ACCURATE NUMERICAL METHOD FOR SINGULARLY PERTURBED REACTION-DIFFUSION EQUATIONS

Awoke Tesfaw Arega

A Thesis Submitted to the Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the

Degree of Master's in Mathematics (Numerical Analysis)

Office of Graduate Studies

Adama Science and Technology University

June, 2022

Adama, Ethiopia

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Declaration

I hereby declare that this Master Thesis entitled "Accurate Numerical Method for Singularly Perturbed Reaction-Diffusion Equations" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citations.

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we, the advisor(s) of this thesis, hereby certify that we have read the revised version of the thesis entitled "Accurate Numerical Method for Singularly Perturbed Reaction-Diffusion Equations" prepared under our guidance by Awoke Tesfaw Arega submitted in partial fulfillment of the requirements for the degree of Mater's of Science in Mathematics (Numerical Analysis). Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
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Approval Page

I/we, the advisors of the thesis entitled "Accurate Numerical Method for Singularly Perturbed Reaction-Diffusion Equations" and developed by Awoke Tesfaw Arega hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by Awoke Tesfaw Arega have read and evaluated the thesis entitled "Accurate Numerical Method for Singularly Perturbed Reaction-Diffusion Equations" and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Mathematics (Numerical Analysis).

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ACRONYMS

CFDM	Compact Finite Difference Method
FDMs	Finite difference methods
FMFDM	Fitted Mesh Finite Difference Method
FOFDM	Fitted Operator Finite Difference Method
ICM	International Congress of Mathematicians
LDG	Local Discontinuous Galerkin
NSFD	Non-Standard Finite Difference
SPDEs	Singularly Perturbed Differential Equations

Abstract

In this thesis, we considered a time dependent one-dimensional singularly perturbed parabolic reaction-diffusion problems. We proposed a parameter-uniform numerical scheme to solve the problems. First, we dicretized the continuous problem of space variable using a non-standard finite difference method via the method of line, then the spatial discretization is transformed to a system of initial value problems. After changing the spatial discretization to initial value problem, the time derivative is solved using second order Fehlberg Runge-Kutta method. The method is second order parameter-uniform convergent in both space and time direction.

Keywords: Parabolic reaction-diffusion problems; Singular perturbations; Non standard finite difference method; Parameter-uniform convergence.

CHAPTER 1

INTRODUCTION

1.1 Background of the Study

Prandtl introduced the singular perturbations at the third International Congress of Mathematicians (ICM) in Heidelberg in 1904, and it was reported in the conference proceedings. Differential equations with big or small parameters are used to represent many practical situations, such as mathematical boundary layer theory or approximation solutions of different problems. Layer areas, which are narrow sections of the domain over which the solution undergoes sudden changes, characterize the solution to these kind of issues. A boundary layer is a thin layer of rapid change that occurs at a very small interval around a boundary. When the reduced problem's features are parallel to the border, a parabolic boundary layer occurs (Natesan & Gowrisankar, 2012).

The name singular perturbation comes from the fact that in the limit case, when the singular perturbation parameter is equal to zero, the character of the differential equations changes completely (Miller et al., 1996). Singularly perturbed problem is defined by (Roos et al., 2008) "ordinary or partial differential equations with a small positive parameter and solutions (or derivatives) that approach a discontinuous limit as the parameter approaches zero. Problems of this nature are referred to as singularly perturbed.

Singularly perturbed differential equations (SPDEs) appear frequently in the mathematical modeling of real-world problems in science and engineering (Kumara Swamy et al., 2018), (Glizer, 2003), (Keller, 1968). (Lange & Miura, 1985), (Lange & Miura, 1994), and (Lange & Miura, 1982) researchers have produced a series of articles to get an approximate solution for singularly perturbed differential equations. For the numerical approximations of singularly perturbed linear delay differential equations, (Kadalbajoo et al., 2006) combined fitted-operator approaches with Micken's non-standard finite difference techniques.

Typical examples include fluid flow (Kreiss & Lorenz, 2004) and (Kreiss & Lorenz, 2004); they appear in the (linearised) Navier-Stokes and Oseen equations, as well as elasticity, quantum mechanics, electrical networks, gas porous electrodes theory, aerodynamics, plasma dynamics, oceanography, diffraction theory, reaction-diffusion processes (Alhumaizi, 2007) and many other fields. The solution of singularly perturbed problems SPPs has a multi-scale character (non-uniform behavior), that is, there are thin layer(s) (Boundary layer area) where the solution changes rapidly, while the solution acts regularly and varies slowly away from the layer(s) (Outer Region).

Mathematical issues are extremely difficult (or even impossible) to answer precisely, approximate solutions are required. Using perturbation methods, an approximate solution can be obtained(Roja, 2017). A singularly perturbed differential equation is one in which the highest order derivative term of differential equations is multiplied by a small positive parameter ε where $0 < \varepsilon << 1$, and the parameter is known as the perturbation parameter. The behavior the solutions of these differential equations is determined by the magnitude of the parameters. More precisely, consider a problem depending on a small parameter ε (the singular perturbation parameter) which we denote by p_ε where ε is multiplied to the highest derivative term(s). Setting $\varepsilon = 0$ in p_ε, we obtain a reduced problem which we denote by p_0. Let us assume further that $u(x,\varepsilon)$ is a solution of p_ε, and u(x,0) is the solution of the reduced problem.

Now, if

$$\lim_{\varepsilon \to 0} u(x,\varepsilon) = u(x,0)$$

then, P_ε is a regular perturbation problem (RPP); otherwise P_ε is a singular perturbation problem (SPP).

Numerical methods for singularly perturbed differential equations have been studied since the early 1970s and the research frontier has been constantly expanding, then a comprehensive exposition of the state of the art in the analysis of numerical methods for singular perturbation problems was published in 2008 by (Roos & Zarin, 2002).

A reaction-diffusion equation comprises a reaction term and a diffusion term, i.e. the typical form is as follows:

$$u_t = D\Delta u(x,t) + f(x,t)$$

u = u(x, t) is a state variable and describes The state variable u = u(x, t) describes the density/concentration of a substance, a population... at position $x \in \Omega \subset R^n$ at time t (Ω is a open set). Δ denotes the Laplace operator. the first term on the right hand side denotes diffusion with D denoting the diffusion coefficient. The second term, f(x,t) is a smooth function

$f : R \to R$ and describes processes with really change the present u, i.e. something happens to it (birth, death, chemical reaction ...), not just diffuse in the space. It is also possible, that the reaction term depends not only on u, but also on the first derivative of u, i.e. ∇u; and/or explicitly on x (Kuttler, 2011).

Reaction–diffusion system are mathematical models which correspond to several physical phenomena. The most common is the change in space and time of the concentration of one or more chemical substances: local chemical reactions in which the substances are transformed into each other, and diffusion which causes the substances to spread out over a surface in space. Reaction–diffusion systems are naturally applied in chemistry. However, the system can also describe dynamical processes of non-chemical nature. Examples are found in biology, geology and physics (neutron diffusion theory) and ecology. Mathematically, reaction–diffusion systems take the form of semi-linear parabolic partial differential equations.

Classic numerical methods for solving singular perturbation problems are well known for being unstable and fail to produce exact solution when the perturbation parameter ε is small, because the solution of singular perturbation problem is not easy, and the solution depends on the perturbation parameter ε and mesh size h (Doolan et al., 1980), the accuracy and convergence of the methods must be properly examined. This shows that the numerical solution of one-dimensional parabolic reaction-diffusion problems with singularly perturbed parameters should be improved. When utilizing traditional numerical methods, the presence of the singular perturbation parameter causes oscillations or divergence in the computed solutions. When ε is very small, an excessively large number of mesh points is required to avoid these oscillations or divergence, which is not possible. To overcome this drawback of classical numerical methods, we develop a method based on the method of line (MOL), which regards the problem without oscillation by combining a non-standard finite difference method in the spatial direction with an explicit Runge-Kutta method of order two in the time direction. As a result, this study proposed a numerical method for solving a singularly perturbed one-dimensional parabolic reaction-diffusion problem that is both accurate and ε uniformly convergent.

1.2 Statement of the Problem

In this study, we consider one-dimensional time dependent singularly perturbed parabolic reaction-diffusion equation.

$$\begin{cases} u_t(x,t) - \varepsilon u_{xx}(x,t) + b(x,t)u(x,t) = f(x,t) \\ (x,t) \in Q \equiv \Omega \times (0,T] \\ with\ \ initial\ \ and\ \ boundary\ \ conditions \\ U(x,0) = \nu(x,t), & \forall x \in \bar{Q} \\ U(0,t) = \gamma_0,\ U(1,t) = \gamma_1 & \forall t \in [0,T]. \end{cases} \tag{1.1}$$

Where ε is the perturbation parameter such that $0 < \varepsilon \ll 1$. When the perturbation parameter is very small, the results of classical or standard numerical method are not stable and it is difficult to get the the accurate solution (result).

In this study, we construct Fehlberg Runge-Kutta methods for time derivative and non-standard finite difference method in space for a singularly perturbed parabolic reaction-diffusion problem.

1.3 Objectives

1.3.1 General Objectives

The main objective of this study is to develop parameter uniform convergent numerical method for solving singularly perturbed reaction-diffusion problems.

1.3.2 Specific Objectives

The specific objectives of this study are:

- apply the non standard FDM for solving singularly perturbed parabolic reaction-diffusion problems.
- analyze the parameter-uniform convergent of the method.
- establish the stability of the scheme.

1.4 Significance of the Study

The result obtained from this study: understand how to solve singularly perturbed parabolic reaction-diffusion problems, used as a reference material for scholars who works on this area and help to provide graduate students with research skills and scientific processes.

CHAPTER 2

REVIEW OF RELATED LITERATURE

In (2003), Madden and Stynes study a uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction–diffusion problems. They considered two singularly perturbed linear reaction–diffusion of two point boundary value problems are investigated as a coupled system. Each equation's leading term is multiplied by a smallest positive parameter, but the magnitudes of these parameters may vary. The boundary layers of the system's solutions overlap and interact. The structure of these layers is investigated, and a piece-wise uniform mesh, a version of the standard Shishkin mesh, is created. On this mesh, the central difference has been shown to be almost first-order accurate in both small parameters.

A fitted mesh method for singularly perturbed reaction-convection-diffusion problems with boundary and interior layers was presented by Shanthi et al., (2006). The problem are solved by applying a robust numerical method and they constructed an appropriate piece-wise uniform mesh. To drive the discrete system of equations classical upwind finite difference schemes are used. They achieved a parameter-uniform error bounded of numerical approximation. To get the error of this numerical solution, they used double mesh principle.

Clavero et al. in (2009) a higher order schemes for reaction-diffusion singularly perturbed systems was presented. On a Shishkin mesh, a non–monotone HODIE finite difference scheme is first developed. At the transition points, the prior approach is adjusted to produce an inverted monotone scheme. They show that a third order uniformly convergent approach exists when the diffusion parameters are equivalent. If the diffusion parameters are changed, numerical evidence suggests that an order more than two uniformly convergent scheme is achieved. Nonetheless, the uniform errors are larger and the orders of uniform convergence are lower than when the diffusion parameters are equal.

Clavero and Gracia, (2010) used a uniform convergence of a finite difference scheme for time dependent singularly perturbed one dimensional reaction-diffusion equations. To approximate the solution of one-dimensional singularly perturbed reaction diffusion equations, they use a numerical scheme, which is a combination of classical backward Euler and central difference method. The scheme of the meshes are defined as the product of uniform mesh in time and a special mesh in space. They used three different types of meshes, which are Bahkvalov (Kellogg et al., 2008), Vulanovic (Vulanović, 2001) and Shishkin (Miller et al., 1997). They discritize the time derivatives using Euler method, to provide uniform convergence of the analysis is based on asymptotic behavior of the solution of the semi-discrete problems. They get a uniform convergent with respect to the parameter of diffusion.

In (2013), Gowrisankar and Natesan used the parameter uniform numerical method for singularly perturbed parabolic reaction–diffusion problems on equidistributed grids. They use a modified backward Euler finite difference scheme on layer adapted non-uniform meshes at each time level, and the central difference for the space derivative to solve this singularly perturbed reaction diffusion problems. The non-uniform meshes are obtained by equidistribution of a positive monitor function, which involves the second-order spatial derivative of the singular component of the solution. The equi-distributing monitor function at each time level allows us to use this technique to non-linear parabolic problems. They provided a parameter-uniform numerical method for the initial and boundary value problem.

$$
\begin{gathered}
u_t(x,y)+\pounds_\varepsilon u(x,t)=f(x,t)\ (x,t)\in\omega=(0,x)\times[0,T],\\
u(x,0)=s(x) \text{ on } S(x)=\{(x,0):0\le x\le 1\}\\
u(0,t)=a_0(t) \text{ on } S(0)=\{(0,t):0\le t\le T\}\\
u(1,t)=a_1(t) \text{ on } S(1)=\{(1,t):0\le t\le T\}.
\end{gathered}
$$

Where $\pounds_\varepsilon u\equiv-\varepsilon u_{xx}+b(x)u, 0<\varepsilon\ll 1$. Where ε is a small parameter, and b, f are sufficiently smooth functions with $b(x)\ge\beta>0$ for $x\in[0,1]$. Under suitable continuity and compatibility conditions on the data, the above initial boundary value problem has a unique solution $u(x,t)$ (Miller et al., 1998). Boundary layers occur in the solution when $\varepsilon\to 0$. In this study, the proposed scheme is parameter-uniform convergent of order $O(\Delta t+N^{-2})$

Wondimu et al., (2016) solve singularly perturbed one dimensional reaction diffusion problems with in Dirichlet boundary conditions. To solve the numerical solution of these problems, they used fourth order finite difference method. To get the solution, first they replace the derivative of the given differential equation by finite difference approximation and then, apply compact finite difference method by using uniform mesh. Compact finite difference method (CFDM) is a finite difference method which employs a linear combination of three consecutive points of derivatives to approximate a linear combination of the same three consecutive values of a function $y(x_j), j=i-1,i,i+1$. The numerical results of the problem with fourth order compact finite difference method approximates the exact solution very well.

Das, (2018) used a higher order finite difference method for singularly perturbed convection-diffusion parabolic partial differential problems. To solve the numerical solution of the given problem, they apply implicit Euler method to discritize the time derivative and an upwind scheme are used for the space discritization. A higher order convergent solution with respect to space and time has obtained using the post-processing based extrapolation approach. as a result they get the convergent of these presented method is independent of the perturbation parameter ε. By applying extrapolation approach, they rises the accuracy of uniform convergence from first order to second order in both space and time variable.

Bullo et al. (2021) used an accelerated fitted operator finite difference method for singularly perturbed parabolic reaction-diffusion problems. To solve the solution of the problem, they introduced fitted parameter with asymptotically solution and develop a fitted operator finite difference method. Richardson extrapolation technique is used to accelerate rate of convergence of the numerical method. They show the method is consistent, stable, and more accurate than some methods that used for solving singularly perturbed parabolic reaction-diffusion initial boundary value problems and as a result, they get second order convergent method and by applying Richardson extrapolation the convergence the method is accelerated to fourth order.

A uniformly convergent higher-order finite difference scheme is constructed and analyzed for solving singularly perturbed parabolic problems with non-smooth data. The scheme involves an average non-standard finite difference with the Richardson extrapolation method for space variables and second-order finite difference approximation for time direction on uniform meshes (Tesfaye et al., 2021). In this scheme they show convergent of second-order in both temporal and spatial directions and the scheme is proven to be uniformly convergent and also confirmed by numerical experiments.

Sumit et al.,(2021) studies for singularly perturbed reaction–diffusion problems, optimal fourth-order parameter-uniform convergence of a non-monotone scheme on equidistributed meshes. They provide an optimal fourth-order parameter-uniform non-monotone strategy for singularly perturbed reaction–diffusion boundary value problems with boundary layers at both ends of the domain on equi-distributed meshes in this study. They use a high-order non-monotone finite difference approach to discretize the problem and show that it is stable in the maximal norm. To discretize the problem, the equi-distribution of an appropriate monitor function is used to build layer-adapted meshes. On these equi-distributed meshes, the approach is shown to be optimal fourth-order uniformly convergent. To validate the theory and demonstrate the efficiency of the method, numerical results are presented.

The method of lines was proposed by Mbroh and Munyakazi, (2021), as a robust numerical scheme for solving singularly perturbed parabolic reaction-diffusion problems. They considered a one-dimensional and two-dimensional singularly perturbed parabolic reaction diffusion equation and proposed a parameter-uniform numerical scheme to solve singularly perturbed parabolic reaction diffusion equations. To answered the solution of this singularly perturbed reaction diffusion problem. They propose a parameter-uniform numerical technique to address the singularly perturbed reaction diffusion problems.

First, they discretized the space variable using a fitted operator finite difference method to build (propose) the numerical strategy for these continuous problems. After discretizing the space variable or continuous problem, they discretized the time derivative, which transforms the partial differential equation into a system initial value problem, then they solve using the Crank-Nicholson finite difference approach. As a result they show the proposed approach is second-order parameter-uniform in both space and time, and that by using the Richard extrapolation methodology on the space variable scheme, the result is changed in to a fourth-order parameter uniform convergent method.

In (2021), Bullo et al., deals with the numerical treatment of singularly perturbed parabolic reaction-diffusion initial boundary value problems. Introducing a fitting parameter into the asymptotic solution and applying average finite difference approximation, a fitted operator finite difference method is developed for solving the problem. To accelerate the rate of convergence of the method, Richardson extrapolation technique is applied. The consistency and stability of the method have been established very well to ensure the convergence of the method. Generally, the formulated method is consistent, stable, and more accurate than some methods existing.

Efficient discretization and reconditioning of the singularly perturbed reaction diffusion was studied by Bacuta et al. (2022). They investigate the reaction diffusion problems and show how to discretize and precondition when the reaction term dominates the equation in the singular perturbed situation. They give effective discretization procedures for uniform and non-uniform meshes using the principles of optimal test norm and saddle point reformulation. They provide a preconditioning technique that is effective across a wide range of perturbation parameters. For a problem on the unit square, numerical examples are supplied to demonstrate the method's effectiveness.

Cheng et al. (2022) investigated the balanced-norm error estimation of the local discontinuous Galerkin technique on layer-adapted meshes for reaction-diffusion issues. For singularly perturbed problems, we investigate the local discontinuous Galerkin (LDG) approach on layer-adapted meshes. The balanced norm is appropriate for capturing the contribution of the boundary layer component of the solution in these reaction-diffusion situations. In the broader case, however, obtaining an optimal balanced-norm error estimate for the local discontinuous Galerkin approach is problematic. That is, using the usual procedure, one can only get a sub optimal error estimate of the local discontinuous Galerkin method in the specified "artificial" norm. They proposed a new projection that combines the local Gauss-Radau projection and the local L^2 projection in a subtle way. They get a more accurate balanced-error estimate.

Computational method for for singularly perturbed parabolic reaction diffusion equations with robin boundary conditions was studied by (Wendimu & Duressa, 2022). The non-standard finite difference method has been presented in this paper for the numerical solution of singularly perturbed parabolic reaction-diffusion subject to Robin boundary conditions. They used the implicit Euler and non-standard finite difference methods on a uniform mesh to discretize temporal and spatial variables, respectively. They demonstrated that, regardless of the perturbation parameter, the suggested technique exhibits uniform convergence in time with first-order and in space with second-order. They designed three numerical examples to back up their theoretical conclusions.

Generally, we have seen several parameter-uniform convergent numerical method for singularly perturbed parabolic reaction-diffusion problems. Motivated by looking high order uniform convergent and the work of (Mbroh & Munyakazi, 2021),using non-standard finite difference method in space direction and second order Fehlberg Rugge-Kutta method in time derivative, we develop a parameter-uniform convergent method of second order both in space and time. Non Standard finite difference (NSFD) approaches for numerical integration of differential equations were first described in 1989 (Mickens, 1989). Patidar presents an almost full listing and overview of articles employing non standard finite difference (NSFD) methods up to 2004 in his paper (Patidar, 2005). "Using the known solution of the differential equation or 'ad hod experimentation..." is how the non standard finite difference (NSFD) method approach works (García-López, 2003). A thorough evaluation of the results obtained thus far clearly demonstrates that this interpretation of what has been accomplished utilizing non standard finite difference method (NSFD) approaches is incorrect. Non-standard finite difference techniques offer the possibility of producing finite difference models that are free of the standard numerical instabilities induced by discretizing differential equations with traditional approaches. The non-standard finite difference scheme is defined by Anguelov and Lubuma

CHAPTER 3

FORMULATION OF THE METHOD

3.1 Qualitative properties of continuous problem

We consider a simple singularly perturbed time dependent reaction diffusion equation

$$L_\varepsilon u(x,t) = u_t - \varepsilon u_{xx} + b(x,t)u = f(x,t), (x,t) \in Q \equiv \Omega \times (0,T] \tag{3.1}$$

subject to the initial and boundary conditions

$$\begin{cases} u(x,0) = \nu(x), & \forall x \in \bar{\Omega} \\ u(0,t) = \gamma_0, u(1,t) = \gamma_1, & \forall t \in [0,T] \end{cases} \tag{3.2}$$

where γ_0 and γ_1 are given constants and $0 < \varepsilon \ll 1$. The functions $b(x,t)$ and $f(x,t)$ are assumed to be sufficiently smooth such that $b(x,t) \geq \beta > 0$, $\forall (x,t) \in Q$ and $b, f \in C^{(4,2)}(\bar{Q})$. Also, we impose the following compatibility conditions between data of problem (3.1)-(3.2) so that the exact solution $u(x,t) \in C^{(4,2)}(\bar{Q}$:
$f(0,0) = f(1,0) = 0, f_{xx}(0,0) + b(0,0)f(0,0) = f_t(0,0), f_{xx}(1,0) + b(1,0)f(1,0) = f_t(1,0)$,
for the solution of problem (3.1)-(3.2) to be compatible at the corners of $\bar{Q}$.

Lemma 3.1. (Continuous maximum principle). Let κ be a sufficiently smooth function defined on Q which satisfies $\kappa(x,t) \geq 0$, $\forall (x,t) \in \partial Q$. Then $L\kappa(x,t) \geq 0, \forall (x,t) \in Q$ implies that $\kappa(x,t) \geq 0, \forall (x,t) \in \bar{Q}$.

Proof. Let $(x^{'},t^{'})$ be such that

$$\kappa(x^{'},t^{'}) = \min_{(x,t)\in\bar{Q}} \kappa(x,t)$$

and suppose $\kappa(x^{'},t^{'}) < 0$. It is clear that $\kappa(x^{'},t^{'}) \notin \partial Q$. We have

$L_\varepsilon \kappa(x^{'},t^{'}) = \kappa_t(x^{'},t^{'}) - \varepsilon\kappa_{xx}(x^{'},t^{'}) + b(x^{'},t^{'})\kappa(x^{'},t^{'})$.

Since $\kappa_{xx}(x^{'},t^{'}) \geq 0$ and $\kappa_t(x^{'},t^{'}) = 0$, we obtain $L_\varepsilon(x^{'},t^{'}) < 0$, which contradicts with the initial assumption that $L_\varepsilon(x,t) > 0, \forall(x,t) \in Q$. Therefore, $\kappa(x,t) \geq 0, \forall(x,t) \in \bar{Q}$

Lemma 3.2. (Stability estimate). Let $u(x,t)$ be the solution of the continuous problem(3.1)-(3.2). Then we have the bound

$$\|u\| \leq \beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)).$$

Proof. We define two comparison functions $\omega^{\pm}$ as

$$\omega^{\pm}(x,t) = \beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)) \pm u(x,t).$$

At the initial stage we have

$$\begin{aligned}
\omega^{\pm}(x,0) &= \beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)) \pm u(x,0), \\
&= \beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)) \pm \nu(x), \\
&\geq 0.
\end{aligned}$$

at the boundaries we obtain

$$\begin{aligned}
\omega^{\pm}(0,t) &= \beta^{-1}\|f\| + max(\gamma_0,\gamma_1)) \pm u(0,t), \\
&= \beta^{-1}\|f\| + max(\nu(0), max(\gamma_0,\gamma_1)) \pm \gamma_0, \\
&\geq 0.
\end{aligned}$$

$$\begin{aligned}
\omega^{\pm}(1,t) &= \beta^{-1}\|f\| + max(\nu(1), max(\gamma_0,\gamma_1)) \pm u(1,t), \\
&= \beta^{-1}\|f\| + max(\nu(1), max(\gamma_0,\gamma_1)) \pm \gamma_1, \\
&\geq 0.
\end{aligned}$$

and at the boundaries we obtain

$$
\begin{aligned}
L_\varepsilon \omega^{\pm}(x,t) &= \omega_t^{\pm}(x,t) - \varepsilon \omega_{xx}^{\pm}(x,t) + b(x,t)\omega^{\pm}(x,t), \\
&= b(x,t)\beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)) \pm L_\varepsilon u(x,t), \\
&= b(x,t)\beta^{-1}\|f\| + max(\nu(x), max(\gamma_0,\gamma_1)) \pm f(x,t), \\
&\geq 0, \qquad since \;\; b(x,t) \geq \beta.
\end{aligned}
$$

Therefore $\omega^{\pm}(x,t) \geq 0, \forall (x,t) \in \bar{Q}$. This show that it is bounded.

3.2 Spatial Discretization

Let N be a positive number that sub-divide the domain [0, 1] of the space x direction with in a uniform mesh size $\Delta x = h$ such that

$$x_0 = 0, x_i = x_0 + i\Delta x, i = 1:(N-1), \Delta x = x_i - x_{i-1}, x_n = 1,$$

where N be the number of mesh point in the space discretization. For a continuous problem of in the form of Eq.(3.1)-(3.2), whilst keeping the time direction domain (0,T], we solve the differential part using non-standard finite difference method. To construct the exact finite difference method, we follow Mickens rules. Consider the constant coefficient sub-equations given by:

$$-\varepsilon \frac{d^2u}{dx^2} + bu = 0, \tag{3.3}$$

where $b > 0$. The solution of Eq.(3.3) has two linearly independent solutions namely $e^{(\lambda_1 x)}$ and $e^{(\lambda_2 x)}$ with

$$\lambda_{1,2} = \pm\sqrt{\frac{b}{\varepsilon}}.$$

We denote the approximate solution of $u(x)$ at x_i by U_i. The solution a difference equation which has the same general solution as the differential equation in (3.3) has at the grid point x_i given by $U_i = A_1 e^{(\lambda_1 x_i)} + A_2 e^{(\lambda_2 x_i)}$. Using the theory of difference equations for second order linear difference equations have

$$\begin{vmatrix} U_{i-1} & e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ U_i & e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \\ U_{i+1} & e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} = 0,$$

then we have,

$$U_{i-1}\begin{vmatrix} e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \\ e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} - U_i \begin{vmatrix} e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ e^{(\lambda_1 x_{i+1})} & e^{(\lambda_2 x_{i+1})} \end{vmatrix} + U_{i+1}\begin{vmatrix} e^{(\lambda_1 x_{i-1})} & e^{(\lambda_2 x_{i-1})} \\ e^{(\lambda_1 x_i)} & e^{(\lambda_2 x_i)} \end{vmatrix} = 0,$$

$$U_{i-1}\left(e^{(\lambda_2 h)} - e^{(\lambda_1 h)}\right) - U_i\left(e^{((\lambda_2 - \lambda_1)h)} - e^{((\lambda_1-\lambda_2)h)}\right) + U_{i+1}\left(-e^{(-\lambda_2 h)} + e^{(-\lambda_1 h)}\right) = 0 \tag{3.4}$$

substituting the values of λ_1 and λ_2 in Eq.(3.4),

$$U_{i-1}\left(e^{(\sqrt{\frac{b}{\varepsilon}}h)} - e^{(-\sqrt{\frac{b}{\varepsilon}}h)}\right) - U_i\left(e^{(2\sqrt{\frac{b}{\varepsilon}}h)} - e^{(-2\sqrt{\frac{b}{\varepsilon}}h)}\right) + U_{i+1}\left(-e^{(-\sqrt{\frac{b}{\varepsilon}}h)} + e^{(\sqrt{\frac{b}{\varepsilon}}h)}\right) = 0$$

dividing both sides by $e^{(\sqrt{\frac{b}{\varepsilon}}h)} - e^{(-\sqrt{\frac{b}{\varepsilon}}h)}$ we have:

$$U_{i-1} - \left(e^{(\sqrt{\frac{b}{\varepsilon}}h)} + e^{(-\sqrt{\frac{b}{\varepsilon}}h)}\right)U_i + U_{i+1} = 0,$$

$$U_{i-1} - 2\cosh\left(\sqrt{\frac{b}{\varepsilon}}h\right)U_i + U_{i+1} = 0. \tag{3.5}$$

Using half angle identity, $\sinh^2\left(\frac{\sqrt{\frac{b}{\varepsilon}}h}{2}\right) = \frac{\cosh\left(\sqrt{\frac{b}{\varepsilon}}h\right)-1}{2}$, we have

$$2\cosh\left(\sqrt{\frac{b}{\varepsilon}}h\right) = 2 + 4\sinh^2\left(\frac{\sqrt{\frac{b}{\varepsilon}}h}{2}\right),$$

then Eq.(3.5) can be written as:

$$U_{i-1}-\left(2+4\sinh^2\left(\frac{\sqrt{\frac{b}{\varepsilon}}h}{2}\right)\right)U_i+U_{i+1}=0,$$
$$U_{i-1}-2U_i+U_{i+1}=4\sinh^2\left(\frac{\sqrt{\frac{b}{\varepsilon}}h}{2}\right)U_i,$$
$$\frac{U_{i-1}-2U_i+U_{i+1}}{4\sinh^2\left(\frac{\sqrt{\frac{b}{\varepsilon}}h}{2}\right)}-U_i=0,$$

multiplying both sides by $-\varepsilon\rho^2$, where $\rho=\sqrt{\frac{b}{\varepsilon}}$ we obtain

$$-\varepsilon\frac{U_{i-1}-2U_i+U_{i+1}}{\frac{4}{\rho^2}\sinh^2(\frac{\rho h}{2})}+bU_i=0, \tag{3.6}$$

is an exact difference scheme for Eq.(3.3). The denominator function becomes $\phi^2=\frac{4}{\rho^2}\sinh^2(\frac{\rho h}{2})$. Adopting this denominator function for the variable coefficient problem we can write as:

$$\phi_i^2=\frac{4}{\left(\sqrt{\frac{b_i}{\varepsilon}}\right)^2}\sinh^2\left(\frac{\sqrt{\frac{b_i}{\varepsilon}}h}{2}\right). \tag{3.7}$$

Using ϕ_i^2 and Eq.(3.6) into the discretization of the main equation in Eq.(3.1), we obtain

$$L_\varepsilon^N U_i(t)=U_i'(t)-\varepsilon\left[\frac{U_{i+1}-2U_i+U_{i-1}}{\phi_i^2}\right]+b_iU_i(t)=f_i(t), \tag{3.8}$$

together with the semi-discrete boundary and initial conditions

$$U_0(t)=\gamma_0,\ \ U_N(t)=\gamma_1\ \ and\ \ U_i(0)=\nu_i,\qquad i=1,...,N-1. \tag{3.9}$$

Here the denominator function ϕ^2 is given by

$$\phi_i^2=\frac{4}{\rho_i^2}\sinh^2\left(\frac{\rho_i h}{2}\right),$$

where $\rho_i=\sqrt{b_i/\varepsilon}$, i = 1,2,...,N-1. Similar to the previous section, we write the difference scheme (3.8)-(3.9) as

$$U^{'}(t) + B(t)U(t) = F(t), \tag{3.10}$$

where $U(t), F(t) \in R^{N-1}$ and $B(t) \in R^{N-1} \times R^{N-1}$. The entries of $B(t)$ and $F(t)$ are given by

$$\begin{aligned}
B_{i,i}(t) &= \frac{2\varepsilon}{\phi_i^2} + b_i, && i = 1,2,3,...,N-1,\\
B_{i,i+1}(t) &= -\frac{\varepsilon}{\phi_i^2}, && i = 1,2,3,...,N-2,\\
B_{i,i-1}(t) &= -\frac{\varepsilon}{\phi_i^2}, && i = 2,3,...,N-1\\
F_1(t) &= f_1(t) + \frac{\varepsilon}{\phi_i^2}\gamma_0, &&\\
F_i(t) &= f_i(t), && i = 2,3,...,N-1\\
F_{N-1}(t) &= f_{N-1} + \frac{\varepsilon}{\phi_{i-1}^2}\gamma_1 &&
\end{aligned}$$

When we consider the entries of equation (3.8)-(3.9), it is clear that the semi-discrete matrix is a tri-diagonal matrix, which implies that it satisfies a semi-discrete maximum principle as well as the uniform stability estimate which is given in the next section.

3.3 Convergence Analysis

We follow (Clavero & Jorge, 2015) to give some properties of the semi-discrete problem (3.10) which show that the solution of this problem exist and also unique. Then we use these results to analyze the convergence of the spatial discretization.

Lemma 3.3. (Semi-discrete maximum principle). Suppose the operator defined by the semi-discrete scheme in Eq.(3.8)-(3.9) satisfies a semi-discrete maximum principle. That is, $\kappa_0(t) \geq 0$, $\kappa_N(t) \geq 0$, $\forall$ $(x_i, t) \in \bar{\Omega}^N \times [0, T]$. Then $L_\varepsilon^N \kappa_i(t) \geq 0, \forall (x_i, t) \in \Omega^N \times [0, T]$ implies $\kappa_i(t) \geq 0, \forall (x_i, t) \in \bar{\Omega} \times [0, T]$.

Proof. Note that $\kappa_i(t)$ denotes $\kappa(x_i, t)$. Let $t^{'} \in [0, T]$ be such that

$$\kappa_l(t^{'}) = \min_{x_i \in \bar{\Omega}, t \in [0,T]} \kappa_i(t) \quad and \quad \kappa_i(t^{'}) < 0.$$

Then we have $0 < l < N$. It follows that $(\kappa_s(t))_t = 0, \quad \kappa_{s+1}(t) - \kappa_s(t) \geq 0$ and $\kappa_s(t) - \kappa_{s-1}(t) \geq 0$. Therefore, $L_\varepsilon^N \kappa_s(t) < 0$, which is a contradiction. Hence $\kappa_i(t) \geq 0, \forall (x_i, t) \in \bar{\Omega}^N \times [0, T]$.

Lemma 3.4. If $u_i(t)$ is the solution of the semi-discrete problem (3.8)-(3.9) then, it follow the bound

$$\|u_i(t)\| \leq \beta^{-1} \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} \|L_\varepsilon^N u_i(t)\| + \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} (\nu(x), max(\gamma_0, \gamma_1)).$$

Proof. We consider the functions $\omega_i^{\pm}(t)$ defined by

$$\omega_i^{\pm}(t) = d \pm u_i(t),$$

where

$$d = \beta^{-1} \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} \|L_\varepsilon^N u_i(t)\| + \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} (\nu(x), max(\gamma_0, \gamma_1)).$$

At the boundaries we have

$$\begin{aligned} \omega_0^{\pm}(t) &= d \pm u_0(t) = d \pm \gamma_0 \geq 0 \\ \omega_N^{\pm}(t) &= d \pm u_n(t) = a \pm \gamma_1 \geq 0 \end{aligned}$$

Now for $\Omega \times [0, T]$ we have

$$\begin{aligned} L_\varepsilon^N \omega_i^{\pm}(t) &= (d \pm u_i(t))t - \varepsilon \left(\frac{d \pm u_{i+1}(t) - 2(d \pm u_i(t)) + d \pm u_{i-1}(t)}{\phi^2} \right) + b_i(t)(d \pm u_i(t)) \\ &= (b_i(t))d \pm L_\varepsilon u_i(t) \\ &= (b_i(t)) \left[\beta^{-1} \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} |L_\varepsilon u_i(t)| + \max_{(x_i,(t) \in \bar{\Omega} \times [0,T]} (|\nu(x)|, max(\gamma_0, \gamma_1)) \right] \pm f_i(t) \\ &\geq 0, \ \ since \ \ b_i(t) \geq \beta. \end{aligned}$$

From Lemma 3.3 it follows that $\omega_i^{\pm}(t) \geq 0, \forall (x_i, t) \in \bar{\Omega} \times [0, T]$, this completes the proof. Before we proceed, let us state a Lemma that will be used in the error analysis. This Lemma was proved in Munyakazi and Patidar (2010).

Lemma 3.5. For all integers k on a fixed mesh, we have that

$$\lim_{\varepsilon\to 0}\max_{i<i<n-1}\frac{exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}}=0$$

and

$$\lim_{\varepsilon\to 0}\max_{i<i<n-1}\frac{exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{k/2}}=0$$

where $x_i = ih, h = 1/N, \forall i = 1 : N-1$.

Proof. Let's take partition of the domain

$[0,1] := 0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1.$

For the interior grid points of the domain, we have

$$\max_{i<i<N-1}\frac{exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}}\leq\frac{exp(-Cx_1/\sqrt{\varepsilon})}{\varepsilon^{k/2}}=\frac{exp(-Ch/\sqrt{\varepsilon})}{\varepsilon^{k/2}}$$

and

$$\max_{i<i<N-1}\frac{exp(-C(1-x_i)/\sqrt{\varepsilon})}{\varepsilon^{k/2}}\leq\frac{exp(-C(1-x_N)/\sqrt{\varepsilon})}{\varepsilon^{k/2}}=C\frac{exp(-Ch/\sqrt{\varepsilon})}{\varepsilon^{k/2}}$$

as $x_1 = h, 1-x_{N-1} = 1-(N-1)h = h$. By applying the L'Hospital's rule the gives

$$\lim_{\varepsilon\to 0}\frac{exp(-Cx_i/\sqrt{\varepsilon})}{\varepsilon^{k/2}}=\lim_{p=1/\varepsilon\to\infty}\frac{p^k}{exp(Chp)}\equiv\lim_{p\to\infty}\frac{k!}{(Ch)^k exp(Chp)}=0.$$

This is proof of the above lemma.

The truncation error of the spatial discretisation is

$$\begin{aligned}
L_\varepsilon u_i(t)-U_i(t) &= \frac{du_i(t)}{dt}-\varepsilon(u_{xx}(t))i+b_i(t)u_i(t)- \\
&\quad\left[\frac{du_i(t)}{dt}-\varepsilon\left(\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\phi_i^2}\right)+b_i(t)u_i(t)\right] \\
&= -\varepsilon(u_{xx}(t))+\left[\varepsilon\left(\frac{u_{i+1}(t)-2u_i(t)+u_{i-1}(t)}{\phi_i^2}\right)\right].
\end{aligned}$$

Since this truncation error does not include the time derivative term, it's estimate is equivalent to that of its corresponding stationary problem. Using a truncated Taylor series expansions of $u_{i+1}(t)$ and $u_{i-1}(t)$ yields

$$L_\varepsilon(u_i(t)) - U_i(t) \leq -\varepsilon(u_{xx}(t))_i + \frac{\varepsilon}{\phi_i^2}\left[h^2(u_{xx}(t))_i + \frac{h^4}{12}(u_{xxxx}(t))_i\right] + \ldots.$$

Again we use the Taylor series expansion of the denominator function $1/h^2 - \beta/(12\varepsilon) + \beta^2h^2/(240\varepsilon^2)$ and this results into

$$\begin{aligned} L_\varepsilon(u_i(t)) - U_i(t) &\leq -\varepsilon(u_{xx}(t))_i + \left(\frac{\varepsilon}{h^2} - \frac{\beta}{12} + \frac{\beta^2h^2}{240\varepsilon} + \ldots\right) \\ &\quad \times \left[h^2(u_{xx}(t))_i + \frac{h^4}{12}(u_{xxxx}(t))_i\right] \\ &\leq \left(\frac{\varepsilon}{12}(u_{xxxx}(t))_i - \frac{\beta}{12}(u_{xx}(t))_i\right)h^2 + \left(\frac{\beta^2}{240\varepsilon}(u_{xx}(t))_i - \frac{\beta^2}{144}(u_{xxxx}(t))_i\right)h^4 \\ &\quad + \frac{\beta^2h^6}{2880\varepsilon}(u_{xxxx}(t))_i + \ldots. \end{aligned}$$

Applying the bound (3.2) and Lemma 3.5 gives

$$\begin{aligned} L_\varepsilon(u_i(t)) - U_i(t) &\leq \left(\frac{\varepsilon}{12} - \frac{\beta}{12}\right)h^2 + \left(\frac{-\beta}{144} + \frac{\beta}{240\varepsilon}\right)h^4 + \frac{\beta^2h^6}{2880\varepsilon} + \ldots \\ &\leq Ch^2, \end{aligned}$$

where we have also used the relation $h^2 > h^4 > \ldots$. The error becomes

$$|u_i(t) - U_i(t)| \leq Ch^2 \tag{3.11}$$

when we apply Lemma 3.4.

We have thus proved the following theorem.

Theorem 3.6. Let $U_i(t)$ be the numerical solution of equation (3.10) and $u_i(t)$ the exact solution, then we have

$$\max_{0\leq i\leq n; t\in[0,T]} |u_i(t) - U_i(t)| \leq Ch^2,$$

as the error associated with the spatial discretization for problem (3.1)-(3.2).

3.4 Time Discretization

The discritization of a time t with the domain [0,T] with a step size $\Delta t = t_{j+1} - t_j$, j=1:M, such that Q^M be discretized domain where M is the number of mesh in the time direction. We use numerical scheme for discretizing the system of initial value problems in Eq.(3.10), and use second order Runge-Kutta-Fehlberg method (Fehlberg, 1969). First we write Eq. (3.10) as follow

$$\frac{dU_i(t)}{dt} = f(t, U_i(t)), i = 0 : N \tag{3.12}$$

with initial condition $U_i(0) = v_i$, for i = 0:N. where $f(t, U_i(t)) = -B(t)U_i(t) + F_i(t)$. Then an approximation to the solution of the I.V.P. is made using a Runge-Kutta method of order two is given by

$$U_{i,j+1} = U_{i,j} + \frac{1}{512}K_1 + \frac{255}{256}K_2 + \frac{1}{512}K_3 \tag{3.13}$$

Where

$$\begin{aligned} K_1 &= hf(t_j, U_{i,j}), \\ K_2 &= hf(t_j + \frac{1}{2}\Delta t, U_{i,j} + \frac{1}{2}K_1) \\ K_3 &= hf(t_j + \Delta t, U_{i,j} + \frac{1}{256}K_1 + \frac{255}{256}K_2) \end{aligned}$$

Let $\tilde{U}_i(t_{j+1})$ be the approximation solution of $U_i(t_{j+1})$ at the mesh point t_j. For j=1:M the local approximation of $U_{i,j+1}$ to $\tilde{U}_{i,j+1}$ has second order accuracy (i.e $O(\Delta t)^3$).

Lemma 3.6. The global error estimates of the above approximation method in the time discritization are given by:

$$\|E_{j+1}\| = \|\tilde{U}_i(t_{j+1}) - U_i(t_{j+1})\| \leq C\Delta t^2,$$

where E_{j+1} is the global error in the time discritization at the time step $(j+1)^{th}$.

Proof. By using the estimate of local error e_j up to the time step j^{th}, we get the global error estimate of time step at $(j+1)^{th}$ grid point.

$$\begin{aligned}
\|E_{j+1}\| &= \sum_{i=1}^{j}\|e_j\|, \;\; j \leq M \\
&\leq \|e_1\|+\|e_2\|+\|e_3\|+\cdots+\|e_j\|, \;\; \|e_j\| = C(\Delta t^3) \\
&\leq d(j\Delta t)(\Delta t)^2 \\
&\leq dT(\Delta t)^2, \;\; j\Delta t \leq T \\
&\leq C(\Delta t)^2
\end{aligned}$$

By using the boundedness of the solution, the above Lemma 3.6 implies

$$\sup_{0<\varepsilon\leq 1}\|\tilde{U}_i(t_{j+1}) - U_i(t_{j+1})\|_{\Omega^M} \leq C(\Delta t)^2 \tag{3.14}$$

This shows that the time direction in the above discritiazation is consistent and global error is bounded. Then, by using Eq.(3.14) we can prove the parameter uniform convergence of the fully discrete scheme as follow

$$\sup_{0<\varepsilon\leq 1}\|u_i(t_j) - \tilde{U}_i(t_j)\| \leq \sup_{0<\varepsilon\leq 1}\|u_i(t_j) - U_i(t_j)\|_{\Omega^N} + \sup_{0<\varepsilon\leq 1}\|U_i(t_j) - \tilde{U}_i(t_j)\|_{Q^M} \tag{3.15}$$

Applying Theorem (3.6), boundedness of the solution, and Eq.(3.9), we get:

$$\sup_{0<\varepsilon\leq 1}\|u_i(t_j) - \tilde{U}_i(t_j)\| \leq C((\Delta x)^2 + (\Delta t)^2). \tag{3.16}$$

CHAPTER 4

NUMERICAL EXPERIMENT AND DISCUSSION

In this section we test the performance of the proposed method with numerical examples. We use the double mesh principle to estimate the errors because the exact solutions are not known for the above equation. The maximum error $E^{N,M}$ at each mesh points are computed as follow.

$$E^{N,\Delta t} = \max_{0 \leq n \leq N; 0 \leq m \leq M} |U_{n,m}^{N,\Delta t} - U_{n,m}^{2N,\Delta t/2}|$$

where $U_{n,m}$ are the numerical solutions of the one-dimensional problems, N is number of mesh points in the space direction and M is the mesh point of time direction. Furthermore, we compute the rate of convergence by using the following formula.

$$r = log2(E^{N,\Delta t} - E^{2N,\Delta t/2}) \tag{4.1}$$

Example 4.1. Consider the problem (Munyakazi & Patidar, 2013)

$$u_t - \varepsilon u_{xx} + \tfrac{2+x^2}{3} u = 1 + t^2, \ \ (x,t) \in (0,1) \times (0,1].$$
$$u(x,0) = 0, x \in [0,1]; u(0,t) = u(1,t) = 0, t \in (0,1].$$

Example 4.2. Consider (Clavero & Gracia, 2005)

$$u_t - \varepsilon u_{xx} + \tfrac{1+x^2}{2} u = (1+t)^3, \ \ (x,t) \in (0,1) \times (0,1];$$
$$u(x,0) = 0, x \in [0,1]; u(0,t) = u(1,t) = 0, t \in (0,1].$$

Example 4.3 Consider ()

$$u_t - \varepsilon u_{xx} + \tfrac{1+x^2}{2} u = e^t - 1 + sin(\pi x)$$
$$u(x,0) = 0, x \in \bar{\Omega}$$
$$u(0,t) = u(1,t) = 0, t \in (0,1].$$

Table 4.1: Maximum point-wise error of Example 4.1 and its convergent rate

ε	N=16 M=16	32 32	64 64	128 128	256 256	512 512
10^{-8}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 1.9998	1.5146e-07
10^{-9}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-10}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-11}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-12}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-50}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-51}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-52}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07
10^{-53}	1.6243e-04 2.0353	3.9627e-05 2.0180	9.7839e-06 2.0091	2.4306e-06 2.0046	6.0573e-07 2.0023	1.5119e-07

Table 4.2: Maximum point-wise error of Example 4.2 and its convergent rate

ε	N=16 M=16	32 32	64 64	128 128	256 256	512 512
10^{-8}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9999	6.8611e-07
10^{-9}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
10^{-10}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
10^{-11}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
10^{-12}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
10^{-22}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07
10^{-23}	6.6370e-04 1.9535	1.7136e-04 1.9791	4.3465e-05 1.9902	1.0941e-05 1.9952	2.7442e-06 1.9976	6.8717e-07

Table 4.3: Maximum point-wise error and convergent rate of Example 4.3

ε	N=16 M=16	32 32	64 64	128 128	256 256	512 512
10^{-8}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9949	2.2572e-07
10^{-9}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
10^{-10}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
10^{-11}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
10^{-12}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
10^{-21}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07
10^{-22}	1.9335e-04 1.8593	5.3287e-05 1.9348	1.3937e-05 1.9687	3.5607e-06 1.9847	8.9970e-07 1.9924	2.2611e-07

Table 4.4: The comparison of Example 4.1 with recent articles

ε	Schemes	N=16 M=16	32 32	64 64	128 128	256 256	512 512
10^{-8}	A B	1.6243e-04 5.12E-04	3.9627e-05 1.28E-04	9.7839e-06 3.20E-05	2.4306e-06 8.00E-06	6.0573e-07 2.00E-06	1.5146e-07 5.00E-07
10^{-12}	A B	1.6243e-04 5.12E-04	3.9627e-05 1.28E-04	9.7839e-06 3.20E-05	2.4306e-06 8.00E-06	6.0573e-07 2.00E-06	1.5119e-07 5.00E-07
10^{-16}	A B	1.6243e-04 5.12E-04	3.9627e-05 1.28E-04	9.7839e-06 3.20E-05	2.4306e-06 8.00E-06	6.0573e-07 2.00E-06	1.5119e-07 5.00E-07

Where, A is proposed scheme and B is a scheme proposed by (Mbroh & Munyakazi, 2021)

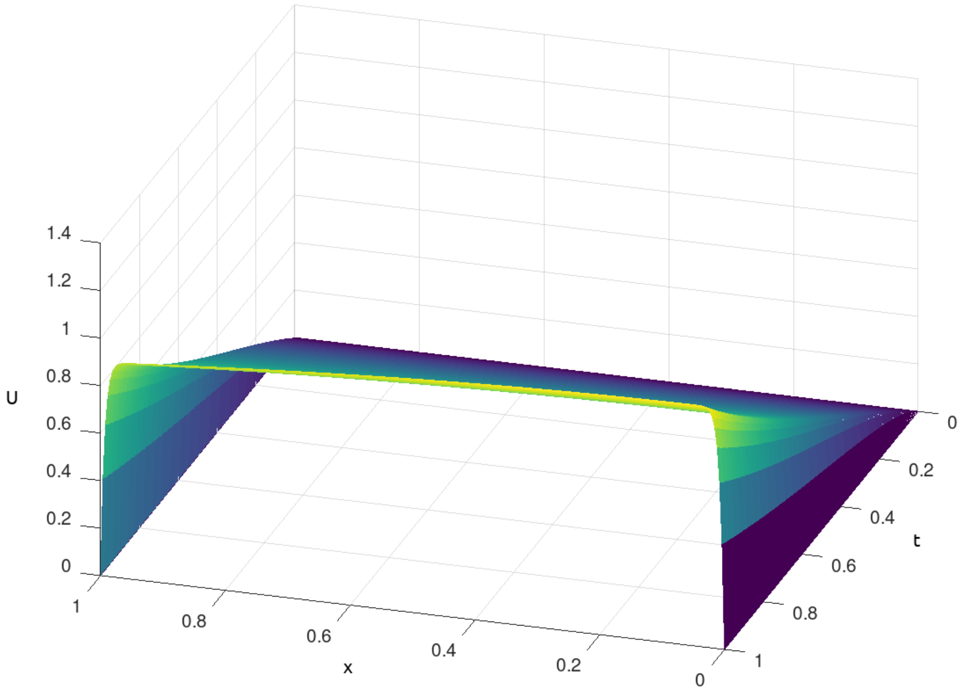

Figure 4.1: Graph of the numerical solution of Example 4.1 when M=N=256 and $\varepsilon = 10^{-5}$

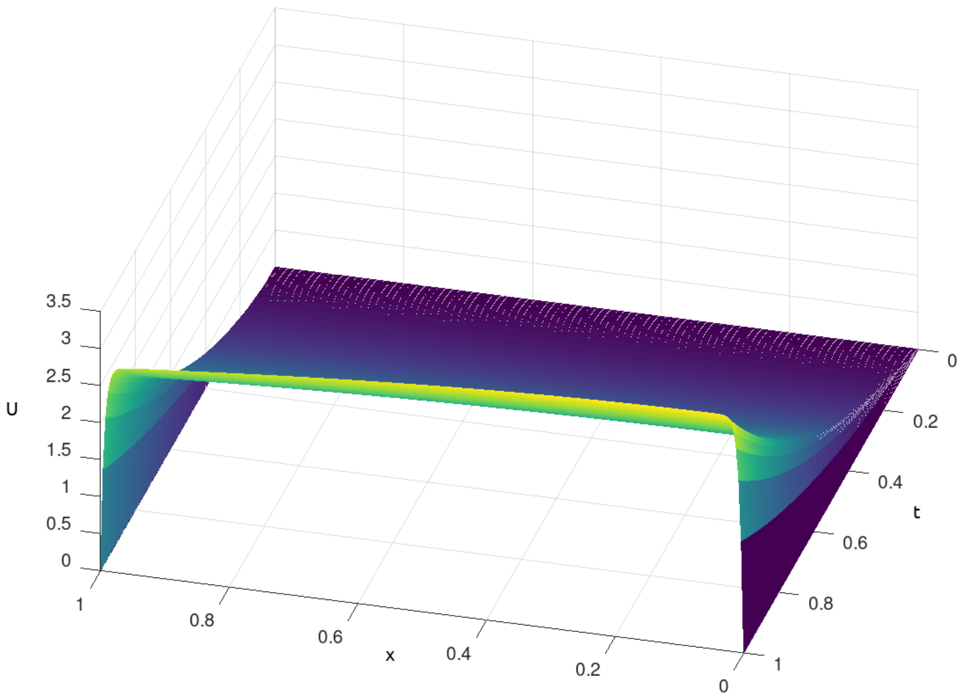

Figure 4.2: Graph of the numerical solution of Example 4.2, when M=N=256 and $\varepsilon = 10^{-6}$

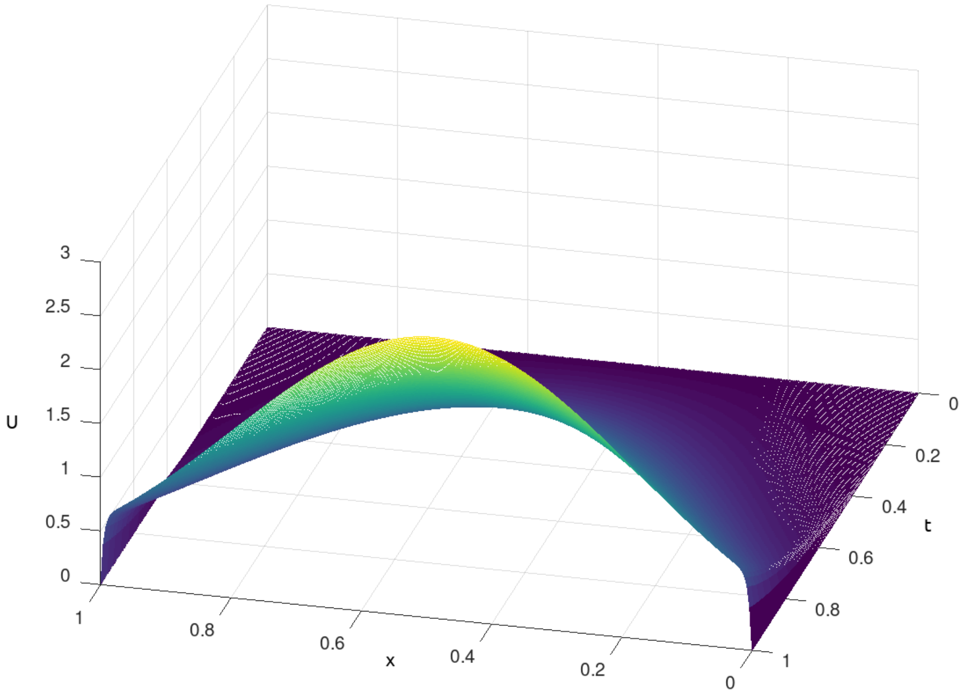

Figure 4.3: Graph of the numerical solution of Example 4.3, when M=N=256 and $\varepsilon = 10^{-5}$

The solution of the problems given in Example 4.1, 4.2 and 4.3 has two boundary layer at the right and left side (see Figures 4.1, 4.2 and 4.3). To validate the theoretical analysis of this thesis, three examples have been considered by taking different value of meshes size in both space and time derivative for different value of perturbation parameter ε. As we see from the numerical result of table 4.1, 4.2 and 4.3, the proposed method is second order parameter uniformly convergent in both space and time derivatives and from these table we, observe the maximum point wise error is decreasing as the number of meshes increase for each ε. By testing the result of the presented method with in examples, we show this method is stable, accurate and parameter uniform convergent. When we, compare this method with in Munyakazi and Patidar, (2013) the presented method is better. In this paper, we use non standard finite difference method in the space direction and explicit Fehlberg Runge-Kutta in the time direction. In the case of using explicit Fehlberg Runge-Kutta the numerical solution is converge, when the step size very small.

CONCLUSION AND RECOMMENDATION

Conclusion

In this study, we prepared a time-dependent singularly perturbed reaction-diffusion problem in one dimension. We develop non-standard finite difference method based on method of line in space discritization and second order Fuhlberg Runge-Kutta method for the system of initial value problems in time discritiazation after the results of space discritization. The proposed scheme's stability and convergence are investigated. In this work, we obtained second order parameter-uniformly convergent in both space direction and time derivative. To validate the theoretical study of this work, we have considered three examples by taking different value of perturbation parameter and mesh size. The point-wise error of these proposed method very small, when we are compared with a previous published articles.

Recommendation

We recommended for the future researchers, who studies in this area to choice implicit method. If, we select an explicit methods, the solution may not be converge, when the step size is large due to this case implicit method is better than explicit method to converge the solution quickly.

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