Mathematical Modeling of the dynamics of Prey-Predator interaction with Scavenger

By

Adem Aman Hassen

A Thesis Submitted to

Program of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

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As a thesis research advisor, I here by certify that I have read and evaluated this thesis prepared under my guidance by Adem Aman Hassen entitled **Mathematical Modeling of the dynamics of Prey-Predator interaction with Scavenger**. I recommend that it will be submitted as fulfilling thesis the requirement.

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Abstract

In this study we develop a mathematical model which describes the dynamics of prey-predator interaction with scavenger. We develop the model based on Holling type II functional responses. We solved the equilibrium points and their existence. The positivity and boundedness of the solution of the model are also determined. Conditions for local and global stability analysis are studied both analytically and numerically. The thesis also addresses the effect of extinction of a population and mechanism that three species coexist. As a result the mechanism that three species become coexist if there is large number of prey population compute with small number of predator and average number of scavenger population. The scavenger species also has a great role in stabilizing as well as for coexistence of three species. Numerical simulation are carried out to illustrate the analytical findings. Finally the biological implication of analytical and numerical are discussed critically.

Keywords: *Prey-Predator, Scavenger, Lyapunov function, Local stability, Global stability*

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Chapter 1

Introduction

1.1 Background of Study

Mathematical models are often used to examine the dynamics of complex interacting populations. In mathematical ecology a widely known and applied model is the classical Lotka-Volterra model. This model was formulated in 1926 by Vito Volterra and around the same time Alfred Lotka independently studied similar equations. The Lotka-Volterra model, also known as predator-prey model, is a cornerstone in the field of mathematical ecology and a significant amount of literature devoted to studying variants of these equations has been established. It consists of two coupled nonlinear differential equations and illustrates the interactions of one predator and one prey populations. These nonlinear differential equations are:

$$\begin{cases} \frac{dx}{dt} = x(a - by) \\ \frac{dy}{dt} = y(-c + dx) \end{cases} \tag{1.1.1}$$

where y(t) and x(t) represent, respectively, the predator and prey population as functions of time. The parameters $a, b, c, d > 0$ are interpreted as follows:

- a represents the natural growth rate of the prey in the absence of predators,
- b represents the effect of predation on the prey,
- c represents the death rate of the predator due to lack of food,
- d represents the efficiency and propagation rate of the predator in the presence of prey.

After three changes of variables, we can assume without loss of generality that a = 1, b = 1, and d = 1 making the system,

$$\begin{cases} \frac{dx}{dt} = x - xy \\ \frac{dy}{dt} = -cy + xy, \end{cases} \tag{1.1.2}$$

where c is possibly a different constant than the original system (1.1.1). The dynamics of the two systems, before and after the coordinate changes, are identical. With the exception of the fixed point (c, 1), all trajectories in positive space are closed curves which are given by the family of equations

$$\ln(y) - y + c\ln(x) - x = C,$$

where C corresponds to initial conditions.

The populations are represented as a pair of lagging, oscillating curves over time. The predator population will grow according to the amount of prey. If the prey population is large, the predators will have more food to support a larger population. However, when the predator population grows too large, the prey begins to die off. This will result in a decrease in the predators. This trend continues as time goes on, implying a stable coexistence of the two populations.

The modified two dimensional Lotka-Volterra predator-prey model also uses a nonlinear system of equations that includes logistic growth of two species, a carrying capacity of the prey, and a predatory factor. The modified Lotka-Volterra predator-prey model is given by:

$$\begin{cases} \frac{dx}{dt} = x(1 - bx - y) \\ \frac{dy}{dt} = y(-c + x) \end{cases} \tag{1.1.3}$$

where b is the carrying capacity of the prey and c is the death rate of the predator.

In the modified model, we find that the populations start as a pair of oscillating curves, but over time the amplitude of the oscillations decrease with each period of time until the curves flatten out. This implies that the populations experienced a stable coexistence that saturates after sometime such that the populations will remain constant.

The Lotka-Volterra model indeed may be the simplest possible predator-prey model. It has been criticized as being unrealistic mainly for its structural instability and the assumption of the unlimited growth of the prey population x(t) in the absence of a predator. Nevertheless, it is a useful tool containing the basic properties of the real predator-prey dynamics, and serves as a robust basis from which it is possible to develop more sophisticated and realistic models.
The model (1.1.1) can be naturally generalized for the multi-species case. The generalization of the Lotka-Volterra model (1.1.1) for the multi-species case retains the basic features of real ecological systems and, allows us to obtain valuable results that are easy to be interpreted. For the three-species predator-prey interaction two possibilities arise: the two prey-one predator system

$$\begin{cases} \frac{dx_1}{dt} = x_1(a_1 - b_1 y) \\ \frac{dx_2}{dt} = x_2(a_1 - b_2 y) \\ \frac{dy}{dt} = y(-c + d_1 x_1 + d_2 x_2) \end{cases} \tag{1.1.4}$$

and a two predator-one prey system

$$\begin{cases} \frac{dx}{dt} = x(a_1 - b_1 y_1 - b_2 y_2) \\ \frac{dy_1}{dt} = y_1(-c_1 + d_1 x) \\ \frac{dy_2}{dt} = y_2(-c_2 + d_2 x) \end{cases} \tag{1.1.5}$$

Here $x_1,\ x_2$ are prey populations, $y_1,\ y_2$ are predator populations, and $a_i,\ b_i,\ c_i,\ d_i$ are positive constants. The equations (1.1.4) assume that the prey populations $x_1,\ x_2$ have not reached, and are not able to reach, their media capacity because of predation or for some other reasons, and that interspecies competition is absent or negligible. As a result of the artificial introduction of alien species there are some examples of such systems in New Zealand. The interaction of kiwi-rabbits-stoats is an example. The populations of both species are far from reaching their media capacity as a result of predation and artificial controlling of the rabbit population. Furthermore, both the kiwi and the rabbits have the same predators: stoats, cats, minks, or any others. Hence, the kiwi–rabbit–stoat interaction can be adequately described by the equations (1.1.4).

The equations (1.1.5) describe the situation when two predator species depend on a common prey and furthermore, these two predator species do not interact directly, they do not fight directly, do not predate one the other, and do not depend one on the other as a food source. Such situations are not unusual, and can arise in many different cases.

We consider a three species, the prey, predator and scavenger where the scavenger is a predator of the prey and scavenges the carcasses of the predator. There is the case where the scavenger has no negative effects on the population that it scavenges. A possible triple of such species are hyena/lion/antelope, where the hyena scavenges lion carcasses and preys upon antelope. But, in our study the scavenger affect the population it scavenges and also eaten by predator. An example of such a triple can be a lion, zebra and hyena, where the lion is considered as predator, zebra the prey and the hyena represents scavenger. To understand more let us see the diagram below.

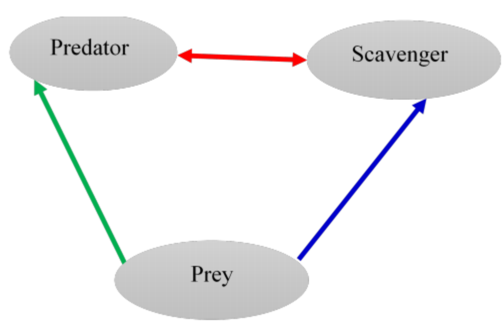

Figure 1.1: Schematic diagram for the dynamics of the prey, predator and scavenger in an ecosystem.

As we observe from the above diagram both species predator and scavenger use a variety of different resources from two trophic level, so we called as generalist.

Interaction among these three species can be observed by learning the population growth. To investigate these growth we develop the mathematical model which represent the dynamics of the prey, predator and scavenger. In the model we use functional response Holling type II to make the system more real. Then we determine equilibrium points, positivity and boundedness of the solution and also both local as well as global stability of the system.

The thesis is organized as follows. In chapter 2 we gather the review literature concerning the prey predator interaction with scavenger. In chapter 3 we develop and then analyse the model of three species system by using Holling type II functional response. We explain why it is important to analyse this three species predator-prey system and how we desire to analyze this system. The equations that model this system are explained in their relationship to biological realities. Under this chapter; we describe the assumption and parameter of model, we prove for the boundedness of the non-dimensionlized model, we find out the existence of the equilibrium points, local stability analysis for the equilibrium points is presented and global stability by constructing suitable Lyapunov function is presented. In chapter 4 we study numerical simulation in order to verify analytical calculations as well as to study the stability. Chapter 5 provides a result and discussion. We briefly review the results of the model analysis in relation to possible ecological situations.

1.2 Statement of the Problem

This research work will address the following basic questions;

- ✓ what are the assumptions to develop prey predator model with scavenger?
- ✓ what are the conditions for the co-existence of the three species in an ecosystem?
- ✓ what are the role of scavengers in stabilizing the ecosystem?
- ✓ does the extinction of one population lead to the extinction of the two remaining populations?

1.3 Objectives

1.3.1 General Objective of the Study

✓ The general objectives of this research work is to understand /or investigate mathematical modeling of the dynamics of the interaction between the three species prey, predator and scavenger.

1.3.2 Specific Objectives of the Study

The specific objectives of this study are to:

- ✓ develop a mathematical model which describes the interaction between the three species.
- ✓ determining positivity of the solutions and existence equilibrium points of the system.
- ✓ show boundedness of solution of the prey predator model with scavenger.
- ✓ determine the local and global stability analysis prey predator model with scavenger.
- ✓ investigate the conditions in which the three species coexist.
- ✓ study the conditions that one of the species extinct, and then study dynamics of the survive species.

1.4 Significance of the Study

Benefits of the Study are to:

- ✓ make awareness that scavengers have been neglected in population ecology, relative to herbivores, predators and omnivores.
- ✓ establish a mechanism for the co-existence of the populations.
- ✓ show effect of the scavenger population on the predator-prey population that it scavenges.

Beneficiaries of the Study

- ✓ the result from this research work will broaden the knowledge of scientific community by understanding dynamics of the prey, predator and scavenger populations as well as their coexistence.
- ✓ moreover the result obtained from this research work will be serves as a basis for further research on modeling populations with scavengers.
- ✓ policy makers.

1.5 Research Methodology

In this thesis we developed mathematical model which describes prey-predator model with scavenger using non-linear differential equations. Since non-linear differential equation are not solvable analytically, we solved the model numerically by using Runge-Kutta order 4^{th} and 5^{th} method. To develop the model we used functional response type II to show the interaction between them. In addition to this we included the assumption which states that the scavenger can be eaten by predator species. After we developed we studied the positivity and boundedness of the solutions. We also determined the local stability analysis of the model using Jacobian matrix, especially by using Routh-Hurwith method. Global stability analysis of the model is also determined by choosing a suitable Lyapunov function. Moreover, a number of numerical simulations was performed using the software a graphical Matlab package for the analysis of dynamical systems, in order to verify analytical calculations. Under this we showed the dynamics survive species when one or two of them extinct. We showed also the role of the scavenger species in stabilizing the prey predator ecosystem. Finally, we discussed the result of both local and global stability numerically.

Chapter 2

Review of the Literature

We start by giving a bit of history of the Lotka-Volterra model which was founded by Vito Volterra (1860-1940) and Alfred J. Lotka (1880-1949). The equations which model the struggle for existence of two species (prey and predators) bear the name of two scientists: Lotka and Volterra. They lived in different countries, had distinct professional and life trajectories, but they are linked together by their interest and results in mathematical modeling. Vito Volterra was an Italian mathematician who retired from a distinguished career in pure mathematics in the early 1920s. Vito Volterras son in law, Humberto DAncona, who was a biologist, studied populations of various species of fish in the Adriatic Sea.

The work of Lotka and Volterra overlapped in the discussion of predator- prey interaction. The problem was discussed by Lotka in 1920 and by Volterra in 1926, their conclusion being the same, that the interaction of the two species would give rise to periodic oscillation in their populations. Volterra acknowledged Lotkas priority, but he mentioned the differences in their papers. They even exchanged some respectful letters. In the case of the predator-prey interaction, the priority of Lotka was firmly established, and the equations with periodic solutions are called Lotka-Volterra equations. Volterra produced more general equations, for more than two species and considering also their interactions in the past.

Lotka (1925) from a hypothetical chemical reaction which he said could exhibit periodic behavior in the chemical concentrations. With this motivation the dependent variables represent chemical concentrations.

Volterra (1926) first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. If N (t) is the prey population and P(t) that of the predator at time t then Volterras model is

$$\frac{dN}{dt} = N(a - bP) \tag{2.0.1}$$

$$\frac{dP}{dt} = P(cN - d) \tag{2.0.2}$$

where a, b, c and d are positive constants. The model (2.0.1) and (2.0.2) is known as the LotkaVolterra model.

Parrish and Saila (1970) formulated a mathematical model of a two-prey- one-predator system, motivated primarily by the work of Paine (1966). They were not able to show stability in the three-species system with their numerical analyses, although, in some cases, the presence of a predator delayed the extinction of one prey species when compared to the case without predator.
The model discussed by Parrish and Saila (1970) consists of the following three differential equations using their original notation:

$$\frac{dN_1}{dt} = (\varepsilon_1 - \alpha_{11}N_1 - \alpha_{12}N_2 - \alpha_{13}N_3)N_1,$$

$$\frac{dN_2}{dt} = (\varepsilon_2 - \alpha_{21}N_1 - \alpha_{22}N_2 - \alpha_{23}N_3)N_2,$$

$$\frac{dN_3}{dt} = (-\varepsilon_1 - \alpha_{31}N_1 + \alpha_{32}N_2)N_3,$$

where N_1, N_1 and N_3 are the population sizes of two prey and one predator species, respectively, ε_i (i = 1, 2, 3) are the intrinsic rates of increase or decrease, and α_{ij} (i = 1, 2, 3; j = 1, 2, 3) are coefficients representing intra- specific and inter-specific checks on rates of increase. All the constants are considered to be strictly positive.

Cramer and May (1972) discussed the Parrish and Saila (1970) model from a more analytical viewpoint and showed that the presence of a predator may stabilize the system even when the same two-prey species system without the predator is unstable. They also gave the parametric condition which, when satisfied, produces the stable co-existence of three species, even though the two prey species cannot co-exist without a predator. In their paper, they examined the sufficient conditions for the stability around the equilibrium point of the two-prey-one-predator system, dealt with by Parrish and Saila (1970) and Cramer and May (1972) and also investigate the existence of diversity-stability relationship in two-prey-one-predator ecosystem on a global scale.

Kolmogoroffs theorem has been a very versatile tool for the qualitative analysis of two species interactions. It was used very successfully by Rescigno and Richardson (1973) and May (1974) in their analyses of two species interactions. Goh (1977) proposed a two- sided energy principle for the construction of Lyapunov functions for population models. This principle states that a viable single species population must on balance absorb energy at low densities and dissipate energy at high densities. In a viable multispecies community there must be a balance in the energy exchanges between the member species and between the species and their environment. The construction of biologically meaningful Lyapunov and Lyapunov-like functions (Goh, 1978) provides a new tool which is more versatile than Kolmogoroffs theorem.

Freedman (1980) came up with a generalised prey-predator model represented by the system of differential equations:

$$\dot{x} = xg(x) - yf(x)$$

$$\dot{y} = y[-e + p(x)],$$

where x and y are the densities of the prey and predator populations respectively, g(x) is the growth rate of the prey in absence of the predators, f(x) is the functional response of the predators with respect to prey x, and p(x) is the numerical response of the predator. In most cases, p(x) is a product of a constant and f(x). The functions g(x), f(x) and p(x) are continuous and differentiable.

Chaudhuri and Kar (2004) proposed and analysed a fishery model of a two prey-one predator system in which the prey were being harvested and the feeding rate of the predator increases linearly with prey density. They derived conditions for global stability of the system using a Lyapunov function. Using Pontryagin's maximal principal, they established the conditions for optimal harvest. However, the prey dependent-linear functional response used in their model does not represent the feeding patterns of most species as compared to Holling Type II or ratio-dependent responses which are used in our model.

Vlastmil and Eisner (2006) studied a one consumer-two resource population dynamics system in which the resource was spatially distributed between two patches. The study showed that when resources grow exponentially, handling times are zero and apparent competition always leads to extinction of the weaker resource. However, with logistic growth and Holling Type II functional response included in the model, species permanence was guaranteed. This showed the importance of incorporating logistic growth in prey-predator models.

Akugizibwe Edwin (2010) discussed modeling and analysis of a two prey-one predator system with harvesting, Holling Type II and ratio-dependent responses. He considered a prey predator model in which the lion is the predator specie while the Uganda Kob and buffalo are the prey species, where N_1(t), N_2(t) and P(t) represent the population of the Uganda Kobs, buffaloes and lions at any time t. The main feature of the model is that two different functional responses of the predator are incorporated in the model to represent the difference in the way the predator feeds on each of the prey species. According to his assumptions the Uganda Kob is the easy-to-capture prey and the predators response to the easy-to-capture prey is Holling Type II. The buffalo has adopted anti-predator behavior and is hard-to-capture prey and this behavior is represented by the ratio-dependent response of the predator. Constant effort harvesting of the prey is incorporated in the model to cater for the effects of human poaching on the Uganda Kobs. Terms representing logistic growth of the prey species in absence of the predator are included in the prey equations. It's model has three non linear autonomous ordinary differential equations describing how the population densities of the three species would vary with time, as given below.

$$\frac{dN_1}{dt} = r_1N_1(1 - \frac{N_1}{K_1}) - \alpha_1N_1N_2 - (\frac{a_1N_1}{1 + b_1N_1})P - EN_1$$

$$\frac{dN_2}{dt} = r_2N_2(1 - \frac{N_2}{K_2}) - \alpha_2N_1N_2 - (\frac{c_1N_1}{1 + d_1N_2 + d_2P})P \qquad (2.0.3)$$

$$\frac{dP}{dt} = \lambda_1(\frac{a_1N_1}{1+b_1N_1})P + \lambda_2(\frac{c_1N_1}{1+d_1N_2+d_2P})P - eP,$$

where all parameters in the model are positive.

Several authors; Ben Nolting et al (2008), Previte, J. P. and Hoffman, K.A. in (2010) and (2013), have studied the model with different assumptions in relation to the presence of the scavenger.

Ben Nolting et al(2008) analyzed a three species population model namely a predator, its prey and a scavenger. An equation for the scavenger population is introduced into the classical Lotka-Volterra equations. In their model the scavenger doesn't influence the predator and prey population. This assumption makes the mathematical analysis easier. That does not mean that the model is completely unrealistic because there are real life situations where scavengers have no impact on predators and preys. The authors also stated that with scavengers they meant carrion feeders such as hyenas, vultures, ravens etc. Conditions for persistence and extinction of the scavenger population are calculated.

Ben Nolting et al introduced a scavenger species, z, into the two species predator prey system. He assumed that the scavenger species, z, was die exponentially in the absence of any other species and will directly benefit in proportion to the number of deaths of x and y that occur naturally, as well as those caused by the predation of y on x (i.e. the kills). Its model is:

$$\begin{cases} \frac{dx}{dt} = ax - bxy, \\ \frac{dy}{dt} = dxy - cy, \\ \frac{dz}{dt} = fxyz + gxz + hyz - ez - iz^2 \end{cases} \tag{2.0.4}$$

where,

- e represents the natural death rate of the scavengers,
- f represents the benefit to the scavenger by scavenging carcases of the prey killed by the predator,
- g represents the benefit to the scavenger by scavenging carcases of the prey who die naturally,
- h represents the benefit to the scavenger by scavenging carcases of the predator who die naturally,
- i is associated with the carrying capacity of the scavenger.

Notice that the cubic interaction term makes this system a non-quadratic system (i.e., not a generalized Lotka-Volterra system). The quadratic term iz^2 is necessary, otherwise the scavenger population can grow without bound.

On the other hand Previte, J. P. and Hoffman, K.A. in (2010) discussed two scenarios of a predator prey model with the introduction of a third species, a scavenger of the prey. In the first scenario the scavenger is also a predator of the original prey and consumes the carcasses of the predator and benefits from both interactions, while in the second scenario the scavenger benefit from the interaction with the predator and inhibits the prey but has no benefit from that interaction.

In a later study Previte, J. P. and Hoffman, K.A. in (2013) introduced and studied a three species model with a scavenger added to a predator prey system, namely: the scavenger eats the prey and also consumes the carcasses of the predator. Their model has the biologically relevant property of bounded trajectories for positive initial conditions. Furthermore their model exhibits Hopf bifurcations, bistability, and chaos, which indicates the complexity of the dynamics due to adding a scavenger to a classical predator prey system.

In both cases Previte, J. P. and Hoffman, K.A. studied a third species z which is a predator of the prey x, and consumes the carcasses of the predator y, but has no direct inhibitory effects on the population of the predator. Scavenger species was studied where the scavenger had no effect on the predator or the prey. Thus, the predator and prey were independent of the scavenger. In their study, the scavenger species scavenges the predator, while also being a predator of the common prey species. They put the model that represent these three species are as follows:

$$\begin{cases} \frac{dx}{dt} = ax - bx^2 - cxy - dxz, \\ \frac{dy}{dt} = fxy - ey, \\ \frac{dz}{dt} = hxy + ixz - gz - jz^2 \end{cases} \tag{2.0.5}$$

where x is the prey, y is the predator, and z scavenges y and preys upon x. The parameters can be interpreted as follows:

- a is the net growth rate associated with x.
- b and j are implicitly related to the carrying capacities of x and z, respectively.
- c and d are the rates of change of the prey population in response to the presence of y and z, respectively.
- e and g are the natural death rate of y and z in the absence of prey.
- f and h are the rates of change in the predator and scavenger populations, respectively, in the presence of the prey x.
- i is the rate of change in the scavenger population z in the presence of the population y.

All parameters are assumed to be positive, and all variables non-negative.

In this research work we are going to develop the model proposed by Previte, J. P. and Hoffman, K.A. (2.0.5). Our model incorporates both the Holling type II and addition the assumption says that the scavenger can be eaten by predators, something none of the models above have done. Hence the existing model (2.0.5) is not enough to describe the real dynamics between prey, predator and scavenger population. Because, one species cannot get all of the other species. Additionally, as we discussed above no one of the authors studied the global stability of predator-prey model with a scavenger up to know. Therefore, in this paper first we will develop the mathematical model which represent the dynamics of this three species by using Holling type II functional response. Then we determine both local and global stability analysis for predator-prey model with a scavenger.

Chapter 3

Model Formulation and Analysis

In prey-predator system prey populations take various strategies for their defensive purpose or food resources and nature of dynamics of any prey-predator system depends on the terms of interaction. In this work, we interested to analyze the dynamics of prey-predator system with scavenger species and single predator population in which prey as well as scavenger populations are fighting for the same resources. The interaction will be happened between prey and scavenger species and also between prey and predator population.

In this chapter, the development of the mathematical model, its analysis and the findings will be discussed in order to study possible conditions for a stable coexistence and global stability analysis of the predator, prey and scavenger populations.

The development of the mathematical model consists of the construction of the equations based on the intended assumptions and the transformation of the system in the dimensionless form. In order to develop and analyse the model, a brief literature review was carried out, followed by a study of theorems and concepts of non-linear dynamical systems to achieve a clear understanding of:

i) how the equations should be written and the model be constructed;

ii) the importance of writing the model in the dimensionless form;

iii) the positivity and boundedness of the solutions;

iv) the equilibrium points and criteria for their existence;

v) the local stability analysis of equilibrium points;

vi) the global stability analysis of equilibrium points.

The analysis includes the calculation and determination of the global stability analysis of all equilibrium points. The analysis of prey predator with scavenger model also involves numerical simulations. The findings consist of an interpretation of the analytical and numerical outcomes.

3.1 Assumptions and Parameters of the Model

3.1.1 Assumptions of the Model

In formulation of mathematical model we assume the following basic assumptions:

- ✓ the prey will grow logistically in the absence of predators and scavenger population. The logistic growth model is illustrated by the term $AX(1 - \frac{B}{A}X)$.
- ✓ the rate of predation upon the prey is proportional to the encounters of predators and prey or the effect on prey population due to interaction with some of predator populations. This assumption is represented by the term $\frac{CXY}{A_1+X}$.
- ✓ the rate of predation upon the prey is proportional to the encounters of scavenger and prey or interaction between prey and scavenger populations. This assumption is represented by the term $\frac{DXZ}{A_1+X}$.
- ✓ predators will die out exponentially in the absence of prey and scavengers or natural death rates of predator populations. This assumption is represented by the term EY.
- ✓ the predator population increases, due to predation upon prey or benefit of predator population from prey population. The terms $\frac{FXY}{A_1+X}$ is represents this assumption.
- ✓ the predator population increases, due to predation upon scavenger i.e there is a predator population which eat scavenger population. The terms $\frac{GYZ}{A_3+Z}$ is represents this assumption.
- ✓ without predators and prey the scavengers will also almost goes to extinct. The terms that represent this assumption are HZ and IZ^2.
- ✓ the term HZ represent natural death rate of the scavenger and IZ^2 ensures that the interaction within scavenger species itself for the same resource.
- ✓ the scavenger population benefits from prey and predator that die naturally. The terms representing that are $\frac{JXZ}{A_1+X}$ and $\frac{KYZ}{A_2+Y}$ respectively.
- ✓ predators prey and scavengers will come across each other randomly in the environment.
- ✓ the populations live in a closed environment, so there will be neither immigration nor emigration and there will be no change benefiting a particular species.

3.1.2 Parameters of the Model

The parameters of model are presented in Table 2.1.

Table 3.1: **Parameters of the Model**

Parameter	Interpretation
A	natural growth rate of X
A_1	half saturation constants for Y and Z
A_2	half saturation constants for Z
A_3	half saturation constants for Y
B	interaction between prey population itself
C	effect on x due to predation of Y
D	rate changes on the X population in due to presence of S
E	natural death rate of Y
F	benefit to Y from X
G	benefit to Y from Z
H	natural death rate of Z
I	interaction between S population itself
J	benefit to S from X
K	benefit to S from Y

3.2 Description of the Model

A three species predator-prey model, where the third species are scavenger are considered in this paper. This model is constructed by assuming that there are only three species in such an isolated ecosystem. The first species, called prey, acting as the prey for the second and the third species. The second species called predator, feeds on the first and third species and can extinct with the absence of them. The third species, namely scavenger eat prey and the carcasses of predator. Consequently, the scavenger predation reduces the prey population and also affect the predator growth indirectly. Based on these assumptions, the mathematical model representing those three population dynamics is governed by a nonlinear system ordinary differential equations, namely

$$\begin{aligned}
\frac{dX}{dt} &= AX - BX^2 - \frac{CXY}{A_1 + X} - \frac{DXZ}{A_1 + X} \\
\frac{dY}{dt} &= \frac{FXY}{A_1 + X} + \frac{GYZ}{A_3 + Z} - EY \qquad (3.2.1) \\
\frac{dZ}{dt} &= \frac{JXZ}{A_1 + X} + \frac{KYZ}{A_2 + Y} - HZ - IZ^2
\end{aligned}$$

where, X(t),Y(t), and Z(t) stand for the dynamics of prey, predator, and scavenger populations, respectively. All parameters of model (3.2.1) are positive.

3.2.1 The Dimensionless form of the Model

A model can be transformed into a dimensionless form. That is rewriting the system in terms of dimensionless quantities. One of the advantages of a system in the dimensionless form is that the number of parameters is reduced to a minimum and it makes the analysis easier. Furthermore parameters can be better compared with each other, in terms of small and large and thus one gets more insight into the system. It is also possible to make a comparison between different systems. The dynamics of the new model will be the same as in the original system.
System (3.2.1) can be written as

$$\begin{aligned}
\frac{dX}{dt} &= AX(1-\frac{B}{A}X) - \frac{CXY}{A_1(1+\frac{X}{A_1})} - \frac{DXZ}{A_1(1+\frac{X}{A_1})} \\
\frac{dY}{dt} &= -EY + \frac{FXY}{A_1(1+\frac{X}{A_1})} + \frac{GYZ}{A_3(1+\frac{Z}{A_3})} \\
\frac{dZ}{dt} &= -HZ - IZ^2 + \frac{JXZ}{A_1(1+\frac{X}{A_1})} + \frac{KYZ}{A_2(1+\frac{Y}{A_2})}.
\end{aligned} \tag{3.2.2}$$

The variable changes are as follows:

$$Let \quad \tau = At,\ x = \frac{X}{A_1},\ y = \frac{Y}{A_2}\ and\ z = \frac{Z}{A_3}. \tag{3.2.3}$$

Substituting (3.2.3) in each equations of system (3.2.2), the dimensionless form of the system (3.2.1) becomes

$$\begin{aligned}
\frac{dx}{dt} &= x(1-\alpha_1 x) - \frac{\beta_1 xy}{1+x} - \frac{\theta_1 xz}{1+x} \\
\frac{dy}{dt} &= \frac{\alpha_2 xy}{1+x} + \frac{\beta_2 yz}{1+z} - \theta_2 y \\
\frac{dz}{dt} &= \frac{\alpha_3 xz}{1+x} + \frac{\beta_3 yz}{1+y} - \theta_3 z - \theta_4 z^2
\end{aligned} \tag{3.2.4}$$

where,

$\alpha_1 = \frac{A_1 B}{A}$; $\beta_1 = \frac{CA_2}{AA_1}$; $\theta_1 = \frac{DA_3}{AA_1}$; $\alpha_2 = \frac{F}{A}$; $\beta_2 = \frac{G}{A}$; $\theta_2 = \frac{E}{A}$; $\alpha_3 = \frac{J}{A}$; $\beta_3 = \frac{K}{A}$; $\theta_3 = \frac{H}{A}$; $\theta_4 = \frac{IA_3}{A}$ and τ is substituted by t for simplicity.

3.3 Positivity and Boundedness of the Solutions

3.3.1 Positivity of Solution

Theorem 3.1.1. All solutions of system (3.2.4) that start in R^3_+ remain positive forever if the initial conditions $x_0 > 0,\ y_0 > 0,\ z_0 > 0$.

Proof: We can write from equation (3.2.4)

$$x(t) = x_0 exp[\int_0^t (1 - \alpha_1 x(s) - \frac{\beta_1 y(s)}{1 + x(s)} - \frac{\theta_1 z(s)}{1 + x(s)})ds]$$

$$y(t) = y_0 exp[\int_0^t (\frac{\alpha_2 x(s)}{1 + x(s)} + \frac{\beta_2 z(s)}{1 + z(s)} - \theta_2)ds]$$

$$z(t) = z_0 exp[\int_0^t (\frac{\alpha_3 x(s)}{1 + x(s)} + \frac{\beta_3 y(s)}{1 + y(s)} - \theta_3 - \theta_4 z(s))ds].$$

Showing that $x(t) \geq 0$, $y(t) \geq 0$ and $z(t) \geq 0$ whenever $x_0 > 0$, $y_0 > 0$ and $z_0 > 0$. Hence all solutions remain within the first quadrant of the x-y-z plane starting from an interior point of it. Further we can easily establish that solution trajectories starting from $(x_0,\ 0)$ with $x_0 > 0$, remain within the positive x-axis at all future time and similar result holds for trajectories starting from a point on the positive y-axis and z-axis. Hence, $R^3_+ = \{(x, y, z) : x, y, z \geq 0\}$ is an invariant set.

3.3.2 Boundedness of Solution

Theorem 4.3.1. All solutions of the (3.2.4) that starts in R^3_+ are uniformly bounded if the initial conditions $x_0 > 0,\ y_0 > 0,\ z_0 > 0$.

Proof: From the first equation of the system (3.2.4), we have

$$\frac{dx}{dt} \leq x(1 - \alpha_1 x) \Rightarrow x(t) \ \leq \ \frac{1}{\frac{e^{-t}}{x_0} + \alpha_1}$$

for all t $\geqslant$ 0, which implies that x(t) <1 for sufficiently large t.

Let (x(t), y(t), z(t)) be any solution of the system (3.2.4) with positive initial conditions and define that

$$w = \alpha_2\alpha_3 x(t) + \beta_1\alpha_3 y(t) + \theta_1\alpha_2 z(t).$$

$$Then \qquad \frac{dw}{dt} = \alpha_2\alpha_3\frac{dx}{dt} + \beta_1\alpha_3\frac{dy}{dt} + \theta_1\alpha_2\frac{dz}{dt}.$$

Therefore,

$$\frac{dw}{dt} = \alpha_2\alpha_3[x(1-\alpha_1 x)] + \beta_1\alpha_3[\frac{\beta_2 z_0 y}{1+z_0} - \theta_2 y] + \theta_1\alpha_2[\frac{\beta_3 y_0 z}{1+y_0} - \theta_3 z - \theta_4 z^2].$$

Now, choose $\alpha_1 = 1,\ \beta_2 = \frac{1+z_0}{z_0}\ \ and\ \ \beta_3 = \frac{1+y_0}{y_0}$ then,

$$\frac{dw}{dt} \leq \alpha_2\alpha_3 x(1-x) + \beta_1\alpha_3 y + \theta_1\alpha_2 z.$$

Moreover,

$$\frac{dw}{dt} \leq \alpha_2\alpha_3 - \gamma(x+y+z)$$

where $\gamma = min\{-\alpha_2\alpha_3,\ \beta_1\alpha_3,\ \theta_1\alpha_2\} > 0$. This implies that

$$\frac{dw}{dt} + \gamma w \leq \alpha_2\alpha_3.$$

Applying the theory of differential inequality we obtain

$$0 < w < \frac{\alpha_2\alpha_3(1-e^{-\gamma t})}{\gamma} + w(x_0,\ y_0,\ z_0)e^{-\gamma t}.$$

For $t \longrightarrow \infty$, we have $0 < w < \frac{\alpha_2\alpha_3}{\gamma}$.
Hence all the solutions of the system (3.2.4) that initiate in R_3^+ are confined in the region

$$S = (x,\ y,\ z) \in R_3^+ : w = \frac{\alpha_2\alpha_3}{\gamma} + \eta, \quad for\ any\ \ \eta > 0,$$

which means that all species are uniformly bounded for any initial value in R_3^+.

According to the above theorem we assume that there exists real numbers $(\eta_1,\ \eta_2,\ \eta_3) > 0$ such that $\Omega(x_0,\ y_0,\ z_0) \subset R_3^+ = \{(x,\ y,\ z) : 0 \leqslant x \leq \eta_1, 0 \leq y \leq \eta_2,\ 0 \leq z \leq \eta_3\}\ \ for\ all\ (x_0,\ y_0,\ z_0) > 0$, where $\Omega(x_0,\ y_0,\ z_0)$ is the omega limit set of the orbit initiating at $(x_0,\ y_0,\ z_0)$.

Therefore, the system (3.2.4) is bounded. ■

3.4 Equilibrium Points and Criteria for Their Existence

In this section, we find equilibrium points and then provide some sufficient conditions that guarantee the existence of equilibrium points of system (3.2.4). The equilibrium points of the system in equation (3.2.4) can be found by setting all equations equal to zero and solving the system for x, y and z.

To find equilibrium points of the system, we solve the following system equations simultaneously

$$\frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\beta_1 xy}{1+x} - \frac{\theta_1 xz}{1+x} = 0$$

$$\frac{dy}{dt} = \frac{\alpha_2 xy}{1+x} + \frac{\beta_2 yz}{1+z} - \theta_2 y = 0 \tag{3.4.5}$$

$$\frac{dz}{dt} = \frac{\alpha_3 xz}{1+x} + \frac{\beta_3 yz}{1+y} - \theta_3 z - \theta_4 z^2 = 0.$$

That is equivalent to solving eight equilibrium points of the system (3.4.5), namely $E_0(0,\ 0,\ 0)$, $E_1(x^*,\ 0,\ 0)$, $E_2(0,\ 0,\ z^*)$ $E_3(x^*,\ y^*,\ 0)$, $E_4(0,\ y^*,\ z^*)$ $E_5(x^*,\ 0,\ z^*)$ and the co-existence equilibrium $E_6(x^*,\ y^*,\ z^*)$, To accommodate biological meaning, the existence conditions for the equilibria require that they are nonnegative.

i) The Trivial Equilibrium

It is obvious that the trivial equilibrium $E_0(0,\ 0,\ 0) = (0,\ 0,\ 0)$ always exist.

ii) The Predator and Scavenger-Free (i.e. Prey-only) Equilibrium

Let $y = 0$ and $z = 0$. Then equations (3.4.5) gives: $x(1-\alpha_1 x) = 0$; from which we have $x = \frac{1}{\alpha_1}$. Thus, $E_1(x^*,\ 0,\ 0) = (\frac{1}{\alpha_1},\ 0,\ 0)$. Therefore the equilibrium E_1 always exists. Hence, in the absence of predator y and scavenger z, the populations of prey x will increase until it reaches the carrying capacity α_1 of the prey population.

iii) The Scavenger-Free (i.e. Prey and Predator-only) Equilibrium

Let $z = 0$, then equations (3.4.5) gives $x(1-\alpha_1 x - \frac{\beta_1 y}{1+x}) = 0$ *and* $y(\frac{\alpha_2 x}{1+x} - \theta_2) = 0$. From these after solving simultaneously we have $x^* = \frac{\theta_2}{\alpha_2-\theta_2}$ if $\alpha_2 > \theta_2$ and $y^* = \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2)}{\beta_1(\alpha_2-\theta_2)^2}$ if $\alpha_2 > \theta_2$ *and* $\alpha_2 - \theta_2 > \alpha_1\theta_2$.

Thus

$$E_2(x^*,\ y^*,\ 0) = (\frac{\theta_2}{\alpha_2-\theta_2}, \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2)}{\beta_1(\alpha_2-\theta_2)^2},\ 0).$$

Therefore the equilibrium E_2 exists if,

$$\theta_2 < \alpha_2 \ \ and \ \ \alpha_1\theta_2 < \alpha_2 - \theta_2. \tag{3.4.6}$$

iv) The Prey-Free (i.e. Predator and Scavenger-only) Equilibrium

When $x = 0$, then equations (3.4.5) becomes

$$\frac{\beta_2 yz}{1+z} - \theta_2 y = 0 \tag{3.4.7}$$

$$\frac{\beta_3 yz}{1+y} - \theta_3 z - \theta_4 z^2 = 0.$$

Solving simultaneously we get

$$z^* = \frac{\theta_2}{\beta_2 - \theta_2} \quad and \quad y^* = \frac{\theta_3(\beta_2 - \theta_2) + \theta_2\theta_4}{(\beta_3 - \theta_3)(\beta_2 - \theta_2) - \theta_2\theta_4}.$$

Thus

$$E_3(0,\ y^*,\ z^*) = (0,\ \frac{\theta_3(\beta_2 - \theta_2) + \theta_2\theta_4}{(\beta_3 - \theta_3)(\beta_2 - \theta_2) - \theta_2\theta_4},\ \frac{\theta_2}{\beta_2 - \theta_2}).$$

Therefore the equilibrium E_3 exists if,

$$\theta_2 < \beta_2,\ \theta_3 < \beta_3 \ \ and \ \ \theta_2\theta_4 < (\beta_3 - \theta_3)(\beta_2 - \theta_2). \tag{3.4.8}$$

Hence for the existence of predator and scavenger populations in the absence prey the following has to be occurs.

✓ Natural death rate of predator populations is less than the benefits of predator from the interaction with some of the scavenger populations.

✓ Natural death rate of scavenger populations is less than benefits of scavenger populations from dead body of some predator populations.

v) The Predator-Free (i.e. Prey and Scavenger-only) Equilibrium

When $y = 0$, then equations (3.4.5) is reduced to

$$x(1 - \alpha_1)x - \frac{\theta_1 xz}{1+x} = 0 \tag{3.4.9}$$

$$\frac{\alpha_3 xz}{1+x} - \theta_3 z - \theta_4 z^2 = 0.$$

Solving simultaneously we get cubic equation,

$$P(x) = \alpha_1\theta_4 x^3 + (2\alpha_1\theta_4 - \theta_4)x^2 + (\alpha_3\theta_1 - 2\theta_4 + \alpha_1\theta_4 - \theta_3)x - \theta_3 - \theta_4.$$

Since P(x) has only one variation in sign changes, Descartes' Rule of Sign implies that there is precisely one positive real zeros. To determine the number of possible negative real zeros, replace x with $-x$ in P(x) to produce

$$P(-x) = -\alpha_1\theta_4 x^3 + (2\alpha_1\theta_4 - \theta_4)x^2 + (2\theta_4 - \alpha_3\theta_1 + \theta_3 - \alpha_1\theta_4)x - \theta_3 - \theta_4.$$

There are two variations of signs in $P(-x)$, so Descartes' Rule of Sign implies that there are either 2 or 0 negative zeros.

We will solve the equilibrium points E_4 by using numerical simulations.

vi) Positive Interior Equilibrium Point $\mathbf{E_5(x^*, y^*, z^*)}$

We solve equations (3.4.5) simultaneously

$$1 - \alpha_1 x - \frac{\beta_1 y}{1+x} - \frac{\theta_1 z}{1+x} = 0 \tag{3.4.10}$$

$$\frac{\alpha_2 x}{1+x} + \frac{\beta_2 z}{1+z} - \theta_2 = 0 \tag{3.4.11}$$

$$\frac{\alpha_3 x}{1+x} + \frac{\beta_3 y}{1+y} - \theta_3 - \theta_4 z = 0. \tag{3.4.12}$$

Now from 3.4.11 equation we have

$$x = \frac{\theta_2(1+z) - \beta_2 z}{(\alpha_1 - \theta_2)(1+z) + \beta_2 z}.$$

Again from 3.4.11 and 3.4.12 equation we get

$$y = \frac{\alpha_2\theta_4 z^2 + (\alpha_3\beta_2 + \alpha_2\theta_4)z + \alpha_2\theta_3 - \alpha_3\theta_2}{(\alpha_3 - \alpha_2\theta_4)z^2 + (\alpha_2\beta_2 - \alpha_3\beta_2 - \alpha_2\theta_4 - \alpha_2\theta_3)z + \alpha_2\beta_2 - \alpha_2\theta_3}.$$

Substituting these two values in one of (3.4.5) equation we get the value of z. But it is difficult to solve analytically, we use numerical simulation to solve.

3.5 Local Stability Analysis of the Equilibrium Points

The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues evaluated at each equilibrium point. We continue the analysis of the equilibria by computing the Jacobian J of the three-dimensional system given by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}$$

where $f(x,\ y,\ z) = \acute{x},\ g(x,\ y,\ z) = \acute{y}\ \ and\ \ h(x,\ y,\ z) = \acute{z}$. The Jacobian of system (3.2.4) is as follows:

$$J(E_i) = \begin{bmatrix} 1 - 2\alpha_1 x - \frac{\beta_1 y}{(1+x)^2} - \frac{\theta_1 z}{(1+x)^2} & -\frac{\beta_1 x}{1+x} & -\frac{\theta_1 x}{1+x} \\ \frac{\alpha_2 y}{(1+x)^2} & \frac{\alpha_2 x}{1+x} + \frac{\beta_2 z}{1+z} - \theta_2 & \frac{\beta_3 z}{(1+z)^2} \\ \frac{\alpha_3 z}{(1+x)^2} & \frac{\beta_3 z}{(1+y)^2} & \frac{\alpha_3 x}{1+x} + \frac{\beta_3 y}{1+y} - 2\theta_4 z - \theta_3 \end{bmatrix}. \tag{3.5.13}$$

The local asymptotic stability for each equilibrium point is analysed as below:

3.5.1 Local Stability of E_0

By finding the Jacobian of system (3.5.13) at the equilibrium point E_0 we have

$$J(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\theta_2 & 0 \\ 0 & 0 & -\theta_3 \end{bmatrix}. \tag{3.5.14}$$

Since $J(0,\ 0,\ 0)$ is a diagonal matrix, the eigenvalues are $1,\ -\theta_2\ \ and\ \ -\theta_3$. Because the eigenvalues are real and differ in sign, we have unstable at $(0,\ 0,\ 0)$.

3.5.2 Local Stability of E_1

The Jacobian matrix evaluated at E_1 gives

$$J(E_1) = \begin{bmatrix} -1 & -\frac{\beta_1}{1+\alpha_1} & -\frac{\theta_1}{1+\alpha_1} \\ 0 & \frac{\alpha_2}{1+\alpha_1} - \theta_2 & 0 \\ 0 & 0 & \frac{\alpha_3}{1+\alpha_1} - \theta_3 \end{bmatrix}. \tag{3.5.15}$$

The eigenvalues of matrix $J(E_1)$ are $-1,\ \frac{\alpha_2}{1+\alpha_1} - \theta_2\ \ and\ \ \frac{\alpha_3}{1+\alpha_1} - \theta_3$. Since $\mathrm{J}(\frac{1}{\alpha_1},\ 0,\ 0)$ is an upper triangular matrix. When $\theta_2 < \frac{\alpha_2}{1+\alpha_1}$ and $\theta_3 < \frac{\alpha_3}{1+\alpha_1}$, all of the eigenvalues are negative and, we have a stable node at $(\frac{1}{\alpha_1},\ 0,\ 0)$, thus E_1 is locally asymptotically stable if this condition satisfied. But if either of these conditions or both of these conditions do not hold, then the eigenvalues are real and differ in sign and the equilibrium point is unstable.

3.5.3 Local Stability of E_2

The Jacobian matrix evaluated at E_2

$$J(E_2) = \begin{bmatrix} \frac{\alpha_2\theta_2-\alpha_1\alpha_2\theta_2-\alpha_1\theta_2^2-\theta_2^2}{\alpha_2(\alpha_2-\theta_2)} & -\frac{\beta_1\theta_2}{\alpha_2} & -\frac{\theta_1\theta_2}{\alpha_2} \\ \frac{\alpha_2-\theta_2-\beta_1\theta_2}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{\alpha_2\beta_3(\alpha_2-\theta_2)-\alpha_1\alpha_2\theta_2\beta_3}{\beta_1(\alpha_2-\theta_2)^2+\alpha_2(\alpha_2-\theta_2)-\alpha_1\alpha_2\theta_2} + \frac{\theta_2\alpha_3}{\alpha_2} - \theta_3 \end{bmatrix}. \tag{3.5.16}$$

The characteristic equation is given by

$$\lambda^3 - (a_{11} + a_{33})\lambda^2 + (a_{11}a_{33} - a_{12}a_{21})\lambda + a_{12}a_{21}a_{33} = 0$$

where;

$$a_{11} = \frac{\alpha_2\theta_2 - \alpha_1\alpha_2\theta_2 - \alpha_1\theta_2^2 - \theta_2^2}{\alpha_2(\alpha_2 - \theta_2)}, \;\; a_{12} = -\frac{\beta_1\theta_2}{\alpha_2}, \;\; a_{13} = -\frac{\theta_1\theta_2}{\alpha_2}, \;\; a_{21} = \frac{\alpha_2 - \theta_2 - \beta_1\theta_2}{\beta_1} \;\; \textit{and}$$

$$a_{33} = \frac{\alpha_2^2\beta_3(\alpha_2-1)+(2\beta_1\alpha_2\theta_2-\beta_1\alpha_2^2)(\theta_3-\theta_2\alpha_3)+\alpha_1\alpha_2\theta_2(\alpha_3-\theta_2)(\theta_3-\alpha_2\beta_2)+\alpha_2\theta_2^2(\beta_1\theta_2-\beta_3)+\alpha_2\theta_2\alpha_3(\alpha_2-\theta_2)+\theta_2\theta_3(\alpha_2-\beta_1\theta_2)}{\beta_1\alpha_2(\alpha_2-\theta_2)^2+\alpha_2^2(\alpha_2-\theta_2)-\alpha_1\alpha_2\theta_3}.$$

Thus it is the form of, $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. Since $a_3 = 1$ which is positive then by Routh-Hurwitz criteria, the λ's are negative if, $a_2 > 0,\; a_0 > 0,\; a_2a_1 - a_0 > 0$. Each of these conditions are considered next as follows:

(a) $a_2 > 0 \quad \Longrightarrow \quad -(a_{11} + a_{33}) > 0 \quad \textit{or} \quad a_{11} + a_{33} < 0.$

This can be satisfied if, $a_{11} < 0$ and $a_{33} < 0$.

(i) $a_{11} < 0$ implies $\alpha_2\theta_2(1 - \alpha_1) - \theta_2^2(1 + \alpha_1) < 0$. This will hold if,

$$\alpha_1 < 1. \tag{3.5.17}$$

In terms of original parameters, (3.5.17) represents that $F < A$. This implies that, the benefits of predator populations from interaction with prey population is less than natural growth rates of prey population in the absence of predator and scavenger populations.

(ii) $a_{33} < 0$ implies $\alpha_2^2\beta_3(\alpha_2 - 1) + (2\beta_1\alpha_2\theta_2 - \beta_1\alpha_2^2)(\theta_3 - \theta_2\alpha_3) + \alpha_1\alpha_2\theta_2(\alpha_3 - \theta_2)(\theta_3 - \alpha_2\beta_2) + \alpha_2\theta_2^2(\beta_1\theta_2 - \beta_3) + \alpha_2\theta_2\alpha_3(\alpha_2 - \theta_2) + \theta_2\theta_3(\alpha_2 - \beta_1\theta_2) < 0$.

Therefore, $a_{33} < 0$ if,

$$\alpha_1 < 1,\; \theta_3 < \theta_2\alpha_3,\; \alpha_3 < \theta_2,\; \alpha_2\beta_2 < \theta_3,\; \beta_1\theta_2 < \beta_3,\; \alpha_2 < \theta_2 \;\; \textit{and} \;\; \alpha_2 < \beta_1\theta_2. \tag{3.5.18}$$

(b) $a_0 > 0 \quad \Longrightarrow \quad a_{12}a_{21}a_{33} > 0.$

This is satisfied if, for $a_{33} < 0$ and $a_{21} > 0$.

(i) Clearly $a_{12} = -\frac{\beta_1\theta_2}{\alpha_2}$ which is negative. And $a_{33} < 0$ if it satisfies the above condition (a) part (ii).

(ii) $a_{21} > 0$ implies $\alpha_2 - \theta_2(1+\alpha_1) > 0$. This will hold if,

$$\theta_2(1+\alpha_1) < \alpha_2. \tag{3.5.19}$$

(c) $a_2a_1 - a_0 > 0 \implies -(a_{11}+a_{33})(a_{11}a_{33} - a_{12}a_{21}) - a_{12}a_{21}a_{33} > 0$. This is satisfied if,

$$a_{11} < 0, \quad a_{12} < 0, \quad a_{33} < 0 \quad and \quad a_{11}a_{33} - a_{12}a_{21} < 0.$$

This all have been prior established.

Therefore, $\mathrm{E}_2(\frac{\theta_2}{\alpha_2-\theta_2}, \ \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2)}{\beta_1(\alpha_2-\theta_2)^2}, \ 0)$ is locally asymptotically stable if conditions (3.5.17), (3.5.18) and (3.5.19) hold.

3.5.4 Local Stability of E_3

The Jacobian matrix evaluated at E_3 gives

$$J(E_3) = \begin{bmatrix} 1 - \frac{\beta_1\theta_3(\beta_2-\theta_2)+\beta_1\theta_2\theta_4}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4} - \frac{\theta_1\theta_2}{\beta_2-\theta_2} & 0 & 0 \\ \frac{\alpha_2\theta_3(\beta_2-\theta_2)+\alpha_2\theta_2\theta_4}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4} & \frac{\theta_2^2}{\alpha_2-\theta_2} & \frac{\alpha_2\beta_3}{\beta_2} \\ \frac{\theta_2\alpha_3}{\beta_2-\theta_2} & \frac{\theta_2(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2^2\theta_4}{(\beta_2-\theta_2)^2} & -\frac{\theta_2\theta_4}{\beta_2-\theta_2} \end{bmatrix}. \tag{3.5.20}$$

The eigenvalues of $J(E_3)$ are obtained by solving

$$det \begin{bmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0$$

where;

$$a_{11} = \frac{(\beta_3-\theta_3)(\beta_2-\theta_2) + \theta_2\theta_4(\beta_1-1) + \beta_1\theta_3(\theta_2-\beta_2) + \theta_1\theta_2(\theta_2\theta_4+\theta_3-\beta_3)}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \quad a_{22} = \frac{\theta_2^2}{\alpha_2-\theta_2},$$

$$a_{21} = \frac{\alpha_2\theta_3(\beta_2-\theta_2)+\alpha_2\theta_2\theta_4}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \quad a_{23} = \frac{\alpha_2\beta_3}{\beta_2}, \quad a_{31} = \frac{\theta_2\alpha_3}{\beta_2-\theta_2}, \quad a_{33} = -\frac{\theta_2\theta_4}{\beta_2-\theta_2}$$

$$and \qquad a_{32} = \frac{\theta_2(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2^2\theta_4}{(\beta_2-\theta_2)^2}.$$

Now solving the above determinant gives

$$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) = 0.$$

So it simplifies to a characteristic equation,

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})\lambda - a_{11}a_{22}a_{33} = 0$$

which is again of the form, $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. By Routh-Hurwitz criteria the λ's are negative if, $a_2 > 0,\ a_0 > 0,\ a_2a_1 - a_0 > 0$. And we considered each of these conditions as follows:

(a) $a_2 > 0 \quad \Longrightarrow \quad -(a_{11} + a_{22} + a_{33}) > 0 \quad or \quad a_{11} + a_{22} + a_{33} < 0.$

This can be satisfied if, $a_{11} < 0$, $a_{22} < 0$ and $a_{33} < 0$.

(i) $a_{11} < 0$ implies $(\beta_3 - \theta_3)(\beta_2 - \theta_2) + \theta_2\theta_4(\beta_1 - 1) + \beta_1\theta_3(\theta_2 - \beta_2) + \theta_1\theta_2(\theta_2\theta_4 + \theta_3 - \beta_3) < 0$. This will hold if,

$$\beta_1 < 1, \quad \theta_2 < \beta_2, \quad \beta_3 < \theta_3 \quad and \quad \theta_2\theta_4 < \beta_3. \tag{3.5.21}$$

(ii) $a_{22} < 0$ implies $\frac{\theta_2^2}{\alpha_2 - \theta_2} < 0$. This will hold if,

$$\alpha_2 < \theta_2. \tag{3.5.22}$$

In terms of original parameters, (3.5.22)represents that $F < E$. This shows that, benefits of predator populations from some of prey populations is less than natural death rates of predator populations in the absence of other two species.

(ii) $a_{33} < 0$ implies $-\frac{\theta_2\theta_4}{\beta_2 - \theta_2} < 0$. This will hold if,

$$\theta_2 < \beta_2. \tag{3.5.23}$$

(b) $a_0 > 0 \quad \Longrightarrow \quad -a_{11}a_{22}a_{33} > 0 \quad or \quad a_{11}a_{22}a_{33} < 0.$
This is satisfied if, for $a_{11} < 0$, $a_{22} < 0$ and $a_{33} < 0$.

Therefore, if all conditions **(a)** are satisfied then $a_0 > 0$.

(c) $a_2a_1 - a_0 > 0 \quad \Longrightarrow \quad -(a_{11} + a_{22} + a_{33})(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}) + a_{11}a_{22}a_{33} > 0.$

This is satisfied if $a_{11} < 0$, $a_{22} < 0$ and $a_{33} < 0$ which have been prior established.

Therefore, E_3 is locally asymptotically stable if conditions (3.5.21), (3.5.22) and (3.5.23) hold.

3.5.5 Local Stability of E_4

By finding the Jacobian of system (3.5.13) at the equilibrium point $\mathrm{E}_4(x^*,\ 0,\ z^*)$ we have

$$J(E_4) = \begin{bmatrix} \frac{(1+x^*)^2-2\alpha_1 x^*(1+x^*)^2-\theta_1 z^*}{(1+x)^{*2}} & -\frac{\beta_1 x^*}{1+x^*} & -\frac{\theta_1 x^*}{1+x^*} \\ 0 & \frac{\alpha_2 x^*(1+z^*)+\beta_2 z^*(1+x^*)-\theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)} & \frac{\beta_3 z^*}{(1+z^*)^2} \\ \frac{\alpha_3 z^*}{(1+x^*)^2} & \beta_3 z^* & \frac{\alpha_3 x^*-2\theta_4 z^*(1+x^*)-\theta_3(1+x^*)}{1+x^*} \end{bmatrix}. \tag{3.5.24}$$

The eigenvalues of $J(E_4)$ are obtained by solving

$$det \begin{bmatrix} a_{11}-\lambda & -a_{12} & -a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0$$

where; $a_{11} = \frac{(1+x^*)^2-2\alpha_1 x^*(1+x^*)^2-\theta_1 z^*}{(1+x)^{*2}}$, $a_{12} = \frac{\beta_1 x^*}{1+x^*}$, $a_{13} = \frac{\theta_1 x^*}{1+x^*}$, $a_{22} = \frac{\alpha_2 x^*(1+z^*)+\beta_2 z^*(1+x^*)-\theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}$, $a_{31} = \frac{\alpha_3 z^*}{(1+x^*)^2}$, $a_{32} = \beta_3 z^*$, $a_{33} = \frac{\alpha_3 x^*-2\theta_4 z^*(1+x^*)-\theta_3(1+x^*)}{1+x^*}$.
The characteristic equation becomes

$$\lambda^3-(a_{11}+a_{22})\lambda^2+(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{13}a_{31})\lambda+a_{12}a_{23}a_{31}+a_{22}a_{13}a_{31}-a_{11}a_{22}a_{33}=0$$

which is the form, $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. By Routh-Hurwitz criteria the λ's are negative if, $a_2 > 0,\ a_0 > 0,\ a_2 a_1 - a_0 > 0$. And we considered each of these conditions as follows:

(i) $a_2 > 0 \implies -(a_{11}+a_{22}+a_{33}) > 0 \ \ or \ \ a_{11}+a_{22}+a_{33} < 0.$

(ii) $a_0 > 0 \implies a_{12}a_{23}a_{31} + a_{22}a_{13}a_{31} > a_{11}a_{22}a_{33}.$

(iii) $a_1 a_2 - a_0 > 0 \implies -(a_{11}+a_{22})(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{13}a_{31}) + a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{22}a_{13}a_{31} > 0.$

Therefore, E_4 is locally asymptotically stable if conditions (i), (ii) and (iii) hold.

3.5.6 Local Stability of E_5

Similar to E_4 after solving Jacobian matrix of system (3.5.13) at the equilibrium point $\mathrm{E}_5(x^*,\ y^*,\ z^*$, the eigenvalues o are obtained by solving

$$det \begin{bmatrix} a_{11}-\lambda & -a_{12} & -a_{13} \\ a_{12} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0$$

where; $a_{11} = \frac{(1+x^*)^2 - 2\alpha_1 x^*(1+x^*)^2 - \beta_1 y^* - \theta_1 z^*}{(1+x)^{*2}}$, $a_{12} = \frac{\beta_1 x^*}{1+x^*}$, $a_{13} = \frac{\theta_1 x^*}{1+x^*}$, $a_{21} = \frac{\alpha_2 y^*}{(1+x^*)^2}$, $a_{22} = \frac{\alpha_2 x^*(1+z^*) + \beta_2 z^*(1+x^*) - \theta_2(1+x^*)(1+z^*)}{(1+x^*)(1+z^*)}$, $a_{23} = \frac{\beta_3 z^*}{(1+z^*)^2}$, $a_{31} = \frac{\alpha_3 z^*}{(1+x^*)^2}$, $a_{32} = \frac{\beta_3 z^*}{(1+x^*)^2}$, $a_{33} = \frac{\alpha_3 x^* - 2\theta_4 z^*(1+x^*) - \theta_3(1+x^*)}{1+x^*}$.

The characteristic equation becomes

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{12}a_{21})\lambda + a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} = 0$$

which is the form, $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$. By Routh-Hurwitz criteria the λ's are negative if, $a_2 > 0,\ a_0 > 0,\ a_2a_1 - a_0 > 0$. And we considered each of these conditions as follows:

(i) $a_2 > 0 \implies -(a_{11} + a_{22} + a_{33}) > 0 \ \ or \ \ a_{11} + a_{22} + a_{33} < 0.$

(ii) $a_0 > 0 \implies a_{12}a_{21}a_{33} > a_{11}a_{22}a_{33}.$

(iii) $a_1a_2 - a_0 > 0 \implies -(a_{11} + a_{22} + a_{33})(a_{11} + a_{22} + a_{33}) + a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} > 0.$

Therefore, E_5 is locally asymptotically stable if conditions (i), (ii) and (iii) hold.

3.6 Global Stability Analysis of the Equilibrium Points

It is a well-known fact that the local stability of an equilibrium point in a system of ordinary differential equations does not necessarily imply its global stability. An important technique in stability theory for differential equations is known as the direct method of Liapunov. A function with particular properties known as a Liapunov function is constructed to prove stability or asymptotic stability of an equilibrium point in a given region. The construction of Liapunov functions is often difficult for particular systems, but for Lotka-Volterra systems, there has been some success.

In section 3.5 we discussed the local stability of all four equilibrium points. From which E_1, E_2 and E_3 are locally asymptotically stable conditionally and E_0 locally asymptotically unstable. We now examine the global stability of dynamical system (3.2.4) at E_2 by suitable Liapunovs functions.

Theorem (3.6.1) The equilibrium point $E_2(x^*,\ y^*,\ 0)$ is globally asymptotically stable.

proof: Let us consider the following Liapunovs function

$$V(x,\ y,\ z) = x - x^* - x^* \ln(\frac{x}{x^*}) + y - y^* - y^* \ln(\frac{y}{y^*}) + \frac{z^2}{2}.$$

Now, the time derivative of V, along solutions of (3.2.4) can be written as

$$\frac{dV}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + \frac{y - y^*}{y}\frac{dy}{dt} + z\frac{dz}{dt}$$

$$= (x - x^*)(1 - \alpha_1 x - \frac{\beta_1 y}{1+x}) + (y - y^*)(\frac{\alpha_2 x}{1+x} - \theta_2)$$

which is simplified to

$$\frac{dV}{dt} = -\alpha_1 (x - x^*)^2 - (\beta_1 - \alpha_2)\frac{(x - x^*)(y - y^*)}{(1+x)(1+x^*)} - \beta_1 \frac{(x - x^*)(xy - x^* y^*)}{(1+x)(1+x^*)}$$

which is negative definite, when

$$\alpha_2 < \beta_1. \tag{3.6.25}$$

Therefore, with these condition, $E_2(x^*,\ y^*,\ 0)$ is globally asymptotically stable. $\square$

Chapter 4

Numerical Simulation

In this work, we are interested to analyze the dynamics of prey-predator system with scavenger population. Beside analytical findings, numerical simulations are also important; because simulations can be used to validate the analytical findings. Numerical computing environments such as MATLAB are not intended to solve differential equations models symbolically. In other words, given a differential equation such as $\frac{dx}{dt} = f(t, x)$, one should not expect to get a solution that is an equation of the form x = g(t). We will focus on one of its most rudimentary solvers, ode45, which implements a version of the Runge-Kutta 4^{th} and 5^{th} order algorithm. In this section we will allow for one or more than two populations where they depend on each other. One population could be the prey such as zebra, predator such as a lion, and the third could be the scavenger such as a hyenas. The populations will be modeled by three differential equations. The Matlab command ode45 can be used to solve such systems of differential equations. Let us start with the dynamics of the survive populations when one or two of the species dies out.

4.1 Dynamics of Prey Population

The predator species is totally dependent on the prey species as its only food supply. The prey species has enough food supply and no threat to its growth other than the specific predator and scavenger. Without predator and scavenger population the dynamics of prey population is given by the following logistic growth model:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}). \tag{4.1.1}$$

If there were no predator and scavenger the prey population grow logistically i.e if x = x(t) is the size of prey population at time t.

4.2 Dynamics of Prey and Predator Populations

If the scavenger population dies out, the remaining model reverts back to the classical prey predator model. The only difference is functional response included between them.

The model which describes their dynamics are

$$\frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\beta_1 xy}{1+x}$$

$$\frac{dy}{dt} = \frac{\alpha_2 xy}{1+x} - \theta_2 y. \tag{4.2.2}$$

By using $\alpha_1 = 0.1,\ \alpha_2 = 0.25,\ \beta_1 = 0.3\ \ and\ \ \theta_2 = 0.2$ we can plot the above system of equation (4.2.2) as follows.

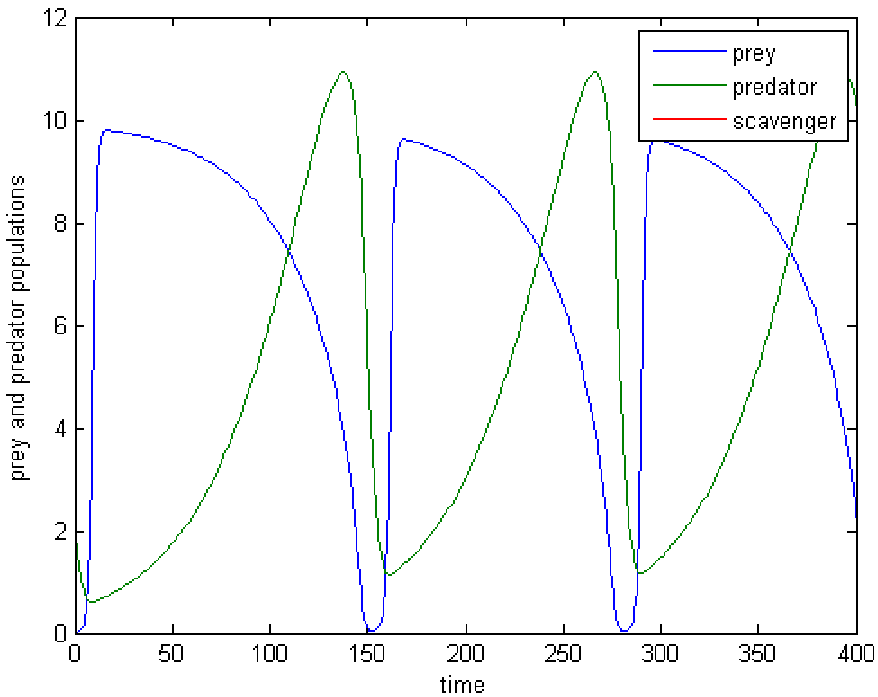

Figure 4.1: Dynamics of prey and predator without scavenger populations

In the initial due to less number of prey species the predator species fails downward and prey species dominated for a low rate. The prey abundances are not strongly positively correlated then as one prey species becomes scarce, the predator can continue to feed and increase its population size. As we observe from the above figure when the predator population decreases the prey population conversely increased and vise versa.

4.3 Dynamics of Predator and Scavenger Populations

In the absence of prey population the system of differential equation which describes the dynamics between predator and scavenger populations is given by

$$\frac{dy}{dt} = \frac{\beta_2 yz}{1+z} - \theta_2 y$$

$$\frac{dz}{dt} = \frac{\beta_3 yz}{1+y} - \theta_3 z - \theta_4 z^2. \tag{4.3.3}$$

One thing that differentiate this paper from other is the benefits of predator species from scavenger species are included. Now we see their dynamics without prey species by using $\alpha_2 = 0.25,\ \beta_3 = 0.6,\ \theta_3 = 0.3\ \ and\ \ \theta_4 = 0.1.$

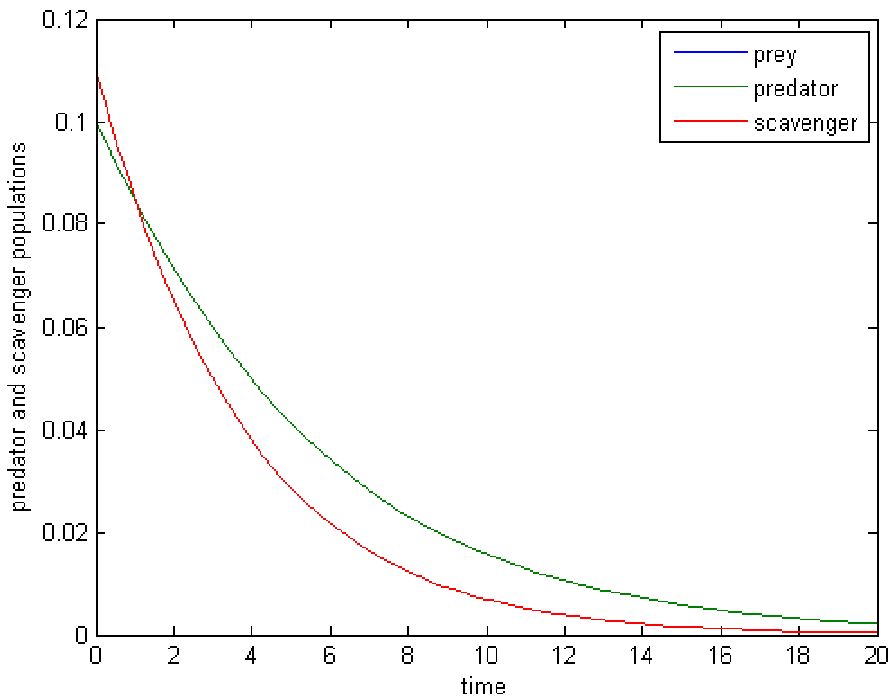

Figure 4.2: Dynamics of predator and scavenger without prey populations

As we observe from the graph without prey population predator as well as scavenger population can exist only for a few time. The scavenger benefit more from interaction between prey and predator, but here the only its source is natural death rates of predator. The predator population benefits from the interaction with scavenger population but not enough for them. In the case of initial both species almost the same and the scavenger species dominates the predator species. But, after a low rates the predator species dominates the scavenger species always, even-though their population decreases from time to time. Furthermore after a long period of time both species almost goes to extinct. From this we conclude that predator and scavenger populations cannot live without prey population for long time.

4.4 Dynamics of Prey and Scavenger Populations

Similarly, the non-linear differential equation system which shows the dynamics between prey and scavenger populations without predator population is represented as follows,

$$\frac{dx}{dt} = x(1-\alpha_1 x) - \frac{\theta_1 xz}{1+x}$$

$$\frac{dz}{dt} = \frac{\alpha_3 xz}{1+x} - \theta_3 z - \theta_4 z^2. \tag{4.4.4}$$

We use the parameters $\alpha_1 = 0.1,\ \theta_1 = 0.1,\ \alpha_3 = 0.01,\ \theta_3 = 0.3,\ \theta_4 = 0.1$ to show the dynamics of two species.

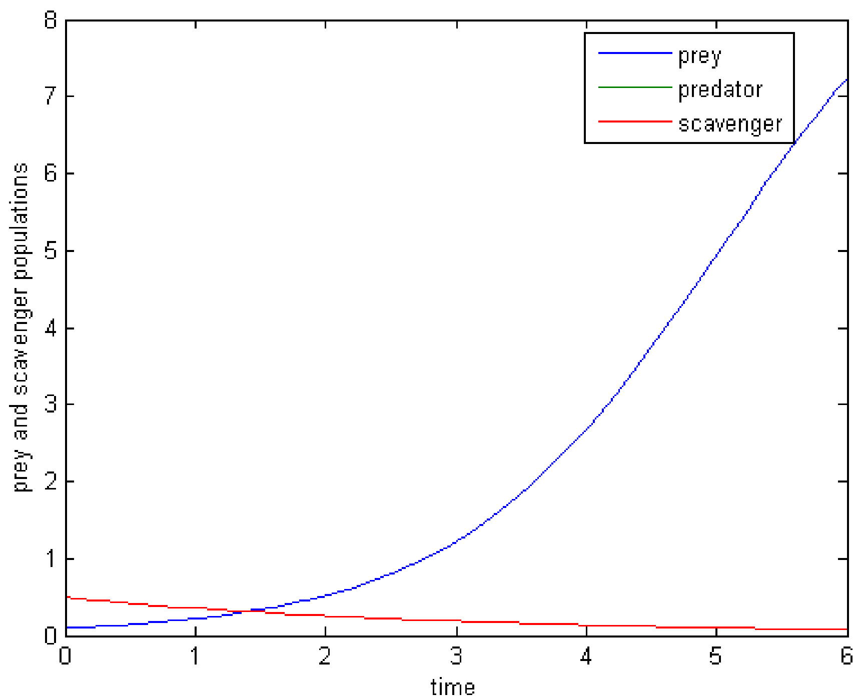

Figure 4.3: Dynamics of prey and scavenger without predator populations

As we explain in the previous chapters scavenger is animal that predates the prey and eat the carcases of predator. We are considering the case of no predator extinct but prey population nothing benefit from the scavenger. As we observe from the graph the scavenger population declines downward as time increases. Because scavenger population get their food more from carcases or death due to predator population. Only some of the scavengers get their food from eating prey population. As a result the number of scavenger population becomes fails downward without predator population. Conversely, if see we the prey population in the absence of predator, their number steep rise up from time to time. Even-though some of scavenger affect prey population but they cannot stop the prey population to grow. As a result, prey species dominates scavenger species within low rate, and the above figure also shows this.

4.5 Coexistence of Three Populations

One of the most important question in mathematical biology is concerns the long term survival or coexistence of all the species in multi-species community. So, to show the coexistence of prey, predator and scavenger population we shall use the parameters $\alpha_1 = 0.1,\ \alpha_2 = 0.25,\ \alpha_3 = 0.01,\ \beta_1 = 0.3,\ \beta_2 = 0.35,\ \beta_3 = 0.003,\ \theta_1 = 0.04,\ \theta_2 = 0.21,\ \theta_3 = 0.0035,\ \theta_4 = 0.1$ and initial point $(x_0,\ y_0,\ z_0) = (15.2,\ 4.6,\ 10.4)$.

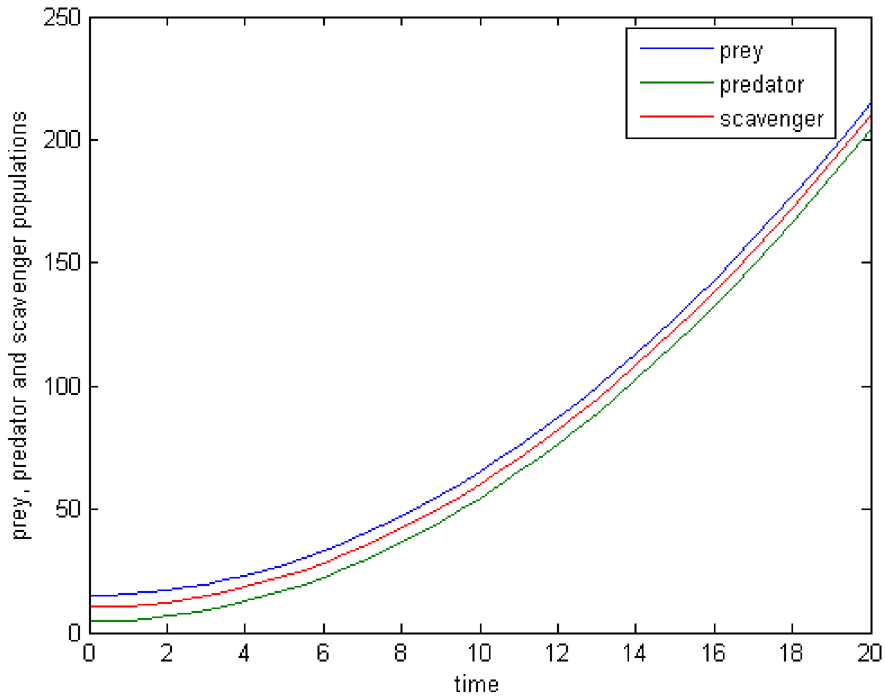

Figure 4.4: Coexistence of three species

The mechanism that prey, predator and scavenger species coexist are if low predation occur in an ecosystem. Further more, three species become coexist if large number of prey population computes with less number of predator and average number of scavenger populations. As their figure shows all three species increases constantly. Even-though both predator and scavenger depends on prey species due to large number of prey species their number rise up constantly. Additionally, the main thing that maintain their coexistence is both predator and scavenger species cannot get all of the prey species. The scavenger species benefits from the large number prey species as well as from their dead bodies results to increase constantly from time to time. The predator species also gets enough food from other two and their population flows the prey and scavenger species.

4.6 Equilibrium Points

In the previous section we solved four of equilibrium point analysis. Now for these and left two we determine the equilibrium points numerically. We start by setting values of parameters in (3.2.4). Let $\alpha_1 = 0.1,\ \alpha_2 = 0.25,\ \alpha_3 = 0.01,\ \beta_1 = 0.3,\ \beta_2 = 0.4,\ \beta_3 = 0.6,\ \theta_1 = 0.5,\ \theta_2 = 0.2,\ \theta_3 = 0.3,\ \ and\ \ \theta_4 = 0.01.$

i) Trivial equilibrium point $E_0 = (0,\ 0,\ 0)$.

ii) $E_1 = (\frac{1}{\alpha_1},\ 0,\ 0) = (10,\ 0,\ 0)$.

iii) $E_2 = (\frac{\theta_2}{\alpha_2-\theta_2}, \frac{\alpha_2((\alpha_2-\theta_2)-\alpha_1\theta_2)}{\beta_1(\alpha_2-\theta_2)^2}, 0) = (4,\ 10,\ 0)$.

iv) $E_3 = (0, \frac{\theta_3(\beta_2-\theta_2)+\theta_2\theta_4}{(\beta_3-\theta_3)(\beta_2-\theta_2)-\theta_2\theta_4}, \frac{\theta_2}{\beta_2-\theta_2}) = (0,\ 2,\ 1)$.

v) $E_4 = (x^*, 0, z^*)$

In previous section we made difficulty to solve this equilibrium point analytically. Now easily by numerical

$$E_4(x^*, 0, z^*) = (1.9854,\ 0,\ 7.2185).$$

vi) Positive interior equilibrium point $\mathbf{E_5(x^*,\ y^*,\ z^*)}$

Similar to E_4 it is difficult to solve analytically, but by using MATLAB computer program

$$E_5(x^*, y^*, z^*) = (0.0321,\ 1.8874,\ 0.9251).$$

4.7 Application of Stability Analysis

The recent decades have witnessed development of theory of non linear dynamical systems having applications in various fields including population biology. Due to the complexity in analyzing non linear equations, a comprehensive theory of dynamics of non linear systems still awaits development. An important technique for analyzing nonlinear systems qualitatively is the analysis of the behavior of the solutions near equilibrium points using linearization. One of the most urgent problems demanding the attention of researchers is the stability of ecosystems.

In this section, we present the time plots of prey, predator and scavenger and phase portraits diagrams for the system (3.2.4) obtained in previous sections. The numerical experiments are designed to show the rich complex dynamical behavior of the system with different set of parameters. We need the following Lemma to discuss the stability of the equilibrium points of (3.2.4).

Lemma 1. Let

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{4.7.5}$$

be the characteristic equation for a matrix defined by (3.5.13). Then the following statements are true:

1. If every root of equation (4.7.5) has less than one, then the equilibrium point of the system (3.2.4) is locally asymptotically stable and equilibrium point is called a sink.
2. If at least one of the roots of equation (4.7.5) has absolute value greater than one, then the equilibrium point of the system (3.2.4) is unstable and equilibrium point is called a saddle.
3. If every root of equation (4.7.5) has absolute value greater than one, then the equilibrium point of the system (3.2.4) is a source.

Using the lemma, we have the following results for the system (3.2.4).

4.7.1 Local Stability of $\mathbf{E}_0$

We shall use the parameters $\alpha_1 = 0.1,\ \alpha_2 = 0.35,\ \alpha_3 = 0.01,\ \beta_1 = 0.3,\ \beta_2 = 0.4,\ \beta_3 = 0.06,\ \theta_1 = 0.5,\ \theta_2 = 0.2,\ \theta_3 = 0.3\ ,\theta_4 = 0.01\ and\ \ E_0(0.2, 0.1, 0.5)$. Then substituting in Jacobian matrix (3.5.14) this set of parameter values the eigenvalues becomes $\lambda_1 = 1,\ \lambda_2 = -0.2\ \ and\ \ \lambda_3 = -0.3$. Therefore E_0 is unstable. Now by using above parameters we show their graph as follows

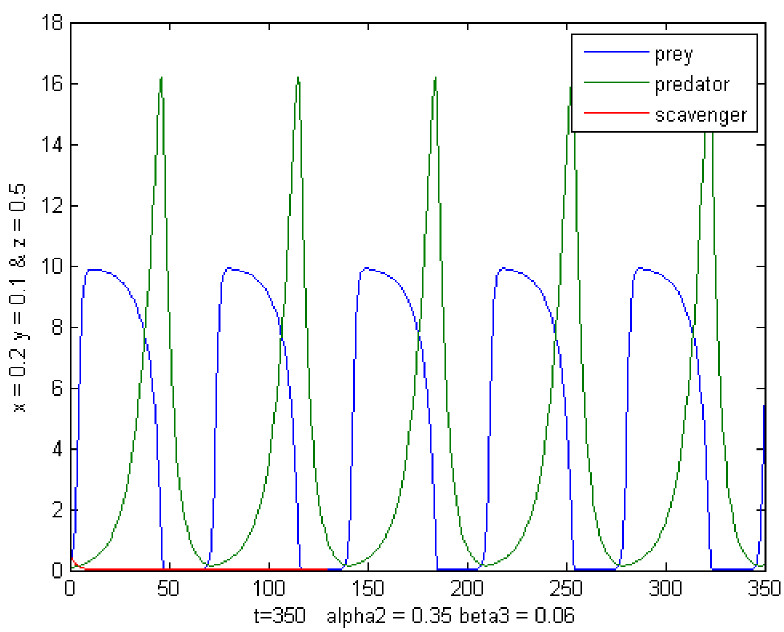

Figure 4.5: Time series plot for unstable equilibrium point E_0

4.7.2 Local Stability of $\mathbf{E_1}$

For the stability of E_1 we shall use $\beta_3 = 0.006,\ \theta_2 = 0.25,\ \theta_3 = 0.3$ and other are as before. Similarly substituting these values of parameters in Jacobian matrix (3.5.15), we get the eigenvalues $\lambda_1 = -1,\ \lambda_2 = -0.023\ \ and\ \ \lambda_3 = -0.29$ and $E_1(9.7,\ 0.5,\ 0.6)$. Hence, with above parameters E_1 is locally asymptotically stable and their graph is given in figure 4.6.

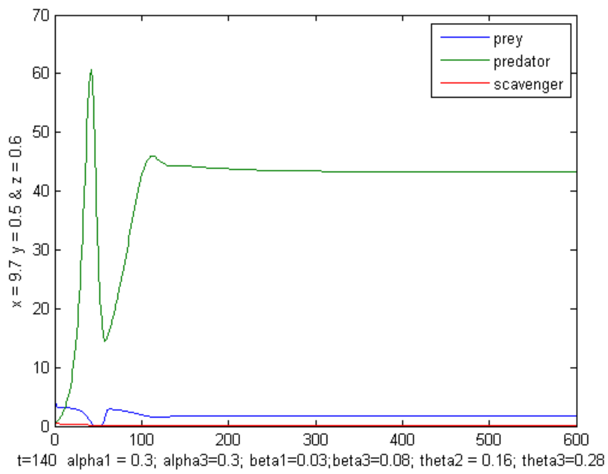

Figure 4.6: Time series plot for stable equilibrium point E_1

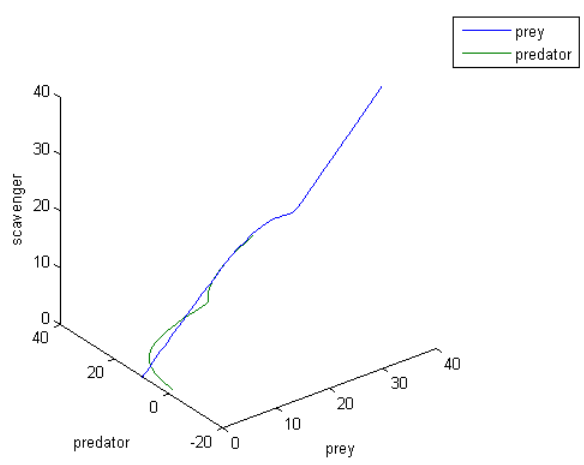

Figure 4.7: Phase Portrait at stable equilibrium point E_1

4.7.3 Local Stability of $\mathbf{E}_2$

The stability of E_2 depends on the the conditions (3.5.17), (3.5.18) and (3.5.19) as we explain in previous chapter. Depending on this conditions we set the parameter values as $\alpha_1 = 0.01$, $\alpha_2 = 0.025$, $\beta_1 = 0.1$, $\beta_3 = 0.2$, $\theta_2 = 0.01$. With this parameter values $E_2(4.1, 10.02, 0.5)$ and the corresponding eigenvalues are $\lambda_1 = -0.8314$, $\lambda_2 = -0.3756$ *and* $\lambda_3 = -0.0047$.

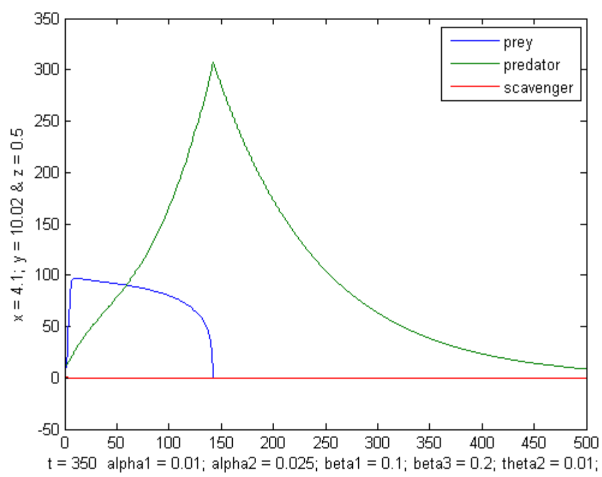

Figure 4.8: Time series plot for Stability at equilibrium point E_2

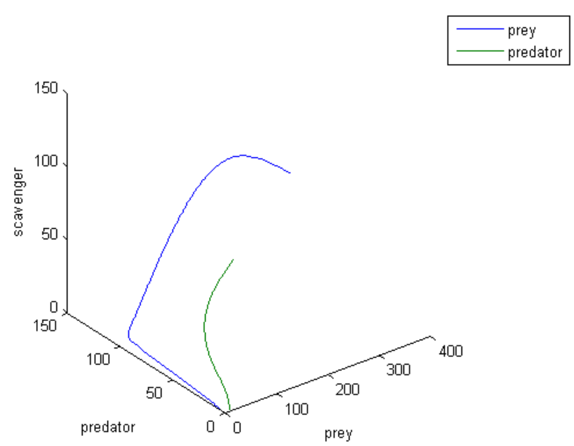

Figure 4.9: Phase Portrait at stable equilibrium point E_2

4.7.4 Local stability of E_3

We shall consider the values $\alpha_2 = 0.02$, $\beta_2 = 0.35$, $\beta_3 = 0.003$, $\theta_1 = 0.04$, $\theta_3 = 0.035$ in order to satisfies conditions (3.5.21), (3.5.22) and (3.5.23). With these set parameter values we obtain $E_3(0.4,\ 1.7,\ 1.6)$ and the the corresponding eigenvalues are $\lambda_1 = -0.1086$, $\lambda_2 = -0.1703 + 0.4365i$ *and* $\lambda_3 = -0.1703 - 0.4365i$. Therefore, with these sets of parameter values equilibrium point E_3 is locally asymptotically stable. From the time plot and phase portrait the trajectory towards to the E_3, see figure below

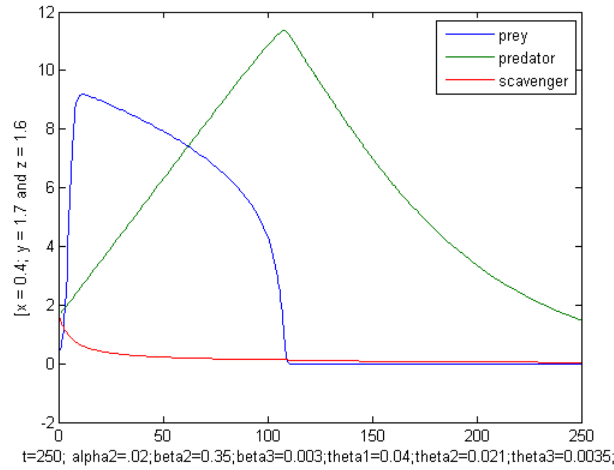

Figure 4.10: Time series plot for Stability at equilibrium point E_3

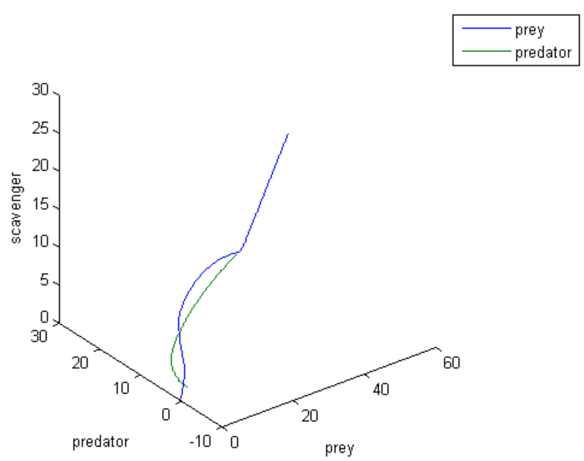

Figure 4.11: Phase Portrait at equilibrium point E_3

4.7.5 Local Stability of E_4

Let $\alpha_1 = 0.1,\ \alpha_2 = 0.25,\ \alpha_3 = 0.01,\ \beta_1 = 0.3,\ \beta_2 = 0.4,\ \beta_3 = 0.6,\ \theta_1 = 0.5,\ \theta_2 = 0.05,\ \theta_3 = 0.06,\ \ and\ \ \theta_4 = 0.01.$
Then Jacobian matrix evaluated at E_4 gives

$$J(E_4) = \begin{bmatrix} 0.198 & -0.1997 & -1.209 \\ 0 & 0.3178 & 0.0641 \\ 0.008 & 4.3311 & -1.737 \end{bmatrix}.$$

Then the characteristic polynomial becomes

$$\lambda^3 + (1.0103)\lambda^2 + (0.119)\lambda + 0.0735.$$

The eigenvalues are more complex, so we will use the Routh-Hurwitz method.
The Routh-Hurwitz method used for a third-degree polynomial

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

can be applied to our system, where

$$a_3 = 1,\ a_2 = 1.0103,\ a_1 = 0.119,\ \ and\ \ a_0 = 0.0735.$$

Now the eigenvalues are negative if, $a_2 > 0,\ a_0 > 0,\ a_2a_1 - a_0 > 0$. And we considered each of these conditions as follows:

i) Clearly $a_2 = 0.6268$ and $a_0 = 0.0683$ which are non-negative.

ii) $a_2a_1 - a_0 = 0.0467$ which is again positive.

Therefore all of the first column of the Routh array is positive and the corresponding eigenvalues are $\lambda_1 = -0.9659,\ \lambda_2 = -0.0222 + 0.275i\ \ and\ \ \lambda_3 = -0.0222 - 0.275$. Then the equilibrium point $E_4(x^*,\ 0,\ z^*)$ is locally asymptotically stable. Figure 4.12 and 4.13 are also shows this.

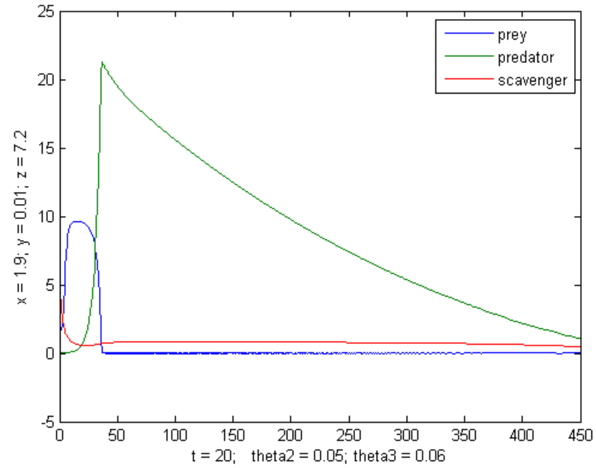

Figure 4.12: Time series plot for stable equilibrium point E_4

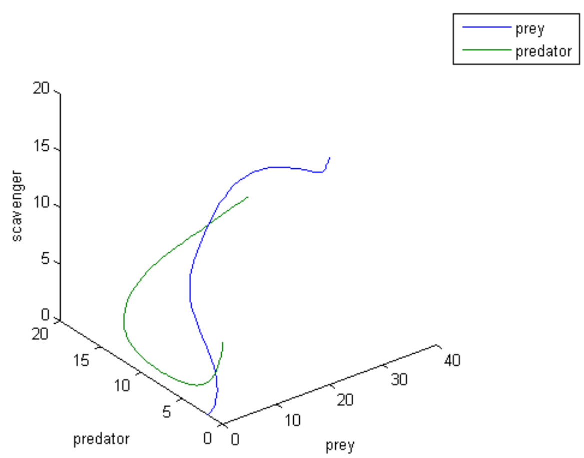

Figure 4.13: Phase Portrait at stable equilibrium point E_4

4.7.6 Local Stability of $\mathbf{E_5}$

Let $\alpha_1 = 0.1,\ \alpha_2 = 0.25,\ \alpha_3 = 0.01,\ \beta_1 = 0.3,\ \beta_2 = 0.4,\ \beta_3 = 0.6,\ \theta_1 = 0.5,\ \theta_2 = 0.05,\ \theta_3 = 0.06,\ \ and\ \ \theta_4 = 0.01.$
Then Jacobian matrix evaluated at E_5 gives

$$J(E_5) = \begin{bmatrix} 0.9405 & -0.0093 & -0.0160 \\ 0.4430 & -0.0001 & 0.0749 \\ 0.0087 & 0.0023 & 0.0925 \end{bmatrix}.$$

The characteristic polynomial becomes

$$\lambda^3 + 1.033\lambda^2 + 0.31\lambda + 0.0874$$

which is of the form,

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \quad with \;\; a_2 = 1.033, \; a_1 = 0.31, \quad and \;\; a_0 = 0.0874.$$

Now the eigenvalues are negative if, $a_2 > 0, \; a_0 > 0, \; a_2a_1 - a_0 > 0$. And we considered each of these conditions as follows:

i) Clearly $a_2 = 1.033$ and $a_0 = 0.0874$ which are non-negative.

ii) $a_2a_1 - a_0 = 0.0042$ which is again positive.

Therefore all of the first column of the Routh array is positive and the corresponding eigenvalues are $\lambda_1 = -0.7791, \; \lambda_2 = -0.127 + 0.3099i \;\; and \;\; \lambda_3 = -0.127 - 0.3099$. Then, Routh-Hurwith Criteria the equilibrium point $E_5(x^*, \; y^*, \; z^*)$ is locally asymptotically stable. See figure 4.14

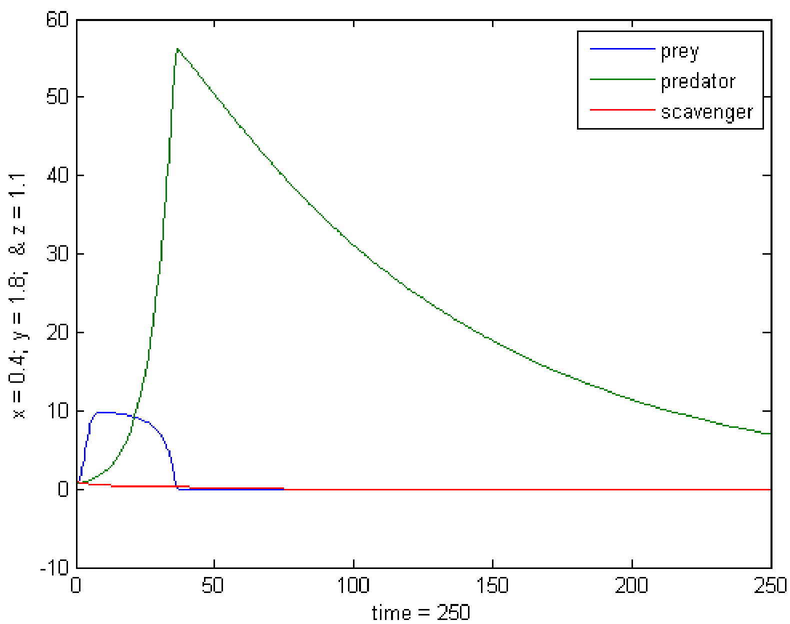

Figure 4.14: Time series plot for stable equilibrium point E_5

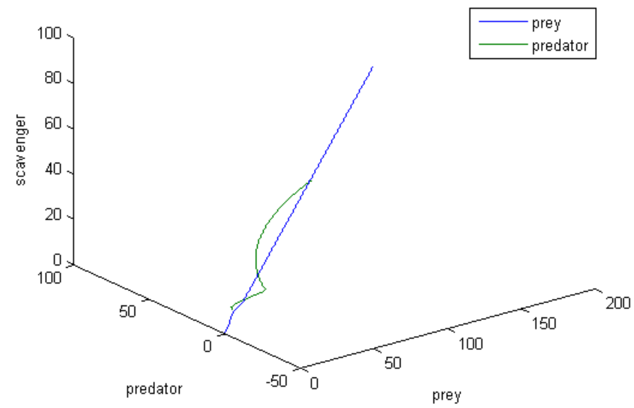

Figure 4.15: Phase Portrait at stable equilibrium point E_5

4.8 Global Stability of E_2

To show the global stability of E_2 we shall uses $\alpha_1 = 0.01$, $\alpha_2 = 0.32$, $\beta_2 = 0.35$, $\theta_2 = 0.21$, $\theta_3 = 0.01$, *and* $\theta_4 = 0.99$ and we take the initial point from a different corner (100.3, 40.3, 40.4).

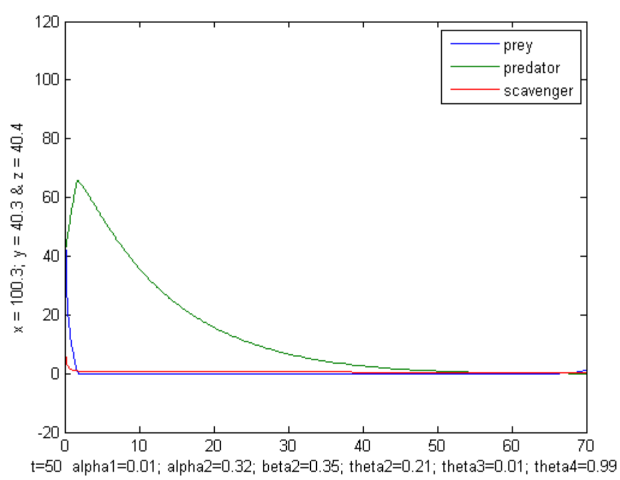

Figure 4.16: Time series plot for global stability of E_2

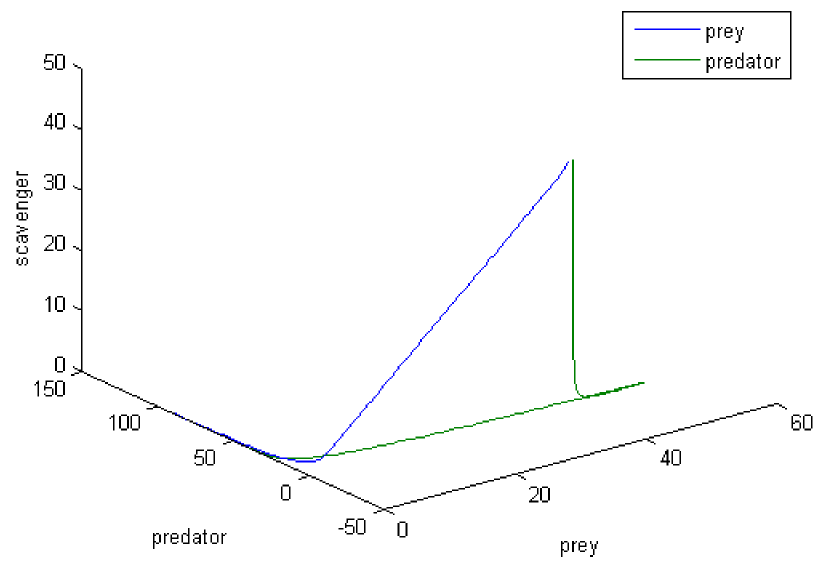

Figure 4.17: Phase Portrait at global stability of E_2

Chapter 5

Result and Discussion

5.1 Result

In this section we investigate the result we obtained in section four. At the origin $E_0(0.2,\ 0.1,\ 0.5)$ (figure 4.5 and 4.6) the prey and predator population increases while scavenger population fails automatically. In the initial all the species are almost the same and prey species dominates other two species till the time instant $t^* = 45$ and thereafter predator species dominates for at a low rate. As figure 4.5 shows, again the dominance changes between the prey and predator species from time to time. Therefore, this shows that when the predator population increases conversely, prey population decreases and vice versa. The scavenger species are not benefit at this dynamics. Even-though they benefit from the interaction between prey and predator, but due to interaction with predator species their population stays only till $t^* = 140$ and it is almost extinct as seen in figure (4.5).

Next let us explain the result from figure (4.6 and 4.7). We consider here the stability when large number of prey population compute with small number of predator and scavenger populations. Initially the prey species dominates other two species due to their large number at a low rate. Since there is enough resource exist the dominance is changed to predator species after a few time. As we observe from figure (4.6) predator species increase directly until $t^* = 40$. In this case the prey species almost goes to extinct and predator species also fails downward. While predator species reduce prey species are try to rise up but, due to interaction between them both remain constant and dominated by predator species. The scavenger species almost goes to extinct from time to time as figure (4.6) shows. It is evident that all the species asymptotically converges to equilibrium point.

The stability of small number of scavenger species, average number of prey species and large number of predator species is shown by figure (4.8 and 4.9). Initially the prey species are less than half of predator species. Even-though the prey species dominate for a few time, due to large number of predator species their population fails from time to time. The predator species grow immediately and dominates other two species, but after $t^* = 140$ due to extinction of prey species their number decreases following prey species.

The third species scavenger are remain constant from initial up to end as figure 4.8 shows.

Figure (4.10 and 4.11) shows the stability when small number of prey population computes with large number of predator and scavenger populations. At the initial predator species dominates other two species, even-though after certain time the prey takes place the dominance but, due to high interaction with predator they become fails within a low rate. Then the predator species dominates always. Depending on the conditions (3.5.21), (3.5.22) and (3.5.23) the system become stable for the $\alpha_2 = 0.02$, $\beta_2 = 0.35$, $\beta_3 = 0.003$, $\theta_1 = 0.04$, $\theta_3 = 0.035$. On the other hand, when the competition between scavenger population with other species become high then the system changes to unstable.

Figure (4.12 and 4.13) shows the stability of less number of predator population with large number of prey and scavenger populations. The prey population increase more due to less number of predator population and dominates over other two for a long period of time. But when predator becomes survive and also due to competition with scavenger their population decreases from time to time. As the figure (4.12) shows at the initial the scavenger population dominates over prey and predator populations for a low rate and thereafter fails downward due to less death occur on prey population. But when predator starts to survive and interact with prey population their number increases from time to time. On the other hand the predator population dominates always.

As we observe from figure (4.14 and 4.15) at the initially the three species are almost the same and prey species dominate over both prey and scavenger species for a few time. Thereafter the predator species dominate always due its large number. But, as a resource become scarce the predator species also fails downward following other two species.

Figure (4.16 and 4.17) shows the global stability at E_2. In the initially case all the three species are far apart and the predator species dominate over both prey and scavenger species. The scavenger species play a great role in stabilizing this system. As a result the parameter value $\theta_4 = 0.99$ implies there is high interaction within scavenger species. Furthermore, as the scavenger species interact within each other for the same resources as well as with predator species, the prey and predator species attain their stability. As their graph shows all the three species asymptotically converges to equilibrium point.

5.2 Discussion

A mathematical model was proposed and analysed to study the dynamics of a prey, predator and scavenger systems in which Holling type II response is included between them. Functional response is the key component in prey-predator ecosystems. It determines the basic characteristics of a system since both the prey death rate and predator growth rate are maintained by this. All the six possible equilibrium points were analysed for local stability and E_2 were analysed for global stability. What makes this model different from other studies is that the scavenger species is also a victim of the predator, this can happen in nature. The purpose of the study was to formulate a mathematical model which describes the dynamics of three populations and set conditions for the continued existence of all three populations. Furthermore, it examined what happens to the remaining populations when one or more of the species dies out.

For the local stability of E_2 we used the parameter value which satisfies the conditions(3.5.17)-(3.5.19). From this condition we conclude that, in order to attain its stability ; the benefit of predator from prey species is less than death rate of predator species due to lack of resources. Again it implies that interaction within prey species each other is less than natural death rates of predator species. On the other hand natural death rates of predator species is greater than benefit of scavenger from prey species. Moreover, natural growth rates of prey and death rates of scavenger species is also greater than benefits of predator from prey and scavenger species.

For the local stability of E_3 similarly we choosed a suitable parameter values which satisfy the conditions (3.5.21)-(3.5.23). From these conditions, for the stability of low prey species with large number of predator and scavenger species; the effect of predator species on prey species must be less than natural growth rates of prey species at low predator and scavenger species. Additionally, natural death rates of predator species is less than benefits of predator species from scavenger species. But as result shows natural death rates of scavenger species is greater than benefits of scavenger species from dead bodies of predator species.

At the end of previous section we determined the stability of E_4 and E_5 numerically. stabiliy of E_5 shows that for large number of predator and scavenger species with low number of prey species, their population varies from time to time. Therefore, for the stability of the three species there must be low number of predator species, average number of scavenger species and large number of prey species.

We also discussed the global stability of E_2 both analytically and numerically. The equilibrium point E_2 is globally stable if it satisfies the condition (3.6.25). This implies that the interaction between prey and predator species is less than the effect on prey species due to predator species. In this, the scavenger species play a great role. As their graph shows all the species asymptotically converges to the equilibrium point.

Chapter 6

Conclusion and Recommendations

6.1 Conclusion

Our aim in this research work is to develop the model which describes the dynamics of prey, predator and scavenger population in an ecosystem. To do this we used functional response type II and the assumption which differentiate from other such that predator species can benefit also from scavenger species.

The existence of non-negative equilibria of the system (3.2.4) has been investigated. A complete classification of the equilibria, with respect to the various parameters based on their existence conditions, is provided. The boundedness of solution is also proved. We provide a complete exploration and classification for the local stability conditions of each equilibrium.

Also, conditions for the local and global asymptotic stability of the equilibrium points were established. The conditions for local asymptotic stability of the equilibrium points were in most cases found to be similar to those for the existence of the equilibrium points. Moreover, we used the Routh-Hurth Criteria to determine the local stability of equilibrium points. Conditions for the global asymptotic stability of the equilibrium point E_2 were established by choosing a suitable Lyapunov function.

The predator prey model with a scavenger demonstrates the possible population trends when a predator, a prey and a scavenger population interact. We have found that the predator and the prey can coexist in the absence of the scavenger, and the scavenger and the prey can coexist in the absence of the predator. However, the scavenger and the predator coexist only for few time without the prey. Biologically this is reasonable, because without the prey, the predator will have food only from some of the scavenger population and after a few year will die off. The scavenger will then lose all sources of food and will almost existent. We have also found that the three populations can coexist with each other.

Experimenting numerically with this model will be easier and from the simulations a variety of interesting questions can arise. For example, one can try to further refine the study in this thesis and find a range of parameter values that lead to the stablity or instabilty of the three species. Or one can try to know parameter values that play a great role for the stabilty of the system. This is biologically relevant because no population grows endlessly. Another interesting issue is to find conditions that lead to the extinction of the prey or predator or scavenger populations. Dynamical behavior of three species model is investigated at equilibrium points. Numerical study shows the rich and interesting complicated dynamics of the model.

6.2 Recommendations

The study in this thesis can be viewed as a first step in modeling a prey, predator and scavenger where the scavenger is consumed by the predator. The model can be further improved and numerous interesting research questions can be studied. The model can also further developed when carrying capacity of predator included. On the other hand, one can use functional response type III in place of type II. Additionally bifurcation analysis can be carried out.

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Appendix

In order to develop and analysis the model that represent the dynamics the three species prey, predator and scavenger in an ecosystem we have to use the following mathematical preliminaries.

Preliminaries and Basic Concepts

Basic Concepts

This paragraph contains some basic concepts, definitions and theorems to better understand the calculations made in future chapters. The interaction types between the species are explained. The most commonly used tropic functions in ecological modeling have been described.

Definition 1.2.1.1. Equilibrium Points
We consider a system of n differential equations

$$\frac{d\tilde{x}}{dt} = f(\tilde{x}) \tag{6.2.1}$$

where $\tilde{x} = (x_1, x_2, \ldots\ldots\ldots\ldots, x_n)$ *and* $f(\tilde{x}) = (f_1(\tilde{x}), f_2(\tilde{x}), \ldots\ldots\ldots\ldots, f_n(\tilde{x}))$.

The steady state solution(s) or equilibrium point solution(s) is a solution of the system of equations

$$f(\tilde{x}) = 0.$$

In ecology, the balance of nature can be divided into three categories:

1. the claim that natural populations have a more or less constant number or individuals,
2. the claim that natural systems have a more or less constant number of species,
3. and the claim that communities of species maintain a delicate balance of relation ships, where the removal of one species could cause the collapse of the whole community.

Definition 1.2.1.2. Local Stability

An equilibrium point x* of $\dot{x}$= f(x) (i.e., f(x*) = 0) is locally stable if for every R > 0 there exists r > 0, such that
$\|x_0 - x^*)\| < r \Rightarrow \|x(t)) - x^*\| < R,\ t \geq 0$. if x* is not stable it is said to be unstable.

Definition 1.2.1.3. Locally Asymptotically Stable

An equilibrium point is locally asymptotically stable if all eigenvalues of the Jacobian matrix have negative real parts and it is unstable if at least one eigenvalue has positive real part. An equilibrium point is locally asymptotically stable if solutions with initial conditions sufficiently close to the equilibrium point are attracted to the equilibrium in forward time.

Moreover, a system is locally asymptotically stable(L.A.S.) if it is locally stable and
$\|x(0) - x^*)\| < r \Rightarrow \lim_{t\to\infty}$ x(t)= x^*.

Definition 1.2.1.4. Globally Asymptotically Stable

Global stability, is concerned with whether initial conditions far from the equilibrium point are also attracted to the equilibrium point in forward time. For global stability analysis, we use a Lyapunov-like function. For a differential equation or system of differential equations, L is called a Lyapunov function if it is a differentiable function defined in a neighborhood of a equilibrium point P with an absolute minimum at P with the property that for any solution with initial conditions not equal to P, $\frac{dL}{dt} < 0$. The existence of such a function implies that the equilibrium point is asymptotically stable for all solutions with initial conditions in the domain of L. Thus a system is globally asymptotically stable(G.A.S.) if it is asymptotically stable for all x(0)$\in R^n$.

Lyapunov Functions

Consider a nonlinear autonomous system

$$\frac{dx}{dt} = f(x, y, z),\ \frac{dy}{dt} = g(x, y, z)\ \ and\ \ \frac{dz}{dt} = h(x, y, z)$$

such that the system has an isolated equilibrium point (0, 0, 0) and the function f(x, y, z), g(x, y, z) and h(x, y, z) have continuous first order partial derivatives for all (x, y, z). Let E(x, y, z) be a function with first order partial derivatives at all point (x, y, z) in the domain D containing the point (0, 0, 0).

i) The function E(x, y, z)is said to be **positive definite** in D if and only if E(x, y, z) > 0, $\forall$(x, y, z) $\in$D/{0, 0, 0} and E(0, 0, 0)=0.

ii) The function E(x, y, z)is said to be **negative definite** in D if and only if E(x, y, z) < 0, $\forall$(x, y, z) $\in$D/{0, 0, 0} and E(0, 0, 0)=0.

iii) The function E(x, y, z)is said to be **negative semi definite** in D if and only if E(x, y, z) $\leq$ 0, $\forall$(x, y, z) $\in$D/{0, 0, 0} and E(0, 0, 0) = 0.

Considering the above system such that the system has an isolated initial point (0, 0, 0) and the functional of f and g have first order partial derivatives for all (x, y, z) in domain D containing the point (0, 0, 0). Let E(x, y, z) be positive definite function in the domain D such that

$$\frac{dE}{dt} = E_x f(x,\ y,\ z) + E_y g(x,\ y,\ z) + E_z h(x,\ y,\ z),$$

is negative semi definite in D. Then the function E is called **Lyapunov function** for the system in D.

Routh-Hurwitz Criteria

Given the polynomial,

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n$$

where the coefficients a_i are real constants, for i = 1,... , n, define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1 = \begin{bmatrix} a_1 \end{bmatrix},\quad H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},\quad H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}$$

and

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & ... & 0 \\ a_3 & a_2 & a_1 & 0 & ... & 0 \\ a_5 & a_4 & a_3 & a_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots \\ 0 & 0 & 0 & 0 & ... & a_n \end{bmatrix},$$

where $a_j = 0$ if j > n. All of the roots of the polynomial P(λ) are negative or have negative real part iff the determinants of all Hurwitz matrices are positive:

$$detH_j > 0,\quad for\ all\ \ j = 1, 2, ., ., n.$$

When n = 2, we have $P_2(\lambda) = \lambda^2 + a_1\lambda + a_2$ and hence the Routh-Hurwitz criteria simplify to $detH_1 = a_1 > 0$ and

$$detH_2 = det\begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} = a_1 a_2 > 0$$

or $a_1 > 0$ and $a_2 > 0$. For polynomials of degree n = 3, 4 and 5, the Routh-Hurwitz criteria are summarized.

Routh-Hurwitz criteria for n = 3, $P_3(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ and hence,

$$n = 3 : a_1 > 0, a_3 > 0,\ and\ a_1 a_2 > a_3.$$

$$n = 4 : a_1 > 0, a_3 > 0, a_4 > 0,\ and$$

.

$n = 5 : a_i > 0,\ i = 1, 2, 3, 4, 5, a_1a_2a_3 > a_3^2 + a_1^2a_4$ *and* $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_5^2.$

(6.2.2)

Accordingly, an application of Routh-Hurwitz criterion gives a number of constraints on the coefficients $a_1, a_2, ..., a_n$, which are necessary and sufficient to ensure all the eigenvalues (roots of the characteristic polynomial equation of the Jacobian matrix $J = dF(x^*)$ lie in left half complex plane. Hence, if the Routh-Hurwitz constraints are simultaneously satisfied, then system (6.2.1) will be asymptotically stable at x^*. However, violation of any one of these conditions implies unstable point.

Functional Responses

A functional response is the change in the density of prey attacked per unit of time per predator as the prey density changes. A functional response is described as a predator's instantaneous per capita feeding rate as a function of prey abundance (Holling, 1959a). This means that the consumption rate of an individual predator depends on the prey density. Understanding and clearly quantifying functional responses is at the heart of ecological modeling.

According to Abrams and Ginzburg (2000), functional responses are generally categorized as; prey density-dependent, f(N), ratio-dependent, $f(N/P)$ and prey-predator density dependent, $f(N, P)$.

For prey density-dependent responses, the consumption rate of the predator varies with the prey density alone. Holling (1959b) categorized the prey-dependent responses into three types;

(a) Holling Type I functional response which is the standard mass action or linear response

$$f(N) = aN. \tag{6.2.3}$$

Here $a > 0$ is the attack rate of the predator. This type of response is found in passive predators like spiders. The number of flies caught in the net is proportional to fly density.

(b) Holling Type II, also called the cyrtoid functional response, is represented by the function

$$f(N) = \frac{bN}{1 + cN} \tag{6.2.4}$$

where b (units; 1/time) and c (units; 1/prey) are positive constants that describe the effects of capture rate and handling time on the feeding rate of the predator (Skalski and Gillian, 2001). The Holling Type II response is the most common type of functional response and is well documented. According to Sharov (1996), at low prey densities, the predator spends more time in searching the prey while at high prey densities, the predator spends more time handling the prey. Relating this to equation (6.2.4), we see that at low prey densities, b is greater than c. However, at high prey densities, c is greater than b. Either way, the number of prey that a predator can consume is limited and consequently the predator reaches a saturation level. Predators of this type cause maximum mortality at low prey density. For example, small mammals destroy most of gypsy moth pupae in sparse populations of gypsy moth. However in high-density defoliating populations, small mammals kill a negligible proportion of pupae (Sharov, 1996).

c) Holling Type III functional response is represented by the equation below

$$f(N) = \frac{dN^x}{F + N^x} \tag{6.2.5}$$

where $x > 1$ is the encounter rate between predator and prey before the predator reaches maximum efficiency. According to Sharov (1996), Holling Type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines. If predator density is constant (e.g., birds, small mammals) then they can regulate prey density only if they have a Holling Type III functional response because this is the only type of functional response for which prey mortality can increase with increasing prey density. However, regulating effect of predators is limited to the interval of prey density where mortality increases. If prey density exceeds the threshold value of this interval, then mortality due to predation starts declining, and predation will cause a positive feed-back. As a result, the number of prey will get out of control. They will grow in numbers until some other factors (diseases or food shortage) will stop their reproduction.

Ratio-dependency is obtained by substituting the prey-predator ratio (N/P) for prey density (N) in the Holling Type II equation;

$$f(N/P) = \frac{dN/P}{m + N/P} = \frac{dN}{mP + N}. \tag{6.2.6}$$

Here d and m are positive constants that stand for capturing rate and half saturation constant for predator P respectively (Xiao and Ruan, 2001). Beddington (1975) derived and DeAngeles et al . (1975) proposed independently a more general form of ratio dependent functional response as

$$f(N/P) = \frac{aN}{1 + bN + c(P - 1)} \tag{6.2.7}$$

where $c > 0$ describes the magnitude of interference among predators. Thus, this functional response takes into account the delay in time incurred by the predators as a result of interspecific competition for the same prey species.

For prey-predator-dependency, the consumption rate of the predator depends on both the prey and predator density. Examples of prey-predator-dependency functional response are the Crowley-Martin model and the Hassel-Varley model, both documented in (Skalski and Gilllian, 2001). The Crowley-Martin model is as below;

$$f(N/P) = \frac{aN}{1 + bN + c(P-1) + bcN(P-1)}. \tag{6.2.8}$$

An important distinction between the Beddington-DeAngelis model and the Crowley-Martin model is that the former predicts that the effects of predator interference (competition/ infighting among predators) on the feeding rate becomes negligible under conditions of high prey density while the latter assumes that the interference remains important even at high prey density (Skalski and Gilllian, 2001). This can be seen by letting $N(t) \to \infty$ in both models. It is noticed that the Beddington-DeAngelis model gives an expression

$$\lim_{N\to\infty} f(N/P) = \frac{a}{b}, \tag{6.2.9}$$

independent of predator interference parameter c, while the Crowley-Martin model gives an expression

$$\lim_{N\to\infty} f(N/P) = \frac{a}{b + bc(P-1)}, \tag{6.2.10}$$

dependent on predator interference and density.

The Classical Lotka-Volterra System

In this section, we review well-known results of the planar Lotka-Volterra model. The differential equations that model the population dynamics of a predator Y and its prey X with a carrying capacity, the maximum population size that is sustainable due to limited environmental resources, are:

$$\begin{cases} \frac{dX}{dt} = X(A - BX - CY) \\ \frac{dY}{dt} = Y(-D + EX) \end{cases} \tag{6.2.11}$$

where,

- ✓ A is the natural growth rate of prey populations.
- ✓ B is the interaction between prey population itself.
- ✓ C is the rate of change of the prey population in response to the presence of a predator.
- ✓ D is the natural death rate of the predator in the absence of prey.

✓ E is the rate of change of the predator population in response to the presence of prey.

All parameters are positive, and all variables are non-negative. In the classical Lotka-Volterra model, it is customary to take B = 0 as the trajectories with positive initial conditions are bounded, but in order to maintain bounded solutions in models where additional species are added requires a logistic terms on X. This quadratic term in X corresponds to the effective carrying capacity of the prey population.

The system cannot be solved analytically, and thus numerical methods must be used to find the solution of the model. For autonomous systems, solutions in the xy-plane are analyzed. The xy-plane is called the phase plane and the solution curve is called trajectory or orbit of the system. Volterra has proven geometrically that solutions of system (6.2.11) in the positive space are closed curves with the exception of the equilibrium point. Closed orbits correspond to periodic solutions. Figure 1.1 is an example of closed curves in the phase space.

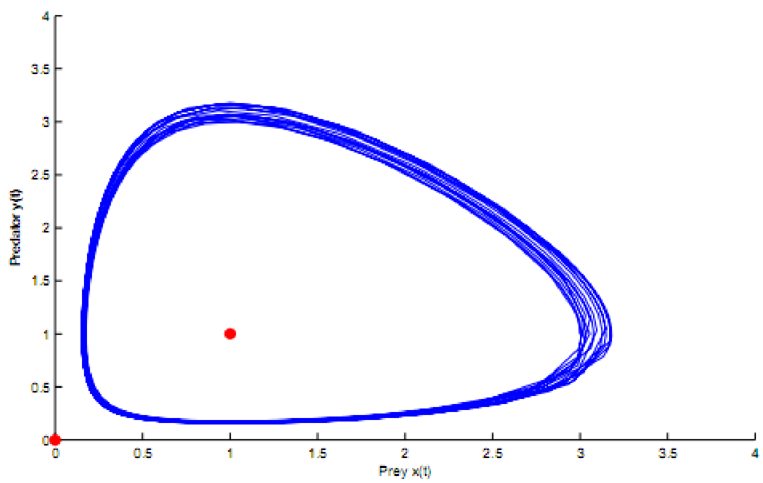

Figure 6.1: Trajectories in the phase plane.

Figure 1.1: Trajectories in the phase plane. The red dots are the equilibrium points. (0,0) is a saddle and the other equilibrium point is a center. A center is neutrally stable.

Descartes Rule of sign

Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ . \ . \ . \ + a_1 x + a_0$ with real coefficients, the Rational Zero Test provides an easy method for isolating the possible zeros of the polynomial that are rational numbers. Let p(x) define a polynomial function with real coefficients and nonzero constant term, with terms in descending power of x . Therefore,

(i) The number of positive real zeros of p(x) = 0 either equals the number of variation in sign occurring in the coefficients of p(x), or less than the number of variations by a positive even integer.

(ii) The number of negative real zeros of $p(x) = 0$ either equals the number of variation in sign occurring in the coefficients of $p(-x)$, or less than the number of variations by a positive even integer.

Matlab Code

Provided below is the code needed to produce the three-dimensional prey predator model with scavenger solutions and diagrams using the MATLAB program

m-file ade.m:

```
function ade = ade(t,y)
alpha1 = 0.1;
alpha2 = 0.25;
alpha3 = 0.01;
beta1 = 0.3;
beta2 = 0.4;
beta3 = 0.6;
theta1 = 0.5;
theta2 = 0.2;
theta3 = 0.3;
theta4 = 0.1;
ade(1) = y(1) * (1 - alpha1 * y(1) - beta1 * y(2)/(1 + y(1)) - theta1 * y(3)/(1 + y(1)));
ade(2) = y(2) * (alpha2 * y(1)/(1 + y(1)) + beta2 * y(3)/(1 + y(3)) - theta2);
ade(3) = y(3) * (alpha3 * y(1)/(1 + y(1)) + beta3 * y(2)/(1 + y(2)) - theta4 * y(3) - theta3);
ade = [ade(1) ade(2) ade(3)];
end
```

```
clear;
to = 0;
tf =500;
yo = [0.2 0.6 0.4];
[t y] = ode45('ade',[to tf],yo);
plot(t,y(:,1),t,y(:,2),t,y(:,3));
legend('prey','predator','scavenger')
ylabel('prey, predator and scavenger populations')
xlabel('time')
```