Mathematical Modeling and Optimal Control of Corruption Dynamics

By

Legesse Lemecha (Ph.D)
Shiferaw Feyissa (Ph.D)

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Abstract

Several research reports have shown that corruption is an impediment to growth, as it mainly constitutes hindrance to investment. It has adverse impacts on the economy and on well-being of a society. In this study, we propose a mathematical model for corruption dynamics by considering awareness created by anti-corruption and counciling in jail. The model is proved to be both epidemiologically and mathematically well posed. We have shown that all solutions of the model are positive and bounded with initial conditions in a certain meaningful set. The existence of unique corruption-free and endemic equilibrium points are investigated and the basic reproduction number is computed. Then we study the local asymptotic stability of these equilibrium points. The analysis shows that the system has a locally asymptotically stable corruption-free equilibrium point when the reproduction number is less than one and locally asymptotically stable endemic equilibrium point for the reproduction number is greater than one. The simulation result shows the agreement with the analytical results. Further, we apply optimal control techniques to a corruption controlled mathematical model to determine the optimal control strategy in order to minimize the number of susceptible and corrupt populations. The control strategies are based on education campaign (awareness creation) and law enforcement. We then proved the existence of optimal control problem, determined the necessary conditions for optimality, and then performed numerical simulations. The numerical results showed that the control strategy that involve two control measures have significant impact in reducing corruption dynamics.

Keywords: Mathematical model; Corruption; Basic reproduction number; Stability; Optimal Control.

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Notation and Abbreviations

PMP	Pontryagin's maximum principle
OP	Optimal Control
FBS	Forward-backward sweep
TICPI	Transparency International Corruption Perception Index

Chapter 1
Introduction

1.1 Background of the study

Corruption is a pressing global problem and no country is absolutely free of its menacing grip. It is a cancer to economic, social, and political development of a country. In the developing countries, most of the central government institutions are overtly threatened by corruption which is both deliberate and unconscious break down in human morals by public servants. These in turn gravitate to undermine democratic processes in the governance structures and also challenging situation to governance both at centralized and devolved governance (Le Van, C. and M. Maurel., 2006).

Corruption remains one of the major obstacles to economic prosperity in many countries. It is known to distort incentives, impede investment and divert the allocation of productive resources to rent-seeking activities as cited in (Shleifer. A and Vishny, 1993) and references therein. Broadly, corruption can be classified into three categories which are mutually exclusive. To briefly summarize:

(a) **Petty and Grand corruption**

- **Petty corruption** is practised on a smaller scale. It is defined as the use of public office for private benefit in the course of delivering a pubic service. The direct victim of this abuse of power is the citizen.
- **Grand corruption** is the most dangerous and covert type of corruption. It occurs at financial, political and administrative centres of power.

(b) **Political and Business corruption**

- **Political corruption** occurs predominantly in developing and less developed countries. Usually associated with the electoral process such as voting irregularities, holding on to power against the will of the people and false political promises.

- **Business corruption** is often not regarded as a crime, rather as a means to accelerate business processes. Usually it includes bribery, insider trading, money laundering, embezzlement, tax evasion and accounting irregularities.

(c) **Chaotic and Organized corruption**

- **Chaotic corruption** is a disorganized system where there is no clarity regarding whom to bribe and how much payment should be offered. As main feature, there is no:
  - guarantee that further bribes will not have to be paid to other officials;
  - reasonable assurance that the favor will be delivered;
  - coordination between the recipients of benefits, with the result
- **Organized corruption** is often perpetrated by crime gangs and syndicates which includes white-collar crime and identity. A well-organized system of corruption in which there is a confident that they will receive the favor in return, how much should be offered and of whom to bribe.

Different scholars from social sciences, psychology, political sciences, and religious studies have attempted to give a working definition for corruption from their various disciplines. However all these definition are interwoven (Olagunju, 2012). In everyday use, corruption is a term which conveys an element of moral disapproval. The World Bank has defined corruption as "the abuse of public office for private gain" (World Bank, 1997). Further, Transparency International (TI) defined corruption as "the misuse of entrusted power for private benefit" (Pope, 2000). In a more legal form, (Ogbu, 2008) defined corruption as "an abuse of position or inducement for an undeserved benefit, advantage or relief".

According to the Global Integrity Report of 2006, corruption is considered as norm of social, economic and political intercourse in Ethiopia. Currently, corruption is extremely rampant in Ethiopia and challenging its politics, governance and public sector. Obtaining robust measure of corruption is very difficult task due to the illicit and secretive nature of corruption practices.

Numerous studies on corruption have been carried out due to increasing public interest and concern over its universal threat to humanity. Some studies (Leff, 1964; Huntington,1968; Friedrich, 1972; Lui,1986; Beck and Marker , 1986; Lein,1986) have pointed out the desirability of some corruption on the ground that it can be beneficial to the functioning of the economy. However, most of the study see for example, Myrdal (1991), Rose-Ackerman (1975), Gould and Amaro-Reyes (1983), Baumol (1990), Murphy et al., (1991) and Klitgaard (1991) concluded that corruption undermines the economic, political as well as social development. It is both a major cause and a result of poverty around the world.

There are several sectors where businesses are highly vulnerable to corruption. For example, sectors such as tax and land administrations are susceptible to corruption. It also occurs while processing business permits and licenses as a result of complicated bureaucracy. Public procurement is also seriously hampered by corruption, and different types of irregularities such as non-transparent tender processes and awarding contracts to people with close connection to the government.

Mathematical models have significant role in understanding the spread of corruption and its control. For instance, the epidemiological corruption modelling approach was reported by many authors including(Starkermann,1989; Le Van and Maurel, 2006; Becker et al., 2008; Brianzoni et al., 2011; Sayaji R. Waykar, 2013 and Abdulrahman, S. 2014). In all these literatures, the effects of corruption on national development was studied by modelling corruption as a diseases. In particular, Hathroubi, S. (2013), investigated an epidemiological corruption threshold based on the approximation of honest population by dividing the total population $N(t)$ into three compartments Susceptible $S(t)$, Corrupt $C(t)$, and Honest $H(t)$ individual without giving stability analysis of the equilibrium points. However, the author didn't consider well the case of losing immunity of jailed corrupts.

The aim of this work is to model mathematically the transmission process of corruption, which can be defined generally as follows: when a reasonable amount of corrupted individuals are introduced into a population of susceptible, the corruption is passed to non-corrupt individuals and thus spreading in the population. It is assumed to be a non standard epidemic process that rarely emerges out of nothing but is usually related to some already corrupt environment which may affect susceptibles. Hence, in this research report we present a mathematical model for the spread of corruption in the sense of epidemiology to describes the dynamical behaviour of corruption. The study also take into account losing immunity that is gained in jail, the role of anti-corruption and effort rate against corruption. Optimal controlling strategies are also studied by assuming reasonable size of sub-population.

1.2 Statement of the Problems

Corruption is a complex and multifaceted phenomenon (Aidt, 2003) associated with all forms of human organization. It is a symptom and result of institutional weakness, having negative effects on the economic performance of a country (Bardhan, 1997). As cited in Grass et al (2008), the World Bank estimated the annual cost of corruption over US $80 billion worldwide which is more than the total of all economic assistance for development. Corruption distorts the rule of law and weakens the institutional foundations on which economic growth depends (World Bank, 2014) and thus identified among the greatest obstacles to economic and social development. It degrades national security, economic prosperity and international reputation. Corruption is the deep rooted cause of instability and conflict as witnessed in the present situation in Ethiopia.

The epidemiological corruption modelling approach was reported by many authors including(Starkermann,1989; Le Van and Maurel, 2006; Becker et al., 2008; Brianzoni et al., 2011; Sayaji R. Waykar, 2013 and Abdulrahman, S. 2014). The authors studied the effects of corruption on national development by modelling corruption as a diseases. In particular, Hathroubi, S. (2013), investigated an epidemiological corruption threshold based on the approximation of honest population by dividing the total population $N(t)$ into three compartments Susceptible $S(t)$, Corrupt $C(t)$, and Honest $H(t)$ individual without giving stability analysis of the equilibrium points. However, the author didn't considered well the case of losing immunity of jailed corrupts . More recently, Anthithan S. et al, (2018) studied a corruption control model using the theory of differential equation and optimal control without considering jailed class.

Thus, the present research report is motivated to develop a mathematical model using the theory of differential equation and optimal control to minimize the spread of corruption dynamics. Optimal control strategies are used to predict the possible future outcome in terms of resource utilization for corruption control and the effectiveness of awareness on corrupted and susceptible populations. The study fills the existing gap by taking into account losing immunity of jailed and effort rate against corruption.

Therefore, this research work attempt to address the following basic questions:

1. What assumptions are helpful to formulate a mathematical model which describes the dynamics of corruption ?
2. Are the equilibrium points are locally and globally stable?

3. What are the optimal control mechanisms to minimize the expansion of corruption?

1.3 Objective of the Study

1.3.1 General Objectives

This study aims to understand the dynamical nature of corruption mathematically and propose the optimal control strategies to minimize its expansion.

1.3.2 Specific Objectives

The specific objectives of the proposed study are to:

- develop a mathematical model that describe the dynamical nature of corruption.
- propose an optimal control strategies that are used to minimize the expansion of corruption.
- investigate the stability of both corruption free and endemic equilibria.

1.4 Significance

The study use mathematical modeling approaches to precisely figure out the impact of corruption on the economic development. Thus, the outcome of this study benefits policy makers to design optimal corruption control and preventions mechanisms. Also, initiate other researchers to undertake further extension and rigorous mathematical analysis.

Chapter 2
Methodology

In this chapter we briefly present study design and mathematical techniques used in this report.

2.1 Descriptions of Mathematical Procedures

To achieve the objective of this research, analytical and numerical method were employed. We developed a mathematical model which describes the dynamics of corruption using a system of non-linear differential equations. Well possedness of the formulated model is determined both in the mathematical and epidemiological sense. We then perform the qualitative analysis of the model. Local stability analysis of the equilibrium points of the model is investigated. The numerical simulation was conducted using MATLAB to confirm the agreement between analytical and numerical results. Further, optimal control is formulated and then numerical technique was employed in order to optimize the performance index subject to model equations as constraints.

2.1.1 Numerical methods for system of ODEs

The following numerical techniques are summarized from (Lenhart, S., & Workman, J. T. (2007), Amos, G. Vish S. (2014), Rodrigues, H. S., et al (2014)). When a system of ODEs is too extensive or composed by equations that can not be solved analytically, numerical methods are used to compute its approximate solution. For example, Euler's and fourth order Runge-Kutta methods can be used to solve initial-value problems. Suppose we want to solve the following initial value problem on the interval $t_0 \leq t \leq T$:

$$x' = \frac{dx}{dt} = f(t, x(t)), x(t_0) = x_0$$

Then, we divide the interval $t_0 \leq t \leq T$ into N small segments of constant length h, called the step size given by

$$h = \frac{T - t_0}{N}$$

This defines the set of equally spaced discrete times $t_0, t_1, t_2, \cdots, t_N = T$, where $t_n = t_0 + nh, n = 0, 1, 2, \cdots, N$. Note that $h = t_{n+1} - t_n$. We develop a recursive formula that gives the approximation X_{n+1} at time t_{n+1} in terms of the previous approximation X_n at time t_n.

2.1.1.1 Euler's Method

Euler's method approximates the equation $x' = \frac{dx}{dt} = f(t, x(t)), x(t_0) = x_0$ by

$$x_{n+1} = x_n + hf(t_n, x_n), n = 0, 1, 2, \cdots, N.$$

From an equivalent perspective, this method can be described by the following equation:

$$x(t + h) = x(t) + h\frac{dx}{dt},$$

where the h is the step size defined above. This method computes the value at $t + h$ with an error of order $O(h^2)$. The time step h is crucial to determine the accuracy of the approximation. Here, the smaller the value of h, the greater the accuracy of the approximation. However, the smaller the value of h, the greater the computational effort required. As such, for higher order systems of ODEs, due to its dependency on h, the Euler's method can become unstable and, consequently, does not converge to the exact solution.

2.1.1.2 Systems of equations

The numerical methods introduced for a single equation can be easily extended to systems of equations. For example, consider the two-dimensional system

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

with initial conditions

$$x(t_0) = x_0,\ y(t_0) = y_0$$

We want to obtain a numerical solution on the interval $t_0 \leq t \leq T$ .The first step is to discretize the interval as before by defining the time steps by

$$t_n = t_0 + nh, n = 0, 1, 2, \cdots, N,$$

where N is the number of steps. Again, we use the notation X_n and Y_n to denote the approximate values of $x(t_n)$ and $y(t_n)$, respectively. We can summarize the methods as follows.

Euler method

$$\begin{aligned} X_{n+1} &= X_n + hf(t_n, X_n, Y_n), \\ Y_{n+1} &= Y_n + hg(t_n, X_n, Y_n). \end{aligned}$$

Runge-Kutta method

First we compute the values of the eight slopes:

$$\begin{aligned} k_{11} &= f(t_n, X_n, Y_n) \\ k_{21} &= g(t_n, X_n, Y_n) \\ k_{12} &= f\left(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_{11}, Y_n + \frac{h}{2}k_{21}\right) \\ k22 &= g\left(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_{11}, Y_n + \frac{h}{2}k_{21}\right) \\ k13 &= f\left(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_{12}, Y_n + \frac{h}{2}k_{22}\right) \\ k23 &= g\left(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_{12}, Y_n + \frac{h}{2}k_{22}\right) \\ k14 &= f\left(t_n + X_n + hK_{13}, Y_n + hK_{23}\right) \\ k24 &= g\left(t_n + X_n + hK_{13}, Y_n + hK_{23}\right) \end{aligned}$$

Then we compute the next approximation using weighted averages of these slopes,

$$\begin{aligned} X_{n+1} &= X_n + \frac{h}{6}\left(K_{11} + 2K_{12} + 2K_{13} + K_{14}\right) \\ Y_{n+1} &= Y_n + \frac{h}{6}\left(K_{21} + 2K_{22} + 2K_{23} + K_{24}\right) \end{aligned} \tag{2.1}$$

In such process, the continuous time model is reduced to an approximate discrete-time model that is amenable to computer simulation.

There are two major classes of numerical methods for solving OC problems: indirect and direct methods. The first ones indirectly solve the problem by converting the optimal control problem to a boundary-value problem, using the Pontryagin's maximum principle-PMP. On the other hand, in a direct method, the optimal solution is found by transcribing an infinite-dimensional optimization problem to a finite-dimensional optimization problem which we do not consider in this report.

2.1.2 Indirect method

In an indirect method, the Pontryagin's maximum principle(PMP) is used to determine the first order optimality conditions of the original optimal control problem. For an indirect method it is necessary to explicitly get the adjoint equations, the control equations and all the transversality conditions, if they exist. Indirect methods use the PMP to solve OC problems. More precisely, PMP is used to compute the first-order optimality conditions. Thus, the optimal trajectories are derived by solving a multiple-point boundary-value problem.

2.1.2.1 Backward-forward sweep method

At this point, as an indirect numerical approach, forward-backward sweep method (FBS) is presented in (Suzanne L. and Workman J. T., (2007) and references therein. This method can be used to solve problems with the minimization problems , and generates numerical approximations to the optimal piecewise continuous control $u^*(t)$. The process begins with an initial guess on the control variable. Then, the state equations are simultaneously solved forward in time and the adjoint equations are solved backward in time. The control is updated by inserting the new values of states and adjoints into its characterization, and the process is repeated until convergence occurs . It can be implemented, using the following algorithm: Considering $\mathbf{x} = (x_1, x_2, \ldots, x_{N+1})$ and λ = $(\lambda_1, \lambda_2 \ldots, \lambda_{N+1})$ the vector approximations for the state and the adjoint respectively. Hence, this algorithm can be summarized as follows.

Remark 2.1. The convergence speed can be improved when a convex combination between the control values given by the characterization of the optimal control in step 4 and the previous ones is used.

Algorithm 1 Forward-Backward Sweep Method

1: Make an initial guess for $\mathbf{u}$ over the time interval ($\mathbf{u} \equiv 0$ is almost always sufficient)
2: Using the initial condition $x_1 = x(t_0) = a$ and the values for $\mathbf{u}$, solve $\mathbf{x}$ **forward** in time in compliance with its differential equation in the optimality system(using, e.g., RK4).
3: Using the transversality condition $\lambda_{N+1} = \lambda(t_f)$ and the values for $\mathbf{u}$ and $\mathbf{x}$, solve λ **backward** in time according to its differential equation in the optimality system (using, e.g., RK4).
4: Update $\mathbf{u}$ using the new values for $\mathbf{x}$ and λ into the characterization of the optimal control.
5: Check convergence: if the variables are sufficiently close to the corresponding in the previous iteration, then output the current values as solutions, else return to Step 2.

We make a few notes about the algorithm. When making the initial guess, $\mathbf{u}$ is almost always sufficient. For certain problems, where division by u occurs, an alternative initial guess must be used. Occasionally, our initial guess may require adjusting if the algorithm has problems converting. For step 2 and 3, any standard ODE solver can be used. For the purpose of this research report, a Runge-Kutta method of order four is used.

Chapter 3
Literature Review

3.1 Mathematical modelling of corruption dynamics

The World Bank (1997) defined corruption as the abuse of public office for private gain. In this sense, corrupt practices includes: bribery, extortion, fraud, embezzlement, nepotism, cronyism, appropriation of public institutions assets and properties for private use, and influence peddling. It is a symptom and result of institutional weakness, having negative effects on the economic performance of a country (Bardhan, 1997). Corruption is a complex and multifaceted phenomenon (Aidt, 2003) associated with all forms of human organization. As cited in Grass et al (2008), the World Bank estimated the annual cost of corruption over US $80 billion worldwide which is more than the total of all economic assistance for development. Corruption distorts the rule of law and weakens the institutional foundations on which economic growth depends (World Bank, 2014) and thus identified among the greatest obstacles to economic and social development. Corruption degrades national security, economic prosperity and international reputation. It is the deep rooted cause of instability and conflict as witnessed in the present situation in Ethiopia.

Corruption could be characterized as a ”disease” inherent to public power and an indication of bad governance (Tiihonen, 2003). However, as reported in Blanchard Ph. et al (2005) the features of corruption propagation differs from classical epidemic processes due to its dependence on the threshold value of the local transition probabilities and the mean field dependence of the corruption process. By this we mean that a non-corrupt individual gets infected with high probability if the number of corrupt individuals in the social neighbourhood exceeds a certain threshold value whereas in the case of mean field dependence an individual can get corrupt because there is a high prevalence in the society even in the absence of corruption in the local neighbourhood. In contrast to the high prevalence of corruption worldwide and large literature on political, social and economical aspects of corruption there is only a small number of attempts to model the dynamics of corruption in a mathematically quantified way. In this regard, the first mathematical model dedicated to corrupt structures was appeared in (Rose-Ackerman, 1975) and become active research area since then. Several literature showed that corruption smothering economic development across vast stretches of the globe, notably in Africa and Asia countries. For example, see in (Tanzi,1998, Le Van, C. and Maurel M., 2006, Grass et al, 2008) and reference cited therein. It impacts negatively on a variety of economic and social well-being of the society and thus described as a worm within the body of the society and a cancer to economic, social, and political development (Abdulrahman, S., 2014; Tanzi V. 1998; and Sayaji R. Waykar, 2013). Corruption is an unavoidable part of human social interaction, prevalent in every society at any time since the very beginning of human history till today. It cannot be measured directly due to its elusive nature and the difficulty of separating the control of corruption from corruption itself.

The epidemiological corruption modelling approach was reported by many authors including(Starkermann,1989; Le Van and Maurel, 2006; Becker et al., 2008; Brianzoni et al., 2011; Sayaji R. Waykar, 2013 and Abdulrahman, S. 2014). The authors studied the effects of corruption on national development by modelling corruption as a diseases. In particular, Hathroubi, S. 2013, investigated an epidemiological corruption threshold based on the approximation of honest population by dividing the total population $N(t)$ into three compartments Susceptible $S(t)$, Corrupt $C(t)$, and Honest $H(t)$ individual without giving stability analysis of the equilibrium points. However, the author didn't considered well the case of losing immunity of jailed corrupts .

3.2 Optimal control of corruption dynamics

Dieter Grass, et al. (2008), proposed a one-dimensional model for corruption dynamics by classifying the population as : honest (h) and corrupt (c) by denoting the state variable $x(t)$ the proportion of people who are corrupt at time t, i.e., $0 \leq x(t) \leq 1$. According to this model, people move back and forth between states in response to differences in the reward or pay-off offered by the various states. That is, the model assumes that if being honest offers a higher utility, then corrupt people will flow back into the honest state at a per capita rate that is proportional to the difference in utility. However, if being corrupt offers a higher utility, then corrupt people will recruit honest people into corruption at a rate that is proportional to $U_c - U_h$, so that the aggregate rate $\frac{dx}{dt}$ is proportional to the product of the number of corrupt people $x(t)$ and the utility difference $(U_c - U_h)$. In this context, the term utility $U(t)$ denotes the income minus the expected sanction

$$U_c(t) - U_h(t) = w - \frac{u_0 + u(t)}{x(t)}, \tag{3.1}$$

where w is the premium or reward, u is the corruption control parameter and a certain fixed amount u_0 of sanctioning. The governing model was summarized as follows:

$$\dot{x}(t) = k\left[wx(t) - u(t)\right],\ k, w \geq 0, \tag{3.2}$$

and the corresponding dynamic optimization model with the objective function to be minimized, i.e.,

$$\min_{u(\cdot)} \int_0^{\infty} e^{-rt}\left(x(t) + \frac{1}{2}u^2(t)\right)dt, \tag{3.3}$$

together with the state constraints: $0 \leq x \leq 1$. This modeling approach did not considered detail corruption activities stated above.

Anthithan S. et al, (2018) studied a corruption control model using the theory of optimal control. That is, Pontryagin's Maximum Principle was applied to exhibit the effect of optimal control. Their study predicted that the parameter corresponding to the self cure rate as optimal control offers better results than the fixed control. Optimal control theory is a powerful mathematical tool which makes the decision involving complex dynamical systems. As noted in Lenhart, S., & Workman, J. T., (2007), it is a standard method for solving dynamic optimization problems, when those problems are expressed in continuous time. Optimal control is the process of determining control and state trajectories for a dynamic system over a period of time to minimize a performance index . The formulation of an optimal control problem requires a mathematical model of the system to be controlled, a specification of the performance index, and a specification of all boundary conditions on states and constraints to be satisfied by states and controls. Pontryagin's maximum (or minimum) principle of optimal control gives the fundamental necessary conditions for a controlled trajectory to be optimal, for example see (Fleming, W. H., & Rishel, R. W. (2012), Sch*ä*ttler, H., & Ledzewicz, U. (2012), Rodrigues, H., S.et al (2014)) and references cited therein.

Thus, in this research report we present a mathematical model for the spread of corruption dynamics in the sense of epidemiology and extend the model to optimal control problem to suggest possible intervention mechanism.

Chapter 4
Model Formulation

4.1 The Model

In this section we consider a SCJH (Susceptible-Corrupt-Jailed-Honest) type mathematical model for the dynamics of corruption. The total population $N(t)$ is divided into four compartments: Susceptible $S(t)$, Corrupt $C(t)$, in council through Jail $J(t)$ and Honesty $H(t)$ at time $t \geq 0$ as given in Abdulrahman, S. (2014). In this report, we meant by a susceptible individual that have never engage in any corrupt practice or those jailed and then released after taking council. A corrupt is one that have engage in a corruption practice and capable of influencing a susceptible individual to become corrupt. A jailed corrupt is one that is found to be guilty corrupt by law and a honest individual is one that can never be corrupt under any situation.

We assume that there is a positive recruitment Λ into the susceptible class and a positive natural death rate μ for all time under the study. Susceptible individuals can become corrupt at rate $\frac{\beta C(t)}{\theta + C(t)}$ through contact with corrupt individuals which is depend on time. The remaining model parameters are also defined as follows: τ is the effort rate against corruption, p is the corruption transmission probability per contact, θ is the maximum saturation constant of the corrupt population in the society, $\beta = p(1-\tau)$ is the effective corruption contact rate, δ is the rate at which corrupt individuals are caught and imprison, $\frac{1}{\gamma}$ is the average period jailed individuals spent in prison, α is the proportion of individuals that leaves S, C and J to H compartment due to awareness created by anti-corruption agency through mass media, and counselling in jail respectively on the impact of corruption.

The model also depends on the following assumptions: an individual can be corrupted only through contacts with corrupted individuals, a susceptible, corrupt, and jailed individual can be converted to honest due to awareness created by anti-corruption or counciling in jail. Compared to the work of Abdulrahman, S. (2014), an individual who lose immunity gained through council in jail do not directly join corrupt class rather susceptible with the rate of $\gamma(1-\alpha)$ because of human behaviour. Further, our model takes into consideration the impact due to awareness created by anti corruption. The model prosed in Anthithan S. *et al* (2018) do not consider the jailed class which make differ from our model.

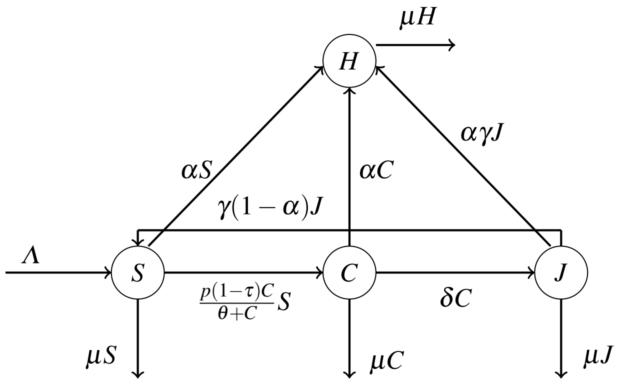


Fig. 4.1: Schematic Diagram of Corruption Dynamics Model.

These assumptions are translated to the following governing mathematical model:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta C}{\theta + C}S - (\alpha+\mu)S + \gamma(1-\alpha)J \\
\frac{dC}{dt} &= \frac{\beta C}{\theta + C}S - (\delta+\alpha+\mu)C \\
\frac{dJ}{dt} &= \delta C - (\gamma+\mu)J \\
\frac{dH}{dt} &= \alpha(S + C + \gamma J) - \mu H.
\end{aligned} \tag{4.1}$$

The model involves human population and hence all the parameter used are positive. Further, we assume that the initial conditions

$$S(0) \geq 0,\ C(0) \geq 0,\ J(0) \geq 0,\ H(0) \geq 0 \tag{4.2}$$

of the governing equations are non-negative throughout this research report.

4.2 Model Analysis

In this section, we study the solution of (4.1) in the epidemiologically feasible region:

$$\Omega = \left\{ (S,C,J,H) \in \mathbb{R}^4_+ : 0 \le S(t) + C(t) + J(t) + H(t) \le \frac{\Lambda}{\mu} \right\}.$$

4.2.1 Positivity and boundedness of solutions

Now we study the positivity and boundedness of solutions of the governing system (4.1).

Proposition 4.1. . *Assume that the initial conditions* (4.2) *holds in* $\mathbb{R}^4_+$. *Then the solutions* $\{S(t), C(t), J(t), H(t)\}$ *of the model equations are also non-negative for all* $t \ge 0$.

Proof. Assume that all the state variables are continuous. Then from the system of equations (4.1) one can easily obtain that

$$\begin{aligned}
\frac{dS}{dt} &\ge -\frac{\beta C}{\theta + C} S - (\alpha + \mu) S; \\
\frac{dC}{dt} &\ge -(\delta + \alpha + \mu) C; \\
\frac{dJ}{dt} &\ge -(\gamma + \mu) J; \\
\frac{dH}{dt} &\ge -\mu H.
\end{aligned}$$

Solving these system of differential equations yields

$$\begin{aligned}
S(t) &\ge S(0) exp\left(\int \left(\frac{\beta C}{\theta + C} - (\alpha + \mu) \right) dt \right) \ge 0; \\
C(t) &\ge C(0) exp(-(\alpha + \delta + \mu) t) \ge 0; \\
J(t) &\ge J(0) exp(-(\gamma + \mu) t) \ge 0; \\
H(t) &\ge H(0) exp(-(\mu) t) \ge 0.
\end{aligned}$$

Thus, we can conclude that all the solutions are non-negative in $\mathbb{R}^4$ for all $t \ge 0$.

The next proposition shows that it is sufficient to study the dynamics of corruption by model equations (4.1) in a region Ω.

Proposition 4.2. . *Assume that all the initial conditions are non-negative in* $\mathbb{R}^4_+$ *for the system* (4.1) *and*

$$\Omega = \left\{ (S,C,J,H) \in \mathbb{R}^4_+ : 0 \leq S(t) + C(t) + J(t) + H(t) \leq \frac{\Lambda}{\mu} \right\}.$$

If $N(0) \leq \frac{\Lambda}{\mu}$, *then the region* Ω *is positively invariant.*

Proof. Note that all the state variables $S,C,J,H \in C(\mathbb{R}^+,\mathbb{R}^+)$ and the total population

$$N(t) = S(t) + C(t) + J(t) + H(t).$$

From this, we can easily deduce that

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t).$$

Using the assumption that $N(0) \leq \frac{\Lambda}{\mu}$ and by integration we can have that $N(t) \leq \frac{\Lambda}{\mu}$ that is, $N(t)$ is bounded for all $t \geq 0$. This implies all the solutions of the model equation (4.1) with initial condition in Ω remains in Ω. This complete the proof.

Remark 4.1. In the region Ω, the proposed mathematical model is mathematically and epidemiologically well posed.

4.2.2 Stability Analysis

The corruption free equilibrium point (CFE) of the system (4.1) is given by

$$\bar{\mathrm{E}} = (\bar{S},\bar{C},\bar{J},\bar{H}) = \left(\frac{\Lambda}{\alpha+\mu}, 0, 0, \frac{\alpha\Lambda}{(\alpha+\mu)\mu} \right) \tag{4.3}$$

Following the approach of Abdulrahman, S. (2014) and Anthithan S. *et al* (2018) we compute the basic reproduction number R_0. In the present work, the basic reproductive number is the expected number of new corrupts from one corrupt individual in a fully susceptible population through contact period. Now we have the following Lemma.

Proposition 4.3. . *The basic reproduction number of the mathematical model* (4.1) *is given by*

$$\mathrm{R}_0 = \frac{\beta\Lambda}{(\alpha+\mu)(\delta+\alpha+\mu)\theta} \tag{4.4}$$

Proof. Let the matrix $\mathscr{F}$ with the (i,j) entry denotes the number of new corrupts at stage j caused by contacts with corrupt individuals in stage i and the transition matrix $\mathscr{V}$ with (i,j) entry denotes the rate individuals in stage j progress to stage i. From model(4.1) we have $(S'(t),C'(t),J'(t),H'(t))^T = \mathscr{F} - \mathscr{V}$ such that

$$\mathscr{F} = \begin{pmatrix} \frac{\beta C}{\theta + C} S \\ 0 \end{pmatrix} \text{ and } \mathscr{V} = \begin{pmatrix} aC \\ -\delta C + bJ \end{pmatrix}.$$

where $a = \delta + \alpha + \mu$, $b = \gamma + \mu$. The Jacobian matrices of $\mathscr{F}$ and $\mathscr{V}$ at equilibrium point are given by $\mathrm{F} = \begin{pmatrix} \frac{\beta \bar{S}}{\theta} & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathrm{V} = \begin{pmatrix} a & 0 \\ -\delta & b \end{pmatrix}$.

Hence the basic reproduction number

$$\mathrm{R}_0 = \rho(\mathrm{FV}^{-1}) = \frac{\beta\Lambda}{(\alpha+\mu)(\delta+\alpha+\mu)\theta}.$$

This concludes the proof.

Theorem 4.1 (Stability of the CFE). . *The corruption-free equilibrium $\bar{E}$ of the dynamical system is*

(a) locally asymptotically stable if $\Lambda\beta < a(\alpha+\mu)\theta$ and unstable otherwise,
(b) neutral if $\Lambda\beta = a(\alpha+\mu)\theta$.

Proof. The characteristic polynomial of the corresponding linearised system of the governing equation is $p(\lambda) = det((\mathrm{F}-\mathrm{V}) - \lambda I_4)$. That is,

$$\begin{vmatrix} -b-\lambda & \frac{\beta\Lambda}{(\alpha+\mu)\theta} & \gamma(1-\alpha) & 0 \\ 0 & \frac{\beta\Lambda}{(\alpha+\mu)\theta} - a - \lambda & 0 & 0 \\ 0 & \delta & -b-\lambda & 0 \\ \alpha & \alpha & \alpha\gamma & -\mu-\lambda \end{vmatrix} = 0$$

Evaluating the roots of this characteristic polynomial we have

$$\lambda = -b, -b, \text{ or } \lambda = -\mu \text{ or } \lambda = \frac{\beta\Lambda}{(\alpha+\mu)\theta} - a.$$

As the model deals with human population we assumed all the model parameters are non-negative. Consequently, it is straightforward that the corruption free-equilibrium $\bar{E}$ is asymptotically stable for $\lambda = -b$ with multiplicity two or $\lambda = -\mu$. On the other hand, the CFE $\bar{\mathrm{E}}$ is locally asymptotically stable only if $\frac{\beta\Lambda}{(\alpha+\mu)\theta} - a < 0$ or equivalently, $\Lambda\beta < a(\alpha+\mu)\theta$.
Trivially, a neutral case occurred if $\Lambda\beta = a(\alpha+\mu)\theta$ and it becomes inconclusive. With this we complete the proof of the theorem.

In the following theorem we shall investigate the existence of corruption endemic equilibrium point and its stability based on the computed reproduction number.

Theorem 4.2. . *If* $\mathrm{R}_0 > 1$*, then the*

i. *model*(4.1) *has a corruption endemic equilibrium point given by*

$$\tilde{E} = (\tilde{S}, \tilde{C}, \tilde{J}, \tilde{H}) \tag{4.5}$$

where

$$\tilde{S} = \frac{(\mu+\alpha+\delta)\left(\mu^2\theta+(\theta\,\alpha+\theta\,\gamma+\delta\,\theta+\Lambda)\,\mu+(\theta\,(\delta+1)\,\alpha+\Lambda)\,\gamma\right)}{\mu^3+(\gamma+2\,\alpha+\beta+\delta)\,\mu^2+(\alpha^2+(\beta+2\,\gamma+\delta)\,\alpha+(\beta+\delta)\,\gamma+\delta\,\beta)\,\mu+(\alpha+(\beta+1)\,\delta+\beta)\,\alpha\,\gamma};$$
$$\tilde{C} = \frac{(\beta\,\Lambda-(\mu+\alpha)\,(\mu+\delta+\alpha)\,\theta)\,(\gamma+\mu)}{\mu^3+(\gamma+2\,\alpha+\beta+\delta)\,\mu^2+\left(\alpha^2+(\beta+2\,\gamma+\delta)\,\alpha+(\beta+\delta)\,\gamma+\beta\,\delta\right)\mu+(\alpha+(\beta+1)\,\delta+\beta)\,\gamma\alpha};$$
$$\tilde{J} = \frac{\delta\,(\beta\,\Lambda-(\mu+\alpha)\,(\mu+\delta+\alpha)\,\theta)}{\mu^3+(\gamma+2\,\alpha+\beta+\delta)\,\mu^2+\left(\alpha^2+(\beta+2\,\gamma+\delta)\,\alpha+(\beta+\delta)\,\gamma+\beta\,\delta\right)\mu+(\alpha+(\beta+1)\,\delta+\beta)\,\alpha\,\gamma};$$
$$\tilde{H} = \frac{((\Lambda-\theta\,\delta\,\gamma+\theta\,\delta)\,\mu^2-\left((\alpha\,\theta\,\delta-\Lambda+\theta\,\delta^2)\,\gamma-\theta\,\delta^2+(-\Lambda-\theta\,\alpha)\,\delta-\Lambda\,(\alpha+\beta)\right)\mu+\gamma\,((1+\beta)\,\delta+\alpha+\beta)\,\Lambda)\,\alpha}{(\mu^3+(\gamma+2\,\alpha+\beta+\delta)\,\mu^2+((2\,\alpha+\beta+\delta)\,\gamma+(\delta+\alpha)\,(\alpha+\beta))\,\mu+\gamma\alpha\,((1+\beta)\,\delta+\alpha+\beta))\,\mu}.$$

ii. *corruption endemic equilibrium is locally asymptotically stable.*

Proof. We set the left hand side of equations (4.1) equal to zero and solve it to obtain the equilibrium (4.5). From the effective corruption induced rate $\frac{\beta C(t)}{\theta + C(t)}$ at equilibrium we have

$$\frac{\beta\tilde{C}}{\theta+\tilde{C}} = -\frac{(\gamma+\mu)\,((\mu+\alpha)\,(\mu+\delta+\alpha)\,\theta-\Lambda\,\beta)}{(\mu^2+(\gamma+\alpha+\delta)\,\mu+\alpha\,\gamma\,(1+\delta))\,\theta+\Lambda\,(\gamma+\mu)}.$$

This is equivalent to

$$\frac{\beta\tilde{C}}{\theta+\tilde{C}} = \frac{(\gamma+\mu)\,(\mu+\alpha)\,(\mu+\delta+\alpha)\,\theta\left(\frac{\beta\Lambda}{(\mu+\alpha)(\mu+\delta+\alpha)\theta}-1\right)}{(\mu^2+(\gamma+\alpha+\delta)\,\mu+\alpha\,\gamma\,(1+\delta))\,\theta+\Lambda\,(\gamma+\mu)}.$$

By Proposition(4.3) $\mathrm{R}_0 = \frac{\beta\Lambda}{(\alpha+\mu)(\delta+\alpha+\mu)\theta}$ and note that all parameters are positive. Hence, if $\mathrm{R}_0 > 1$, then $\frac{\beta\tilde{C}}{\theta+\tilde{C}} > 0$. This infer the existence of corruption in the specified population. With this we can conclude the existence of corruption endemic equilibrium point. To prove (ii) we compute the Jacobian matrix of the system (4.1) at corruption endemic equilibrium $\tilde{E}$ and then evaluate the eigenvalues as given in Theorem(4.1). Finally, we employ Routh-Hurwitz's criterion (Olsder G.J. and Woude J.W., 2005) to show the corruption endemic equilibrium $\tilde{E}$ of the system (4.1) is locally asymptotically stable for $\mathrm{R}_0 > 1$. This conclude the proof of the theorem.

In Section(4.1) we proposed a mathematical model and detailed its analysis in Section (4.2). In Section(4.3) we study the solution of the proposed mathematical model through numerical simulation and compare its properties with analytical results.

4.3 Numerical Simulation

This section presents and discusses numerical results for the system(4.1) for different values of parameters given in the model. The simulation process is carried out using Matlab and Maple for symbolic computations. We start by defining the value of each parameters used in the model developed. Then we illustrate the simulation results graphically.

4.3.1 Parameter Estimations

The values of parameters given in Table 4.1 are borrowed from Abdulrahaman,S. (2014),and TICPI,(2017). The remaining parameters values are also carefully estimated and used in the simulation process. The assumed total population and recruitment rate is related by $\Lambda = \mu N$.

Table 4.1: Values of parameter used in the numerical simulation.

Parameter	Description	Value	Reference
Λ	Recruitment rate	μN	To be computed
p	Corruption transmission probability per contact	0.036	Abdulrahaman,S. (2014)
β	Effective corruption contact rate	0.0234	computed using TICPI, (2017)
τ	Effort rate against corruption	0.35	TICPI, (2017) for Ethiopia
δ	Rate at which corrupt individuals are caught	0.0007	Abdulrahaman,S. (2014)
θ	Maximum saturation of corrupters population	100,000	Assumed
γ	Average rate of jailed individuals spent in prison	0.125	Abdulrahaman,S. (2014)
μ	Natural death rate	0.0160	WHO (2017) for Ethiopia
α	Proportion of individuals that joins H from each compartment due awareness	0.03	Assumed

4.3.2 Simulation results

The dynamics of corruption can be studied through numerical simulations by estimating the values of parameters in the model formulated with appropriate initial conditions. If the total population $N = 700,000$, then $\Lambda = 11,182.108$. The corruption-free equilibrium state is given by $\bar{E} = \left(\frac{\Lambda}{\alpha+\mu}, 0, 0, \frac{\alpha\Lambda}{(\alpha+\mu)\mu}\right) = (243224.4615, 0, 0, 456775.5389)$ and the reproduction number $R_0 = 0.6097$. Figure(4.2) depicts that the susceptible population decreases asymptotically to the corruption-free equilibrium state while the honest population asymptotically increase to corruption-free equilibrium point.

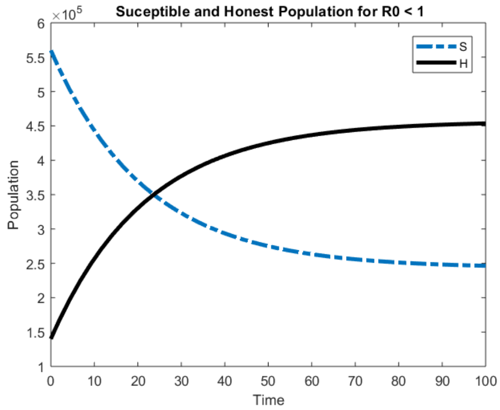


Fig. 4.2: Susceptible and Honest population approaches asymptotically to corruption-free equilibrium state when $R_0 < 1$.

Thus we can observe that an agreement between the numerical simulation of the model (4.1) and the analysis of the local stability of the corruption-free $\bar{E}$ presented in Section (4.2).

The corruption endemic equilibrium point is

$$\tilde{E} = (228590.5911, 14602.76181, 72.50912434, 456734.1384)$$

for the parameter values given in Table 1. The basic reproduction number in this case is $R_0 = 1.22$ which is greater than one. Figure(4.3) clearly shows that all the state variables asymptotically approach to the corruption endemic equilibrium point whenever $R_0 > 1$. Further, the corrupt population increases while the susceptible decrease to ward equilibrium point. This means, the susceptible populations become influenced by corrupt individuals and the size of corrupt population raise in the society. Also, the population of honest significantly increasing due to awareness by anti-corruption and counselling in jail.

As it can be seen in the Figure(4.3), the jailed population slowly growing. This may be due to highly secretive nature of corruption and the corruptors are remain in the community without arrested. For this reason, the number of the susceptible individuals become decreases to the equilibrium which need optimal strategy to interfere. It is presented in the next chapter.

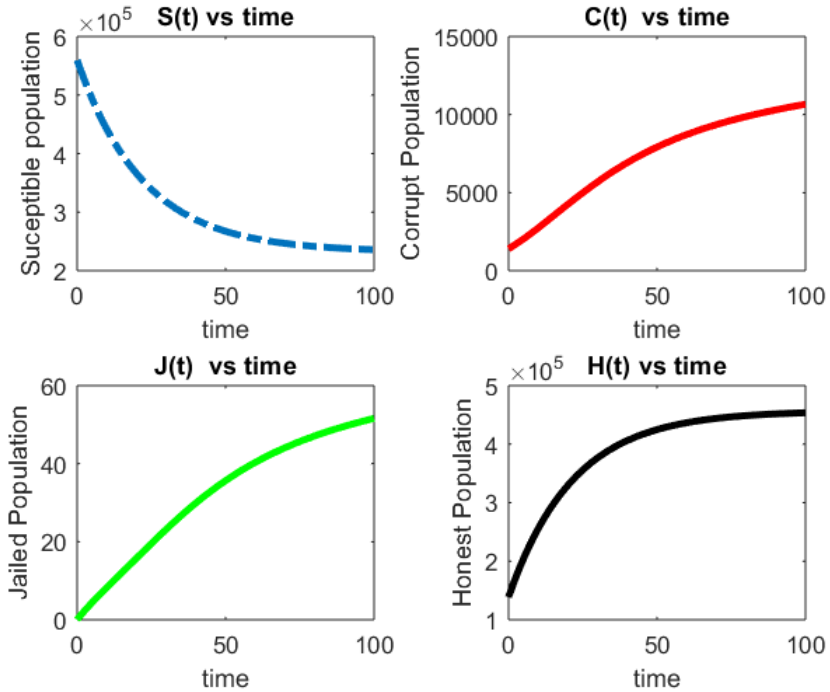


Fig. 4.3: All the state variables approach asymptotically to corruption endemic equilibrium point when $R_0 > 1$.

By its nature corruption is secretive and its level of prevalence in the population are not uniform. Thus, the dynamics of corruption for two countries namely, Ethiopia and New Zealand are comparatively studied based on Transparency International corruption perception index (TICPI, 2017) using the model developed. TICPI use a scale of 0 to 100 where 0 is highly corrupt and 100 is very clean. In this context, New Zealand rocked highest clean with score of 89 and Somalia is the most corrupt country with score 9. Ethiopia scored 35 and ranked 108 out of 180 countries. The penalties for corruption varies from country to country. In some country it ranges from one year to life imprisonment and loss of public office in addition to repaying the amount involved or confiscating property which is gained as a result of corruption. The average period corrupt individuals spent in prison $\left(\frac{1}{\gamma}\right)$ is eight (8) years as cited in Abdulrahaman,S. (2014). The corruption endemic is almost similar between 0 and 60 simulation time. Beyond $t = 60$, the endemic is stronger in the case Ethiopia. This is due to the difference in corruption resistance in the two countries and thus,the proposed model well predicts the difference of corruption dynamics in the two countries.

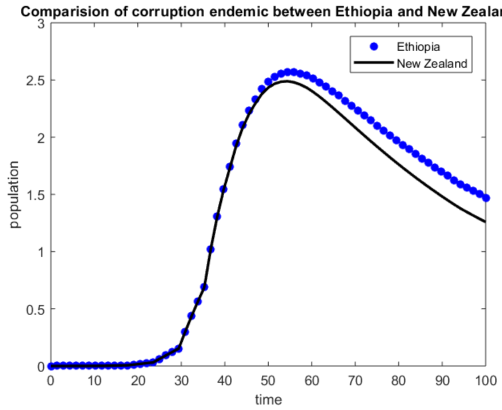


Fig. 4.4: Comparison of corruption endemic in the case of Ethiopia and New Zealand based on TICPI, (2017).

Chapter 5
The optimal control of corruption dynamics

5.1 Introduction

In Section 4.1 we proposed a mathematical model for corruption dynamics and studied both its mathematically and epidemiologically well posedeness. The section present a model to study and understand a certain characteristic of corruption without giving way to interfere and manipulate it. This is done in this chapter, where we introduce a control that allow us to decide how many individuals move to jail. Naturally, the question is then to know how to choose such control in an optimal way. For that, we use the theory of optimal control presented in Lenhart, S., (2007). After the theoretical study of the optimal control problem done, we investigate through numerical simulations the spread of corruption in the community.

5.2 The optimal control problem

In this section, we propose and analyse an optimal control problem applied to corruption dynamics described by model (4.1). We introduce to model (4.1) a control function $u_1(t)$ that represents the fraction of the susceptible that are educated per unit time while the control function $u_2(t)$ is the proportion of the corrupt individuals that are caught and jailed. The aim is to use optimal control technique to minimize the number of corrupt individual and maximize the population of honest with optimal cost over a fixed period of time T. The dynamic corruption transmission model with optimal control is governed by the modified nonlinear system of differential equations:

$$
\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta C}{\theta + C}S - (\alpha u_1 + \mu)S + \gamma(1-\alpha)J \\
\frac{dC}{dt} &= \frac{\beta C}{\theta + C}S - (\delta u_2 + \alpha + \mu)C \\
\frac{dJ}{dt} &= \delta u_2 C - (\gamma + \mu)J \\
\frac{dH}{dt} &= \alpha(u_1 S + C + \gamma J) - \mu H, \\
& 0 \le u_1(t) < 1, \ \ 0 \le u_2(t) < 1, \ t \in [0,T].
\end{aligned} \tag{5.1}
$$

with initial condition as given in 4.2.

The optimal control problem is to minimize the objective functional J considering the costs of educating susceptible and capturing corrupt individuals. Mathematically, the problem is to minimize the objective functional of the form:

$$
J(u_1(\cdot), u_2(\cdot)) = \int_0^T \Big(C(t) + \frac{w_1}{2}u_1^2(t) + \frac{w_2}{2}u_2^2(t) \Big)\, dt \to \min \tag{5.2}
$$

subjected to the system of nonlinear differential equations (5.1). The positive constants w_1 and w_2 are a measure of the cost of intervention associated with control u_1 and u_2 respectively. The parameter T is the duration, in months, of the education campaign program on corruption.

We assume that $0 \le u_1(t) < 1$ since reducing the contact rate between the entire susceptible and corrupt impossible in reality. In this case, $u_1(t)$ is corresponding to the effort made to educate susceptible individuals not to corrupt. In practice, educating entire society is impossible due to many factors such as financial constraint and hence $u_1(t) < 1$. Similarly, $u_2(t)$ is the rate at which corrupt individuals are caught over a fixed period of time and added to jailed people. This is corresponding to law enforcement and thus, $0 \le u_2(t) < 1$. In reality handling all corrupt people is impossible due to many factors such as the illicit and secretive nature of corruption. The law enforcement cost could include the cost of police to capture corrupts, food cost in jails, and others. From this we can conclude that $u_2(t) < 1$.Thus, the control takes values in the set $[0,1) \times [0,1)$. If $u = 0$, then no control measure is done and the model equation (5.1) is equivalent to (4.1). On the other hand $u = 1$ means all susceptible are awared or corrupt people are put in jail. In reality this case is not possible. The control $u_1(t)$ is corresponding to prevention mechanism where as $u_2(t)$ is handling the risk once it occurred. Thus, we are looking for the optimal control pair $(\bar{u}_1, \bar{u}_2)$ such that

$$
J(\bar{u}_1, \bar{u}_2) = \min_{u_1(\cdot), u_2(\cdot)} \{ J(u_1, u_2) : u_1, \ u_2 \in \Omega \} \tag{5.3}
$$

where Ω is the set of admissible controls defined by

$\Omega = \{(u_1, u_2) : 0 \leq u_1(t), u_2(t) \leq 1 - \varepsilon,\ t \in [0, T],\ u_1$ and u_2 are Lebesgue measurable function, $\varepsilon << 1\}$.

The solution of the optimal control problem is the vector function

$$\bar{V}(\cdot) = (\bar{S}(\cdot), \bar{C}(\cdot), \bar{J}(\cdot), \bar{H}(\cdot)) \in \mathscr{V}$$

associated with an admissible control $\bar{u} = (\bar{u}_1, \bar{u}_2) \in \Omega$ on the time interval [0,T] that minimize the cost functional (5.2). Note that the system (5.1) is regular and the integrand of the objective functional is convex on Ω with respect to $u = (u_1, u_2)$. Moreover, the admissible control set Ω is convex and closed. Hence, the optimal control $\bar{u}$ exists on the finite time interval [0, T] which will be proved in the subsequent section.

From the history and human behaviour, elimination of corruption is impossible at once as it requires education(awareness) at higher levels all the time. Thus, we use the optimal control strategies in the form of education (awareness) to decrease (minimize) the number of corrupt individuals and increase the total number of honest population with minimum investment in corruption control.

5.2.1 Characterization of the optimal control

According to the Pontrygin's maximum principle, if $\bar{u}(\cdot) \in \Omega$ is optimal for problem (5.3) with fixed final time T, then there exists a non-trivial absolutely continuous mapping $\lambda : [0, T] \to \mathbb{R}^4, \lambda = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t))$ called the adjoint vector, such that

1. the Hamiltonian function is defined as

$$\mathscr{H} = C(t) + \frac{w_1}{2} u_1^2(t) + \frac{w_2}{2} u_2^2(t) + \sum_{i=1}^{4} \lambda_i(t) g_i(t, S, C, J, H, u_1, u_2), \tag{5.4}$$

   where g_i stands for the right hands of the constraints (5.1) for $i = 1, 2, 3, 4$.
2. the control system

$$S' = \frac{\partial \mathscr{H}}{\partial \lambda_1}, \quad C' = \frac{\partial \mathscr{H}}{\partial \lambda_2}, \quad J' = \frac{\partial \mathscr{H}}{\partial \lambda_3}, \quad H' = \frac{\partial \mathscr{H}}{\partial \lambda_4};$$

3. the adjoint system

$$\lambda_1' = -\frac{\partial \mathscr{H}}{\partial S}, \quad \lambda_2' = -\frac{\partial \mathscr{H}}{\partial C}, \quad \lambda_3' = -\frac{\partial \mathscr{H}}{\partial J}, \quad \lambda_4' = -\frac{\partial \mathscr{H}}{\partial H};$$

4. and the minimality condition

$$\mathscr{H}(\bar{V}(t), \bar{u}(t), \bar{\lambda}(t)) = \min_{u \in \Omega} \mathscr{H}(\bar{V}(t), u(t), \bar{\lambda}(t))$$

holds for almost all $t \in [0, T]$.

5. Moreover, the transversality condition

$$\lambda_i(T) = 0, \qquad i = 1, 2, 3, 4$$

also holds true.

Summarizing, for the optimal control problem (5.2) we prove the existence of the optimal control and then characterize the optimal control for optimality following (Lenhart, S., (2007), Sch*ä*ttler, H., & Ledzewicz, U. (2012)).

Theorem 5.1. *If in the equations* (5.1) *and* (5.2)*:*

(a) the set of controls and corresponding state variables are nonempty.
(b) the control set is convex and closed.
(c) the right-hand side of the state system is bounded by a linearised function in the state and control variables.
(d) the integrand of the objective functional is convex.
(e) the integrand of the objective functional is bounded below by $\xi_1|u_1|^r + \xi_2|u_2|^r - \xi_3$, where $\xi_i > 0$, $\forall i = 1,\ 2,\ 3$ and $r > 1$.

are all satisfied, then there exists an optimal control variables $\bar{u} = (\bar{u}_1,\ \bar{u}_2) \in \Omega$ such that

$$J(\bar{u}_1, \bar{u}_2) = \min_{u_1(\cdot), u_2(\cdot)} \{J(u_1, u_2) : u_1,\ u_2 \in \Omega\} \tag{5.5}$$

subjected to (5.1).

Proof. Condition (a) is easily follow by assumption of the governing model. To prove (b), consider

$$\Omega = \{u \in \mathbb{R}^2 : ||u|| \leq 1 - \varepsilon\}.$$

Let $u_1, u_2 \in \Omega$ such that $||u_1|| \leq 1 - \varepsilon$ and $||u_2|| \leq 1 - \varepsilon$. Then for any $\alpha \in [0, 1]$,

$$||\alpha u_1 + (1 - \alpha) u_2|| \leq \alpha ||u_1|| + (1 - \alpha) ||u_2|| \leq 1 - \varepsilon.$$

This implies that Ω is convex and closed.

Consider the system

$$\begin{aligned}
\frac{dS}{dt} &= F_1 = \Lambda - \frac{\beta C}{\theta + C} S - (\alpha u_1 + \mu) S + \gamma(1-\alpha) J \\
\frac{dC}{dt} &= F_2 = \frac{\beta C}{\theta + C} S - (\delta u_2 + \alpha + \mu) C \\
\frac{dJ}{dt} &= F_3 = \delta u_2 C - (\gamma + \mu) J \\
\frac{dH}{dt} &= F_4 = \alpha(u_1 S + C + \gamma J) - \mu H, \\
\frac{dN}{dt} &= F_5 = \mu N
\end{aligned}$$

Equivalently, this can be expressed as

$$\begin{aligned}
F_1 &\le \mu N - (\alpha u_1 + \mu) S + \gamma(1-\alpha) J \\
F_2 &\le \frac{\beta C}{\theta + C} S - (\delta u_2) C \\
F_3 &\le \delta u_2 C - \gamma J \\
F_4 &\le \alpha(u_1 S + C + \gamma J), \\
F_5 &\le \mu N
\end{aligned}$$

More compactly in matrix form

$$F(t, X, u) \le \begin{pmatrix} \mu & -\mu & 0 & \gamma(1-\alpha) & 0 \\ 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma & 0 \\ 0 & 0 & \alpha k_2 & \alpha\gamma & 0 \end{pmatrix} X + \begin{pmatrix} \alpha S \\ \delta C \\ \delta C \\ \alpha S \end{pmatrix} u$$

where k_1 is the function of C and k_2 is the function of S. Computing its matrix norm yields

$$||F|| \le K|X| + M|u| \le \max(K, M)(|X| + |u|)$$

This follow from the fact that, all the state variables are bounded and $\max(K, M)$ is the upper bound of the matrix. Hence we see that the right hand side is bounded by a sum of the state and the control. With this we can conclude that condition (c) is satisfied. The integrand of the cost functional is

$$C(t) + \frac{w_1}{2} u_1^2(t) + \frac{w_2}{2} u_2^2(t).$$

Note that it is the sum of convex function and hence convex which proves condition (d).

Further,

$$J(c,u) \geq -C(t) + \frac{w_1}{2}u_1^2(t) + \frac{w_2}{2}u_2^2(t)$$

which implies,

$$J(C,u) \geq \xi_1|u_1|^r + \xi_2|u_2|^r - \xi_3$$

where ξ_3 depends on the upper bounds of $C(t)$. Furthermore,setting $\xi_1 = \frac{w_1}{2}$, $\xi_2 = \frac{w_2}{2}$ and $r = 2 > 1$. With this we conclude the assistance of an optimal control $\bar{u}$ that minimizes the objective functional (5.2). This ends the proof.

Next we give necessary conditions for the optimality solutions and the adjoint equations using the theory of Pontryagin's minimum principle.

Theorem 5.2. *The optimal control problem* (5.5) *with fixed final time* T *admits a unique optimal solution* $(\bar{S}(\cdot), \bar{C}(\cdot), \bar{J}(\cdot), \bar{H}(\cdot))$ *associated with an optimal control* $\bar{u} = (\bar{u}_1, \bar{u}_2)$ *for all* $t \in [0,T]$. *Moreover, there exist adjoint function* $\bar{\lambda}_i(\cdot), i = 1,2,3,4$ *such that*

$$\begin{aligned}
\lambda_1' &= \frac{\beta C}{\theta + C}(\lambda_1 - \lambda_2) + \alpha u_1(\lambda_1 - \lambda_4) + \mu\lambda_1 \\
\lambda_2' &= -1 + \frac{\beta S\theta}{(\theta + C)^2}(\lambda_1 - \lambda_2) + \delta u_2(\lambda_2 - \lambda_3) + (\alpha + \mu)\lambda_2 - \alpha\lambda_4 \\
\lambda_3' &= (\alpha - 1)\gamma\lambda_1 + (\mu + \gamma)\lambda_3 - \gamma\lambda_4 \\
\lambda_4' &= \mu\lambda_4
\end{aligned}$$

with transiversality conditions

$$\bar{\lambda}_i(T) = 0, \qquad i = 1,2,3,4.$$

Moreover, the optimal control $\bar{u}(t)$ *is given by*

$$\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t)) = \left(\max\left[\min\left(\frac{(\lambda_1 - \lambda_4)\alpha S}{w_1}, 1 - \varepsilon\right), 0\right], \max\left[\min\left(\frac{(\lambda_2 - \lambda_3)\delta C}{w_2}, 1 - \varepsilon\right), 0\right]\right)$$

where $0 < \varepsilon << 1$.

Proof. The Hamiltonian function associated with the system is given by

$$\mathscr{H} = C(t) + \frac{w_1}{2}u_1^2(t) + \frac{w_2}{2}u_2^2(t) + \sum_{i=1}^{4}\lambda_i(t)g_i(t,S,C,J,H,u_1,u_2),$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are adjoint functions to be determined and g_i is the right hand side of equation (5.1).

Given the optimal control $\bar{u}$ and corresponding solution $(\bar{S}, \bar{C}, \bar{J}, \bar{H})$ of equation (5.2), then following Pontryagin's maximum principle there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3$ and λ_4 that satisfy the following which is easily obtained by differentiating the Hamiltonian (5.4) with respect to each state variables:

$$\lambda_1' = \frac{\partial \mathscr{H}}{\partial S} = \frac{\beta C}{\theta + C}(\lambda_1 - \lambda_2) + \alpha u_1(\lambda_1 - \lambda_4) + \mu\lambda_1;$$
$$\lambda_2' = \frac{\partial \mathscr{H}}{\partial C} = -1 + \frac{\beta S\theta}{(\theta + C)^2}(\lambda_1 - \lambda_2) + \delta u_2(\lambda_2 - \lambda_3) + (\alpha + \mu)\lambda_2 - \alpha\lambda_4;$$
$$\lambda_3' = \frac{\partial \mathscr{H}}{\partial J} = (\alpha - 1)\gamma\lambda_1 + (\mu + \gamma)\lambda_3 - \gamma\lambda_4;$$
$$\lambda_4' = \frac{\partial \mathscr{H}}{\partial H} = \mu\lambda_4.$$

The state variables are not assigned at the final time T. Hence, following Lenhart, S., (2007) the transversality condition yields:

$$\lambda_i(T) = 0, \;\; i = 1, 2, 3, 4 \tag{5.6}$$

For the optimality condition, we differentiate the Hamiltonian $\mathscr{H}$ with respect to $u = (u_1, u_2)$ and equate it to zero by taking into account the boundedness on u. That is, we have the following optimality condition:

$$\frac{\partial H}{\partial u_1} = w_1 u_1 + (\lambda_4 - \lambda_1)\alpha S = 0.$$

Solving we have

$$\bar{u}_1 = \frac{(\lambda_1(t) - \lambda_4(t))\alpha S}{w_1}.$$

Similarly,

$$\frac{\partial H}{\partial u_2} = w_2 u_2 - \lambda_2\delta C + \lambda_3\delta C = 0,$$

and

$$\bar{u}_2 = \frac{(\lambda_2(t) - \lambda_3(t))\delta C}{w_2}.$$

From boundedness on $\bar{u}$ and minimality condition we obtain;

$$\bar{u}_1 = \begin{cases} 0, & \text{if } \frac{\partial H}{\partial u} > 0; \\ \frac{(\lambda_1(t) - \lambda_4(t))\alpha S}{w_1}, & \text{if } \frac{\partial H}{\partial u} = 0; \\ 1 - \varepsilon, & \text{if } \frac{\partial H}{\partial u} < 0 \end{cases}$$

$$\bar{u}_2 = \begin{cases} 0, & \text{if } \frac{\partial H}{\partial u} > 0; \\ \frac{(\lambda_2(t)-\lambda_3(t))\delta C}{w_2}, & \text{if } \frac{\partial H}{\partial u} = 0; \\ 1-\varepsilon, & \text{if } \frac{\partial H}{\partial u} < 0. \end{cases}$$

Concluding, the optimal control $\bar{u}$ can be summarized as

$$\bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t)) = \left(\max\left[\min\left(\frac{(\lambda_1-\lambda_4)\alpha S}{w_1}, 1-\varepsilon\right), 0\right], \max\left[\min\left(\frac{(\lambda_2-\lambda_3)\delta C}{w_2}, 1-\varepsilon\right), 0\right]\right)$$

where $0 < \varepsilon << 1$. With this we complete the proof of the theorem.

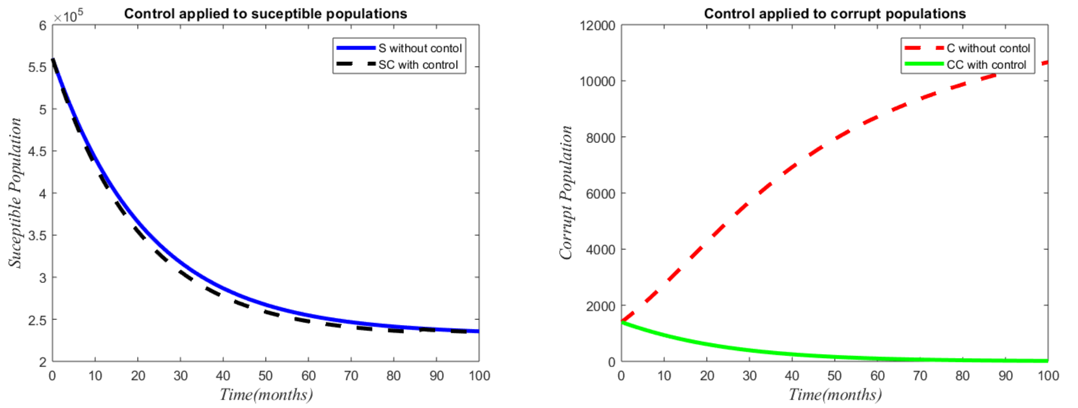
5.3 Numerical approximation and discussion

Applied problem may not easily solvable analytically and instead numerical methods are used to approximate the solutions. To solve the optimal control problem with initial conditions for the states and final time conditions for the adjoints, we used the Runge-Kutta fourth order procedure. It is a two-point boundary problem, because of the initial condition of the state system and the terminal condition for the adjoint system. A Runge-Kutta method is a single-step method, where the solution at time t_{k+1} is obtained from a pre-defined set of values t_k. Particularly, in this research report we employed the technique described in section (2.1.2.1). The process begins with an initial guess on the control variable and given initial conditions for states, we approximate solutions for state equations using Runge-Kutta forward sweep method. Given the state solutions from previous step and the final time conditions for adjoints, we approximate solutions for adjoint equations using Runge-Kutta backward sweep method. The value of control variables is updated by averaging the previous value and the new value arising from the control characterization. The process is repeated for forward numerical scheme and updating of the controls until successive values of all states, adjoints, and controls are sufficiently close or converge.

5.3.1 Simulation results

We conduct numerical simulation in order to investigate the effects of the control strategies on the transmission dynamics of corruption. The simulations are performed using MATLAB, and we set time in months. The estimated initial values of model state variables are $S_0 = 560,000, C_0 = 1400, J_0 = 0, H_0 = 138,600$, and the parameter values are as shown in Table 4.1. For the adjoint system we set terminal conditions $\lambda_i(T) = 0,\ i = 1, \cdots 4$, where $T = 100$ months. The cost coefficients corresponding to control variables are estimated to be $w_1 = 0.5$ and $w_2 = 0.7$.

We preformed the simulation process with and without control strategies and then we compared the results. We considered the numerical value of the control variable u in between zero and one because implementing 100 percent effective control in reality is not possible. Then, the state equations are solved simultaneously forward in time, and the adjoint equations are simultaneously solved backward in time. The control is updated by inserting the new values of states and adjoints into its characterization, and the process is repeated until convergence occurs as detailed in section(2.1.2.1).

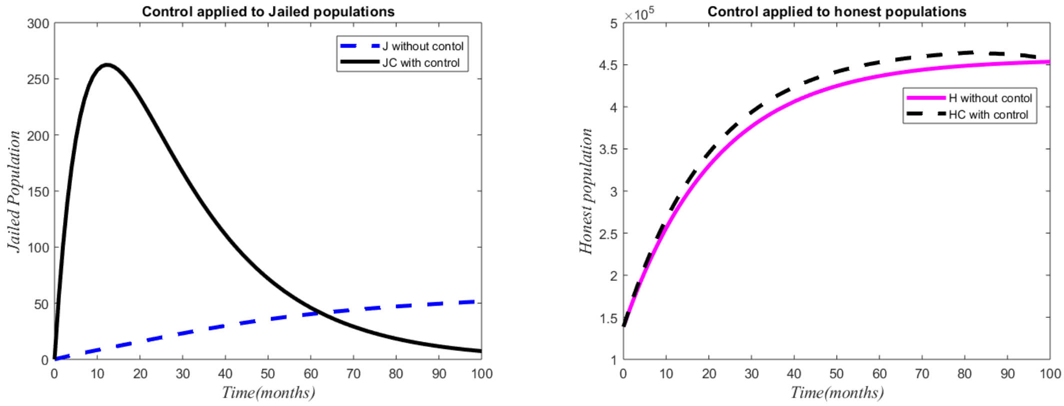


(a) Suscetiple populations with and without Control.

(b) Corrupt populations with and without Control.

Fig. 5.1: Comparing the effectiveness of control strategies on corruption dynamics in the case of susceptible and corrupt population.

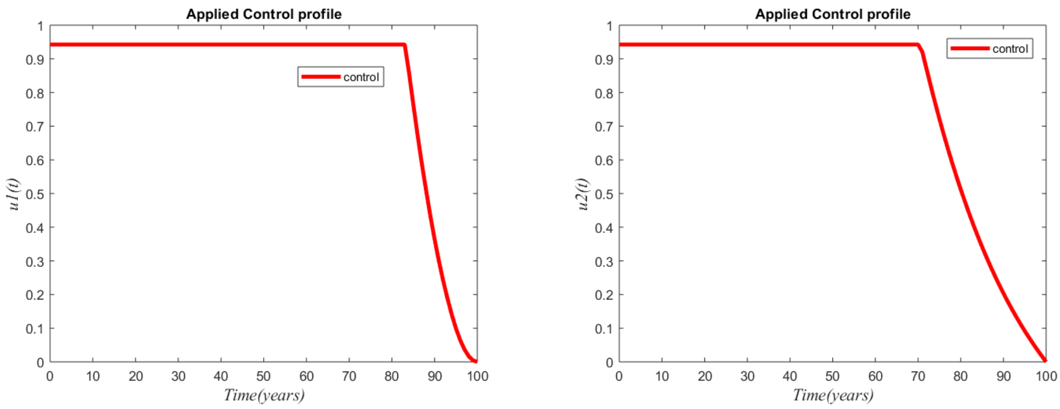
Figure(5.1)(a) presents the impact of control on the susceptible population. It shows the density of susceptible decrease rapidly and gradually converges to the magnitude of the population in the absence of control. This is due to the decreasing in the applied control. On the other hand, Figure(5.1)(b) shows the population of corrupt lastly damped down when control applied.



(a) Jailed populations with and without Control.

(b) Honest populations with and without Control.

Fig. 5.2: Comparing the effectiveness of control strategies on corruption dynamics in the case of jailed and honest population.



(a) Control profile applied to susceptible population as prevention mechanism.

(b) Control profile applied to jailed population.

Fig. 5.3: Applied control strategies on both jailed and susceptible population.

Figures (5.1) and (5.2) show, respectively, the significant difference in the number of susceptible, corrupt, jailed and honest individuals, with and without control, along time. As expected, the number of susceptible individuals $S(t)$ decrease rapidly in case of education campaign (see (5.1)(a)), beginning to lose immunity as we decrease awareness campaign: compare with Figure

(5.3)(a), which represents the optimal control function $u_1(t)$ along time. Figure (5.2) shows that the number of corrupt individuals decreases rapidly in case of control. In the same figure, the curve of corrupt shows that the corrupt individuals are rapidly captured for $0 \leq t \leq 20$ and became decline due to raised in awareness and control impact. Figure(5.3) shows that the number of honest individuals increases rapidly in presence of control and then start decline. In conclusion, one can infer from these Figures that the effectiveness of optimal awareness and law enforcement in controlling corruption expansion is very significant.

Chapter 6
Conclusion and Recommendations

6.1 Conclusion

Corruption challenges good governance and becomes an hinder for investment. In this research report, we developed and analysed a mathematical model to study the dynamical nature of corruption based on a system of non-linear ordinary differential equation with constant recruitment rate from the total population. The domain in which the model is mathematically and epidemiologically well posed was determined. Then the next generation matrix was used to derive the basic reproduction number. The corruption-free equilibrium of the model was proved to be locally asymptotically stable whenever reproduction number is less than one. It is also showed that the corruption epidemic equilibrium is locally asymptotically stable provided that the basic reproduction number is greater than one. Finally, the analytical result was verified using numerical simulations.

In section (5.2), the optimal control techniques have been applied on corruption dynamical model with control strategies. We proved the existence of optimal control problem and determined the necessary conditions for optimality using Pontryagin's maximum principle which converts constrained optimization problem into unconstrained Hamiltonian function then optimality and adjoint equations are obtained. We, lastly, performed numerical simulations of the resulting control problem to investigate the effects of the control strategies under consideration and compare their performances. The numerical results showed that the control strategy that comprises two control measures have significant impact in reducing corruption dynamics.

6.2 Future work

One can investigate sensitivity analysis, global stability and the stochastic nature of corruption dynamics which are not treated in this research report. Furthermore, it is possible to extend the present model to more general form by including time delay and optimal control simultaneously.

6.3 Recommendations

Based on the findings of this research report, we recommend to government and concerned stake holders to implement a combination of intensive awareness creation and law enforcement with optimal resource as a control strategy to minimize the expansion of corruption.

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