Numerical Solution of the Non Linear Integro Differential Equations

By

Megbaru Awoke

A Thesis Submitted to

The Program of Applied Mathematics

School of Applied Natural Science

Submitted in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

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Abstract

This thesis deals with the numerical solution of nonlinear integro-differential equations. The numerical solution described in this thesis is based on the Variational Iteration Method (VIM)and Adomian Decomposition Method (ADM). Therefore, a comparative study of Adomian Decomposition Method (ADM) and Variational Iteration Method (VIM) will be considered to solve non linear integro-differential equations: (Volterra and Fredholm equations). Both methods will be compared interms of efficiency, accuracy and convergence. Various numerical examples will be given to illustrate the reliability, convergence and accuracy of these two methods.

Key Words: *Adomian Decomposition Method, Variational Iteration Method, Non linear integro-differential equations.*

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Notations and Abbreviations

VIM	Variational Iteration Method
ADM	Adomian Decomposition Method
LM	Lagrange Multiplier
ICs	Initial Conditions
BCs	Boundary Conditions
IDE	Integro Differential Equation
NLIDE	Non Liniear Integro Differential Equation
VIDE	Non Voltera Integro Differential Equation
FIDE	Fredholm Integro Differential Equation
ODE	Ordinary Differential Equation
IVP	Initial Value Problem

Chapter 1

Introduction

Numerical methods are mathematical techniques used for solving mathematical problems that cannot be solved or difficult to be solved analytically. An analytical solution is an exact answer in the form of a mathematical expression in terms of the variables associated with the problem that is being solved. A numerical solution is an approximate numerical value (a number) for the solution. Although numerical solutions are approximations, they can be very accurate. In many numerical methods, the calculations are executed in an iterative manner until a desired accuracy is achieved[A.Amos, 2014].

A non linear integro-differential equation is an equation that contains $u^{(n)}(x)$, which is the n^{th} derivative of $u(x)$, and an unknown function $u(x)$ that appears under an integral sign. A standard integro-differential equation is of the form:

$$u^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)F(u(t))dt \tag{1.1}$$

where $F(u(x))$ is a non linear function of $u(x)$, $g(x)$ and $h(x)$ are the limits of the integral , λ is a constant parameter, $k(x,t)$ is a function of two variables x and t, which is called the kernel or the nucleus of the equation and $f(x)$ is a source term.

The limits of integration $g(x)$ and $h(x)$ can be variables, constants or mixed(the combination of variable and constant).

We will describe two distinict ways that depend on the limits of integration. These limits of integration are used to characterize the integral equations. Namely:

- If the upper bound of the region for the integral part is a fixed number then, the integro-differential equation is called a **Fredholm integro-differential equation (FIDE)**.
- If the upper bound of the region for the integral part is a variable, the equation is called a **Voltera integro-differential equation (VIDE)**.

In this study, we focus on a second order non linear integro-differential equation and apply two basic methods(variational iteration method and Adomian Decomposition Method) for effectively solving it.

The general second order non linear fredholm integro differential equation is given by:

$$u''(x) = f(x) + \lambda \int_a^b k(x,t)F(u(t))dt \tag{1.2}$$

and that of volterra type is given by:

$$u''(x) = f(x) + \lambda \int_0^x k(x,t)F(u(t))dt \tag{1.3}$$

It is important to mention that a major shortcoming of the variational iteration method is that the error slowly deteriorates as we increase the values of x over the entire domain; hence, the convergence is local and not uniform. We will also discuss and give a detail review of the Adomian Decomposition Method (ADM). The Adomian decomposition method was first introduced and developed by George Adomian in $1980's$. It has been receiving much attention from researchers in recent years in the field of applied mathematics, in general, and in the area of initial and boundary value problems in particular. The method efficiently handles a wide class of linear/nonlinear ordinary and partial differential equations, linear and nonlinear integral equations, and integro-differential equations. The ADM provides several significant advantages; it demonstrates fast convergence of the solution. It also handles the problem in a direct way without using linearization (a way of choosing our initial guess), or any other restrictive assumptions that may change the physical behavior of the model under study. Furthermore, it provides an efficient numerical solution in the form of an infinite series that is obtained iteratively and usually converges to the exact solution using Adomian polynomials.

In this study, we wil also apply the decomposition method on some problems of the non linear integro-differential equations.

1.1 Statement of the Problem

In the previous work, many researchers have solved the non linear integro-differential equations using different types of numerical techniques, like: laplace transform method, semi analytical method, differential transform method,... and so on. In particular (A.M.Wazwaz, 2010) presented a comparative study between Variational iteration method and Adomian decomposition method for solving the higher order nonlinear integro-differential equations. However, as my investigation he didn't consider the rate and the region of convergence of the approximate solution of the nonlinear integro-differential equations. Therefore, in this study, we will try to investigate on the rate and the region of convergence of the approximate solution of some nonlinear integro-differential equations. So that, the thesis is intended to answer the following basic question:

- What will be the region of convergence of VIM and ADM to solve the non linear integro differential equations?
- What are the necessary conditions for the convergence of the methods?
- What will be the rate of convergence of VIM and ADM to solve the non linear integro differential equations?
- Which method is efficient(VIM or ADM)?
- How to improve the accuracy of the approximate solution?

1.2 Objective of the Study

1.2.1 General objective

The general objective of this thesis is to describe a comparative study to examine the performance of the Variational Iteration Method and Adomians Decomposition Method for solving non linear integro differential equations.

1.2.2 Specific objectives

The specific objectives of this study are:

- To find the region of convergence of VIM and ADM for the solution of the non linear integro differential equation.
- To identify the necessary conditions for the convergence of our approximate solution.
- To find the rate of convergence of VIM and ADM for the solution of the non linear integro differential equation.
- To identify the efficient numerical method(from VIM and ADM).
- To improve the accuracy of the approximate solution.

Chapter 2

Literature Review

Over the recent years, there have been a lot of investigations on developing the numerical solutions of the integro-differential equations in order to model the physical problems. However, some recommendations need to be done for developing more advanced and efficient method for solving integro-differential equations. In previous studies certain methods such as the differential transform method, and semi analytical numerical techniques such as the Adomian decomposition method have been used. In this thesis, we will apply the modified decomposition method for solving nonlinear Volterra Fredholm integro differential equations (A.M. Wazwaz, 2010).

Adomian decomposition method (ADM) was a method for solving integral equations which has been presented by G. Adomian. Then, (A.M.Wazwaz, 2006) extended this method to solve Volterra integral equations and boundary value problems for higher order integro-differential equations. Besides that, Adomian decomposition method (ADM) also had been applied for a wide class of nonlinear equations, such as the ordinary and partial differential equations, integral equations and integro-differential equations.

The VIM has successfully been applied to many situations. For example, He solved the classical Blasius equation using VIM. He used VIM to give approximate solutions for some well-known nonlinear problems.
He used VIM to solve autonomous ordinary differential systems. He coupled the iteration method with the perturbation method to solve the well-known Blasius equation; He solved strongly nonlinear equations using VIM. Soliman(N.H.Soliman, 2006) applied the VIM to solve the KdV-Burgers and Laxs seventh-order KdV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by (M.Wadati, 1972)

The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat et al. Bildik et al (N.H.Sweilam, 2006 and A.konuralp) used VIM for solving different types of nonlinear partial differential equations. Dehghan and Tatari employed VIM to solve a Fokker-Planck equation. Wazwaz presented a comparative study between the variational iteration method and Adomian decomposition method.(A.M Wazwaz,2006) Tamer et al introduced a modification of VIM. Batiha et al used VIM to solve the generalized Burgers-Huxley equation. B.Batiha.et.al, 2008 applied VIM to the generalized Huxley equation. Abbasbandy (S. Abbasbandy, 2009)solved one example of the quadratic Riccati differential equation (with constant coefficient) by Hes variational iteration method by using Adomians polynomials. Wang and He applied VIM to solve integro-differential equations. (N.H.Sweilam, 2006) used VIM solve both linear and nonlinear boundary value problems for fourth order integro-differential equations.
In [S. M. Hosseini and S. Shah morad, 2003], Hosseini and Shahmorad employed Tau method to obtain a numerical solution to the integro-differential equations given by (G.Adomian, 1986). Batiha et al. [B. Batiha, M. S. M. Noorani, 2008] presented the variational iteration method (VIM) and the ADM for solving nonlinear integro-differential equations. Fariborzi Araghi and Sadigh Behzadi [M. A. Fariborzi Araghi, 2009] solved nonlinear Volterra-Fredholm integro-differential equations by using the modified ADM,the VIM, and the homotopy analysis method, respectively. Borhanifar and Abazari [A. Borhanifar and R. Abazari, 2012] implemented the differential transform method for solving nonlinear integro-differential equations with the kernel functions including derivative type of unknown solution. Ben Zitoun and Cherruault [F.BenZitounandY.Cherru 2012] presented a method for solving nonlinear integro-differential equations with constant or variable coefficients with initial or boundary conditions. El-Kalla [I. L. El-Kalla, 2012] introduced a new technique for solving a class of quadratic integral and integro differential equations

Chapter 3

Research Methodology

This study focuse on two important methods called ADM and VIM. This section contains the research source of informations and working procedures.

3.1 Source of information

The applicable source of information for this study are : Reference books, published journals, articles and related studies from the internet.

3.2 Study Design and Produceres

In order to accomplish the stated objectives ,the thesis will follow the following steps:

- ⧫ Method description.
- ⧫ Derivation of the iteration schemes.
- ⧫ Implementation of the Method (coding).
- ⧫ Determination of lagrange's multiplier λ.
- ⧫ Define a correction functional.
- ⧫ Compute the errors to identify the more accurate method.

3.3 Benefits and Beneficiaries

This study has its own advantages, limitations and delimitations .

3.3.1 Benefit of the study

This study provides a reliable information on how can we use the numerical methods, especially VIM and ADM to solve the non linear integro-differential equations. Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields like biological models, industrial mathematics, control theory of financial mathematics, economics, electrostatics, fluid dynamics, heat and mass transfer, oscillation theory, and queuing theory.

3.3.2 Limitation of the study

The limitations of this research work are:

- There was scarcity of time.
- Sometimes the network acces had been limited.
- Shortage of related reference books.
- The error may be magnifying.

3.3.3 Delimitation of the study

This study is delimited to solve the second order non linear integro-differential equations by using the two numerical methods known as Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM) to improve the accuracy of the approximate solution

Chapter 4

Variational Iteration Method for Integro Differential Equation

4.1 Method description

In this section, we are going to discuss a well-known numerical method, namely, the Variational Iteration Method (VIM). The VIM was first proposed by the Chinese mathematician He in 1999 as modification of a general Lagrange multiplier method for solving a wide range of problems whose mathematical models yield differential equations or systems of differential equations. It was and still is utilized by mathematicians to handle a wide variety of applications that arise in engineering and sciences, such as homogeneous and inhomogeneous linear problems, as well as those that are nonlinear.

Essentially, the VIM accurately computes the solutions in a series form that rapidly converges to the exact solution in an iterative fashion with specific features for each scheme. It yields several successive approximations by using the iteration of the correction functional. The VIM is a powerful and efficient method that results in approximations that are highly accurate, and also gives closed form solutions if they exist. This powerful technique handles both linear and nonlinear problems in a unified manner.

In this part, we apply the variational iteration method on some problems of non linear integro-differential equations. The advantages of the method are that it gives highly accurate numerical solutions and reduces the size of computational work. It is important to mention that a major shortcoming of the variational iteration method is that the error slowly deteriorates as we increase the values of x over the entire domain. Variational Iteration Method (VIM) is based on the general Lagranges multiplier method. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution.

If the convergence is assured, the obtained approximations by this technique are of high accuracy level even if some iterations are used. Now, we derive some useful iteration formulas for certain classes of first order and higher order (especially of second order) non linear integro differential equations and determine the Lagrange multiplier $\lambda(s)$ for each class as well. The following procedure is useful to evaluate the value of λ for first order non linear integro differential equations. Now, to illustrate the basic concepts of VIM, we consider the following nonlinear problem:(J.H.He, 2006)

$$Lu(x) + Nu(x) = g(x) \tag{4.1}$$

Where L is a linear operator.
N is a non linear operator.
g is a known analytic function,which is non homogenuous term.
Now, consider the equation:(J.H.He, 1999)

$$u_{n+1}(x_0) = u_n(x_0) + \lambda \int_0^{x_0} (L\tilde{u}_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, n \geq 0 \tag{4.2}$$

Where, u_0 is an initial approximation and $\tilde{u}_n$ is considered as a restricted variation. for arbtrary x_0

$$u_{n+1}(x) = u_n(x) + \lambda \int_0^{x} (L\tilde{u}_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, n \geq 0 \tag{4.3}$$

This equation is called variational iteration method, while the integral in this equation is called correctional function. Where λ is a general Lagrangian multiplier(LM) which can be identified optimally via the variational theory, the subscript n denotes the n^{th}-order approximation , $\tilde{u}_n$ is considered as a restricted variation. that is $\delta\tilde{u}_n = 0$.
Now, we have the solution of the following integro-differential equation.

$$\frac{du}{dx} = f(x) + \int_0^x \psi(t, u(t), u'(t))dt \tag{4.4}$$

where $f(x)$ is a source term.
To do so, first construct a correction functional

$$u_{n+1}(x) = u_n(x) + \lambda(s) \int_0^x \left(u_n'(s) - f(s) - \int_0^s \psi(t, \tilde{u}_n(t), \tilde{u'}_n(t))dt \right) ds \tag{4.5}$$

where $\tilde{u}_n$ is considered as a restricted variation, which means $\tilde{u}_n = 0$.

To find the optimal $\lambda(s)$, we proceed the following step.

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\lambda(s)\int_0^x \left(u_n'(s) - f(s) - \int_0^s \psi(t, \tilde{u}_n(t), \tilde{u'}_n(t))dt\right) ds \tag{4.6}$$

But from (4.4) above,we have

$-\tilde{u'}_n(s) = -f(s) - \int_0^s \psi(t, \tilde{u}_n(t), \tilde{u'}_n(t))dt$

$\Rightarrow \delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)(u_n'(s) - \tilde{u'}_n(s))ds$

$\Rightarrow \delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)u_n'(s)ds - \delta \int_0^x \lambda(s)\tilde{u'}_n(s)ds$

since $\delta\tilde{u_n} = 0$,then the above equation becomes

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta \int_0^x \lambda(s)u_n'(s)ds \tag{4.7}$$

It is clear that the main step of He's variational iteration method is to determine the Lagrange multiplier $\lambda(s)$.

Integration by part is usually used to determine $\lambda(s)$.

Let $u = \lambda(s)$ and $dv = u_n'(s)ds$

$\Rightarrow du = \lambda'(s)ds, v = u_n(s)$

thus , $\delta \int_0^x \lambda(s)u_n'(s)ds = \lambda(s)u_n(x) - \int_0^x \lambda'(s)u_n(x)ds$

$$\Rightarrow \delta u_{n+1}(x) = \delta u_n(x) + \delta\lambda(s)u_n(x) - \int_0^x \delta\lambda'(s)u_n(x)ds \tag{4.8}$$

$\delta u_{n+1}(x) = \delta u_n(x)(1 + \lambda(s) - \int_0^x \lambda'(s)ds)$

Thus , the stationary conditions can be obtained as:

$$\lambda'(s) = 0 \;\; and \;\; 1 + \lambda(s)|_{s=t} = 0 \tag{4.9}$$

The Lagrange multiplier(LM), therefore can be identified as:

$$\lambda(s) = -1 \tag{4.10}$$

and the iteration formula 4.5 can be written as:

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(s) - f(s) - \int_0^s \psi(t, u_n(t), u_n'(t))dt)ds \tag{4.11}$$

For heigher order derivatives of $u(x)$, we can find the value of λ by:

$\lambda = \frac{(-1)^n(s-x)^{n-1}}{(n-1)!}$(M. Inokuti, 1978), where the index n denotes the n^{th} derivative of $u(x)$.

Therefore, the variational iteration formula for n^{th} order integro differential equtions can be expressed as:

$$u_{k+1}(x) = u_k(x) + \int_0^x \frac{(-1)^n(s-x)^{n-1}}{(n-1)!}(u_k^{(n)}(s) - f(s) - \int_0^s \psi(t, u_k(t), u_k'(t), ..., u_k^{(n)}(t))dt)ds \tag{4.12}$$

4.2 Convergence of the Method

The variational iteration formula creates a recurrence sequence $u_n(x)_{n=1}^{\infty}$. Obviously, the limit of the sequence will be the solution $u(x)$, if the sequence is convergent. The successive approximations, $u_{n+1}(x)$ of the solution $u(x)$ will be obtained by applying the obtained lagrange multiplier and a properly choosen $u_0(x)$. However, for fast convergence, the function $u_0(x)$ showld be selected by using the given initial conditions.

Theorem : The sequence $u_n(x)_{n=1}^{\infty}$ defined as the above equation 4.12 of second order integro differential equation with $u_0(x) = a + bx$ (a , b are real constants) converges to the solution $u(x)$ of $u''(x) + u^2(x) = 0, u(0) = a, u'(0) = b$, provided that u and the iterates $u_n's$ are bounded.

Consequently, the solution is given by;

$$u = \lim_{n \longrightarrow \infty} u_n \tag{4.13}$$

Furthermore, the rate and region of convergence are likely difficiencies and limitations. Though in certain situations the series converges very rapidly in a very small region or neighborhood of the boundary points, it has very slow convergence rate in the wider or outer region, where the truncated series solution is an inaccurate solution in that region which will greatly restrict the application area of the method.

4.3 Implementation of the Method

The variational iteration method (VIM) handles nonlinear problems and linear problems in a parallel manner. Unlike Adomian decomposition method, the variational iteration method does not need specific treatment for the nonlinear operator. There is no need for Adomian polynomials. As stated before, the main step in the variational iteration method is to determine the Lagrange multiplier $\lambda(s)$. In this section, we will apply the VIM to solve two nonlinear integro differential equations and show the resulting iterative formula.

Problem 1: Consider the following non linear volterra integro differential equation
$u'(x) = -\frac{1}{2} + \int_0^x u'^2(t)dt$,
for $x \in [0,1]$ with the exact solution $u(x) = -ln(\frac{1}{2}x + 1)$ and boundary condition(BC) $u(0) = 0$
The variational iteration formula is written as:
$u_{n+1}(x) = u_n(x) - \int_0^x (u'_n(s) + \frac{1}{2} - \int_0^s u'^2_n(t)dt)ds,$
In this example , we can find an initial approximation as follows:
$u_0(x) = u(0) + xu'(0)$. but ,$u'(0) = -\frac{1}{2} + \int_0^0 u'^2(t)dt = -\frac{1}{2}$ and using the boundary condition to get $u_0(x) = -\frac{1}{2}x$.

$$\begin{aligned}
u_1(x) &= u_0(x) - \int_0^x (u'_0(s) + \frac{1}{2} - \int_0^s u'^2_0(t)dt)ds \\
&= -\frac{1}{2}x - \int_0^x (-\frac{1}{2} + \frac{1}{2} - \int_0^s \frac{1}{4}dt)ds \\
&= -\frac{1}{2}x + \int_0^x (\int_0^s \frac{1}{4}tdt)ds \\
&= -\frac{1}{2}x + \frac{1}{8}x^2 \\
u_2(x) &= u_1(x) - \int_0^x (u'_1(s) + \frac{1}{2} - \int_0^s u'^2_1(t)dt)ds \\
&= -\frac{1}{2}x + \frac{1}{8}x^2 - \int_0^x (-\frac{1}{2} + \frac{1}{4}x + \frac{1}{2} - \int_0^s (\frac{1}{4} + \frac{1}{16}t^2 - \frac{1}{4}t)dt)ds. \\
&= -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{192}x^4. \\
u_3(x) &= u_2(x) - \int_0^x (u'_2(s) + \frac{1}{2} - \int_0^s u'^2_2(t)dt)ds \\
&= -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 + \frac{1}{240}x^5 + \frac{1}{1152}x^6 - \frac{1}{8064}x^7 + \frac{1}{129024}x^8. \\
&\dots
\end{aligned} \tag{4.14}$$

and so on.
It is obvious that the iterations converge to the power series of the exact solution. In order to show the efficiency and high accuracy of this method we present the curve of the absolute error of the n^{th} iteration, which is defined by:
$E_n(x) = |u(x) - u_n(x)|$.
Where, $u(x)$ be the analytic or exact solution and u_n be the n^{th} iteration approximate solution of the given problem.

Rate of convergences

In numerical analysis, the speed at which a convergent sequence approaches to it's limit is called the rate of convergence. Rates of convergences are what makes numerical analysis is true subject, both computationaly and analyticaly. Without rates of convergence, it would be almost impossible to obtain reliable accurate answers to numerical problems. There are the methods, of course, but their use is dependent on their validity, hence accuracy. The numerical analysis is always conscious of and always consider the error of approximation, and thus we come to rates of convergence. Define the error:

$E_n = |u(x) - u_n(x)|$,

If for larger n , we have the approximate relationship:

$|E_{n+1}| = K|E_n|^p$ with K a positive constant. then, we can say that our numerical method is of order p.

Larger value of p corresponding to faster convergence to the exact solution. Thus. the error is influenced by approximately a factor of p with each iteration.

Convergence and Stability

Convergence refers to as the step size gets smaller value, the numerical method approaches to the true solution.

Stability also refers to as the computation proceeds, the calculated errors get smaller.

The results and the corresponding absolute errors are presented in the following table (with third-order approximation)

Table 1. Numerical results of problem 1.

x	Exact Solution	VIM_{-u_3}	Absolute Error
0.0	0	0	0
0.1	-0.04879016417	-0.04879014498	1.91×10^{-8}
0.2	-0.09531017980	-0.09530961268	5.67×10^{-7}
0.3	-0.1397619424	-0.1397579563	3.98×10^{-6}
0.4	-0.1823215568	-0.1823059759	1.55×10^{-5}
0.5	-0.2231435513	-0.0.2230993543	4.41×10^{-5}
0.6	-0.2623642645	-0.2622618412	1.02×10^{-4}
0.7	-0.3001045925	-0.2998980358	2.06×10^{-4}
0.8	-0.3364722366	-0.3360958171	3.76×10^{-4}
0.9	-0.3715635564	-0.3709284685	6.35×10^{-4}
1.0	-0.4054651081	-0.4044565353	1×10^{-3}

The above table shows that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only consider a third-order approximation, it achieves a high level of accuracy.

The approximate solutions of the non linear integro differential equation have a power series form with center at 0 .

That is $\Sigma_{n=0}^{\infty} a_n x^n$, where, a_n is a function of x and n.

when we come to our series:

$u(x) = \Sigma_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x)+, ... = \dfrac{1}{2}x + \frac{1}{8}x^2 - \dfrac{1}{24}x^3 + \dfrac{1}{64}x^4 + \dfrac{1}{240}x^5 + \frac{1}{1152}x^6 - \dfrac{1}{8064}x^7 + \dfrac{1}{129024}x^8 + ... = \Sigma_{n=1}^{\infty}(-1)^n \dfrac{x^n}{n2^n}$

Now, to determine the region or the interval of convergence of the above power series (that is the approximate solution), we have to remember the undergraduate concept(Ratio test) **Theorem (Ratio test):**

Let $\Sigma_{n=0}^{\infty} a_n$ be a seies and suppose that

$$u = \lim_{n \longrightarrow \infty} |\frac{a_{n+1}}{a_n}| = L \tag{4.15}$$

, then the series converges if $L < 1$ and the series diverges if $L > 1$.

now come to our series, $\Sigma_{n=1}^{\infty}(-1)^n\dfrac{x^n}{n2^n}$, it is simple to find the region of convergence of the above series(approximate solution).
let say $a_n = \dfrac{(-1)^n x^n}{n2^n}$, $\Rightarrow a_{n+1} = \dfrac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}$
Now,

$$\begin{aligned} \lim_{n\longrightarrow\infty}|\frac{a_{n+1}}{a_n}| &= \lim_{n\longrightarrow\infty}|\frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n x^n}{n2^n}}| \\ &= \lim_{n\longrightarrow\infty}|\frac{(-1)^{n+1}x^{n+1}}{(n+1)2^{n+1}}.\frac{n2^n}{(-1)^n x^n}| \\ &= |\frac{x}{2}|\lim_{n\longrightarrow\infty}|\frac{n}{n+1}| \\ &= \frac{|x|}{2}\lim_{n\longrightarrow\infty}(\frac{n}{n+1}) \\ &= \frac{|x|}{2} \end{aligned} \tag{4.16}$$

Thus, based on the above theorem(Ratio test), the given series converges:
$\lim_{n\longrightarrow\infty}|\dfrac{a_{n+1}}{a_n}| < 1 \Leftrightarrow \dfrac{|x|}{2} < 1$
$\Leftrightarrow |x| < 2$

$\Leftrightarrow -2 < x < 2$

To determine the interval of convergence of the approximate solution(which is power series), let us check the convergence of the series at the end points. that is at $x = -2$ and $x = 2$
At $x = 2$; $\Sigma_{n=1}^{\infty}\dfrac{(-1)^n x^n}{n2^n} = \Sigma_{n=1}^{\infty}\dfrac{(-1)^n(2)^n}{n2^n} = \Sigma_{n=1}^{\infty}\dfrac{(-1)^n}{n} = -1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} + ...$
And similarly
At $x = -2; \Sigma_{n=1}^{\infty}\dfrac{(-1)^n x^n}{n2^n} = \Sigma_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n} = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + ...$
Before determining the convergence of these serieses, it is better to study the following definition.

Definition: A series of the form $\Sigma_{n=1}^{\infty}(-1)^n a_n$ or $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$ where, $a_n > 0$ is called an alternating series.

Alternating series test:

The alternating series $\Sigma_{n=1}^{\infty}(-1)^n a_n$ or $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, if the following conditions are satisfied.

- a_n is decreasing as n becomes larger and
- $\lim_{n\rightarrow\infty} a_n = 0$.

Therefor, series converges for both $x = 2$ and $x = -2$, therefore the given power series(approximate solution) converges only if $-2 \leq x \leq 2$

for the series above,the alternating series test determines that the series converges for $|x| \leq 2$ and diverges for $|x| > 2$.

Definition: Let $\Sigma_{n=1}^{\infty} a_n x^n$ be a power series, then there exist a radius R for which the series converges for $|x| < R$ and diverges for $|x| > R$. therefore R is called **the radius of convergence** of the power series.

In this example the radius of convergence $R = 2$.

Problem 2: consider the following fredholm integro differential equation:

$u''(x) = e^x - x + \int_0^1 xtu(t)dt; u(0) = 1, u'(0) = 1.$

To know the analytic solution we can apply direct computation method. if we write the above problem as:

$u''(x) = e^x - x + x\int_0^1 tu(t)dt$

Now, if we define this unknown integral $\int_0^1 tu(t)dt$ and assuming that the solution of this integral exists such that $u(t)$ is a continuous function and finite quantity which is denoted as β.

That is $\int_0^1 tu(t)dt = \beta$

Then, the above equations becomes:

$u''(x) = e^x - x + x\beta = e^x + (\beta - 1)x; u(0) = 1 = u'(0).$

And we can see this is an ODE which is the IVP. Since the functions e^x and x are continuous, the solution of this equation exists.

So, if we integrate both sides of the equation from 0 to x and applying the given conditions, we get:

$u'(x) - 1 = e^x - 1 + (\beta - 1).\dfrac{x^2}{2}$

$\Rightarrow u'(x) = e^x + (\beta - 1).\dfrac{x^2}{2}$. Again integrating both sides of this equation from 0 to x, we get:

$u(x) - 1 = e^x - 1 + (\beta - 1).\dfrac{x^3}{6}$

$\Rightarrow u(x) = e^x + (\beta - 1).\dfrac{x^3}{6}$

Now, once you have this expression for $u(x) = e^x + (\beta - 1).\dfrac{x^3}{6}$. then, in the above equation $\int_0^1 tu(t)dt = \beta$. we can substitute this expression for $u(x)$ and substituting this expression and we can integrate it and solving for β. Then. we can find the particular value of β and the substituting this β in the final expression, we get the solution of the given integro differential equation.

Now, $\beta = \int_0^1 tu(t)dt = \int_0^1 (te^t + (\beta - 1)\dfrac{t^4}{6})dt$

$= \int_0^1 te^t dt + (\beta - 1)\int_0^1 \dfrac{t^4}{6})dt$

$= [te^t - e^t + (\beta - 1).\dfrac{t^5}{30}]_0^1 = 1 + \dfrac{\beta - 1}{30}$
$\Rightarrow \beta = 1.$

Therefore, $u(x) = e^x$ is the analytic solution of the given integro differential equation.

Now, directly go to numerical solutions:

Making $u_{n+1}(x)$ stationary with respect to $u_n(x)$,we can identify the lagrange multiplier, which reads:

$\lambda = s - x$. So we can construct the following variational iteration formula as :
$u_{n+1}(x) = u_n(x) + \int_0^x (s - x)[u_n''(s) - e^s + s - \int_0^1 stu_n(t)dt]ds$
We begin with

$u_0(x) = e^x(a + bx)$ where a and b are unknown constants to be further determined.

By the above iteration formula, we have:
$u_1(x) = (a - 1) + (a + b - 1)x + (\dfrac{-1}{6} + \dfrac{1}{6}a - \dfrac{1}{3}b + \dfrac{1}{6}be)x^3 + e^x$
If the first-order approximate solution is enough, by the aid of the given initial conditions, we can identify the unknown constants as $a = 1$ and $b = 0$.

So we obtain the following first-order approximate solution:

$u_0 = x + 1$, which satisfies the given initial conditions, the we will have below approximations.

$$\begin{aligned} u_1(x) &= e^x - \frac{1}{36}x^3 \\ u_2(x) &= e^x - \frac{1}{1080}x^3 \\ u_3(x) &= e^x - \frac{1}{32400}x^3 \end{aligned} \tag{4.17}$$

and so on.It is obvious that the iterations converges to the exact solution.

Chapter 5

Adomian Decomposition Method for Integro Differential Equations

5.1 Description of the method

An analytic method called the Adomain decomposition method (ADM) proposed by Adomain at the begining of $1980's$ aims to solve nonlinear physical problems. it has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear ,in physics, biology and chemical reactions etc. for non-linear models , the method has shown reliable results in supplying analytical approximations that converge rapidly.
The ADM is a relatively new approach to provide an analytical approximation to linear and nonlinear problems and it is particularly valuable as a tool for scientists and applied mathematicians. consider the following equation:

$$Lu + Nu + Ru = g, \tag{5.1}$$

where L is a linear operator, N represents a non linear operator and R is the remaining linear part. By defining the inverse operator of L as L^{-1}, assuming that it exists, we get:

$$u = L^{-1}g - L^{-1}Nu - L^{-1}Ru, \tag{5.2}$$

The Adomian Decomposition Method assumes that the unknown function u can be expressed by an infinite series of the form
$u(x) = \Sigma_{n=0}^{\infty} u_n(x)$,
it is equivalent to the equation:

$$u(x) = u_0 + u_1 + u_2 + ..., \tag{5.3}$$

where the components u_n will be determined recursively. Moreover, the method defines the nonlinear term by the Adomian polynomials. More precisely, the ADM assumes that the nonlinear operator $N(u)$ can be decomposed by an infinite series of polynomials given by

$$N(u) = \Sigma_{n=0}^{\infty} A_n, \tag{5.4}$$

where $A_n's$ are the Adomians polynomials defined as $A_n = A_n(u_0, u_1, u_2, ..., u_n)$.
Substituting 5.3 into 5.2 and using the fact that R is a linear operator, we obtain:

$$\Sigma_{n=0}^{\infty} u_n(x) = L^{-1}g - L^{-1}(\Sigma_{n=0}^{\infty} R(u_n)) - L^{-1}(\Sigma_{n=0}^{\infty} A_n(u_0, u_1, u_2, ..., u_n)), \tag{5.5}$$

or equivalently

$$u_0 + u_1 + u_2 + ... = L^{-1}g - L^{-1}(\Sigma_{n=0}^{\infty} R(u_n)) - L^{-1}(A_0 + A_1 + ...), \tag{5.6}$$

Therefore the formal recurrence algorithm could be defined by:
$u_0 = L^{-1}g,$
$u_1 = -L^{-1}(R(u_0)) - L^{-1}(A_0(u_0)),$
$u_2 = -L^{-1}(R(u_1)) - L^{-1}(A_1(u_0, u_1)),$
....
The series solution converges to the exact solution if such a solution exists. Like the VIM,in Adomian decomposition method, the convergence is accurate locally, mainly in a neighborhood of the boundary point(s).

$$u_{n+1} = -L^{-1}(R(u_n)) - L^{-1}(A_0(u_0, u_1, ..., u_n)), n \geq 0 \tag{5.7}$$

It is useful to note that the recursive formula is constructed on the basis that the zeroth component $u_0(x)$ is defined by all terms that arise from the initial conditions and from integrating the source terms. The remaining components $u_n(x), n > 0$, can be completely determined such that each term is computed by using the previous term. As a result, the components $u_0(x), u_1(x), u_2(x), ...$are identified, and the series solutions are thus entirely determined.

The $n-$ term approximation Φ_n is defined by:
$\Phi_n = \Sigma_{k=0}^{n-1} u_k(x).$
Which can be used for numerical approximation. Consider the nonlinear function $f(u)$. Then, the infinite series generated by applying the Taylors series expansion of f about the initial function u_0 is given by:

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2!} f''(u_0)(u - u_0)^2 + ... \tag{5.8}$$

By substituting 5.3 into equation 5.8, we have:

$$f(u) = f(u_0) + f'(u_0)(u_1 + u_2 + ...) + \frac{1}{2!}f''(u_0)(u_1 + u_2 + ...)^2 + ... \tag{5.9}$$

Now, we expand equation 5.9. To obtain the Adomian polynomials, we need first to re-order and re-arrange the terms. Indeed, one needs to determine the order of each term in 5.9 which actually depends on both the subscripts and the exponents of the $u_n's$. For instance, u_1 is of order 1; u_1^2 is of order 2; u_2^3 is of order 6 and so on. In general, u_n^k is of order nk. In case a particular term involves the multiplication of $u_n's$, its order is determined by the sum of the terms of the $u_n's$ in each term. For example, $u_2^3u_1^2$ is of order 8 since $(3)(2) + (2)(1) = 8$.
As a result, rearranging the terms in the expansion 5.8 according to the order, we have

$$f(u) = f(u_0)+f'(u_0u_1+f'(u_0u_2+\frac{1}{2!}f''(u_0u_1^2+f'(u_0u_3+\frac{2}{2!}f''(u_0u_1u_2+\frac{1}{3!}f'''(u_0u_1^3+f'(u_0u_4+\frac{1}{2!}f''(u_0u_2^2+... \tag{5.10}$$

The Adomian polynomials are constructed in a certain way so that the polynomial A_1 consists of all terms in the expansion 5.10 of order 1, A_2 consists of all terms of order 2, and so on. In general, A_n consists of all terms of order n. Therefore, the corrosponding terms of Adomian polynomials are listed as follows:
$A_0 = f(u_0)$,
$A_1 = f'(u_0)u_1$,
$A_2 = f'(u_0)u_2 + \frac{1}{2!}f''(u_0)u_1^2$,
$A_3 = f'(u_0)u_3 + \frac{2}{2!}f''(u_0)u_1u_2 + \frac{1}{3!}f'''(u_0)u_1^3$,
... and so on.
The Adomian polynomial's A_n was first introduced by Adomian himself; it was defined via the general formula:
$A_n(u_0, u_1, u_2, ..., u_n)) = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\Sigma_{k=0}^{\infty}u_k\lambda^k)]_{\lambda=0}, n = 0, 1, 2, ...$
From the above equation we can easily calculate the Adomian polynomialsas follows:

$$\begin{aligned}
A_0 &= N(u_0),\\
A_1 &= \frac{d}{d\lambda}N(u_0 + u_1\lambda)|_{\lambda=0}\\
&= N(u_0u_1),\\
A_2 &= \frac{1}{2!}\frac{d}{d\lambda}((u_1 + 2u_2\lambda)N''(u_0 + u_1\lambda))|_{\lambda=0}\\
&= N'(u_0)u_2 + \frac{1}{2!}N''(u_0)u_1^2
\end{aligned} \tag{5.11}$$

,...and so on.

5.2 Method implementation

Like Variational iteration method, this method also handles linear as well as non linear problems. the main step in this method is that finding the Adomian polynomials.Under this section we will try to solve two numerical examples to illustrate the method.

Problem 1: Consider the following non linear volterra integro differential equation

$u'(x) = -\frac{1}{2} + \int_0^x u'^2(t)dt$,

for $x \in [0,1]$ with the exact solution $u(x) = -ln(\frac{1}{2}x + 1)$ and boundary condition $u(0) = 0$

To solve this problem using ADM, we have to follow the following steps:

Integrating both sides of the equation from 0 to x , we get: here the non linear function is $f(t) = u'^2(t)$

$\int_0^x u'(x)dx = \int_0^x(-\frac{1}{2})dx + \int_0^x(\int_0^x u'^2(t)dt)dx$

$\Rightarrow u(x) - u(0) = -\frac{1}{2}x + L^{-1}(\int_0^x u'^2(t)dt)$

Where $L^{-1}(.) = \int_0^x(.)dx$ but, we are given that the initial condition $u(0) = 0$

thus the equation becomes

$u(x) = u(0) - \frac{1}{2}x + L^{-1}(\int_0^x u'^2(t)dt) = -\frac{1}{2}x + L^{-1}(\int_0^x u'^2(t)dt)$

Now, substitute $\Sigma_{n=0}^{\infty}u_n(x) = u_0 + u_1 + u_2 + ...$ instead of $u(x)$,which is:

$u_0 + u_1 + u_2 + ... = -\frac{1}{2}x + L^{-1}(\int_0^x A_n(t)dt)$

Therefore, the formal recurrence relation indicates that:

$$u_0(x) = -\frac{1}{2}x,$$

$u_1(x) = L^{-1}(\int_0^x A_0(t)dt)$, where the adomian polynomial, $A_0(t) = f(u_0) = u_0'^2(t) = \frac{1}{4}$

$\Rightarrow u_1(x) = L^{-1}(\int_0^x \frac{1}{4}dt) = L^{-1}(\frac{1}{4}x) = \frac{x^2}{8}$

$u_2(x) = L^{-1}(\int_0^x A_1(t)dt)$, where $A_1(t) = f'(u_0)u_1 = 2u_0'(t)u_1(t) = 2(-\frac{1}{2})\frac{t^2}{8} = -\frac{t^2}{8}$

Then, $u_2(x) = L^{-1}(\int_0^x(-\frac{t^2}{8})dt) = -\frac{x^3}{24}$

and in similar fashion, we can calculate the next iteration:

$u_3(x) = \frac{1}{480}x^5, ...$ and so on.

Generally, $u_{k+1}(x) = L^{-1}(\int_0^x A_k(t)dt)$

Problem 2: Consider the following fredholm integro differential equation

$u'(x) = 1 - \frac{x}{3} + \int_0^1 xtu(t)dt,\ u(0) = 0.$

$= 1 - \frac{x}{3} + x\int_0^1 tu(t)dt$

Now, integrating both sides of he above equation from 0 to x and applying the given initial

condition we get:
$u(x) = x - \frac{x^2}{6} + \frac{x^2}{2}\int_0^1 tu(t)dt$
And again now, we assume that the solution of the equation can be expressed in the form:
$u(x) = \Sigma_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + ...$
Therefore,

$$\begin{aligned}
u_0(x) &= x - \frac{x^2}{6}, \\
u_1(x) &= \frac{x^2}{2}\int_0^1 tu_0(t)dt. \\
&= \frac{x^2}{2}\int_0^1 (t^2 - \frac{t^3}{6})dt \\
&= \frac{x^2}{2}.\frac{7}{4}. \\
u_2(x) &= \frac{x^2}{2}\int_0^1 tu_1(t)dt. \\
&= \frac{x^2}{2}.\frac{7}{24}\int_0^1 \frac{t^3}{2}dt \\
&= \frac{x^2}{2}.\frac{7}{24}.\frac{1}{8} \\
u_3(x) &= \frac{x^2}{2}\int_0^1 tu_2(t)dt. \\
&= \frac{x^2}{2}.\frac{7}{24}.\frac{1}{8}\int_0^1 \frac{t^3}{2}dt \\
&= \frac{x^2}{2}.\frac{7}{24}.(\frac{1}{8})^2
\end{aligned} \tag{5.12}$$

$$u_{n+1}(x) = \frac{x^2}{2}.\frac{7}{24}.(\frac{1}{8})^n$$

Therefore, $u(x) = u_0(x) + u_1(x) + u_2(x) + ...$
$= x - \frac{x^2}{6} + \frac{x^2}{2}.\frac{7}{24}(1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + ...)$ But, $1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + ...$ is a geometric series that converges to
$\frac{1}{1 - \frac{1}{8}} = \frac{8}{7}$
Thus, $u(x) = x - \frac{x^2}{6} + \frac{x^2}{2}.\frac{7}{24}.(\frac{8}{7}) = x$ is the solution of the given fredholm integro differential equation.

Therefore, the approximate solution ($u(x) = x$) of the above problem converges for all values of x satisfying $|x| < 1$

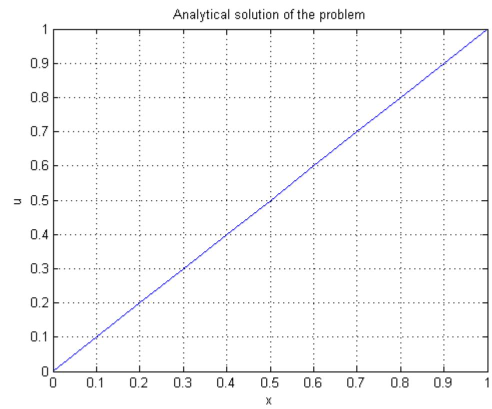

Figure 5.1: The graph for the exact solution of problem 2

To determine the region of convergence of the approximate solution(which has power series form), let us check its convergence at the end points, that is $x = 1$ and $x = -1$:
At $x = 1$, the series becomes $\Sigma_{n=0}^{\infty}(1) = \infty$ and
At $x = -1$, the series also becomes $\Sigma_{n=0}^{\infty}(-1) = -\infty$
Both serieses are divergent serieses at the points.
Therefore, the region or interval of convergence of the approximate solution is $(-1, 1)$.

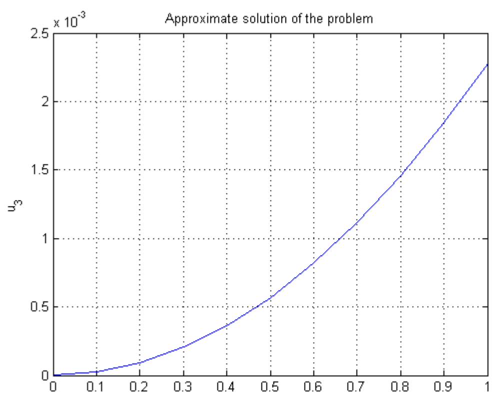

Figure 5.2: The graph for the Approximate solution of problem 2

5.3 Results and Discussions

In the previous two chapters, we have seen that the two important numerical methods for solving some non linear integro differential equations and we tried to get the numerical error as well as the region of convergence for the desired approximate solution. Now, we also compare the accuracy the methods by plotting both the analytical and an approximate solution of a non linear integro differential equation of problem 1 on one coordinate plane as follows:

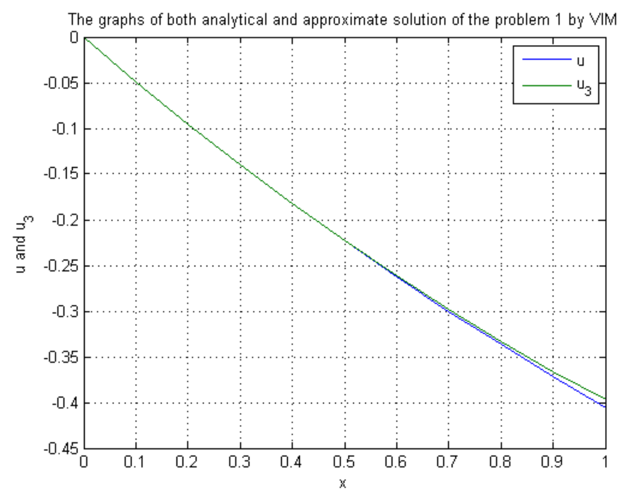

Figure 5.3: Comparison of true solution with the approximate solution using VIM of problem 1

And similarly we can plot these graphs using the other numerical method that is ADM as follows:

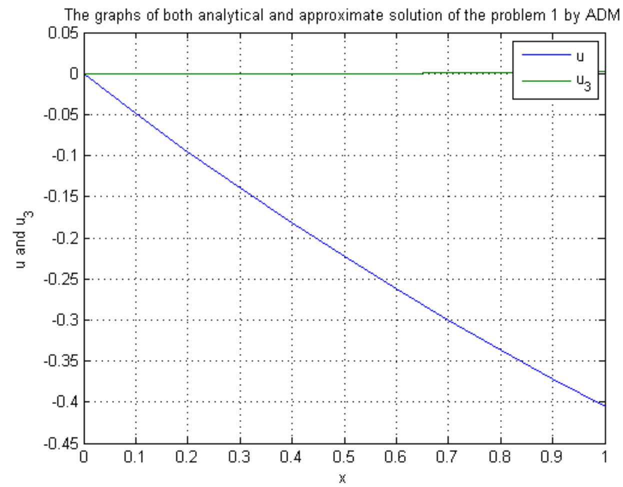

Figure 5.4: Comparison of true solution with the approximate solution using ADM of problem 1

Generally, from the above two graphs, we can understand that the approximate solution of problem 1 is approaches to the given exact solution when we apply the Variational Iteration Method. However, the approximate solution is far away from the given exact solution when we apply the Adomian Decomposition Method.

Therefore, the Variational Iteration Method is more accurate method than the Adomian Decomposition method for solving the non linear integro differential equations. We can also increase the accuracy of the above two numerical methods by adding more terms to the approximate solution.

Chapter 6

Conclusions

In this research we have seen two basic techniques or methods for solving some non linear integro differential equations: namely VIM and ADM.
We introduced the variational iteration method (VIM) for solving non linear integro differential equations. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists. It also provides an approximation with high level of accuracy by applying some iterations only. Actually, the VIM proved to be an effective tool to handle non linear integro differential equations without the use of Adomian polynomials.
However, we noticed that for this method, as we increase the values of x, the error is magnifying or far away from the exact solution.
We also introduced the other method that is Adomian Decomposition Method (ADM) to solve the same problem(non linear integro differential equations), the ADM often converges to the exact solution if such a solution exists. We also noticed that this method gives highly accurate approximations only close to the initial condition but not as we move away from it.
In both methods some examples have been solved to illustrate the methods and demonstrate its reliability and accuracy.

Bibliography

Adomian, G. (1994): Solving frontier problems of physics: the decomposition method. Boston: Kluwer Academic,

Batiha, M. S. M. Noorani, I. Hashim .(2008): Numerical Solutions Of The Nonlinear Integro-Diferential Equations, Open Problems Compt. Math. 34-42.

Cherruault, Y. (1989): Convergence of Adomians method. Kybernetes,18(2),31-38.

Chen, W.(2004): An algorithm for Adomian decomposition method. Appl Math Comput, 159. 221-235.

El-Sayed, S. M, Abdel-Aziz, M.R. (2003): comparison of Adomians decomposition method and wavelet-Galerkin method for solving integro-differential equations. Appl. Math. Comput. 136. 151159

He, J.H. (1997): A new approach to nonlinear partial differential equations. Commun. Nonlin. Sci. Numer. Simul. 2 . 230-235.

He, J.H. (1998): Approximate analytical solution of Blasius equation. Commun. Nonlin. Sci. Numer. Simul. 3. 260-263.

He, J.H. (1999): Variational iteration methoda kind of non-linear analytical technique: some examples. Int. J. Non-Linear Mech. 34. 699-708.

He, J.H. (2000): Variational iteration method for autonomous ordinary differential systems. Appl. Math. Comput. 114. 115-123. Integro-Differential Equations 41

He, J.H. (2003): A simple perturbation approach to Blasius equation. Appl. Math. Comput. 140. 217-222.

He, J.H. (2006): Some asymptotic methods for strongly nonlinear equations. Int. J. Modern Phys. B 20. 1141-1199.

Jeffrey, M.N.B. Mohamad.(1991): Exact solutions to the KdV-Burgers equation, Wave Motion 14

Kaya, D., El-Sayed, S. M. (2003): Numerical soliton-like solutions of the potential Kadomtsev-Petviashvili equation by the decomposition method, Phys Lett A 320. 192-199.

M. Inokuti, H. Sekine and T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics.1978

Senthilvelan, M. (2001): On the extended applications of homogenous balance method, Appl. Math. Comput. 123. 381-388.

Sweilam, N.H. (2006): Fourth order integro-differential equations using variationaliteration method, Comp.Math. Appl., doi:10.1016/j.camwa.

Wadati, M. (1972): The exact solution of the modified Korteweg-de Vries equation. J. Phys. Soc. Jpn. 32. 1681-1687.

Wang, X.Y. (1988): Exact and explicit solitary wave solutions for the generalized Fisher equation Phys. Lett. A 131. 277-279.

Wazwaz, A.M.(2006): A comparison between the variationaliteration method and adomin decomposition method.

Zhang, D.S. Li, H. Q.(2003): New soliton-like solutions to the potential KadomstevPetviashvili (PKP) Petviashvili equation by the decomposition method, Phys Lett A 320. 192-199.

Zhang, D.S. Li, H. Q.(2003): Symbolic computation and various exact solutions of potential Kadomstev-Petviashvili equation, Appl. Math. Comput. 145. 351-359.