NUMERICAL SOLUTION FOR SYSTEMS OF FRACTIONAL INITIAL VALUE PROBLEMS

Mathewos Gizaw Gibo

A Thesis Submitted to

The Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics (Numerical Analysis)

Office of Graduate Studies

Adama Science and Technology University

July, 2023

Adama, Ethiopia

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Declaration

I hereby declare that this Master Thesis entitled "Numerical solution for systems of fractional initial value problems" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation

Name of Student	Signature	Date

Recommendation of Advisors

We, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled "Numerical solution for systems of fractional initial value problems" prepared under our guidance by Mathewos Gizaw Gibo submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled "Numerical solution for systems of fractional initial value problems" and developed by Mathewos Gizaw Gibo, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

____________________	________________	______________
Major Advisor	Signature	Date
____________________	________________	______________
Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by Mathewos Gizaw Gibo have read and evaluated the thesis entitled "Numerical solution for systems of fractional initial value problems" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics (Numerical Analysis).

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Final approval and acceptance of the thesis is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate Council (DGC) and School Graduate Committee (SGC).

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List of Abbreviation

${}^{c}D^{\alpha}$	The generalized Caputo derivative
I^{α}	Riemann–Liouville integral
o(h)	order of h
FDES	fractional differential equations
FIVPS	fractional initial value problems
FFDM	fractional finite difference method
FC	fractional calculus
B-T	Bagley-Torvik equations
IVPS	Initial value problems

Symbols

β	Beta(order)
α	Alpha(order)
Γ	Gamma function
τ	Tau
Σ	Summation
!	factorial
θ	theta
Δ	Delta function
ρ	rho
$\mathbb{R}$	Real numbers
$\in$	an elements

Abstract

This thesis presents a numerical solution for systems of fractional initial value problems (SFIVPs. It play a crucial role in various scientific and engineering fields, offering more accurate models for real-world phenomena involving memory and hereditary effects. To address these challenges, the thesis employs the L_3 type Caputo method, a powerful numerical technique for solving SFIVPs. Additionally, the fractional finite difference method (FFDM) is utilized for discretizing the system. The L_3 type Caputo method is applied to find the numerical solutions for such SFIVPs. Lipschitz continuity is the main condition for Picard-Lindelöf's theorem that guarantees the existence and uniqueness of a system of fractional initial value problems depending on the given discretization. Moreover, we investigate the consistency, stability, and convergence properties of the proposed method. Stability analysis is performed to examine the behavior of the L_3 Caputo-type method. The stability properties are analyzed to ensure that the computed solution remains bounded. Consistency is demonstrated by proving that the numerical approximation error converges to zero as the step size approaches zero. Finally, some numerical examples are given to illustrate the applicability and usefulness of the approximate solution of the obtained results.

Keywords: Fractional initial value problem; Systems of fractional initial value problem; fractional finite difference method; fractional calculus; stability analysis.

CHAPTER 1

INTRODUCTION

1.1 Background of the study

Duan et al. (2016) the study of fractional-order derivatives and integrations is known as fractional calculus (FC). Differential calculus was invented independently by Isaac Newton and Gottfried Leibniz, and it was understood that the notion of the derivative of the n^{th} order, that is, applying the fractional operation n times in succession, was meaningful. In a 1695 letter, L'Hopital asked Leibniz about the possibility that n could be something other than an integer, such as n=$\frac{1}{2}$. Leibniz responded that it will lead to a paradox, from which "one day useful consequences will be drawn". Leibniz was correct, but it would not be centuries before it became clear just how correct he was. By employing fractional calculus rather than classical calculus, some of the significant physical events in nature have been modeled more precisely.

Batiha et al. (2023) the Fractional initial value problems can be found in fluid traffic, airfoil, modeling of earthquake nonlinear oscillation, finance, cancer chemotherapy, electrodynamics, the Zener model, chaos theory, diabetes, Poisson-Nernst-Planck diffusion, and hepatitis B disease model, tuberculosis, fractional COVID-19 model, pine wilt disease, hepatitis B virus, and other applications in various areas of research. Furthermore, fractional initial value problems can contribute significantly to modeling many nonlinear phenomena that can propagate in different plasma models, such as solitary waves, cnoidal waves, shock waves, and roguewaves, and so on.

The theory of fractional initial value problems, which are natural extensions of ordinary derivatives, has become very attractive to scientists because of its applications in a variety of fields. Fractional-order derivatives of a given function include the entire function history; the following state of a fractional order system is not only dependent on its current state but also on all its historical states. Several fractional initial value problem models, where non-locality plays a very important role in different areas, including physics, engineering, mechanics and dynamical systems, signal and image processing, Control theory, biology, environmental sciences, and materials have been introduced. A class of fractional initial value problems, such as Riemann-Liouville, Hadamard, and Caputo derivative, are defined using fractional integrals (Odibat & Baleanu, 2020).

In mathematics, fractional initial value problems have been studied for more than three centuries and have a long history. It had been regarded as one of the best mathematical methods for characterizing the memory qualities of complicated systems and specific materials, according to substantial theoretical and practical studies. Due to their significant applications in numerous scientific fields, such as fluid mechanics, biomathematics, ecology, visco-elastodynamics, aerodynamics, control theory, electro-dynamics of complex media, etc. the study of fractional initial value problems has become an active area of research (S. Zhang & Su, 2021).

It is a branch of mathematics that studies derivative properties and integrals of non-integer orders (called fractional derivatives and integrals), briefly differ-integrals. Fractional calculus allows the operators of integration to have fractional order in initial value problems. The history of fractional initial value problems started almost at the same time as classical calculus was established. Fractional initial value problems are specific parts of initial value problems that replace integer-order derivatives with fractional-order derivatives.

In general, ordinary differential equations are applied when describing dynamic phenomena. In various fields such as physics, biology, and chemistry. However, for some complex systems, the common simple initial value problems cannot provide satisfactory solutions. Therefore, to obtain better models, FIVPs are employed instead of integer-order ones. On the other hand, the FDEs are Analytical methods and the theoretical background of this problem make it difficult to solve. Fractional initial value problems can be viewed as one of the extensions of classical ordinary calculus. In reality, a physical phenomenon may depend not only on the present but also on the future of previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order. Hence, in recent years, mathematicians have discovered new methods of numerical solution. There are several methods to solve FIVPs, such as the A domain decomposition method, fractional iteration method, differential transformation method, fractional finite difference method, and the wavelet method. This subject is known as fractional calculus, and its birth happened in 1695, when L'Hospital asked Leibniz what could be the interpretation of $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}}$ (Sonar & Sonar, 2021).

However, in the past decades, this subject has proven its applicability in many and different natural situations, such as visco-elasticity, anomalous diffusion, stochastic processes, signal and image processing, fractional models, and control theory.

In recent years, the Taylor series has also been employed to solve FIVPs numerically with the Bagley-Torvik (B-T) equation. However, the B-T equation is an FDE with constant coefficients. It was first mentioned in Leibniz's letter to L'Hospital in 1695, where the idea of a semi-derivative was suggested (Ghanim & Fawzi Al-Janaby, 2021). During time Fractional initial value problems were built on formal foundations by many famous mathematicians, E.g Liouville, Grunwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel, etc. The fact that the differ-integral is an operator that includes both integer-order derivatives and integrals as special cases is the reason why, in the present fractional initial value problems, it becomes very popular, and many applications arise. The Caputo derivative and the Riemann-Liouville Integrals are well-known and widely applied to find solutions of fractional initial value problems (Fedorov et al., 2021).

1.2 Statement of the problem

Due to the lack of exact analytic solutions for most systems of fractional initial value problems, the utilization of numerical techniques and approximate solutions becomes crucial (Diethelm & Luchko, 2004). In this context, the L_3 (III) type Caputo method, a modification of the Riemann-Liouville definition, is highly effective in handling these problems. One notable advantage of this method is its ability to appropriately handle the initial value problem by incorporating the initial conditions regarding field variables and their non-integer orders. To address a specific class of linear systems of fractional initial value problems, we employ the L_3-type Caputo method (Tavares et al., 2016) in combination with the finite fractional difference method (FFDM) for discretization (Almeida et al., 2016). This approach facilitates the computation of numerical solutions for systems of initial value problems. Furthermore, the stability, consistency and convergence properties of the method are thoroughly investigated.

1.3 Objectives

1.3.1 General objective

The general objective of this study is to formulate numerical solutions for system of fractional initial value problems.

1.3.2 Specific objectives

The specific objective of this study are:

- To study the consistency analysis of the proposed scheme.
- To study the stability analysis.
- To validate the convergence analysis for the proposed scheme.

1.4 Significance of the study

The result obtained from this study may

- Used as reference material for scholars who work in this research area.
- Offer a numerical solution for systems of fractional order and a solution for system of fractional initial value problems.

1.5 Expected outcomes of the study

The expected outcomes of this study are the formulation of approximate solutions to fractional initial value problems. Finally, the results written in the form of a thesis for degree purposes and published in an indexed journal.

1.6 Delimitation of the study

The study is limited to solutions of system of fractional initial value problem of the form:

$$\begin{cases} D^{\alpha}x(t) = \text{ax(t)+y(t)+z(t)} & x(0) = x_0 \\ D^{\alpha}y(t) = \text{x(t)+by(t)+z(t)} & y(0) = y_0 \\ D^{\alpha}z(t) = \text{x(t)+y(t) +cz(t)} & z(0) = z_0 \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ and a, b, c are known constants, t > 0 and D^{α}, the generalized elliptic curve L_3 type Caputo fractional derivative operator, Γ is the gamma function.

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t^k}(t_n-\tau)^{-\alpha}f^{'}(\tau)d\tau, \qquad 0 \leq n-1 < \alpha < n$$

CHAPTER 2

REVIEW OF RELATED LITERATURE

In (Al-Mdallal et al., 2021) a novel perspective has been presented to address the numerical solution of fractional initial value problems (FIVPs). The approach involves dividing the FIVP into smaller subdomains, allowing iterative solutions to be obtained independently for each subdomain. By doing so, an approximate solution can be derived for the entire domain.

Prakash et al. (2015) the theory and a few numerical methods to resolve this kind of problem under generalized differentiability requirements were described. Initial value problems for differential equations of fractional order with uncertainty were taken into consideration. Additionally, the numerical approximation of a fuzzy fractional initial value problem's solutions utilizing product trapezoidal and product rectangle formulas was researched; the numerical scheme's convergence was carefully examined.

Odibat and Baleanu (2020) a new generalized Caputo-type fractional initial value problem that generalizes the Caputo fractional derivative was introduced. To demonstrate the new generalized derivative features, some characteristics were developed. Then, for the numerical resolution of generalized Caputo-type initial value problems, they introduced an adaptive predictor-corrector approach. The suggested algorithm can be seen as a slight improvement on the traditional Adams-Bashforth-Moulton technique. Numerical discussions are made of several fractional initial value problems' dynamic behavior.

Lyons et al. (2017) approximations to solutions of fractional initial value problems (IVPs) have significant applications in the associated fields due to the growing popularity of fractional differential equations (FDEs) and the ongoing development of methods for their solution. This study establishes an application of Caputo fractional ordinary differential equations to Picard's Iterative Existence and Uniqueness Theorem. Selvam et al. (2021) the application of Newton's second law to several biological processes, including the behavior of vibrating eardrums, was well recognized. As a model for the dynamic of a vibrating eardrum, a nonlinear discrete fractional initial value problem was taken into consideration. Outline adequate requirements for the suggested model's solutions' existence, originality, and Hyers-Ulam stability.

Abdeljawad et al. (2020) solve fractional initial value problems using comparison concepts. involving $0 \leq n-1 \leq n$ Caputo derivatives in the order α. The comparison results using rigorous inequalities for problems involving fractional initial value problems and Caputo derivatives were first published. They next investigated the harsh local conditions ofCaputo derivative solutions are applied to fractional initial value problems. They investigated the comparative outcomes using non-strict inequalities for fractional beginning value issues using the Caputo method in their final analysis.

Pitolli et al. (2022) proposed to use a cubic spline to roughly approximate the Riesz-Caputo derivative of a given function. This is the first time that cubic splines have been applied to the Riesz-Caputo derivative, as far as they are aware. The efficiency of the suggested numerical method was demonstrated by numerical tests that contrasted the numerical solution produced by a spline collocation method with the analytical solution to a number of boundary differential problems that had the Riesz-Caputo derivative in space.

Teodoro et al. (2019) critical to establishing a systematic classification given the rise in research and operator definitions in the context of fractional initial value problems. However, a lot of the definitions of fractional derivatives that have emerged in the literature cannot be taken into consideration. Looked at a number of equations to get a broad understanding of the idea of fractional (integral) derivatives.

Lu and Zhu (2018) they deal with comparison concepts for issues with fractional initial values problemsinvolving $0 \leq n-1 < p \leq n$ and the order p Caputo derivatives. First, they provided comparison results for fractional differential equations with Caputo derivatives with tight inequality. The application of Caputo derivatives to fractional initial value issues was then investigated, as were local existence and extreme solutions. Finally, they took into account comparison results for situations involving fractional initial values, Caputo derivatives, and nonstrict inequalities. Caputo and Fabrizio (2021) investigated a number of characteristics of their kernels' singularity and the fractional derivatives. Then, the properties of the Caputo-Fabrizio fractional derivative were considered.

Tuan et al. (2020) proposed an SEIR epidemic model utilizing Caputo fractional derivatives to explain the spread of COVID-19. Calculations are made on the system's feasibility region and equilibrium points, and the stability of those points is looked into. Using fixed-point theory, they demonstrated that the model had a single solution. The fractional Euler method was used to obtain a rough solution to the model. They offer a numerical simulation based on actual data to forecast COVID-19 transmission in Iran and globally.

Phang et al. (2020) the numerical solution of the time fractional partial differential equations (FPDEs) is the focus of this article. Shifted Chebyshev-Gauss-Lobatto (CGL) collocation points and an operational matrix of Caputo sense derivatives via Genocchi polynomials are used in the suggested method. The main equation, along with the beginning and boundary conditions, are collocated by employing shifted CGL collocation points, and the result is the system of linear algebraic equations. This method's fundamental strategy is to convert the FPDEs into a set of algebraic equations.

Ahmad et al. (2020) the Katugampola-Caputo fractional derivative-connected implicitly coupled fractional differential system's existence and uniqueness are examined in this paper. To obtain the necessary conclusions, various fixed-point theorems are applied. Additionally, they derived certain necessary conditions to ensure that the solutions to our system under consideration are Hyers-Ulam stable. They also offered an illustration that clarifies their findings.

In (Boutiara et al., 2022) the context of Riesz-Caputo operators, they looked at the existence and uniqueness of solutions for a class of fractional starting value problems with boundary conditions. They used functional analysis methods to demonstrate the uniqueness conclusion using the Banach contraction principle and the existence result utilizing the Schaefer and Krasnoslkii fixed point theorems.

Based on this (Bawa'neh, 2019) numerical approximations of systems of fractional initial value problems, the method reduced SFDEs to a system of algebraic equations by taking advantage of their characteristics. Studying the convergence and estimating the accuracy of the proposed approach received particular attention. A potential tool for resolving several systems of non-linear fractional initial value problems was introduced by the methods. To demonstrate the reliability and enormous potential of both proposed methodologies, numerical examples are provided.

Chalishajar et al. (2021) results for the fractional impulsive neutral functional integro-differential evolution equation in Banach spaces were studied for existence, uniqueness, and controllability. The primary methods rely on the sectorial operators and characteristic solution operators' properties for fractional initial value problems. They specifically did not take into account that the mechanism generates a compact semigroup. They therefore asserted that there is phase space for limitless delay and impulse. Finally, a concrete example was given to illustrate the desired outcomes.

Zada et al. (2018) for a linked system made up of Caputo-type fractional starting value problems subject to Riemann-Liouville fractional integral boundary conditions, they investigated the existence and uniqueness of solutions. The Banach contraction principle establishes the uniqueness of solutions. while Leray-Schauder's option leads to the availability of solutions. They also looked at the system's Hyers-Ulam stability. Examples are also provided at the conclusion to further highlight the findings.

Odibat and Baleanu (2022) the complex dynamical motion of a quarter-car suspension system with a sinusoidal road excitation force was to be studied. They started by thinking of an innovative mathematical model made up of fractional-order differential equations. They used an exponential kernel together with the Caputo-Fabrizio fractional operator in the suggested model. Then, they proposed a quadratic numerical approach and demonstrated its stability and convergence to solve the associated equations.

Demirci and Ozalp (2012) they studied Caputo-type fractional initial value issues with orders $0 \leq \alpha \leq 1$ and beginning conditions x (0) = x_0. introduced a method to use integer-order initial value problem solutions to obtain precise solutions to fractional initial value problems. Additionally, a generalization of the method for finite systems is provided. Finally, they provided a few instances to show how their findings may be used.

Delavari et al. (2012) an extension of Lyapunov's direct technique for fractional-order systemsUsing the inequality of Bihari and Bellman-Gronwall, a proof of the comparison theorem for fractional-order systems is presented.

Thuan and Huong (2018) wondering about the systematic construction of robust stabilizing state feedback controllers for fractional orders. Nonlinear systems. By applying the Lyapunov direct technique and a recent finding on the Caputo fractional derivative of a quadratic function, stabilization requirements described in terms of linear matrix inequalities were established. The controllers can then be developed by using known computationally effective convex techniques. Two numerical examples with simulation results were supplied to highlight the usefulness of their outcomes.

Li and Zhang (2011) a quick description of the recent stability results of fractional initial value problems and the analytical methodologies employed is offered. These equations include linear fractional differential equations, nonlinear fractional equations, and fractional initial value problems with time delay. Some conclusions about stability were comparable to those of traditional integer-order initial value problems.

Jafari et al. (2020) by using this technique, the SFIPs were converted into an optimization problem, and the resulting system of nonlinear algebraic equations was created. They approximated solutions and derived the coefficients of polynomial expansions by resolving the system of nonlinear equations using polynomials as basis functions. They disagreed on the proposed technique's convergence analysis.

Bouchama et al. (2022) used the fractional Caputo-Hadamard derivative of order $\alpha > 0$ to identify accurate solutions for linear fractional differential equations. This objective was achieved by developing a revolutionary method known as the finite fractional difference approach. The number response is located using FFDM. Thus, the convergence and stability of the numerical technique are explained and shown by solving two linear fractional differentials. To demonstrate the efficacy of their strategy, they used equational issues of order $0 \leq \alpha \leq 1$.

Almeida et al. (2016) it was shown that fractional initial value problems may be more effective than ordinary differential equations in modeling various challenges using real-world examples and experimental data from several experiments. The order of the fractional operator that best describes real data and other associated parameters are found by a numerical optimization method based on least squares approximation.

Tavares et al. (2016) three Caputo-type fractional operators are taken into consideration, and each one is given an approximate expression in terms of common (integer-valued) functions. Orders should only be for variations. There were also estimates about the estimations' accuracy. Then, contrast the precise fractional derivative of a test function with its numerical approximation. Concluded with an explanation of how partial fractional initial value problems of variable order can be addressed utilizing the strategies.

Odibat and Baleanu (2020) introduced a brand-new generalized Caputo-type fractional derivative that generalizes the Caputo fractional derivative. Some attributes were derivated in order to illustrate the new generalized derivative features. Next, the generalized Caputo-type initial value problems are numerically solved for the adaptive predictor and corrector technique. The proposed approach is regarded as a fractional expansion of the traditional Adams-Bashforth-Molton method. The dynamic behaviors of a variety of fractional derivative models were investigated numerically.

Al-Mdallal and Omer (2018) gave a numerical solution for problems of second order with fractional initial values. This numerical algorithm was based on the fractional Legendre-collocation spectral technique. The governing fractional initial value problem led to the creation of a nonlinear system of algebraic equations. The error analysis of the suggested numerical approach was provided. Comparisons with existing numerical approaches show that the suggested algorithm is more precise and simpler to use.

J. Zhang et al. (2016) introduced a discretization procedure to study the dynamical characteristics of a modified fractional-order optically injected semiconductor laser model. To be more exact, the necessary and sufficient conditions for the stability of the discrete system are examined, and a sufficient condition for the existence and uniqueness of the solution is found. The findings indicated that the fractional parameter of the system affects the discrete system's stability and that the system exhibits a variety of dynamic properties, including Hopf bifurcation, attractor crises, and chaotic attractors.

León et al. (2013) discussed how fractional autonomous order systems can maintain the initial value stability problem. Because their solution in the linear situation is identical to the one known in literature, it establishes the mathematical strategy to solve the initial trajectory problem that will be provided in further works. The presented criteria are related to the computations for the integer case and are simpler to verify than the well-known ones.

Gallegos and Duarte-Mermoud (2016) established standards to guarantee the boundedness and convergence of signals represented by Caputo derivative-based non-integer-order equations. Findings for linear time-varying forced equations and time-varying unforced non-linear equations are provided and discussed after a study of the scenario of linear time-varying unforced equations.

Wang et al. (2021) concerned with the stability of fractional-order systems with randomly time-varying parameters. Two ways are offered to check the stability of such systems in the normal sense. The first technique is based on a suitable Lyapunov functional to assess stability, which is of crucial relevance in the theory of stability. Based on integral inequalities and creative mathematical methods.

Brandibur et al. (2021) it was suggested that linear systems of fractional initial value problems are stable. The inquiry is expanded to include multi-order systems after beginning with the well-known Matignon results on the stability of single-order systems, for which a new proof is offered along with a clarification of a limit case. A thorough investigation of the asymptotic behavior of the Mittag-Leffler function and its derivatives is also proposed due to the crucial role this function plays in expressing the solution of linear systems of FIVPs.

CHAPTER 3

METHODOLOGY OF THE STUDY

3.1 Study Design

Both documentation review and numerical experimentation be used in this study.

3.2 Procedure of the study

To attain the objective of the study, the following procedures are undertaken:

- Define the problem,
- Formulating the model,
- Obtaining a system of equations,
- Applying the the L_3 type left Caputo method for fractional initial value problem,
- Solving the model equation by employing numerical methods,
- Analyzing the consistence, stability and convergence,
- Using Matlab write code to solve a problem,
- Validating the methods using numerical examples,
- Discussing the results with graph.

CHAPTER 4

DESCRIPTION AND ANALYSIS OF THE SCHEME

4.1 Preliminaries

Definition. *Diethelm and Ford (2002) a fractional initial value problems (FIVP) is initial value problems with fractional derivatives or integrals that incorporate non-integer-order. The memory and heredity characteristics of specific materials or processes are related to the physical aspects of systems with fractional initial value problems. In these systems, the past history of the system as well as the recent past affect the current condition. The term "physical feature" of a fractional initial value problem usually refers to how particular characteristics or actions of a mathematical or physical system depend on the fractional nature of the problem under consideration.*

The **Caputo Derivative** and the **Riemann-Liouville integral** are well-known and widely applied to find the solutions of fractional differential equations.

4.1.1 Riemann Liouville Integral

Definition. *If $f(x) \in C([a,b])$ and $a \leq x \leq b$, then*

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-\tau)^{\alpha-1} f(\tau) d\tau$$

where $\alpha \in (0,1)$ called the Riemann-Liouville fractional integral.

4.1.2 The Caputo derivative

The Caputo derivative, named after mathematician Michele Caputo, is a type of fractional derivative used in the field of fractional calculus. It is a non-integer order derivative and is defined as a modification of the Reimann-Liouville fractional derivative.
The Caputo derivatives are denoted by D^{α}, where α is a positive, non-integer real number representing the order of the derivatives (Teodoro et al., 2019).

$$D^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^m(\tau) d\tau, \qquad 0 \leq m-1 < \alpha < m$$

The use of Caputo derivatives provides a notable advantage by allowing the initial conditions of a fractional initial value problem to be easily understood in physical terms, as long as they relate to the integer-order derivatives of the function. Caputo derivatives are extensively employed in diverse fields such as viscoelasticity, fluid dynamics, and control theory, enabling the characterization of systems that display memory or self-similarity characteristics.

Definition. *The Caputo derivative is useful for modeling phenomena that take account of interactions within the past and also have problems with non-local properties. In this sense, one can think of the equation as having "memory." The Caputo derivative with the lower limit t_0 and of order $\alpha \in (0,1)$ is defined as*

$$D^{\alpha}x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^m(\tau) d\tau \qquad 0 \leq m-1 < \alpha < m \tag{4.1}$$

Al-Mdallal et al. (2021) here α is a real constant known as the order of the fractional derivative, and 0 is the initial point, and Γ denotes the gamma function.
The Gamma function*:- is an extension of the factorial function to real numbers, and is defined by*
$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ *is the euler gamma function.*
$\Gamma(k+1) = k\Gamma(k) = k!$

The modified Caputo definition of non-integer order derivatives, which is a variation of the Riemann-Liouville definition, offers an advantage by allowing it to be combined with initial conditions that hold specific physical significance. The primary benefit of the Caputo approach lies in its treatment of initial conditions. It enables fractional initial value problems formulated with Caputo derivatives to follow a similar format as their integer-order counterparts.

Definition. *Tavares et al. (2016) Caputo fractional derivative of order $\alpha(t)$ type I, II, III methods. Given a function $x : [a,b] \to \mathbb{R}$.*
The type I left Caputo derivative of order $\alpha(t)$ is defined by

$${}_a^C D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} [x(\tau) - x(a)] d\tau \tag{4.2}$$

The type II left Caputo derivative of order α_t is defined by

$${}_a^c D_t^{\alpha} x(t) = \frac{d}{dt} \left(\frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} [x(\tau) - x(a)] d\tau \right) \tag{4.3}$$

From this method of Caputo derivative, we can use only Caputo type III left Caputo derivatives of order α_t is defined by

$${}_0^c D_t^{\alpha} x(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x'(\tau) d\tau \tag{4.4}$$

where n is such that $n-1 < \alpha < n$.

Definitions of L_3 Caputo-type method for fractional initial value problems

Srivastava and Singh (2023) the L_3 Caputo-type method is a numerical approach used to solve fractional initial value problems (FIVP). It is an extension of the classical Caputo fractional derivative operator, which is defined for real numbers greater than or equal to 1. The L_3 Caputo-type method allows for the approximation of the Caputo fractional derivative of any positive real order, including non-integer values.

To understand the L_3 Caputo-type method, let's first review the classical Caputo fractional derivative operator. Given a function $f(t)$ that is differentiable n times on the interval $[a,t]$, where $n-1<\alpha\leq n$, the Caputo fractional initial value problems of order α is defined as:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f^{'}(\tau)\,d\tau$$

where Γ denotes the gamma function.

The L_3 Caputo-type method approximates the fractional initial value problems $D^{\alpha}f(t)$ by introducing the L_3 fractional difference operator ${}_{a}^{C}D_{t}^{\alpha}$ defined as:

$${}_{a}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}f(t-k)$$

where $\binom{n}{k}$ represents the binomial coefficient. The L_3 Caputo-type method for solving fractional initial value problems is based on replacing the Caputo fractional initial value problem in the original problem with the L_3 fractional difference operator. This allows the problem to be transformed into a system of algebraic equations that can be numerically solved. The method involves discretizing the time domain, approximating the fractional initial value problems, and solving the resulting algebraic equations to obtain an approximate solution to the fractional initial value problem.

It's worth noting that the L_3 Caputo-type method is just one of many numerical methods available for solving fractional initial value problems. Different methods may have varying degrees of accuracy and computational efficiency depending on the specific problem at hand.

4.2 Analysis of the L_3 Caputo - type

In this subsection, for the sake of showing motivation of the method, we solve the system of fractional initial value problems. From(4.4)

$${}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t_1}(t_n-\tau)^{-\alpha}f^{'}(\tau)d\tau+\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}(t_n-\tau)^{-\alpha}f^{'}(\tau)d\tau+\cdots+ \tag{4.5}$$

$\frac{1}{\Gamma(1-\alpha)}\int_{t_n}^{t_{n+1}}(t_n-\tau)^{-\alpha}f^{'}(\tau)d\tau$

It becomes:

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\alpha}f^{'}(\tau)d\tau \tag{4.6}$$

We discretized the first order space derivative by first-order accurate backward difference formula as follows:

$$ f^{'}(t)=\frac{f(t_k)-f(t_{k-1})}{h}+O(h) \tag{4.7}$$

By using backward difference method.

To discretization this system of fractional order initial value problems, we can use L_3 type Caputo method. The domain of discretization will be represented as a discrete set of time points t_n, where n=0, 1, 2,... and $t_n=t_0+nh$, with h being the the time step size. The L_3 finite difference scheme is the simplest and most popular method for discretizing Caputo fractional derivatives of order $\alpha \in$(0, 1).

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\alpha}\left[\frac{f(t_k)-f(t_{k-1})}{h}\right]d\tau \tag{4.8}$$

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\left[\frac{f(t_k)-f(t_{k-1})}{h}\right]\int_{t_k}^{t_{k+1}}(t_n-\tau)^{-\alpha}d\tau \tag{4.9}$$

Applying the integral operator to the equation (4.9), one obtains Integrating by using power rule, we get:

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{-1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\left[\frac{f(t_k)-f(t_{k-1})}{h}\right]\left[\frac{(t_n-\tau)^{1-\alpha}}{1-\alpha}\right]_{t_k}^{t_{k+1}} \tag{4.10}$$

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{-1}{h(1-\alpha)\Gamma(1-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(t_n-t_{k+1})^{1-\alpha}-(t_n-t_k)^{1-\alpha}\right]$$

We use the uniform grid $t_n=nh$, where n=0, 1,..., M, Mh=T and use the abbreviations x_n, y_n and z_n for approximation of the true solutions $x(t_n)$, $y(t_n)$ and $z(t_n)$ in the grid point t_n where as, $t_n=nh$, $t_{k+1}=(k+1)h$, $t_k=kh$

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{-1}{h(1-\alpha)\Gamma(1-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(nh-(k+1)h)^{1-\alpha}-(nh-kh)^{1-\alpha}\right]$$

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{-h^{1-\alpha}}{h(1-\alpha)\Gamma(1-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(n-k-1)^{1-\alpha}-(n-k)^{1-\alpha}\right] \tag{4.11}$$

But, $\Gamma(2-\alpha)=(1-\alpha)\Gamma(1-\alpha)$

$$ {}_{0}^{c}D_{t}^{\alpha}f(t_n)=\frac{-h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(n-k-1)^{1-\alpha}-(n-k)^{1-\alpha}\right] \tag{4.12}$$

Hence, the type *III* method for the caputo fractional derivatives is obtained below:

$$ {}_{0}^{c}D_{t}^{\alpha}x(t_n)=\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right] \tag{4.13}$$

$$ {}_{0}^{c}D_{t}^{\alpha}y(t_n)=\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right] \tag{4.14}$$

$$
{}_{0}^{c}D_{t}^{\alpha}z(t_{n}) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[f(t_{k})-f(t_{k-1})\right]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right] \tag{4.15}
$$

By using (Al-Mdallal et al., 2021), (J. Zhang et al., 2016), (Matouk & Elsadany, 2016) and (Solís-Pérez et al., 2018).

Lemma 4.1. *For $t \in I_n$, the FIVP $D_{t_{n-1}}^{\alpha}y_n = f(t, y_n)$ on sub-domain I_n is equivalent to the following integral equation:*

$$
y_n = y_{n-1} + \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t}(t-\tau)^{\alpha-1}y'(\tau)d\tau
$$

Thus, after repeating the discretization process n times, we obtain. The systems of equations becomes

$$
x_{n+1} = x_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]\left[f(t_k)-f(t_{k-1})\right] \tag{4.16}
$$

$$
y_{n+1} = y_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]\left[f(t_k)-f(t_{k-1})\right] \tag{4.17}
$$

$$
z_{n+1} = z_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]\left[f(t_k)-f(t_{k-1})\right] \tag{4.18}
$$

4.3 Convergence of systems of fractional initial value problem

4.3.1 Existence and uniqueness of numerical solutions

Accordingly, to the Caputo fractional derivative (1.1) is, given in Eqs. (4.17), eqn(4.18) and eqn(4.19) (Deressa, 2022).

Let

$H_1(x,y,z,t)$=$ax+y+z$

$H_2(x,y,z,t)$=$x+by+z$

$H_3(x,y,z,t)$=$x+y+cz$

$$
\begin{cases}
{}_{0}^{c}D_{t}^{\alpha}x(t) = H_1(x,y,z,t) & x(0) = x_0 \\
{}_{0}^{c}D_{t}^{\alpha}y(t) = H_2(x,y,z,t) & y(0) = y_0 \\
{}_{0}^{c}D_{t}^{\alpha}z(t) = H_3(x,y,z,t) & z(0) = z_0
\end{cases} \tag{4.19}
$$

(Lyons et al., 2017) this section proves the existence and uniqueness of a solution to the system (4.20) via the Banach's fixed-point theorem and the Banach contraction principle. Note that since H_1, H_2, and H_3 defined in (4.20), are all continuous every where in R^3, the existence of at least one solution for system (4.20) is guaranteed. We now show the uniqueness of the solution for the given initial condition shown in (4.20) by defining Picard's operator and the principle of contraction mapping.

Definition. *function f(t,y) satisfies a lipschitz condition in the variable x on a set $D \subset \mathbb{R}^3$ if a constant L >0 exists with*

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2|$$

whenever (t, x_1), (t, x_2) are in D. L is Lipschitz constant.

Theorem 4.1. *(Picard Lindelöf) Suppose $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(\alpha,\beta)} \to \mathbb{R}^3$ is continuous and bounded by M. Suppose, furthermore, that f(t, .) is Lipschitz continuous with Lipschitz constant L for every $t \in [t_0 - \alpha, t_0 + \alpha]$. Then*

$$\begin{cases} D^{\alpha}x(t) = f(t,x) \\ x(t_0) = a \end{cases}$$

has a uniqueness solution defined on $[t_0 - \alpha, t_0 + \alpha]$, where $b = min\{\alpha, \frac{\beta}{M}\}$.

Since Lipschitz continuity is the main condition for Picard-Lindel$\ddot{o}$f's theorem that guarantees the existence and uniqueness of a system of in fractional initial value problem such as (4.20), we show whether each of the equations in (4.20) satisfies Lipschitz continuity was followed by defining Picard's operator. First, it is shown that the first equation of (4.20) satisfies the condition of Lipschitz continuity, as demonstrated in the following procedure:

$$||H_1(x_1,y,z,t) - H_1(x_2,y,z,t)|| = ||(ax_1 + y + z) - (ax_2 + y + z)|| \leq a||x_2 - x_1|| \tag{4.20}$$

$$||H_2(x,y_1,z,t) - H_2(x,y_2,z,t)|| = ||(ax + y_1 + z) - (ax + y_2 + z)|| \leq b||y_2 - y_1|| \tag{4.21}$$

$$||H_3(x,y,z_1,t) - H_3(x,y,z_2,t)|| = ||(ax + y + z_1) - (ax + y + z_2)|| \leq c||z_2 - z_1|| \tag{4.22}$$

Hence, the Lipschitz continuity is satisfied with the Lipschitz constants a, b, and c. This result assures that the first equation of (4.20) has at least one solution. Secondly, Picard's operator is set up as follows: Applying the fundamental theorem of calculus to the first equation of (4.20).

${}_0^cD_t^{\alpha}x(t) = H_1(x,y,z,t)$, ${}_0^cD_t^{\alpha}y(t) = H_2(x,y,z,t)$ and ${}_0^cD_t^{\alpha}z(t) = H_3(x,y,z,t)$ we obtain

$x(t) - x(0) = {}_0^c D_t^{\alpha}\{H_1(x,y,z,t)\}$

$y(t) - y(0) = {}_0^c D_t^{\alpha}\{H_2(x,y,z,t)\}$

$z(t) - z(0) = {}_0^c D_t^{\alpha}\{H_3(x,y,z,t)\}$

Assuming that Q, R and S is Picard's operator, we have

$Qx(t) - x(0) = {}_0^c D_t^{\alpha}\{H_1(x,y,z,t)\}$

$Ry(t) - y(0) = {}_0^c D_t^{\alpha}\{H_2(x,y,z,t)\}$

$Sz(t) - z(0) = {}_0^c D_t^{\alpha}\{H_3(x,y,z,t)\}$

The boundedness of Picard's operator Q, R and S is shown as follows:

$$||Qx(t) - x(0)|| = ||{}_0^cD_t^{\alpha}\{H_1(x,y,z,t)\}|| \leq {}_0^c D_t^{\alpha}||\{H_1(x,y,z,t)\}|| \tag{4.23}$$

$$||Ry(t) - y(0)|| = ||{}_0^cD_t^{\alpha}\{H_2(x,y,z,t)\}|| \leq {}_0^c D_t^{\alpha}||\{H_2(x,y,z,t)\}|| \tag{4.24}$$

$$||Sz(t) - z(0)|| = ||{}_0^cD_t^{\alpha}\{H_3(x,y,z,t)\}|| \leq {}_0^c D_t^{\alpha}||\{H_3(x,y,z,t)\}|| \tag{4.25}$$

Since it is proved that $H_1(x,y,z,t)$, $H_2(x,y,z,t)$ and $H_3(x,y,z,t)$ are Lipschitz continuous, we have $H_1(x,y,z,t)$, $H_2(x,y,z,t)$ and $H_3(x,y,z,t)$ are bounded, and thus there is a constant k_1, k, and r such that $H_1(x,y,z,t) \leq k_1$, $H_2(x,y,z,t) \leq k$ and $H_3(x,y,z,t) \leq r$. Consequently, Eqn(4.14), eqn(4.15) and eqn(4.16) combine together with eqn(4.24), eqn(4.25) and eqn(4.26) it become as follow respectively:

$$||Qx(t)-x(0)|| = ||{}_0^cD_t^\alpha x(t)\{H_1(x,y,z,t)\}|| \leq_0^c D_t^\alpha x(t)||\{H_1(x,y,z,t)\}|| \leq k_1({}_0^cD_t^\alpha x(t))$$

$\leq k_1(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$

$$||Ry(t)-y(0)|| = ||{}_0^cD_t^\alpha y(t)\{H_2(x,y,z,t)\}|| \leq_0^c D_t^\alpha y(t)||\{H_2(x,y,z,t)\}|| \leq k({}_0^cD_t^\alpha)y(t)$$

$\leq k(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$

$$||Sz(t)-z(0)|| = ||{}_0^cD_t^\alpha z(t)\{H_3(x,y,z,t)\}|| \leq_0^c D_t^\alpha z(t)||\{H_3(x,y,z,t)\}|| \leq r({}_0^cD_t^\alpha z(t))$$

$\leq r(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$

The boundedness of the operator Q, R and S is thus proved.

A condition for the operator Q, R and S to be a contraction mapping is developed as follows:

$$||Qx_1(t)-Qx_2(t)|| = ||{}_0^cD_t^\alpha x(t)\{H_1(x_1,y,z,t)\}-\{H_1(x_2,y,z,t)\}||$$

$\leq ||\{H_1(x_1,y,z,t)\}-\{H_1(x_2,y,z,t)\}||{}_0^cD_t^\alpha x(t) \leq a||x_1-x_2||{}_0^cD_t^\alpha x(1)$

$\leq (\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]||x_1-x_2||)$

$$||Ry_1(t)-Ry_2(t)|| = ||{}_0^cD_t^\alpha x(t)\{H_1(x,y_1,z,t)\}-\{H_1(x,y_2,z,t)\}||$$

$\leq ||\{H_2(x,y_1,z,t)\}-\{H_2(x,y_2,z,t)\}||{}_0^cD_t^\alpha y(t) \leq b||y_1-y_2||{}_0^cD_t^\alpha y(1)$

$\leq b(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]||y_1-y_2||)$

$$||Sz_1(t)-Sz_2(t)|| = ||{}_0^cD_t^\alpha z(t)\{H_3(x,y,z_1,t)\}-\{H_3(x,y,z_2,t)\}||$$

$\leq ||\{H_1(x,y,z_1,t)\}-\{H_1(x,y,z_2,t)\}||{}_0^cD_t^\alpha z(1) \leq c||z_1-z_2||{}_0^cD_t^\alpha z(1)$

$$\leq c(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]||z_1-z_2||)$$

Thus, the operator Q, R and S are a contraction mapping if the following condition are satisfied:

$$(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]) \leq \frac{1}{a} \quad (4.26)$$

$$(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]) \leq \frac{1}{b} \quad (4.27)$$

$$(\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right]) \leq \frac{1}{c} \quad (4.28)$$

The uniqueness of the solution is proved as follows:

Let us assume that x_1 and x_2, y_1 and y_2, z_1 and z_2 are two solutions to the equation of eqn(4.27), eqn(4.28) and eqn(4.29). Then, we have

$$x_1(t)-x_1(0)={}_0^c D_t^\alpha x(t)\{H_1(x_1,y,z,t)\} \tag{4.29}$$

$$x_2(t)-x_2(0)={}_0^c D_t^\alpha x(t)\{H_1(x_2,y,z,t)\}$$

$$y_1(t)-y_1(0)={}_0^c D_t^\alpha y(t)\{H_2(x,y_1,z,t)\} \tag{4.30}$$

$$y_2(t)-y_2(0)={}_0^c D_t^\alpha y(t)\{H_2(x,y_2,z,t)\}$$

$$z_1(t)-z_1(0)={}_0^c D_t^\alpha z(t)\{H_3(x,y,z_1,t)\} \tag{4.31}$$

$$z_2(t)-z_2(0)={}_0^c D_t^\alpha z(t)\{H_2(x,y,z_2,t)\}$$

Considering the difference between the two solutions x(t), y(t) and z(t) assuming the two solutions have the same initial point, and taking the Euclidian norm, we obtain

$$||x_1(t)-x_2(t)||=||{}_0^c D_t^\alpha x(t)\{H_1(x_1,y,z,t)-H_1(x_2,y,z,t)\}||$$

$$\le D_t^\alpha x(t)\{H_1(x_1,y,z,t)-H_1(x_2,y,z,t)\}||$$
$$\le a\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}\textstyle\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$$

$$||y_1(t)-y_2(t)||=||{}_0^c D_t^\alpha y(t)\{H_2(x,y_1,z,t)-H_2(x,y_2,z,t)\}||$$

$$\le D_t^\alpha y(t)\{H_2(x,y_1,z,t)-H_2(x,y_2,z,t)\}||$$
$$\le b\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}\textstyle\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$$

$$||z_1(t)-z_2(t)||=||{}_0^c D_t^\alpha z(t)\{H_3(x,y,z_1,t)-H_3(x,y,z_2,t)\}||$$

$$\le D_t^\alpha z(t)\{H_3(x,y,z_1,t)-H_3(x,y,z_2,t)\}||$$
$$\le c\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}\textstyle\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])$$

Thus, we have the following relation:

$$||x_1(t)-x_2(t)||[1-a\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])]\le 0 \tag{4.32}$$

$$||y_1(t)-y_2(t)||[1-b\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-\alpha}-(n-k-1)^{1-\alpha}\right])]\le 0 \tag{4.33}$$

$$||z_1(t)-z_2(t)||[1-b\frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^n [f(t_k)-f(t_{k-1})]\left[(n-k)^{1-c\alpha}-(n-k-1)^{1-\alpha}\right])]\le 0 \tag{4.34}$$

It then follows that

$||x_1(t)-x_2(t)||\le 0$, $||y_1(t)-y_2(t)||\le 0$ and $||z_1(t)-z_2(t)||\le 0$.

Hence $x_1(t)=x_2(t)$, $y_1(t)=y_2(t)$ and $z_1(t)=z_2(t)$.

We proved that all the operators Q, R and S have well-defined contraction mappings. Thus, the system (4.20) in the context of the L_3 type Caputo fractional derivative has an existence and uniqueness solution for a given initial condition of our system.

4.3.2 Constistence of systems of fractional initial value problems

Khader et al. (2015)

Theorem 4.2. *Numerical approximation (4.16), (4.17) and (4.18) is consistent with fractional order initial value problems (4.13), (4.14) and (4.15)* $x_n - x(t_n) = O(h^{1+\alpha})$, $y_n - y(t_n) = O(h^{1+\alpha})$ *and* $z_n - z(t_n) = O(h^{1+\alpha})$.

Proof. to prove the consistency of the given numerical solutions for fractional initial value problems, we need to show that the approximation error converges to zero as the step size approaches zero.

Let's consider the given numerical approximation:

$$x_{n+1} = x_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] \tag{4.35}$$

$$y_{n+1} = y_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

$$z_{n+1} = z_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

where h is the step size, α is the fractional order, Γ is the gamma function, and f(t) is the function related to the initial value problem.

To prove consistency, we need to show that the error between the numerical approximation and the exact solution, denoted by E, satisfies $E = O(h^{1+\alpha})$ as h approaches zero.

Let's define the exact solution as $x(t_n)$, $y(t_n)$, $z(t_n)$ and the true solution of the numerical approximation as X_n, y_n and z_n.

The error E can be written as:

$$E = |X_n - x(t_n)| \tag{4.36}$$

$$E = |y_n - y(t_n)|$$

$$E = |z_n - z(t_n)|$$

Now, we can express X_n, y_n and z_n in terms of the exact solution $x(t_n)$, $y(t_n)$ and $z(t_n)$ and the error term E:

$$X_n = x(t_n) + E \tag{4.37}$$

$$y_n = y(t_n) + E$$

$$z_n = z(t_n) + E$$

Substituting this back into the numerical approximation, we have:

$$x_{n+1} = x_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

$$y_{n+1} = y_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

$$z_{n+1} = z_n + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

Expanding the summation and rearranging the terms, we obtain:

$$x_{n+1} - x_n = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

$$y_{n+1} - y_n = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

$$z_{n+1} - z_n = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E$$

Now, we can subtract the true solution equation from both sides:

$$(x_{n+1} - x(t_{n+1})) - (x_n - x(t_n)) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E - E$$

$$(y_{n+1} - y(t_{n+1})) - (y_n - y(t_n)) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E - E$$

$$(z_{n+1} - z(t_{n+1})) - (z_n - z(t_n)) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})] + E - E$$

Using the definition of the difference quotient for the derivative, we have:

$${}^{c}D_t^{\alpha} x(t_n) - \frac{x(t_{n+1}) - x(t_n)}{h} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

$${}^{c}D_t^{\alpha} y(t_n) - \frac{y(t_{n+1}) - y(t_n)}{h} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

$${}^{c}D_t^{\alpha} z(t_n) - \frac{z(t_{n+1}) - z(t_n)}{h} = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

Rearranging the terms and multiplying both sides by $-h^{\alpha}$, we get:

$$\frac{x(t_{n+1}) - x(t_n)}{h^{\alpha}} - {}^{c}D_t^{\alpha} x(t_n) = -\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

$$\frac{y(t_{n+1}) - y(t_n)}{h^{\alpha}} - {}^{c}D_t^{\alpha} y(t_n) = -\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

$$\frac{z(t_{n+1}) - z(t_n)}{h^{\alpha}} - {}^{c}D_t^{\alpha} z(t_n) = -\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} [(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}][f(t_k) - f(t_{k-1})]$$

Now, taking the absolute value on both sides and applying the triangular inequality, we obtain:

$$|\frac{x(t_{n+1})-x(t_n)}{h^\alpha} -^c D_t^\alpha x(t_n)| \leq \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} |(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}||f(t_k) - f(t_{k-1})|$$

$$|\frac{y(t_{n+1})-y(t_n)}{h^\alpha} -^c D_t^\alpha y(t_n)| \leq \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} |(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}||f(t_k) - f(t_{k-1})|$$

$$|\frac{z(t_{n+1})-z(t_n)}{h^\alpha} -^c D_t^\alpha z(t_n)| \leq \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} |(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}||f(t_k) - f(t_{k-1})|$$

We can further simplify the right-hand side using the mean value theorem:

$$|(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}| \leq C|\alpha|(n-k)^{-\alpha}$$

where C is a constant.

Substituting this back into the inequality, we have:

$$|\frac{x(t_{n+1})-x(t_n)}{h^\alpha} -^c D_t^\alpha x(t_n)| \leq C|\alpha|\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} (n-k)^{-\alpha}|f(t_k) - f(t_{k-1})|$$

$$|\frac{y(t_{n+1})-y(t_n)}{h^\alpha} -^c D_t^\alpha y(t_n)| \leq C|\alpha|\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} (n-k)^{-\alpha}|f(t_k) - f(t_{k-1})|$$

$$|\frac{z(t_{n+1})-z(t_n)}{h^\alpha} -^c D_t^\alpha z(t_n)| \leq C|\alpha|\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} (n-k)^{-\alpha}|f(t_k) - f(t_{k-1})|$$

The sum on the right-hand side can be bounded by a constant M:

$$|\frac{x(t_{n+1})-x(t_n)}{h^\alpha} -^c D_t^\alpha x(t_n)| \leq CM \sum_{k=1}^{n} (n-k)^{-\alpha}$$

$$|\frac{y(t_{n+1})-y(t_n)}{h^\alpha} -^c D_t^\alpha y(t_n)| \leq CM \sum_{k=1}^{n} (n-k)^{-\alpha}$$

$$|\frac{z(t_{n+1})-z(t_n)}{h^\alpha} -^c D_t^\alpha z(t_n)| \leq CM \sum_{k=1}^{n} (n-k)^{-\alpha}$$

Now, we need to analyze the behavior of the sum on the right-hand side. As h approaches zero, the number of terms in the sum increases. We can approximate the sum by an integral:

$$\sum_{k=1}^{n} (n-k)^{-\alpha} \approx \int_0^n (n-x)^{-\alpha} dx$$

$$\sum_{k=1}^{n} (n-k)^{-\alpha} \approx \int_0^n (n-y)^{-\alpha} dy$$

$$\sum_{k=1}^{n} (n-k)^{-\alpha} \approx \int_0^n (n-z)^{-\alpha} dz$$

Evaluating this integral, we get:

$$\int_0^n (n-x)^{-\alpha}dx = [\frac{-(n-x)^{1-\alpha}}{(1-\alpha)}]_0^n = \frac{n^{1-\alpha}}{(1-\alpha)}$$

$$\int_0^n (n-y)^{-\alpha}dy = [\frac{-(n-y)^{1-\alpha}}{(1-\alpha)}]_0^n = \frac{n^{1-\alpha}}{(1-\alpha)}$$

$$\int_0^n (n-z)^{-\alpha}dz = [\frac{-(n-z)^{1-\alpha}}{(1-\alpha)}]_0^n = \frac{n^{1-\alpha}}{(1-\alpha)}$$

Substituting this back into the inequality, we have:

$$|\frac{x(t_{n+1})-x(t_n)}{h^\alpha} -^c D_t^\alpha x(t_n)| \leq \frac{CMn^{1-\alpha}}{(1-\alpha)}$$

$$|\frac{y(t_{n+1})-y(t_n)}{h^\alpha} -^c D_t^\alpha y(t_n)| \leq \frac{CMn^{1-\alpha}}{(1-\alpha)}$$

$$|\frac{z(t_{n+1})-z(t_n)}{h^\alpha} -^c D_t^\alpha z(t_n)| \leq \frac{CMn^{1-\alpha}}{(1-\alpha)}$$

Since n = $\frac{t}{h}$, we can rewrite the inequality as:

$$|\frac{x(t+h)-x(t)}{h^\alpha} -^c D_t^\alpha x(t)| \leq CM\frac{(\frac{t}{h})^{1-\alpha}}{(1-\alpha)}$$

$$|\frac{y(t+h)-y(t)}{h^\alpha} -^c D_t^\alpha y(t)| \leq CM\frac{(\frac{t}{h})^{1-\alpha}}{(1-\alpha)}$$

$$|\frac{z(t+h)-z(t)}{h^\alpha} -^c D_t^\alpha z(t)| \leq CM\frac{(\frac{t}{h})^{1-\alpha}}{(1-\alpha)}$$

Finally, multiplying both sides by h^α, we get:

$$|x(t+h)-x(t)-h^\alpha\ {}^cD_t^\alpha x(t)| \leq \frac{CMt^{1-\alpha}}{(1-\alpha)} \tag{4.38}$$

$$|y(t+h)-y(t)-h^\alpha\ {}^cD_t^\alpha y(t)| \leq \frac{CMt^{1-\alpha}}{(1-\alpha)} \tag{4.39}$$

$$|z(t+h)-z(t)-h^\alpha\ {}^cD_t^\alpha z(t)| \leq \frac{CMt^{1-\alpha}}{(1-\alpha)} \tag{4.40}$$

From this inequality, we can see that the error E is bounded by a constant times $t^{1-\alpha}$, which shows that $E = O(h^{1+\alpha})$. Therefore, the given numerical approximation is consistent with the fractional order initial value problem. □

4.3.3 The stability analysis of the system

To analyze the stability of a systems of fractional initial value problems (FIVP) using the L_3 type Caputo method, we begin by approximating the fractional derivatives using the L_3 method. This allow us to rewrite the problem as a set of three difference equations that can be analyzed for stability. After approximating the

fractional derivatives and writing the difference equations, we analyze the stability of the system by looking at the eigenvalue of the matrix of coefficients.
The L_3 type Caputo method is given by:

$$D^{\alpha} f(t) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] \left[f(t_k) - f(t_{k-1})\right] \tag{4.41}$$

where $f(t)$ is any one of x(t), y(t) and z(t).

Now we approximate the derivative using the L_3 method for each components:

$$D^{\alpha} x(t) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] x(t_n) - \tag{4.42}$$

$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] x(t_{n-1})=$
$ax(t_n) + y(t_n) + z(t_n)$

$$D^{\alpha} y(t) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] y(t_n) - \tag{4.43}$$

$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] y(t_{n-1})=$
$x(t_n) + by(t_n) + z(t_n)$

$$D^{\alpha} z(t) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] z(t_n) - \tag{4.44}$$

$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right] z(t_{n-1})=$
$x(t_n) + y(t_n) + cz(t_n)$
Let $A = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=1}^{n} \left[(n-k)^{1-\alpha} - (n-k-1)^{1-\alpha}\right]$.
(Khader et al., 2015)

Theorem 4.3. *The numerical scheme eqn(4.16), eqn(4.17) and eqn(4.18) of the system of fractional initial value problems eqn(4.13), eqn(4.14) and eqn(4.15) is stable.*

Let (x_n, y_n, z_n) and $(\bar{x_n}, \bar{y_n}, \bar{z_n})$ be three solutions of the numerical scheme(4.17), (4.18) and (4.19). Let e_n=$x_n - \bar{x_n}$, E_n=$y_n - \bar{y_n}$ and ε_n=$z_n - \bar{z_n}$
Next, rewrite the systems of equations as follows:

$$(A-a)e_n = Ae_{n-1} - E_n - \varepsilon_n \leq -Ae_{n-1} \tag{4.45}$$

$$(A-b)E_n = AE_{n-1} - e_n - \varepsilon_n \leq -AE_{n-1} \tag{4.46}$$

$$(A-c)\varepsilon_n = A\varepsilon_{n-1} - e_n - E_n \leq -A\varepsilon_{n-1} \tag{4.47}$$

Thus from eqn(4.46), eqn(4.47) and eqn(4.48), we have
$\| e_n \| \leq \max(\| e_0 \|, \| e_1 \|, \| e_2 \|, \ldots, \| e_{n-1} \|) \leq \cdots \leq \| e_0 \|$,

$\| E_n \| \leq \max(\| E_0 \|, \| E_1 \|, \| E_2 \|, \ldots, \| E_{n-1} \|) \leq \cdots \leq \| E_0 \|,$

$\| \varepsilon_n \| \leq \max(\| \varepsilon_0 \|, \| \varepsilon_1 \|, \| \varepsilon_2 \|, \ldots, \| \varepsilon_{n-1} \|) \leq \cdots \leq \| \varepsilon_0 \|.$

Therefore, the numerical approximation for solving systems of fractional initial value problems Eqn (1.1) is stable. According to theorem 4.2, based on equations (4.38), (4.39), and (4.40), the systems mentioned are deemed consistent. Additionally, equations (4.45), (4.46), and (4.47) establish the stability of these systems.

Theorem 4.4. *The Lax-Richtmyer theorem is the fundamental results in the theory of numerical methods for solving fractional initial value problem. It provides a criterion for establishing the convergence of a numerical method based on the consistency and stability properties of the method.*

Therefore, applying the above theorem 4.4 we can infer that the conbination of consistency and stability leads to convergence.

CHAPTER 5

RESULTS AND DISCUSSION

In this chapter, the fractional initial value problems are solved using the L_3 Caputo method, which employs Caputo derivatives (1.1). The outcomes obtained from our method are presented, along with the corresponding general numerical solutions. Significant qualitative findings are reported for systems involving fractional initial value problems and Caputo-type initial value problems.

By utilizing the L_3 Caputo approach, we address the fractional initial value problem within System (1.1). This is achieved by adjusting the non-uniform grid N, the control parameter order α, and the step size h. Assuming x(0) = x_0, y(0) = y_0, and z(0) = z_0, and considering $\alpha > 0$ as the generalized Caputo-type fractional derivative operator, we tackle the problem.

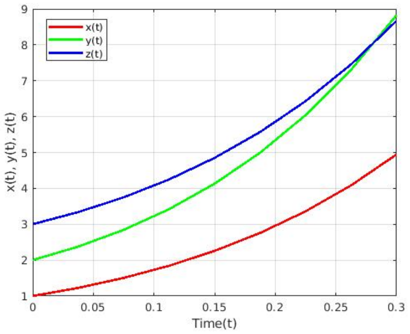
In summary, the chapter outlines the utilization of the L_3 Caputo method to solve fractional initial value problems, discusses the obtained results, and provides insights into the qualitative aspects of systems involving such problems and Caputo-type derivatives.

5.1 Numerical Examples

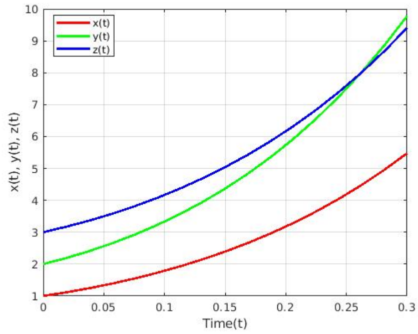
Example 1. *The first problem is to show our method.*

$$\begin{cases} D^{\alpha}x(t) = ax + y + z & x_0 = 1 \\ D^{\alpha}y(t) = x + by + z & y_0 = 3 \\ D^{\alpha}z(t) = x + y + cz & z_0 = 2 \end{cases} \tag{5.1}$$

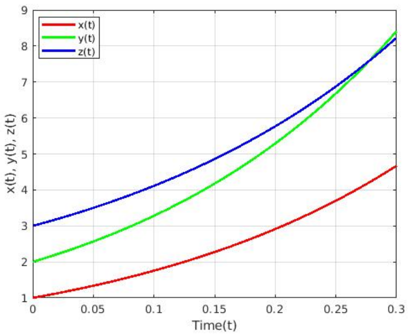
We can obtain numerical solutions for the fractional initial value problems by utilizing the parameter values (a, b, c) = (2, 3, 1) and adjusting the order parameter α as well as using an extremely small step size h.



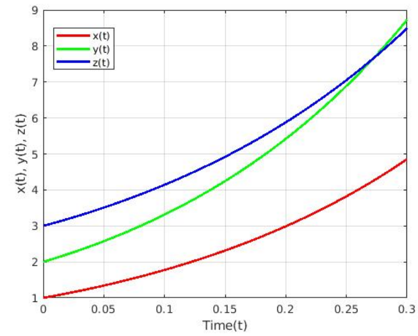
(a) α=0.3



(b) α=0.4



(c) α=0.8



(d) α=1

Figure 5.1: Solution of Example 1,with different α, step size h

Table 5.1: The result of our method for the systems

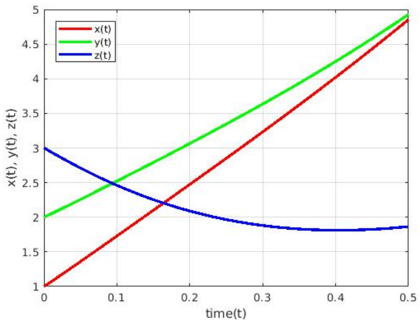
h	N	α	x	y	z
$\frac{0.3}{2}$	2	0.1	4.0000	7.0000	7.5000
$\frac{0.3}{4}$	4	0.2	4.1	7.4	7.6
$\frac{0.3}{8}$	8	0.3	5.00	8.82	8.7
$\frac{0.3}{16}$	16	0.4	5.4757	9.7768	9.4056
$\frac{0.3}{32}$	32	0.5	5.4815	9.8060	9.3974
$\frac{0.3}{64}$	64	0.6	4.0814	7.3104	7.4486
$\frac{0.3}{128}$	128	0.7	4.84	8.71	8.45
$\frac{0.3}{256}$	256	0.8	4.6	8.4	8.22
$\frac{0.3}{512}$	512	0.9	4.6	8.3	8.13
$\frac{0.3}{1024}$	1024	1	4.6	8.3	8.1

The numerical solutions to systems of fractional initial value problems (5.1) are shown in table 5.1 and figure 5.1. When (a, b, c) = (2, 3, 1) and α = 0.1, 0.2, 0.8, and 1, respectively. Where (x_0, y_0, z_0) = (1, 3, 2) when employing Caputo derivatives of type L_3. The numerical solutions produced using our technique are in excellent agreement with those found using the L_3 type Caputo method when the step size h is too small, as can be seen from the numerical results provided in table 5.1. Our technique replicates problem (1.1) on non-uniform grid sets N whose step size varies on the parameters a, b, and c, as can be observed using the graphical representations provided in figure 5.1. And also, by looking at table 5.1, the result shows that, depending on the non-uniform grid point, the order α converges to the same limit point. Based on the above discrepancies, figure (5.1) and table 5.1 show that our numerical solutions are very stable and converge to the same limit point.

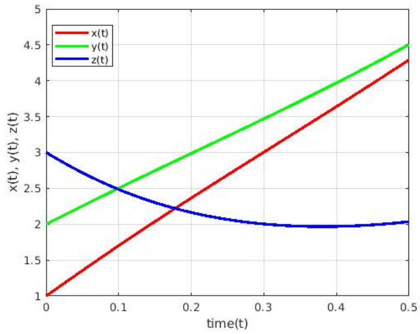
Example 2. *The second problem is to show our method.*

$$\begin{cases} D_0^{\alpha}x(t) = a(y-x)+cz \\ D_0^{\alpha}y(t) = (c-a)x-z+cy \\ D_0^{\alpha}z(t) = x+y-bz \end{cases} \tag{5.2}$$

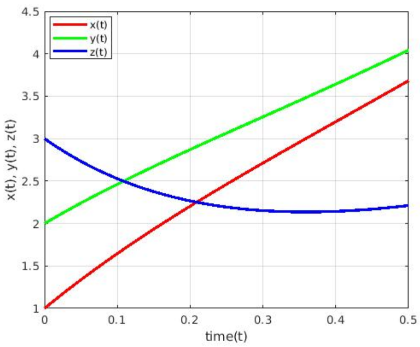
The system of fractional initial value problems is considered with the following initial conditions: $x_0 = 1$, $y_0 = 2$, *and* $z_0 = 3$. *The system has the parameters (a, b, c) = (1, 3, 2). The parameter* α *is varied, and a very small step size h is used for* $t > 0$. *The fractional derivative operator* D^{α} *represents the generalized Caputo-type fractional derivative operator. In this setup, the goal is to numerically solve the system of fractional initial value problems by employing a numerical method such as the L3 Caputo method. By adjusting the parameter* α *and choosing a small step size h, the solutions can be computed for the given time interval* $t > 0$.



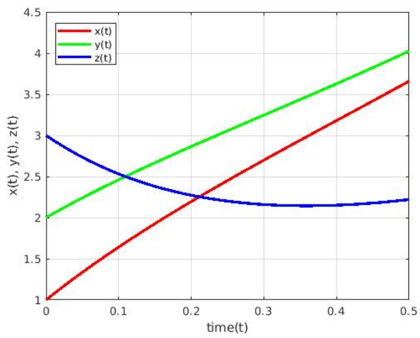
(a) α=0.5



(b) α=0.6



(c) α=0.9



(d) α=1

Figure 5.2: Solution of Example 2,with different α, step size h

Table 5.2: The result of Example 2 by using our method for the systems

h	N	α	x	y	z
$\frac{0.5}{2}$	2	0.1	4.5000	4.5000	0
$\frac{0.5}{4}$	4	0.2	3.8093	4.0067	1.7170
$\frac{0.5}{8}$	8	0.3	4.1044	4.2893	1.7928
$\frac{0.5}{16}$	16	0.4	4.4268	4.5648	1.8260
$\frac{0.5}{32}$	32	0.5	4.5171	4.6544	1.8906
$\frac{0.5}{64}$	64	0.6	4.3130	4.5120	1.9992
$\frac{0.5}{128}$	128	0.7	4.0159	4.2921	2.1046
$\frac{0.5}{256}$	256	0.8	3.8025	4.1323	2.1729
$\frac{0.5}{512}$	512	0.9	3.6997	4.0551	2.2052
$\frac{0.5}{1024}$	1024	1	3.6633	4.0279	2.2171

Table 5.2 displays the numerical solutions obtained through the FFDMs L3 type Caputo method algorithm for the fractional initial value problems described in system (5.2). The table 5.2, along with figure 5.2, presents the solutions for different values of the order parameter α, namely 0.5, 0.6, 0.9, and 1. The parameters (a, b, c) are fixed at (1, 3, 2), and the initial conditions (x_0, y_0, z_0) are set to (1, 2, 3).

The numerical results clearly indicate that as the step size h decreases and the order parameter α increases, the solutions obtained using our algorithm closely match those obtained through the L3 type Caputo method, as demonstrated in figure 5.2.

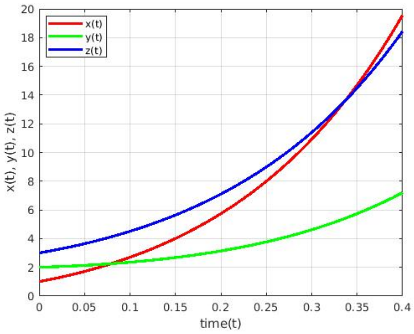
It is worth noting that despite utilizing a small step size and a non-uniform grid N, the obtained solutions exhibit smoothness and continuity. This agreement between the approximate solutions and the expected results provides strong validation for the effectiveness of our algorithm. Additionally, Figure 5.2 strengthens this conclusion by illustrating the convergence of the solutions x(t), y(t), and z(t) towards a common limiting point for various values of α.

Example 3. *The third problem is to show our method.*

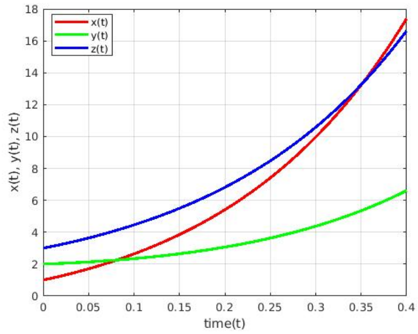
$$\begin{cases} D_0^{\alpha}x(t) = b(x+y)+az \\ D_0^{\alpha}y(t) = ax-y+z \\ D_0^{\alpha}z(t) = az+cx+by \end{cases} \tag{5.3}$$

The given system of fractional initial value problems is subject to the initial conditions $x_0 = 1$, $y_0 = 4$, and $z_0 = 3$. The parameters a, b, and c are specified as 0.2, 0.02, and 0.002 respectively, for $t > 0$. The generalized Caputo-type fractional derivative operator is denoted by D^{α} in this context.

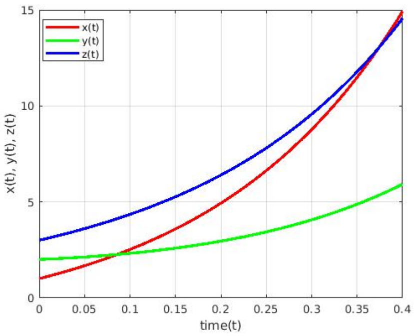
To solve these fractional initial value problems numerically, we can vary the parameter α and utilize a very small step size h. By systematically adjusting α and choosing a small h, we can obtain numerical approximations of the solutions for the given time interval $t > 0$.



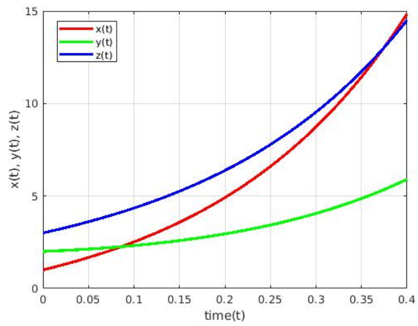
(a) α=0.5



(b) α=0.6



(c) α=0.9



(d) α=1

Figure 5.3: Solution of Example 3,with different α, step size h

Table 5.3: The result of our method for Example 3

h	N	α	x	y	z
$\frac{0.4}{2}$	2	0.1	8.0623	3.6637	9.1704
$\frac{0.4}{4}$	4	0.2	11.1516	4.6295	11.6637
$\frac{0.4}{8}$	8	0.3	14.8231	5.7311	14.6497
$\frac{0.4}{16}$	16	0.4	17.6839	6.5986	16.9623
$\frac{0.4}{32}$	32	0.5	18.3014	6.8219	17.4220
$\frac{0.4}{64}$	64	0.6	17.2217	6.5469	16.4928
$\frac{0.4}{128}$	128	0.7	15.9566	6.2041	15.4266
$\frac{0.4}{256}$	256	0.8	15.2149	6.0008	14.8040
$\frac{0.4}{512}$	512	0.9	15.000	6.00	15.000
$\frac{0.4}{1024}$	1024	1	15.000	6.000	15.000

Table 5.3 presents the numerical solutions obtained for the fractional Chen system (5.3) using both the FFDMs and L_3 type Caputo method algorithms. The solutions are computed for different values of the order parameter α =0.1, 0.2, 0.9, and 1 with fixed parameters (a, b, c) set to (0.2, 0.02, 0.002). The initial conditions are specified as $(x_0, y_0, z_0) = (1, 4, 3)$.

The numerical results, as shown in table 5.3 and figure 5.3, reveal that as the step size h becomes smaller and the order parameter α increases, the solutions obtained using our algorithm align closely with those obtained using the L_3 type Caputo method, as evidenced by their agreement in figure 5.3.

It is important to note that for very small step sizes and high values of α, the computed solutions are smoother and more continuous. However, as the step size h decreases and α increases, the approximate solutions for $y(t)$, $x(t)$, and $z(t)$ tend to increase. Despite this, the overall behavior of the solutions remains stable and converges to the same limit point for different α values, as observed in figure 5.3 and table 5.3.

The system exhibits stability, smoothness, continuity, and convergence properties, which validate the reliability and effectiveness of our algorithm.

Conclusion and Recommendations

Conclusion

This study focuses on the numerical solutions of systems of fractional initial value problems using the L_3 type Caputo method of Caputo derivatives. The study introduces the concept of fractional initial value problems and proposes the use of the L_3 type Caputo method to obtain numerical solutions for such problems. Furthermore, the explores the existence and uniqueness of solutions by applying the Lipschitz continuity condition. To facilitate the numerical computation, a discretization approach is adopted, and a Picard operator is defined based on this discretization. The numerical results demonstrate the consistence, stability and convergence of the suggested approach, which utilizes a limited number of basic functions. These findings contribute to the understanding and practical implementation of the L_3 type Caputo method for solving systems of fractional initial value problems.

Recommendations

This thesis specially focused on numerical solution for systems of fractional initial value problems. Systems of fractional initial value problems is very interesting and easy to find the approximate numerical and analytical solutions on L_3 type caputo method. Hence, we recommended that, future researchers should try to compare the L_3 type caputo method with other method to generalize the efficiency, validity of this method.

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