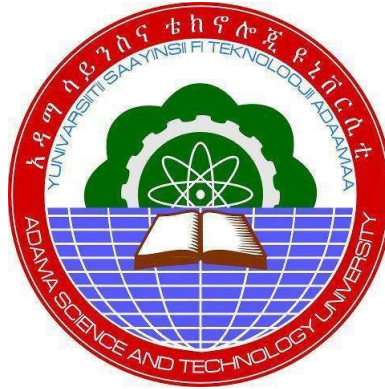


**FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED
NON-LOCAL BOUNDARY VALUE PROBLEMS ON
BAKHVALOV-SHISHKIN MESH**



Beserat Bulti Bekele

A Thesis Submitted to

The Department of Applied Mathematics

School of Applied Natural Science

**Presented in Partial Fulfillment of the Requirement for the Degree of
Master's in Applied Mathematics (Numerical Analysis)**

Office of Graduate Studies

Adama Science and Technology University

July, 2023

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Declaration

I here by declare that this Master Thesis entitled “**Finite Difference Method for Singularly Perturbed Non-local Boundary Value Problems on Bakhvalov-Shishkin Mesh**” is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

Name of Student

Signature

Date

Recommendation of Advisors

we, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled “**Finite Difference Method for Singularly Perturbed Non-local Boundary Value Problems on Bakhvalov-Shishkin Mesh**” prepared under our guidance by Beserat Bulti Bekele submitted in partial fulfillment of the requirements for the degree of Master’s of Science in Applied Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

_____ Major Advisor	_____ Signature	_____ Date
_____ Co-advisor	_____ Signature	_____ Date

Approval Page

we, the advisors of the thesis entitled “**Finite Difference Method for Singularly Perturbed Non-local Boundary Value Problems on Bakhvalov-Shishkin Mesh**” and developed by Beserat Bulti Bekele, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

_____	_____	_____
Major Advisor	Signature	Date
_____	_____	_____
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by Beserat Bulti Bekele have read and evaluated the thesis entitled “**Finite Difference Method for Singularly Perturbed Non-local Boundary Value Problems on Bakhvalov-Shishkin Mesh**” and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Numerical Analysis).

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Final approval and acceptance of the thesis is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

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Table of Contents

Declaration	i
Recommendation of Advisors	ii
Approval page	iii
Acknowledgment	v
Table of Contents	vii
List of Tables	viii
List of Figures	ix
List of Acronyms	x
Abstract	xi
CHAPTER 1: INTRODUCTION	1
1.1 Background of the Study	1
1.2 Statement of the Problem	4
1.3 Objectives	4
1.3.1 General Objective	4
1.3.2 Specific Objectives	4
1.4 Significance of the Study	5
1.5 Delimitation of the Study	5
CHAPTER 2: LITERATURE REVIEW	6
CHAPTER 3: METHODOLOGY OF THE STUDY	9
3.1 Study Design	9
3.2 Mathematical Procedures	9
CHAPTER 4: DESCRIPTION AND ANALYSIS OF THE METHOD	10
4.1 Properties of Continuous Problems	10
4.2 Construction of the Difference Scheme	13



4.3 Stability and Uniform Convergence Analysis	16
CHAPTER 5: RESULTS AND DISCUSSIONS	29
CONCLUSIONS AND RECOMMENDATIONS	36
References	39

List of Tables

Table:5.1	Example 1's maximum absolute error and rate of convergence for the proposed method.	31
Table:5.2	Comparison of results for Example 1's maximum absolute error. . . .	32
Table:5.3	Example 2's maximum absolute error and rate of convergence for the proposed method.	34
Table:5.4	Comparison of results for Example 2's maximum absolute error. . . .	35

List of Figures

Figure 5.1	Solutions profile for Example 1 with boundary layer formation as ϵ goes small with mesh size $N = 2^8$	31
Figure 5.2	Solutions profile for Example 2 with boundary layer formation as ϵ goes small with mesh size $N = 2^{10}$	33
Figure 5.3	Solutions profile for Example 2 with boundary layer formation as ϵ goes small with mesh size $N = 2^{10}$	34

LIST OF ACRONYMS

FDM	Finite difference method
FFDM	Fitted finite difference method
CDDE	Convection Diffusion differential equation
FMFDM	Fitted mesh finite difference method
SPCDDEs	Singularly perturbed convection diffusion differential equations
SPPs	Singularly perturbed problems

Abstract

This study aims to achieve a precise computational solution for a linear singularly perturbed problem with a non-local boundary condition using discretization of the Bakhvalov-Shishkin mesh. A conventional finite difference scheme was developed to approximate the problem, while the composite Simpson's 1/3 rule was used to approximate the boundary portion of the integral. By analyzing the ε -perturbation parameter, achieved a fitted mesh finite difference method and it was determined that the first-order uniform convergence adhered to the discrete maximum norm. To showcase the effectiveness and accuracy of the proposed method, a numerical experiment was conducted and the results were validated using appropriate tables and figures.

Keywords: Singular perturbation; Finite difference scheme; Bakhvalov-Shishkin mesh; Uniformly convergence; Non-local boundary condition

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

The differential equations that have their highest order derivative terms multiplied by a small positive parameter ε are known as singularly perturbed differential equations. These equations exhibit thin transition layers in their solutions around the points 0 and 1 due to the sudden and rapid changes caused by the parameter, while away from layers it behaves regularly and varies slowly ε (Nhan & Vulanović, 2021).

The presence of singular perturbations in mathematical models poses significant challenges to the application of classical numerical methods. As such, there is a need to develop appropriate numerical techniques that can effectively tackle these problems. It is important that these methods are not dependent on the value of the parameter ε , and their accuracy should be maintained regardless of its value. In essence, there is a need for convergence ε – uniform methods to ensure the efficient and accurate computation of singular perturbation problems (Amiraliyev & Çakir, 2002).

To achieve convergence ε – uniformity, numerical methods need to be designed such that they can effectively handle the delicate balance between the different scales involved in the problem. Classical numerical methods often struggle with this balance, leading to significant errors and inaccuracies (Kadalbajoo & Awasthi, 2011). Therefore, specialized numerical methods such as the boundary layer methods have been developed to tackle these problems. The aim of such methods is to capture the solution's behavior in the boundary layer regions accurately. These methods typically utilize asymptotic expansions and perturbation techniques to construct approximations that maintain their accuracy regardless of the value of the perturbation parameter (Kadalbajoo & Awasthi, 2008). Overall, the development of convergence- ε uniform methods is crucial for the accurate and efficient computation of singular perturbation problems, and it requires the use of specialized numerical techniques that can handle the delicate balance between the different scales involved in these problems (Çakır & Arslan, 2016).



Research on singular perturbation problems began in the 1900s and was found to be prevalent in various fields such as natural sciences, engineering, medical sciences, fluid mechanics, aerodynamics, magnetic dynamics, diffusion theory, and many others (Singh & Natesan, 2020). These problems have been observed in areas such as reaction diffusion, communication networks, plasma dynamics, and mass transport, among others. The study of singular perturbation problems has helped to advance physical modeling techniques in fields such as plastics, chemical reactor theory, oceanography, meteorology, and electricity current (Arslan, 2020b). Lately, singularly perturbed problems, particularly with the nonlocal boundary condition and boundary layers have been studied by several researchers (Amiraliyev & Cakir, 2000), (Bugajev & Čiegis, 2012), (Cakir, 2016).

Nonlocal boundary singularly perturbed convection-diffusion differential equations (SPCDDEs) arise in many physical and engineering applications, such as chemical reaction kinetics, fluid dynamics, and heat transfer (Zahra & El Mhlawy, 2013). These equations are characterized by a small parameter, typically denoted by ε , that multiplies the highest order derivative term. This small parameter causes the boundary layer near the boundary of the domain to be much thinner than the interior layer. In addition, nonlocal boundary conditions are often imposed, which involve integrals over the entire domain (Raja, Sekar, Priya, & Unyong, 2023).

The general ODE form of a nonlocal boundary SPCDDE can be written as:

$$\begin{aligned} \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) &= f(x), 0 < x < 1, \\ u(0) &= \phi(u), \quad \int_0^1 g(x)u(x)dx = \alpha. \end{aligned}$$

where $u(x)$ is the unknown function, $a(x)$ and $b(x)$ are the coefficients of the first and zeroth order derivatives, respectively, and $f(x)$ is a forcing term. The function $\phi(u)$ is a nonlinear boundary condition at $x = 0$, and $g(x)$ is a weight function that appears in the nonlocal boundary condition at $x = 1$. The parameter α is a constant. The solution to such equations typically exhibits sharp boundary layer behavior near $x = 0$, and careful numerical techniques are required to accurately capture this behavior (Sekar & Tamilselvan, 2018).

There are various methods to solve singularly perturbed problems, including fitted finite difference methods, finite element methods with special elements like exponential elements, and fitted mesh methods with a priori refined or uniform grids that condense in boundary lay-



ers. One approach to developing parameter-uniform methods is using Shishkin type meshes, which are constructed before solving the problem and depend on the parameter ε , problem data, and number of mesh points. Research on numerical methods for solving singularly perturbed problems has been extensive over the past 20 years (Cakir & Arslan, 2021).

The Bakhvalov-Shishkin mesh is a type of computational mesh commonly used in numerical methods for solving singularly perturbed differential equations. It is formed by dividing the solution domain into three regions: a boundary layer region, a transition region, and a regular region (Kellogg et al., 2008). The mesh is designed to have a higher density of nodes in the boundary layer and transition regions, where the solution changes rapidly, and a lower density in the regular region. This helps to improve the accuracy and efficiency of numerical methods, as it allows for a better resolution of the solution near the boundary layer without requiring an excessive number of nodes in the regular region. The Bakhvalov-Shishkin mesh has applications in a wide range of fields, including fluid mechanics, heat transfer, and chemical engineering, among others (Linss, 2000), (Zheng, Li, & Gao, 2015).

The Bakhvalov-Shishkin mesh is an improvement over the Shishkin mesh, providing a more robust error bound for values of $\varepsilon \leq CN^{-1}$. While the original Bakhvalov mesh utilized a nonlinear equation to determine the transition point from coarse to fine mesh, the Bakhvalov-Shishkin mesh simplifies this process by using the transition points from the Shishkin mesh (Arslan, 2020b).

Through the implementation of an upwind difference scheme on a Bakhvalov-Shishkin mesh, the researcher demonstrated that this approach produces more precise outcomes than utilizing an upwind scheme on a Shishkin mesh. Our team has further improved this technique by integrating Bakhvalov's concept of condensing mesh in the boundary layers (Gupta, Kadalbajoo, & Dubey, 2019). This innovation is achieved by inverting the boundary layer terms, leading to the development of our novel mesh. However, we have made the process simpler by eliminating the need to solve a nonlinear equation to determine the exact location of the mesh's transition from coarse to fine. Instead, we have retained the transition points as they exist in the Shishkin mesh.

This mesh uses Bakhvalov's idea of inverting $\exp(-\beta x / \sigma \varepsilon)$ on $[0, \sigma]$, while using a uniform mesh (Shishkin mesh) on $[\sigma, 1]$.



1.2 Statement of the Problem

The numerical analysis of singular perturbation cases has always been far from trivial because of the boundary layer behavior of the solution. Such problems undergo rapid changes within very thin layers near the boundary or inside the problem domain. It is well known that standard numerical methods for solving such problems are unstable and fail to give accurate results when the perturbation parameter ε is small (Doolan et al., 1980) (Mbroh & Munyakazi, 2019). Therefore, it is important to develop suitable numerical methods to these problems, whose accuracy does not depend on the parameter value ε . i.e methods that are convergence ε — uniformly (Turuna et al., 2020). To approximate their solution, It is well known that classical numerical methods cannot be applied to uniform distributions (Farrell et al., 2000). The reason is that the error is unbounded for arbitrary values of the singularly perturbed parameter, so when solving such problems numerically, one would like to adjust the discretization to the solution. In terms of mesh generation, we want to develop many points in the area where the solution has strong variations and a few points in the area where the solution has weak variations, the strategy being known as well as fitted mesh finite difference methods. The majority of recent Singularly perturbed problems papers make use of widely finite differences or finite element methods on piecewise uniform meshes. As we understand, the mesh plays a very crucial role in solving singularly perturbed problems accurately, therefore we will apply a kind of piecewise uniform mesh Bakhvalov-Shishkin mesh(a combination of Bakhvalov mesh and Shishkin mesh) on standard FDM.

1.3 Objectives

1.3.1 General Objective

The general objective of this study is to formulate ε -uniform numerical method for solving singularly perturbed non local boundary problems on Bakhvalov-Shishkin mesh.

1.3.2 Specific Objectives

The specific objective of this study are:

- to construct a fitted mesh finite-difference scheme for the numerical solution of singularly perturbed problem with non local boundary condition problems.



- to establish the stability and ε – uniform convergence of the proposed difference method.

1.4 Significance of the Study

The result obtained from this study may:

- used as reference material for scholars who works on this research area.
- offer a numerical method for solving singularly perturbed with non local boundary problems.

1.5 Delimitation of the Study

The study is delimited to singularly perturbed with non local boundary condition problems of the form:

$$\varepsilon u''(x) + a(x)u'(x) = g(x), \quad 0 < x < 1, \quad (1.1)$$

$$u'(0) = \frac{A}{\varepsilon}, \quad (1.2)$$

$$\int_0^1 b(x)u(x)dx = B, \quad (1.3)$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants; $a^* > a(x) \geq \alpha > 0$; $g(x)$ and $b(x)$ are assumed to be sufficiently smooth functions in $[0, 1]$. Furthermore, the solution $u(x)$ has a boundary layer near $x = 0$. This problem is done by finite difference method on Bakhvalov-Shishkin mesh.

CHAPTER 2

LITERATURE REVIEW

The numerical handling of singularly perturbed problems poses formidable challenges that must be tackled to attain precise numerical solutions. Consequently, a plethora of works pertaining to the numerical handling of singularly perturbed integral boundary equations have been documented in recent years in the literature. These endeavors aim to offer effective numerical techniques that mitigate the negative impact of small perturbations on the accuracy of the solutions to these types of equations.

Arslan (2019) presented an absolutely accurate computational solution of a linear singularly perturbed problem with integral boundary condition on Bakhvalov mesh. They constructed and obtained finite difference scheme and the approximation of the presented problem based on the ε perturbation parameter. In their work, they established that the first-order uniform convergence scheme lies within the discrete maximum norm.

Arslan and Cakir (n.d.) solved a singularly perturbed problem with non-local boundary conditions numerically. The finite difference method was used to discretize this problem on the Bakhvalov-Shishkin mesh, and some properties of the exact solution were examined. They obtained the error in first order in the discrete maximum norm in their work. Cakir and Amiraliev (2005) presented a finite difference method for numerical solutions of singularly perturbed boundary value problems for second-order ordinary differential equations with non-local boundary conditions. By the method of integral identities with the use of exponential basis functions and interpolating quadrature rules with the weight and remainder term in integral form, an exponentially fitted difference scheme on an uniform mesh is developed, which is shown to be original ε -uniformly first-order accurate in the discrete maximum norm for the original problem.

Cakir and Arslan (2021) applied the finite difference method on a Shishkin mesh to solve the singularly perturbed problem with integral boundary conditions and obtained properties of the exact solution. They constructed a finite difference scheme on this mesh. In their work, they showed the stability and convergence analysis of the method as first-order convergent at the discrete maximum norm, regardless of the perturbation parameter ε .



Arslan (2020a) studied about the finite difference method for the solution of the singularly perturbed problem with the boundary layer and integral boundary condition. They determined the behaviour of the exact solution and its derivative. They constructed the finite difference scheme on Shishkin mesh.

Arslan (2020c) presented a singularly perturbed multipoint boundary value problem with a rightboundary layer. This problem is discretized using the finite difference method on a Bakhvalov-Shishkin-type mesh. They developed a uniform error estimate within a discrete maximum norm and presented a first-order estimate of accuracy difference schemes for the approximate solutions to the problem.

Musa, Gabil M, et al. (2010) presented a uniform finite difference method for the numerical solution of a nonlinear singularly perturbed convection-diffusion problem with nonlocal and third type boundary conditions and constructed the numerical method on a piecewise uniform Shishkin type mesh. The method was shown to be convergent, uniformly in the diffusion parameter ε , of first order in the discrete maximum norm.

Debela et al. (2022) presented singularly perturbed differential equations of convection diffusion type with non-local boundary conditions. The proposed numerical scheme was a combinations of convection diffusion type with non-local boundary conditions. The proposed numerical scheme was a combination of the classical finite difference method for the initial boundary condition and the nonstandard finite difference method for the differential equations at the interior points. Maximum absolute errors and rates of convergence for different values of the perturbation parameter and mesh sizes The method was shown to be first-order convergence independent of the perturbation parameter ε .

Debela and Duressa (2020) considered a class of singularly perturbed differential equations of convection diffusion type with non-local boundary conditions. A uniformly convergent numerical method is constructed via non-standard finite difference and numerical integration methods to solve the problem. The non-local boundary condition is treated using numerical integration techniques. Maximum absolute errors and rates of convergence for different values of perturbation parameter and mesh sizes are tabulated for the numerical example considered. The method is shown to be ε -uniformly convergent.

Debela and Duressa (2019) considered the fitted finite difference method for solving singularly perturbed delay differential equations with integral boundary conditions. To treat the integral boundary condition, they applied Simpson's rule. In their work, they proved the



stability and parameter ε -uniform convergence of the proposed method.

Debela and Duressa (2023) presented a fitted operator numerical method for solving singularly perturbed boundary value problems with integral boundary conditions. They proved the stability and parameter-uniform convergence of the proposed method. In their work, they considered validating the applicability of the scheme, a model problem for numerical experimentation, and solved it for different values of the perturbation parameter, ε , and mesh size, h , and presented maximum absolute errors and rates of convergence. Their method is more accurate and ε -uniformly convergent for $h \geq \varepsilon$, where the classical numerical methods fail to give good results.

The literature presented focuses on the numerical solution of singularly perturbed boundary value problems with integral or non-local boundary conditions. Various authors have utilized finite difference methods on different types of meshes, such as Bakhvalov, Shishkin, and Bakhvalov-Shishkin meshes. The works highlight the construction of finite difference schemes, approximation of the problems based on perturbation parameters, and analysis of stability and convergence. Most methods proposed in these studies demonstrate first-order convergence in the discrete maximum norm, regardless of the perturbation parameter ε . Some authors also incorporate non-standard finite difference and numerical integration techniques to handle specific types of problems. These methods exhibit ε -uniform convergence and are shown to be more accurate compared to classical numerical methods in certain scenarios. Overall, the literature contributes to the development of numerical techniques for solving singularly perturbed boundary value problems with various types of boundary conditions.

The majority of researchers have opted not to employ the Bakhvalov, Shishkin, or Bakhvalov-Shishkin mesh in conjunction with conventional or typical finite difference techniques. They have utilized alternative approaches such as operator-based, non-standard, exponentially fitted, or other methods that impact the formation of the boundary layer in their methodology. Consequently, our objective is to utilize a standard finite difference scheme on a Bakhvalov-Shishkin mesh in order to achieve improved boundary layer formation.

CHAPTER 3

METHODOLOGY OF THE STUDY

3.1 Study Design

The goal of this research is to develop a suitable numerical model that uses standard finite difference in a specialised piecewise uniform mesh by separating the space into two sections and using the Bakhvalov mesh for the first discretization and the Shishkin mesh for the rest. In order to explore the boundary solution if the non-local boundary is an integral boundary treated by Simpson's rule of integration, we created the scheme and proved its stability and convergence, as well as the specified mesh stability and boundedness. To demonstrate our numerical conclusions, we will run numerical simulations in MATLAB. Finally, the solution of the system of singularly differential equations with its graph is discussed.

3.2 Mathematical Procedures

To attain the objective of the study, the following procedures will be undertaken:

- Defining the problem,
- Formulating the fitted mesh finite difference model,
- Analyzing the stability and convergence,
- Solving the discrete problem by iterative method,
- Discussing the results with their graph and table.

CHAPTER 4

DESCRIPTION AND ANALYSIS OF THE METHOD

4.1 Properties of Continuous Problems

Numerical aspects of singularly perturbed problems, their analytical aspects play an important role. Problems (1.1) satisfy the following lemma.

Lemma 4.1. *If $a(x), g(x) \in C^2[0, 1], b(x) \in C^1[0, 1]$,*

$$||b_1|| = \int_0^1 b(x)dx \neq 0, \quad (4.1)$$

the exact solution and its derivative for the problem yielded the following properties

$$|u(x)| \leq C_0, \quad (4.2)$$

and

$$|u'(x)| \leq C \left\{ 1 + \frac{1}{\varepsilon} e^{\frac{-\alpha x}{\varepsilon}} \right\}. \quad (4.3)$$

Proof. Let us apply $u'(x) = v(x)$ in the equation (1.1) and the solution of the problem (1.1) - (1.3) become

$$\varepsilon v(x)' + a(x)v(x) = g(x), \quad (4.4)$$

$$v(0) = \frac{A}{\varepsilon}. \quad (4.5)$$

we can use the method of integrating factors

Let us define the integrating factor as $e^{\frac{1}{\varepsilon} \int a(x)dx}$ then, we can rewrite the differential equation as :

$$e^{\frac{1}{\varepsilon} \int a(x)dx} v'(x) + e^{\frac{1}{\varepsilon} \int a(x)dx} a(x)v(x) = e^{\frac{1}{\varepsilon} \int a(x)dx} g(x). \quad (4.6)$$

We can recognize the left-hand side of the equation as the derivative of the product of the



integrating factor and $v(x)$:

$$\left(e^{\frac{1}{\varepsilon} \int a(x) dx} v(x) \right)' = e^{\frac{1}{\varepsilon} \int a(x) dx} g(x). \quad (4.7)$$

Now we can integrate both sides of the equation with respect to x , from 0 to x

$$\int_0^x \left(e^{\frac{1}{\varepsilon} \int a(\zeta) d\zeta} v(t) \right)' dt = \int_0^x e^{\frac{1}{\varepsilon} \int a(\eta) d\eta} g(s) ds. \quad (4.8)$$

Using the fundamental theorem of calculus on the left hand-side, we get:

$$e^{\frac{1}{\varepsilon} \int a(\zeta) d\zeta} v(x) - v(0) = \int_0^x e^{\frac{1}{\varepsilon} \int a(\eta) d\eta} g(s) ds. \quad (4.9)$$

Substituting $v(0) = \frac{A}{\varepsilon}$, we get

$$e^{\frac{1}{\varepsilon} \int a(\zeta) d\zeta} v(x) - \frac{A}{\varepsilon} = \int_0^x e^{\frac{1}{\varepsilon} \int a(\eta) d\eta} g(s) ds. \quad (4.10)$$

Now we can solve for $v(x)$ by multiplying both sides of the equation by $e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta}$

$$v(x) = \frac{A}{\varepsilon} e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} + \int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau. \quad (4.11)$$

Integrating equation (4.11) over 0 to x , we obtained

$$u(x) = u(0) + \frac{A}{\varepsilon} \int_0^x e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} ds + \frac{1}{\varepsilon} \int_0^x \left(\int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right) ds. \quad (4.12)$$

Once the equation (4.12) is written in equation (1.3) we obtained:

$$\int_0^1 b(x) \left(u(0) + \frac{A}{\varepsilon} \int_0^x e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} ds + \frac{1}{\varepsilon} \int_0^x \left(\int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right) ds \right) dx = B. \quad (4.13)$$

Now we can evaluate the inner integral first

$$\int_0^1 b(x) u(0) dx + \frac{A}{\varepsilon} \int_0^1 b(x) \int_0^x e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} ds dx + \frac{1}{\varepsilon} \int_0^1 b(x) \int_0^x \left(\int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right) ds dx = B, \quad (4.14)$$

$$\int_0^1 b(x) u(0) dx = B - \frac{A}{\varepsilon} \int_0^1 b(x) \int_0^x e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} ds dx - \frac{1}{\varepsilon} \int_0^1 b(x) \int_0^x \left(\int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right) ds dx. \quad (4.15)$$



Then, we get:

$$u(0) = \frac{1}{\int_0^1 b(x)} \left(B - \frac{A}{\varepsilon} \int_0^1 b(x) \int_0^s e^{-\frac{1}{\varepsilon} \int a(\zeta) d\zeta} ds dx - \frac{1}{\varepsilon} \int_0^1 b(x) \int_0^x \left(\int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right) ds dx \right). \quad (4.16)$$

The evaluation of this equality resulted in:

$$|u(0)| \leq \frac{|B|}{\|b_1\|} - \frac{|c|}{\|b_1\|} \leq C. \quad (4.17)$$

Finally from (4.12) and (4.16), we obtained

$$|u(x)| \leq C_0. \quad (4.18)$$

and it proved (4.2) There after, it is essential to examine the inequality (4.3)

$$|u'(x)| \leq |u'(0)| \left| e^{-\frac{1}{\varepsilon} \int_0^x a(\zeta) d\zeta} + \int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right|.$$

using triangle inequality , we get:

$$|u'(x)| \leq |u'(0)| \left| e^{-\frac{1}{\varepsilon} \int_0^x a(\zeta) d\zeta} \right| + \left| \int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau \right|.$$

We can apply the mean value theorem of integration to the integral term to obtain:

$$|u'(x)| \leq |u'(0)| \left| e^{-\frac{1}{\varepsilon} \int_0^x a(\zeta) d\zeta} \right| + \left| g(c) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} \right|.$$

For some $c \in [0, x]$

Using the fact that $|g(c)|$ is bounded by some constant M .

We can further simplify the inequality

$$|u'(x)| \leq |u'(0)| \left| e^{-\frac{1}{\varepsilon} \int_0^x a(\zeta) d\zeta} \right| + M \left| e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} \right|.$$

Now let $c = \frac{A}{\varepsilon} \int_0^1 b(x) \int_0^x e^{-\frac{1}{\varepsilon} \int a(x) dx} ds - \frac{1}{\varepsilon} \int_0^1 b(x) \int_0^x \int_0^x g(\tau) e^{-\frac{1}{\varepsilon} \int a(\eta) d\eta} d\tau ds dx$, then we can



rewrite :

$$|u'(x)| \leq |u'(0)| \left| e^{-\frac{\alpha x}{\varepsilon}} \right| + C. \quad (4.19)$$

where $\alpha = \frac{1}{\int_0^1 b(x) dx}$, and we have used the fact that the maximum value of the exponential term is $e^{-\frac{\alpha x}{\varepsilon}}$ at $x=0$. Rearranging the terms we get:

$$|u'(x)| \leq c \left\{ 1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}} \right\}. \quad (4.20)$$

□

4.2 Construction of the Difference Scheme

A non-uniform mesh on the interval $[0,1]$ is introduced that have strong variation.

$$W_n = \{0 < x_1 < x_2 < \dots < x_{N-1} < 1\}, \quad u_i = u(x_i)$$

Before delving into the numerical method used in this work, it is necessary to define a few terms related to mesh functions. For each mesh function, the following finite difference was defined (Arslan, 2019).

$$u_{\bar{x},i} = \frac{u_i - u_{i-1}}{h_i}, \quad u_{x,i} = \frac{u_{i+1} - u_i}{h_{i+1}}, \quad u_{o_{x,i}} = \frac{u_{x,i} + u_{\bar{x},i}}{2}, \quad (4.21)$$

$$u_{\hat{x},i} = \frac{u_{i+1} - u_i}{\hat{h}_i}, \quad u_{\bar{x}\hat{x},i} = \frac{u_{x,i} - u_{\bar{x},i}}{\hat{h}_i}, \quad \hat{h}_i = \frac{h_i + h_{i+1}}{2}. \quad (4.22)$$

$$\|u\|_{\infty} \equiv \|u\|_{\infty, \bar{\omega}_N} := \max_{0 \leq i \leq N} |u_i|$$

To construct a fitted mesh finite difference scheme for the given problem, we can discretize the domain into a set of grid points and approximate the derivatives using finite difference approximations.

Let's consider a uniform mesh with grid points given by ,
Bakhvalov-Shishkin mesh

$$\delta = \min \left\{ \frac{1}{2}, \alpha^{-1} \varepsilon \ln N \right\}. \quad (4.23)$$

where δ is transition parameter.



A set of mesh $\bar{\omega}_N = \{x\}_{i=0}^N$ were introduced

$$x_i = \begin{cases} -(\alpha)^{-1} \varepsilon \ln[1 - (1 - \varepsilon) \frac{2i}{N}], & \text{for } x_i \in [0, \delta], \delta < \frac{1}{2}, \\ -(\alpha)^{-1} \varepsilon \ln[1 - (1 - e^{-\frac{\alpha}{2\varepsilon}}) \frac{2i}{N}], & \text{for } x_i \in [0, \delta], \delta = \frac{1}{2}, \\ \delta + (i - N/2)h^{(1)}, & \text{for } x_i \in [\delta, 1], h^{(1)} = \frac{2(1-\delta)}{N}. \end{cases}$$

where h is the mesh size and $i = 0, 1, 2, \dots, N$. Here, N is the total number of grid points, and $h_i = x_i - x_{i-1}$.

To approximate the second derivative $u''(x)$, we can use the centered difference formula:

$$u''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h_i^2}.$$

where u_i represents the approximation of $u(x)$.

Similarly, we can approximate the first derivative using a backward difference formula:

$$u'(x_i) \approx \frac{u_i - u_{i-1}}{h_i}.$$

To discretize the differential equation using finite differences, we substitute the finite difference approximations into equation (1.3):

$$\varepsilon \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \right) + a_i \left(\frac{u_i - u_{i-1}}{h} \right) = g_i. \quad (4.24)$$

Now, let's introduce the quantity $\hat{h}_i$ as the average of h_i and h_{i+1} (4.22). We can rewrite the equation in terms of $\hat{h}_i$:

$$\varepsilon \left(\frac{\frac{u_{i+1} - u_i}{h_{i+1}} - \frac{(u_i - u_{i-1})}{h_i}}{\hat{h}_i} \right) + a_i \left(\frac{u_i - u_{i-1}}{h_i} \right) = g_i + R_i.$$

Rearranging the equation, we get:

$$\varepsilon \left(\frac{u_{i+1} - u_i}{\hat{h}_i h_{i+1}} - \frac{(u_i - u_{i-1})}{\hat{h}_i h_i} \right) + a_i \left(\frac{u_i - u_{i-1}}{h_i} \right) = g_i + R_i.$$

Simplifying the equation further:

$$\left(\frac{\varepsilon}{\hat{h}_i h_{i+1}} \right) u_{i+1} - \left(\frac{\varepsilon}{\hat{h}_i h_{i+1}} + \frac{\varepsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i} \right) u_i + \left(\frac{\varepsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i} \right) u_{i-1} = g_i + R_i. \quad (4.25)$$



Thus neglecting the error terms (1.1) become,

$$\left(\frac{\varepsilon}{\hat{h}_i h_{i+1}}\right) y_{i+1} - \left(\frac{\varepsilon}{\hat{h}_i h_{i+1}} + \frac{\varepsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i}\right) y_i + \left(\frac{\varepsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i}\right) y_{i-1} = g_i. \quad (4.26)$$

Finally, we can rewrite the equation in the desired form:

$$\varepsilon y_{\bar{x}\bar{x},i} + a_i y_{\bar{x},i} = g_i. \quad (4.27)$$

The boundary condition (1.2) can be discretized using a forward difference:

$$\frac{u_1 - u_0}{h_1} = \frac{A}{\varepsilon} + r_0. \quad (4.28)$$

where $r_0 = \frac{1}{2!}u''(0) + \frac{h_1^2}{3!}u'''(x) + \dots$

Simplifying the equation with neglecting the error term r_0 (4.28), we have:

$$y_0 = y_1 - \frac{h_1 A}{\varepsilon}. \quad (4.29)$$

Finally, we get the desired form

$$\varepsilon y_{x,0} = A. \quad (4.30)$$

To handle this boundary condition on (1.3), we can introduce an auxiliary point and use the following ghost point value: The composite Simson's rule approximates the integral of $b(x)u(x)$ by

$$\int_0^1 b(x)u(x)dx = \frac{h}{3} \left(b(0)u(0) + b(1)u(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(x_{2i})u(x_{2i}) + 4 \sum_{i=1}^{\frac{N}{2}} b(x_{2i-1})u(x_{2i-1}) \right). \quad (4.31)$$

Substitute in equation (1.3) gives

$$\begin{aligned} \int_0^1 b(x)u(x)dx = \\ \frac{h}{3} \left(b(0)u(0) + b(1)u(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)u(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)u((2i-1)h) \right) \\ = B + r_1. \end{aligned} \quad (4.32)$$

In case R_i , r_0 and r_1 were neglected in equations (4.27), (4.30) and (4.32), respectively, the following difference scheme for the approximation of the problem (1.1)-(1.3) could be proposed:

$$\varepsilon y_{\bar{x}\bar{x},i} + a_i y_{\bar{x},i} = g_i \quad (4.33)$$

$$\varepsilon y_{x,0} = A. \quad (4.34)$$

$$\frac{h}{3} \left(b(0)y(0) + b(1)y(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)y(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)y((2i-1)h) \right) = B. \quad (4.35)$$

4.3 Stability and Uniform Convergence Analysis

The convergence of the method for the problem (1.1) - (1.3) on Bakhvalov-Shishkin mesh was examined in this section. The error function $z_i = y_i - u_i, i = 0, 1, \dots, N$, where z_i was the solution to the discrete problem, was provided:

$$\varepsilon z_{\bar{x}\bar{x},i} + a_i z_{\bar{x},i} = R_i \quad (4.36)$$

$$\varepsilon z_{x,0} = r_0. \quad (4.37)$$

$$\frac{h}{3} \left(b(0)z(0) + b(1)z(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)z(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)z((2i-1)h) \right) = r_1. \quad (4.38)$$

Lemma 4.2. *Let z_i be the solution of the discrete problem (4.36) - (4.38), then the estimate*



would hold:

$$\|z\|_{\infty, \bar{\omega}_N} \leq C \{ \|R\|_{\infty, \omega_N} |r_0| + |r_1| \}. \quad (4.39)$$

where C is a constant equal to $\max\{1, \frac{3}{h|b(1)|}\}$.

Proof. To prove the lemma, we will use the maximum principle for elliptic equations. Let $M = \max_{i \in \bar{\omega}_N} |z_i|$. We will show that M is bounded by a constant multiple of $|R|_{\infty, \omega_N} |r_0| + |r_1|$.

Consider the grid points in the interior of $\bar{\omega}_N$, i.e., $i = 1, 2, \dots, N-1$. For these points, we have the equation:

$$\varepsilon z_{\bar{x}\hat{x}, i} + z_{\bar{x}, i} = R_i. \quad (4.40)$$

Taking the absolute value of both sides on (4.40) and using the triangle inequality, we get:

$$\varepsilon |z_{\bar{x}\hat{x}, i}| + |z_{\bar{x}, i}| \leq |R_i|. \quad (4.41)$$

Now, let i_0 be the index where $|z_{i_0}| = M$. Then, we have:

$$\varepsilon |z_{\bar{x}\hat{x}, i_0}| + |z_{\bar{x}, i_0}| \leq |R_{i_0}| \leq |R|_{\infty, \omega_N}. \quad (4.42)$$

we can rewrite it as:

$$\varepsilon |z_{\bar{x}\hat{x}, i_0}| \leq |R|_{\infty, \omega_N} - |z_{\bar{x}, i_0}|.$$

Since $|z_{\bar{x}, i_0}| \leq M$, we have:

$$\varepsilon |z_{\bar{x}\hat{x}, i_0}| \leq |R|_{\infty, \omega_N} - M. \quad (4.43)$$

Similarly, for the equation at the point $i = 0$, we have:

$$\varepsilon |z_{x, 0}| \leq |r_0|. \quad (4.44)$$



Now, let's consider the equation at the point $i = N$:

$$\frac{h}{3} \left(b(0)z(0) + b(1)z(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)z(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)z((2i-1)h) \right) = r_1. \quad (4.45)$$

Taking the absolute value of both sides and applying the triangle inequality, we obtain:

$$\frac{h}{3} \left| b(0)z(0) + b(1)z(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)z(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)z((2i-1)h) \right| \leq |r_1|. \quad (4.46)$$

Using the fact that $|z_i| \leq M$ for all i , we can further simplify the above inequality as:

$$\frac{h}{3} \left(|b(0)z(0)| + |b(1)z(1)| + 2 \sum_{i=1}^{\frac{N}{2}-1} |b(2ih)z(2ih)| + 4 \sum_{i=1}^{\frac{N}{2}} |b((2i-1)h)z((2i-1)h)| \right) \leq |r_1|.$$

Applying the triangle inequality again, we have:

$$\begin{aligned} & \frac{h}{3} \left(|b(0)||z(0)| + |b(1)||z(1)| + 2 \sum_{i=1}^{\frac{N}{2}-1} |b(2ih)||z(2ih)| + 4 \sum_{i=1}^{\frac{N}{2}} |b((2i-1)h)||z((2i-1)h)| \right) \\ & \leq |r_1|. \end{aligned}$$

Using the fact that $|z(0)| = M$, we can write:

$$\begin{aligned} & \frac{h}{3} \left(|b(0)||z(0)| + |b(1)||z(1)| + 2 \sum_{i=1}^{\frac{N}{2}-1} |b(2ih)||z(2ih)| + 4 \sum_{i=1}^{\frac{N}{2}} |b((2i-1)h)||z((2i-1)h)| \right) \\ & \geq \frac{h}{3} (|b(0)|M + |b(1)||z(1)|). \end{aligned}$$

Since $|b(0)| \geq 0$, we have:

$$\frac{h}{3} (|b(0)|M + |b(1)||z(1)|) \geq \frac{h}{3} (|b(1)||z(1)|) = \frac{h}{3} |b(1)||z(1)|.$$

Therefore, we can rewrite the inequality as:

$$\frac{h}{3} |b(1)||z(1)| \leq |r_1|. \quad (4.47)$$



Dividing both sides by $\frac{h}{3}|b(1)|$ gives:

$$|z(1)| \leq \frac{3}{h|b(1)|} |r_1|. \quad (4.48)$$

Now, combining Equations (4.43), (4.44), and (4.48), we can bound M as follows:

$$\begin{aligned} M &\leq \max \left\{ \varepsilon |z_{\bar{x}\bar{x}, i_0}|, |z_{\bar{x}, i_0}|, |z(1)| \right\} \\ &\leq \max \left\{ \varepsilon |z_{\bar{x}\bar{x}, i_0}|, |z_{\bar{x}, i_0}|, \frac{3}{h|b(1)|} |r_1| \right\} \\ &\leq \max \left\{ |R|_{\infty, \omega_N} - M, M, \frac{3}{h|b(1)|} |r_1| \right\}. \end{aligned} \quad (4.49)$$

Note that the above inequality holds for any index i_0 where $|z_{i_0}| = M$. Taking the maximum of both sides over all such indices i_0 , we have:

$$\begin{aligned} M &\leq \max \left\{ \max_{i_0} \{ |R|_{\infty, \omega_N} - M \}, \max_{i_0} \{ M \}, \frac{3}{h|b(1)|} |r_1| \right\} \\ &\leq \max \{ |R|_{\infty, \omega_N} - M, M \} + \frac{3}{h|b(1)|} |r_1|. \end{aligned} \quad (4.50)$$

Using the fact that $\max_{a,b} \leq \max_{a+c,b}$ for any non-negative value c , we can further simplify the inequality as:

$$M \leq \max \left\{ |R|_{\infty, \omega_N} + \frac{3}{h|b(1)|} |r_1|, M \right\}. \quad (4.51)$$

Finally, applying the maximum principle for equations, we conclude that:

$$M \leq |R|_{\infty, \omega_N} + \frac{3}{h|b(1)|} |r_1|. \quad (4.52)$$

Therefore, we have proven that:

$$|z|_{\infty, \bar{\omega}_N} \leq C \{ |R|_{\infty, \omega_N} |r_0| + |r_1| \}, \quad (4.53)$$

where C is a constant equal to $\max \{ 1, \frac{3}{h|b(1)|} \}$.

This completes the proof of the lemma. $\square$

Lemma 4.3. *Based on the assumptions Lemma 4.2, the solution of the problem (1.1) - (1.3)*



fulfilled the following estimates for the remainder terms R_i , r_0 and r_1 ,

$$\|R\|_{\infty, \omega_N} \leq CN^{-1}, \quad (4.54)$$

$$|r_0| \leq CN^{-1}, \quad (4.55)$$

$$|r_1| \leq CN^{-1}. \quad (4.56)$$

Where

$$R_i = \varepsilon \left(\frac{h_i}{\hat{h}_i h_{i+1}} u'(x) + \frac{h_i^2}{2! \hat{h}_i h_{i+1}} u''(x) + \dots \right) + \varepsilon \left(\frac{-1}{\hat{h}_i} u'(x) + \frac{h_i}{2! \hat{h}_i} u''(x) - \dots \right) + a_i \left(u'(x) - \frac{h_i}{2!} u''(x) + \frac{h_i^2}{3!} u'''(x) \dots \right),$$

$$r_0 = \frac{1}{2!} u''(0) + \frac{h_2}{3!} u'''(x) + \dots, \quad (4.57)$$

$$r_1 = -\frac{h^4}{180} \left(b^{(4)}(\xi) u^{(4)}(\xi) \right). \quad (4.58)$$

Proof. To prove the Lemma, we'll start by analyzing the remainder term R_i .

Let's rewrite the expression for R_i :

$$\begin{aligned} R_i &= \varepsilon \left(\frac{h_i}{\hat{h}_i h_{i+1}} u'(x) + \frac{h_i^2}{2! \hat{h}_i h_{i+1}} u''(x) + \dots \right) \\ &+ \varepsilon \left(\frac{-1}{\hat{h}_i} u'(x) + \frac{h_i}{2! \hat{h}_i} u''(x) - \dots \right) \\ &+ a_i \left(u'(x) - \frac{h_i}{2!} u''(x) + \frac{h_i^2}{3!} u'''(x) \dots \right). \end{aligned} \quad (4.59)$$

Now, let's estimate the terms in R_i one by one.

For the first term, we have:

$$\left| \frac{h_i}{\hat{h}_i h_{i+1}} u'(x) + \frac{h_i^2}{2! \hat{h}_i h_{i+1}} u''(x) + \dots \right|.$$

Since the derivatives of $u(x)$ are bounded on ω_N , we can estimate each term as follows:

$$\begin{aligned} \left| \frac{h_i}{\hat{h}i h_{i+1}} u'(x) \right| &\leq \frac{C|h_i|}{|\hat{h}i| h_{i+1}}, \\ \left| \frac{h_i^2}{2! \hat{h}i h_{i+1}} u''(x) \right| &\leq \frac{C|h_i|^2}{2! |\hat{h}i| h_{i+1}}, \\ &\vdots \end{aligned}$$

Since $|\hat{h}i|$ and h_{i+1} are constants, we can combine the terms and obtain:

$$\left| \frac{h_i}{\hat{h}i h_{i+1}} u'(x) + \frac{h_i^2}{2! \hat{h}i h_{i+1}} u''(x) + \dots \right| \leq \frac{C|h_i|}{|\hat{h}i| h_{i+1}} \leq \frac{C}{N}, \quad (4.60)$$

where we have used the assumption that $|h_i| \leq N$ (Shishkin et al., 2004). Similarly, we can estimate the second term in R_i :

$$\left| \frac{-1}{\hat{h}_i} u'(x) + \frac{h_i}{2! \hat{h}_i} u''(x) - \dots \right| \leq \frac{C}{N}. \quad (4.61)$$

For the third term in R_i , we have:

$$\left| a_i \left(u'(x) - \frac{h_i}{2!} u''(x) + \frac{h_i^2}{3!} u'''(x) \dots \right) \right|.$$

Since a_i dominated by α we can take a_i is a constant, we can estimate each term as follows:

$$\left| u'(x) - \frac{h_i}{2!} u''(x) + \frac{h_i^2}{3!} u'''(x) \dots \right| \leq C, \quad (4.62)$$

where C is a constant.

Therefore, combining the estimates for each term, we have:

$$|R_i|_{\infty, \omega_N} \leq C \left(\frac{1}{N} + \frac{1}{N} + 1 \right) = C \left(\frac{2}{N} + 1 \right). \quad (4.63)$$

This proves the estimate for the remainder term R_i :

$$|R_i|_{\infty, \omega_N} \leq CN^{-1}. \quad (4.64)$$

Next, let's analyze the estimates for r_0 and r_1 .



For r_0 , we have:

$$r_0 = \frac{1}{2!}u''(0) + \frac{h_2}{3!}u'''(x) + \dots$$

Since $|u''(0)|$ and $|u'''(x)|$ are bounded on ω_N , we can estimate each term as follows:

$$\left| \frac{1}{2!}u''(0) \right| \leq \frac{C}{2!}, \quad \left| \frac{h_2}{3!}u'''(x) \right| \leq \frac{C|h_2|}{3!}.$$

Using the assumption from Lemma 4.2 that $|h_2| \leq N$, we can further simplify the estimate:

$$|r_0| \leq \frac{C}{2!} + \frac{C}{3!} = \frac{C}{2}. \quad (4.65)$$

Therefore, we have the estimate for r_0 :

$$|r_0| \leq CN^{-1}. \quad (4.66)$$

case 1 : For $x_i \in [0, \delta]$ and $\delta < \frac{1}{2}$ We can use the mean value theorem

$$f(a) - f(b) = f'(c)(b - a).$$

Applying this theorem to each interval in the Bakhvalov-Shishkin mesh, we get For $x_i \in [0, \delta]$ and $\delta < \frac{1}{2}$

$$x_{i+1} - x_i = -(\alpha)^{-1}\varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i+1)}{N} \right] - (-\alpha)^{-1}\varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i)}{N} \right]. \quad (4.67)$$

Using the mean value theorem, we can write

$$x_{i+1} - x_i = \frac{-\alpha^{-1}\varepsilon}{1 - \varepsilon} \frac{2}{N} - \frac{-\alpha^{-1}\varepsilon}{1 - \varepsilon} \frac{2i+1}{N}.$$

Thus, we have

$$h_i = |x_{i+1} - x_i| \leq \frac{4\alpha^{-2}\varepsilon^2}{N(1 - \varepsilon)^2} \leq CN^{-1}. \quad (4.68)$$



Consequently, it was possible to write from (4.64) and (4.68):

$$|R_i| \leq CN^{-1}. \quad (4.69)$$

Case 2 : For $x_i \in [0, \delta]$ and $\delta = \frac{1}{2}$

$$x_{i+1} - x_i = -(\alpha)^{-1} \varepsilon \ln \left[1 - \left(1 - e^{-\frac{\alpha}{2\varepsilon}} \right) \frac{2(i+1)}{N} \right] - (-\alpha)^{-1} \varepsilon \ln \left[1 - \left(1 - e^{-\frac{\alpha}{2\varepsilon}} \right) \frac{2(i)}{N} \right]. \quad (4.70)$$

Using the mean value theorem, we can write

$$x_{i+1} - x_i = \frac{-\alpha^{-1} \varepsilon}{1 - e^{-\frac{\alpha}{2\varepsilon}}} \frac{2}{N} \frac{-\alpha^{-1} \varepsilon}{1 - e^{-\frac{\alpha}{2\varepsilon}}} \frac{2i+1}{N}.$$

Thus, we have

$$h_i = |x_{i+1} - x_i| \leq \frac{4\alpha^{-2} \varepsilon^2}{N(1 - e^{-\frac{\alpha}{2\varepsilon}})^2} \leq CN^{-1}. \quad (4.71)$$

Consequently, it was possible to write from (4.64) and (4.71):

$$|R_i| \leq CN^{-1}. \quad (4.72)$$

Case 3: For $x_i \in [\delta, 1]$

$$x_i = \sigma + \left(1 - \frac{N}{2} \right) h. \quad (4.73)$$

where

$$h = \frac{2(1 - \sigma)}{N}.$$

Following equations (4.64) and (4.73), we acquired

$$|R_i| \leq Ch \leq CN^{-1}. \quad (4.74)$$



Similar to the approach described above in (4.66) mesh points for $[0, \delta]$ and $[\delta, 1]$ were

$$|r_0| \leq Ch_1 \leq CN^{-1}. \quad (4.75)$$

The error term for composite Simpson's rule is given by:

$$r_1 = -\frac{h^4}{180} \left(b^{(4)}(\xi) u^{(4)}(\xi) \right), \quad (4.76)$$

where $h = \frac{1}{N}$ is the step size, N is the number of subintervals, $b(x)$ and $u(x)$ are functions being integrated, and ξ is some unknown point in the interval $(0, 1)$. The notation $b^{(4)}(\xi)$ and $u^{(4)}(\xi)$ denote the fourth derivatives of $b(x)$ and $u(x)$, respectively, evaluated at the point ξ . Subsequently, the error function r_1 was evaluated as

$$|r_1| \leq \left| -\frac{h^4}{180} \left(b^{(4)}(\xi) u^{(4)}(\xi) \right) \right|, \quad (4.77)$$

Since $b(x)$ and $u(x)$ are both smooth functions, so that their fourth derivatives exist and are continuous.

Apply the Mean Value Theorem to $b^{(4)}(\xi)$ and $u^{(4)}(\xi)$, where ξ is some point in the interval $[x - h, x + h]$. This gives us

$$b^{(4)}(\xi) u^{(4)}(\xi) = \frac{1}{2} b^{(4)}(\xi_1) u^{(4)}(\xi_2). \quad (4.78)$$

where ξ_1 and ξ_2 are some points in the interval $[x - h, x]$ and $[x, x + h]$, respectively.

Substitute this into the first equation, to get

$$|r_1| \leq -\frac{h^4}{360} \left| b^{(4)}(\xi_1) \right| \left| u^{(4)}(\xi_2) \right|.$$

Let M_1 and M_2 be upper bounds on $|b^{(4)}(x)|$ and $|u^{(4)}(x)|$, respectively, over the interval of interest. Then we have

$$|r_1| \leq -\frac{h^4}{360} M_1 M_2.$$

Let h_1 be the maximum step size used in the numerical method, and let N be the number of steps taken to cover the interval of interest. Then we have $h_1 = \frac{1}{N}(b - a)$, where a and b are



the endpoints of the interval. Substituting this into the previous equation, we get

$$|r_1| \leq -\frac{(b-a)^4}{360N^4} M_1 M_2.$$

Let $C = \frac{M_1 M_2}{360}$, so that we have

$$|r_1| \leq -\frac{C}{N^4} (b-a)^4 \leq \frac{C}{N} \leq CN^{-1}. \quad (4.79)$$

Hence, the proof of Lemma 4.3 was achieved $\square$

Consistency and truncation error analysis of the Bakhvalov-shishkin mesh in each interval using (Gunes & Duru, 2022)

i) $x_i \in [0, \delta]$ and $\delta < \frac{1}{2}$ In this interval, we have

$$x_i = -(\alpha)^{-1} \varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i)}{N} \right]. \quad (4.80)$$

The mesh spacing is given by

$$\begin{aligned} h_i = x_i - x_{i-1} &= -(\alpha)^{-1} \varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i)}{N} \right] - (-\alpha)^{-1} \varepsilon \ln \left[1 - (1 - \varepsilon) \frac{2(i-1)}{N} \right], \\ &= -(\alpha)^{-1} \varepsilon \ln \left(\frac{1 - (1 - \varepsilon) \frac{2i}{N}}{1 - (1 - \varepsilon) \frac{2(i-1)}{N}} \right). \end{aligned} \quad (4.81)$$

Using Taylor series expansion of $\ln(1+x)$, we have

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

For $x = (1 - \varepsilon) \frac{2i}{N}$, we have

$$\ln \left(1 - (1 - \varepsilon) \frac{2i}{N} \right) = -(1 - \varepsilon) \frac{2i}{N} + \frac{1}{2} (1 - \varepsilon)^2 \left(\frac{2i}{N} \right)^2 + \dots \quad (4.82)$$

Ignoring terms of order greater than $(\frac{1}{N})^2$, we obtain

$$\ln \left(1 - (1 - \varepsilon) \frac{2i}{N} \right) = -(1 - \varepsilon) \frac{2i}{N} + \frac{1}{2} (1 - \varepsilon)^2 \left(\frac{2i}{N} \right)^2. \quad (4.83)$$



Substituting the approximation in to the expression for h_i , we get

$$\begin{aligned} h_i &= -\frac{\alpha^{-1}\varepsilon}{2i/N - (1-\varepsilon)} \ln \left(\frac{1 - (1-\varepsilon)\frac{2i}{N}}{1 - (1-\varepsilon)\frac{2(i-1)}{N}} \right), \\ &\approx -\frac{\alpha^{-1}\varepsilon}{2i/N - (1-\varepsilon)} \left[-(1-\varepsilon)\frac{2}{N} \right], \\ &= \frac{\alpha^{-1}\varepsilon}{i} + O(N^{-2}), \quad (i = 1, \dots, N/2). \end{aligned} \quad (4.84)$$

Therefore, the mesh is consistent with first-order accuracy in this interval, and the truncation error is of order $O(N^{-1})$. ii) $x_i \in [0, \delta]$ and $\delta = \frac{1}{2}$ In this interval, we have

$$x_i = -\frac{\alpha^{-1}\varepsilon}{2} \ln \left[1 - (1 - e^{-\alpha/(2\varepsilon)}) \frac{2(i)}{N} \right]. \quad (4.85)$$

Similar to the previous case, we have

$$h_i = \frac{\alpha^{-1}\varepsilon}{i} + O(N^{-2}) \quad (i = 1, \dots, N/2). \quad (4.86)$$

Therefore, the mesh is consistent with first-order accuracy in this interval, and the truncation error is of order $O(N^{-1})$. iii) $x_i \in [\delta, 1]$ In this interval, we have

$$h^{(1)} = \frac{2(1-\delta)}{N}. \quad (4.87)$$

The mesh is uniformly spaced in this interval, so we have

$$h_i = h^{(1)} = \frac{2(1-\delta)}{N}, \quad (i = N/2 + 1, \dots, N). \quad (4.88)$$

Therefore, the mesh is consistent with second-order accuracy in this interval, and the truncation error is of order $O(N^{-2})$. Overall, the Bakhvalov-Shishkin mesh is consistent with first-order accuracy and has a truncation error of $O(N^{-1})$ in the boundary layer region $[0, \delta]$ (Linß, 2007), and it is consistent with first-order accuracy and has a truncation error of $O(N^{-2})$ outside the boundary layer region $[\delta, 1]$ (Gregory & Wickramasinghe, 2023).

Theorem 4.1. Given $u(x)$ as the solution of the problem (1.1) - (1.3) and y_i as the solution



of the difference scheme (4.33) - (4.35) the following uniform error estimate fulfills

$$\|y - u\|_{\infty, \bar{\omega}_N} \leq CN^{-1}. \quad (4.89)$$

Proof. since $z_i = y_i - u_i$ be the solution of the discrete problem (4.33) - (4.35). Applying (Nhan & Vulcanović, 2021), we have the following estimate for z :

$$|z|_{\infty, \bar{\omega}_N} \leq C \{ |R|_{\infty}, \omega_N |r_0| + |r_1| \}. \quad (4.90)$$

Now, let's substitute the estimates from Lemma 4.2 into equation (4.90):

$$|z|_{\infty, \bar{\omega}_N} \leq C \{ CN^{-1} \cdot CN^{-1} + CN^{-1} \}.$$

Simplifying the expression:

$$|z|_{\infty, \bar{\omega}_N} \leq C \{ C^2 N^{-2} + CN^{-1} \}.$$

Since C is a constant, we can rewrite the expression as:

$$|z|_{\infty, \bar{\omega}_N} \leq C^3 N^{-2} + CN^{-1}. \quad (4.91)$$

Now, we want to establish an error estimate for $|y - u|_{\infty, \bar{\omega}_N}$. Using the definition of the supremum norm, we have:

$$|y - u|_{\infty, \bar{\omega}_N} = \max_{i=0, \dots, N} |y_i - u_i|. \quad (4.92)$$

Since $z_i = y_i - u_i$, we can rewrite the expression as:

$$|z|_{\infty, \bar{\omega}_N} = \max_{i=0, \dots, N} |z_i|.$$

Therefore, $|z|_{\infty, \bar{\omega}_N}$ represents the maximum error at any point in the discrete domain (Linß & Madden, 2004). Using inequality (4.91), we can write:

$$|z|_{\infty, \bar{\omega}_N} \leq C^3 N^{-2} + CN^{-1}$$



Taking the maximum on both sides, we have:

$$|z|_{\infty, \bar{\omega}_N} \leq \max\{C^3 N^{-2}, CN^{-1}\}. \quad (4.93)$$

Since C is a constant, we can further simplify:

$$|z|_{\infty, \bar{\omega}_N} \leq CN^{-1}. \quad (4.94)$$

This proves the uniform error estimate:

$$|y - u|_{\infty, \bar{\omega}_N} \leq CN^{-1}. \quad (4.95)$$

which is what we wanted to show. Thus, the theorem is proven. $\square$

CHAPTER 5

RESULTS AND DISCUSSIONS

In this section, an efficient algorithm for solving the difference scheme (1.1)-(3) was described, and numerical results were presented in tables and graphs.

The algorithm for resolving the difference scheme (3.37)-(3.39) is as follows:

$$\begin{aligned}
 \epsilon y_{\bar{x}\bar{x},i} + a_i y_{\bar{x},i} &= g_i, \\
 \epsilon y_{x,0} &= A, \\
 \frac{h}{3} \left(b(0)y(0) + b(1)y(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)y(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)y((2i-1)h) \right) &= B.
 \end{aligned} \tag{5.1}$$

$$\begin{aligned}
 \left(\frac{\epsilon}{\hat{h}_i h_{i+1}} \right) y_{i+1} - \left(\frac{\epsilon}{\hat{h}_i h_{i+1}} + \frac{\epsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i} \right) y_i + \left(\frac{\epsilon}{\hat{h}_i h_i} - \frac{a_i}{h_i} \right) y_{i-1} &= g_i, \\
 y_0 = y_1 - \frac{h_1 A}{\epsilon}, \\
 \frac{h}{3} \left(b(0)y(0) + b(1)y(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)y(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)y((2i-1)h) \right) &= B.
 \end{aligned} \tag{5.2}$$

On the mesh point

$$x_i = \begin{cases} -(\alpha)^{-1} \epsilon \ln[1 - (1 - \epsilon) \frac{2i}{N}], & \text{for } x_i \in [0, \delta], \delta < \frac{1}{2}, \\ -(\alpha)^{-1} \epsilon \ln[1 - (1 - e^{-\frac{\alpha}{2\epsilon}}) \frac{2i}{N}], & \text{for } x_i \in [0, \delta], \delta = \frac{1}{2}, \\ \delta + (i - N/2)h^{(1)}, & \text{for } x_i \in [\delta, 1], h^{(1)} = \frac{2(1-\delta)}{N}. \end{cases}$$

The maximum absolute error is calculated using the formula (Doolan et al., 1980)

$$E_\epsilon^N = \max_{0 \leq i \leq N} |U^N(x_i) - U^{2N}(x_{2i})|,$$

and rate of convergence by formula

$$Roc^N = \log 2 \left(\frac{E_r^N}{E_r^{2N}} \right).$$



Example 1. Consider the singularly perturbed problem (Arslan, 2019)

$$\begin{aligned}\varepsilon u''(x) + u'(x) &= 1, \\ u'(0) &= \frac{1}{\varepsilon}, \\ \int_0^1 u(x) &= \frac{1}{2}.\end{aligned}$$

The exact solution is (Arslan, 2019)

$$u(x) = x + \varepsilon(\varepsilon - 1)(1 - \exp(-1/\varepsilon)) + (\varepsilon - 1) * (1 - \exp(-x/\varepsilon)).$$

In this Example $a(x) = 1$, $b(x) = 1$, $g(x) = 1$ and $A = 1$ and $B = \frac{1}{2}$. Therefore, the difference scheme for Example 1 becomes

$$\begin{aligned}\left(\frac{\varepsilon}{\hat{h}_i h_{i+1}}\right) y_{i+1} - \left(\frac{\varepsilon}{\hat{h}_i h_{i+1}} + \frac{\varepsilon}{\hat{h}_i h_i} - \frac{1}{h_i}\right) y_i + \left(\frac{\varepsilon}{\hat{h}_i h_i} - \frac{1}{h_i}\right) y_{i-1} &= 1, \\ y_0 &= y_1 - \frac{h_1 A}{\varepsilon}, \\ \frac{h}{3} \left(b(0)y(0) + b(1)y(1) + 2 \sum_{i=1}^{\frac{N}{2}-1} b(2ih)y(2ih) + 4 \sum_{i=1}^{\frac{N}{2}} b((2i-1)h)y((2i-1)h) \right) &= \frac{1}{2}.\end{aligned}$$

where $\hat{h}_i$ defined as in (3.26) and x_i defined as in the above

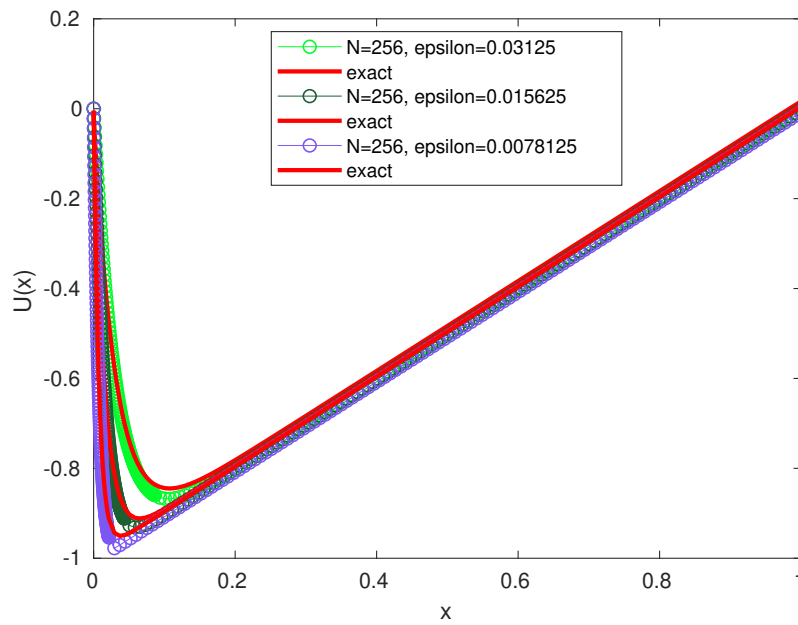


Figure 5.1: Solutions profile for Example 1 with boundary layer formation as ε goes small with mesh size $N = 2^8$.

Table 5.1: Example 1's maximum absolute error and rate of convergence for the proposed method.

$N \rightarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
$\varepsilon \downarrow$						
2^{-3}	9.8719e-02	4.6418e-02	2.2386e-02	1.0640e-02	5.9821e-03	2.9959e-03
	1.07	1.05	1.07	0.83	1.01	
2^{-4}	1.2285e-01	6.0487e-02	3.5199e-02	1.6958e-02	8.7752e-03	4.8147e-03
	1.02	0.78	1.05	0.95	0.86	
2^{-5}	1.4654e-01	8.3284e-02	4.7501e-02	2.8209e-02	1.4791e-02	9.0034e-03
	0.81	0.81	0.77	0.93	0.71	
2^{-6}	1.5199e-01	8.8559e-02	5.5855e-02	2.7207e-02	1.5051e-02	8.5414e-03
	0.77	0.66	1.03	0.85	0.81	
2^{-7}	1.1348e-01	7.3219e-02	4.3250e-02	1.7517e-02	7.6491e-03	4.7105e-03
	0.63	0.75	1.30	1.19	0.69	
E^N	1.5199e-01	8.8559e-02	5.5855e-02	2.8209e-02	1.5051e-02	9.0034e-03
R^N	0.77	0.66	0.98	0.90	0.74	

Table 5.2: Comparison of results for Example 1's maximum absolute error.

N → ε ↓	2 ⁵	2 ⁶	2 ⁷	2 ⁸	2 ⁹	2 ¹⁰
Proposed scheme						
2 ⁻²	3.5055e-01	1.6594e-01	8.7584e-02	6.7979e-02	4.8195e-02	2.8301e-02
2 ⁻³	9.8719e-02	4.6418e-02	2.2386e-02	1.0640e-02	5.9821e-03	2.9959e-03
2 ⁻⁴	1.2285e-01	6.0487e-02	3.5199e-02	1.6958e-02	8.7752e-03	4.8147e-03
3 ⁻²	9.3304e-02	6.8774e-02	4.1478e-02	2.6641e-02	1.2614e-02	9.9971e-03
3 ⁻³	9.6009e-02	6.5484e-02	3.2851e-02	1.4508e-02	9.5119e-03	5.5388e-03
Result obtained in Arslan (2019)						
2 ⁻²	0.0291769	0.0144542	0.0071922	0.0010958	0.0035876	0.0008990
2 ⁻³	0.0725140	0.0354105	0.0174912	0.0086917	0.0043356	0.0021556
2 ⁻⁴	0.1659372	0.0784342	0.0380923	0.0187665	0.0093175	0.0046408
3 ⁻²	0.1758537	0.0835009	0.0045713	0.0099388	0.0049515	0.0024707
3 ⁻³	0.2008266	0.0946524	0.0225589	0.0111871	0.0055690	0.0027710

Example 2. Consider singularly perturbed problem (Debela, 2021)

$$\begin{aligned}\varepsilon u''(x) + 2u'(x) &= (\varepsilon - 2)e^{-x}, \\ u'(0) &= \frac{1}{\varepsilon}, \\ u(0) + \frac{1}{3}u\left(\frac{1}{4}\right) + u(1) &= 1.\end{aligned}$$

The exact solution is (Debela, 2021)

$$\begin{aligned}u(x) &= d1 + d2 * \exp\left(\frac{-2x}{\varepsilon}\right) + \exp(-x), \\ d2 &= \frac{-(1 + \varepsilon)}{2}, \\ d1 &= -\frac{3}{7}[\exp(-1) + \frac{1}{3}\exp(-1/4) + (1 + \exp(-2x/\varepsilon) + \frac{1}{3}\exp(-x/(2\varepsilon)))d2].\end{aligned}$$

In this Example $a(x) = 2$, $g(x) = (\varepsilon - 2)e^{-x}$ and $A = 1$ and $B = 1$. Therefore, the difference

scheme for Example 2 becomes

$$\left(\frac{\varepsilon}{\hat{h}_i h_{i+1}}\right) y_{i+1} - \left(\frac{\varepsilon}{\hat{h}_i h_{i+1}} + \frac{\varepsilon}{\hat{h}_i h_i} - \frac{2}{h_i}\right) y_i + \left(\frac{\varepsilon}{\hat{h}_i h_i} - \frac{2}{h_i}\right) y_{i-1} = (\varepsilon - 2)e^{-x_i},$$

$$y_0 = y_1 - \frac{h_1}{\varepsilon},$$

$$u(0) + \frac{1}{3}u\left(\frac{1}{4}\right) + u(1) = 1.$$

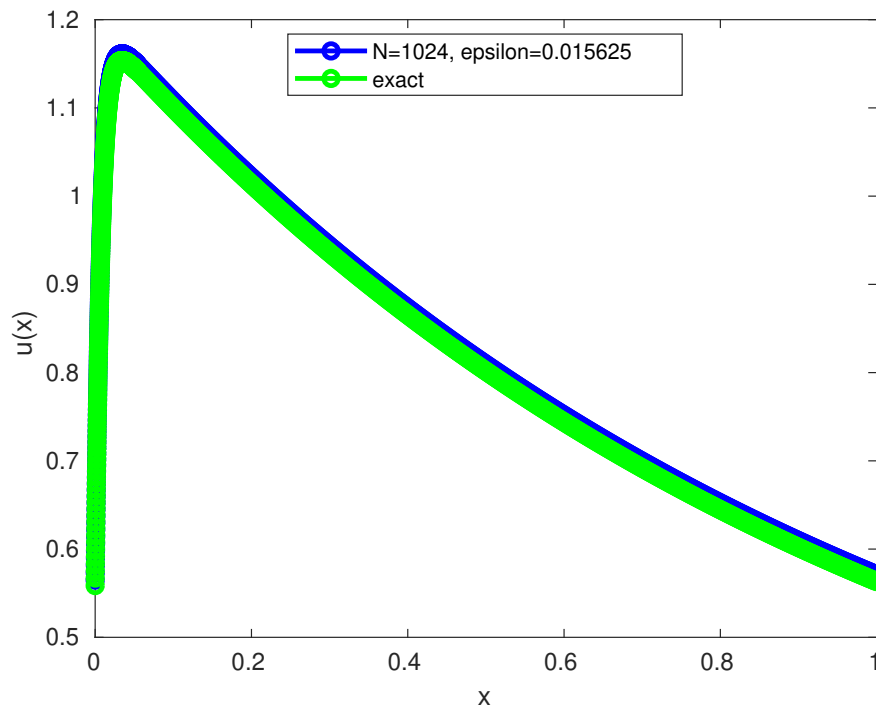


Figure 5.2: Solutions profile for Example 2 with boundary layer formation as ε goes small with mesh size $N = 2^{10}$.

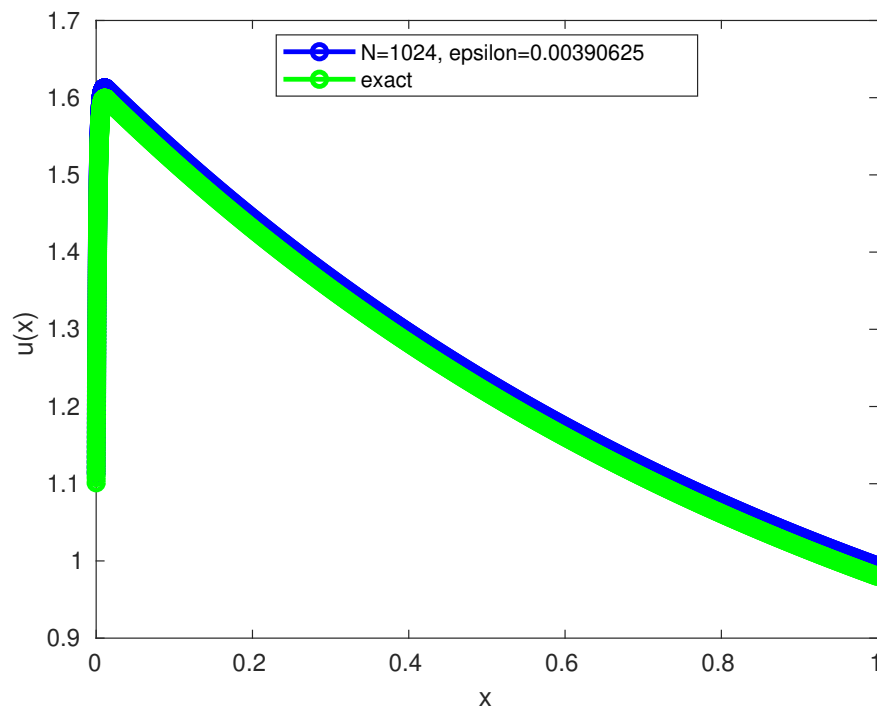


Figure 5.3: Solutions profile for Example 2 with boundary layer formation as ε goes small with mesh size $N = 2^{10}$.

Table 5.3: Example 2's maximum absolute error and rate of convergence for the proposed method.

$N \rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8	2^9	2^{10}
2^{-3}	1.7804e-01 1.15	7.9735e-02 0.94	4.1523e-02 0.87	2.2585e-02 0.86	1.24421e-03 0.86	6.8295e-03
2^{-4}	2.9412e-01 1.35	1.1502e-02 1.25	4.8110e-02 0.99	2.4157e-02 0.91	1.2830e-03 0.88	6.9389e-03
2^{-5}	3.3264e-01 1.35	1.5699e-01 1.15	7.0470e-02 1.22	2.8070e-02 1.03	1.3734e-02 0.93	7.1685e-03
2^{-6}	3.9851e-01 0.98	2.0168e-01 1.30	8.1685e-02 0.97	4.1685e-02 1.37	1.5986e-02 1.05	7.6766e-03
2^{-10}	2.0481e-02 1.08	9.6841e-03 0.46	7.0412e-03 1.13	3.2164e-03 1.04	1.2830e-03 1.02	7.16412e-04

Table 5.4: Comparison of results for Example 2's maximum absolute error.

$N \rightarrow$ $\varepsilon \downarrow$	2^5	2^6	2^7	2^8
Proposed scheme				
E^N	3.9851e-01	2.0168e-01	8.1685e-02	4.1685e-02
R^N	0.98	1.30	0.97	
Result obtained in Debela (2021)				
E^N	2:2115e-03	1:0920e-03	5:4258e-03	2:7044e-04
R^N	1.0181	1.0091	1.0045	

Discussions

We perform calculations on two different examples, each with varying values of ε , using a finite difference method on Bakhvalov-Shishkin meshes. The numerical solutions obtained are then compared with the exact solutions, and the maximum norm error is recorded in tables (5.1) and (5.3). To validate our results against Theorem (4.1), we determine the convergence rates of the numerical solutions obtained from a series of meshes. In order to handle the boundary layer near $x = 0$, we employ an upwinding Bakhvalov-Shishkin mesh. The convergence rates closely match those described in Theorem (4.1). By comparing the boundary layer resolving results of our proposed finite difference method on Bakhvalov-Shishkin meshes with the results obtained for the boundary layer region, we conclude that the proposed scheme yields a more accurate outcome in the boundary layer region. This conclusion is supported by the observations made in figures (5.1), (5.2) and (5.3).

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

We have constructed numerical methods by utilizing the standard finite difference method (FDM) on piecewise uniform Bakhvalov-Shishkin meshes. Our investigation focused on a singularly perturbed problem that includes non-local boundary conditions, which we solved using the finite difference method and composite Simpson's rule for integral boundary. The solution to the equation under consideration displays a boundary layer with a width of $O(\varepsilon)$. For spatial discretization, we employed the developed numerical methods based on the standard FDM and Bakhvalov-Shishkin meshes. We thoroughly examined and discussed the stability of these methods. Our analysis demonstrated that the developed scheme exhibits uniform convergence in the discrete maximum norm with respect to the perturbation parameter ε , and the convergence is of first order. To assess the applicability of the methods, we considered several test examples. The results revealed that the developed schemes uniformly converge with a linear order of convergence, providing accurate, stable, and reliable numerical outcomes. Notably, the proposed scheme successfully resolves boundary layers. In conclusion, we can confidently state that this method shows promise for tackling more complex nonlinear singularly perturbed problems.



Recommendations

This study primarily focused on utilizing the Bakhvalov-Shishkin Mesh in combination with the finite difference method to address the treatment of non-local boundaries while solving linear second-order singularly perturbed convection diffusion equations on a layer-adapted mesh. The Bakhvalov-Shishkin mesh, when applied to the finite difference method, proves highly effective in generating boundary layer resolving result and stable numerical solutions for singular perturbation problems. Consequently, we propose that future researchers explore the extension of the Bakhvalov-Shishkin mesh, together with the corresponding quadrature rule, to solve both linear and nonlinear singularly perturbed problems with delay, thereby obtaining numerical solutions. Additionally, the use of the Bakhvalov-Shishkin mesh in conjunction with the fitted finite difference method should also be considered for further investigation.

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