ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

SCHOOL OF APPLIED NATURAL SCIENCES

DEPARTEMENT OF MATHEMATICS

MATHEMATICAL MODEL OF COMPLEX BEHAVIOUR IN THREE SPECIES FOOD – WEB

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR

THE DEGREE OF MASTER'S IN MATHEMATICS

BY: Dereje Gelete

Advisor: Shiferaw Feyissa (Ph.D)

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Adama, Ethiopia

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Abstract

A biologically feasible continuous time food web model consisting of a bottom prey, a middle predator and top predator which is an extension of Hasting and powell three species food chain model that incorporates the nonlinear functional response was studied. The predation follows modified Holling type II functional responses as the up take for both predators. It is shown that under certain restrictions on the parameter space the model has bounded solutions for all positive initial conditions .Which eventually enters an invariant attracting set as an initial step. Towards the analysis of the model the governing equation have been splined in to kolmogorov type systems. Based on the Jacobean matrix associated to the sub system, the dynamical behavior about the semi-trivial equilibra is analyzed and analytic results show that any one of the boundary prey predator planes has a stable equilibrium point. More over the dynamical of the sub systems in the boundary planes changes as some parameter values vary. Criteria's for the existence, local stability of the nonnegative equilibra are analyzed algebraically. Based on numerical simulations the system is numerically solved for suitable combinations of parameters in such a manner that Kolomorov conditions are satisfied. It is also observed that the Routh-Hurwitz criteria the necessary and sufficient conditions for stability are not satisfied. More over the coexistence of the three biological species is observed from the oscillation nature of the three solution curves. Finally it is observed that there is a change on the qualitative behavior of the system when trophic function is of prey dependent than ratio-dependent.

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CHAPTER ONE

1.1. Background of the study

Ecology is the field of study of mutual relationships among plants, animals, and their environment. Population ecology is the study of inter specific interactions among species. The interaction between two species is mainly classified into three, namely mutualism, competition and prey-predator. Mutualism is the interaction between two species where the interaction enhances both species, whereas competition is where the interaction of the two species affect each other negatively. Prey-predator type of interaction is where one species use the other species as a food source [4].

Mathematical modeling is the process of developing sets of equation (or inequalities) and formula describing some process of interest and the main goal of mathematical modeling is to predict how things will turn of some point of interest. Specific observations that have motivated ecologists to seek about mathematical models of populations growth is the situation in the Adriatic sea during world war I, that resulted ‘‘the growth of one population to be followed by its decline’’[1,2,3].

In 1926 the Italian Mathematician Vito -Volterra [14] proposed differential equation model to describe the way we analyze the fluctuation of populations of different species (population dynamics) in a closed area that involve interaction of many species of animals and plants life (echo system). Independently, in united-states the equation studied by Volterra were described by Alfered-Lotka [13] to describe a hypothetical chemical reaction in which chemical concentrations oscillate. The mathematical model that describes the predation of one species by another is known as Lotka-Volterra model.

In ecological system with only two interacting species can account for only a small number of phenomena that commonly exhibited in nature. The building block which multi species food chains and food web model developed is Lotka-Volterra type prey predator model. An echo system includes all of the living organisms in specific area. Food chains and food webs describe the feeding relationships between species in a biotic community. Both are common in nature. They show the transfer of energy from one species to the other with in the echo system. A food chain describes a single path way that energy or nutrients may flow in an echo system. There is one organism per trophic level. They usually start with the primary producer and end up with top predator. Most food chains have no more than four or five links. Food chains are overly simplistic as representatives of what typically happens in nature. Food sources of most species in an echo system are much more diverse, resulting in a complex food web of relationships. Most consumers feed on multiple species for their food and energy requirements (more than one prey population). They are fed up on by multiple other species. Food webs are networks that reflect the feeding relationships among ecological populations [8].

One aspect to the dynamics of the ecological community is its food web and interaction among constituent species, the dynamical behavior of food web (the network of who eats whom) are studied with the help of mathematical models governed by non–linear differential equation or difference equations. In order to explain more and more new observation such model thus extended subsequently by introducing additional variables as well as varieties of a non – linear interaction terms coupled with some parameters which control evolution of the system[5].

Some of the aspects that need to be critically considered in a realistic and plausible mathematical model to describe the dynamics of the interaction between the prey and predator in a given ecosystem include ;carrying capacity which is the maximum number of prey that the echo system can sustain in absence of predators and functional responses of predators.

A functional response is described as a predator's instantaneous per capita feeding rate as a function of prey abundance [17].this means that the consumption rate of an individual predator depends on the prey density .There are mainly three Holling types of functional responses.

Holling type I; given by $F(x) = ax$

Holling type II; given by $F(x) = \frac{a_i x}{1+b_i x}$, where a_i and b_i are parameters that are specific for each predator-prey system and determine the saturation level and the steepness of the response.

Holling type III is called the learning functional response since the function is a sigmoid type curve with an inflection point. More recent analysts use a Holling type II functional response which increases monotonously but saturates with increasing prey density. If one predator feeds more than one prey population, n-species functional responses are mostly widely used.

The behavior of nonlinear mathematical models for food chain/ food web is much more complex than the simple two species model. The understanding of nonlinear equations has complex dynamical behavior limit cycle, causi-periodic behavior and chaos. The food web models are more complex and intractable as compared to food chain model as more complex multilevel interaction is possible in food web [3, 4].

Following this sprit Hastings and powell (HP) [10] examined the complex nonlinear behavior of three species continuous time ecological model. His study initiated a concerted effort among theoretical ecologists who sought to analyze the subtle dynamics of three nonlinear tropic (predation) model. He produced an example of chaotic system in three species food chain modeled with type II functional responses and found to be most widely accepted for defining the movement of aquatic echo system from its equilibrium state to chaotic state. He demonstrated the existence of a ' tea cup' strange attractor &Maiti.et.al [15], Upadhyay and his coworkers [16] studied the stability bifurcation behavior of models describing evolution of three species food chain consisting of Holling type response function . From this study it appear that the presence of increasing number of variables and additional parameters raise considerable problems for analysis of the model both for the experimenter and theoretician.

1.2. Statement of the problem

From several investigations of the complex dynamical behavior of multi – species food web, the three level food web consists of the prey, middle predator and top predator, Owaidy and Ammar [21] mathematical model deals with the dynamics around the stability of various isolated Singularitiess. But it raises certain doubts about various functional responses in the model.

Michael Freeze [7] proposed and analyzed a nonlinear mathematical model to study the complex dynamics for three ecological species the prey, the predator that consumes the prey with ratio dependent functional response and the supper predator that consumes both with ratio dependent functional response. Biological equilibras of the system are obtained and criteria for local stability and global stability of the system derived. He indicated the coexistence of the steady state of the three species and the complexity of the interaction is also discussed by the help of numerical simulations.

Keeping the above in view in this thesis a mathematical model of one prey a middle predator and generalist predator that consumes both system in which the predator interference is of prey dependent is proposed and analyzed. It may be pointed here that results on one prey a middle predator and top predator that consumes both system with ratio-dependent trophic function are well known [7,18]. Here we are interested to investigate and see the change in the qualitative behavior of the same system as [7] when trophic function is prey dependent.

Therefore this research work addresses the following basic questions.

1. How a mathematical model can describe the dynamics of the three species food – web?
2. What are the dynamical behaviors of the interaction between the three species?
3. What are the effects of the interaction between the three species on the eco – system?
4. What are the possible intervention mechanism for the co–existence of the interacting species?
5. Is there a change in the qualitative behavior with the investigation of Michael Freeze who implemented ratio-dependent interaction function?

1.3. Objective of the study

1.3.1. General objective

The general objective of this research work is to study the complex behavior of the dynamics of the three interacting species in the food web.

1.3.2. Specific objective

The specific objectives of the study are to:

(i) Use a mathematical food web model which describes the interaction between three species.

(ii)Explain the dynamics of the interaction between the three species in an echo system,

(iii)Explore the effect of the interaction between the three species in the echo-system,

(iv)Determine conditions under which the species co – exist.

(v)Identify the changes in qualitative behavior with which investigated by Michael Freeze.

1.4. Significance of the study

The purposes (significance) of this study are to

(i)Contribute a mechanism on the co – existence of the three species for a longer time.

(ii) Apply biological/ ecological mathematical model for analyzing and predicting the long term predation relationship among the three species.

(iii) Point out conditions that our nature resource species could not extinct.

(iv) See the existence of the complex behavior in the interaction.

(v) See the existence of the change in qualitative behavior with the one investigated in Michael Freeze.

1.5 Review of Related Literature

In 2002 ERICA [6] used a Lotka – Volterra three – species food chain model which is an extension of two species model. Example of such echo system include (mouse – snake-owl)where the lowest level prey (x) is preyed upon by a middle level species (y) , which in turn ,is preyed upon by a top level predator (z).

He characterized the qualitative behavior of the linear three species food chain model where the dynamics are given by classic (non-logistic) Lotka-Volterra type equation and studied non logistic system, which the model exhibits degeneracy's that makes it an excellent instructional tool whose analysis involves advanced topics such as trapping region nonlinear analysis, in variant set, Lyapunov type function, the stable / center manifold theorem and the Poincare-Bendixon theorem. And he concluded that the overall long –term persistence of the top species (z) hinges (depends) solely on parameter values a, b, f and g. In particular if $ag < fb$ then (z) dies out. While if $ag \geq fb$ he quite interestingly noted that the parameter most directly related to mid-level specie y, (namely c, d and e) in no way affect whether species (z) will become extinct or not. Furthermore the model does not allow for the possibility of extinction of species(y) while species (x) persist.

In (2007) Sunita jakkhar [19] used the dynamical equation of food web consisting of two independent predators and a prey. The model considered direct competition between the two predator populations and incorporates nonlinear functional response. The model has a unique and bounded solutions. Local and global analysis has been carried out.

The phase diagram of the food web have been obtained for selected range of different parameters and the analysis shows that the persistence is not possible for two competing predators sharing prey species in the cases when any one of the boundary prey-predator planes has a stable equilibrium point. The principle of competitive exclusion holds in such cases. However, numerical simulation exhibit persistence in the presence of periodic solutions in the boundary planes. The system exhibit quase- periodic behavior in the positive octant. More over the coexistence in the form of a limit cycle is also possible in some cases.

In (2012) Jawdat Alebraheem[9] used a model of two competing predators sharing one prey in homogeneous environment with holling type II functional response and logistic growth for each of the species. He showed the system equations are of Kolmogorov type and he proved the solutions of the system of three species is bounded. He restricted conditions with parameter values to show the local stability of the nontrivial equilibrium points of the subsystems and the existence and stability analysis of the three dimensional system using variational matrix of the equilibrium points. Moreover he showed the nonexistence of a limit cycle and conditions for persistence. Finally by considering different values of parameters (for those measuring the efficiency conversion of predators) he presented numerically the existence and extinction of one of the predators in a non-periodic solution (i.e. no limit cycle).Numerical simulations have investigated that the three species can coexist when the value of efficiency conversions for the two predators are near to each other.

In (2014) Michael Freeze[7] implemented the food web model consisting a prey with logistic growth ,a predator that consumes the prey with ratio dependent functional response and has the ability to consume both the predator and the prey and super predator that has the ability to consume both the predator and the prey with ratio dependent functional response.

The thesis analyzed and demonstrated the complexity of the population dynamics for three ecological species. In that he gave a necessary condition for the existence of equilibrium point with presence of at least two species and the stability analysis of these semi trivial equilibrium points. Moreover he explored the condition for the permanence (existence of positive global attractor) in the model and studied the dynamics in relation to the component wise positive equilibrium , which indicates the coexistence of steady state of three biological species . Finally concluded as it is quite difficult to express and analysis the co-existence equilibrium in terms of all the ecological parameters, because of the complexity of the interaction. So he used simulation for the population solution. As a result numerical simulation for biologically feasible ecological parameters, he showed the coexistence equilibrium point exists and is locally asymptotically stable.

1.6 Research methodology

In this research work we developed a mathematical model which describes the interaction between three species population. That has the prey species (x) with logistic growth, a predator(y) that consumes prey (x) with Holling type II functional response and a super predator (z) that has an ability to consume both predator and the prey with Holling type II functional responses using a non –linear differential equation. we studied the local behavior of various singularities for the system and we developed sufficient conditions for local stability of equilibrium points using Jacobean matrix and Routh-Hurwitz criteria. Bifurcation analysis of the sub systems was studied to determine the parameter that plays role to change the dynamical behavior from stability of equilibrium point in their plane and admits a limit cycle. And analytic solution of the model was supplemented by numerical simulation using 'MATLAB'.

1.6.1 Definition: Ordinary Differential Equations

A differential equation of order n is an equation of the form

$f(y, \frac{dy}{dt}, \frac{d^2y}{dt^2}, \frac{d^ny}{dt^n}, t) = 0.$

A typical n^{th} order linear differential equation is given by

$$\frac{d^ny}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + a_2(t)\frac{d^{n-2}y}{dt^{n-2}} + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

1.6.2 Definition: System of ordinary differential equations

A system of n differential equation is defined as

$\frac{dx}{dt} = F(x(t), t)$Where

$x(t) = \left(x_1(t), x_2(t), x_3(t), \dots, x_n(t)\right)^T, F = (f_1, f_2, f_3, f_4, \dots, f_n)^T$ And

$f_i = f_i(x_1(t), x_2(t), \dots, x_n(t), t)$

1.6.3 Definition: equilibrium points

The equilibrium points to a system of the first order differential equation are the points at which the first derivatives are equal to zero. That is for the system

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

The equilibrium points are the solution of algebraic equation

$$f(x, y) = 0$$

$$g(x, y) = 0$$

1.6.4 Stability of the equilibrium point

1.6.4.1Stability by linearization

For most dynamical systems the equilibrium point of a system of nonlinear differential equations plays an important role in the analysis of the models [3], we therefore give the definition of equilibrium point and describe the analysis of the equilibrium point below.

Let $f: R^n \to R^n$ be a C^1 map and suppose that p is a point such that$f(p) = 0$. That is p is an equilibrium point for the differential equation$y'(t) = f(y(t))$. The linear part of $f$at p denoted by $Df(p)$ is the matrix of the partial derivative at$p$.

For $y \in R^n$ we write

$$f(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \\ \vdots \\ f_n(y) \end{pmatrix}$$

The function f_i are called the components of$f$. We define

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(p) & \frac{\partial f_1}{\partial y_2}(p) & \cdots & \frac{\partial f_1}{\partial y_n}(p) \\ \frac{\partial f_2}{\partial y_1}(p) & \frac{\partial f_2}{\partial y_2}(p) & \cdots & \frac{\partial f_2}{\partial y_n}(p) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1}(p) & \frac{\partial f_n}{\partial y_2}(p) & \cdots & \frac{\partial f_n}{\partial y_n}(p) \end{pmatrix}$$

Called Jacobean matrix and the stability of a flow of a nonlinear system can be studied using different approaches if the algebraic sign of the Eigen values of the Jacobean matrix is easily identified. But if the algebraic sign of these Eigen values are not determined easily we can use and we restrict our self to Ruth – Hurwitz stability criterion

1.6.4.2 Ruth – Hurwitz stability criterion

The Ruth – Hurwitz criterion is a method for determining whether a linear system is stable or not by examining the locations of the roots of the characteristics equation of the system. In fact, the method determines only if there are roots that lie outside of the left half plane: it does not actually compute the roots. Consider the characteristics equation

$$P_n(\lambda)=a_n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda^1 + a_0$$

To determine whether this system is stable or not, we have to check the two necessary but not sufficient conditions that all the roots have negatives real parts:

I. All the polynomial coefficients must be the same sign and

II. All the polynomial coefficients must be nonzero. Thus; if these condition are satisfied, then compute the Ruth-Hurwitz array as follows:

Ruth-Hurwitz array as follows:

$$\begin{array}{c|llll} \lambda^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} \cdots \\ \lambda^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} \cdots \\ \lambda^{n-2} & b_1 & b_2 & b_3 \cdots & \\ \lambda^{n-3} & c_1 & c_2 & c_3 \cdots & \\ & & & & \end{array}$$

λ^{n-4} d_1 $\dots$

λ^1

λ^0

Where the $a_{i`s}$ the polynomial coefficients and the coefficients in the rest of the table are computed using the following pattern: To determine whether this system is stable or not, check the following conditions:

1. Two necessary but not sufficient conditions that all the roots have negative real parts are: All the polynomial coefficients must have the same sign and all the polynomial coefficients must be nonzero. If this condition is satisfied then compute the Ruth-Hurwitz array as follows

$$b_1 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-3} - a_{n-2} a_{n-1})$$

$$b_2 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-5} - a_{n-4} a_{n-1})$$

$$b_3 = \frac{-1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} = \frac{-1}{a_{n-1}}(a_n a_{n-7} - a_{n-1} a_{n-6})$$

$$c_1 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix} = \frac{-1}{b_1}(a_{n-1} b_2 - b_1 a_{n-3})$$

$$c_2 = \frac{-1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} = \frac{-1}{b_1}(a_{n-1} b_3 - b_1 a_{n-5})$$

2. The necessary condition that all roots have negative real parts is that all elements of the first column of the array have the same sign. The number of changes of sign equals the number of roots with positive real parts.
3. Special case 1: The first element of a row is zero, but some other elements in the row are nonzero. In this case, simply replace the zero element by $''\epsilon''$,complete the table

development, and then interpret the results assuming that ϵ is a small number of the same sign as the element above it. The results must be interpreted in the limit as $\epsilon \to 0$.

4. Special case 2: All the elements of a particular row are zero. In this case, some of the roots of the polynomial are located symmetrically about the origin of the s plane. e.g: a pair of purely imaginary roots. The zero row will always occur in a row associated with an odd power of s. The row just above the zero row holds the coefficients of the auxiliary polynomial. The roots of the auxliry polynomial are the symmetrically placed roots. Be careful to remember that the coefficients in the array skip powers of s from one coefficent to the next.

CHAPTER TWO

2. Model formulation and Analysis

In this section we are going to develop a mathematical model which describes the dynamics of the interaction between three species food web. Which is continuous time model. For the three species food web interaction which consists of a bottom prey, middle predator and top predator population we used a system of nonlinear differential equations to describe the dynamics. We assumed that the prey population follow logistic growth model. The middle predator consumes the prey population with Holling type II functional response and the top predator consumes both the predator and the prey with Holling type II functional response. The dynamics of the three species are governed by the following system of differential equations.

$$\frac{dX}{dt} = g(X) - YP(X) - Zq_1(X)$$

$$\frac{dY}{dt} = -rY + cYP(X) - Zq_2(Y) \qquad (2\ .1)$$

$$\frac{dZ}{dt} = -sZ + d_1Zq_1(X) - d_2Zq_2(Y)$$

The following assumptions are made for the growth and interaction functions.

i. $g(X) = a_0\left(1 - \frac{X}{K}\right), where\ g(0) = a_0 > 0$
ii. $p(X) = \frac{a_1X}{1+b_1X}$ $,(holling\ type\ II\ functional\ response\ of\ Y\ on\ X).$
iii. $q_1(X) = \frac{a_2X}{1+b_2X+b_3Y}$ $,(holling\ type\ II\ functional\ response\ of\ Z\ on\ X).$
iv. $q_2(Y) = \frac{a_3Y}{1+b_3Y+b_2X}$ $,(holling\ type\ II\ functional\ response\ of\ Z\ on\ Y).$

We assume that g, p, q_1 and q_2 are continuously differentiable functions.

System (2.1) has the following form

$$\frac{dX}{dt} = a_0X\left(1-\frac{X}{K}\right) - \frac{a_1XY}{1+b_1X} - \frac{a_2XZ}{1+b_2X+b_3Y}$$

$$\frac{dY}{dt} = -rY + \frac{a_1cXY}{1+b_1X} - \frac{a_3YZ}{1+b_2X+b_3Y} \tag{2. 2}$$

$$\frac{dZ}{dt} = -sZ + \frac{a_2d_1XZ}{1+b_2X+by} + \frac{a_3d_2YZ}{1+b_2X+b_3Y}$$

Where t is the time

a_0- is the intrinsic growth rate of the bottom prey.

K is the carrying capacity of the prey population.

C - is the conversion factor of middle predator by consuming the bottom prey.

d_1- is the conversion factor of the top predator by consuming the bottom prey.

d_2 - is the conversion factor of the top predator by consuming the middle predator.

r - Is the intrinsic death rate of the middle predator in the absence of the prey.

S -Is the intrinsic death rate of the top predator in the absence of both the prey and middle-predator.

a_i and b_i determine the saturation level and the steepness of the response.

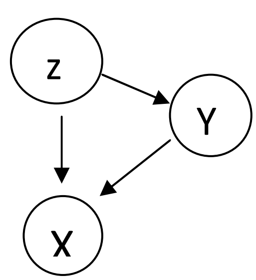

Fig1. Energy flow diagram of the model.

The model we have just specified has 13 parameters, the larger number of parameters in the system further complicates the interpretations of the mathematical result in ecological situations and decrease applicability of the model. In addition to make the algebraic manipulation simple a suitable non-dimensional variables and parameters are introduced to transform the given system to the corresponding non-dimensional system (with less number of parameters). By choosing

$$t = \frac{T}{a_0}, w_1 = b_1 k, \qquad w_2 = b_2 k\,, \qquad w_3 = \frac{a_0 b_3}{a_1}, \quad w_4 = \frac{a_1 cK}{a_0},$$

$$w_5 = \frac{-r}{a_0}, \quad w_6 = \frac{a_3}{a_2}, \quad w_7 = \frac{k d_1 a_2}{a_0}, \qquad w_8 = \frac{a_3 d_2}{a_1}, \quad w_9 = \frac{-s}{a_0}$$

$$x = \frac{X}{k}, \quad y = \frac{a_1}{a_0} Y\,, \qquad z = \frac{a_2}{a_0} Z$$

Making this substitution and simplifying yields the following system of equations. Where the variables x, y and z are rescaled measures of the population sizes and t the new non- dimensional time variable. We obtain a system of the form

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{1+w_1 x} - \frac{xz}{1+w_2 X + w_3 Y} = \text{x F (x, y, z)}$$

$$\frac{dy}{dt} = -w_5 y + \frac{w_4 xy}{1+w_1 x} - \frac{w_6 yz}{1+w_3 y + w_2 x} = \text{y G (x, y, z)} \tag{2.3}$$

$$\frac{dz}{dt} = -w_9 z + \frac{w_7 xz}{1+w_2 x + w_3 y} + \frac{w_8 yz}{1+w_3 y + w_2 x} = \text{z H(x, y, z)}$$

In system (2.3) all the model parameter are positive and have usual meaning in ecological context. The system is associated with the initial conditions.

$$X = x_0 \geq 0, \quad Y = y_0 \geq 0 \;\; , and \;\; Z = z_0 \geq 0$$

2.1. Kolmogorov analysis

The food web system (2.3) is feasible if it is possible to split the system into feasible prey-predator subsystems, in the absence of other predator[11]. So we have two systems namely x - y and x-z, this two dimensional subsystems are formulated as follows.

1. **In the absence of z**

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{1+w_1 x} \quad = x\, f_1(x,y) \tag{2.4}$$

$$\frac{dy}{dt} = \frac{w_4 xy}{1+w_1 x} - \; w_5 y \quad = y\, g_1(x,y)$$

2. **In the absence of y**

$$\frac{dx}{dt} = x(1-x) - \frac{xz}{1+w_2 x} \quad = x\, f_2(x,z) \tag{2.5}$$

$$\frac{dz}{dt} = \frac{w_7 xz}{1+w_2 x} \; - \; w_9 z \; = z\, g_2(x,z)$$

The subsystem (2.4) is said to be kolmogorov system, if f_1 and g_1 satisfies the following - conditions.

a. $\frac{\partial f_1}{\partial y} < 0$; $\quad x\,\frac{\partial f_1}{\partial x} + y\,\frac{\partial f_1}{\partial y} \; < 0$

b. $\frac{\partial g_1}{\partial y} \; < 0$; $\quad x\,\frac{\partial g_1}{\partial x} + y\,\frac{\partial g_1}{\partial y} \; > 0$

c. $f_1\,(0,0) > 0$ d. $f_1\,(0, \mathrm{A}) = 0$, A > 0 e. $f_1\,(\mathrm{B}, 0) = 0$, B>0 f. $g_1(\mathrm{C},0) = 0$ g. B>C

Since the conditions from a – g holds, system (2.4) is kolmogorov in the domain

$$D_1 = \{\, (x, y) : \; 0 \; < \; x \; < \; \bar{x} \; < \; 1 \;\; , y > 0 \,\}$$

We now find the equilibrium solution of the Kolmogorov system of (2.4) by solving the system of equation

$$x(1-x) - \frac{xy}{1+w_1 x} \; = 0$$

$$\frac{w_4 xy}{1+w_1 x} - \; w_5 y \quad = 0$$

Simultaneously.

The equilibrium solutions are solutions of

$$x\left((1-x)-\frac{y}{1+w_1x}\right)=0$$
$$y\left(\frac{w_4x}{1+w_1x}-w_5\right)=0$$

Solving simultaneously we get x=0, y = 0, $(1-x)-\frac{y}{1+w_1x}=0$, $\frac{w_4x}{1+w_1x}-w_5=0$

Therefore the three non-negative equilibrium points for the Kolmogorov system (2.4) are

$E_0=(0,0)$ $E_1=(1,0)$ $E_2=(\bar{x},\bar{y})=(\frac{w_5}{w_4-w_1w_5},(1-\bar{x})(1+w_1\bar{x}))$
$=(\frac{w_5}{w_4-w_1w_5},\frac{w_4(w_4-w_5-w_1w_5)}{(w_4-w_1w_5)^2})$ the equilibrium points exist if

$$w_4-w_5-w_1w_5>0 \tag{2.6}$$

Note that $\bar{x}\geq 1$ is not feasible equilibrium point for logistically growing prey species x.

Further $\bar{x}<1$

$$\frac{w_5}{w_4-w_1w_5}<1 \tag{2.7}$$

The equilibrium points E_0 and E_1 in xy-plane always exists and the linearized system about E_0 has zero eigenvalues and about E_1 has one zero and one negative eigenvalue. These are non-hyperbolic saddle points.

The variational matrix of system (2.4) about $E_2=(\bar{x},\bar{y})=(\frac{w_5}{w_4-w_1w_5},(1-\bar{x})(1+w_1\bar{x}))$ is given by

$$J=\begin{bmatrix}-\bar{x}+\frac{w_1\bar{x}\,\bar{y}}{(1+w_1\bar{x})^2} & \frac{-\bar{x}}{(1+w_1\bar{x})}\\ \frac{w_4\bar{y}}{(1+w_1\bar{x})^2} & 0\end{bmatrix}$$

$$J = \begin{bmatrix} -\bar{x} + \dfrac{w_1\bar{x}\,(1-x)}{(1+w_1\bar{x})} & \dfrac{-\bar{x}}{(1+w_1\bar{x})} \\ \dfrac{w_4(1-x)}{(1+w_1\bar{x})} & 0 \end{bmatrix}$$

The eigenvalues of the characteristics equation about E_2 has a negative real part if $a_{11} < 0$

$$-\bar{x} + \frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})} < 0$$

$$\bar{x} > \frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})}$$

$$1 > \frac{w_1\ (1-\bar{x})}{(1+w_1\bar{x})}$$

$$\bar{x} > \frac{w_1-1}{2w_1} > 0 \tag{2.8}$$

Combining (2.7) and (2.8)

$$0 < \frac{w_1-1}{2w_1} < \bar{x} < 1$$

$$0 < \frac{w_1-1}{2w_1} < \frac{w_5}{w_4 - w_1w_5} < 1$$

$$\frac{w_1-1}{2w_1} < \frac{w_5}{w_4 - w_1w_5}$$

$$w_1(w_4 - w_1w_5) - w_4 - w_1w_5 < 0 \tag{2.9}$$

Therefore the equilibrium point $E_2 = (\bar{x}, \bar{y})$ is locally asymptotically stable under

(i) $\bar{x} > \frac{w_1-1}{2w_1}$ (ii) $n_1 = w_1(w_4 - w_1w_5) - w_4 - w_1w_5 < 0$

The characteristics equation about E_2 is

P (λ) = $\lambda^2 - \left(-\bar{x} + \frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})}\right)\lambda + \frac{w_4\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})^2} = 0$ solving for the eigienvalues we get

$$\lambda = \frac{-\bar{x}+\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})} \pm \sqrt{(-\bar{x}+\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})})^2-4\frac{w_4\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})^2}}}{2}$$

Let the determinant of the quadratic equation be 'D'.

$$D = (-\bar{x} + \frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})})^2 - 4\frac{w_4\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})^2}$$

$$D = (\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})})^2 - 2\bar{x}(\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})} + \bar{x}^2 - 4\frac{w_4\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})^2}$$

We get a complex eigenvalue for 'λ' if $D < 0$ that is if

$$2\bar{x}(\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})} + 4\frac{w_4\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})^2} > \left(\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})}\right)^2 + \bar{x}^2$$

$$2(\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})} + 4\frac{w_4\,(1-\bar{x})}{(1+w_1\bar{x})^2} - \bar{x}\left(\frac{w_1\,(1-\bar{x})}{(1+w_1\bar{x})}\right)^2 - \bar{x} > 0$$

$$\frac{2w_1\bar{x}\,(1-\bar{x})(1+w_1\bar{x}) + 4w_4\,(1-\bar{x}) - \bar{x}(w_1\,(1-\bar{x}))^2 - \bar{x}(1+w_1\bar{x})^2}{(1+w_1\bar{x})^2} > 0$$

$$2w_1\bar{x}\,(1-\bar{x})(1+w_1\bar{x}) + 4w_4\,(1-\bar{x}) - \bar{x}\big(w_1\,(1-\bar{x})\big)^2 - \bar{x}(1+w_1\bar{x})^2 > 0$$

$$2w_1\bar{x}\,(1-\bar{x})(1+w_1\bar{x}) + 4w_4\,(1-\bar{x}) - w_1\bar{x}\,(1-\bar{x})(w_1\,(1-\bar{x})) - \bar{x}(1+w_1\bar{x})^2 > 0$$

$$w_1\bar{x}\,(1-\bar{x})\,[\,2(1+w_1\bar{x}) - w_1\,(1-\bar{x})] + 4w_4\,(1-\bar{x}) - \bar{x}(1+w_1\bar{x})^2 > 0$$

$$[\,2(1+w_1\bar{x}) - w_1\,(1-\bar{x})] - \frac{(1+w_1\bar{x})^2}{w_1\,(1-\bar{x})} + \frac{4w_4}{w_1\bar{x}} > 0$$

$$(1+w_1\bar{x})\left[2 - \frac{(1+w_1\bar{x})}{w_1\,(1-\bar{x})}\right] - w_1\,(1-\bar{x}) + \frac{4w_4}{w_1\bar{x}} > 0$$

$$(1+w_1\bar{x})\left[\frac{2w_1\,(1-\bar{x}) - (1+w_1\bar{x})}{w_1\,(1-\bar{x})}\right] - w_1\,(1-\bar{x}) + \frac{4w_4}{w_1\bar{x}} > 0$$

$$\left[\frac{2w_1\,(1-\bar{x}) - (1+w_1\bar{x})}{w_1\,(1-\bar{x})}\right] - \frac{w_1\,(1-\bar{x})}{(1+w_1\bar{x})} + \frac{4w_4}{w_1\bar{x}\,(1+w_1\bar{x})} > 0$$

$$2w_1\,(1-\bar{x})-(1+w_1\bar{x})-\frac{w^2_{1(1-\bar{x})^2}}{(1+w_1\bar{x})}+\frac{4w_4\,(1-\bar{x})}{\bar{x}\,(1+w_1\bar{x})}>0$$

$$w_1\,(1-\bar{x})\left[1-\frac{w_1\,(1-\bar{x})}{(1+w_1\bar{x})}\right]+w_1\,(1-\bar{x})-(1+w_1\bar{x})+\frac{4w_4\,(1-\bar{x})}{\bar{x}\,(1+w_1\bar{x})}>0$$

$$w_1\,(1-\bar{x})\left[\frac{-w_1\ +(1+2w_1\bar{x})}{(1+w_1\bar{x})}\right]-(-w_1\ +(1+2w_1\bar{x}))\ +\frac{4w_4\,(1-\bar{x})}{\bar{x}\,(1+w_1\bar{x})}>0$$

$$(-w_1\ +(1+2w_1\bar{x}))\left[\frac{w_1\,(1-\bar{x})}{(1+w_1\bar{x})}-1\right]+\frac{4w_4\,(1-\bar{x})}{\bar{x}\,(1+w_1\bar{x})}>0$$

The inequality satisfied if $(-w_1\ +(1+2w_1\bar{x})=0$ or $\frac{w_1\,(1-\bar{x})}{(1+w_1\bar{x})}-1=0$ that is

When $\bar{x}=\frac{w_1-1}{2w_1}$, the eigenvalues of the characteristic equation will be a complex number.

Let $\lambda=a_0\pm b_0 i$ then the real part of the eigenvalues which is a_0 can be zero if $a_{11}=0$

$$a_0\ =-\bar{x}+\frac{w_1\bar{x}\,(1-\bar{x})}{(1+w_1\bar{x})}\ =0=a_{11}$$

$$a_0\ =0\quad \text{If}\quad \bar{x}=\frac{w_1-1}{2w_1}$$

Therefore the Kolmogorov subsystem changes its dynamical behavior from stability at equilibrium point E_2, and admits a limit cycle in domain D_1 under the condition

(i) $\bar{x}<\frac{w_1-1}{2w_1}$ and (ii) $n_1\ =\ w_1(w_4-w_1w_5)-w_4-w_1w_5\ >0$

Hence the two purely imaginary eigenvalues crosses the imaginary axis when the parameter w_1 vary this leads us to the formation of Hopf- Bifurcation.

Similarly for the kolmogorov subsystem (2.5) we get the following results.

The three non-negative equilibrium points for system (2.5) are

$E_0=(0,0)$ $E_1=(1,0)$ $E_3=(\hat{x}\,,\hat{z}\,)$=($\frac{w_9}{w_7-w_2w_9}$ $,(1-\bar{x})(1+w_1\bar{x})$)=($\frac{w_9}{w_7-w_2w_9}$, $\frac{w_7(w_7-w_9-w_2w_9)}{(w_7-w_2w_9)^2}$) with these we have the following conditions.

The equilibrium pointE_3 exists if $w_7 - w_9 - w_2 w_9 > 0$ (2.10)

$$\hat{x} = \frac{w_9}{w_7 - w_2 w_9} < 1 \quad (2.11)$$

$$\hat{x} > \frac{w_2 - 1}{2w_2} \quad (2.12)$$

Combining (2.11) and (2.12) we get,

$$0 < \frac{w_2 - 1}{2w_2} < \frac{w_9}{w_7 - w_2 w_9} < 1$$

So $n_2 = w_2(w_7 - w_2 w_9) - w_7 - w_2 w_9 < 0$ (2.13)

Therefore the equilibrium point $E_3 = (\hat{x}, \hat{z})$ is locally asymptotically stable under the conditions

(i) $\hat{x} > \frac{w_2 - 1}{2w_2}$ (ii) $n_2 = w_2(w_7 - w_2 w_9) - w_7 - w_2 w_9 < 0$

More over the sub system (2.5) changes its dynamical behavior from stability of E_3 to admitting a limit cycle in the domain $D_2 = \{(x, z): 0 < x < \hat{x} < 1, z > 0\}$ under the conditions

(i) $\hat{x} < \frac{w_2 - 1}{2w_2}$ (ii) $n_2 = w_2(w_7 - w_2 w_9) - w_7 - w_2 w_9 > 0$

Hence the two purely imaginary eigenvalues crosses the imaginary axis when the parameter w_2 vary this leads us to the formation of Hopf- Bifurcation.

Further analysis of a system (2.3) carried out under conditions (2.6) and (2.10)

$$w_4 - w_5 - w_1 w_5 > 0$$

$$w_7 - w_9 - w_2 w_9 > 0$$

2.2 Model Analysis

It is observed that all the interacting functions F(x, y, z), G(x, y, z) and H(x, y, z) of system (2.3) are continues and have continues derivatives on the nonnegative octant.

$\overline{\Omega} = \{(x, y, z)\ \varepsilon\ R^3\ ; x, y, z \geq 0\}$

Theorem 2.1

The solution of system (2.3) initiating in $\overline{\Omega}$ for $t \geq 0$ is uniformly bounded.

Proof

The first equation of the system that represents the prey equation is bounded through

$$\frac{dx}{dt} = x(1-x) - \frac{xy}{1+w_1x} - \frac{xz}{1+w_2X+w_3Y}$$

$\frac{dx}{dt} \leq x(1-x)$ since $x(t) \geq 0$, $y(t) \geq 0$,and $z(t) \geq 0$ for all $t \geq 0$.

$$\frac{dx}{dt} = x(1-x)$$

$$\left(\frac{1}{x} + \frac{1}{(1-x)}\right) dx = dt$$

$$\frac{1}{x}dx - \frac{1}{(x-1)}dx = dt$$

$$\int_0^t \left(\frac{1}{x}dx - \frac{1}{(x-1)}dx\right) = \int_0^t dt$$

$\ln|x| - \ln|(x-1|$ = t + constant

The constant enables the initial value problem $x(0) = x_0$

$$\ln|x| - \ln|(x-1| = t + \ln|x_0| - \ln|(x_0-1|$$

$$\ln\left|\frac{x}{x_0}\right| + \ln\left|\frac{x_0-1}{x-1}\right| = t$$

$$\ln\left(\frac{x}{x_0}\right)\left(\frac{x_0-1}{x-1}\right)=t$$

$$\left(\frac{x}{x_0}\right)\left(\frac{x_0-1}{x-1}\right)=e^t$$

$$x\,(x_0-\ 1)=x_0(x-1)e^t$$

$$x\,(x_0-\ 1)=x_0xe^t-\ \ x_0e^t$$

$$x\,(x_0-\ 1)-x_0xe^t=-\ \ x_0e^t$$

$$x\big((x_0-\ 1)-x_0e^t\big)=-\ x_0e^t$$

$$x=\frac{x_0e^t}{\big((x_0-\ 1)-x_0e^t\big)}$$

$$x=\frac{1}{1+\left(\frac{1-x_0}{x_0}\right)e^{-t}}$$

$$x=\frac{1}{1+ce^{-t}}\qquad where\quad c=\frac{1-x_0}{x_0}$$

$\rightarrow$ x (t) = $\frac{1}{1+ce^{-t}}$ for $t\geq 0$ and constant c. Hence for a very large value of t we get x (t)$\leq$ 1.

The bounded ness of y(t) and z(t) follows from the bounded ness of x(t),hence all the species are uniformly bounded for any initial value in R_+^3.

2.2.1 Local behavior of the system

Local stability of the equilibrium points is done by using the variational (Jacobean) matrix.

$$J=\begin{bmatrix}F_x & G_x & H_x\\ F_y & G_y & H_y\\ F_z & G_z & H_z\end{bmatrix}$$

$$J=\begin{bmatrix} 1-2x-\frac{y}{(1+w_1x)^2}-\frac{z(1+w_3y)}{(1+w_2x+w_3y)^2} & \frac{-x}{1+w_1x}+\frac{w_3xz}{(1+w_2x+w_3y)^2} & \frac{-x}{1+w_2x+w_3y} \\ \frac{w_4y}{(1+w_1x)^2}+\frac{w_2w_6yz}{(1+w_2x+w_3y)^2} & -w_5+\frac{w_4x}{1+w_1x}-\frac{w_6z+w_2w_6xz}{(1+w_2x+w_3y)^2} & \frac{-w_6y}{1+w_2x+w_3y} \\ \frac{w_7z+w_3w_7yz-w_2w_8yz}{(1+w_2x+w_3y)^2} & \frac{w_8z+w_2w_8xz-w_3w_7xz}{(1+w_2x+w_3y)^2} & -w_9+\frac{w_7x+w_8y}{1+w_2x+w_3y} \end{bmatrix}$$

Which is evaluated at the equilibrium points. Then find the eigenvalues of the Jacobean matrix. According to the sign of the eigenvalues, we can determine the local stability of the equilibrium points.

I. Behavior of the system or stability analysis of trivial singularities

A. The variational matrix at $E_1(0,0,0)$ is obtained by substituting it in general variational matrix and we get

$$J_1=\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -w_5-\lambda & 0 \\ 0 & 0 & -w_9-\lambda \end{bmatrix}$$

The corresponding characteristics polynomial of the equilibrium point E_1 is

P (λ)= (1- λ) [(- w_5- λ)(- $w_9-\lambda$)]=0

The eigenvalues of J_1 are $\lambda_1=1,\quad \lambda_2 = -w_5\ ,\ \lambda_3=-w_9$

B. The variational matrix at E_2 (1,0,0) is obtained by substituting it in general variational matrix and we get

$$J_2=\begin{bmatrix} -1-\lambda & \frac{-1}{1+w_1} & \frac{-1}{1+w_2} \\ 0 & -w_5+\frac{w_4}{1+w_1}-\lambda & 0 \\ 0 & 0 & -w_9+\frac{w_7}{1+w_2}-\lambda \end{bmatrix}$$

The corresponding characteristics polynomial of the equilibrium point E_2 is

$$\text{P}\ (\lambda)=(-1-\lambda)\left(-w_5+\frac{w_4}{1+w_1}-\lambda\right)\left(-w_5+\frac{w_4}{1+w_1}-\lambda\right)=0$$

The eigenvalues of J_2 are $\lambda_1 = -1$, $\lambda_2 = -w_5 + \frac{w_4}{1+w_1}$ and $\lambda_3 = -w_9 + \frac{w_7}{1+w_2}$

λ_2, $\lambda_3 < 0$ If $\lambda_2 = w_4 - w_5 - w_1 w_5 < 0$ and $\lambda_3 = (w_7 - w_9 - w_2 w_{9)} < 0$

We observe that all the eigenvalues are negative.

Theorem 2.2

a. The trivial singularity $E_1 = (0,0,0)$ of system (2. 3) always exist and hyperbolic saddle having one dimensional unstable manifold along the x direction.
b. The axial singularities $E_2 = (1,0,0)$ of a system (2.3) always exists and hyperbolic saddle having one dimensional stable manifold along x direction.

These results are evident from the eigenvalues of the varitational matrix J_1 and J_2.

II. Behavior of the system or stability analysis of non- trivial singularities

A. variational matrix at $E_3 = (\bar{x}, \bar{y}, 0) = (\frac{w_5}{w_4 - w_1 w_5}, \frac{w_4(w_4 - w_5 - w_1 w_5)}{(w_4 - w_1 w_5)^2}, 0)$ is obtained by substituting it in general variational matrix and we get

$$J_3 = \begin{bmatrix} 1 - 2\bar{x} - \frac{\bar{y}}{(1+w_1\bar{x})^2} & \frac{-\bar{x}}{1+w_1\bar{x}} & \frac{-\bar{x}}{1+w_2\bar{x}+w_3\bar{y}} \\ \frac{w_4\bar{y}}{(1+w_1\bar{x})^2} & -w_5 + \frac{w_4\bar{x}}{1+w_1\bar{x}} & \frac{-w_6\bar{y}}{1+w_2\bar{x}+w_3\bar{y}} \\ 0 & 0 & -w_9 + \frac{w_8 y}{1+w_2\bar{x}+w_3\bar{y}} \end{bmatrix}$$

$$J_3=\begin{bmatrix} \bar{x}\left[\frac{w_1(1-\bar{x})}{1+w_1\bar{x}}-1\right]-\lambda & \frac{-\bar{x}}{1+w_1\bar{x}} & \frac{-\bar{x}}{1+w_2\bar{x}+w_3\bar{y}} \\ \frac{w_4\bar{y}}{(1+w_1\bar{x})^2} & 0-\lambda & \frac{-w_6\bar{y}}{1+w_2\bar{x}+w_3\bar{y}} \\ 0 & 0 & -w_9+\frac{w_7\bar{x}+w_8\bar{y}}{1+w_2\bar{x}+w_3\bar{y}}-\lambda \end{bmatrix}$$

The corresponding characteristics polynomial of

E_3 *is* p(λ) $= |A-\lambda I| = 0$ *has eigen values* λ_1, λ_2 *and* λ_3

$$\mathrm{p}(\lambda)=\left[\left(-w_9+\frac{w_7\bar{x}+w_8\bar{y}}{1+w_2\bar{x}+w_3\bar{y}}\right)-\lambda\right]\left[\lambda^2-\left[\bar{x}\left(\frac{w_1(1-\bar{x})}{1+w_1\bar{x}}-1\right)\right]\lambda+\frac{w_4\bar{x}\,\bar{y}}{(1+w_1\bar{x})^3}\right]=0$$

$$\lambda_3=-w_9+\frac{w_7\bar{x}+w_8\bar{y}}{1+w_2\bar{x}+w_3\bar{y}}$$

$$\lambda_1+\lambda_2=\ \bar{x}\left(\frac{w_1(1-\bar{x})}{1+w_1\bar{x}}-1\right)=trace\ of\ A<0$$

$$\lambda_1\lambda_2=\ \frac{w_4\bar{x}\,\bar{y}}{(1+w_1\bar{x})^3}=\det A>0$$

For stability we need Re $\lambda<0$

Theorem 2.3

a. The singularities $E_3=(\bar{x},\bar{y},0)$ *is hyperbolic attracting focus in the hyper plane* $\Omega_1=\overline{\Omega_1}\cap\{(\mathrm{x,y,z})\varepsilon\mathrm{IR}^3;\mathrm{z}=0\}$, provided

$\cap_1= w_1(\,w_4-w_1w_5)-w_4-w_1w_5<0$ ……………………. (2.14)

$\cap_2=(\,w_4-w_1w_5)(w_5w_7+w_4w_8+w_1w_5w_9-w_4w_9-w_3w_4w_9-w_2w_5w_9)-w_4w_5\,)(w_8-w_3w_9)<0$ ………………………………. (2.15)

proof

From the eigenvalues λ_1 *and* λ_2 we want to show λ_1 *and* λ_2 are both negative with $\cap_1<0$

$\lambda_1 + \lambda_2 = \bar{x}\left(\frac{w_1(1-\bar{x})}{1+w_1\bar{x}} - 1\right)$

$\lambda_1 + \lambda_2 = 1 - \frac{2w_5}{q} - \frac{p}{w_4}, where \quad q = w_4 - w_1 w_5 > 0 \; and \; P = q - w_5$

$\lambda_1 + \lambda_2 = \frac{w_4 q - 2w_4 w_5 - pq}{w_4 q}$

$\lambda_1 + \lambda_2 < 0$

If $w_4 q - 2w_4 w_5 - pq < 0$

q$(w_4 - p) - 2w_4 w_5 < 0$

q$(w_4 - (w_4 - w_1 w_5 - w_5)) - 2w_4 w_5 < 0$

q$(w_1 w_5 + w_5) - 2w_4 w_5 < 0$

q$(w_1 + 1) - 2w_4 < 0$

$(w_4 - w_1 w_5)(w_1 + 1) - 2w_4 < 0$

$w_1 w_4 - {w_1}^2 w_5 - w_4 - w_1 w_5 < 0$

$\Rightarrow \quad w_1(w_4 - w_1 w_5) - w_4 - w_1 w_5 < 0$

Therefore the sum of eigenvalues λ_1 *and* λ_2 is negative for $n_1 < 0$. More over their product is positive.

$\lambda_1 \lambda_2 = \frac{w_4 \bar{x}\,\bar{y}}{(1+w_1\bar{x})^3} > 0 \; , for \; w_4 > 0, \;\; \bar{x} > 0, \;\; \bar{y} > 0 \; , \;\; 1 + w_1\bar{x} > 0$

Therefore, λ_1 *and* λ_2 *are both negative*.

For the other eigenvalue $\lambda_3 = -w_9 + \frac{w_7\bar{x} + w_8\bar{y}}{1 + w_2\bar{x} + w_3\bar{y}}$

$\lambda_3 = -w_9 + \frac{w_7\bar{x} + w_8\bar{y}}{R}$, where (R $= 1 + w_2\bar{x} + w_3\bar{y} > 0$)

λ_3 *is less than zero if*

$w_7\bar{x} + w_8\bar{y} - w_9\, R < 0$

$w_7\bar{x} + w_8\bar{y} - w_9 - w_2w_9\bar{x} - w_3w_9\bar{y} < 0$

$L\bar{x} - w_9 + K(1 + w_1\bar{x} - \bar{x} - w_1\bar{x}^2) < 0$ where $L = w_5 - w_2w_9$ and $K = (w_8 - w_3w_9)$

$[L + w_1k - k - w_1k\bar{x}]\,\bar{x} - w_9 + k < 0$

$\frac{w_5}{q}\left[w_7 - w_2w_9 + w_1k - k - w_1k\frac{w_5}{q}\right] - w_9 + k < 0$, where $\bar{x} = \frac{w_5}{q}$

$\frac{w_5w_7q - w_2w_5w_9q + w_1w_5kq - w_5kq - w_1{w_5}^2k - w_9q^2 + kq^2}{q^2} < 0$

$q[w_5w_7 - w_2w_5w_9 + w_1w_5k - w_5k - w_9q + kq] - w_1{w_5}^2k < 0$

$q[w_5w_7 - w_2w_5w_9 + w_1w_5(w_8 - w_3w_9) - w_5(w_8 - w_3w_9) - w_9(w_4 - w_1w_5) + (w_8 - w_3w_9)(w_4 - w_1w_5)] - w_1{w_5}^2(w_8 - w_3w_9) < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] + q(w_3w_5w_9 - w_5w_8) - w_1{w_5}^2k < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - w_5qk - w_1{w_5}^2k < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - [w_5q - w_1{w_5}^2]k < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - [w_5(w_4 - w_1w_5) + w_1{w_5}^2]k < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - [w_4w_5 - w_1{w_5}^2 + w_1{w_5}^2]k < 0$

$q[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - [w_4w_5]k < 0$

$(w_4 - w_1w_5)[w_5w_7 + w_4w_8 + w_1w_5w_9 - w_5w_8 - w_4w_9 - w_3w_4w_9] - w_4w_5(w_8 - w_3w_9) < 0$

Hence $\lambda_3 < 0$, *for* $\cap_2 < 0$.

Here the sum of λ_1 *and* λ_2 is negative, the product of λ_1 *and* λ_2 is positive, so we can say that $E_3 = (\bar{x}, \bar{y}, 0)$ *exists and is asymotoically stable in* xy plane provided condition (2.14) holds. More over E_3 will be stable in x- y- z, plane. Provided condition (2.15) holds.

B. The variational matrix at $E_4 = (\hat{x}, 0, \hat{z}) = \left(\frac{w_9}{w_7 - w_2w_9}, 0, \frac{w_7(w_7 - w_9 - w_2w_9)}{(w_7 - w_2w_9)^2}\right)$ is obtained by-

Substitution of E_4 in the general variational matrix and we get

$$J_4 = \begin{bmatrix} \left(1-2\hat{x}-\frac{\hat{z}}{(1+w_2\hat{x})^2}\right)-\lambda & \frac{-\hat{x}}{1+w_2\hat{x}} & \frac{-\hat{x}}{1+w_1\hat{x}}+\frac{w_3\hat{x}\hat{z}}{(1+w_2\hat{x})^2} \\ \frac{w_7\hat{z}}{(1+w_2\hat{x})^2} & 0-\lambda & \frac{w_8\hat{z}+w_2w_6\hat{x}\hat{z}-w_3w_7\hat{x}\hat{z}}{(1+w_2\hat{x})^2} \\ 0 & 0 & -w_5+\frac{w_4\hat{x}}{1+w_1\hat{x}}-\frac{w_6\hat{z}}{1+w_2\hat{x}}-\lambda \end{bmatrix}$$

The characteristics polynomial of J_4 is

$$\text{P}(\lambda) = \left[-w_5+\frac{w_4\hat{x}}{1+w_1\hat{x}}-\frac{w_6\hat{z}}{1+w_2\hat{x}}-\lambda\right]\left[\lambda^2-\left(1-2\hat{x}-\frac{\hat{z}}{(1+w_2\hat{x})^2}\right)\lambda+\frac{w_7\hat{x}\hat{z}}{(1+w_2\hat{x})^3}\right]=0$$

$$\lambda_1+\lambda_2= 1-2\hat{x}-\frac{\hat{z}}{(1+w_2\hat{x})^2}=\hat{x}\left[\left(\frac{w_2-w_2\hat{x}}{1+w_2\hat{x}}\right)-1\right]=\hat{x}\left[\left(\frac{w_2(1-\hat{x})}{1+w_2\hat{x}}\right)-1\right]$$

$$\lambda_1\lambda_2=\frac{w_7\hat{x}\hat{z}}{(1+w_2\hat{x})^3}$$

$$\lambda_3=\frac{w_4\hat{x}}{1+w_1\hat{x}}-w_5-\frac{w_6\hat{z}}{1+w_2\hat{x}}$$

Theorem 2.4

a. The singularity $E_4=(\hat{x},0,\hat{z})$ is hyperbolic attracting focus in the hyper plane

$\Omega_2=\overline{\Omega}\cap\{(\text{x},\text{y},\text{z})\varepsilon\text{IR}^3;\text{y}=0\}$, provided

$\cap_{1'}= w_2(w_7-w_2w_9)-w_7-w_2w_9<0$ (2.16)

$\cap_{2'}=\cup(w_9q-w_5\cup)-w_6(\cup-w_9)(\cup+w_1w_9)<0$ (2.17)

Where $\cup= w_7-w_2w_9$ and $q= w_4-w_1w_5$

We want to show $\lambda_1+\lambda_2<0$, $\lambda_1\lambda_2>0$, $\lambda_3<0$ if $\cap_{1'}<0$ and $\cap_{2'}<0$

Proof

$$\left[\lambda_1+\lambda_2=\hat{x}\left(\frac{w_2(1-\hat{x})}{1+w_2\hat{x}}\right)-1\right]$$

Let

$\hat{x} = \frac{w_9}{U}, for \;\; u = w_7 - w_2 w_9 \;\; and \;\; \hat{z} = \frac{w_7 V}{U^2} \;\; , \qquad for \;\; V = U - w_9$

$$\lambda_1 + \lambda_2 = \frac{w_9}{U}\left[\frac{w_2(U - w_9)}{w_7} - 1\right] = \frac{w_9}{U}\left[\frac{w_2 V}{w_7} - 1\right] = \frac{w_9}{U}\left[\frac{w_2 V - w_7}{w_7}\right]$$

But $\lambda_1 + \lambda_2 < 0 \qquad if \qquad w_9(w_2 V - w_7) < 0$

$\Longrightarrow \quad w_2 V - w_7 < 0$

$\Longrightarrow w_2(\, w_7 - w_2 w_9 - w_9) - \, w_7 < 0$

And $\lambda_1 \lambda_2 = \frac{w_7 \hat{x} \hat{z}}{(1+w_2\hat{x})^3} > 0, for \; w_7 > 0 \quad \hat{x} > 0, \;\; \hat{z} > 0$

For the third eigenvalue

$\lambda_3 = -w_5 + \frac{w_4 \hat{x}}{1+w_1\hat{x}} - \frac{w_6 \hat{z}}{1+w_2\hat{x}} \;\; < 0$

$-w_5 + \frac{w_4 \hat{x}}{1+w_1\hat{x}} - \frac{w_6(1-\hat{x})(1+w_2\hat{x})}{1+w_2\hat{x}} \;\; < 0$

$\frac{w_4 \hat{x}}{1+w_1\hat{x}} - w_6\,(1 - \hat{x}\,) - w_5 \;\; < 0$

$\frac{w_4\hat{x} + w_6\hat{x} - w_6 - w_5 + w_1 w_6 \hat{x}^2 - w_1 w_6 \hat{x} - w_1 w_5 \hat{x}}{1+w_1\hat{x}} \;\; < 0$ if

$w_1 w_6 \hat{x}^2 + (w_4 + w_6 - w_1 w_6 - w_1 w_5)\hat{x} - w_6 - w_5 < 0$

Let $\hat{x} = \frac{w_9}{w_7 - w_2 w_9} \quad where \;\; U = w_7 - w_2 w_9 \rightarrow \qquad \hat{x} = \frac{w_9}{U}$

$\Rightarrow \frac{w_1 w_6 {w_9}^2 + w_4 w_9 U + w_6 w_9 U - w_1 w_6 w_9 U - w_1 w_5 w_9 U - w_6 U^2 - w_5 U^2}{U^2} < 0$

$-w_5 U^2 + w_4 w_9 U - w_1 w_5 w_9 U - w_6 U^2 - w_1 w_6 w_9 U + w_6 w_9 U + w_1 w_6 {w_9}^2 < 0$

$-w_5 U^2 + w_9 U(\, w_4 - w_1 w_5) - \, w_6 U(U + w_1 w_9) + \, w_6 w_9 (U + w_1 w_9) < 0$

$U(w_9 q - w_5 U) - w_6(U - w_9)\,(U + w_1 w_9) < 0$

$(w_7 - w_2 w_9)(\ w_9(\ w_4 - w_1 w_5) - \ w_5(w_7 - w_2 w_9)) - \ w_6((w_7 - w_2 w_9) - w_9)(\quad (w_7 - w_2 w_9)$ $+w_1 w_9 < 0$

2.2.2 Behavior of three dimensional system

a. Existence of E_5 = (x*,y*,z*)

Here x*, y* and z* are the positive solutions of the system of algebraic equation given by

$$1 - \boldsymbol{x}^* - \frac{\boldsymbol{y}*}{1+w_1\boldsymbol{x}*} - \frac{\boldsymbol{z}*}{1+w_2\boldsymbol{x}*+w_3\boldsymbol{y}*} = 0 \qquad (2.18)$$

$$-w_5 + \frac{w_4\boldsymbol{x}*}{1+w_1\boldsymbol{x}*} - \frac{w_6 z^*}{1+w_2\boldsymbol{x}*+w_3\boldsymbol{y}*} = 0 \qquad (2.19)$$

$$-w_9 + \frac{w_7\boldsymbol{x}*}{1+w_2\boldsymbol{x}*+w_3\boldsymbol{y}*} + \frac{w_8\boldsymbol{y}*}{1+w_2\boldsymbol{x}*+w_3\boldsymbol{y}*} = 0 \qquad (2.20)$$

From (2.20) we get

y*= $\frac{(w_2 w_9 - w_4)\boldsymbol{x}*+w_9}{w_8 - w_3 w_9} = \frac{M\boldsymbol{x}*+w_9}{K}$ where M = $w_2 w_9 - w_4 \ \ and \ \ K = \ w_8 - w_3 w_9$

Substituting y* in (2.19) we get

z*= $\frac{(w_4\boldsymbol{x}* - w_1 w_5\boldsymbol{x}* - \ w_5)(k + w_2\boldsymbol{kx}* + w_3\boldsymbol{Mx}* + w_3 w_9)}{w_6 k(1+w_1\boldsymbol{x}*)}$

z*= $\frac{(q\boldsymbol{x}* - w_5)(k + w_2\boldsymbol{kx}* + w_3\boldsymbol{Mx}* + w_3 w_9)}{w_6 k(1+w_1\boldsymbol{x}*)}$ where $q = w_4 - w_1 w_5$

Subtitling y* and z* in (2.18) $we\ get$

$1 - \mathbf{x*} - \frac{M\boldsymbol{x}*+w_9}{K(1+w_1\boldsymbol{x}*)} - \frac{(q\boldsymbol{x}* - w_5)}{w_6(1+w_1\boldsymbol{x}*)} = 0$

⇨$w_1 w_6 \boldsymbol{k}\ \boldsymbol{x}^{*\mathbf{2}} \ + (w_6 k + w_6 M + kq - w_1 w_6 k)\mathbf{x*} + w_6 w_9 - w_6 k - w_5 k = 0$

x* can be obtained from the quadratic equation of the form

$A^*\ x^{*2} \ - \ B\ ^* x^* + 1 = 0$

Where

$$A^*= \frac{w_1 w_6 (w_8 - w_3 w_9)}{w_6 w_9 - (w_5 + w_6)\,(w_8 - w_3 w_9)}$$

$$B^*= \frac{(w_8 - w_3 w_9)(w_4 - w_1 w_5) + w_6[(w_8 - w_7 - w_1 w_8) + w_9 (w_2 + w_1 w_3 - w_3)]}{(w_5 + w_6)(w_8 - w_3 w_9) - w_6 w_9}$$

The vector field $X(x, y, z)$ admits unique singularity when

$$A^* < 0 \tag{2.2.1}$$

$$x^* = \frac{-B^* \pm \sqrt{B^{*2} - 4A^*}}{2A^*}$$

$$y^* = \frac{(w_2 w_9 - w_4)\boldsymbol{x}* + w_9}{w_8 - w_3 w_9}$$

$$z^* = \frac{(q\boldsymbol{x}* - w_5)(k + w_2 \boldsymbol{kx}* + w_3 \boldsymbol{Mx}* + w_3 w_9)}{w_6 k(1 + w_1 \boldsymbol{x}*)}$$

where $q = w_4 - w_1 w_5$, $k = w_8 - w_3 w_9$ and $M = w_2 w_9 - w_4$

b. Local stability of $\boldsymbol{E_5} =$ (x*,y*,z*)

The characteristics polynomial of the general variational matrix about $\boldsymbol{E_5} =$ **(x*,y*,z*) is**

$P(\lambda) = det(A - \lambda I) = 0$, where A is an operator and λ is an eigenvalue.

$$\begin{vmatrix} \boldsymbol{m_{11}} - \lambda & \boldsymbol{m_{12}} & \boldsymbol{m_{13}} \\ \boldsymbol{m_{21}} & \boldsymbol{m_{22}} - \lambda & \boldsymbol{m_{23}} \\ \boldsymbol{m_{31}} & \boldsymbol{m_{32}} & \boldsymbol{m_{33}} - \lambda \end{vmatrix} = \boldsymbol{0}$$

$P(\lambda) = \lambda^3 + a\,\lambda^2 + b\,\lambda + c = 0$, where

$$\boldsymbol{a} = -(\boldsymbol{m_{11}} + \boldsymbol{m_{22}} + \boldsymbol{m_{33}})$$

$$\boldsymbol{b} = (\boldsymbol{m_{11} m_{22}} + \boldsymbol{m_{11} m_{33}} + \boldsymbol{m_{22} m_{33}}) - (\boldsymbol{m_{32} m_{23}} + \boldsymbol{m_{12} m_{21}} + \boldsymbol{m_{13} m_{31}})$$

$$\boldsymbol{c} = (\boldsymbol{m_{11} m_{32} m_{23}} + \boldsymbol{m_{22} m_{13} m_{31}} + \boldsymbol{m_{33} m_{12} m_{21}}) - (\boldsymbol{m_{11} m_{22} m_{33}} + \boldsymbol{m_{12} m_{31} m_{23}} + \boldsymbol{m_{13} m_{21} m_{32}})$$

$$\boldsymbol{ab-c} = [-(\boldsymbol{m_{11}} + \boldsymbol{m_{22}} + \boldsymbol{m_{33}})(\boldsymbol{m_{11}m_{22}} + \boldsymbol{m_{11}m_{33}}) + \boldsymbol{m_{11}}\,(\boldsymbol{m_{12}m_{21}} + \boldsymbol{m_{13}m_{31}})$$
$$- \boldsymbol{m_{22}}(\boldsymbol{m_{22}m_{33}} - \boldsymbol{m_{12}m_{21}}) - \boldsymbol{m_{33}}(\boldsymbol{m_{22}m_{33}} - \boldsymbol{m_{32}m_{23}} - \boldsymbol{m_{13}m_{31}})$$
$$+ \boldsymbol{m_{22}m_{32}m_{23}}\,] + \boldsymbol{m_{12}m_{31}m_{23}} + \boldsymbol{m_{13}m_{21}m_{32}}$$

Where $\boldsymbol{m_{11}} = 1 - 2x - \frac{y}{(1+w_1x)^2} - \frac{z(1+w_3y)}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{12}} = \frac{-x}{1+w_1x} + \frac{w_3xz}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{13}} = \frac{-x}{1+w_2x+w_3y}$

$\boldsymbol{m_{21}} = \frac{w_4y}{(1+w_1x)^2} + \frac{w_2w_6yz}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{22}} = -w_5 + \frac{w_4x}{1+w_1x} - \frac{w_6z+w_2w_6xz}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{23}} = \frac{-w_6y}{1+w_2x+w_3y}$

$\boldsymbol{m_{31}} = \frac{w_7z+w_3w_7yz-w_2w_8yz}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{32}} = \frac{w_8z+w_2w_8xz-w_3w_7xz}{(1+w_2x+w_3y)^2}$ $\boldsymbol{m_{33}} = -w_9 + \frac{w_7x+w_8y}{1+w_2x+w_3y}$

According to Routh – Hurwitz criterion the necessary and the sufficient condition for stability are a > 0, b > 0, c > 0 and a b > c

Using the Routh –Hurwitz criteria which gave rise to intractable mathematical condition for local stability of E_5 for $m_{13} < 0$ and $m_{23} < 0$ we have the following theorem.

Theorem

Suppose $E_5 ==$ (x*,y*,z*) exists and let m_{11} < 0, m_{22} < 0 and $m_{33} \le 0$ if any one of the following conditions satisfied.

I. If $0 < (\boldsymbol{m_{11}m_{22}m_{33}} + \boldsymbol{m_{12}m_{31}m_{23}} + \boldsymbol{m_{13}m_{21}m_{32}}) < (\boldsymbol{m_{11}m_{32}m_{23}} + \boldsymbol{m_{22}m_{13}m_{31}} + \boldsymbol{m_{33}m_{12}m_{21}})$

II. If $(\boldsymbol{m_{11}m_{22}m_{33}} + \boldsymbol{m_{12}m_{31}m_{23}} + \boldsymbol{m_{13}m_{21}m_{32}}) = \mathbf{0}$

III. If $\mathbf{0} < -\ \boldsymbol{m_{12}m_{31}m_{23}} - \boldsymbol{m_{13}m_{21}m_{32}} < -(\boldsymbol{m_{11}m_{22}m_{33}})(\boldsymbol{m_{11}m_{22}} + \boldsymbol{m_{11}m_{33}}) + \boldsymbol{m_{11}}(\boldsymbol{m_{12}m_{21}} + \boldsymbol{m_{13}m_{31}}) - \boldsymbol{m_{22}}(\boldsymbol{m_{22}m_{33}} - \boldsymbol{m_{12}m_{21}}) - \boldsymbol{m_{33}}(\boldsymbol{m_{22}m_{33}} - \boldsymbol{m_{32}m_{23}} - \boldsymbol{m_{13}m_{31}}) + \boldsymbol{m_{22}m_{32}m_{23}}$**. Then** $\boldsymbol{E_5}$ **is asymptotically stable. If not** $\boldsymbol{E_5}$ **is unstable.**

Assuming that the vector field has unique singularity. Condition for existence of an invariant compact set in Ω is proved in [20] and showed that for a maximal invariant set Γ 'which is non empty because it is an intersection of decreasing family of compact set and a path connected set that doesn't reduce to a single point [12]' there exist an invariant compact set $\wedge_1 \subset \Gamma$, which contains all possible orbits of $X(x, y, z)$.Therefor with this attracting compact set the coexistence of the three species with time guaranteed.

CHAPTER THREE

3. 1 Numerical simulation Results

Analytical studies can never be completed without numerical verification of the results. in this section I present computer simulation of some solution of the system (2.3).Beside verification of an out analytical findings, these numerical solutions are very important from practical point of view .The system is numerically solved for suitable combination of parameters in such a manner that kolmogorov condition (2.6) and (2.10) satisfied.

We choose parameters for the vector field X(x, y, z) as

$W_1 = 1.4$ $w_2 = 5.0$ $w_3 = 8.0$ $w_4 = 1.0$ $w_5 = 0.16$ $w_6 = 0.1$ $w_7 = 1.1$
$w_8 = 2.0$ $w_9 = 0.163$ (3.1)

And find the equilibrium values E_3, E_4, and E_5

(i). In the absence of the top predator Z, The equilibrium point $E_3 = (\bar{x}, \bar{y}, 0)$ where

$$\bar{x} = \frac{w_5}{w_4 - w_1 w_5} = \frac{0.16}{0.776} = 0.2061856 \qquad and$$

$$\bar{y} = (1 - \bar{x})(1 + w_1\bar{x}) = (0.7938144)(1.2886598) = 1.0229567$$

Therefore $E_3 = (0.2061856\ ,1.0229567,0)$ is an attracting focus in the x y plane and has an unstable manifold along z axis.

(ii) In the absence of the middle predator Y, The equilibrium point $E_4 = (\hat{x}, 0, \hat{y})$ where

$$\hat{x} = \frac{w_9}{w_7 - w_2 w_9} = \frac{0.163}{0.285} = 0.5719298 \qquad and$$

$$\hat{z} = (1 - \hat{x})(1 + w_2\hat{x}) = (0.4280702)(3.859649) = 1.6522007$$

Therefore $E_4 = (0.5719298, 0\ ,1.6522007)$ is also an attracting focus in x z plane and has an unstable manifold along y –axis.

Thus the equilibrium points E_3 and E_4 have one dimensional unstable manifold in axial directions.

(iii) In the presence of all the species the equilibrium point $E_5 = (x^*, y^*, z^*)$ is obtained from the quadratic equation

$A^* \, x^{*2} \; - \; B^* x^* + 1 = 0$

$x^* = \frac{-B^* \pm \sqrt{B^{*2} - 4A^*}}{2A^*}$ Where A^* $and \; B^*$ are

A*= $\frac{w_1 w_6 (\, w_8 - w_3 w_9)}{w_6 w_9 - (w_5 + w_6)\,(w_8 - w_3 w_9)} = \frac{(0.14)(0.696)}{0.0163 - 0.18096} = -0.5917648$

B*= $\frac{(\, w_8 - w_3 w_9)(\, w_4 - w_1 w_5) + \, w_6 [(\, w_8 - w_7 - w_1 w_8) + w_9 (w_2 + w_1 w_3 - w_3)]}{(\, w_5 + w_6)(\, w_8 - w_3 w_9) - w_6 w_9}$

B*= $\frac{0.696)(0.776) + \, 0.1[-1.9 + (0.163)(8.2)]}{(0.26)(0.696) - 0.163}$

B*= $\frac{0.483756}{0.16466} = 2.9379084, \quad so$

$x^* = \frac{2.9379084 \pm \sqrt{10.99836493}}{2(-0.5917648)}$

The positive root of the quadratic equation is

$x^* = 0.3197807$

Next we get y^* and z^* by substituting x^* in

y*= $\frac{(w_2 w_9 - w_4)\boldsymbol{x}{*} + w_9}{w_8 - w_3 w_9} = \frac{(-0.285)(0.3197807) + 0.163}{0.696}$

y*= 0.1032507 and

z*= $\frac{(q\boldsymbol{x}{*} - w_5)(k + w_2 \boldsymbol{kx}{*} + w_3 \boldsymbol{Mx}{*} + w_3 w_9)}{w_6 k (1 + w_1 \boldsymbol{x}{*})}$

z*=2.0854223 where q $= w_4 - w_1 w_5 = $ 0.776 , k $= w_8 - w_3 w_9 = 0.696$ and M $= w_2 w_9 - w_4 =$ -0.285

More over the kolmogorove conditions (2.6) and (2.10)

(i) $w_4 - w_5 - w_1 w_5 = 0.616 \ > 0$

$(ii) \ \ w_7 - w_{9-} \ {}_{w_2} w_9 = 0.122 \ > 0$

Since A*$<$ 0 and kolmogorove conditions satisfied the vector field $X(x, y, z)$ admits unique solution, and hence the equilibrium point $E_5 = (0.3197807{,}0.1032507{,}2.0854223)$ istheonly singularity in the interior of Ω. To get the characteristics polynomial associated with the positive equilibrium point E_5we substitute x^*, y^*, z^* in the general variational matrix of system (2.3) where

$m_{11} = \ -0.0134637 \quad m_{12} = 0.2339287 \quad m_{13} = -0.0933691$

$m_{21} = 0.0586889 \quad m_{22}$=0.0146852 $\quad m_{23} = -0.0030147$

$m_{31} = 0.1735361 \quad m_{32} = 0.4237935 \quad m_{33} = 0$

$$\begin{vmatrix} \boldsymbol{m_{11}} - \lambda & \boldsymbol{m_{12}} & \boldsymbol{m_{13}} \\ \boldsymbol{m_{21}} & \boldsymbol{m_{22}} - \lambda & \boldsymbol{m_{23}} \\ \boldsymbol{m_{31}} & \boldsymbol{m_{32}} & \boldsymbol{m_{33}} - \lambda \end{vmatrix} = \mathbf{0}$$

p(λ) =$\lambda^3 + a\,\lambda^2 + b\,\lambda + c = 0$ where

$\boldsymbol{a} = -(\boldsymbol{m_{11}} + \boldsymbol{m_{22}} + \boldsymbol{m_{33}})$= **-(0.0012215)**$<$ 0

$\boldsymbol{b} = (\boldsymbol{m_{11}m_{22}} + \boldsymbol{m_{11}m_{33}} + \boldsymbol{m_{22}m_{33}}) - (\boldsymbol{m_{32}m_{23}} + \boldsymbol{m_{12}m_{21}} + \boldsymbol{m_{13}m_{31}})$

=-0.0001977+0.0037515

=0.0035538 $>$ 0

$$\boldsymbol{c} = (\boldsymbol{m_{11}m_{32}m_{23}} + \boldsymbol{m_{22}m_{13}m_{31}} + \boldsymbol{m_{33}m_{12}m_{21}}) \\ - (\boldsymbol{m_{11}m_{22}m_{33}} + \boldsymbol{m_{12}m_{31}m_{23}} + \boldsymbol{m_{13}m_{21}m_{32}})$$

=0.002224$>$ 0

ab-c= -0.0022283 < 0. It is observed that for the Routh-Hurwitz criterion the necessary and sufficient condition for stability are not satisfied for at least a<0 thus the real part of the three roots of the polynomial are not all negative.

$$P(\lambda) = \lambda^3 - (0.0012215)\lambda^2 + (0.0035538)\lambda + 0.002224 = 0$$

The three Eigen values λ_x , λ_y , λ_z associated with singularity E_5 are given by (λ_x=0.060983+0.120909i, $\lambda_y = 0.060963 - 0.120909i$,$\lambda_z = -0.120744$). the existence of the non- trivial invariant attracting compact set in the interior Ω is now possible due to [20] this implies that the prey population X and both the predator population Y and Z co-exist in time which tends to the invariant set Λ_1 .

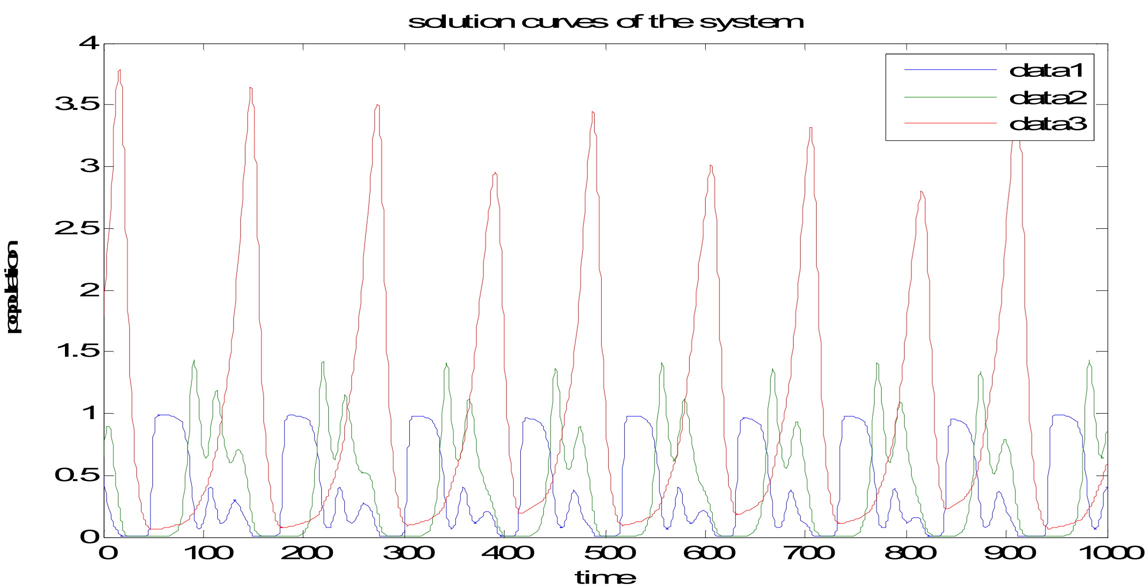

Fig-1 The solution curves of the dynamical system of the three species for initial population density (0.2 , 0.65 , 0.15) and selected parameters $W_1 = 1.4$ $w_2 = 5.0$ $w_3 = 8.0$ $w_4 = 1.0$ $w_5 = 0.16$ $w_6 = 0.1$ $w_7 = 1.1$ $w_8 = 2.0$ $w_9 = 0.163$; shows that the system is unstable .

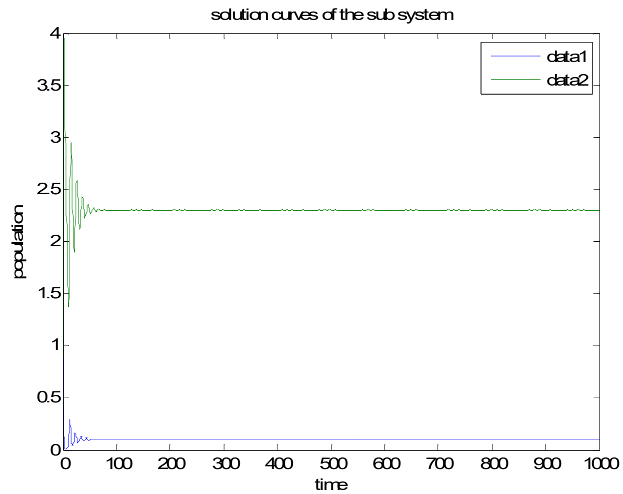

Fig-2 The solution curves of the dynamical system of the two species for initial population density (0.5, 0.4) and selected parameters $W_1 = 1.4$ $w_4 = 1.0$ $w_5 = 0.16$; shows that the system is stable.

CHAPTER FOUR

Conclusion

In this thesis a mathematical model is developed for three dimensional food web system consisting of one prey population one middle predator population feeding on the prey and one generalist population feeding up on both populations. The complex dynamic behavior of the system incorporporating prey dependent functional responses is investigated. The solution initiating the positive octant is uniformly bounded. The system is decomposed in to two Kolomogrov sub systems. Conditions for the asymptotic stability the nonzero positive equilibrium points in boundary planes of x-y and x-z has been established. The instability of the existing equilibrium point in three dimensional space shown by Routh-Hurstz criteria. Numerical simulations on fig-2 shows that the solution curves for the chosen initial populations' approaches their equilibrium values. So the equilibrium points are stable. More over simulation results on fig-1 shows that the solution curves are oscillatory for initial population (0.2, 0.65, 0.15) , so the equilibrium point of system 2.3 is unstable. The simulation also suggested that the coexistence of the three biological species in a longer time is due to the oscillation of the curves. No one of the species come to extinction. Finally it is observed that there is a change on the qualitative behavior of the system when trophic function is prey dependent in that the coexistence equilibrium point investigated in this paper is unstable. Whereas the results on the paper of Michael-Freeze in which the predators interference is ratio-dependent shows the coexistence equilibrium point is asymptotically stable for selected parameters and initial conditions.

Recommendation

It is recommended that the unstable coexistence of the three species may be stable by choosing appropriate values to the parameters, more over the stability analysis done in this thesis does not guarantee the coexistence for longer time so finding the global stability is crucial.

APPENDIX

User defined function file of the main system.

```
ydot=stud(t,y,par)

par =[1.4 5.0 8.0 1.0 0.16 0.1 1.1 2.0 0.163];

w1=par(1);

w2=par(2);

w3=par(3);

w4=par(4);

w5=par(5);

w6=par(6);

w7=par(7);

w8=par(8);

w9=par(9);


ydot=[y(1)*(1-y(1))-(y(1)*y(2))/(1+w1*y(1))-(y(1)*y(3))/(1+w2*y(1)+w3*y(2));
  (w4*y(1)*y(2))/(1+w1*y(1))-(w6*y(2)*y(3))/(1+w2*y(1)+w3*y(2))-w5*y(2);
  (w7*y(1)*y(3)+w8*y(2)*y(3))/(1+w2*y(1)+w3*y(2))-w9*y(3)];
```

Script file of the main system.

```
 t=[0 1000];

y0=[0.2;0.65;0.15];

[t,y]=ode45(@stud,t,y0);
```

```
% figure;plot(t,y(:,1));

% figure;plot(t,y(:,2));

% figure;plot(t,y(:,3));

plot(t,y(:,1),t,y(:,2),t,y(:,3));
```

User defined function file of the sub system.

```
function [ ydot ] = tem( t,y,par )

par=[1.4 1.0 0.16];

w1=par(1);

w4=par(2);

w5=par(3)

ydot=[y(1)*(1-y(1))-y(1)*y(2)/1+w1*y(1);

  -w5*y(2)+ w4*y(1)*y(2)/1+w1*y(1)];
```

Script file of the sub system

```
t=[0 1000];

y0=[0.5;0.4];

[t,y]=ode45(@tem,t,y0);

%figer;plot(t,y(:,1));

%figer;plot(t,y(:,2));

plot(t,y(:,1),t,y(:,2));
```

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