ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

NUMERICAL SOLUTION OF BURGER-FISHER EQUATION USING VARIATIONAL ITERATION METHOD (VIM)

A thesis submitted in partial fulfillment of the Requirements for the Degree of Master's in Applied Mathematics (Numerical Analysis)

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ID, No. GSU/0431/06

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Advisor; Lemi Guta (Dr)

Co-Advisor Yadeta Chimdessa

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ABREVIATION AND ACRONYMS

VIM	Variational Iteration Method
DE	Differential Equation
PDE	Partial Differential Equation
NLPDE	Non Linear Partial Differential Equation
ADM	Adomian Decomposition Method
ODE	Ordinary Differential Equation
HPM	Homotopy Perturbation Method
MADM	Modified Adomian Decomposition Method
MVIM	Modified Variational Iteratiom Method
BFE	Burger-Fisher equation

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ACKNOWLEDGEMENTS

First and foremost I would like to give my thanks to the Lord who gave me the chance to live and have all such experience in learning up to this level of I didn't think of. Secondly, my heartfelt gratitude goes to my advisor Dr. Lemi Guta and my co-advisor Ato Yadeta Chimdessa for their constructive comments and friendly approaches all through the work. Next I would like thank my wife Mrs. Rehima Ahmed and my class mates for their constant support in sharing ideas. Finally yet importantly, my sincere thanks go to my family and everyone who helped me directly or indirectly and gave me supports

Abstract

The Burger-Fisher equations occur in various areas of applied sciences and physical applications, such as modeling of gas dynamics, financial Mathematics and fluid mechanics. This equation (BFE) is a non-linear partial differential equation. In this thesis the variational iteration method (VIM) has been used to obtain the solutions of Burger Fisher Equations (BFE). This method (VIM) is based on General Lagrange multiplier, restricted variation and correction functional which are the main concepts of VIM. Using this method creates a sequence which tends to the exact solution of the problem. The advantage of VIM so that, it does not require linearization, a small parameter and Adomian polynomial in an equation as perturbation technique needs. The VIM is used to solve effectively, and easily a large class of non-linear problems with approximations which converge rapidly to accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. Two numerical examples present in this thesis by using tables and graphs suggest that this method is a powerful series approach to find numerical solutions that are in good agreement with the exact solution.

CHAPTER ONE

1. INTRODUCTON

1.1 Back Ground of the study

The Variational iteration method (VIM) was first developed by Chinese mathematician Ji-Huan He, professor at Donghua University. The Variational iteration method (VIM) was initially proposed toward the end of the most recent century/in later 1990s and completely grew in 2006 and 2007.

The method VIM is used to solve effectively, and easily, a large class of non-linear problems with approximations, which converge rapidly to the accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified.

This method (VIM) introduces a reliable and efficient process for a wide variety of scientific and engineering applications like linear or non-linear, homogeneous or inhomogeneous, equations and systems of equations, etc. Also this method is more powerful than other existing techniques, such as the Adomian decomposition method (ADM). The method (VIM) gives rapid convergent successive approximations of the exact solution if such a solution exists, otherwise a few approximations can be used for numerical purpose. The existing numerical techniques suffer from the restrictive assumptions that are used to handle non-linear terms. The VIM has no specific requirements, such as linearization, small parameters, Adomian polynomials etc. for non-linear operations. Another important advantage of VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of numerical solution. Moreover, the power of the method gives it a wide applicability in handling a huge number of analytical and numerical applications in real life problems.

Since most non-linear differential equations do not have exact solutions, approximate and numerical techniques, therefore, are used extensively. The Variational Iteration Method (VIM) is relatively new approache to provide approximate solutions to linear and non-linear problems. The Variational Iteration Method (VIM) was successfully applied to find the solution of several classes of variational problems.

The VIM is an iterative method based on the use of Lagrange multiplier, restricted variations and correction functional which has found a wide application for the solution of non-linear Ordinary and Partial differential equations. In this method (VIM), it required first determining the Lagrange multiplier λ optimally; then the successive approximation $u_{n+1}, n \geq 0$ of the solution of the correction functional will be readily obtained up on using the determined Lagrange multiplier λ and the first approximation. Consequently the solution is given by $u = \lim_{n\to\infty} u_n$. This method does not require the presence of small parameters in the DE, and provides the solution (an approximation to it) as a sequence of iterates. The method does not require that the non-linearity be differentiable with respect to the dependent variable and its derivatives. Being different from the other non-linear analytical methods, such as perturbation methods, this method (VIM) does not depend on small parameters and initial gauss that can easily chosen even also with unknown parameters. So, it gives a wide application in non-linear problems without linearization or small perturbations. Therefore, it reduces the computational size and number of iterations.

Differential equations are commonly used for mathematical modeling in science and engineering. Many problems of mathematical physics can be started in the form of differential equations. These equations are also occurring as reformulations of other mathematical problems such as ordinary differential equations (linear or non-linear) and partial differential equations (linear or non-linear). In most real life situations the differential equations that model the problem is too complicated to solve exactly and one of two approaches is taken to approximate the solution. The first approach is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, which we will examine in this thesis, uses methods for approximating the solution of original problems. This is the approach that is most commonly taken since the approximation methods give more accurate results and realistic error information than the first approach. Numerical methods are generally used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Even though there are many analytical methods for finding the solution of differential equations, but only a limited number of differential equations can be solved analytically. Even then there exist a large number of differential equations whose solutions cannot be obtained in closed form by using well known analytical methods, where we have to use the numerical methods to get the approximate solution of differential equations under the prescribed initial condition or conditions.

The non-linear Partial differential equation (NLPDE) have important applications in various fields of financial mathematics, gas dynamic, traffic flow, applied mathematics and physics applications. Non-linear Partial differential equations (NLPDE) are equations that involve rate of changes with respect to two or more independent variables usually representing time. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect, and diffusion transport. The Burger's-Fisher equations (BFEs) are among those non-linear partial differential equations (NLPDE). The Burger-Fisher equation is a mixed hyperbolic and parabolic non-linear partial differential equation (NLPDE). The Burger-Fisher equation (BFE) discovers are (Johannes Matrinus Burgers, 1895-1981) and (Ronald Aylmen Fisher, 1890-1962). First time Sir. Ronald Fisher wrote the Burger-Fisher equation in 1937. The Fisher equation deliberated the Mathematical model in population Biology. (Zhang et al. 2012) derived the solitary wave solution for the generalized Burger equation. (Abdul-Majid Wazwaz, 2005) applied tanh method for the Burger's Fisher equation and found verity of travelling wave solutions. (Guo-cheng Wu, 2009) proposed solitary solutions and periodic solutions of Burger-Fisher equation. (Hassan Ismael, 2004) developed approximate solution of Burger-Fisher equation by using ADM. (H. Chen and H. Zhang, 2004) developed generalized tanh function method for constructing exact travelling wave solution for Burger-Fisher equation. Since there does not exist general technique for finding analytical solution of those non-linear diffusion equations so far numerical solutions of non-linear equations are of great importance in physical problem. Then recently, the non-linear equations have attracted much attention of researchers. Therefore there are various powerful types of practical numerical methods for solving differential equations (Ordinary Differential Equations and Partial Differential Equations), including VIM, the homotopic mapping method, the tanh method, and the other methods have been proposed to obtain numerical (approximate) solutions for the non-linear equations. In this thesis we present Variational Iteration Method to solve the generalized Burger-Fisher equations of the form.

$u_t + \alpha u^\delta u_x - D u_{xx} = \beta u\left(1 - u^\delta\right), 0 \leq x \leq 1, t \geq 0.$ --------------------------------- (1.1)

Where $\alpha,\ \beta,\ D\ and\ \delta$ are parameters/positive constants.

With the initial conditions $u(x,0) = (\frac{1}{2} + \frac{1}{2}\tanh(\frac{-\alpha\delta}{2(\delta+1)}x))^{\frac{1}{\delta}} = f(x)$ ------------------ (1.2)

Boundary conditions $u(0,t) = (\frac{1}{2} + \frac{1}{2}\tanh[\frac{-\alpha\delta}{2(\delta+1)}(\frac{\alpha}{\delta+1} + \frac{\beta(\delta+1)}{\alpha})t])^{\frac{1}{\delta}},\ t \geq 0$ and

$$u(1,t) = (\frac{1}{2} + \frac{1}{2}tanh[\frac{-\alpha\delta}{2(\delta+1)}(1 - \left(\frac{\alpha}{\delta+1} + \frac{\beta(\delta+1)}{\alpha}\right)t)])^{\frac{1}{\delta}}, t \geq 0 \text{ -(1.3)}$$

The exact solution is $u(x,t) = (\frac{1}{2} + \frac{1}{2}\tanh(\theta_1(x - \theta_2 t)))^{\frac{1}{\delta}}, 0 \leq x \leq 1, t \geq 0$ --------(1.4)

Where $\theta_1 = \frac{-\alpha\delta}{2(1+\delta)}$ and $\theta_2 = \frac{\alpha}{1+\delta} + \frac{\beta(1+\delta)}{\alpha}$

If $\delta = \alpha = D = \beta = 1$ equation (1.1) reduces to the Burger-Fisher equation. It is a non-linear partial differential equation of the form,

$u_t + u u_x - u_{xx} = u(1-u)$-- (1.5)

From the literature review we may realize that several works in numerical solutions of Burger-Fisher equations using Variational iteration method have been carried out. Many authors have attempted to solve Burger-Fisher equation to obtain high accuracy rapidly by using numerous methods, such as Variational iteration method (VIM) and some other methods.

(A. A. Hemeda, 2008) studied VIM for solving wave equation. (Abdul-Majid wazwaz, 2007), discussed a comparison between the VIM and ADM. (Dragan escu GF, 2006) studied Application of a Variational Iteration Method to linear and non-linear viscoelastic models with fractional derivatives. (J. H. He, 2003) studied generalized variational principles in fluids. (Ji-Huan He, Guo-Cheng, Wu, F. Austin, 2010) studied the Variational Iteration Method which should be followed. (Ji-Huan, He, 2007) studied

Variational Iteration Method-some recent results and a new interpretation. (Ji-Huan, He, 2007) studied VIM a kind of non-linear analytical technique. (J. H. He Wu XH, 2007) studied Variational Iteration Method: new development and applications. (M. A. Abdou, A. A. Soliman, 2005) discussed new applications of Variational Iteration Method. (M. A. Abdou, A. A. Soliman, 2005) studied Variational Iteration Method for solving Burger's and coupled Burger's equations. (Odibat ZM, Momani S, 2006) discussed Application of the VIM to non-linear differential equations of fractional order. (Mokhtari R, 2008) studied VIM for solving non-linear difference differential equations. (M. Matinfar and M. Ghanbari, 2009) discussed solving the Fisher's equation by means of VIM. (Tame A. Abassy, Magdy AEI-Jawil, H. EI Zoheiry, 2007) discussed a modified variational iteration method (MVIM).

1.2 Statement of the Problem

The Burger-Fisher equation has important applications in various fields of financial mathematics, gas dynamic, traffic flow, applied mathematics and physics applications. This equation shows a prototypical model for describing the interaction between the reaction mechanism, convection effect and diffusion transport. There are several numerical methods to solve Burger-Fisher equation. Such as Adomian Decomposition Method (ADM), Modified Adomian Decomposition Method (MADM), Variational Iteration Method (VIM), Modified Variational Iteration Method (MVIM), Modified Homotopy perturbation Method (MHPM), Homotopy Analysis Method (HAM) and etc. Among those methods VIM is chosen by the investigator because the method has relatively less error and more convergent than the other numerical methods. Then the thesis will be answering the following questions.

- How can we determine the approximate solution?
- How can we describe the VIM?
- How can we apply VIM to find approximate solution of Burger-Fisher equation?
- How can we choose the initial approximation u_0?
- How can we determine the Lagrange multiplier λ?

1.3. Objectives of the Study

1.3.1. General Objective

The general objective of this thesis is to find the numerical solution of Burger-Fisher equation using Variational iteration method**.**

1.3.2. Specific Objective

Specific objectives of this study are:

- ✓ To describe the VIM.
- ✓ To apply VIM for approximate solution of Burger-Fisher equation.

- ✓ To determine the unknown function of VIM.
- ✓ To choice the initial approximation u_0.
- ✓ To determine the Lagrange multiplier λ.

1.4. Significant of the Study

Variational iteration method (VIM) for solving Burger-Fisher equation (BFE) is an interesting area of research with abundant real application. There are many works about the Variational iteration method for solving Burger-Fisher equation of numerical analysis. The researcher hopes that the result obtained in this study will contribute to world knowledge.

1.5. Delimitation/Scope of the Study

This thesis is mainly delimited to apply Variational iteration method (VIM) for solving Burger-Fisher equation (BFE). The thesis has been done under numerical analysis stream in mathematics department in 2017 at Adama Science and Technology University.

1.6. Limitation of the study

The limitation of this study was lack of adequate literature materials at work place. It was also difficult to get reference in our surrounding. The following are other limitations of the study.

- ➢ Time constraint/shortage of time.
- ➢ Shortage of materials.
- ➢ Shortage of sufficient budget.
- ➢ Lack of internet services at the work place.
- ➢ Shortage of net-work to communicate my advisor to get a constructive idea
- ➢ Shortage of reference to write review of related literature on the case.

1.7. Organization of the Study

This thesis consist the following sequences. i.e. it was organized in to five chapters, the first chapter contains the introduction part: it deals with the statement of the problem, objective of the study, significance of the study, delimitation of the study, and organization of the study, chapter-2 gives a brief review of related literature, in chapter-3 the methods and procedures of the study is described, chapter-4 presents numerical results and discussions, chapter-5 deals on Conclusion and Recommendation and at the end References are included.

CHAPTER TWO

2. REVIEW OF RELATED LITERATURE

In this chapter the researcher tries to present some related literatures from different sources such as published journals and Books. A review of related literature is a classification and evaluation of what different scholars and researchers have written on a topic.

The Burger's Fisher equation is a mixed hyperbolic-parabolic non-linear partial differential equation (NLPDE).The Burger's-Fisher equation discovers are (Johannes Matrinus Burgers, 1895-1981) and (Ronald Aylmen Fisher, 1890-1962). Since there does not exist general technique for finding analytical solution of non-linear diffusion equations so far numerical solutions of non-linear equations are of great importance in physical problem. Therefore different researchers used various powerful mathematical methods, including Variational iteration method (VIM) to find the numerical (approximate) solution of Burger-Fisher equation (BFE).

From the review of different authors work in the area the researcher realize that several works in numerical solutions of Burger's-Fisher equations using Variational iteration method have been carried out. Many authors have attempted to solve Berger Fisher Equation by using different numerical methods, such as Variational Iteration Method, Adomian Decomposition Method, and some other iterative methods

(A. A. Hemeda, 2008) studied on Variational Iteration Method for solving wave equation, (Abdul-Majid wazwaz, 2007), discussed on a comparison between the VIM and ADM. (Dragan escu GF, 2006) studied on Application of a Variational Iteration Method to linear and non-linear viscoelastic models with fractional derivatives. (J. H. He, 2003) studied on generalized variational principles in fluids. (Ji-Huan He, Guo-Cheng, Wu, F. Austin, 2010) studied on the Variational Iteration Method which should be followed. (Ji-Huan, He, 2007) studied on Variational Iteration Method-some recent results and a new interpretation. (Ji-Huan, He, 2007) studied on VIM a kind of non-linear analytical technique. (J. H. He Wu XH, 2007) studied on Variational Iteration Method: new development and applications. (M. A. Abdou, A. A. Soliman, 2005) discussed on new applications of Variational Iteration Method. (M. A. Abdou, A. A. Soliman, 2005) studied on Variational Iteration Method for solving Burger's and coupled Burger's equations. (Odibat ZM, Momani S, 2006) discussed on Application of the VIM to non-linear differential equations of fractional order. (Mokhtari R, 2008) studied on VIM for solving non-linear difference differential equations. (M. Matinfar and M. Ghanbari, 2009) discussed on solving the Fisher's equation by means of VIM. (Tame A. Abassy, Magdy AEI-Jawil, H. EI Zoheiry, 2007) discussed on a modified VIM. (Zhang et al. 2012) derived the solitary wave solution for the generalized Burger equation. (Abdul-Majid Wazwaz, 2005) applied tanh method for the Burger's Fisher equation and found verity of travelling wave solutions. (Guo-cheng Wu. 2009) proposed solitary solutions and periodic solutions of Burger-Fisher equation. (Hassan Ismael, 2004) developed approximate solution of Burger-Fisher equation by using ADM. (H. Chen and H. Zhang, 2004) developed generalized tanh function method for constructing exact travelling wave solution for Burger-Fisher equation.

From those related literatures results the researcher have obtained that the method VIM is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations, which converge rapidly to the accurate solutions than the other methods. For linear problems its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. And Variational Iteration Method (VIM) is also more powerful than other existing techniques, such as the Adomian Decomposition Method (ADM), Homotopy-Perturbation Method (HPM) etc. And also has no specific requirements such as linearization, small parameters, Adomian polynomials etc.

CHAPTER THREE

3. METHODS AND PROCEDURES

In this chapter the researcher mainly focus on the study design and study procedures

3.1. Study Design

The design of this study is to find numerical (approximation) solution of Burger's-Fisher equation by using Variational Iteration Method (VIM). Then the researcher uses different schemes such as General Lagrange multiplier, restricted variation and correction functional which are the main concepts of Variational Iteration Method.

3.2. Study procedure

For this study, the researcher will follow the following procedures.

- Construct a correction functional.
- Make correction functional stationary with respect to u_n.
- By Using integration by part and Making correction functional stationary determine the Lagrange multiplier λ.
- Using the determined Lagrange multiplier λ and the first approximation u_0 the solution u of the correction functional will be obtained.
- Use tables and graphs to compare the exact and approximate solution by the help of MATLAB.
- Summarize the result and conclusion.
- Finally the approximate solution u is obtained by $u = \lim_{n \to \infty} u_n$

CHAPTER FOUR

4. RESULTS AND DISCUTION

4.1. RESULTS

4.1.1 Description of the VIM

To illustrate the basic concepts of VIM and obtain the approximate solution of Burge's-Fisher Equation (BFE) by means of VIM, we consider the following non-linear Differential Equation (DE).

$Lu + Nu = g(x)$--- (4.1)

Where L is a linear operator, N is a non-linear operator and $g(x)$ is an inhomogeneous term; Lu_n is a linear term, Nu_n is a non-linear term. According to the Variational Iteration Method (VIM) first we can construct a correction functional as follows:

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{Lu_n + N\tilde{u}_n - g(x)dt,\ t \geq 0$------------------------------ (4.2)

Where t and x are time and space respectively, λ is a general Lagrange multiplier, which can be identified optimally via integration by part, the subscript n denotes the n^{th} order approximation; u_0 is an initial approximation which can be known according to the initial conditions or the boundary conditions, the second term on the right is called correction and $\tilde{u}_n$ is considered as a restricted variation. $i.e\ \tilde{u}_n\delta = 0$.

So, first we will determine the Lagrange multiplier λ. The successive approximations, $u_{n+1}(x,t),\ n \geq 0$. of the solution $u(x,t)$ will be readily obtained using the obtained λ and by using any selective function u_0. Consequently, the approximate solution u will be given by $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. According to iterations of this sequence; we can determine approximations of the solution.

4.1.3. Application of VIM to Burger-Fisher equation

In this section, we will present examples to illustrate the efficiency and accuracy of the variational iteration method (VIM) for solving non-linear Burger's-Fisher equation (BFE).

Numerical examples

Consider the Generalized Burger-Fisher Equation of the form

$u_t + \alpha u^{\delta} u_x - D u_{xx} = \beta u\left(1 - u^{\delta}\right), 0 \le x \le 1, t \ge 0.$ ----------------------------------(4.3)

Where $\alpha,\ \beta,\ D\ and\ \delta$ are non-negative parameters/positive constants,

With the initial conditions $u(x,0) = (\frac{1}{2} + \frac{1}{2}\tanh(\frac{-\alpha\delta}{2(\delta+1)}x))^{\frac{1}{\delta}} = f(x)$ ------------------ (4.4)

Boundary conditions $u(0,t) = (\frac{1}{2} + \frac{1}{2}\tanh[\frac{-\alpha\delta}{2(\delta+1)}(\frac{\alpha}{\delta+1} + \frac{\beta(\delta+1)}{\alpha})t])^{\frac{1}{\delta}},\ t \ge 0$ and

$$u(1,t) = (\frac{1}{2} + \frac{1}{2}tanh[\frac{-\alpha\delta}{2(\delta+1)}(1 - \left(\frac{\alpha}{\delta+1} + \frac{\beta(\delta+1)}{\alpha}\right)t)])^{\frac{1}{\delta}}, t \ge 0 \text{ -(4.5)}$$

The exact solution is $u(x,t) = (\frac{1}{2} + \frac{1}{2}\tanh(\theta_1(x - \theta_2 t)))^{\frac{1}{\delta}}, 0 \le x \le 1, t \ge 0$ ------- (4.6)

Where $\theta_1 = \frac{-\alpha\delta}{2(1+\delta)}$ and $\theta_2 = \frac{\alpha}{1+\delta} + \frac{\beta(1+\delta)}{\alpha}$

If $\delta = \alpha = D = \beta = 1$, equation (4.3) reduces to the Burger-Fisher equation of the form,

$u_t + u u_x - u_{xx} = u(1-u)$-- (4.7)

If $\alpha = 0,\ \delta = D = \beta = 1$, equation (4.3) is reduces to the Fisher Equation of the form,

$u_t - u_{xx} = u(1-u)$--- (4.8)

If $D = 0$, equation (4.3) is reduces to the diffusion less Burger equation with non-linear reaction, of the form, $u_t + u u_x - u_{xx} = u(1-u)$-- (4.9)

If $\alpha = D = 0$, equation (4.3) reduces to logistic Ordinary Differential Equation (ODE) of the form,

$u_t = u(1-u)$-- (4.10)

If $\beta = D = 0$, equation (4.3) reduces to the diffusion less Burger equation of the form,

$u_t + u u_x = 0$-- (4.11)

If $\delta = \alpha = D = 1,\ \beta = 0$, equation (4.3) is reduced to Burger Equation of the form,

$u_t + u u_x - u_{xx} = 0$------------------------------ --(4.12)

If $\alpha = \beta = 0,\ D = \delta = 1$, equation (4.3) is reduced to Heat Equation (linear diffusion equation) of the form, $u_t = u_{xx}$--- (4.13)

To solve equation (4.3) by means of the Variational Iteration Method (VIM), we substitute it in equation (4.2) by;

$Lu_n = \frac{\partial u_n}{\partial t}$, $R(u_n) = -\frac{\partial^2 u_n}{\partial x^2}$, $Nu_n = \alpha u^{\delta}{}_n \frac{\partial u}{\partial x} - \beta u(1-u^{\delta})$. To get

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[\frac{\partial u}{dt} - \frac{\partial^2 u}{\partial x^2} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \beta u\left(1-u^{\delta}\right)\right] dt$----------------- (4.14)

Let $N_n(u) = \alpha u^{\delta}{}_n \frac{\partial u}{\partial x} - \beta u_n(1-u^{\delta})$ is a non-linear term.

$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left[\frac{\partial u_n}{ds}(x,s) - \frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,s) + N_n(\tilde{u}_n(x,s))\right] ds$ -------(4.15)

Where $\tilde{u}(x,s)$ is considered as a restricted variation. ($i.e\ \ \delta\tilde{u}_{\mathrm{n}}(x,s) = 0$).

The variation of equation (4.15) is;

$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s) \left[\frac{\partial u_n}{ds}(x,s) - \frac{\partial^2 \tilde{u}_n}{\partial x^2}(x,s) + N_n(\tilde{u}_n(x,s))\right] ds$

By using integration by parts we have

$\delta u_{n+1} = \delta u_n + \delta \Big|_{s=t} \lambda(s) u_n(x,s) - \delta \int_0^t \lambda'(s) u_n(x,s) ds - 0 + 0 = 0$..

We obtain $\delta u_{n+1} = 1 + \lambda(s)\Big|_{s=t} \delta u_n - \delta \int_0^t \lambda'(s) u_n ds$.

This yields the stationary conditions

$1 + \lambda(t) = 0$ and

$\lambda'(t) = 0, \ \Longrightarrow \ \lambda(t) = -1$.-- (4.16)

From this equation we have $\lambda = -1$, substituting the value of λ in eq. (4.14), we obtain;

$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial u}{dt} - \frac{\partial^2 u}{dx^2} + \alpha u^{\delta} \frac{\partial u}{dx} - \beta u(1-u^{\delta}))\, dt$ -------------------- (4.17)

Thus we obtain the numerical solution (an approximate solution) for $u(x,t)$, considering boundary conditions and insert $u\,(x,t) \cong u_n(x,t)$. The solution of this equation (the Generalized Burger Fisher Equation) can be found by using MATLAB. Then we have the following successive approximations;

$u_0 = \frac{1}{2} + \frac{1}{2}\tanh(\frac{-\alpha}{4})x$, Initial condition (If t=0).

$u_1 = \quad \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x + \frac{1}{2\alpha}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x \tanh\left(\frac{-\alpha}{4}\right)t + \beta(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt$, (If n=0).

$u_2 =$

$\frac{1}{2}+\frac{1}{2}tanh\left(\frac{-\alpha}{4}\right)x+\frac{1}{2\alpha}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))\,x\tanh\left(\frac{-\alpha}{4}\right)t+\beta\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x\right)\left(\frac{1}{2}+\right.$
$\left.\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt+\ \frac{\alpha}{4}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x\tanh\left(\frac{-\alpha}{4}\right)+\beta(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2}+$
$\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt-\frac{1}{4}\beta\tanh(\frac{-\alpha}{4})^2t^2+\frac{1}{3}(\frac{-\alpha}{4})(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt$, (If n=1).

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u_n for n-iterations, - - -

Special cases

Case-I

For computational work, we have taken ($\alpha = D = \delta = 1$ and $\beta = 0$) in the generalized Burger-Fisher equation, it reduced to Burger equation which is non-linear partial differential equation (PDE) of the form, $u_t + uu_x - u_{xx} = 0$. We can find approximate solution, and make comparison between the exact solution and an approximate solution. In this case we can obtain the following successive approximation

$u_0 = \frac{1}{2}+\frac{1}{2}tanh(\frac{-x}{4})$, Initial condition (If t=0).

$u_1 = \frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)+\ \frac{1}{2}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-t}{4}\right))$, (If n=0).

$u_2 = \frac{1}{2}+\frac{1}{2}tanh\left(\frac{-x}{4}\right)+\frac{1}{2}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-t}{4}\right)+\frac{1}{4}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-1}{4}\right)+$
$\frac{1}{3}(\frac{-1}{4})(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right))t$, (If n=1).

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u_n for n-iterations, - - -

In this case the exact solution is $u(x,t) = \frac{1}{2}+\frac{1}{2}\tanh(\frac{t}{8}-\frac{x}{4})$.

Table (1)

Results of VIM and their errors for different values of x and t assuming ($\alpha = \delta = D = 1$ and $\beta = 0$) by using MATLAB.

x	t	Exact solution	Approx. VIM	Absolute error
0.1	0.0001	0.4875	0.4576	0.0299
	0.0005	0.4875	0.4576	0.0299
	0.001	0.4876	0.4576	0.0300
	0.005	0.4878	0.4571	0.0307
	0.01	0.4881	0.4566	0.0315
	0.05	0.4906	0.4526	0.0380
	0.1	0.4938	0.4475	0.0463
	0.5	0.4906	0.4070	0.0836
0.5	0.0001	0.4378	0.4110	0.0268
	0.0005	0.4379	0.4110	0.0269
	0.001	0.4379	0.4109	0.0270
	0.005	0.4381	0.4106	0.0275
	0.01	0.4384	0.4101	0.0283
	0.05	0.4409	0.3656	0.0753
	0.1	0.4440	0.4019	0.0421
	0.5	0.4688	0.3656	0.1032
0.9	0.0001	0.3894	0.3655	0.0239
	0.0005	0.3894	0.3655	0.0239
	0.001	0.3894	0.3654	0.0240
	0.005	0.3897	0.3651	0.0246
	0.01	0.3900	0.3647	0.0253
	0.05	0.3923	0.3254	0.0669
	0.1	0.3953	0.3574	0.0379
	0.5	0.4195	0.3851	0.0344

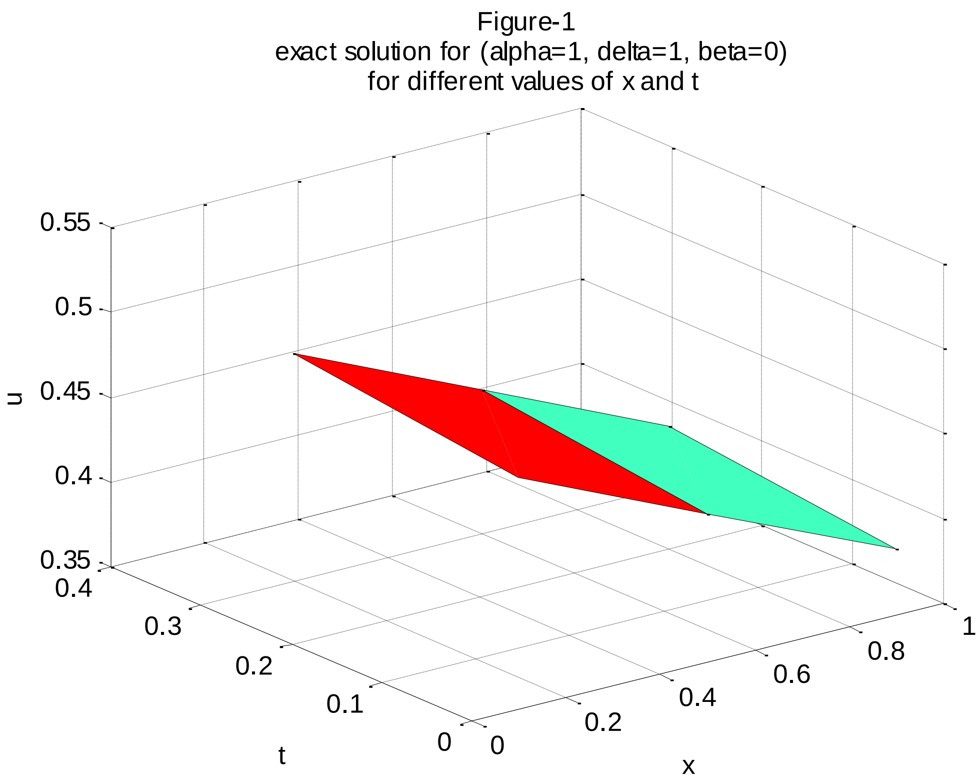
Figure-1
exact solution for (alpha=1, delta=1, beta=0)
for different values of x and t
u
0.55
0.5
0.45
0.4
0.35
t
0.4
0.3
0.2
0.1
0
x
0
0.2
0.4
0.6
0.8
1

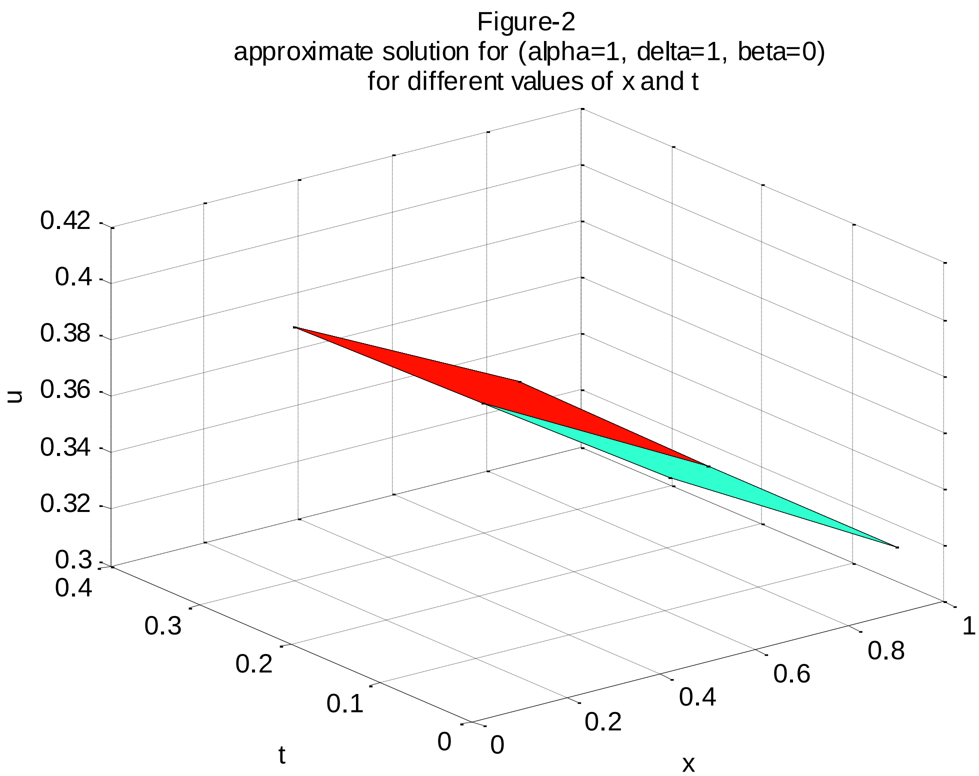
Figure-2
approximate solution for (alpha=1, delta=1, beta=0)
for different values of x and t
u
0.42
0.4
0.38
0.36
0.34
0.32
0.3
t
0.4
0.3
0.2
0.1
0
x
0
0.2
0.4
0.6
0.8
1

Case-II

For computational work, we have taken ($\alpha = 0.001,\ \delta = D = 1\ and\ \beta = 0.1$) in the generalized Burger-Fisher equation, it reduced to the Burger-Fisher equation which is a non-linear partial differential equation (NLPDE) of the form, $u_t + (0.001)uu_x - u_{xx} = 0.1u(1-u)$. We can find an approximate solution and make comparison between the exact and approximate solution. In this case we can obtain the following successive approximation

$u_0 = \frac{1}{2} + \frac{1}{2}\tanh(\frac{-\alpha}{4})x$, initial condition (if t=0).

$$u_1 = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x + \frac{1}{2\alpha}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x\tanh\left(\frac{-\alpha}{4}\right)t + \beta(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt,$$ (if n=0).

$$u_2 = \frac{1}{2} + \frac{1}{2}tanh\left(\frac{-\alpha}{4}\right)x + \frac{1}{2\alpha}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))\,x\tanh\left(\frac{-\alpha}{4}\right)t + \beta\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x\right)\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt + \frac{\alpha}{4}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x\tanh\left(\frac{-\alpha}{4}\right) + \beta(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt - \frac{1}{4}\beta\tanh(\frac{-\alpha}{4})^2t^2 + \frac{1}{3}(\frac{-\alpha}{4})(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt,$$ (if n=1).

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u_n for n-iterations, - - -

In this case the exact solution is $u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{400001}{8000000}t - \frac{x}{4000})$

Table (2)

Results of VIM and their errors for different values of x and t assuming ($\alpha = 0.001, \beta = 0.1, \delta = D = 1$), by using MATLAB.

X	T	Exact solution	Approx. VIM	Absolute error
0.1	0.0001	0.5000	0.5250	0.0250
	0.0005	0.5001	0.5250	0.0249
	0.001	0.5002	0.5250	0.0248
	0.005	0.5012	0.5251	0.0239
	0.01	0.5025	0.5252	0.0227
	0.05	0.5125	0.5262	0.0137
	0.1	0.5250	0.5275	0.0125
	0.5	0.6224	0.5375	0.0849
0.5	0.0001	0.5000	0.5249	0.0249
	0.0005	0.5001	0.5249	0.0248
	0.001	0.5002	0.5250	0.0248
	0.005	0.5012	0.5251	0.0239
	0.01	0.5024	0.5252	0.0228
	0.05	0.5124	0.5262	0.0138
	0.1	0.5249	0.5274	0.0025
	0.5	0.6224	0.5374	0.0850
0.9	0.0001	0.4999	0.5249	0.0250
	0.0005	0.5000	0.5249	0.0249
	0.001	0.5001	0.5249	0.0248
	0.005	0.5001	0.5250	0.0249
	0.01	0.5024	0.5251	0.0227
	0.05	0.5124	0.5261	0.0137
	0.1	0.5249	0.5274	0.0025
	0.5	0.6224	0.5374	0.0850

Figure-3
exact solution for (alpha=0.001, beta=0.1, delta=1)
for different values of x and t

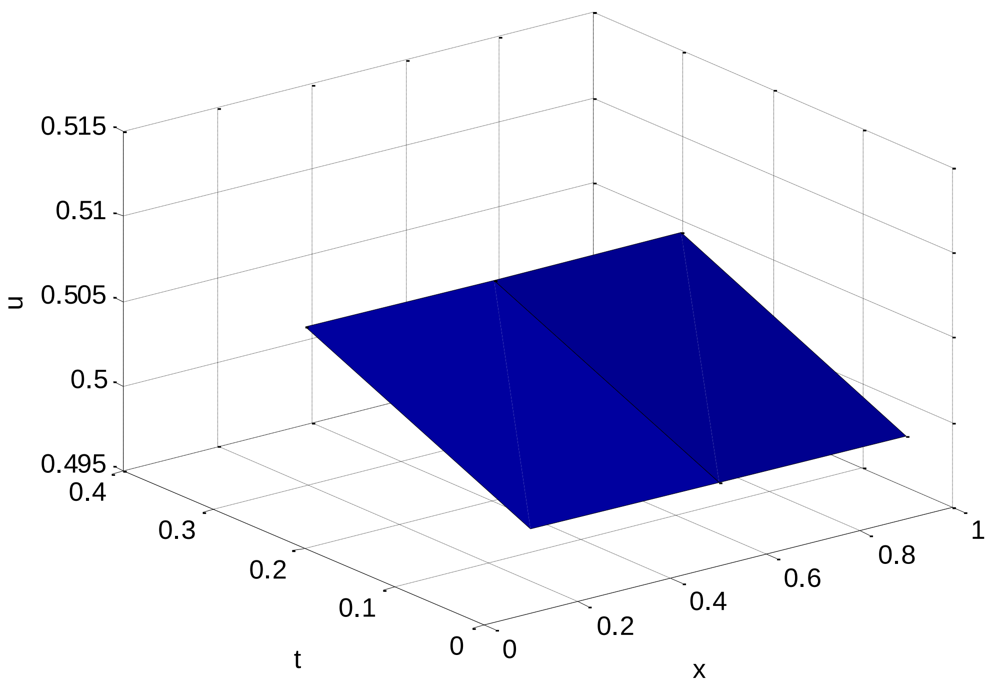
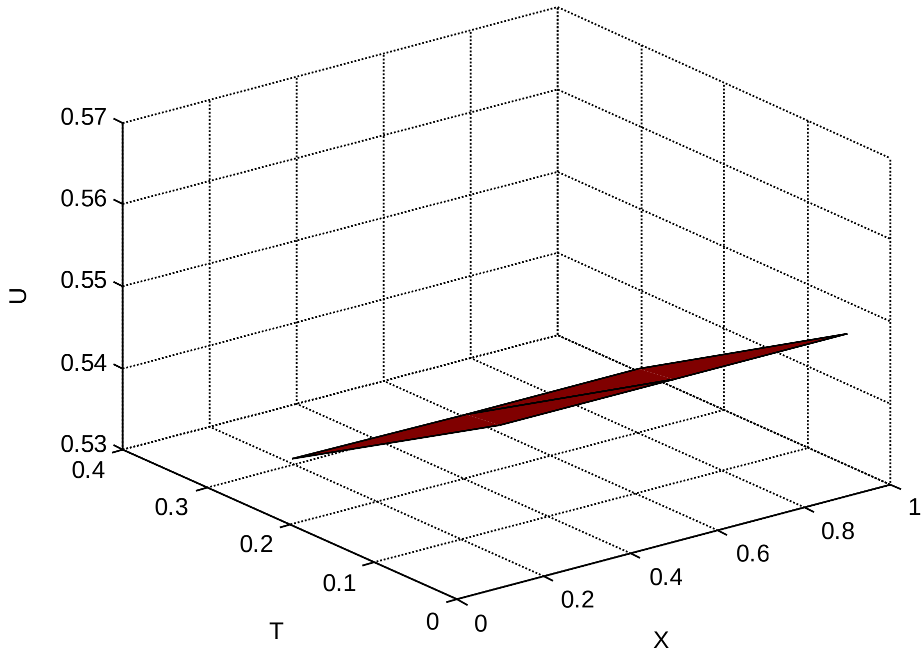


Figure-4
Graph of approximate solution for (alpha=0.001, beta=0.1, delta=1)



4.2. DISCUSION

In this section we compare the exact solution and approximate solution for different values of x and t using two examples, two tables and four graphs up to three iterations of VIM by using the error formula:

$Error = |u_{exact} - u_{approx}|$

In the first example assuming that ($\alpha = \delta = D = 1, and\ \ \beta = 0$) in the generalized Burger-Fisher equation, it reduced to Burger equation which is non-linear partial differential equation of the form, $u_t + uu_x - u_{xx} = 0$. the error analysis is indicated in table-1 and the result shown in figure (1 and 2). Again in the second example assuming that ($\alpha = 0.001,\ \delta = D = 1$ & $\beta = 0.1$) in the generalized Burger-Fisher equation, it reduced to the Burger-Fisher equation which is a non-linear partial differential equation (NLPDE) of the form, $u_t + 0.001uu_x - u_{xx} = 0.1u(1-u)$. the error analysis is indicated in table-2 and the result is shown in figure (3 and 4).

As we can see from results in tables 1 and 2 and figures 1-4: the error is minimum, this indicates that Variational Iteration Method (VIM) is converged to the exact solution. From this we can conclude that this method is one of the best methods for solving Burger-Fisher equation (BFE) even using only a few iterations. This indicates that as the number of iteration increases the error becomes minimizing and then the method is more converges to the exact solution.

Therefore from this information we can conclude that the variational iteration method (VIM) is more efficient method for solving Burger-Fisher Equations.

CHAPTER FIVE

5. CONCLUSION AND RECOMMENDATION

5.1. CONCLUSION

In this thesis Variational Iteration Method (VIM) has been applied to find the solution of Burger-Fisher equation (BFE). The Burger-Fisher Equation (BFE) is the non-linear partial differential equation (NLPDE) and a mixture of parabolic hyperbolic second order equation. Variational iteration method is a powerful tool/method which is capable of handling linear/non-linear partial differential equations and therefore can widely apply in engineering. The Variational iteration method reduces the volume of calculations without requiring the Adomian polynomials, linearization and small parameters in equation as the perturbation technique do. The main part of the method (VIM) that is construction of correction functional which can be constructed by the Lagrange multiplier, and the Lagrange multiplier can optimally identified by variational theory. The application of restricted variation in correction functional makes it much easier to determine the Lagrange multiplier. In this method the initial approximation can be easily determined from initial conditions and boundary conditions. From results in table 1 and 2 and figure 1-4, we can conclude that this method is one of the best method for solving higher order non-linear problems even using only few successive approximations of iterations. The approximate solution obtained by the method is good agreement with the exact solution. Also we conclude that variational iteration method (VIM) is more efficient method in finding approximate solutions of Burger-Fisher equations (BFEs).

5.2. RECOMMENDATION

We recommended that for future work we will be solving the Burger Fisher Equation by using other numerical methods such as Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Modified Adomian Decomposition Method (MADM), and Modified Variational Iteration Method (MVIM) and make comparison between them

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Appendix-A

The MAT LAB codes to construct tables

For table-1

>>Syms x t

>>u_1=in line $(\frac{1}{2}+\frac{1}{2}\tanh(\frac{t}{8}-\frac{x}{4}))$, (For exact solution)

>>u_2=inline$(\frac{1}{2}+\frac{1}{2}tanh\left(\frac{-x}{4}\right)+\frac{1}{2}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-t}{4}\right)+\frac{1}{4}(\frac{1}{2}+$
$\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-1}{4}\right)+\frac{1}{3}\left(\frac{-1}{4}\right)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{4}\right)\right)t)$,

(For approximate solution)

For table-2

>>Syms x t

>>u_1=in line $(\frac{1}{2}+\frac{1}{2}\tanh(\frac{400001}{8000000}t-\frac{x}{4000}))$, (For exact solution)

>>u_2=inline$(\frac{1}{2}+\frac{1}{2}tanh\left(\frac{-\alpha}{4}\right)x+\frac{1}{2\alpha}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))\,x\tanh\left(\frac{-\alpha}{4}\right)t+\quad\beta\left(\frac{1}{2}+\right.$
$\left.\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x\right)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt+\frac{\alpha}{4}(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x\tanh\left(\frac{-\alpha}{4}\right)+$
$\beta(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt-\frac{1}{4}\beta\tanh(\frac{-\alpha}{4})^2t^2+$
$\frac{1}{3}\left(\frac{-\alpha}{4}\right)\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt)$. (For approximate solution)

Where $\alpha=0.001$, $\beta=0.1$, $\delta=1$.

Appendix-B

The MAT LAB codes to sketch graphs (figures)

Figure-1

>>x = (0.1:0.4:0.9)

>>t = (0.0001:0.24995:0.5)

>>[x, t] = mesh grid(x, t)

$\gg u = \frac{1}{2} + \frac{1}{2}\tanh(\frac{t}{8} - \frac{x}{4})$, (For exact solution)

>>Surf (x, t, u)

>>Insert: label-x, label-y, and label-z

>>Edit: safe figure

Figure-2

>>x = (0.1:0.4:0.9)

>>t = (0.0001:0.24995:0.5)

>>[x, t] = mesh grid(x, t)

>>$u = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{-x}{4}\right) + \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-t}{4}\right) + \frac{1}{4}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-x}{4}\right)\tanh\left(\frac{-1}{4}\right) + \frac{1}{3}\left(\frac{-1}{4}\right)\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-x}{4}\right)\right) t$, (For approximate solution)

>Surf (x, t, u)

>>Insert: label-x, label-y, and label-z

>>Edit: safe figure

Figure-3

>>x = (0.1:0.4:0.9)

>>t = (0.0001:0.24995:0.5)

>>[x, t] = mesh grid(x, t)

$\gg u = \frac{1}{2} + \frac{1}{2}\tanh(\frac{400001}{8000000}t - \frac{x}{4000}$, (For exact solution)

>>Surf (x, t, u)

>>Insert: label-x, label-y, and label-z

>>Edit: safe figure

Figure-4

>>x = (0.1:0.4:0.9)

>>t = (0.0001:0.24995:0.5)

>>[x, t] = mesh grid(x, t)

$$_{>>}u = \frac{1}{2} + \frac{1}{2}tanh\left(\frac{-\alpha}{4}\right)x + \frac{1}{2\alpha}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))\,x\tanh\left(\frac{-\alpha}{4}\right)t + \beta\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x\right)\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt + \frac{\alpha}{4}(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))x\tanh\left(\frac{-\alpha}{4}\right) + \beta(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)x)(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right))xt - \frac{1}{4}\beta\tanh(\frac{-\alpha}{4})^2t^2 + \frac{1}{3}\left(\frac{-\alpha}{4}\right)\left(\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{-\alpha}{4}\right)\right)xt).$$

(For approximate solution), Where $\alpha = 0.001$, $\beta = 0.1$, $\delta = 1$.

>>Surf (x, t, u)

>>Insert: label-x, label-y, and label-z

>>Edit: safe figure