over many years and his book "Asymptotic expansion for ordinary differential equation" Wasow (2018), attracted much interest in the area of singular perturbed boundary value problems . A brief survey for the historical development of perturbation and singular perturbation problems is covered in a recent books by Nayfeh (2011), O'Malley Jr (1991) respectively.

More precisely, a perturbation problem is a problem that contain a small parameter $\varepsilon$, called perturbation parameter. If the solution of the problem can be approximated by setting the value of the perturbation parameter equals to zero, then the problem is called regular perturbation problem, otherwise it is called singular perturbation problem. That is, if it is impossible to approximate the solution by an asymptotic expansion as the perturbation parameter tends to zero, then the problem is called singular. Singular perturbed differential equation is a differential equation in which the highest order derivative term is multiplied by a small positive parameter $\varepsilon$ called perturbation parameter(Turuna et al., 2020). Singular perturbation problems (SPPs) arise very frequently in fluid dynamics, elasticity, aerodynamics, plasma dynamics,magneto hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain (Mbroh & Munyakazi, 2019).

Thus, typically there are thin transition layers where the solution varies rapidly or jumps abruptly while away from the layers, the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior . Time-dependent convection–diffusion problems have been largely explored by researchers. They either use Rothe's method (time discretization followed by the spatial discretization) or discretize all the variables at once. A brief discussion is given below.

In one-dimensional case, Clavero et al. (2003) used the classical implicit Euler to discretize the time variable and the upwind scheme on a non-uniform mesh for the spatial variable. The approximation was done via the Rothe method and so was the analysis. The scheme was shown to be first order accurate in time and almost first order accurate in space. To obtain a higher order method in time, (2006) used the Crank Nicolson finite difference method to discretize the time variable along with the classical finite difference method on a piecewise uniform mesh for the space variable. Convergence analysis proved that the method is first order accurate except for a logarithmic factor in space and second order in time. Gowrisankar and Natesan (2014a) employed the backward Euler to discretize the time variable and the classical upwind finite difference method on a layer adapted mesh for the space variable.

To construct the mesh, the authors used equidistribution of a positive monitor function which is a linear combination of a constant and the second order spatial derivative of the singular component of the solution at each time level. Note that the monitor function which was used to construct the mesh led to problems with different meshes at each time level. Their analysis gave a first order accuracy in time and almost first order accuracy in space. In Gowrisankar and Natesan (2014b), Gowrisankar and Natesan again constructed a layer adapted mesh together with the classical upwind finite difference method to discretize the spatial variable.

Like in Avijit and Natesan (2020) they used a positive monitor function to construct the mesh. However, this positive monitor function was the second order spatial derivative of the singular component of the solution. The authors used the backward Euler method to discretize the time variable. Their analysis gave a first order accuracy in time and almost first order accuracy in space. Likewise, Chen and Liu (2016) used the backward Euler and the classical upwind finite difference method on a non-uniform mesh. In the framework of adaptive grid methods, the authors provided an efficient monitor function which is characterized with a simple mesh generating algorithm. They analysed their method for convergence and obtained a first order accuracy in time and an almost first order accuracy in space. Ramos(2005) proposed a numerical method which is first order accurate in both space and time variables.

The author used an exponentially fitted finite difference method to discretize the spatial variable and the backward Euler for the time variable. To obtain a higher order accuracy in space, Kadalbajoo et al (2008) used the B-spline collocation method on a piece wise uniform mesh to discretize the spatial variable and the classical implicit Euler for time variable. They performed the temporal discretization followed by the spatial discretization and proved that the method is first order accurate in time and second order in space.

Mukherjee and Natesan (2011) employed Richardson extrapolation method to post process a uniformly convergent method to a second order accuracy in both variables. They used the implicit Euler and simple upwind finite difference methods on a piece wise uniform mesh for the time and space variables respectively.
Mickens(1991) presented novel FD schemes used to numerically integrate the time-dependent Shrödinger equation. The schemes are explicit and use Euler-type expressions for the discrete time derivative. Also in (Mickens, 1994) constructs a new class of NSFD schemes for the Fisher partial differential equation. These schemes have the property that, in the appropriate limits, the discrete models obtained are either exact or the “best” FD schemes for the corresponding differential equation.

Anguelov and Lubuma (2000) formalized the transfer of essential properties of the solution of a differential equation to the solution of a discrete scheme as qualitative stability with respect to the properties. This helped to motivate some rules used in the design of nonstandard finite difference schemes. They considered extensions of some models, and numerical examples confirming the efficiency of the nonstandard approach are provided.

Lubuma and Patidar (2005) considered singular perturbation problems defined by first-order (systems of) ordinary differential equations, second-order ordinary dif- ferential equations, advection-reaction equations and reaction-diffusion equations. Lubuma and Patidar provide some complements to the theory of non-standard finite difference method. They use this theory to design non-standard schemes which replicate the physical properties of the exact solution. They provide several numerical simulations that support the theory.

We apply nonstandard finite difference method which preserve the analogs continuous properties of the original problem with Fehlberg Runge Kutta method of fourth and fifth order for solving singularly perturbed parabolic convection diffusion equation. We use this method in our research to get the more accurate and non oscillating solution, that has minimum point-wise absolute error than that reviewed above.

# CHAPTER 3

# NUMERICAL SCHEME

In this study we considered 1-D time dependent singularly perturbed parabolic convection-diffusion equation of the form :

$$L_u(x,t) \equiv u_t - \varepsilon u_{xx} + a(x,t)u_x + b(x,t)u = f(x,t), \tag{3.1}$$

$$(x,t) \in Q \equiv \Omega \times (0,T], \Omega = (0,1).$$

subject to the boundary and the initial condition

$$u(0,t) = u_o, u(1,t) = u_1, u(x,0) = u(x), x \in (\bar{\Omega}), t \in [0,T].$$

where $\varepsilon$ is the perturbation parameter such that $0 < \varepsilon << 1$. The presence of the singular perturbation parameter $\varepsilon$, leads to oscillations or divergence in the computed solutions while using classical numerical methods.

To avoid these oscillations or divergence, an unacceptably large number of mesh points are required when $\varepsilon$ is very small, which is not practical (Woldaregay et al., 2021). So, to overcome this drawback associated with classical numerical methods, we use nonstandard finite difference method in a spatial direction together with Runge kutta fourth and fifth order(Fehlberg method) in the temporal direction, which treat the problem without creating an oscillation.

After presenting some qualitative results relating to the convection-diffusion problems under study, we present the methods, their convergence analysis and some numerical results to illustrate the performance of the algorithms relates to accuracy and $\varepsilon$-uniformly convergence's of numerical scheme for solving singular perturbed 1-D parabolic convection-diffusion problem.

## 3.1 Properties of the continuous problem

Here we need to present some properties of the continuous problem (3.9) which guarantee the existence and uniqueness of the exact solution.
Let $u_0(x) \in C^2[0,1]$ and $(\varphi_0(0), \varphi_1(1)) \in C^1[0,t]$ by considaring boundary and assuming $u_0(0) = \varphi_0(0)$ and $u_0(1) = \varphi_1(1)$ then from equation(3.2) we can see that:

$$\frac{\partial \varphi_0(0)}{\partial t} - \frac{\varepsilon \partial^2 u_0(0)}{\partial x^2} + a(0)\frac{\partial u_0(0)}{\partial x} + b(0)u_0(0) = f(0,0)$$

$$\frac{\partial \varphi_1(0)}{\partial t} - \frac{\varepsilon \partial^2 u_0(1)}{\partial x^2} + a(1)\frac{\partial u_0(1)}{\partial x} + b(1)u_0(1) = f(1,0)$$

Considering the right boundary layers and using this compitability condition we can assume that there is constant $k$ independant of $\varepsilon$ $\forall u(x,t) \in \tilde{Q}$

$$|u(x,t) - u_0(x)| \leq kt$$
$$|u(x,t) - \varphi_1(t)| \leq k(1-x)$$

with out losing generality assume initial condition to be zero then we can show the boundedness of the solution $u(x,t)$ of equation (3.2).
**lemma 3.1**:-The solution $u(x,t)$ of the continuous problem of equation (3.2) satisfies $|u(x,t)| \leq k, \forall (x,t) \in \tilde{Q}$
**proof:** From inequality we can show that

$$\begin{aligned} |u(x,t) - u(x,0)| - |u(x,t) - u_0(x)| &\leq kt \\ |u(x,t)| - |u_0(x)| \leq |u(x,t) - u(x,0)| &\leq kt \\ \Rightarrow |u(x,t)| \leq kt + |u_0(x)|, &\forall (x,t) \in \tilde{Q} \end{aligned}$$

since $t \in [0,T]$ and $u_0(x)$ is bounded implies that $|u(x,t)| \leq k$
**Lemma 3.2**:- **(Continuous maximum principle).** Let $\psi \in C^{2,1}(Q)$ and be such that $\psi \geq 0, \forall (x,t) \in \partial Q$. Then $L_\varepsilon(x,t) \geq 0, \forall (x,t) \in Q$ implies that $\psi(x,t) \geq 0, \forall (x,t) \in \bar{Q}$

**proof:** assume there exist $(x^*, t^*) \in \bar{Q}$ such that $\psi(x^*, t^*) = \min_{\forall (x,t) \in Q}(\psi(x,t) < 0$ then $L_\varepsilon \psi(x^*, t^*) = \psi_t(x^*, t^*) - \varepsilon \psi_{xx}(x^*, t^*) + b(x^*, t^*)\psi(x^*, t^*) \leq 0$ which is contradict with $L_\varepsilon(x,t) \geq 0$ on Q therefore $\psi(x^*, t^*) \geq 0$
**Lemma 3.3**:- **(Discrete maximum principle).** let $U_0(t) \geq 0$ and $U_N(t) \geq 0$ then $L_\varepsilon U_i(t) \geq 0, \forall i = 1, 2, ..., N-1$ implies that $U_i(t) \geq 0, \forall i = 1, 2, ..., N$

**proof:** assume there exist $r \in \bar{Q}$ and $r \neq 0, N$ such that $U_r(t) = \min_{\forall(x,t)\in Q} U_i(t) < 0$ then we have that $U_{r+1}(t) - U_r(t) > 0$ and $U_r(t) - U_{r-1}(t) < 0$

$$L_\varepsilon U_r(t) = U_t(r_i,t) - \varepsilon\left(\frac{U_{r+1,(t)} - 2U_{r(t)} + U_{r-1(t)}}{\psi^2}\right) + \alpha_r(t)\frac{(U_r(t) - U_{r-1}(t)}{h} + \beta_r(t)U_r(t) \leq 0$$

by this assumption we conclude that $L_\varepsilon U_i(t) < 0, \forall i = 1, 2, ..., N-1$ the considaretion $U_i(t) < 0, \forall i = 1, 2, ...N$ is false therefore $U_i(t) \geq 0, \forall i = 1, 2, ...N$

**Lemma 3.4 :** Under the smoothness and compatibility conditions,Kadalbajoo and Awasthi (2006)proved that the exact solution and its derivatives satisfy $||\frac{\partial^{i+k}u(x,t)}{\partial x^i \partial t^k}|| \leq c(1 + \varepsilon^{-1}e^{\frac{-\alpha(1-x)}{\varepsilon}})$
$\forall(x,t) \in \bar{Q}, 0 \leq i + j \leq 3$

**Proof :** suppose $u(x,t) = v(x,t) + w(x,t)$, which represents the regular and singular components respectively.The regular compo- nent is the solution to the non-homogeneous problem.$L_v(x,t) = f(x,t), (x,t) \in Q, \quad v(0,t) = u(0,t), \quad t \in [0,T], \quad v(x,0) = u(x), \quad x \in \bar{\Omega}$ and the layer component is the solution to the homogeneous problem $L_w(x,t) = 0, \quad (x,t) \in Q, \quad w(x,0) = 0, \quad x \in \Omega, \quad w(0,t) = 0, \quad t \in [0,T], \quad w(1,t) = u(1,t) - v(1,t), \quad t \in [0,T]$.

Further, the regular component can be written in the form:
$v(x,t) = v^0(x,t) + \varepsilon v^1(x,t) + \varepsilon^2 v^2(x,t), \quad (x,t) \in \bar{Q}$.
Where $v_0$ is the solution to the reduced problem
$L_{v^0} \equiv v_t^0(x,t) + a(x,t)v_x^0(x,t) + b(x,t)v^0(x,t) = f(x,t), \quad (x,t) \in Q$
$v^0(x,0) = u(x), \quad x \in \Omega, \quad v^0(0,t) = 0, \quad t \in [0,T]$,
and $v^1, v^2$ are the respective solutions to the problems
$L_{v^1} \equiv v_t^1(x,t) + a(x,t)v_x^1(x,t) + b(x,t)v^1(x,t) = v_{xx}^0(x,t), \quad (x,t) \in Q$,
$v^1(x,0) = 0, \quad x \in \bar{\Omega}, \quad v^1(0,t) = 0, \quad t \in [0,T]$,

$L_{v^2} \equiv v_t^2(x,t) - \varepsilon v_{xx}^2(x,t) + a(x,t)v_x^2(x,t) + b(x,t)v^2(x,t) = v_{xx}^1(x,t), \quad (x,t) \in Q$,
$v^2(x,t) = 0, \quad (x,t) \in \bar{Q}$.

The functions $v^j$ , j = 0, 1, are solutions to a problem which is independent of $\varepsilon$, hence they satisfy the bound $||\frac{\partial^{i+k}u^j(x,t)}{\partial x^i \partial t^k}|| \leq C$

For $v^2$ it is the solution to a problem similar to the original problem therefore it satisfies $||\frac{\partial^{i+k}u^2(x,t)}{\partial x^i \partial t^k}|| \leq C(1 + \varepsilon^{-i}e^{\frac{-\alpha(1-x)}{\varepsilon}})$

When we add these two results, we obtain

$$\begin{aligned}||\frac{\partial^{i+k}u^2(x,t)}{\partial x^i\partial t^k}|| \leq ||\frac{\partial^{i+k}u^0(x,t)}{\partial x^i\partial t^k}|| + \varepsilon||\frac{\partial^{i+k}u^1(x,t)}{\partial x^i\partial t^k}|| + \varepsilon^2||\frac{\partial^{i+k}u^2(x,t)}{\partial x^i\partial t^k}|| \\ \leq C + C\varepsilon + C\varepsilon^2[1 + \varepsilon^{-i}e^{\frac{-\alpha(1-x)}{\varepsilon}}] \\ \leq C[1 + \varepsilon^{2-i}e^{\frac{-\alpha(1-x)}{\varepsilon}}] \\ since \qquad e^{\frac{-\alpha(1-x)}{\varepsilon}} \leq 1\end{aligned}$$

Also the layer part w(x, t) satisfies the bound as
$||w(x,t)|| \leq C(e^{\frac{-\alpha(1-x)}{\varepsilon}})$

**Lemma 3.5 Stability estimate** Let $u(x,t)$ be the solution of problem (3.2). Then we have $||u|| \leq \beta^{-1}||f|| + \max(u(x), max(u_0, u_1))$.

**Proof :** Let we define the barrier functions $\psi\pm(x,t)$ as

$$\psi\pm(x,t) = \beta^{-1}||f|| + max[u_0(x), max(\varphi_0(t), \varphi_1(t))] \pm u(x,t)$$

when time is zero at initial value

$$\psi\pm(x,0) = \beta^{-1}||f|| + max[u_0(x), max(\varphi_0(0), \varphi_1(0))] \pm u(x,0) \geq 0$$

at the two boundary also we have that

$$\begin{aligned}\psi\pm(0,t) = \beta^{-1}||f|| + max[u_0(0), max(\varphi_0(t), \varphi_1(t))] \pm u(0,t) \geq 0 \\ \psi\pm(1,t) = \beta^{-1}||f|| + max[u_0(1), max(\varphi_0(t), \varphi_1(t))] \pm u(1,t) \geq 0\end{aligned}$$

Again for the differential operator also:

$L_\varepsilon\psi\pm(x,t) = \psi\pm_t(x,t) - \varepsilon\psi\pm_{xx}(x,t) + a(x,t)\psi\pm_x(x,t) + b(x,t)\psi\pm(x,t)$
$= max[u_{0t}(t), max(\varphi_{0t}(t), \varphi_{1t}(t))] \pm u_t(x,t) - \varepsilon[max[u_{0xx}(t), max(\varphi_{0xx}(t), \varphi_{1xx}(t))] \pm u_{xx}(x,t)] + a(x,t)[max[u_{0x}(t), max(\varphi_{0xx}(t), \varphi_{1x}(t))] \pm u_x(x,t)] + b(x,t)[\beta^{-1}||f|| + max[u_0(t), max(\varphi_0(t),$
$u(x,t)] \geq 0$

since in our assumption $\varepsilon \geq 0, a(x,t) \geq \alpha > 0$, and $b(x,t) \geq \beta > 0$ This show that $L_\varepsilon\psi\pm(x,t) \geq 0, \forall(x,t) \in Q$ from maximum principle we have that, $\psi\pm(x,t) \geq 0, \forall(x,t) \in \tilde{Q} \Rightarrow |u(x,t)| \leq \beta^{-1}||f|| + max[u_0(t), max(\varphi_0(t), \varphi_1(t))]$

## 3.2 Nonstandard finite difference method

Let us consider the parabolic convection-diffusion problem

$$\frac{\partial u}{\partial t} + L_\varepsilon u = f, (x,t) \in Q \equiv \Omega \times (0,T], \tag{3.2}$$

where $0 < \varepsilon << 1$ and spacial difference operator

$$L_\varepsilon u = -\varepsilon u_{xx} + a(x,t)u_x + b(x,t)u, \quad \Omega = (0,1)$$

where $a \geq \alpha \geq 0$ and $b \geq \beta \geq 0$
Subject to the boundary and the initial condition

$$u(0,t) = u_o, u(1,t) = u_1, u(x,0) = u(x), x \in (\bar{\Omega}), t \in [0,T],$$

where $\varepsilon$ is the perturbation parameter such that $0 < \varepsilon << 1$, the coefficient functions a(x,t), b(x,t) and the source function f (x, t) are assumed sufficiently smooth(Cheng et al., 2015) and (Mbroh and Munyakazi, 2019). This condition guarantees the existence of a unique solution for the problem in Eq.(3.2). To dirive the exact scheme let us consider the equation

$$L_\varepsilon u = -\varepsilon u_{xx} + a(x,t)u_x + b(x,t)u = 0, \quad \Omega = (0,1). \tag{3.3}$$

Thus the equation in (3.3) has two independent solutions namely $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$ with

$$\lambda_{1,2} = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\varepsilon\beta}}{-2\varepsilon}$$

On the spatial domain $[0,1]$, we introduce the equidistant meshes with uniform mesh length $\triangle x = h$ such that $\Omega_x^N x = x_i = x_0 + ih, i = 1,2,...,N, x_0 = 0, x_N = 1, h = 1/N$ where N is the number of mesh points in spatial direction. Let approximate solution of $u(x)$ at $x_i$ by $U_i$ . Now to calculate a difference equation which has the same general solution as the differential equation in (3.3) has at the grid point $x_i$ given by
$U_i = C_1 \exp\,(\lambda_1 x_i) + C_2 \exp\,(\lambda_2 x_i)$. Using the theory of difference equations for second order linear difference equations in (Mickens, 2005) we have that:

$$det \begin{bmatrix} U_{i-1} & U_i & U_{i+1} \\ e^{\lambda_1 x_{i-1}} & e^{\lambda_1 x_i} & e^{\lambda_1 x_{i+1}} \\ e^{\lambda_2 x_{i-1}} & e^{\lambda_2 x_i} & e^{\lambda_2 x_{i+1}} \end{bmatrix} = 0$$

then the determinant of the matrix calculated as

$$U_{i-1}\begin{vmatrix} e^{\lambda_1 x_i} & e^{\lambda_1 x_{i+1}} \\ e^{\lambda_2 x_i} & e^{\lambda_2 x_{i+1}} \end{vmatrix} - U_i \begin{vmatrix} e^{\lambda_1 x_{i-1}} & e^{\lambda_1 x_{i+1}} \\ e^{\lambda_2 x_{i-1}} & e^{\lambda_2 x_i} \end{vmatrix} + U_{i+1}\begin{vmatrix} e^{\lambda_1 x_{i-1}} & e^{\lambda_1 x_i} \\ e^{\lambda_2 x_{i-1}} & e^{\lambda_2 x_i} \end{vmatrix} = 0$$

this equation is equivalent to:

$$U_{i-1}[e^{\lambda_1 x_i+\lambda_2 x_{i-1}} - e^{\lambda_2 x_i+\lambda_1 x_{i+1}}] - U_i[e^{\lambda_1 x_{i-1}+\lambda_2 x_{i+1}} - e^{\lambda_2 x_{i-1}+\lambda_1 x_{i+1}}] + U_{i+1}[e^{\lambda_1 x_{i-1}+\lambda_2 x_i} - e^{\lambda_2 x_{i-1}+\lambda_1 x_i}] = 0 \tag{3.4}$$

from the above we have the value of

$$\lambda_1 = \frac{-\alpha+\sqrt{\alpha^2+4\varepsilon\beta}}{-2\varepsilon}$$

and

$$\lambda_2 = \frac{-\alpha-\sqrt{\alpha^2+4\varepsilon\beta}}{-2\varepsilon}$$

by substituting the value of $\lambda$ in equation (3.4) we obtain

$$\begin{aligned} U_{i-1}[e^{\frac{\alpha(x_i+x_{i+1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_i-x_{i-1})} - e^{\frac{\alpha(x_i+x_{i+1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_i-x_{i+1})}] - \\ U_i[e^{\frac{\alpha(x_{i+1}+x_{i-1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_{i+1}-x_{i-1})} - e^{\frac{\alpha(x_{i-1}+x_{i+1})}{2\varepsilon}} e^{\frac{-\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_{i+1}-x_{i-1})}] + \\ U_{i+1}[e^{\frac{\alpha(x_{i-1}+x_i)}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_i-x_{i-1})} - e^{\frac{\alpha(x_{i-1}+x_i)}{2\varepsilon}} e^{\frac{-\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(x_i-x_{i-1})}] = 0 \end{aligned}$$

then we obtain

$$\begin{aligned} U_{i-1}[e^{\frac{\alpha(x_i+x_{i+1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(h)} - e^{\frac{\alpha(x_i+x_{i+1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(-h)}] - \\ U_i[e^{\frac{\alpha(x_{i+1}+x_{i-1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(2h)} - e^{\frac{\alpha(x_{i-1}+x_{i+1})}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(-2h)}] + \\ U_{i+1}[e^{\frac{\alpha(x_{i-1}+x_i)}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(h)} - e^{\frac{\alpha(x_{i-1}+x_i)}{2\varepsilon}} e^{\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon}(-h)}] = 0 \end{aligned}$$

$$e^{\frac{\alpha h}{2\varepsilon}} U_{i-1} - 2cosh(h\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon})U_i + e^{\frac{\alpha h}{2\varepsilon}} U_{i+1} = 0 \tag{3.5}$$

wich is an exact scheme derived from equation (3.2) as $\varepsilon \to 0$ let we use the approximation $h\frac{\sqrt{\alpha^2+4\varepsilon\beta}}{2\varepsilon} \approx \frac{\alpha h}{2\varepsilon}$ to dirive the denominator function let us consider the spacial derivative of convection-diffusion part

$$\frac{-\varepsilon d^2u(x,t)}{dx^2} + a(x,t)\frac{du(x,t)}{dx} = 0 \tag{3.6}$$

then by replacing derivative value by finite difference equivalence as

$$-\varepsilon\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{\psi^2}+a(x_i,t)\frac{U_i(t)-U_{i-1}(t)}{h}=0 \tag{3.7}$$

let us reduce equation (3.6) to firist order differential equation

$$\begin{cases}\frac{\partial u(x_i,t)}{\partial x}=z\\ \frac{\partial z}{\partial x}=\frac{a(x_i,t)z}{\varepsilon}\end{cases}$$

by using integration we obtain

$$z_i=e^{\frac{a(x_i,t)x_i}{\varepsilon}}$$

then by solving for $\psi^2$ from equation (3.7) we get

$$\psi^2=\frac{h\varepsilon}{a(x_i,t)}(e^{\frac{(ha(x_i,t))}{\varepsilon}}-1) \tag{3.8}$$

## 3.3 Discretization in spatial direction

Here $U_it$ is the approximation of of $u(x_i,t)$ using the theory of non standard finite difference scheme Mickens (2005) we discritize space to obtain semi-discrite problem

$$L_\varepsilon U\equiv U_t(x_i,t)-\varepsilon(\frac{U_{i+1,(t)}-2U_{i(t)}+U_{i-1(t)}}{\psi^2})+\alpha_i(t)\frac{(U_i(t)-U_{i-1}(t)}{h}+\beta_i(t)U_i(t)=f_i(t) \tag{3.9}$$

for i=1,2,...n-1 Subject to the boundary and the initial condition

$$u(0,t)=u_o,u(1,t)=u_1,u(x,0)=u(x),x\in(\bar{\Omega}),t\in[0,T].$$

using the denominator function $\psi^2$ which given by

$$\psi^2=\frac{h\varepsilon}{\alpha(x_i,t)}(e^{\frac{h\alpha(x_i,t))}{\varepsilon}}-1) \tag{3.10}$$

The above system of initial value problems in Eqn.(3.9) can be written in compact form as:

$$\frac{dU_i(t)}{dt}=AU_i(t)+F_i(t) \tag{3.11}$$

where A is tridiagonal matrix of size $N-1 \times N-1$ and $U_i(t)$ and $F_i(t)$ are column vectors of $N-1$ entries. The entries of A and F are given as follows:

$$\begin{cases} A_{ii} = \frac{2\varepsilon}{\psi^2} + \frac{\alpha(x_i)}{h} + \beta(x_i), & i = 1, 2, ...N-1 \\ A_{ii+1} = \frac{-\varepsilon}{\psi^2}, & i = 1, 2, ...N-2 \\ A_{ii-1} = \frac{-\varepsilon}{\psi^2} - \frac{\alpha(x_i)}{h}, & i = 2, 3, ...N-1 \end{cases}$$

$$\begin{cases} F_1(t) = f_1(t) + (\frac{\varepsilon}{\psi^2} + \frac{\alpha(x_i)}{h})u(0,t) \\ F_i(t) = f_i(t), & i = 2, 3, ...N-2 \\ F_{N_{-1}}(t) = f_{N_{-1}}(t) + \frac{2\varepsilon}{\psi^2_{N_{-1}}}u(1,t) \end{cases}$$

Next to this we need to show the semi-discrete operator is bounded.

## 3.4 Convergence analysis of semi-discrete space

Using appropriate Taylor series expansions
the truncation error of the scheme of equation (3.9) is given by

$$L^h(u_i(t) - U_i(t)) = L^h u(x_i,t) - L^h U(x_i,t)$$

$$= \frac{\partial u}{\partial t}(x_i,t) - \varepsilon\frac{\partial^2 u(x_i,t)}{\partial x^2} + \alpha_i\frac{\partial u}{\partial x}(x_i,t) + \beta_i u(x_i,t) - [\frac{\partial u}{\partial t}(x_i,t) - \varepsilon(\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\psi_i^2(\varepsilon,h,t)}) + \alpha_i(\frac{u_i(t) - u_{i-1}(t)}{h}) + \beta_i u(x_i,t)]$$

$$= -\varepsilon\frac{\partial^2 u(x_i,t)}{\partial x^2} + \alpha_i\frac{\partial u}{\partial x}(x_i,t) - [-\varepsilon(\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\psi_i^2(\varepsilon,h,t)}) + \alpha_i(\frac{u_i(t) - u_{i-1}(t)}{h})]$$
$$= \varepsilon - \frac{\partial^2 u(x_i,t)}{\partial x^2} + (\frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)}{\psi_i^2(\varepsilon,h,t)}) + \alpha_i[\frac{\partial u}{\partial x}(x_i,t) - (\frac{u_i(t) - u_{i-1}(t)}{h})]$$

Using Taylor series expansion of $u_{i+1}(t)$ and $u_{i-1}(t)$ leads to

$$= -\varepsilon(u_{xx}))_i + \frac{\varepsilon}{\psi_i^2(\varepsilon,h,t)}(h^2(u_{xx}(t))_i + \frac{h^4(u_{xxxx}(t))_i\eta_i}{12}) + \alpha_i(t)\frac{h(u_{xx}(t))_i}{2}$$

where $\eta_i \in (u_{i+1}(t), u_{i-1}(t))$ again using a truncated Taylor series expansions of $\psi^{-2} = \frac{1}{h^2} - \frac{\alpha_i(t)}{2h\varepsilon} + \frac{\alpha_i^2(t)}{12\varepsilon^2}$ we get that

$$= -\varepsilon(u_{xx}))_i + [h^2(u_{xx}(t))_i + \frac{h^4(u_{xxxx}(t))_i\eta_i}{12}][\frac{\varepsilon}{h^2} - \frac{\alpha_i(t)}{2h} + \frac{\alpha_i^2(t)}{12\varepsilon}] + \alpha_i(t)\frac{h(u_{xx}(t))_i}{2}$$

$$= -\varepsilon(u_{xx}))_i + [h^2(u_{xx}(t))_i + \frac{h^4(u_{xxxx}(t))_i\eta_i}{12}][\frac{\varepsilon}{h^2} - \frac{1}{2}(\frac{\alpha_i(t) - \alpha_{i-1}(t)}{h}) + \frac{\alpha_i^2(t)}{12\varepsilon}] \\ +\alpha_i(t)\frac{h(u_{xx}(t))_i}{2}$$

Using Taylor series expansion of $\alpha_{i-1}$ and rearrenging we obtain

$$= \frac{\alpha_i(t)h(u_{xx}(t))_i)}{2} + [-\frac{\alpha_x(t))_i(u_{xxxx}(t))_i}{24} + \frac{\alpha_i^2(t)(u_{xxxx}(t))_i}{144\varepsilon}\eta_i]h^4 \\ +[\frac{\varepsilon(u_{xxxx}(t))_i}{12}\eta_i - (\frac{\alpha_x(t))_i(u_{xx}(t))_i}{2}) + \frac{\alpha_i^2(t)(u_{xx}(t))_i}{12\varepsilon}]h^2$$

Applying the bounds of the solution and its derivatives of lemma 3.1 and Lemma 3.4 gives that

$$|L^h u(x_i,t) - L^h U(x_i,t)| \leq C|\frac{\alpha_i(t)h}{2} + [-\frac{\alpha_x(t))_i}{24} + \frac{\alpha_i^2(t)}{144\varepsilon}\eta_i]h^4| + C|[\frac{\varepsilon}{12}\eta_i - (\frac{\alpha_x(t))_i}{2}) + \frac{\alpha_i^2(t)}{12\varepsilon}]h^2|$$

from this we obtain that the error in spacial direction is bounded. Under the hypothesis of boundedness of semi-discrete solution (i.e it satisfies)discrete solution satisfy the following bound.

$$\sup_{0<\varepsilon<<1} \max_{1\leq i\leq N} |u(x_i,t) - U_i(t)| \leq CM^{-1}$$

## 3.5 Discretization in temporal direction

We use a higher order numerical scheme for discretizing the system of initial value problems, and use a Runge-Kutta method developed by Fehlberg to discretize problem in Eqn.(3.11) on a uniform mesh. With this in mind, we first rewrite Eqn.(3.11) in the form $\frac{dU_i(t)}{dt} = f(t, U_i(t))$ , $i = 0(1)M$ with initial condition $U(x_i, 0) = U(x_i)$ ,$i = 0(1)M$ here $f(t, U_i(t)) = -AU_i(t) + F_i(t)$ then for every $j = 1(1)N$.Then an approximation of the solution of the I.V.P. is made using a Runge-Kutta Fehlberg method of order four is given by:
$U_{ij+1} = U_{ij} + \frac{25k_1}{216} + \frac{1408k_3}{2565} + \frac{2197k_4}{4104} - \frac{1k_5}{5}$.
However better value for the solution is determined using a Runge-Kutta method of order five given by:
$\bar{U}_{ij+1} = U_{ij} + \frac{16k_1}{135} + \frac{6656k_3}{12825} + \frac{28561k_4}{56430} - \frac{9k_5}{50} + \frac{2k_6}{55}$

Where the $k$ values given as folws:

$$\begin{cases} k_1 = \triangle t f(t_j, U_{ij}) \\ k_2 = \triangle t f(t_j + \frac{\triangle t}{4}, U_{ij} + \frac{k_1}{4}) \\ k_3 = \triangle t f(t_j + \frac{3\triangle t}{8}, U_{ij} + \frac{3k_1}{32} + \frac{9k_2}{32}) \\ k_4 = \triangle t f(t_j + \frac{12\triangle t}{13}, U_{ij} + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197}) \\ k_5 = \triangle t f(t_j + \triangle t, U_{ij} + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104}) \\ k_6 = \triangle t f(t_j + \frac{\triangle t}{2}, U_{ij} - \frac{9k_1}{27} + 2k_2 - \frac{3544k_3}{2565} - \frac{1859k_4}{4104} - \frac{11k_5}{40}) \end{cases}$$

the cofficient value of Fehlberg method derived from Butcher tableau of Seen et al. (2014).

The difference between $U_{ij+1}$ and $\bar{U}_{ij+1}$ can be used to adapt the step size. Where $U_{ij}$ represent approximation of $U_i(t)$ at grid point $t_j$ and the local approximation $U_{ij+1}$ to $U_i(j+1)$ has fourth order accuracy $O(\triangle t)^4$

From the above approximation method in the temporal direction the global error estimates in two approximation techniques (Sundari & Vaithiyalingam, n.d.). The first is an $n^{th}$-order method obtained from an $n^{th}$-order Taylor method of the form:

$U_i(t_{j+1}) = U_i(t_j) + \triangle t f(t_j, U_i(t_j, \triangle t)) + O(\triangle t)^{n+1}$ produce approximations $U_0 = \sigma$

$U_{ij+1} = U_{ij} + \triangle t f(t_j, U_{ij}, \triangle t))$ for $j > 0$ with local truncation error $\tau_{j+1}(\triangle t) = O(\triangle t)^n$.

In general, the method is generated by applying a Runge-Kutta modification to the Taylor method, but the specific derivation is unimportant. The second method is similar but one order higher; it comes from an $(n+l)^{th}$-order Taylor method of the form:

$U_i(t_{j+1}) = U_i(t_j) + \triangle t \bar{f}(t_j, U_i(t_j, \triangle t)) + O(\triangle t)^{n+2}$ in which approximations given by $\bar{U}_0 = \sigma$

$\bar{U}_{ij+1} = \bar{U}_{ij} + \triangle t \bar{f}(t_j, \bar{U}_{ij}, \triangle t))$ for $j > 0$ with local truncation error $\bar{\tau}_{j+1}(\triangle t) = O(\triangle t)^{n+1}$.

Now first make the assumption that $U_{ij} \approx U_i(t_j)$ and choose a fixed step size $\triangle t$ to generate the approximations $U_{ij+1}$ to $U_i(t_{j+1})$. Then

$$\begin{aligned} \tau_{i+1} &= \frac{U_i(t_{j+1}) - U_i(t_j)}{\triangle t} - f(t_j, U_i(t_j), \triangle t) \\ &= \frac{U_i(t_{j+1}) - U_{ij}}{\triangle t} - f(t_j, U_i(t_j), \triangle t) \\ &= \frac{U_i(t_{j+1}) - (U_{ij} + \triangle t f(t_j, U_i(t_j), \triangle t))}{\triangle t} \\ &= \frac{1}{\triangle t}[U_i(t_{j+1}) - U_{ij+1}] \end{aligned}$$

in similar way $\bar{\tau}_{i+1} = \frac{1}{\triangle t}[U_i(t_{j+1}) - \bar{U}_{ij+1}]$
as consequence

$$\begin{aligned}\tau_{i+1} &= \frac{1}{\triangle t}([U_i(t_{j+1}) - \bar{U}_{ij+1}) + (\bar{U}_{ij+1} - U_{ij+1})] \\ &= \bar{\tau}_{i+1}(\triangle t) + \frac{1}{\triangle t}[(\bar{U}_{ij+1} - U_{ij+1})]\end{aligned}$$

but the order of $\tau_{i+1}(\triangle t)$ is $O(\triangle)t^n$ and $\bar{\tau}_{i+1}(\triangle t)$ is $O(\triangle t)^{n+1}$ so the significant portion of $\tau_{i+1}(\triangle t)$ must come from $\frac{1}{\triangle t}[(\bar{U}_{ij+1} - U_{ij+1})]$. This gives us an easily computed approximation for the local truncation error of the $O(\triangle t)^n$ as $\tau_{i+1}(\triangle t) \approx \frac{1}{\triangle t}[(\bar{U}_{ij+1} - U_{ij+1})]$. The object is not simply to estimate the local truncation error but to adjust the step size to keep it within a specified bound. We now assume that since $\tau_{i+1}(\triangle t)$ is $O(\triangle t)^n$ a number $k$ independent of $\triangle t$ exist with $\tau_{i+1}(\triangle t) \approx k(\triangle t)^n$. **Note** if $\tau_{i+1}(\triangle t) < \varepsilon$ keep $U_{ij}$ as the current step solution and move to the next step with step size $q\triangle t$ if $\tau_{i+1}(\triangle t) > \varepsilon$ recalculate the current step with step size $q\triangle t$. Then the local truncation error produced by applying the $n^{th}$-order method with a new step size $q\triangle t$ can be estimated using the original approximations $\bar{U}_{ij+1}$ and $U_{ij+1}$ .
$\tau_{i+1}(q\triangle t) \approx k(q\triangle t)^n = q^n(k\triangle t^n) \approx q^n\tau_{i+1}(\triangle t) \approx \frac{q^n}{\triangle t}[\bar{U}_{ij+1} - U_{ij+1}]$ to bound $\tau_{i+1}(q\triangle t)$ by $\varepsilon$ we choose q so that $\frac{q^n}{\triangle t}|[\bar{U}_{ij+1} - U_{ij+1}]| \approx |\tau_{i+1}(q\triangle t)| \leq \varepsilon$ that means
$q \leq [\frac{\varepsilon\triangle t}{|\bar{U}_{ij+1} - U_{ij+1}|}]^{\frac{1}{n}}$ one popular technique that uses this inequality for error control is the Runge-Kutta-Fehlberg method.

**Lemma 3.6.** From the above approximation method in the temporal direction, the global error estimates in this direction are given by:
$||E_{g.error}|| = ||U_i(t_{j+1}) - U_{i,j+1}|| \leq c(\triangle t)^4$,
where $E_{g.error}$ is the global error in the temporal direction at time step $(j+1)^{th}$.

**Proof:** Using the local error estimate $\tau_j$ up to the $j$ th time step, we obtain the global error estimate at the $(j+1)^{th}$ time step. Suppose that we have $\tau_j$ local error.

$$\begin{aligned}
||E_{g.error}|| &= \sum_{i=1}^{j} ||\tau_j|| \leq N \\
\leq ||\tau_1|| + ||\tau_2|| + ... + ||\tau_j|| & \qquad\qquad \textit{where} \\
||\tau_j|| &= C(\triangle t)^5 \\
&\leq C_1(j\triangle t)(\triangle t)^4 \\
&\leq C_1 T(\triangle t)^4 \qquad \textit{becuase} \\
j\triangle t &\leq T \\
&\leq C(\triangle t)^4
\end{aligned}$$

$$\sup_{0\leq\varepsilon<<1} ||U_i(t_{j+1}) - U_{i,j+1}|| \leq C(\triangle t)^4$$

Suppose that $C(\triangle t)^4 \leq M^{-1}$ when we comparable number of mesh points, then using the boundedness of solution of above lemma3.4 and lemma 3.5:

$$\sup_{0\leq\varepsilon<<1} ||U_i(t_{j+1}) - U_{i,j+1}|| \leq CM^{-1}$$

This indicate that the discretization in temporal direction is consistent and global error is bounded. Then using this to prove the $\varepsilon$-uniform convergence of the fully discrete scheme as:

$$\begin{aligned}
\sup_{0\leq\varepsilon<<1} ||u(x_i,t_j) - U_{i,j}||_{\Omega^M\times[0,T]^N} &\leq \sup_{0\leq\varepsilon<<1} ||U(x_i,t_j) - U_{i,}(t_j)||_{\Omega^M} \\
&+ \sup_{0\leq\varepsilon<<1} ||U_{i,}(t_j) - U(x_i,t_j)||_{[0,T]^N}
\end{aligned}$$

Using boundedness of the solution, we obtain:

$$\sup_{0\leq\varepsilon<<1} ||U_i(t_{j+1}) - U_{i,j+1}|| \leq C(M^{-1} + \triangle t)^4 \leq CM^{-1}$$

# CHAPTER 4

# NUMERICAL RESULTS AND DISCUSSION

To confirm the established theoretical results in this thesis, we perform experments using the proposed numerical scheme on the problem of the form given in Eq.(3.1). Below we present two test examples to illustrate the main result of this thesis. In the example we compute the maximum point-wise error and the rate of convergence. These results are displayed in tables for different values of N, M, $\triangle t$ and $\varepsilon$ for the sake of simplicity, we use $N = M$ in example 1 and $M \neq N$ for example 2 to compare result with other paper.

**Example 1** Consider the singularly perturbed parabolic initial-boundary value problem (Mbroh & Munyakazi, 2019).:

$$\begin{aligned} u_t - \varepsilon u_{xx} + [1 + x^2 + \frac{1}{2}sin(\pi x)]u_x + [1 + x^2 + sin(\frac{\pi t}{2})]u = f \\ where \quad f = x^3[1-x]^3 + t[1-t]sin(\pi t) \;\; and \\ (x,t) \in \Omega \times (0,1], \;\; u(x,0) = u(0,t) = u(1,t) = 0 \end{aligned}$$

**Example 2** Consider the problem (Turuna et al., 2020)

$$u_t - \varepsilon u_{xx} + [2 - x^2]u_x + xu = f$$

$$f = 10t^2e^{-t}x(1-x), \quad (x,t) \in \Omega \times (0,1], \;\; u(x,0) = u(0,t) = u(1,t) = 0$$

Since the exact solutions of the considered test problems are not known, we compute the maximum point wise error, which calculated using the double mesh principle given in (Woldaregay & Duressa, 2022)

$$E_{\varepsilon}^{M,\triangle t} = \max_{0\leq i\leq M-1, 0\leq j\leq N,} |U_{ij}^{M,\triangle t} - U_{ij}^{2M,\frac{\triangle t}{2}}|$$

where $U_{ij}^{M,\triangle t}$ is the numerical solution over $Q^{M,N}$ and $U_{ij}^{2M,\frac{\triangle t}{2}}$ is corresponding approximate solution on the mesh $Q^{2M,2N}$. However in Fehlberg Runge Kutta we use instead $U_{ij}^{M,\triangle t}$ is the numerical solution over $Q^{M,N}$ at fourth order and $\bar{U}_{ij}^{M,\triangle t}$ is corresponding approximate solution on the mesh $Q^{M,N}$ at fifth order so that maximum pointwise error calculated as:

$$E_{\varepsilon}^{M,\triangle t} = \max_{0\leq i\leq M-1, 0\leq j\leq N,} |\bar{U}_{ij+1}^{M,\triangle t} - U_{ij+1}^{M,\triangle t}|$$

For any values M and N the $\varepsilon$-uniform error estimate are calculated using the formula.

$$E^{M,\triangle t} = \max_{0\leq \varepsilon << 1} |E_{\varepsilon}^{M,\triangle t}|.$$

And also we compute the $\varepsilon$-uniform rate of convergence with

$$R = log_2[E_{\varepsilon}^{M,\triangle t} / E_{\varepsilon}^{2M,\frac{\triangle t}{2}}]$$

The solution of the problems given in Example 1 and 2 has a boundary layer at the right side of the $x$-domain (see Figures 4.1 ,4.2,4.3 and 4.4). The computed solutions $U_{i,j}$ for different values of perturbation parameters are also shown in these figures. The numerical results displayed in tables 4.1 and 4.3 clearly indicate that the proposed method is $\varepsilon$-uniform convergent. From the results in these tables, we observe that the maximum point-wise error decreases as $M$ increases for each value of $\varepsilon$. In addition, the maximum point-wise error is stable as $\varepsilon \to 0$ for each $M$ ,$N$ and $\triangle t$. Using computed results in these two examples, we confirm that the proposed numerical method is more accurate, stable and $\varepsilon$-uniform convergent with the rate of conver- gence one. The results in the proposed method are better than those given in (Turuna et al., 2020) and (Mbroh & Munyakazi, 2019).

Table 4.1: Maximum point-wise error and rate of convergence proposed scheme for Example 1.

| $\varepsilon$ | M =32<br>N =32 | 64<br>64 | 128<br>128 | 256<br>256 | 512<br>512 | 1024<br>1024 |
|---|---|---|---|---|---|---|
| $10^{-4}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |
| $10^{-5}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |
| $10^{-6}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |
| $10^{-13}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| $10^{-25}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |
| $E_{\varepsilon}^{M,N}$ | 8.5273e-05 | 4.5726e-05 | 2.3843e-05 | 1.2188e-05 | 6.1615e-06 | 3.0978e-06 |
| $R$ | 0.8991 | 0.9394 | 0.9681 | 0.9841 | 0.9920 | |

Table 4.2: Maximum point-wise error and rate of convergence of result for Example 1. Mbroh and Munyakazi(2019)

| $\varepsilon$ | M=32<br>N=32 | 64<br>64 | 128<br>128 | 256<br>256 | 512<br>512 | 1024<br>1024 |
|---|---|---|---|---|---|---|
| $10^{-4}$ | 1.25E-03 | 6.62E-0.4 | 3.41E-0.4 | 1.74E-0.4 | 8.84E-0.5 | 4.41E-0.5 |
| | 0.92 | 0.96 | 0.97 | 0.98 | 1.00 | |
| $10^{-5}$ | 1.25E-03 | 6.62E-0.4 | 3.41E-0.4 | 1.74E-0.4 | 8.84E-0.5 | 4.41E-0.5 |
| | 0.92 | 0.96 | 0.97 | 0.98 | 0.99 | |
| $10^{-6}$ | 1.25E-03 | 6.62E-0.4 | 3.41E-0.4 | 1.74E-0.4 | 8.84E-0.5 | 4.41E-0.5 |
| | 0.92 | 0.96 | 0.97 | 0.98 | 0.99 | |
| $10^{-13}$ | 1.25E-03 | 6.62E-0.4 | 3.41E-0.4 | 1.74E-0.4 | 8.84E-0.5 | 4.41E-0.5 |
| | 0.92 | 0.96 | 0.97 | 0.98 | 0.99 | |

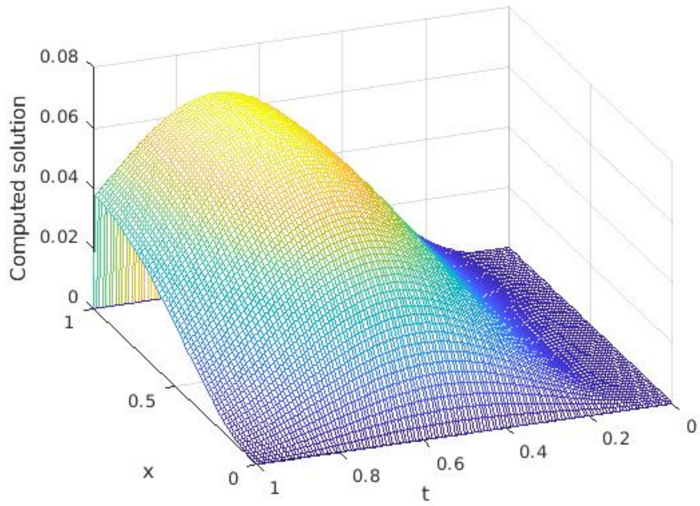


Figure 4.1: 3-D plot solutions profile for Example 1 with boundary layer formation as $\varepsilon$ goes small with mesh size $N = 128, M = 64$ and $\varepsilon = 10^{-4}$.

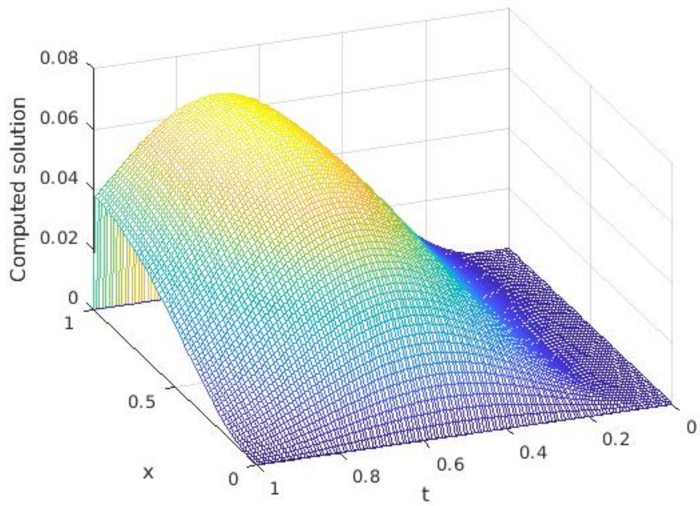


Figure 4.2: 3-D plot solutions profile for Example 1 with boundary layer formation as $\varepsilon$ goes small with mesh size $N = 128, M = 64$ and $\varepsilon = 10^{-10}$.

From the above two figures we can see that our solution are bounded and $\varepsilon$-uniform convergent.

Table 4.3: Maximum point-wise error and rate of convergence of proposed scheme for Example 2.

| $\varepsilon$ | M =8<br>$\triangle t$=1/10 | 16<br>1/20 | 32<br>1/40 | 64<br>1/80 | 128<br>1/160 | 256<br>320 |
|---|---|---|---|---|---|---|
| $10^{-4}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| $10^{-5}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| $10^{-6}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| $10^{-8}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| $10^{-12}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| . | . | . | . | . | . | |
| . | . | . | . | . | . | |
| . | . | . | . | . | . | |
| $10^{-20}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |
| $E_{\varepsilon}^{M,N}$ | 5.5990e-04 | 2.9086e-04 | 1.5086e-04 | 7.6286e-05 | 3.8427e-05 | 1.9278e-05 |
| $R$ | 0.9448 | 0.9471 | 0.9837 | 0.9893 | 0.9952 | |

Table 4.4: Maximum point-wise error and rate of convergence of result for Example 2.

| | $\varepsilon$ | M =32<br>$\triangle t$=1/10 | 64<br>1/20 | 128<br>1/40 | 256<br>1/80 |
|---|---|---|---|---|---|
| Turuna et al.(2020) | $10^{-4}$ | 6.1336e-03 | 3.5748e-03 | 1.9245e-03 | 9.9750e-04 |
| | | 0.7789 | 0.8934 | 0.9481 | 0.9745 |
| | $10^{-6}$ | 6.1336e-03 | 3.5748e-03 | 1.9245e-03 | 9.9750e-04 |
| | | 0.7789 | 0.8934 | 0.9481 | 0.9745 |
| | $10^{-8}$ | 6.1336e-03 | 3.5748e-03 | 1.9245e-03 | 9.9750e-04 |
| | | 0.7789 | 0.8934 | 0.9481 | 0.9745 |

In above comparison between proposed method and Turuna et al.(2020) the proposed method is better because, even if the proposed method is not use the parallel mesh size the ratio of mesh length $M/\triangle t$ of the proposed method is very small than Turuna et al.(2020) to get the corresponding accuracy.

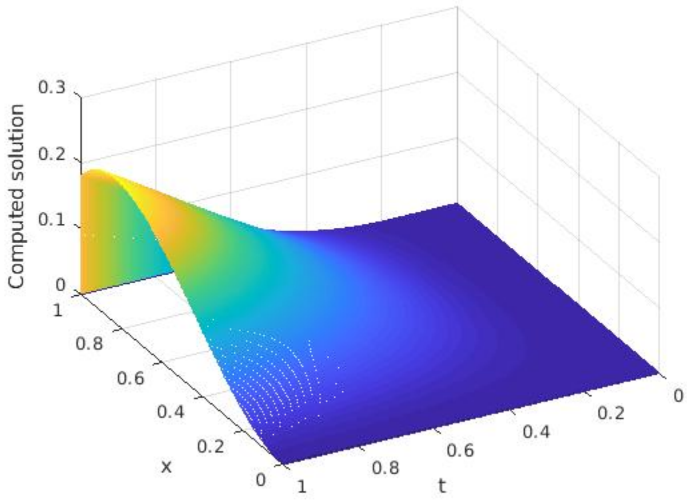


Figure 4.3: 3-D plot solutions profile for Example 2 with boundary layer formation as $\varepsilon$ goes small with mesh size $N = 320, M = 256, \varepsilon = 0.000000011$.

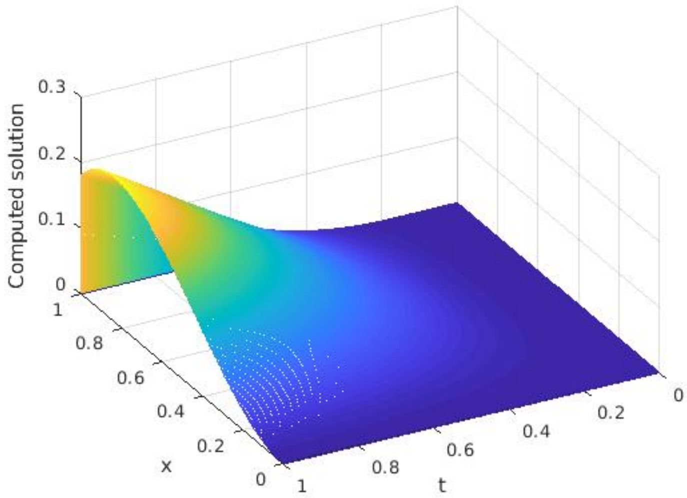


Figure 4.4: 3-D plot solutions profile for Example 2 with boundary layer formation as $\varepsilon$ goes small with mesh size $M = 256, N = 320, \varepsilon = 0.00001$.

From the above (3.3)and (3.4) figures we can see that our solutions bounded and $\varepsilon$-uniform convergent.

# CONCLUSION AND RECOMMENDATIONS

## CONCLUSION

In this thesis,a numerical method is developed for solving singularly perturbed 1-D parabolic convection-diffusion differential equation that the solution exhibit a boundary layer behavior on the right side of the domain. The method consists of a step by step discretization of the space and the time variables. We employ the nonstandard finite difference method(NFDM) for the space discretization and the Fehlberg Runge Kutta method of order four and five on a uniform mesh was used for the time discretization, for the system of initial value problem resulting from the spatial discretization.

We prove that the proposed numerical scheme converges uniformly with respect to the perturbation parameter,$\varepsilon$. To confirm the convergence and the efficiency of the method, we perform numerical experiments on test examples.The method is shown to be uniformly convergent with order of convergence $O(M^{-1} + (\triangle t)^4)$. The performance of the proposed scheme is investigated by comparing with the results in (Turuna et al., 2020) and (Mbroh & Munyakazi, 2019). The proposed method gives more accurate, stable and $\varepsilon$-uniform numerical results. From the two proposed tables, we observe that the numerical results are reflections of the theoretical findings.

# RECOMMENDATION

The scheme proposed in this thesis yields more accurate, stable and $\varepsilon$-uniform numerical results. However in above comparison between proposed method and Turuna et al.(2020) I not use the parallel mesh to show the corresponding accuracy because proposed method not bounded as $\varepsilon$ large approaches to one and bounded as $\varepsilon$ tends to zero. This shows, there are still a number of concerns with singularly perturbed problems.So the new researcher find why explicit numerical scheme not give bounded solution as $\varepsilon$ tends to one for singularly perturbed problems, which is the next topic of research.

Furthermore,one can examine the accuracy between explicit and implicit methods in singularly perturbed problems, and so on.

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# References


Anguelov, R., & Lubuma, J. M.-S. (2000). *Contributions to the mathematics of the nonstandard finite difference method with applications to certain discrete schemes.* UCL.

Avijit, D., & Natesan, S. (2020). Sdfem for singularly perturbed parabolic initial-boundary-value problems on equidistributed grids. *Calcolo*, *57*(3), 1–25.

Chen, Y., & Liu, L.-B. (2016). An adaptive grid method for singularly perturbed time-dependent convection-diffusion problems. *Communications in Computational Physics*, *20*(5), 1340–1358.

Clavero, C., Jorge, J., & Lisbona, F. (2003). A uniformly convergent scheme on a nonuniform mesh for convection–diffusion parabolic problems. *Journal of Computational and Applied Mathematics*, *154*(2), 415–429.

Djebali, R., Mebarek-Oudina, F., & Rajashekhar, C. (2021). Similarity solution analysis of dynamic and thermal boundary layers: further formulation along a vertical flat plate. *Physica Scripta*, *96*(8), 085206.

Farrell, P., Hegarty, A., Miller, J. M., O'Riordan, E., & Shishkin, G. I. (2000). *Robust computational techniques for boundary layers.* Chapman and hall/CRC.

Friedrichs, K. O., & Wasow, W. R. (1946). Singular perturbations of non-linear oscillations. *Duke Mathematical Journal*, *13*(3), 367–381.

Gerald, C. F. (2004). *Applied numerical analysis.* Pearson Education India.

Gie, G.-M., Hamouda, M., Jung, C.-Y., & Temam, R. M. (2018). *Singular perturbations and boundary layers.* Springer.

Gowrisankar, S., & Natesan, S. (2014a). Robust numerical scheme for singularly perturbed convection–diffusion parabolic initial–boundary-value problems on equidistributed grids. *Computer Physics Communications*, *185*(7), 2008–2019.

Gowrisankar, S., & Natesan, S. (2014b). Uniformly convergent numerical method for singularly perturbed parabolic initial-boundary-value problems with equidistributed grids. *International Journal of Computer Mathematics*, *91*(3), 553–577.

Gracia, J., & Lisbona, F. (2007). A uniformly convergent scheme for a system of reaction–diffusion equations. *Journal of computational and applied mathematics*, *206*(1), 1–16.

Hofmanova, M., Koley, U., & Sarkar, U. (2020). Measure-valued solutions to the stochastic compressible euler equations and incompressible limits. *arXiv preprint arXiv:2012.07391*.

Kadalbajoo, M. K., & Awasthi, A. (2006). A parameter uniform difference scheme for singularly perturbed parabolic problem in one space dimension. *Applied mathematics and computation*, *183*(1), 42–60.

Kadalbajoo, M. K., Gupta, V., & Awasthi, A. (2008). A uniformly convergent b-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional time-dependent linear convection–diffusion problem. *Journal of Computational and Applied Mathematics*, *220*(1-2), 271–289.

Liao, C., & Ding, X. (2013). Nonstandard finite difference variational integrators for nonlinear schrödinger equation with variable coefficients. *Advances in Difference Equations*, *2013*(1), 1–22.

Lubuma, J. M.-S., & Patidar, K. C. (2005). Contributions to the theory of non-standard finite difference methods and applications to singular perturbation problems. In *Advances in the applications of nonstandard finite difference schemes* (pp. 513–560). World Scientific.

Mbroh, N. A., & Munyakazi, J. B. (2019). A fitted operator finite difference method of lines for singularly perturbed parabolic convection–diffusion problems. *Mathematics and Computers in Simulation*, *165*, 156–171.

Mickens, R. E. (1991). Novel explicit finite-difference schemes for time-dependent schrödinger equations. *Computer physics communications*, *63*(1-3), 203–208.

Mickens, R. E. (1994). A best finite-difference scheme for the fisher equation. *Numerical Methods for Partial Differential Equations*, *10*(5), 581–585.

Mickens, R. E. (2000). *Applications of nonstandard finite difference schemes.* World Scientific.

Mickens, R. E. (2005). *Advances in the applications of nonstandard finite difference schemes.* World Scientific.

Mondal, S. P., Roy, S., & Das, B. (2016). Numerical solution of first-order linear differential equations in fuzzy environment by runge-kutta-fehlberg method and its application. *International Journal of Differential Equations*, *2016*.

Mukherjee, K., & Natesan, S. (2011). Richardson extrapolation technique for singularly perturbed parabolic convection–diffusion problems. *Computing*, *92*(1), 1–32.

Nayfeh, A. H. (2011). *Introduction to perturbation techniques.* John Wiley & Sons.

O'Malley Jr, R. E. (1991). *Singular perturbation methods for ordinary differential* *equations*. Springer-Verlag.

Ramos, J. I. (2005). An exponentially-fitted method for singularly perturbed, one-dimensional, parabolic problems. *Applied mathematics and computation*, *161*(2), 513–523.

Seen, W. M., Gobithaasan, R., & Miura, K. T. (2014). Gpu acceleration of runge kutta-fehlberg and its comparison with dormand-prince method. In *Aip conference proceedings* (Vol. 1605, pp. 16–21).

Smith, G. D., Smith, G. D., & Smith, G. D. S. (1985). *Numerical solution of partial differential equations: finite difference methods*. Oxford university press.

Sundari, M., & Vaithiyalingam, R. (n.d.). Solving partial differential equations with matlab.

Turuna, D. A., Woldaregay, M. M., & Duressa, G. F. (2020). Uniformly convergent numerical method for singularly perturbed convection-diffusion problems. *Kyungpook Mathematical Journal*, *60*(3), 629–645.

Wasow, W. (2018). *Asymptotic expansions for ordinary differential equations*. Courier Dover Publications.

Woldaregay, M. M., Aniley, W. T., & Duressa, G. F. (2021). Novel numerical scheme for singularly perturbed time delay convection-diffusion equation. *Advances in Mathematical Physics*, *2021*.

Woldaregay, M. M., & Duressa, G. F. (2022). Uniformly convergent numerical method for singularly perturbed delay parabolic differential equations arising in computational neuroscience. *Kragujevac Journal of mathematics*, *46*(1), 65–84.