BIFURCATION ANALYSIS OF THE MODIFIED LOTKA-VOLTERRA PREY-PREDATOR MODEL

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
SCHOOL OF APPLIED NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

By

Fatuma Zeyinu

Adama,Ethiopia

June,2017

BIFURCATION ANALYSIS
OF
THE MODIFIED LOTKA-VOLTERRA PREY-PREDATOR MODEL

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
SCHOOL OF APPLIED NATURAL SCIENCE

DEPARTMENT OF MATHEMATICS

By

Fatuma Zeyinu

Advisor: Shiferaw Feyissa(Ph.D)

Adama,Ethiopia

June,2017

Abstract

In this study we presented a mathematical model which describes the interaction between prey and predator populations. We considered a modified Lotka-Volterra prey-predator model. The existence and stability of the equilibrium points were studied both analytically and numerically. The existence and positive equilibrium of the system and stability of the equilibrium of the system studied under various conditions. The phase portraits are obtained for different sets of parameter values and initial conditions. Also bifurcation diagram are presented for selected range of growth parameter. Finally, numerical simulation has been used to specify the control parameters and confirm the obtained results.

Acknowledgements

Best praise to Allah, most merciful for helping me and giving the patience along the life. first I must graciously thanks to my advisor Dr.Shiferaw Feyissa for his willingness to see and comments all my works, and for his advice, guidance, precious time and invaluable suggestions throughout the entire period of my work. He has been a constant source of inspiration and encouragement.

Next, I would also extend my thanks to Minister of Education for giving me the opportunity to pursuer my graduate work, and providing me with financial assistance for the completion of my thesis work.

Finally my deepest thanks go to my family for everything they have done to enable me to be as ambitious as I wanted. Thanks for being proud of me even without knowing very well about what I have been working on during these years. At the same time, I have to thank all my friends of the department of mathematics for their support. Especially, I want to thank those friends of mine I used to discuss to mathematical concepts and ideas, Mr.Adem Aman and Mr.Miliyon Tilahun for their unconditional supports with enjoyable moments.

Contents

1 Introduction **1**
- 1.1 **Background of the study** 1
- 1.2 **Bifurcation:Analyzing changes in a parameter value** 3
- 1.3 **Statement of the problem** 5
- 1.4 **Objective of the study** 5
  - 1.4.1 **General objective of the study** 5
  - 1.4.2 **Specific objectives of the study** 5
- 1.5 **Significance of the study** 6
- 1.6 **Research methodology** 6

2 Review of Literature **7**

3 Model analysis **10**
- 3.1 **The dimensionless form of the model** 12
- 3.2 **positivity and boundedness of solution** 13
  - 3.2.1 **Positivity of solution** 13
  - 3.2.2 **Boundedness of solution** 14
- 3.3 **Stability analysis** 14
  - 3.3.1 **Equilibrium points** 14
  - 3.3.2 **Local stability analysis of equilibria of the system** 15
  - 3.3.3 **Global stability analysis of equilibria** 18
- 3.4 **Bifurcation analysis** 19

4 Numerical simulation and Results **22**
- 4.1 **Results** 26
- 4.2 **Discussion** 27

5 Conclusion **29**

Reference **30**

Appendix **32**

List of Tables

3.1 Parameters of the Model 11
3.2 Conditions on parameters for existence of equilibrium points of the system (3.1.9). 15

List of Figures

4.1 For this graph we use the parameter $\alpha = 0.001$, $\beta = 0.15$, $\delta = 1$ and the initial conditions $u_0 = 0.2$ and $v_0 = 0.75$ 23
4.2 Local stability of E_1 23
4.3 Local stability of E_2 24
4.4 Global stability of E_2 24
4.5 Bifurcation diagram for u(t) with α as the bifurcation parameter for u(0)=0.3, v(0)=0.4, when β=0.85, δ=0.5. 25

Chapter 1

Introduction

1.1 Background of the study

Mathematical population models have been used to study the dynamics of prey predator systems since Lotka (1925) and Volterra (1927) proposed the simple model of prey-predator interactions now called the Lotka-Volterra model. Since then, many mathematical models, some reviewed in this study, have been constructed based on more realistic explicit and implicit biological assumptions.
In the 1920s, the Italian mathematician Vito Volterra (1926) proposed a differential equation model to describe the population dynamics of two interacting species, a predator and its prey. He hoped to explain the observed increase in predator fish and decrease in prey fish in the Adriatic Sea during World War I. Such mathematical models have long proven useful in describing how populations vary over time. Data about the various rates of growth, death, and interaction of species naturally lead to models involving differential equations.

Independently, in the United States, the very equations studied by Volterra were derived by Alfred Lotka [1925] to describe a hypothetical chemical reaction in which the chemical concentrations oscillate. The Lotka-Volterra model consists of the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = rx - axy \\ \frac{dy}{dt} = -by + cxy \end{cases} \tag{1.1.1}$$

where y(t) and x(t) represent, respectively, the predator population and the prey population as functions of time. The parameters r,a,b,c are positive constants.
The idea of the first equation is that in the absence of predators, the prey would grow at a constant rate r, but decreases linearly as a function of the density y of the predators. Similarly, in the absence of prey, the density of predators would decrease but the rate increases proportional to the density of the prey. As the number of prey increases, the rate of the predator grows increases.
In such system there are always fluctuation in the number of predators and prey and around the value $(x_0 = \frac{b}{c}, y_0 = \frac{r}{a})$ with the amplitudes and period determined by the initial conditions.

This model has two major disadvantages, one of which is a biological character and the other is a mathematical one. From biological view point a linear dependence of the rate of predation and predator reproduction on the number of prey is considered by many authors to be a very crude approximation of reality which is true only with in some very narrow limits .

From the mathematical view point, the main drawback of the Lotka-Volterra system is the fact that this system is not "robust." that is the account of factors or effects not take in to consideration when building the model, results in qualitative modification of the systems behavior.

The Lotka-Volterra model (1.1.1), from many points of view is unstable and unrealistic. Firstly, in the absence of predators, the population of prey would grow exponentially towards infinity. This feature is easily corrected, one way is to introduce a competition rate within the prey species.

And also in the absence of predators the growth of prey population follows the logistic growth model.The growth rate of prey population is determined by the equation;

$$\left\{ \frac{dx}{dt} = rx(1 - \frac{x}{k}) \right. \tag{1.1.2}$$

where r and k are positive constants. The constant r is the internisic growth rate of the prey while the constant k is the carrying capacity of the environment that limits the number of prey population that can be considered as the competition rate as well. This logistic model was first proposed by Verhulst in 1838 to adjust the exponential growth of the population model at that time.

We then can add another term into the model (1.1.1) as an intraspecific competition and if we wish, we may also allow an intraspecific competition with in the predators. The latter is less crucial, as their population does not explode anyway. This leads to a more general Lotka-Volterra model as follows,

$$\begin{cases} \frac{dx}{dt} = rx - kx^2 - axy \\ \frac{dy}{dt} = -by + cxy - dy^2 \end{cases} \tag{1.1.3}$$

Thus the classical model (1.1.1) is one special case of the above model.

Another reason why the classical model is unrealistic is the fact that the solution oscillates periodically in the same periodic solution all the time. If a prey population increases, it encourages growth of its predator. More predators however consume more prey, the population of which starts to decline. With less food around, the predator population declines and when it is low enough, this allows prey population to increase and the whole cycle starts over again. Depending on the detailed system, such oscillations can grow or decay or go into a stable limit cycle oscillation, which does not occur in either (1.1.1) or (1.1.3).

In this research, we are also study several types of bifurcations, saddle node, transcritical, pitchfork and Hopf bifurcation. The first three types of bifurcation occur in scalar and in systems of differential equations. The fourth type called Hopf bifurcation does not occur in scalar differential equations because this type of bifurcation involves a change to a periodic solution. Scalar autonomous differential equations can not have periodic solutions. Hopf bifurcation occurs in systems of differential equations consisting of two or more equations. This type is also referred to as a Poincare-Andronov-Hopf bifurcation.

1.2 Bifurcation:Analyzing changes in a parameter value

The word bifurcation was introduced by Poincar, a French mathematician, in the field of non-linear dynamics. He used this to indicate the qualitative changes in the behavior of systems where one or more system parameters values varied.
In particular equilibrium points may created or destroyed or their stability may change, and in any system, as a parameter value is varied, there is some change in the dynamical behavior. Most of the time these changes are only quantitative in nature. But there may also be situations where one or more parameter values change may result in a qualitative change in steady state behavior of a dynamical system. These qualitative changes in the dynamical system is called bifurcation. The parameter value at which these changes occur are called bifurcation points.
If one of the parameters values in the model changes, the value of the steady state(s), the corresponding eigenvalues and hence the stability characteristics may change as well. However, these changes are in general continuous, which means that plotted as a function of the parameter value the value of the steady state(s) form a smooth and continuous curve. The same holds for the value of the corresponding eigenvalues (both the imaginary and real parts!).

Mathematically, this means that when the parameter λ crosses some point $\lambda = \lambda^*$, the phase portrait of the system for $\lambda < \lambda^*$ is topologically different from the phase portrait of the system for $\lambda > \lambda^*$. The point $\lambda = \lambda^*$ is called a bifurcation point at which the system undergoes a bifurcation. If the qualitative change occurs in a neighborhood of a fixed point or periodic solution, then it is called a local bifurcation. Any other qualitative change that occurs is considered as a global bifurcation. The local bifurcations of fixed points (equilibrium points) are classified into static bifurcations or dynamic bifurcations.

Saddle-node bifurcation, pitchfork bifurcation and transcritical bifurcation are examples of static bifurcations as only branches of fixed points or static solutions meet. Hopf bifurcation is an example of dynamic bifurcation. Saddle-node bifurcation is a collision and disappearance of two equilibria in dynamical systems.
In autonomous systems, this occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation.
A saddle node bifurcation of a fixed point of the system $x^{'} = f(x, \lambda)$ where $x \in R^n$ and $\lambda \in R$ occurs at (x_0, λ_0) if,
(i) There is an equilibrium point at $x = x_0$ for $\lambda = \lambda_0$ i.e f(x_0, λ_0)=0 and
(ii) The Jacobian matrix $D_x f(x_0, \lambda_0)$ has zero eigenvalue.

The other types of bifurcation is transcritical bifurcation which occurs when the equilibrium points change their stability as the bifurcation parameter value is varied. The fixed points of the system exist for all parameter values and can never be destroyed. In a transcritical bifurcation, two families of fixed points collide and exchange their stability properties. The family that was stable before the bifurcation is unstable after it. The other fixed point goes from being unstable to being stable.
A pitchfork bifurcation is a particular type of local bifurcation (possible in dynamical), that is one family of fixed point transfers its stability properties to two families after or before the bifurcation point. If this occurs after the bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial fixed points occur for values of the parameter lower than the bifurcation value. Now that we have introduced bifurcations in one dimension, it is instructive to discuss some key bifurcations in two dimensions. The essential idea is the same; to find parameter values that lead to changes in the system's behavior.
A Hopf or Poincare-Andronov-Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses their stability as a pair of complex conjugate eigenvalues of linearization around the fixed point cross the imaginary axis of the complex plane. Hopf bifurcation is the appearance or disappearance of aperiodic orbit (solution) through a local change in the stability properties of the equilibrium point. It has two parts; supercritical Hopf bifurcation(results in a stable limit cycle) and a sub critical Hopf bifurcation (results in an unstable limit cycle.) A Hopf bifurcation at which the positive equilibrium becomes unstable and the system starts to oscillate around that equilibrium when a parameter is varied. The stable limit cycle grows when a parameter value is varied further. Then at some point the periodic orbit touches the saddle boundary equilibria, collides with these and disappears abruptly.

A bifurcation parameter value is sensitive to very small changes so that even a slight change can lead to differences in the system's behavior. In addition to changing the number of fixed points, bifurcations can also change some characteristics of the fixed point as well, while not necessarily destroying them. For instance, for certain parameter values, the system may have two fixed points, but the stability of each may vary depending on the parameter value. Bifurcation plots describe which parameter values yield oscillatory behavior and allows for a global picture of dynamics.

The main purpose of this research work is to study the qualitative and quantitative changes in a mathematical model which describes the population dynamics of prey and predator interaction. In this study we will discuss two-dimensional Lotka-Volterra systems. The discussion involves the theory of bifurcation analysis. Although some aspect of the two-dimensional Lotka-Volterra system have been discussed, such as the stability of the positive equilibrium, the stability of the origin. In bifurcation theory, the presence of special structure complicates the analysis, since it usually leads to certain degeneracies that one does not normally have. This leads us to then analyze a general dynamical system with the same special structure as the Lotka-Volterra system.

1.3 Statement of the problem

Predator-prey models were used by scientists to predict or explain trends in population dynamics. The goal of this research work will be to explore the behavior of a simple model and to understand the basic concept of prey-predator dynamics using the qualitative and quantitative behavior of the model.
In this research work we will focuses on behavior of the solution of the system,

$$\begin{cases} \frac{dx}{dt} = rx - kx^2 - axy \\ \frac{dy}{dt} = -by + cxy - dy^2 \end{cases} \tag{1.3.4}$$

in order to answer the biological questions;

★ How can the model properties change when the parameter value in the model changed.

★ How the dynamics of prey-predator interaction changes as the parameter value changes.

★ How to predict qualitative and quantitative changes in system's behavior(bifurcation) occurring at the equilibrium points.

1.4 Objective of the study

1.4.1 General objective of the study

The main objective of this study is to understand the dynamics of prey-predator interactions using mathematical model.

1.4.2 Specific objectives of the study

The specific objectives of this study are to:

► explain the qualitative behavior of the mathematical model for prey-predator interaction.

► investigate the effect of change in values of parameter on the dynamics of the system.

► investigate the biological implications of the qualitative and quantitative changes of the system.

► identify the parameter which plays greater role in the qualitative behavior of the system.

1.5 Significance of the study

Predators and prey can influence one another's evolution. Traits that enhance a predator's ability to find and capture prey will be selected for the predator, while traits that enhance the prey's ability to avoid being eaten will be selected for the prey. The "goals" of these traits are not compatible, and it is the interaction of these selective pressures that influences the dynamics of the predator and prey populations. Predicting the outcome of species interactions is also of interest to biologists trying to understand how the ecosystem are structured and sustained.
This studies is expected to have the following significance to

- It provides reliable information about the quantitative and qualitative changes in the long-term dynamics of the interaction between prey and predator.
- To maintain the ecosystem.
- The result of this research work will benefit a scientific community by understanding the bifurcation analysis of a predator-prey model. Moreover the result obtained from this research work will be used as an input for other researchers.

1.6 Research methodology

In this research work we are going to study the qualitative and quantitative changes in a mathematical model which describes the a population dynamics of prey and predator interaction using a system of non-linear ordinary differential equations. Usually, non-linear differential equation are not solvable analytically. We solve the model equation numerically. We will try to determine the local stability of the system using Jacobian matrix. The bifurcation analysis will be done, which helps as to understand the quantitative and qualitative changes in the system. The analytically solutions to the mathematical problems are supplemented by numerical simulations using biologically meaningful system of parameter values. Numerical simulations was performed using the software a graphical MATLAB package for the analysis of dynamical system.

Chapter 2

Review of Literature

In 1925 Alfred Lotka explained deeply the purpose of the new area called physical biology, which consists of the application of physical principles in the study of life-bearing systems as a whole in contrast with biophysics that treats the physics of individual life processes. Among some examples employed in this area he proposed a model to describe chemical reactions in which the concentrations oscillates. One year hence, using the same equations, different studies were carried out by Vito Volterra. The main purpose of his work was to describe the observed variations in some species of fish in the Upper Adriatic sea. The system of equations proposed by Lotka and Volterra describes the change of prey or host density with time, and assumed, for this purpose, that the number attacked per predator was directly proportional to prey density. This model became known as Lotka-volterra predator-prey equation.

Volterra's study showed that the steady state for the co-existence of the prey fish and predatory fish was periodic and that a pause of fishery would indeed lead to an increase of the predators and a decrease in the prey. Since then, many ecological models have been formulated and analyzed to study various phenomena. Volterra (1926) closed his short paper with the modest statement: Seeing that a great number of biological phenomena are characteristic of associations of species, it is to be hoped that this theory may receive further verification and may be of some use to biologists.
Despite the brevity of the paper and the modest closing sentence, Volterra established with his study a corner-stone for the theory about predator-prey interactions. The Lotka-Volterra predator-prey model, as it has been referred to since Volterra's contribution, forms a basis on which most if not all models of such interactions have been founded. In addition, it has become clear that predator-prey interactions are one of the most important causes of oscillations in species abundance. Hence, not only the model that Volterra (1926) studied, but also the fluctuations he reported are to the present day important focal points for studying the dynamics of interacting species.

Freedman (1980) came up with a generalised prey-predator model represented by the system of differential equations given by :

$$x^{'} = xg(x) - yf(x)$$

$$y^{'} = y(-e + p(x))$$

where x and y are the densities of the prey and predator respectively, g(x) is the growth rate of the prey in absence of the predators, f(x) is the functional response of the predators with respect to prey x, and p(x) is the numerical response of the predator. In most cases, p(x) is a product of a constant and f(x).
Another improvement of predator and prey population models was done by Kolmogorov. After noticing population models in Volterras work, Kolmogorov in 1936 studied predator-prey models of the general form:

$$x^{'} = xS(x, y) \tag{2.0.1}$$
$$y^{'} = yW(x, y) \tag{2.0.2}$$

Conditions must be put on S and W to make x and y a prey and a predator respectively. The first group of conditions requires that, if the number of predators increases, then the rates of increase of the two populations decrease;

$$\frac{\partial S}{\partial y} < 0 \text{ and}$$
$$\frac{\partial W}{\partial y} < 0$$

In addition, the rate of increase of the predator population increases with respect to the increase of population of prey, while the rate of the prey population decreases,

$$\frac{\partial S}{\partial x} < 0 \text{ and}$$
$$\frac{\partial W}{\partial x} > 0$$

Of course, there are more conditions we need in order to approximate the model closely to the reality.
The Lotka-Volternra predator-prey model and the other population models are special cases of models in population dynamics. This is because in ecology, interactions between species can be very complex, even when only two species are considered.
As in the predator-prey models, the simplest model that can represent all the above two species interactions is of the Lotka-Volterra type. Thus, using a modern notation we have the general two-dimensional Lotka-Volterra model that can represent all interactions between two species:

$$x^{'} = x(b_1 + a_{11}x + a_{12}y)$$
$$y^{'} = y(b_2 + a_{21}x + a_{22}y)$$

The signs of b_i and $a_{i,j}$ (i,j = 1,2) determine what population model we have. For example, we can construct a competition model here by letting b_1, b_2 be positive and a_{11}, a_{12}, a_{21} and a_{22} be negative. Among all two-species interactions, predator-prey and competition relationships are the most studied models in population dynamics. Besides the Lotka-Volterra model, the Kolmogorov model can also be used to represent not only predator-prey interaction but also competition and cooperative relationships using the freedom in the

conditions of functions S and W. In the Gause model, one has to generalise some assumptions for this model to be able to model not only predator-prey relationship but all two-species interactions as well.
Turchin et al. (2000) have argued convincingly that to the cycles in abundance in voles are predator-prey cycles, in which the vole are the prey of their specialized predator, the weasel. However, data on fluctuations in abundance of the weasel are generally not available, hence it is not possible to compare the cycle characteristics with those of the limit cycles predicted by the Rosenzweig-MacArthur model.
Kesh et al. (2000) proposed and analyzed a mathematical model of two competing prey and one predator species where the prey species follow Lotka-volterra dynamics and predator uptake functions are ratio dependent. They derived conditions for the existence of different boundary equilibria and discussed their global stability. They also obtain sufficient conditions for the permanence of the system.
Hsu et al. (2001) studied the qualitative properties of a ratio dependent predator-prey model. They showed that the dynamical outcome of interactions depend upon parameter values and initial data.
Chaudhuri and Kar (2004) proposed and analyzed a fishery model of a two prey-one predator system in which the prey were being harvested and the feeding rate of the predator increases linearly with prey density. They derived conditions for global stability of the system using a Lyapunov function.
Vlastmil and Eisner (2006) studied a one consumer-two resource population dynamics system in which the resource was spatially distributed between two patches. The study showed that when resources grow exponentially, handling times are zero and apparent competition always leads to extinction of the weaker resource. However, with logistic growth and Holling Type II functional response included in the model, species permanence was guaranteed.
Braza (2008) considered a two predator; one prey model in which one predator interferes significantly with the other predator is analyzed. The analysis centers on bifurcation diagrams for various levels of interference in which the harvesting is the primary bifurcation parameter.
Lotka-Volterra developed the prey-predator model, but after developed the model does not studies bifurcation analysis of prey-predator model. And also no one of the above authors studies bifurcation analysis of the lotka-volterra prey-predator model. In this research work the bifurcation analysis of the system will be done, and discuss the types of changes in dynamics that can occur in continuous-time models of biological populations when a single model parameter value is varied. Several types of such bifurcations can occur and these types posses particular characteristics that hold independent of the model they occur in. This research work will essentially not introduce any new techniques or methods, as the discussion below depends entirely on the computation of steady states, their eigenvalues and stability properties. A new perspective will be added, because we will focus on the question how these model properties change when a parameter value in the model is changed.

Chapter 3

Model analysis

In this section, we analysis a mathematical model which describes the dynamics of interaction between prey and predator. The model equation we consider a modified Lotka-Volterra prey-predator model. Reduce the model to non-dimensional form and determine the existence of solution, positivity of solution and boundedness of the solution will be caried out.
The main feature of the model is that the bifurcation analysis incorporated in the model to represent the properties in the way the predators feeds on each of the prey species. Terms representing logistic growth of the prey species in the absence of the predator and the predator species with it self included in the model. The model has two non-linear autonomous ordinary differential equations describing how the population densities of the two species would vary with time t.
The analysis includes the local and global stability analysis and also determining all the bifurcation type, i.e saddle,pitchfork,transcritical and Hopf bifurcation.
Lastly numerical simulation is followed in order to better understand the dynamics of modified lotka-volterra model.
The model consists of two species, prey and predators and the lotka-volterra assumes logistic growth for the population of each species.The mathematical model for two interacting species are the non linear differential equation which represent the modified lotka-volterra system is as follows;

$$\begin{cases} \frac{dx}{dt} = rx - axy - kx^2 \\ \frac{dy}{dt} = -by + cxy - dy^2 \end{cases} \tag{3.0.1}$$

where x and y are respectively, the size of prey and predator population at a time t, and the interaction coefficients,k,d>0, measures the carrying capacity of the environment that limits the number of prey and predator population,respectively that can be considered as the interaction rate as well.There are six parameters on this model,we non-dimensionalize to reduce this number to three.

Assumption of the model

■ The prey population in the absence of predator to be growing logistically.

■ The effect of the predation is to reduce the preys per capita growth rate of prey by a term proportional to the prey and predator populations.

■ In the absence of prey population the predators death rate results in logistically decay.

■ The preys contribution to the predators growth rate is cxy ; that is it is proportional to the available prey as well as to the size of the predator population.

■ Death rate due to interaction between prey population within itself computing for the same resources,the kx^2 term.

■ Death rate due to interaction between predation population within itself computing for the same resources, the $\mathrm{d}y^2$ term.

Parameters of the model

The parameters of model are presented in Table 3.1. The parameter r represents the natural growth rate of prey in the absence of predators. The parameter a represents the effect of predation on the prey population because of the predation of the predator. k represents the implicitly related to the carrying capacities of prey population or the interaction between prey population with in itself. In the second equation parameter b represents the natural death rate of the predators in the absence of prey. The predator population benefits from the prey and that result in an increase in the predator population represents by parameter c. Parameter d represents the benefit of predator due to interaction between predator with it self.

Parameter	Interpretation
r	Natural growth rate of prey in the absence of predator
a	The effect of predation due to prey
k	Interaction between prey population with itself
b	The natural death rate of the predator in the absence of prey.
c	The efficiency and propagation rate of the predator in the presence of prey.
d	Interaction between predator population with itself

Table 3.1: Parameters of the Model

3.1 The dimensionless form of the model

A model can be transformed in to a dimensionless form,i.e rewriting the system interms of dimensionless quantities. One of the advantages of the system in dimensionless form is that the number of parameter is reduced to a minimum, and that makes also the analysis easier. Further more parameters can be better compared with each other, in terms of small and large and thus one gets more insight in to the system. It is also possible to make a comparison between different systems. The dynamics of the transformed model will be the same as in the original system,

$$\begin{cases} \frac{dx}{dt} = rx - axy - kx^2 \\ \frac{dy}{dt} = -by + cxy - dy^2 \end{cases} \tag{3.1.2}$$

system(3.1.2) becomes

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{a}{r}y - \frac{k}{r}x) \\ \frac{dx}{dt} = by(-1 + \frac{c}{b}x - \frac{d}{b}y) \end{cases} \tag{3.1.3}$$

Let

$$\begin{cases} \tau = rt \Rightarrow d\tau = rdt \Rightarrow dt = \frac{d\tau}{r} \end{cases} \tag{3.1.4}$$

$$\begin{cases} v = \frac{a}{r}y \Rightarrow y = \dfrac{r}{a}v \Rightarrow \frac{dy}{dt} = \frac{r}{a}\frac{dv}{dt} \end{cases} \tag{3.1.5}$$

$$\begin{cases} u = \frac{c}{b}x \Rightarrow x = \dfrac{b}{c}u \Rightarrow \frac{dx}{dt} = \frac{b}{c}\frac{du}{dt} \end{cases} \tag{3.1.6}$$

Now substituting (3.1.4), (3.1.5) and(3.1.6) in to first and in the second equation of(3.1.3), we have

$$\begin{cases} \frac{du}{d\tau} = u(1 - v - \frac{kb}{rc}u) \\ \frac{dv}{d\tau} = \frac{b}{r}v(-1 + u - \frac{rd}{ba}v) \end{cases} \tag{3.1.7}$$

Thus equation(3.1.3) becomes

$$\begin{cases} \frac{du}{d\tau} = u(1 - v - \alpha u) \\ \frac{dv}{d\tau} = \delta v(-1 + u - \beta v) \end{cases} \tag{3.1.8}$$

with $\alpha = \frac{kb}{rc}, \beta = \frac{rd}{ab}$ and $\delta = \frac{b}{r}$ Therefore the dimensionless form of the model(3.1.2) is

$$\begin{cases} \frac{du}{d\tau} = u(1 - v - \alpha u) \\ \frac{dv}{d\tau} = \delta v(-1 + u - \beta v) \end{cases} \tag{3.1.9}$$

3.2 positivity and boundedness of solution

3.2.1 Positivity of solution

Theorem 3.2.6.1 All solution of system(3.1.9) that starts in R_+^2 with positive initial conditions stay positive for all time, i.e.if u(0),v(0)> 0, then u(τ),v(τ)> 0 for all $\tau > 0$.
proof; Let u(τ) and v(τ) be the solution of system(3.1.9).

$$\begin{cases} \frac{du}{u} = (1 - v - \alpha u)d\tau \\ \frac{dv}{v} = \delta(-1 + u - \beta v)d\tau \end{cases} \tag{3.2.10}$$

Integrating system(3.2.10) with initial conditions(u_0, v_0), we have,

$$\int_{u_0}^{u(\tau)} \frac{du}{u} = \int_0^{\tau} (1 - v - \alpha u)d\tau$$

$$\int_{v_0}^{v(\tau)} \frac{dv}{v} = \int_0^{\tau} \delta(-1 + u - \beta v)d\tau$$

$$\Rightarrow u(\tau) = u_0 exp \int_0^{\tau} (1 - v - \alpha u)d\tau$$

$$\Rightarrow v(\tau) = v_0 exp \int_0^{\tau} \delta(-1 + u - \beta v)d\tau$$

showing that $u(\tau) \geq 0$ and $v(\tau) \geq 0$ whenever $u(0) > 0$ and $v(0) > 0$. Hence all solutions remain within the first quadrant of the uv-plane starting from an interior point of it. Further we can easily establish that solution trajectories starting from $(u_0, 0)$ with $u_0 > 0$, remain within the positive u-axis at all future time and similar result holds for trajectories starting from a point on the positive v-axis. Hence, $R_+^2 = \{(u, v) : u, v \geq 0\}$ is an invariant set.

3.2.2 Boundedness of solution

Theorem: If u(τ),v(τ) are the solutions of the system (3.1.9) then u(τ),v(τ) will be bounded for any initial condition u(0)$\geq$0,v(0)$\geq$0.

proof;consider u(τ),v(τ) satisfying the following initial value problem. From the first equation of system(3.1.9), we have

$$\frac{du}{d\tau} \leq u(1-v)$$

then we can write this as,

$$\frac{du}{u} \leq (1-v)d\tau$$

Integrating both side we have,

$$\ln u + \ln u_0 \leq (1-v)\tau$$

$$\Rightarrow uu_0 \leq e^{(1-v)\tau}$$

$$\Rightarrow u(\tau) \leq \frac{1}{u_0 e^{(v-1)\tau}}$$

$$\Rightarrow u(\tau) \leq 1$$

Thus we conclude that $u(\tau)$ will be bounded for any non negative initial condition.
To the same manner,

$$\frac{dv}{d\tau} \leq \delta v(1-\beta v)$$

$$\Rightarrow \frac{dv}{v(1-\beta v)} = \delta d\tau$$

using integration by partial fraction,

$$\Rightarrow \ln v - \ln(1-\beta v) = \delta\tau$$

$$\Rightarrow \ln \frac{v}{(1-\beta v)} = \delta\tau$$

$$\Rightarrow v = e^{\delta\tau(1-\beta v)}$$

$$\Rightarrow v(\tau) \leq \frac{e^{\delta\tau}}{1+\beta e^{\delta\tau}}$$

$v(\tau)$ bounded. Therefore u and v are bounded.

3.3 Stability analysis

3.3.1 Equilibrium points

The dynamical system(3.1.9) has the following three fixed or equilibrium points. The origin(E_0), a boundary fixed point (E_1) and a fixed point for with both populations survive(E_2):

$$E_0 = (0,0)$$
$$\mathrm{E}_1 = (u^*, 0) = (\tfrac{1}{\alpha}, 0)$$
$$\mathrm{E}_2 = (u^*, v^*) = (\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta}), \text{ if } 1-\alpha > 0$$

Biologically the first equilibrium points corresponds to extinction of both species, the second equilibrium points corresponds to extinction of one species,and the last equilibrium points corresponds to steady state co existence of both species.

Equilibrium point	Existential conditions
$E_0(u^*, v^*)$=(0,0)	-
$E_1(u^*, v^*) = (\frac{1}{\alpha}, 0)$	-
$E_2(u^*, v^*) = (\frac{1+\beta}{1+\alpha\beta} \frac{1-\alpha}{1+\alpha\beta})$	$1-\alpha > 0$

Table 3.2: Conditions on parameters for existence of equilibrium points of the system (3.1.9).

3.3.2 Local stability analysis of equilibria of the system

The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues of the jacobian matrix evaluated equilibrium points.

If all the real parts of our eigenvalues are negative,then the equilibrium point is asymptotically stable.
If at least one eigenvalue of the Jacobian matrix is positive, then the equilibrium point is unstable.

From systems (3.1.9), the Jacobian matrix of the system is given by:

$$E_i = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{bmatrix}$$

which gives $J(E_i) = \begin{bmatrix} 1 - v - 2\alpha u & -u \\ \delta v & -\delta + \delta u - 2\delta\beta v \end{bmatrix}$

The local asymptotically stability for each equilibrium point is analyzed as below;

A.Local stability analysis of the point $E_0 = (0,0)$

The jacobian matrix of system(3.1.9) evaluated at E_0 gives the matrix,

$$J(E_0) = \begin{bmatrix} 1 & 0 \\ 0 & -\delta \end{bmatrix}$$

It follows that the origin(0,0) is unstbale, since the eigenvalue of the jacobian matrix evaluated at the origin is negative($\lambda_1 = -\delta$) and the other eigenvalue is positive $\lambda_2 = 1$

B.Local stability analysis of the point $E_1 = (u^*, 0) = (\frac{1}{\alpha}, 0)$
The jacobian matrix of system(3.1.9) evaluated at E_1 gives the matrix

$$J(E_1) = \begin{bmatrix} -1 & \frac{-1}{\alpha} \\ 0 & -\delta + \frac{\delta}{\alpha} \end{bmatrix} = \begin{bmatrix} -1 & \frac{-1}{\alpha} \\ 0 & \delta(-1 + \frac{1}{\alpha}) \end{bmatrix}$$

The eigenvalues of the matrix J(E_1) are $\lambda_1 = -1$ and $\lambda_2 = \delta(-1 + \frac{1}{\alpha}) \Rightarrow \frac{\delta}{\alpha}(-\alpha + 1) = \frac{-\delta}{\alpha}(1 - \alpha)$. The eigenvalues of the

above are negative for $\frac{-\delta}{\alpha}(1 - \alpha) < 0 \Rightarrow 1 - \alpha > 0 = \alpha < 1$

Hence the equilibrium point $E_1(\frac{1}{\alpha}, 0)$ is locally asymptotically stable, if the condition

$$\alpha < 1 \tag{3.3.11}$$

hold.
But if the condition $1 - \alpha > 0 \Rightarrow \alpha > 1$ then one of the eigenvalue become positive, therefore the equilibrium point is unstable. This equilibrium point represents the limiting population of the prey as $\frac{1}{\alpha}$ in the absence of the predator.
C.Local stability analysis of the point $E_2 = (u^*, v^*) = (\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$.

The jacobian matrix of system(3.1.9) evaluated at E_2 gives the matrix,

$$J(E_2) = \begin{bmatrix} \frac{-\alpha\beta-\alpha}{1+\alpha\beta} & \frac{-1-\beta}{1+\alpha\beta} \\ \frac{\delta-\alpha\delta}{1+\alpha\beta} & \frac{\alpha\beta\delta-\beta\delta}{1+\alpha\beta} \end{bmatrix} = \begin{bmatrix} \frac{-\alpha(\beta+1)}{1+\alpha\beta} & \frac{-1-\beta}{1+\alpha\beta} \\ \frac{\delta(1-\alpha)}{1+\alpha\beta} & \frac{\delta\beta(\alpha-1)}{1+\alpha\beta} \end{bmatrix}$$

The eigenvalues of J(E_2) are obtained by solving;

$$\det \begin{bmatrix} \frac{-\alpha(\beta+1)}{1+\alpha\beta} - \lambda & \frac{-1-\beta}{1+\alpha\beta} \\ \frac{\delta(1-\alpha)}{1+\alpha\beta} & \frac{\delta\beta(\alpha-1)}{1+\alpha\beta}) - \lambda \end{bmatrix} = 0$$

This gives

$$(\frac{-\alpha(\beta+1)}{1+\alpha\beta} - \lambda)(\frac{\delta\beta(\alpha-1)}{1+\alpha\beta}) - [(\frac{-1-\beta}{1+\alpha\beta})(\frac{\delta(-\alpha)}{1+\alpha\beta})] = 0$$

which simplifies to a characteristic equation,

$$= \lambda^2 + (\frac{-\alpha(\beta+1)}{1+\alpha\beta})(\frac{\delta\beta(\alpha-1)}{1+\alpha\beta}) - \frac{\delta\beta(\alpha-1)}{1+\alpha\beta}\lambda + \frac{\alpha(\beta+1)}{1+\alpha\beta}\lambda + (\frac{1+\beta}{1+\alpha\beta})(\frac{\delta(1-\alpha)}{1+\alpha\beta})$$

$$\lambda^2 - \frac{(\delta\beta(\alpha-1))\lambda}{1+\alpha\beta} + \frac{\alpha(\beta+1)\lambda}{1+\alpha\beta} - \frac{\alpha(\beta+1)}{1+\alpha\beta}(\frac{\delta\beta(\alpha-1)}{1+\alpha\beta}) + [\frac{(1+\beta)}{1+\alpha\beta}\frac{\delta(1-\alpha)}{1+\alpha\beta}]$$

$$= \lambda^2 - \tfrac{1}{1+\alpha\beta}(\delta\beta(\alpha-1) - \alpha(\beta+1))\lambda - \tfrac{\delta(\beta+1)}{(1+\alpha\beta)^2}(\alpha\beta(\alpha-1) - (1-\alpha))$$

$$=\lambda^2 - \tfrac{1}{1+\alpha\beta}(\delta\beta(\alpha-1) - \alpha(\beta+1))\lambda - \tfrac{\delta(\beta+1)}{(1+\alpha\beta)^2}(\alpha\beta(\alpha-1) + (\alpha-1))$$

Therefore, the characteristic equations becomes,

$$\lambda^2 - \tfrac{1}{1+\alpha\beta}(\delta\beta(\alpha-1) - \alpha(\beta+1))\lambda - \tfrac{\delta(\beta+1)}{(1+\alpha\beta)^2}(\alpha\beta+1)(\alpha-1))=0$$

Hence, the eigenvalues of the jacobian matrix evaluated at E_2 are given by,

$$\lambda_{1,2} = \frac{\frac{1}{1+\alpha\beta}(\delta\beta(\alpha-1)-\alpha(\beta+1))\pm\sqrt{(\frac{\delta\beta\alpha-\delta\beta-\alpha\beta-\alpha}{1+\alpha\beta})^2-4(\frac{-(\beta+1)(\alpha\beta+1)(\alpha-1)}{(1+\alpha\beta)^2})}}{2}$$

$$= \tfrac{1}{2}[\tfrac{1}{1+\alpha\beta}(\delta\beta(\alpha-1) - \alpha(\beta+1)) \pm \sqrt{(\tfrac{\delta\beta\alpha-\delta\beta-\alpha\beta-\alpha}{1+\alpha\beta})^2 + 4\tfrac{(\beta+1)(\alpha\beta+1)(\alpha-1)}{(1+\alpha\beta^2)}}]$$

But in the existence condition $1-\alpha > 0$, therefore

$$\lambda_{1,2} = \tfrac{-1}{2}[\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))}{1+\alpha\beta} \pm \sqrt{\tfrac{(\delta\beta(1-\alpha)-\alpha(\beta+1))^2}{(1+\alpha\beta)^2} - 4(\tfrac{(\beta+1)(\alpha\beta+1)(1-\alpha)}{(1+\alpha\beta)^2})}]$$

Hence,

$$\lambda_1 = \tfrac{-1}{2}[\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))}{1+\alpha\beta} - i\sqrt{\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))^2}{(1+\alpha\beta)^2} + 4(\tfrac{(\beta+1)(\alpha\beta+1)(1-\alpha)}{(1+\alpha\beta)^2})}]$$

$$= \tfrac{-1}{2}[\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))}{1+\alpha\beta} - \tfrac{i}{1+\alpha\beta}\sqrt{(\delta\beta(1-\alpha)+\alpha(\beta+1))^2 + 4(\beta+1)(\alpha\beta+1)(1-\alpha)}]$$

and

$$\lambda_2 = \tfrac{-1}{2}[\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))}{1+\alpha\beta} + i\sqrt{\tfrac{(\delta\beta(1-\alpha)-\alpha(\beta+1))^2}{(1+\alpha\beta)^2} - 4(\tfrac{(\beta+1)(\alpha\beta+1)(1-\alpha)}{(1+\alpha\beta)^2})}]$$

$$= \tfrac{-1}{2}[\tfrac{(\delta\beta(1-\alpha)+\alpha(\beta+1))}{1+\alpha\beta} + \tfrac{i}{1+\alpha\beta}\sqrt{(\delta\beta(1-\alpha)+\alpha(\beta+1))^2 + 4(\beta+1)(\alpha\beta+1)(1-\alpha)}]$$

Therefore, the positive interior equilibrium point is locally asymptotically stable as the real part of the eigenvalues are negative.

3.3.3 Global stability analysis of equilibria

In this section,the global stability analysis of system(3.1.9) in the R_+^2 is investigated by using Lypunov function. Actually the first lypunov function in population biology was contracted by Vito Volterra 1920. He considered the predator-prey model which describes the fluctuation of predator species(selachians(sharkas)) and prey species(the food fish) in Mediterranean sea.

Theorem: The positive interior coexistence equilibrium point E_2 is globally asymptotically stable.

proof: Consider a lypunov function,

$$H(u,v) = u - u^* - u^* \tfrac{\ln u}{u^*} + \tfrac{1}{\delta}(v - v^* - v^* \tfrac{\ln v}{v^*}).$$

Differentiating H with respect to time t,thus we get,

$$\frac{dH(u,v)}{dt} = \frac{(u-u^*)}{u}\frac{du}{dt} + \frac{1}{\delta}\frac{(v-v^*)}{v}\frac{dv}{dt}$$

substituting in the expression for $\frac{du}{dt}$ and $\frac{dv}{dt}$, we get

$$\frac{dH(u,v)}{dt} = \frac{(u-u^*)}{u}(u(1-v-\alpha u)) + \frac{1}{\delta}\frac{(v-v^*)}{v}(\delta v(-1+u-\beta v))$$

$$= (u-u^*)(1-v-\alpha u) + (v-v^*)(-1+u-\beta v)$$

$$= (u-u^*)(-(v-v^*) - \alpha(u-u^*)) + (v-v^*)((u-u^*) - \beta(v-v^*))$$

$$= -\alpha(u-u^*)^2 - (u-u^*)(v-v^*) + (u-u^*)(v-v^*) - \beta(v-v^*)^2$$

$$= -\alpha(u-u^*)^2 - \beta(v-v^*)^2 < 0$$

$\Rightarrow \frac{dH}{dt} < 0$, hence H(u,v) is a Lypunov function. Therefore E_2 is globally asymptotically stable. Note that $\frac{dH}{dt}$ does not change sign and is not identically zero in the Int.R_+^2 of the xy-plane.

3.4 Bifurcation analysis

Bifurcation analysis focuses on the dependency of the long-term dynamics behavior on the model parameters of non linear dynamical systems. In this section we study the occurrences of different types of local bifurcations due to the change in stability properties of various equilibrium points of the system. Local bifurcations are those which can be previewed studying and the above vector field in the neighborhood of a fixed point or a closed trajectory.
An application of the Sotomayor's theorem is used to investigate the occurrence of the local bifurcation near the equilibrium points of system (3.1.9).
The requirement is how we vary parameters and what typical bifurcations can occur.

The equilibrium points of system(3.1.9) are;

$$(0,0),(\tfrac{1}{\alpha},0) \text{ and } (\tfrac{1+\beta}{1+\alpha\beta},\tfrac{1-\alpha}{1+\alpha\beta}),\text{if } (1-\alpha>0), \text{ or } (\alpha<1)$$

The jacobian matrix of system(3.1.9) is;

$$J(u,v)=\begin{bmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{bmatrix}=\begin{bmatrix}1-v-2\alpha u & -u\\ \delta v & -\delta+\delta u-2\delta\beta v\end{bmatrix} \tag{3.4.12}$$

we now determine the nature of equilibrium points using linearization

1. The jacobian matrix evaluated at E_1, J$(\frac{1}{\alpha},0)$=$\begin{bmatrix}-1 & -\frac{1}{\alpha}\\ 0 & \delta(-1+\frac{1}{\alpha})\end{bmatrix}$,

the eigenvalue of the jacobian matrix at E_1 given by $\lambda_1=-1$, and $\lambda_2=\delta(-1+\frac{1}{\alpha})$, using stability condition, if $\alpha<1$, then the equilibrium point $(\frac{1}{\alpha},0)$ is stable, if $\alpha>1$ then the equilibrium point$(\frac{1}{\alpha},0)$ is unstable. Hence $\alpha=1$ is a bifurcation curve called a transcritical bifurcation curve, in the parameter space $(\frac{1}{\alpha},0)$.

Theorem: Suppose that $\delta=1$, then the system undergoes the transcritical bifurcation at equilibrium point $(\frac{1}{\alpha},0)$ as the parameter α passes through the bifurcation point α_0, where $\alpha_0=1$
Proof: Let us consider the jacobian matrix of the system at the equilibrium point J(u,v)

given by J(u,v)=$\begin{bmatrix}1-v-2\alpha u & -u\\ v & -1+u-2\beta v\end{bmatrix}$, since $\delta=1$.
The jacobian matrix J(u,v) evaluated at the equilibrium point $(\frac{1}{\alpha},0)$,we obtain
$B=J(\frac{1}{\alpha},0)=\begin{bmatrix}-1 & \frac{-1}{\alpha}\\ 0 & -1+\frac{1}{\alpha}\end{bmatrix}$
Here observe that the matrix B has a single zero eigenvalue as it's determinant is zero and we assumed that the trace $\neq 0$. This ensures that the Jacobian matrix B has simple zero eigenvalue. We shall establish the existence of transcritical bifurcation at the

equilibrium $(\frac{1}{\alpha}, 0)$ of system (3.1.9) by using Sotomayor theorem. Below we compute all the needed terms to verify the conditions for existence of transcritical bifurcation at $\alpha = 1$.

Let $V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an eigenvector of the matrix B corresponding to the eigenvalue $\lambda = 0$. Hence using the standard computation of the Eigenvector and choosing an arbitrary vector $v_1 = 1$ we obtain, $v_2 = \alpha - 1$.

$$v = \begin{bmatrix} 1 \\ \alpha - 1 \end{bmatrix} \Rightarrow (1, \alpha - 1)^T$$

Now let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be an eigenvector of the matrix

$$B^T = \begin{bmatrix} -1 & 0 \\ \frac{-1}{\alpha} & -1 + \frac{1}{\alpha} \end{bmatrix}.$$

corresponding to the eigenvalue $\lambda = 0$ Thus using similar computation of the eigenvectors w and choosing an arbitrary vector $w_2 = 1$, we obtain $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Differentiating the system(3.1.9) with respect to the bifurcation parameter α and evaluating at the point $(\frac{1}{\alpha}, 0)$ we obtain

$$f_\alpha(\tfrac{1}{\alpha}, 0) = \begin{bmatrix} -u^2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\alpha^2} \\ 0 \end{bmatrix}$$

and hence

$$w^T f_\alpha(\frac{1}{\alpha}, 0) = (0, 1) \begin{bmatrix} \frac{-1}{\alpha^2} \\ 0 \end{bmatrix} = 0 \tag{3.4.13}$$

$$w^T [D_u f_\alpha(\frac{1}{\alpha}, 0) v] = (0, 1) \begin{bmatrix} 1 \\ \alpha - 1 \end{bmatrix} = \alpha - 1 \neq 0 if \alpha \neq 1 \tag{3.4.14}$$

$$w^T [D^2 f_\alpha(\frac{1}{\alpha}, 0)(v, w)] = (0, 1) \begin{bmatrix} -2\alpha & 0 \\ 0 & -2\beta \end{bmatrix} \begin{bmatrix} 1 \\ \alpha - 1 \end{bmatrix} = -2\beta(\alpha - 1) \neq 0, \ if \ \alpha \neq 1. \tag{3.4.15}$$

Therefore Sotomayor theorem together with equations (3.4.13), (3.4.14) and (3.4.15) imply that the system (3.1.9) experiences transcritical bifurcation at $\alpha = 1$.

Theorem2 The system (3.1.9) undergoes transcritical bifurcation at the equilibrium point $E_2(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$ when the parameters δ passes through the bifurcation point $\delta = \delta_0$ where $\delta_0 = \frac{\alpha(\beta+1)}{\beta(\alpha-1)}$.

Proof: Let us consider (3.4.12) which is the Jacobian of system (3.1.9) evaluated at an equilibrium point (u,v) given by;

$$J(u,v) = \begin{bmatrix} 1 - v - 2\alpha u & -u \\ \delta v & -\delta + \delta u - 2\delta\beta v \end{bmatrix}$$

Since the equilibrium point of study is $(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$, evaluating the Jacobian matrix J(u,v) at $(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$,we obtain

$$A = J(\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta}) = \begin{bmatrix} \frac{-\alpha\beta-\alpha}{1+\alpha\beta} & \frac{-1-\beta}{1+\alpha\beta} \\ \frac{\delta-\delta\alpha}{1+\alpha\beta} & \frac{\alpha\beta\delta-\beta\delta}{1+\alpha\beta} \end{bmatrix}$$

Here observe that the matrix A has a single zero Eigenvalue as its determinant is zero and we assumed that the trace 0. This ensures that the Jacobian matrix B has simple zero Eigenvalue. We shall establish the existence of Trans-Critical bifurcation at the equilibrium $(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$ of system (3.1.9) by using Sotomayor theorem. Below we compute all the needed terms to verify the conditions for existence of Trans-Critical bifurcation at $(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$.

Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an Eigenvector of the matrix A corresponding to the Eigenvalue $\lambda = 0$. Hence using the standard computation of the Eigenvector and choosing an arbitrary vector $v_1 = 1$, we obtain $v_2 = -\alpha$

$v = \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}$. Now let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be an eigenvector of the matrix

$$A^T = \begin{bmatrix} \frac{-\alpha(\beta+1)}{1+\alpha\beta} & \frac{\delta(\alpha-1}{1+\alpha\beta} \\ \frac{-1-\beta}{1+\alpha\beta} & \frac{\delta(\alpha-1)}{1+\alpha\beta} \end{bmatrix}$$

Corresponding to the eigenvalue $\lambda = 0$. Thus using similar computation of the eigenvectors w and choosing an arbitrary vector $w_1 = 1$, we obtain $w = \begin{bmatrix} 1 \\ \frac{\alpha(\beta+1)}{\beta(\alpha-1)} \end{bmatrix}$ Differentiating the system (3.1.9) with respect to the bifurcation parameter δ and evaluating at the point $(\frac{1+\beta}{1+\alpha\beta}, \frac{1-\alpha}{1+\alpha\beta})$ we obtain

$$f_\delta(\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta}) = \begin{bmatrix} 0 \\ v(-1+u-\beta v) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and hence

$$w^T f_\delta(\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta}) = (1, \tfrac{\alpha(\beta+1)}{\beta(\alpha-1)}) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$w^T[Df_\delta(\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta})(v,w)] = (1, \tfrac{\alpha(\beta+1)}{\beta(\alpha-1)} \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} = 1 - \tfrac{\alpha^2(\beta+1)}{\beta(\alpha-1)} = 1 + \tfrac{\alpha^2(\beta+1)}{\beta(1-\alpha)} \text{ since } 1-\alpha > 0$$

$$w^T[D^2 f_\delta(\tfrac{1+\beta}{1+\alpha\beta}, \tfrac{1-\alpha}{1+\alpha\beta})(v,w)] = (1, \tfrac{\alpha(\beta+1)}{\beta(\alpha-1)} \begin{bmatrix} -2\alpha & 0 \\ 0 & -2\beta\delta \end{bmatrix} \begin{bmatrix} 1 \\ -\alpha \end{bmatrix} = -2\alpha(1 + \tfrac{\alpha\delta(\beta+1)}{\alpha-1}) = -2\alpha(1 - \tfrac{\alpha\delta(\beta+1)}{1-\alpha})$$

Therefore, Sotomayor's theorem together with the above three conditions imply that the system(3.1.9) experiences transcritical bifurcation at $\delta = \frac{\alpha(\beta+1)}{\beta(\alpha-1)}$.

Chapter 4

Numerical simulation and Results

Numerical simulations help us to analyze the phase diagram of dynamical systems depending up on parameters. The numerical solution of the growth rate equation of(3.1.9) computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and initial conditions. For this MATLAB has been used. It is possible to find more precise numerical solutions to a given set of differential equations using various methods in MATLAB. The Runga-Kutta methods, used by MATLAB's ode45 solvers, are frequently used in systems of ordinary differential equations to estimate the values of an equation at a particular point.

In this section we provide the numerical simulations to illustrate some results of the previous sections. Mainly, we present the time plots of the solutions u and v with the phase diagrams (around the positive equilibrium points) for the predator-prey systems. Dynamical nature of the systems (3.1.9) about the equilibrium points with different sets of parameter values are presented. And also bifurcation diagram for the system(3.1.9), about the equilibrium point under different sets of parameter values are presented. The result are illustrated in the following figures and the analysis of result are presented below.

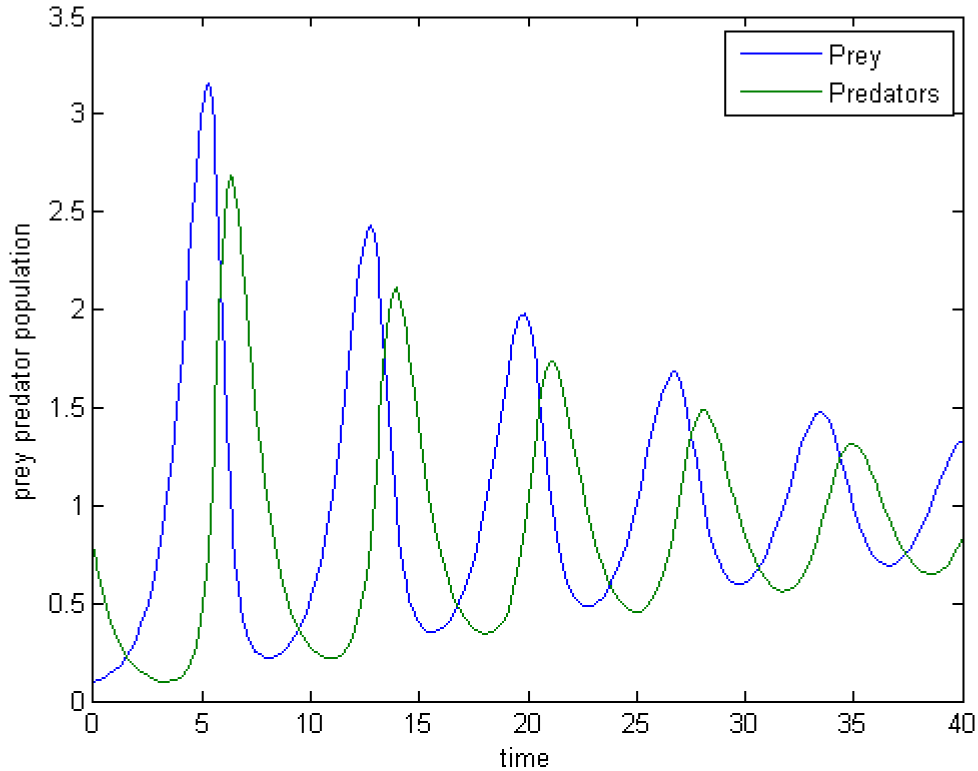

Figure 4.1: For this graph we use the parameter $\alpha = 0.001$, $\beta = 0.15$, $\delta = 1$ and the initial conditions $u_0 = 0.2$ and $v_0 = 0.75$

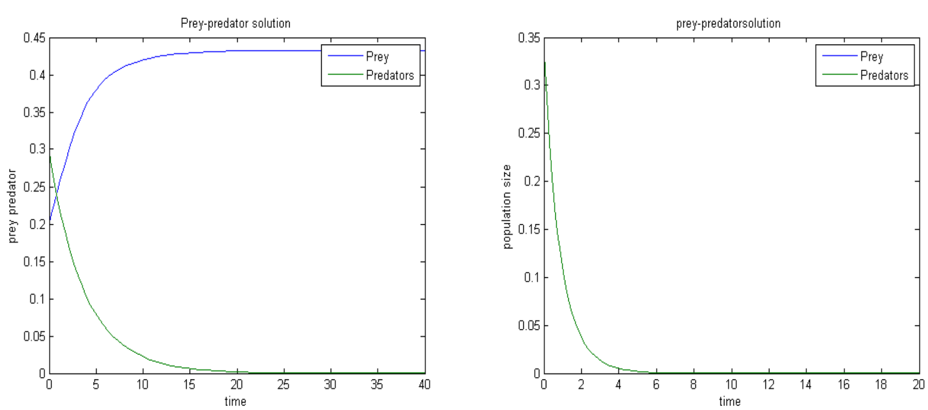

Figure 4.2: Local stability of E_1

For u(0) = 0.2, v(0) = 0.3, $\alpha = 0.85$, $\beta = 0.2$, $\delta = 0.8$, for this choice of parameters values the equilibrium point $E_1 = (\frac{1}{\alpha}, 0)$ is locally asymptotically stable. Eigenvalues are $\lambda_1 = -1$ and$\lambda_2 = -0.1417749$. Hence the equilibrium point is stable. So that $|\lambda_{1,2}| < 1$ Clearly u approaches 1 and v approaches 0 in finite time. The time plot and the phase diagram illustrate the result and the predator is extinction.

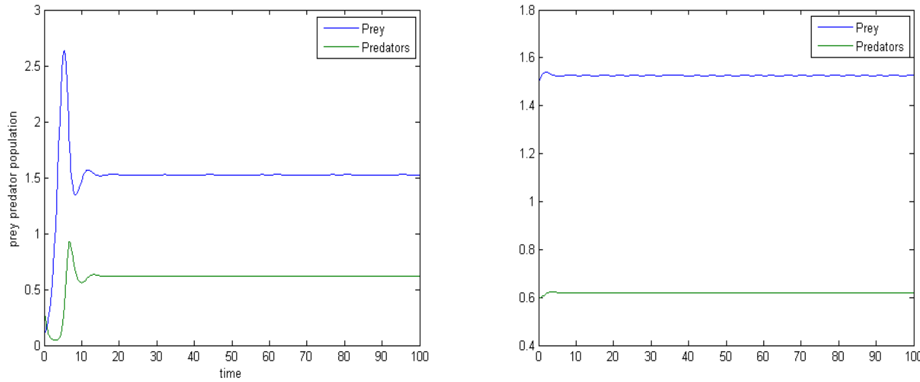

Figure 4.3: Local stability of E_2

Here u(0)=0.1, v(0)=0.35, and we take the parameter $\alpha = 0.25, \beta = 0.85$ and $\delta = 1$. Computations yield $E_2 = 1.5257732, 0.61855617$. The eigenvalues are,

$\lambda_{1,2} = 0.90721649 \pm \frac{i}{1.2125}\sqrt{6.81630784}$ and $|\lambda_{1,2}| < 1$. Hence the criteria for stability are satisfied.

Evidently both species converge to their equilibrium. The result of this graph of system (3.1.9) showing that E_2 is locally asymptotically stable.

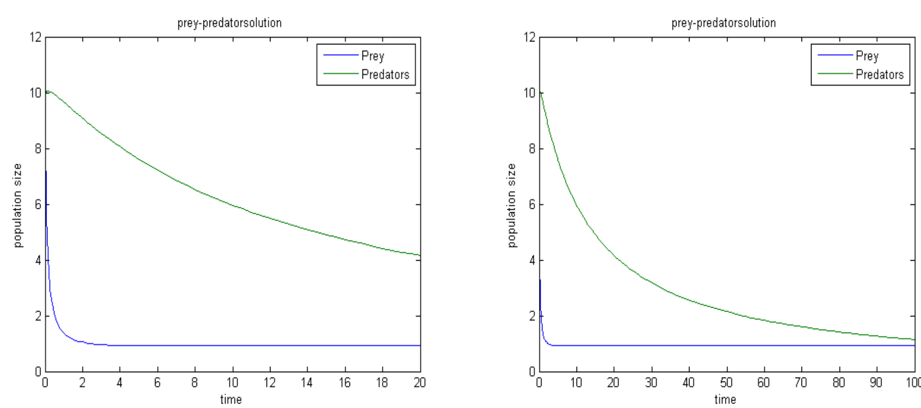

Figure 4.4: Global stability of E_2

Here we consider the model with $\alpha = 0.09$, $\beta = 0.35$ and $\delta = 0.02$ and the initial condition $u_0 = 7.2, v_0 = 10$.

The initial strength of both the species are different and the second species would always dominates the first species. Further the first species has falls down at a very low rate and then become constant after t=1. Eventhough the second species dominates the first species, but their population decreases from time to time, as seen in figure(4.4)

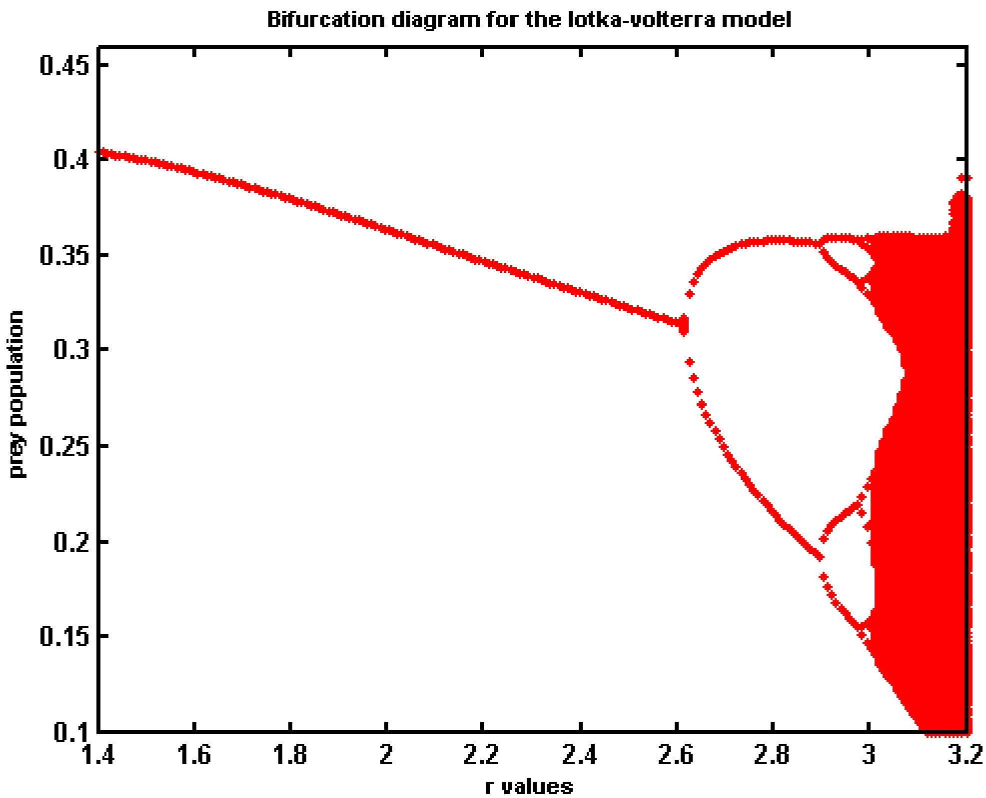

Figure 4.5: Bifurcation diagram for u(t) with α as the bifurcation parameter for u(0)=0.3, v(0)=0.4, when β=0.85, δ=0.5.

A bifurcation with respect to a parameter α in a specific solution at α_0 where there is no neighborhood around α_0 such that in this neighborhood the branch can be uniquely extended. At a bifurcation something happens. Qualitative changes are tied with bifurcation. For this bifurcation diagram for prey population, Figure(4.5), we consider the values, $\beta = 0.85$ $\delta = 0.5$ and varying r from 1.4 to 3.2.

4.1 Results

As we explain in the previous section E_1 is locally asymptotically stable if condition(3.3.11) hold. In terms original parameter(3.1.2) $kb < rc$. Hence the dynamics of prey and predator becomes locally stable if the competition with in prey population and natural death rate of population is less than natural growth rate of prey population with interaction between prey and predator populations.
Dynamic nature of the system (3.1.9) about the stability equilibrium point at E_1, we take $\alpha = 0.85$, $\beta = 0.2$, and $\delta = 0.8$. The initial conditions for prey and predator populations are respectively u(0) = 0.2 and v(0)=0.3. The conditions $\alpha < 1$ satisfied. Since the prey density is small compared to predator population density, the entire predator population does not get enough food which leads to the decline in the predator population. Even though prey population increases continuously subject to the logistic growth, the predator population struggles to recover from the decline. Finally the predator becomes extinct, see Figure(4.2,).

From the figure(4.1) we can conclude that a large enough increase in the number of predators leads to a decrease in the number of prey. This is logical from a biological standpoint, since a larger population of predators leads to increased interactions between predators and prey, and therefore increased prey death. This is also logical based on the Lotka-Volterra model. As the prey population decreases, the predator population also begins to decrease, since the increased predator population can no longer be supported by the shrinking number of prey available. As the predator population decreases, the prey population begins to recover. Once the prey population has sufficiently recovered, the predator population once again increases. This brings us back to where we started: once again, an increase in the predator population leads to a decrease in the number of prey. This periodic pattern is common to all predator prey relationships.
We analyse the complete behavior of the solutions by varying the parameter value of α. We start with $E(\frac{1}{\alpha}, 0)$. Point $E_1(\frac{1}{\alpha}, 0)$ changes its stability when it passes through $\alpha = 1$, from anode (when $\alpha > 1$). The unstable saddle point coexist with the non trivial stable node obtained from E_1 when $0 < 1 < \alpha$.

The bifurcation diagram is another useful numerical analysis tool that illustrate how the characteristics of fixed points change as the varying parameter increases in a specific intervals. We plot the bifurcation diagram as the systems parameter increases in the interval [0,1] for various values of parameter.

Biologically Interpretation

All of the results discussed above is in agreement with the theorems explained in the paper. However, in this work we are able to give a real case from which we can easily interpret the biological phenomena of the solutions of the system.
The trivial solution $E_0(0, 0)$ is always unstable that means the absence of predator will let prey grows following logistic equation (see first equation of (3.0.1)).
As predators death rate increase, the absence of predator will maintain the existence of prey. Meanwhile, for decreasing predator death rate $(\frac{1}{\alpha}, 0)$ becomes unstable meaning at that death rate the absence of predator cannot be sustained, as the predator still can

reproduce and grow. This is the reason why that unstable solution co-exists with a stable nontrivial solution E_1. In this interval, as the death rate decreasing of prey density decreases while predator density increases.

As we said before, for mathematical reasons we studied the population dynamics for the entire parameter plane. However, for biology purposes we give the biological interpretation for the positive quadrant. As a consequence, as we already established by convention, if u_0 and/or v_0 are negative we say that the corresponding equilibrium u_0 does not exist. If one or both initial populations do not exist, they will not exist for ever. In fact, this is a consequence of the fact that the axes of coordinates in the phase plane are separatrix. If the initial population are at any equilibrium point, then the populations remain constant at any subsequent time. If the initial populations are situated on a limit cycle, then the time evolution of populations will be cyclical (periodic). For all other initial values the subsequent populations vary in various manners, depending on the values of the parameters α, β and δ. The subsequent populations are oscillatory but not periodic (first both populations increase, then only v increases and u decreases, then only u increases and v decreases with the amplitudes smaller and smaller, and again both populations increase and so on) until they reach the equilibrium point E(0, 0) and the subsequent populations are slowly oscillating until they come to the equilibrium point E$(\frac{1}{\alpha}, 0)$, i.e. v becomes extinct. In all other cases the equilibrium points are unstable, namely one or both populations go to infinity.
Finally, we can conclude that the prey u(t) flourish in the absence of the predator. Theoretically, the predator can destroy all the prey so that the latter becomes extinct. However,if this happens the predator v(t) will also become extinct since, as we assume, it depends on the prey for its existence. In order for the predation to take place there must be a fight between a predator and a prey. In addition, our parametric portrait shows precisely where all these phenomena occur.

4.2 Discussion

The proposed model is shown biologically well-posed in the sense that any positive solution starts in the first quadrant remains both non-negative and bounded. The local stability of the system in different equilibrium point and the global stability of the existence equilibrium point has been discussed. The existence of equilibrium point imply that the dynamics of the system is very sensitive to the variation of the initial conditions. Prey-predator models with logistic growth rate for the prey and predator population and constant death rate of predator results in either stable steady-state coexistence or oscillatory coexistence of prey and predator when death rate of the predator species is not very high mortality of the predator species results in the extinction of predator and the prey population survive at level of their carrying capacity. The dynamical behavior of the proposed model represented by system(3.1.9) has been investigated locally as well as globally. Local bifurcation near the equilibrium points has been investigated. The proposed system has the equilibrium point $(\frac{1}{\alpha}, 0)$ through the transcritical bifurcation as the bifurcation parameter ($\alpha = 1$), crosses it's critical value and also the system can have a positive equilibrium point through the transcritical as the bifurcation parameter ($\delta = \frac{\alpha(\beta+1)}{\beta(1-\alpha)}$), crosses it's critical value. The Sotomayar's theorem is applied to ensures the existence of saddle-node, transcritical and pitchfork bifurcations.

Chapter 5

Conclusion

This thesis provides a detailed stability and bifurcation analysis of the modified Lotka-Volterra model. In this model, we have find an equilibrium points of the system and discuss their stability nature both analytically and numerically. We analyze the general model for biological reasonable parameters and found that we have the trivial equilibrium point $E_0(0,0)$, as well as the semi-trivial equilibria $E_1(u^*,0)$ and the coexistence equilibrium $E_2(u^*,v^*)$. The trivial equilibrium is always unstable, while the others are stable or unstable depending on the choice of parameters. $(\frac{1}{\alpha},0)$ is the extinction equilibrium point in which the predator go extinct and the prey go to the carrying capacity,α, it is stable for birthing rate of prey, which we defined as our bifurcation parameter. (u^*,v^*) is the coexistence equilibrium point of prey and predator. This equilibrium point is stable after it bifurcates from $(\frac{1}{\alpha},0)$ equilibrium. We simulate the model using MATLAB specifically ode45. These numerical simulations demonstrated again the three equilibria depending on the initial parameter value. We have drown the phase diagrams and bifurcation diagrams of the system for different values of the model parameter. The results are illustrated with suitable parameter values. Numerical simulations are to show that the dynamical behavior of the system(3.1.9).

Reference

Berryman,AA. The origins and evolutions of predatorprey theory. Ecology 1992;73:15305.

Chen, J.B. Existence of positive periodic solution for a delayed periodic predator-prey model of Ivlev type. Ann. Differential Equations, 22(2): 134143 (2006)

Chow,S.-N. J.K. Hale, Methods of Bifurcation Theory, Springer, NewYork, 1982.

Cushing,J.M. Two species competition in a periodic environment, J. Math. Biol., 10, p. 385-400, 1980.

Dubey, B. and Upadhay, R. K. (2004). Persistence and extinction of one prey and two Predator system. Non Linear Modeling and Control, 9(4): 307-329.

Elik C., The stability and Hopf bifurcation for a predator-prey system with time delay, Chaos, Solitons and Fractals, 2008, (37), 87-99.

Gholizade,S. An Analysis of System of Three-species Food Chain Equations, Ilam University, (2012).

Green, E. (2004). The eects of a smart predator in a one predator-two prey system. Un published. (http/green e/University of chicago)

J. Guckenheimer and P. Holmes,Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Sciences 42, Springer-Verlag, New York, 1983.

Hale J.,Koak H., Dynamics and Bifurcations.Texts in Applied Mathematics, New York: Springer-Verlag,1991.

Kar,T.K. Modeling the analysis of a harvested prey-predator system incorporating a prey refuge, J. Comp. Appl. Math., 185, p.19-33, 2006.

Kuznetsov, Y. A. (1995). Elements of Applied Bifurcation Theory. Springer-Verlag. New York.

Lotka, J. Elements of Physical Biology. Williams and Wilkins, Baltimore, MD, USA, 1925; New ed.,Elements of Mathematical Biology, Dover, New York, 1956

Marsden J., McCracken M., Hopf Bifurcation and Its Applications, Springer, 1976.

Murray,J.D. Mathematical Biology: An Introduction (Volume I) (3rd ed.), Springer, New York, (2002).

Paul Georgescu and Ying-Hen Hsieh, Global Dynamics of a prey- predator model with stage structure for the predator, Siam J. Appl. Math., 5, p. (67) 1379-1395 (2007).

Perko,L. Differential equations and dynamical systems, Springer-Verlag, 2001.

Scheurle,J. and J.Marsden, Bifurcation to quasi-periodic tori in the interaction of steady state and Hopf bifurcations, SIAM J. Math. Anal. 15 (1984), 10551074.

Sotomayor, J. [1973] Generic bifurcations of dynamical systems, Dynamical systems , 549560.

Vadasz, V. (2007). Economic Motion: An Economic Application of the Lotka-Volterra Predator- Prey Model. Retrieved from http://dspace.nitle.org/bitstream/handle/10090/

Volterra, V. [1926] Fluctuations in the abundance of a species considered mathematically, Nature 118, 558560.

Xiao,D. and S. Ruan, Global dynamics of a ratio-dependent predator-prey system, Journal of Mathematical Biology, vol. 43, no. 3, pp. 268-290, 2001.

Xiao,Y. N. L. chen, Global stability of a predator-prey system with stage structure for predator, Acta Mathematica Sinica, 19, p.1- 11 (2003).

Appendix

A.Mathematical preliminaries

Lypunov function

Consider a nonlinear autonomous system

$$\frac{dx}{dt} = f(x, y) \tag{5.0.1}$$

$$\frac{dy}{dt} = g(x, y \tag{5.0.2}$$

such that the system has an isolated point (0,0) and the function f(x, y) and g(x, y) have continuous first order partial derivatives for all (x, y). Let E(x, y) be a function with first order partial derivatives at all point (x, y) in the domain D containing the point (0,0).

i. The function E(x, y)is said to be positive definite in D if and only if $E(x, y) > 0$, $\forall(x, y) \in D| \{0, 0\}$ and E(0, 0)=0.

ii. The function E(x, y)is said to be positive semi definite in D if and only if $E(x, y) \geq 0$, $\forall(x, y) \in D| \{0, 0\}$ and E(0, 0)=0.

iii. The function E(x,y)is said to be negative definite in D if and only if $E(x, y) < 0$, $\forall(x, y) \in D| \{0, 0\}$ and E(0,0)=0.

iv. The function E(x, y) is said to be negative semi definite in D if and only if $E(x, y) \leq 0$ $\forall(x, y) \in D| \{0, 0\}$ and E(0, 0) = 0.

Considering the above system such that the system has an isolated critical point (0, 0) and the functional of f and g have first order partial derivatives for all (x, y) in domain D containing the point (0, 0). Let E(x, y) be positive definite function in the domain D such that $\frac{dE}{dt} = E_x f(x, y) + E_y g(x, y)$, is negative semi definite in D. Then the function E is called **Lyapunov function** for the system in D.

Sotomayor's theorem

Suppose that $f(x_0, \mu_0) = 0$ and that the $n * n$ matrix $A = Df(x_0, \mu_0)$ has a simple eigenvalue $\lambda = 0$ with eigenvector v and A^T has an eigenvector w corresponding to the eigenvalue $\lambda = 0$. Furthermore, suppose that A has k eigenvalues with negative real parts and (n-k-1) eigenvalues with positive real parts and that the following conditions are satisfied.

$$w^T f\mu_0(x_0, \mu_0) \neq 0$$

$$\mathrm{w}^T[D^2 f(x_0, \mu_0)(v, v)] \neq 0$$

Then there is a smooth curve of equilibrium points of $x' = f(x, \mu)$ in $R^n * R$ passing through (x_0, μ_0) and tangent to the hyperplane $R^n * \{\mu_0\}$. Depending on the sign of the expression in the above condition there are no equilibrium points of $x' = f(x_0, \mu_0)$ near x_0, when $\mu < \mu_0$ or $(\mu > \mu_0)$ and there are two equilibrium points of $x' = f(x_0, \mu_0)$, when $(\mu > \mu_0)$ or $(\mu > \mu_0)$. The two stable manifolds of dimensional k and k+1 respectively, that is the system experiences a saddle node bifurcation at the equilibrium point x_0 as the parameter μ passes through the bifurcation value $\mu = \mu_0$. The set of c^∞ vector fields satisfying the above condition is an open dense subset in the Banach space of all c in f. One parameter vector fields with an equilibrium point x_0 having a simple zero eigenvalue.

► If the above conditions are changed to,

$$w^T f\mu(x_0, \mu_0) = 0$$

$$\mathrm{w}^T[Df_\mu(x_0, \mu_0)v] \neq 0$$

$$\mathrm{w}^T[D^2 f_\mu(x_0, \mu_0)(v, v)] \neq 0$$

Then the system $x' = f(x_0, \mu_0)$ experiences a transcritical bifurcation at the equilibrium point x_0 as the parameter μ varies through the bifurcation parameter value $\mu = \mu_0$.

► If this conditions are also changed to,

$$w^T f\mu(x_0, \mu_0) = 0$$

$$\mathrm{w}^T[Df_\mu(x_0, \mu_0)v] \neq 0$$

$$\mathrm{w}^T[D^2 f_\mu(x_0, \mu_0)(v, v)] = 0$$

$$\mathrm{w}^T[D^3 f_\mu(x_0, \mu_0)(v, v, v)] \neq 0$$

Then the system experiences pitchfork bifurcation at the equilibrium point x_0 as the parameter μ varies through the bifurcation value $\mu = \mu_0$.

B. Matlab Codes

Lotka-Volterra model

function du = lotka-volterra(t, u)

du = [0;0];

$\alpha = 0.25$;

$\beta = 0.85$;

$\delta = 1$;

du(1) = u(1)*(1-u(2)-alpha * u(1));

du(2) = delta*u(2)*(-1+u(1)-beta*u(2));

[t,u] = ode45('lotka-volterra',[0 20], [0.5 0.45]);

plot(t,u);

ylabel('prey predator population')

xlabel('time')

legend('Prey', 'Predators');

Lotka-Volterra model

```
clear; to = 0;

tf =100;

yo = [1.01 .85];

[t y] = ode45('yprf',[to tf],yo);

plot(t,y(:,1),t,y(:,2))

title('Prey-predator solution')

ylabel('prey predator')

xlabel('time')

legend('Prey', 'Predators');

function yprf =yprf(t,y)
```

$\alpha = 0.99;$

$\beta = 0.35;$

$\delta = 0.2;$

```
yprf(1) = y(1)*(1-y(2)-alpha*y(1));

yprf(2) = delta*y(2)*(-1+y(2)-beta*y(2));
```

yprf = [yprf(1) yprf(2)]';

Bifurcation diagram

```
c=2;

r=0.25;

b=0.25;

tmin=1.4 ;tmax=3.2;

No = 0.4;

Po = 0.3;

n = 10000;

jmax = 500;

t = zeros(jmax + 1, 1);

z = zeros(jmax + 1, 250);

del =(rmax-rmin)/jmax;

for j = 1:jmax + 1

N = zeros(n + 1, 1);

P = zeros(n + 1, 1);

N(1) = N0;
```

```
    P(1) = P0;
    t(j) = (j-1)*del + rmin;
    r = t(j);
for i = 1 : n
    N(i + 1) = N(i)*(1-P(i)-r*N(i));
    P(i + 1) = b*P(i)*(-1+N(i)-c*P(i));
    if (i > 750)
    z(j,i-250) = N(i + 1);
    end
    end
    end
    plot(t,z,'r.','linewidth',2,'MarkerEdgeColor','r','MarkerSize',5)
    xlabel ('t values','fontsize',8,'fontweight','b'),
    ylabel('prey population','fontsize',8,'fontweight','b')
    title ('Bifurcation diagram for the lotka-volterra model','fontsize',8,'fontweight','b')
    set(gca,'fontsize',10,'fontweight','b','linewidth',1.75)
    axis([1.1 3.5 0.1 0.46]);
```