MATHEMATICAL MODELING OF PREY-PREDATOR WITH PREY REFUGE USING HOLLING TYPE II FUNCTIONAL RESPONSE

BY

GELETA BACHA DERESSA

A THESIS SUBMITTED TO
THE PROGRAM OF APPLIED MATHEMATICS
SCHOOL OF APPLIED NATURAL SCIENCE

PRESENTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR
THE DEGREE OF MASTER'S OF SCIENCE IN APPLIED MATHEMATICS

OFFICE OF GRADUATE STUDIES
ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

ADAMA, ETHIOPIA
DECEMBER, 2017

Mathematical Modeling of prey-predator with prey refuge using Holling Type II Functional response

By

Geleta Bacha

Advisor

Legesse Lemecha (Ph.D)

A Thesis Submitted to

The Program of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

Adama, Ethiopia

December, 2017

Approved by the Examination Committee

(Advisor)	Signature	Date
(Internal Examiner)	Signature	Date
(External Examiner)	Signature	Date
(Chair Person)	Signature	Date
(Dep't. Head)	Signature	Date
(Sch. Dean)	Signature	Date
(Sch. Grad. Studies Dean)	Signature	Date

Abstract

Predator-Prey models have been studied for a long time. The most famous ones are the Lotka Volterra model, the Lotka-Volterra model with logistic function, and the Rosen Zweig-MacArthur model. The properties of these models are well known. We analyze the population consequences of refuge use in the Lotka-Volterra model when the area where occupies the preys and predators declines.What physical capacity of refuge it influences in the existence and stability the unique equilibrium point at interior of the first quadrant. We analyze the consequences of such function.

Keywords: *Anti-predator behavior, Refuge, Functional response,Stability analysis.*

Acknowledgements

First of all, I would like to thank the almighty God for his permission to do my daily activity.

Second, I would like to express my deepest appreciation and sincere gratitude to my advisor, Dr. Legesse Lemecha for his kind encouragement, guidance, support and supervisions, starting from seminar presentation to the completion of this thesis, of my graduate study at Adama Science and Technology University,Adama, Ethiopia.

Third, my greatest thanks and appreciation go to my family for their permanent love and support during my graduate thesis work and beyond. I realize how lucky I am to have them.

Fourth,I would like to thank Oromia Development association and Adama science and Technology University for their sponsorship to complete my Ms c.
Finally, I would also like to thank most of my colleagues, discussions with whom might help me in this thesis work and those who supports me for their efficient coordination of the process of writing this thesis.

Adama Science and Technology University — Geleta Bacha
August, 2017

Table of Contents

Chapter 1

Introduction

1.1 Background of the study

In population dynamics the models proposed for interactions of prey-predator have considered diverse suppositions to simplify their Mathematical descriptions such as: The populations homogeneity, homogeneity of environmental distribution spatial uniform, constant rates of growth, encounters between the species predators and equally probable prey, sizes population prey clerks exclusively of the time, the species predators feeds exclusively of the species prey, while this feeds of a resource that is in the habitat in big quantities the one which alone it intervenes passively, they are not considered as behaviors of the species of physiologic, morphological, social type, neither reintroduction of species, etc. The behavior of the species is affected by the effect of ecological variables (readiness of refuges, formation of defense groups, difficulty of mating, appearance of other strategies anti-predator, etc.) The behavior of anti-predator can be:

Physiologic: Emission of chemical substances or pheromones, etc.

Morphological: Patterns of coloration, mimicry with the atmosphere, adaptability of some parts of the body etc.

Imitation of codes: Emission of sounds, to be made the dead, etc.

Habitat adaptation: Uses of refuges, in the nature, many responds to the attacks of the predators looking for such space refuges. The effect itself of refuge use on the population growth is complex nature, but for modeling purposes it can be understood as the reduction of prey mortality due to reduction in predation success. The refuges affect positively the population growth of preys and negatively that of predators. A more relevant behavior trait that affects the dynamics of predator-prey systems is the use of spatial refuges by the prey. Spatial refuges are where environmental heterogeneity provides less-accessible sites for predators in which a number of preys can stay at least temporarily. In this way, some fraction of the prey population is partially protected against predators and we assume that the refuge is physical location in which prey either live or temporarily hide [1].

In earlier works it has been claimed that the prey refuge use exerts a stabilizing effect in the

dynamics of the interacting populations.The above statement is true assuming that the quantity, since we demonstrate that when we include a refuge to the preys we can maintain the equilibrium points and to maintain the stability. Population ecology has given emphasis on the introduction of natural complexity and realism into the basic Lotka-Volterra framework [29].The initial steps were to include density-dependent effects on the endogenous dynamic of predators and prey, and to develop non-linear functions for consumption of prey by predators, the so-called functional response [31, 32].
So, the predator behavior was explicitly considered in predator-prey models. More recently, the different behavior of prey and its consequences at the population level has been worked out and incorporated into the predation theory or in the growth prey function due the Allee effect [30]. In this context, a more relevant behavioral trait that affects the dynamics of predator-prey systems is the use of spatial refuges or covers by the prey.
The majority of the works show the refuge conclusion that stabilizes predator interactions [32]. However, Gonzalez-Olivares and Ramos-Jiliberto, discard the common conclusion that the use of the shelter by the population of prey always leads to stability as considering the same assumptions in the model Rosenzweig-McArthur obtained which trajectories can oscillate for some parameter values [6] .
In the present work, we analyze the physical capacity of refuge it influences and the population consequences of refuge when the part where occupies the preys and predators decreases using Holling type II functional response.
We evaluate the effects of refuge use by the prey through the analysis of local stability of equilibrium points and equilibrium density values in the Lotka-Volterra model.

1.2 Statement of the Problems

The use of refuge has a great role in prey-predator interaction whenever there is predation pressure. So that many researchers studied prey refuge mechanism to solve the problem of extinction and facilitate the co-existence of predators and their preys [6,17,32]. Therefore, in this study, we analyze the population consequences of refuge use in the Lotka-Volterra model, assuming that the amount of prey in refuge is given by the function response type II Holling. Hence, the study raises the following leading research questions:

1. What are the assumptions to modify a mathematical model which describes the dynamics of the prey-predators interaction in Lotka- Voltera model?

2. Can we analyze a mathematical model which efficiently describes the dynamics of the prey-predators interaction in Lotka- Voltera model?

3. What are the ecological interpretations of the solution of the mathematical problem?

1.3 Objective of the Study

1.3.1 General Objective

This study aims to understand the effect of prey refuge on the dynamics of the prey-predator interaction by applying Holling type II functional response to predators in Lotka- Voltera model

1.3.2 Specific Objectives

The specific objectives of the proposed study are:

- modify a mathematical model which describes effect of prey refuge on the dynamics of the prey-predator interaction in Holling type II functional response.
- determine local and global stability analysis
- examine the ecological implications of the analysis.

1.4 Research Method

In order to achieve the stated objectives of the present research analytical and numerical simulation was employed. The sources of information for this research were different Journals and standard books.

1.4.1 Mathematical procedures

This research was based on the following procedure

- Construct Mathematical model that describes the influence of prey refuge on the dynamics of the prey-predator interaction in Holling type II functional response.
- Determine equilibrium points and study its local stability.
- Investigate global stability analysis of the formulated Mathematical model.
- Test the correctness of the formulated Mathematical model whether it reflects the existing reality between the interaction of prey and predator using numerical simulation.

1.5 Significance and Beneficiaries

- To develop awareness to policy makers about effect of prey refuge on the dynamics of the prey-predator interaction.
- To suggest the possible intervention mechanisms.
- Essential for conservation of endangered species creating protected areas to preserve them.
- Initiates other researchers to undertake further extension and rigorous mathematical analysis.

1.6 Expected Outcomes

The expected outcomes of the present study are:

- a mathematical model that governs the effect of prey refuge on the dynamics of the prey-predator interaction,
- the effect of refuge on prey-predator when the area occupy the the species is intervened,

1.7 Limitation of the study

To carry out a research in prey-predator interaction it is vital to visit National Parks and other Protected areas frequently to collect data from the concerned body through different methods to make the study complete. But, this was not be possible due to different reasons such as shortage of time,finance and so on. Hence, the study limited to analytic study without model validation.

Chapter 2

Literature Review

Mathematical population model have been used to study the dynamics of prey-predator system since[19]and [33] proposed a simple model of prey-predator interaction now called Lotka-Volterra model. Since then many mathematical model have been constructed based on more realistic, explicit and implicit biological assumptions. Over the past decades, mathematics has made a considerable impact as a tool to model and understand biological phenomena. In return, biologists have confronted the mathematicians with variety of challenging problems, which have stimulated developments in the theory of nonlinear differential equations. Such differential equations have long played important role in the field of theoretical population dynamics, and they will, no doubt, continue to serve as indispensable tools in future investigations.

Functional responses.

Functional Response is the relationship between the average number of prey eaten by each predator per unit time versus prey density There are many examples in nature which demonstrate that predators control the numbers of their prey. When predators are faced with increasing local density of their prey, they often respond by changing their consumption rate. This relationship of an individual predator's rate of food consumption to prey density was termed the "functional response" as given in [12].The author analyzed the act of predation and broke it onto behavioral units termed the components of predation. There are many factors that might influence predation but we will consider only four major components: search, capture, handling and digestion. Theory and laboratory testing suggest that the functional response of an animal may take one of three forms called the Type I, Type II and Type III functional responses. Mathematical modeling in population dynamics has gained a lot of attention during the last few Decades and among these models predator-prey systems play an important role. The most important element in predator-prey models is the predator functional response on prey population[5, 15, 21], which describes the number (density) of prey consumed per predator per unit time for given quantities (densities) of prey and predator. There are various forms of functional responses among which the most popular and useful functional responses are Lotka-Volttera functional response (Holling type I functional response)[3, 5, 21]and Holling type II functional response [3, 5, 21].

As referred in [3,5, 21], Holling type I functional response is a linear relationship, where the predator is able to keep up with the increasing density of prey. If the predators eat 10 percent of the prey at low density, they continue to eat 10 percent of them at high densities.
Each individual predator takes a fixed fraction of the prey population each time period. If prey density double, the number each predator eats per day doubles. Encounter and kill rate is simple fraction of prey population, No saturation or satiation of predator. No handling or processing time No switching from one prey species to another. Likely valid for the lower range of prey densities for some species.
Holling type II functional response [3,5, 21] describes a situation in which the number of prey consumed per predator initially rises quickly as the density of prey increases but then level off with further increase in prey density. Many researchers have concentrated on the stability of the predator-prey systems with such functional responses.
It is Saturating functional response. As prey density increases, the number of prey each predator eats per day reaches a constant number. Encounter rate initially limits predation, but processing/handling time (including digestion) or satiation eventually limits food intake. No switching from one prey species to another. Likely applies to most predators,and a nearly Type I response occurs over lower range of prey density.
Type III functional response [3,5, 21] is similar to type II in that at high levels of prey density, saturation occurs. But now, at low prey density levels, the graphical relationship of number of prey consumed and the density of the prey population is a more than linearly increasing function of prey consumed by predators. This accelerating function is largely descriptive, and often justified by learning time, prey switching, or a combination of both phenomena, but the type III functional response lacks the rigorous theoretical underpinning of the type II functional response
Type III functional response is called Sigmoid functional response. Initially predation on this prey species is low, but then, at a threshold, increases sharply. It then saturates or levels off, as in the Type II Functional response. Encounter rate initially increases, but processing/handling time (including digestion) or satiation eventually limits food intake.
If a predator switches from another prey species to the focal one. This could result from developing a search image or from changing foraging locations or style to more efficiently capture focal prey species.
Type III functional response occurs in predators which increase their search activity with increasing prey density. For example, many predators respond to kairomones (chemicals emitted by prey) and increase their activity. Polyphagous vertebrate predators (e.g., birds) can switch to the most abundant prey species by learning to recognize it visually. Mortality first increases with prey increasing density, and then declines.
If predator density is constant (e.g., birds, small mammals) then they can regulate prey density only if they have a type III functional response because this is the only type of functional response for which prey mortality can increase with increasing prey density. However, regulating effect of predators is limited to the interval of prey density where mortality increases.

If prey density exceeds the upper limit of this interval, then mortality due to predation starts declining, and predation will cause a positive feed-back. As a result, the number of prey will get out of control. They will grow in numbers until some other factors (diseases of food shortage) will stop their reproduction. This phenomenon is known as "escape from natural enemies" discovered first by Takahashi.

Prey refuge

In nature, prey populations often have access to area where they are safe from their predators. In addition to this the existence of prey refuge can clearly have important effects on the co-existence of predators and their preys. According to R.J Taylor[27] the different kinds of refuge can be arranged into three types. Those who provide:

- Permanent spatial protection for small subset of the prey population.
- Temporarily spatial protection.
- A temporal prey refuge in numbers, that decrease the risk of predation by increasing the abundance of vulnerable prey.

The study of the consequence of prey refuge on the dynamics of predator-prey interactions can be recognized as a major but rather challenging issue in applied mathematics and theoretical ecology[11, 14,16]. Some of the empirical and theoretical works based on prey refuge have concluded that the refuge used by prey has stabilizing effect on predator-prey interactions. In the frame of predator prey systems, most of the empirical and theoretical work has considered the behavior of predators and its implications on the population dynamics [1, 10, 12, 22, 30]. At a comparatively smaller extent, the hiding behavior of preys has been incorporated as a new ingredient of simple predator - prey models and its major consequences on the system stability have been studied. This was initially done by modifying the original LotkaVolterra predator - prey equations and the most widely reported conclusion was the community equilibrium being stabilized by the addition of refuges for preys, and prey extinction being prevented [1, 10, 12, 16,22]. The traditional ways in which the effect of refuge use by the preys has been incorporated in predator-prey models is to consider either a constant number or a constant proportion of the prey population being protected from predation [27] More complex models incorporating prey refuges have been introduced and analyzed [1,16, 30, 31], among others, which implicitly assume the existence of cost on the prey growth rate, and other secondary effects such as predator dependence of refuge use. The conclusions commonly are referred to as changes in the stability properties of the system explained by the addition of refuge. Nevertheless, the results are often ambiguous and difficult to interpret in biological terms. An accepted strategy is to study the most simple but plausible models before to move toward more complex ones if empirical or theoretical evidence justifies this. On the basis of the LotkaVolterra formulation, more realism has been added to obtain simple predator prey models, through including self-limitation in the lowest-level population, and by making saturating the functional and numerical responses [23]. Nevertheless, to our

knowledge there are surprisingly few studies in which the primary effect of refuge is incorporated into simple continuous predator prey models other than the original LotkaVolterra.
An exception is the work of [33], which introduces and analyzes a model, attributed to [20] in its original form, where the population growth of both preys and predators is logistic in absence of predation, and the functional response is hyperbolic. They incorporate the refuge as a constant fraction of the prey, by which the predator carrying capacity and the prey mortality due to predation are affected. The logistic model used by Collings belongs to a family that does not conform to the principle of biomass conversion. That is, the functional and numerical responses are not explicitly related.The predator reproductive rate responds only to the rate of prey killed per predator, thus obeying the principle of biomass conversion [2].This model has a long tradition in theoretical ecology [2,15,16,20,34] and a systematic study of the effects of prey behavior on the system dynamics should not disregard its use as a starting point.
Recognizing that there is a huge variety of predator prey models in the ecological literature, those best known and understood in mathematical and biological terms are likely to be the LotkaVolterra model, the May model, and the Rosenzweig MacArthur model [29] which we use here. For consistency with the previous works on this field, we will consider refuge as an environmental place where predation rate is lower. Likewise, we will consider, as usual, the effect of having a constant number of preys and a constant proportion of preys using refuges. The mathematical analysis will be done separately in each case.
The majority of the works show the refuge conclusion that stabilizes predator interactions [32]. However, Gonzalez-Olivares and Ramos-Gilberto, discard the common conclusion that the use of the shelter by the population of prey always leads to stability as considering the same assumptions in the model Rosenzweig Mc Arthur obtained which trajectories can oscillate for some parameter values[6].
According to [16, 27] two refuge types have usually been considered in the ecological literature.

1. Those that protect a constant fraction of the preys. The fraction of hidden preys is a growing linear function; this implies that the refuge readiness is bigger while bigger it is the population size. Then the system is topologically equivalent to the original one [6], changing only the coordinates of the positive equilibrium point.

2. Those that protect a fixed quantity of preys. Quantity of the refuge prey does not depend on physical capacity of refuge. The occurrence of a constant number or constant proportions of the prey in refuge seem to be very unlikely in the nature [1].

Recognizing that there is a huge variety of predator prey models in the ecological literature, those best known and understood in mathematical and biological terms are likely to be the Lotka - Volterra model, the May model, and the Rosenzweig MacArthur model [27].
In the present work, we analyze the physical capacity of refuge it influences and the population consequences of refuge when the part where occupies the preys and predators decreases using

Holling type II functional response, assuming that the amount of prey in refuge is given by $X_r = \frac{\delta X}{X + \eta}$ where δ represents the maximum physical capacity of refuge and where the population's fraction in cover is falling in the way and η is the quantity of necessary preys to reach half of the maximum capacity δ

Chapter 3

Model Formulation and Analysis

In this chapter we detail model that describes the dynamics of interaction between prey and predator. The model equation we consider is a modified Lotka-volterra model with Holling type II Functional response. The analysis includes the local and global stability and also determining the saddle bifurcation. Lastly numerical simulation is followed in order to better understand the dynamics of modified lotka-volterra model.

3.1 Model assumption and parameters of the model

3.1.1 Model assumption

- The prey population in the absence of predator population grows logistically.
- The effect of predation is to reduce the preys per capita growth rate of prey by term proportional to the prey and predator populations
- In the absence of prey population the predators death rate results in logistically decay
- The quantity of prey population that occupies a refuge is X_r
- The amount of prey in refuge is given by $X_r = \dfrac{\delta X}{X + \eta}$ where δ represents the maximum physical capacity of refuge and where the population's fraction in cover is falling in the way and η is the quantity of necessary preys to reach half of the maximum capacity δ

3.1.2 Parameters of the Model

We denote $X(t) = X$ and $Y(t) = Y$ the population sizes of preys and predators,respectively for $t \geq 0$. The common Lotka Volterra predator-prey model is

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K})X - qXY \\ \frac{dY}{dt} &= (bX - c)Y \end{aligned} \tag{3.1}$$

Usually the parameters have the following biological meanings:
r is the intrinsic per capita prey growth rate;
K is the prey environmental carrying capacity;
q is the maximal per capita predator consumption rate;
b is the efficiency with which predators convert consumed prey into new predators,
c is the natural per capita death predator rate.

3.2 Model Formulation

If $X_r(t) = X_r$,a quantity of prey population that occupies a refuge, then the quantity of preys that interact with the predators is $X - X_r$
Then the model (3.1) is transformed into

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K})X - q(X - X_r)Y \\ \frac{dY}{dt} &= (b(X - X_r) - c)Y \end{aligned} \tag{3.2}$$

If the amount of prey in refuge is given by the functional response type II Holling $X_r = \dfrac{\delta X}{X + \eta}$ where δ represents the maximum physical capacity of refuge and where the population's fraction in cover is falling in the way and η is the quantity of necessary preys to reach half of the maximum capacity δ,then the model (3.2) is transformed into

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K})X - q(X - \frac{\delta X}{X + \eta})Y \\ \frac{dY}{dt} &= (b(X - \frac{\delta X}{X + \eta}) - c)Y \end{aligned} \tag{3.3}$$

3.3 The dimensionless form of the model

A model can be transformed in to a dimensionless form i.e rewriting the system in terms of dimensionless quantities. One of the advantages of the system in dimensionless form is that the number of parameter is reduced to a minimum and that also makes analysis easier. Further more parameters can be better compared with each other in terms of small and large and thus one gets more insight in to the system. It is also possible to make a comparison between different systems. The dynamics of the transformed model will be the same as in the original system,

$$\begin{aligned} \frac{dX}{dt} &= r(1 - \frac{X}{K})X - qXY \\ \frac{dY}{dt} &= (bX - c)Y \end{aligned} \tag{3.4}$$

The following non-dimensional state variables and parameters are chosen.
$\tau = \frac{rK}{Ku+\eta}t$, $X = Ku$, $Y = \frac{r}{q}v$ with $detJ(u,v,\tau) = \frac{Ku+\eta}{q} > 0$ and
$\alpha = \frac{\delta}{K}, \omega = \frac{\eta}{K}, \beta = \frac{Kb}{r}, \lambda = \frac{c}{r}$
The system (3.3) takes the following non-dimensional form.

$$\begin{aligned} \frac{du}{d\tau} &= u[(1-u)(u+\omega) - (u+\omega-\alpha)v] = F(u,v) \\ \frac{dv}{d\tau} &= v[\beta(u+\omega-\alpha)u - \lambda(u+\omega)] = G(u,v) \end{aligned} \tag{3.5}$$

The system (3.3) is ecologically equivalent to the system (3.5)

3.4 Model Analysis

3.4.1 Existence and Positivity of solution

3.3.1.1 Existence of solution

In this section,we establish conditions for the existence of the equilibrium points of the system,and we show that all solutions of the system (3.5) with initial values are bounded.The equilibrium points of the system (3.5) can be found by setting all equations equal to zero and solving the system for u and v
Thus, the system

$$\begin{aligned} \frac{du}{d\tau} &= u[(1-u)(u+\omega) - (u+\omega-\alpha)v] = 0 \\ \frac{dv}{d\tau} &= v[\beta(u+\omega-\alpha)u - \lambda(u+\omega)] = 0 \end{aligned}$$

must be solved.
If $\alpha \geq \frac{(\beta-\lambda)(1+\omega)}{\beta}$ there are two equilibrium points of the system (3.5),
namely $E_0 = (0,0), E_1 = (1,0)$.

- It is obvious that $E_0 = (0,0)$ always exist.
- $E_1 = (1,0)$ also always exist.

If $\alpha < \frac{(\beta-\lambda)(1+\omega)}{\beta}$ there are three equilibrium points of the system (3.5),
namely $(u_1^*, v_1^*) = (0,0), (u_2^*, v_2^*) = (1,0)$ and
$(u_3^*, v_3^*) = (\frac{R+\sqrt{R^2+4\omega\beta\lambda}}{2\beta}, \frac{R\beta-(R^2+R\sqrt{R^2+4\omega\beta\lambda}+2\omega\beta\lambda)}{2\beta\lambda})$ where $R = \lambda+(\alpha-\omega)\beta$.
If $\omega > \alpha$ then $u_3^* \to 0, v_3^* \to \frac{\omega}{\omega-\alpha}$
If $\omega \leq \alpha$ then $u_3^* \to \alpha-\omega, v_3^* \to \infty$

3.3.1.2 Positivity of solution

Theorem 3.4.1. *All solutions of system* (3.5) *that start in* $\Re^2_+$ *with positive initial conditions stay positive for all time,i.e if* $u(0), v(0) > 0$*, then* $u(\tau), v(\tau) > 0$ *for all* $\tau > 0$*.*

Proof. Let $u(\tau)$ and $v(\tau)$ be one of the solution of the system (3.5)

$$\begin{aligned} \frac{du}{u} &= (1-u)(u+\omega) - (u+\omega-\alpha)v \\ \frac{dv}{v} &= \beta(u+\omega-\alpha)u - \lambda(u+\omega) \end{aligned} \tag{3.6}$$

Integrating system (3.5) with initial conditions (u_0, v_0),we have

$$\begin{aligned} &\frac{du}{u} = (1-u)(u+\omega) - (u+\omega-\alpha)v \\ &\int_{u_0}^{u(\tau)} \frac{du}{u} = \int_0^{\tau} (1-u)(u+\omega) - (u+\omega-\alpha)v d\tau \\ &u(t) = u_0 exp \int_0^{\tau} (1-u)(u+\omega) - (u+\omega-\alpha)v d\tau \\ &\frac{dv}{v} = \beta(u+\omega-\alpha)u - \lambda(u+\omega) \\ &\int_{v_0}^{v(\tau)} \frac{dv}{v} = \int_0^{\tau} \beta(u+\omega-\alpha)u - \lambda(u+\omega) d\tau \\ &v(\tau) = v_0 exp \int_0^{\tau} \beta(u+\omega-\alpha)u - \lambda(u+\omega) d\tau \end{aligned}$$

Showing that $u(\tau) > 0$ and $v(\tau) > 0$ whenever $u(0) > 0$ and $v(0) > 0$.Hence all solutions remain within the first quadrant of the uv-plane starting from an interior point of it. Further we can easily establish that the solution trajectories starting from $(u_0, 0)$ with $u_0 > 0$,remain within the positive u-axis at all future time and similar result holds for trajectories starting from a point on the positive v-axis. □

Theorem 3.4.2. *The set* $\Omega = \{(u, v) \in (R_0^+)^2 | 0 \leq u \leq 1, v \geq 0\}$ *is an invariant region.*

Proof. If $u = 1$, We have that $\frac{du}{d\tau} = -(1+\omega-\alpha)v < 0$, and the trajectories point into the region Ω. If $\omega > \alpha$, then $(u, v) \in [0, 1] \times [0, \frac{\omega}{\omega-\alpha}] \subset \Omega$. If $\omega \leqslant \alpha$, then $(u, v) \in [\alpha-\omega, 1] \times [0, \infty] \subset \Omega$ Hence,$\Re^2_+ = (u, v) : u, v \geq 0$ is an invariant set. □

3.4.2 Stability analysis

To determine stability condition of an equilibrium point, the following steps are followed

1. Calculate an equilibrium point
2. Evaluate the Jacobian matrix
3. Evaluate the eigen values of the Jacbian matrix

3.4.3 Equilibrium points

The dynamical system has the following three fixed or equilibrium points. The origin $(u_1^*, v_1^*) = (0, 0)$ and a boundary fixed point $(u_2^*, v_2^*) = (1, 0)$ and a fixed point for which both populations survive $(u_3^*, v_3^*) = (\frac{R + \sqrt{R^2 + 4\omega\beta\lambda}}{2\beta}, \frac{R\beta - (R^2 + R\sqrt{R^2 + 4\omega\beta\lambda} + 2\omega\beta\lambda)}{2\beta\lambda})$ where $R = \lambda + (\alpha - \omega)\beta$.
Ecologically,the first equilibrium point, $(u_1^*, v_1^*) = (0, 0)$ corresponds to extinction of both species,the second equilibrium point, $(u_2^*, v_2^*) = (1, 0)$ corresponds to extinction of one species while the last equilibrium point, $(u_3^*, v_3^*) = (\frac{R + \sqrt{R^2 + 4\omega\beta\lambda}}{2\beta}, \frac{R\beta - (R^2 + R\sqrt{R^2 + 4\omega\beta\lambda} + 2\omega\beta\lambda)}{2\beta\lambda})$ where $R = \lambda + (\alpha - \omega)\beta$ corresponds to steady state co existence of both species.

3.4.4 Local stability of equilibria of the system

The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues of the Jacobian matrix evaluated at equilibrium points.If all the real parts are negative, then the equilibrium point is asymptotically stable.If at least one eigenvalues of the Jacobian matrix is positive, then the equilibrium point is unstable. The Jacobian matrix of the system (3.5) is

$$J(u, v) = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{pmatrix} = \begin{pmatrix} -3u^2 + 2(1 - \omega - v)u + (\omega - \omega v + \alpha v) & -u^2 + (\alpha - \omega)u \\ (2\beta u + (\omega - \alpha)\beta - \lambda)v & \beta u^2 + (\omega\beta - \alpha\beta - \lambda)u - \omega\lambda \end{pmatrix} \tag{3.7}$$

Theorem 3.4.3. *(i) The singularity $(u_1^*, v_1^*) = (0, 0)$ is saddle point for all parameter values. (ii) The singularity $(u_2^*, v_2^*) = (1, 0)$ is saddle point,if $\alpha < \frac{(\beta - \lambda)(1 + \omega)}{\beta}$.*

Proof. (i) Evaluating the system (3.7) at $(0, 0)$.We have,

$$J(0, 0) = \begin{pmatrix} \omega & 0 \\ 0 & -\omega\lambda \end{pmatrix}$$

The eigenvalues are $\lambda_1 = \omega > 0$ and $\lambda_2 = -\omega\lambda < 0$. This way $(0, 0)$ is saddle point. (ii)Evaluating the system (3.7) at $(1, 0)$.We have,

$$J(1, 0) = \begin{pmatrix} -(1 + \omega) & \alpha - (1 + \omega) \\ 0 & \beta(1 + \omega - \alpha) - \lambda(1 + \omega) \end{pmatrix}$$

The eigenvalues are, $\lambda_1 = -(1 + \omega) < 0$ and $\lambda_2 = \beta(1 + \omega - \alpha) - \lambda(1 + \omega)$
Then the sign of λ_2 depends on the sign of $\beta(1 + \omega - \alpha) - \lambda(1 + \omega)$. It follows ,If $\alpha < \frac{(\beta - \lambda)(1 + \omega)}{\beta}$ then $\lambda_2 > 0$ and $(1, 0)$ is saddle point. □

3.4.5 Global stability analysis of equilibria

Theorem 3.4.4. *The singularity* $(u_2^*, v_2^*) = (1, 0)$ *is globally asymptotically stable, if* $\alpha \geq \dfrac{(\beta - \lambda)(1 + \omega)}{\beta}$

Proof. Evaluating the system (3.7) at $(1, 0)$.We have,

$$J(1,0) = \begin{pmatrix} -(1+\omega) & \alpha - (1+\omega) \\ 0 & \beta(1+\omega-\alpha) - \lambda(1+\omega) \end{pmatrix}$$

The eigenvalues are, $\lambda_1 = -(1+\omega) < 0$ and $\lambda_2 = \beta(1+\omega-\alpha) - \lambda(1+\omega)$
Then the sign of λ_2 depends on the sign of $\beta(1+\omega-\alpha) - \lambda(1+\omega)$.
If $\alpha \geq \dfrac{(\beta - \lambda)(1+\omega)}{\beta}$ then $\lambda_2 < 0$ and $(1, 0)$ is globally asymptotically stable by applying the Poincare Bendixon theorem.

□

Theorem 3.4.5. *The singularity*
$(u_3^*, v_3^*) = (\dfrac{R + \sqrt{R^2 + 4\omega\beta\lambda}}{2\beta}, \dfrac{R\beta - (R^2 + R\sqrt{R^2 + 4\omega\beta\lambda} + 2\omega\beta\lambda)}{2\beta\lambda})$ *is:*
(i) Center, if $\alpha = \dfrac{(u^* + \omega)^2}{2u^* + \omega - 1}$
(ii) Spiral unstable , if $\alpha > \dfrac{(u^* + \omega)^2}{2u^* + \omega - 1}$
(iii)Spiral stable $\alpha < \dfrac{(u^* + \omega)^2}{2u^* + \omega - 1}$

Proof. Evaluating the jacobian matrix at
$(u^*, v^*) = (\dfrac{R + \sqrt{R^2 + 4\omega\beta\lambda}}{2\beta}, \dfrac{R\beta - (R^2 + R\sqrt{R^2 + 4\omega\beta\lambda} + 2\omega\beta\lambda)}{2\beta\lambda}$, We have

$$J(u^*, v^*) = \begin{pmatrix} -3u^{*^2} - 2u^*\omega + 2u^* - 2u^*v^* + \omega - v^*\omega + v^*\alpha & -(u^* + \omega - \alpha)u^* \\ (2\beta u^* + (\omega - \alpha)\beta - \lambda)v^* & 0 \end{pmatrix}$$

As $v^* = \dfrac{(1 - u^*)(u^* + \omega)}{u^* + \omega - \alpha}$, then
$-3u^{*^2} - 2u^*\omega + 2u^* - 2u^*v^* + \omega - v^*\omega + v^*\alpha = -u^*(\dfrac{u^{*^2} + 2u^*\omega - 2u^*\alpha + \omega^2 - \omega\alpha + \alpha}{u^* + \omega - \alpha})$ and

$$J(u^*, v^*) = \begin{pmatrix} -u^*(\dfrac{u^{*^2} + 2u^*\omega - 2u^*\alpha + \omega^2 - \omega\alpha + \alpha}{u^* + \omega - \alpha}) & -(u^* + \omega - \alpha)u^* \\ (2\beta u^* + \beta\omega - \beta\alpha - \lambda)v^* & 0 \end{pmatrix}$$

Where $DetJ(u^*, v^*) = (u^* + \omega - \alpha)u^*(2\beta u^* + \beta\omega - \beta\alpha - \lambda)v^* > 0$
With $\beta u^2 + (\beta\omega - \beta\alpha - \lambda)u - \lambda\omega = 0$ then $\beta\alpha - \beta\omega + \lambda = \dfrac{\beta u^2 - \lambda\omega}{u}$,
$(2\beta u + \beta\omega - \beta\alpha - \lambda)v = \dfrac{1}{u}(\beta u^2 + \lambda\omega)v > 0$ and $u + \omega - \alpha > 0$ this way
$(u^* + \omega - \alpha)u^*(2\beta u^* + \beta\omega - \beta\alpha - \lambda)v^* = (u^* + \omega - \alpha)(\beta u^* + \lambda\omega)v^* > 0$
And the behavior of singularity depends on the trace
$TraceJ(u^*, v^*) = -u^*(\dfrac{u^{*^2} + 2u^*\omega - 2u^*\alpha + \omega^2 - \omega\alpha + \alpha}{u^* + \omega - \alpha})$

If $\alpha = \dfrac{(u^*+\omega)^2}{2u^*+\omega-1}$, then the Trace is zero and (u^*, v^*) is center.
If $\alpha > \dfrac{(u^*+\omega)^2}{2u^*+\omega-1}$ then the Trace is positive and (u^*, v^*) is unstable spiral.
If $\alpha < \dfrac{(u^*+\omega)^2}{2u^*+\omega-1}$ then the Trace is negative and (u^*, v^*) is spiral attractor. □

Effect of the refuges when We intervene the area.

The behavior of the species is affected by the effect of ecological variables (readiness of refuges, formation of defense groups, difficulty of mating, appearance of other strategies anti-predator, etc.) It is shown the influence that has the refuge when the area has to be intervened by a social work, comparing the stability and values of the equilibrium points of the systems generated when analyzing all the refuges functions presented in the literature.

As $q = \theta\dfrac{A_y}{A}$ where θ is the fraction of encounters prey-predator where the prey dies, A_y is the area where each predator looks for the preys and A is the region where the preys are distributed.

And $b = \mu q$ where μ is the quantity of new predators taken place by each consumed prey.

If we reduce the area, let us say that it exists $\gamma \in (0,1)$ such that the not intervened area is $\gamma A < A$,the consumption average changes to $\dfrac{1}{\gamma}q > q$ and the rate of conversion of preys in new predators rate changes to $\dfrac{1}{\gamma}b > b$

Now it is shown the influences that have the refuges functions X_r on the variation of the preys and predators species, in the system of Lotka-Volterra.

The refuges functions increase the variation of the preys, positively and negatively to the predators.

It is considered that $a = \dfrac{1}{\gamma}$. And the refuge function in the system (3.2)

The refuge function proposed by Maynard-Smith [16] is represented by

$$\begin{aligned} \frac{dX}{dt} &= r(1-\frac{X}{K})X - qXY - (a-1)qXY + aqX_rY \equiv F(X,Y) \\ \frac{dY}{dt} &= bXY - cY + (a-1)bXY - abX_rY \equiv G(X,Y) \end{aligned} \tag{3.8}$$

(i) The system of linear equations when the area is modified from A to γA and the $X_r = \delta$,the above equation is modified to

$$\begin{aligned} \frac{dX}{dt} &= r(1-\frac{X}{K})X - qXY - (a-1)qXY + \delta aqY \equiv F(X,Y) \\ \frac{dY}{dt} &= bXY - cY + (a-1)bXY - \delta abY \equiv G(X,Y) \end{aligned} \tag{3.9}$$

The equilibrium points of (3.9) are solved as follows

$E_0 = (0,0)$ satisfies both equation

By using the second equation of (3.9)

$$\frac{dY}{dt} = bXY - cY + (a-1)bXY - \delta abY \equiv G(X,Y)$$
$$bXY - cY + (a-1)bXY - \delta abY = 0$$
$$bXY - cY + abXY - bXY - \delta abY = 0$$
$$-cY + abXY - \delta abY = 0$$
$$Y(-c + abX - \delta ab) = 0$$
$$Y = 0, -c + abX - \delta ab = 0$$
$$Y = 0, abX = \delta ab + c$$
$$Y = 0, X = \frac{c}{ab} + \delta$$
$$Y = 0, X = \gamma X_0^* + \delta$$

where $X_0^* = \frac{c}{b}, \gamma = \frac{1}{a}$ Substituting $Y = 0$ or $X = \frac{c}{ab} + \delta$ into the first equation of (3.9)
When $Y = 0$ then from $\frac{dX}{dt} = r(1 - \frac{X}{K})X - qXY - (a-1)qXY + \delta aqY \equiv F(X,Y) = 0$
We have

$$r(1 - \frac{X}{K})X - qXY - (a-1)qXY + \delta aqY = 0$$
$$r(1 - \frac{X}{K})X - qX(0) - (a-1)qX(0) + \delta aq(0) = 0$$
$$r(1 - \frac{X}{K})X = 0$$
$$X = 0, r(1 - \frac{X}{K}) = 0$$
$$X = 0, 1 - \frac{X}{K} = 0$$
$$X = 0, \frac{X}{K} = 1$$
$$X = 0, X = K$$

When $X = \frac{c}{ab} + \delta$ then from $\frac{dX}{dt} = r(1 - \frac{X}{K})X - qXY - (a-1)qXY + \delta aqY \equiv F(X,Y) = 0$

We have

$$r(1-\frac{X}{K})X - qXY - (a-1)qXY + \delta aqY \equiv F(X,Y) = 0$$
$$r(1-\frac{X}{K})X - qXY - aqXY + qXY + \delta aqY = 0$$
$$r(1-\frac{X}{K})X - aqXY + \delta aqY = 0$$
$$r(1-\frac{X}{K})X = aqXY + \delta aqY$$
$$r(1-\frac{X}{K})X = aqY(X-\delta)$$
$$aqY(X-\delta) = r(1-\frac{X}{K})X$$
$$aqY(\frac{c}{ab}+\delta-\delta) = rX(1-\frac{X}{K})$$
$$aqY(\frac{c}{ab}) = rX(1-\frac{X}{K})$$
$$aqY(\frac{c}{ab}) = rX(1-\frac{X}{K})$$
$$(\frac{c}{b}qY) = rX(1-\frac{X}{K})$$
$$Y = \frac{br}{cq}X(1-\frac{X}{K})$$
$$Y = \frac{br}{cq}(\frac{c}{ab}+\delta)(1-\frac{\frac{c}{ab}+\delta}{K})$$
$$Y = b(\frac{c}{ab}+\delta)\frac{r}{cq}(1-\frac{\frac{c}{ab}+\delta}{K})$$
$$Y = b(\frac{c}{ab}+\delta)\frac{r}{cq}(1-\frac{1}{K}(\delta+\frac{c}{ab})$$
$$Y = (\frac{c}{a}+b\delta)(\frac{r}{cq}(1-\frac{1}{K}(\delta+\frac{c}{ab}))$$
$$Y = (\gamma c+\delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK})$$

$$\text{where } \gamma = \frac{1}{a} \text{ and } Y_0^* = \frac{r}{q}(\frac{bK-c}{bK})$$

Therefore , the equilibrium points are:
$E_0 = (0,0)$, $E_1 = (K,0)$ and $E_2 = (X_2^*, Y_2^*) = (\gamma X_0^* + \delta, (\gamma c+\delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))$
where the equilibrium points are into the first quadrant Lotka-Volterra model.
The Jacobian matrix of system (3.9) is:

$$\frac{\partial F}{\partial X} = \frac{\partial}{\partial X}(r(1-\frac{X}{K})X - qXY - (a-1)qXY + \delta aqY)$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - qY - (a-1)qY + 0$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - qY - aqY + qYr(1-\frac{2X}{K}) - aqY$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - aqY$$
$$\frac{\partial F}{\partial Y} = \frac{\partial}{\partial Y}(r(1-\frac{X}{K})X - qXY - (a-1)qXY + \delta aqY)$$
$$\frac{\partial F}{\partial Y} = 0 - qX - (a-1)qX + \delta aq$$
$$\frac{\partial F}{\partial Y} = -qX - aqX + qX + \delta aq$$
$$\frac{\partial F}{\partial Y} = -aqX + \delta aq = -aq(X-\delta)$$
$$\frac{\partial G}{\partial X} = \frac{\partial}{\partial X}(bXY - cY + (a-1)bXY - \delta abY)$$
$$\frac{\partial G}{\partial X} = bY - cY + (a-1)bY - \delta)$$
$$\frac{\partial G}{\partial X} = bY + abY - bY = abY$$
$$\frac{\partial G}{\partial Y} = \frac{\partial}{\partial Y}(bXY - cY + (a-1)bXY - \delta abY)$$
$$\frac{\partial G}{\partial Y} = bX - c + abX - bX - \delta ab$$
$$\frac{\partial G}{\partial Y} = -c + abX - \delta ab$$
$$\frac{\partial G}{\partial Y} = -c + abX - \delta ab = ab(X-\delta) - c$$

Therefore ,

$$J(X,Y) = \begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{pmatrix} = \begin{pmatrix} r(1-\frac{2X}{K}) - aqY & -aq(X-\delta) \\ abY & ab(X-\delta) - c \end{pmatrix} \tag{3.10}$$

Theorem 3.4.6. *The nature of the equilibrium points of the system* (3.9). *For all parameter values it has::*
(a) The singularity $E_0(0,0)$ *is saddle point*

Proof. Evaluating the Jacobian matrix of *equation*(3.10) at $(0,0)$ we have that
If $(X,Y) = (0,0)$,then $\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - aqY = r(1-0) - 0 = r$
If $(X,Y) = (0,0)$,then $\frac{\partial F}{\partial Y} = -aq(X-\delta) = -aq(0-\delta) = -aq(-\delta) = aq\delta$

If $(X,Y)=(0,0)$,then $\dfrac{\partial G}{\partial Y}=abY=ab(0)=0$
If $(X,Y)=(0,0)$,then $\dfrac{\partial G}{\partial Y}=ab(X-\delta)-c=ab(0-\delta)-c=ab(-\delta)-c=-ab\delta-c=-(ab\delta+c)$
Therefore,

$$J(0,0)=\begin{pmatrix} r & aq\delta \\ 0 & -(\delta ab+c)\end{pmatrix}$$

The eigenvalues are
$\lambda_1=r>0$ and $\lambda_2=-(\delta ab+c)<0$,
This way $(X^*,Y^*)=(0,0)$ is hyperbolic saddle. □

(b) $(K,0)$ is saddle point, if and only if,$K>\gamma\dfrac{c}{b}+\delta$, an attractor point, if and only if $K<\gamma\dfrac{c}{b}+\delta$, a saddle-node attractor, if and only if,$K\leq\gamma\dfrac{c}{b}+\delta$

Proof. If $(X,Y)=(K,0)$,then $\dfrac{\partial F}{\partial X}=r(1-\dfrac{2K}{K})-aq(0)=r(1-2)=r(-1)=-r$
If $(X,Y)=(K,0)$,then $\dfrac{\partial F}{\partial Y}=-aq(K-\delta)=-aq(K-\delta)=-aq(K-\delta)$
If $(X,Y)=(K,0)$,then $\dfrac{\partial G}{\partial X}=abY=ab(0)=0$
If $(X,Y)=(K,0)$,then $\dfrac{\partial G}{\partial Y}=ab(K-\delta)-c=ab(K-\delta)-c=ab(K-\delta)-c$
Hence,

$$J(K,0)=\begin{pmatrix} -r & -aq(K-\delta) \\ 0 & ab(K-\delta)-c)\end{pmatrix}$$

The eigenvalues are
$\lambda_1=-r<0$ and $\lambda_2=ab(K-\delta)-c)$,then the sign of λ_2 depends on the sign of $ab(K-\delta)-c$
For $ab(K-\delta)-c>0$ hyperbolic saddle and
For $ab(K-\delta)-c<0$ an attractor point □

(c) If $K>\gamma\frac{c}{b}+\delta$,the singularity (X_2^*,Y_2^*) is a locally asymptotically stable equilibrium point

Proof. For the unique equilibrium point at the first quadrant we get:
If $(X,Y)=(\dfrac{c}{ab}+\delta,(\gamma c+\delta b)(Y_0^*+\dfrac{r(1-\gamma)}{q}+\dfrac{\delta r}{qK}))$,then

$$\frac{\partial F}{\partial X}=r(1-\frac{2(\frac{c}{ab}+\delta)}{K})-aq((\gamma c+\delta b)(Y_0^*+\frac{r(1-\gamma)}{q}+\frac{\delta r}{qK}))$$
$$\frac{\partial F}{\partial X}=r(1-2(\frac{c}{Kab}+\frac{\delta}{K}))-aq((\gamma c+\delta b)(Y_0^*+\frac{r(1-\gamma)}{q}+\frac{\delta r}{qK}))$$
$$\frac{\partial F}{\partial X}=r(1-2(\frac{c}{Kab}+\frac{\delta}{K}))-aq((\gamma c+\delta b)(Y_0^*+\frac{r(1-\gamma)}{q}+\frac{\delta r}{qK}))$$
$$\frac{\partial F}{\partial X}=r(1-2(\frac{c+ab\delta}{Kab}))-aq((\gamma c+\delta b)(Y_0^*+\frac{r(1-\gamma)}{q}+\frac{\delta r}{qK}))$$
$$\frac{\partial F}{\partial X}=-r((2(\frac{c+ab\delta}{Kab})-1)+aq((\gamma c+\delta b)(Y_0^*+\frac{r(1-\gamma)}{q}+\frac{\delta r}{qK})))$$

If $(X,Y) = (\frac{c}{ab} + \delta, (\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))$,then

$$\frac{\partial F}{\partial Y} = -aq(X - \delta) = -aq(\frac{c}{ab} + \delta - \delta)$$
$$\frac{\partial F}{\partial Y} = -aq(\frac{c}{ab}) = -q(\frac{c}{b}) = -qX_0^*$$

If $(X,Y) = (\frac{c}{ab} + \delta, (\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))$,then

$$\frac{\partial G}{\partial X} = abY = ab(\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))$$
$$\frac{\partial G}{\partial X} = ab(\frac{c}{a} + \delta b)(Y_0^* + \frac{r(a-1)}{aq} + \frac{\delta r}{qK})$$

If $(X,Y) = (\frac{c}{ab} + \delta, (\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))$,then

$$\frac{\partial G}{\partial Y} = ab(X - \delta) - c = ab(\frac{c}{ab} + \delta - \delta) - c = c - c = 0$$

Therefore, $J(X_2^*, Y_2^*) = \begin{pmatrix} -r((2(\frac{c+ab\delta}{Kab}) - 1) + aq((\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))) & -qX_0^* \\ ab(\frac{c}{a} + \delta b)(Y_0^* + \frac{r(a-1)}{aq} + \frac{\delta r}{qK}) & 0 \end{pmatrix}$

The Trace of the matrix is

$$\begin{aligned} T(J(X_2^*, Y_2^*)) = \lambda_1 + \lambda_2 &= -r((2(\frac{c+ab\delta}{Kab}) - 1) + aq((\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))) + 0 \\ &= -r((2(\frac{c+ab\delta}{Kab}) - 1) + aq((\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))) \\ &< 0 \end{aligned}$$

Determinant of the matrix is

$$\begin{aligned} DJ(X_2^*, Y_2^*) &= (-r((2(\frac{c+ab\delta}{Kab}) - 1) + aq((\gamma c + \delta b)(Y_0^* + \frac{r(1-\gamma)}{q} + \frac{\delta r}{qK}))))(0) \\ &- (-ab(qX_0^*)(\frac{c}{a} + \delta b)(Y_0^* + \frac{r(a-1)}{aq} + \frac{\delta r}{qK} \\ &= abqX_0^*(\frac{c}{a} + \delta b)(Y_0^* + \frac{r(a-1)}{aq} + \frac{\delta r}{qK}) + 0 \\ &= abqX_0^*(\frac{c}{a} + \delta b)(Y_0^* + \frac{r(a-1)}{aq} + \frac{\delta r}{qK}) \\ &> 0 \end{aligned}$$

Then (X_2^*, Y_2^*) is node attractor. □

(ii) The system of linear equations when the area is modified from A to γA and the $X_r = \eta X$. The system that incorporates refuge function uses in proportion to the prey size

is given by:

$$\begin{aligned}\frac{dX}{dt} &= r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY \equiv F(X,Y)\\ \frac{dY}{dt} &= bXY - cY + (a-1)bXY - \eta abXY \equiv G(X,Y)\end{aligned} \tag{3.11}$$

Solving equation (3.11) for X and Y as follows:
Using the second equation of (3.11), we have:

$$\begin{aligned}&\frac{dY}{dt} = bXY - cY + (a-1)bXY - \eta abXY \equiv G(X,Y) = 0\\ &\frac{dY}{dt} = bXY - cY + (a-1)bXY - \eta abXY = 0\\ &bXY - cY + (a-1)bXY - \eta abXY = 0\\ &bXY - cY + abXY - bXY - \eta abXY = 0\\ &-cY + abXY - \eta abXY = 0\\ &Y(-c + abX - \eta abX) = 0\\ &Y = 0, -c + abX - \eta abX = 0\\ &Y = 0, abX - \eta abX = c\\ &Y = 0, Xab(1-\eta) = c\\ &Y = 0, X = \frac{c}{ab(1-\eta)}\end{aligned}$$

Substituting $Y = 0$ into the first equation of (3.11), we have

$$\begin{aligned}&\frac{dX}{dt} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY \equiv F(X,Y) = 0\\ &\frac{dX}{dt} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY = 0\\ &r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY = 0\\ &r(1-\frac{X}{K})X - qXY - aqXY + \eta aqXY = 0\\ &r(1-\frac{X}{K})X - qX(0) - aqX(0) + \eta aqX(0) = 0\\ &r(1-\frac{X}{K})X - 0 - 0 + 0 = 0\\ &r(1-\frac{X}{K})X = 0\\ &X = 0, (1-\frac{X}{K}) = 0\\ &X = 0, \frac{X}{K} = 1\\ &X = 0, X = K\end{aligned}$$

Substituting $X = \dfrac{c}{ab(1-\eta)}$ into the first equation of (3.11), we have

$$\frac{dX}{dt} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY \equiv F(X,Y) = 0$$
$$\frac{dX}{dt} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY = 0$$
$$r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY = 0$$
$$r(1-\frac{X}{K})X - qXY - aqXY + qXY + \eta aqXY = 0$$
$$aqXY - \eta aqXY = r(1-\frac{X}{K})X$$
$$aqXY(1-\eta) = r(1-\frac{X}{K})X$$
$$aqY(\frac{c}{ab(1-\eta)})(1-\eta) = r(1-\frac{X}{K})X$$
$$qY(\frac{c}{b}) = r(1-\frac{X}{K})X$$
$$Y(\frac{c}{b}q) = rX(1-\frac{X}{K})$$
$$Y = (\frac{br}{cq})(X)(1-\frac{X}{K}) = (\frac{br}{cq})(\frac{c}{ab(1-\eta)})(1-\frac{1}{K}(\frac{c}{ab(1-\eta)}))$$
$$Y = \frac{r}{aq(1-\eta)}(1-\frac{1}{K}(\frac{c}{ab(1-\eta)})) = \frac{r}{aq(1-\eta)}(1-\frac{c}{Kab(1-\eta)})$$

Therefore,the equilibrium points of (3.11) are:

$$E_0 = (0,0)$$
$$E_1 = (K,0) \text{ and}$$
$$E_2 = (X_3^*, Y_3^*) = (\frac{c}{ab(1-\eta)}, (\frac{r}{aq(1-\eta)}(1-\frac{c}{Kab(1-\eta)}))$$

The partial derivative $(\dfrac{\partial F}{\partial X}, \dfrac{\partial F}{\partial Y}, \dfrac{\partial G}{\partial X}, \dfrac{\partial G}{\partial Y})$ of (3.11) are:

$$\frac{\partial F}{\partial X} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - qY - (a-1)qY + \eta aqY$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - qY - aqY + qY + \eta aqY$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - aqY + \eta aqY$$
$$\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - aq(1-\eta)Y$$
$$\frac{\partial F}{\partial Y} = r(1-\frac{X}{K})X - qXY - (a-1)qXY + \eta aqXY$$
$$\frac{\partial F}{\partial Y} = -qX - (a-1)qX + \eta aqX = -qX - aqX + qX + \eta aqX$$
$$\frac{\partial F}{\partial Y} = -aqX + \eta aqX = -aq(1-\eta)X$$
$$\frac{\partial G}{\partial X} = bXY - cY + (a-1)bXY - \eta abXY$$
$$\frac{\partial G}{\partial X} = bY + (a-1)bY - \eta abY$$
$$\frac{\partial G}{\partial X} = bY + abY - bY - \eta abY$$
$$\frac{\partial G}{\partial X} = abY - \eta abY = ab(1-\eta)Y$$
$$\frac{\partial G}{\partial Y} = bXY - cY + (a-1)bXY - \eta abXY$$
$$\frac{\partial G}{\partial Y} = bX - c + (a-1)bX - \eta abX$$
$$\frac{\partial G}{\partial Y} = bX - c + abX - bX - \eta abX$$
$$\frac{\partial G}{\partial Y} = -c + abX - \eta abX = ab(1-\eta)X - c$$

Therefore, The Jacobian matrix of system (3.11) is:

$$J(X,Y) = \begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{pmatrix} = \begin{pmatrix} r(1-\frac{2X}{K}) - aq(1-\eta)Y & -aq(1-\eta)X \\ ab(1-\eta)Y & ab(1-\eta)X - c \end{pmatrix} \tag{3.12}$$

Theorem 3.4.7. *For the singularities of the system* (3.11) *one has:*
(a) The singularity $(0,0)$ *is saddle point for all parameter values.*

Proof. Evaluating the Jacobian matrix of (3.12) at $(0,0)$ we have that
If $(X,Y) = (0,0)$,then $\frac{\partial F}{\partial X} = r(1-\frac{2X}{K}) - aq(1-\eta)Y = r(1-0) - 0 = r$

If $(X,Y)=(0,0)$,then $\dfrac{\partial F}{\partial Y}=-aq(1-\eta)X=-aq(1-\eta)(0)=0$
If $(X,Y)=(0,0)$,then $\dfrac{\partial G}{\partial X}=ab(1-\eta)Y=ab(1-\eta)(0)=0$
If $(X,Y)=(0,0)$,then $\dfrac{\partial G}{\partial Y}=ab(1-\eta)X-c=ab(1-\eta)(0)-c=0-c=-c$
Therefore,

$$J(0,0)=\begin{pmatrix} r & 0 \\ 0 & -c) \end{pmatrix}$$

The determinant of the above matrix is
$DetJ(0,0)=(-r)(c)-(0)(0)=-rc-0=-rc<0$,then $E_0=(0,0)$ is saddle point □

(b)The equilibrium point$(K,0)$ *is a saddle-node attractor(a non hyperbolic equilibrium point) only if* ,$ab(1-\eta)K-c>0$ *and globally asymptotically stable if and only if,*$ab(1-\eta)K-c<0$ *. In this case it does not exist an equilibrium point at interior of the first quadrant*

Proof. Evaluating the Jacobian matrix of (3.12) at $(K,0)$ we have that
If $(X,Y)=(K,0)$,then $\dfrac{\partial F}{\partial X}=r(1-\dfrac{2K}{K})-aq(1-\eta)(0)=r(1-2)-0=-r$
If $(X,Y)=(K,0)$,then $\dfrac{\partial F}{\partial Y}=-aq(1-\eta)X=-aq(1-\eta)(K)=-aq(1-\eta)K$
If $(X,Y)=(K,0)$,then $\dfrac{\partial G}{\partial X}=ab(1-\eta)Y=ab(1-\eta)(0)=0$
If $(X,Y)=(K,0)$,then $\dfrac{\partial G}{\partial Y}=ab(1-\eta)X-c=ab(1-\eta)(K)-c=ab(1-\eta)K-c$
Hence,

$$J(K,0)=\begin{pmatrix} -r & -aq(1-\eta)K \\ 0 & ab(1-\eta)K-c) \end{pmatrix}$$

The eigenvalues are:
$\lambda_1=-r<0$ and $\lambda_2=ab(1-\eta)K-c$
Then the sign of λ_2 depends on the sign of $ab(1-\eta)K-c$.

For $ab(1-\eta)K-c>0$ hyperbolic saddle
For $ab(1-\eta)K-c<0$ an attractor point

□

(c)The unique positive equilibrium point (X_3^*,Y_3^*) *,is globally asymptotically stable if and only if,*$ab(1-\eta)K-c>0$
In this case the equilibrium $(K,0)$ *is saddle point*

Proof. Evaluating the Jacobian matrix of (3.12) at (X_3^*,Y_3^*),we have that
If $(X,Y)=(\dfrac{c}{ab(1-\eta)},(\dfrac{r}{aq(1-\eta)}(1-\dfrac{c}{Kab(1-\eta)})))$,then

$$\frac{\partial F}{\partial X} = r(1 - \frac{2}{K}(\frac{c}{ab(1-\eta)})) - aq(1-\eta)(\frac{r}{aq(1-\eta}(1 - \frac{1}{K}(\frac{c}{ab(1-\eta)}))$$
$$\frac{\partial F}{\partial X} = r(1 - \frac{2c}{Kab(1-\eta)}) - r(1 - \frac{c}{Kab(1-\eta)})$$
$$\frac{\partial F}{\partial X} = r(1 - \frac{2c}{Kab(1-\eta)} - 1 + \frac{c}{Kab(1-\eta)})$$
$$\frac{\partial F}{\partial X} = r(-\frac{2c}{Kab(1-\eta)} + \frac{c}{Kab(1-\eta)})$$
$$\frac{\partial F}{\partial X} = r(\frac{-2c+c}{Kab(1-\eta)}) = r(\frac{-c}{Kab(1-\eta)}) = \frac{-rc}{Kab(1-\eta)}$$

If $(X,Y) = (\frac{c}{ab(1-\eta)}, (\frac{r}{aq(1-\eta)}(1 - \frac{c}{Kab(1-\eta)})))$,then
$\frac{\partial F}{\partial Y} = -aq(1-\eta)(\frac{c}{ab(1-\eta)}) = -aq(1-\eta)X_3^*$
If $(X,Y) = (\frac{c}{ab(1-\eta)}, (\frac{r}{aq(1-\eta)}(1 - \frac{c}{Kab(1-\eta)})))$,then
$\frac{\partial G}{\partial X} = ab(1-\eta)(\frac{r}{aq(1-\eta)}(1 - \frac{c}{Kab(1-\eta)}))) = ab(1-\eta)Y_3^*$
If $(X,Y) = (\frac{c}{ab(1-\eta)}, (\frac{r}{aq(1-\eta)}(1 - \frac{c}{Kab(1-\eta)})))$,then
$\frac{\partial G}{\partial Y} = ab(1-\eta)X - c = ab(1-\eta)(\frac{c}{ab(1-\eta)} - c = c - c = 0$
Therefore,

$$J(X_3^*, Y_3^*) = \begin{pmatrix} \frac{-rc}{Kab(1-\eta)} & -aq(1-\eta)X_3^* \\ ab(1-\eta)Y_3^* & 0 \end{pmatrix}$$

The Trace,
$T(J(X_3^*, Y_3^*)) = \lambda_1 + \lambda_2 = \frac{-rc}{Kab(1-\eta)} + 0 = \frac{-rc}{Kab(1-\eta)} < 0$ and
Determinant
$D(J(X_3^*, Y_3^*)) = (\frac{-rc}{Kab(1-\eta)})(0) - (-aq(1-\eta)X_3^*)(ab(1-\eta)Y_3^*) = a^2bq(1-\eta)^2X_3^*Y_3^* > 0$
Then (X_3^*, Y_3^*) is an attractor point. □

(iii) The system of linear equations when the area is modified from A to γA considering the refuge function $X_r = \frac{\delta X}{X+\eta}$
The system is expressed by:

$$\begin{aligned} \frac{dx}{dt} &= r(1 - \frac{X}{K})X - qXY - (a-1)qXY + \delta aq\frac{XY}{X+\eta} \equiv F(x,y \\ \frac{dy}{dt} &= bXY - cY + (a-1)bXY - \delta ab\frac{XY}{X+\eta} \equiv G(x,y \end{aligned} \tag{3.13}$$

The equilibrium points of (3.13) are

$$E_0 = (0,0)$$

$$E_1 = (K, 0) \text{ and}$$
$$E_2 = (X_4^*, Y_4^*) = (\frac{1}{2ab}(H + \sqrt{H^2 + 4abc\eta}), \frac{1}{2c}H - \frac{1}{2abcK}(H^2 + H\sqrt{H^2 + 4abc\eta} + 2abc\eta)$$
$$\text{where } H = ab(\delta - \eta) + c \text{ and } \delta - \eta > 0$$

The Jacobian matrix of system (3.12) is:

$$J(X,Y) = \begin{pmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} \end{pmatrix} = \begin{pmatrix} r(1 - \frac{2X}{K}) - (aq - \frac{\eta}{(X+\eta)^2})Y & -aq(1 - \frac{\delta}{X+\eta})X \\ ab(1 - \frac{\delta\eta}{(X+\eta)^2})Y & ab(1 - \frac{\delta}{X+\eta})X - c \end{pmatrix}$$

Theorem 3.4.8. *The nature of the equilibrium points of the system* (3.13) *For all parameter values it has:*
(a) The singularity $(0,0)$ *is saddle point.*

Proof. Evaluating the above Jacobian matrix at $(0,0)$ we have that

$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -c) \end{pmatrix}$$

The determinant of the above matrix is
$DetJ(0,0) = (-r)(c) - (0)(0) = -rc - 0 = -rc < 0$,then $E_0 = (0,0)$ is saddle point □

(b) $(K,0)$ *is saddle point,if and only if* $ab(1 - \frac{\delta}{K+\eta})K - c > 0$*: an attractor point if and only if,* $ab(1 - \frac{\delta}{K+\eta})K - c < 0$*,and a saddle node attractor if and only if* $ab(1 - \frac{\delta}{K+\eta})K - c \lneq 0$

Proof.

$$J(K,0) = \begin{pmatrix} -r & -aq(1 - \frac{\delta}{K+\eta})K \\ 0 & ab(1 - \frac{\delta}{K+\eta})K - c \end{pmatrix}$$

The eigenvalues are: $\lambda_1 = -r < 0$ and $\lambda_2 = ab(1 - \frac{\delta}{K+\eta})K - c$
Then the sign of λ_2 depends on the sign of $ab(1 - \frac{\delta}{K+\eta})K - c$
If $ab(1 - \frac{\delta}{K+\eta})K - c > 0$,then $(K,0)$ is saddle point.
If $ab(1 - \frac{\delta}{K+\eta})K - c < 0$,then $(K,0)$ is saddle point.
If $ab(1 - \frac{\delta}{K+\eta})K - c \leq 0$,then $(K,0)$ is saddle point.

□

(c) For $Kab(1-\frac{\delta}{K+\eta})-c>0$ *the singularity* (X_4^*, Y_4^*) *one has:*

(i) If $c=\frac{(X_4^*+\frac{\eta}{K})^2}{2X_4^*+\frac{\eta}{K}-1}$ *the system* (3.11) *has a unique limit cycle, surrounding the* (X_4^*, Y_4^*) *unique equilibrium point at the first quadrant.*

(ii) Unstable focus,if $c>\frac{(X_4^*+\frac{\eta}{K})^2}{2X_4^*+\frac{\eta}{K}-1}$

(iii) If $c<\frac{(X_4^*+\frac{\eta}{K})^2}{2X_4^*+\frac{\eta}{K}-1}$*,the singularity* (X_4^*, Y_4^*) *of system* (3.12) *is a locally asymptotically stable equilibrium point.*

Chapter 4

Numerical Simulation

Numerical simulation help us to analyze the phase diagram of dynamical systems depending up on parameters. It is possible to find more precise numerical solutions to a given to a given set of differential equations using various methods in MATLAB.The method used by MATLAB is ode45 solvers.

In this section we provide the numerical simulations to illustrate some results of the previous sections.Mainly, we present the time plots of the solution X and Y with the phase diagrams (around the positive equilibrium points) for the predator-prey system.Dynamical nature of the system about the equilibrium points with different sets of parameter values are presented.The result are illustrated in the following figures and analysis of result are presented below.

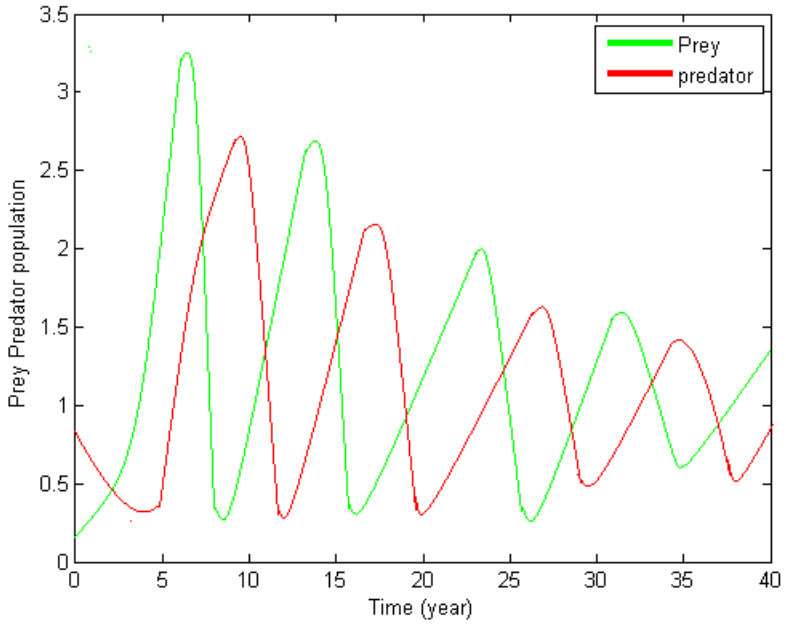

Figure 4.1: simple prey predator model

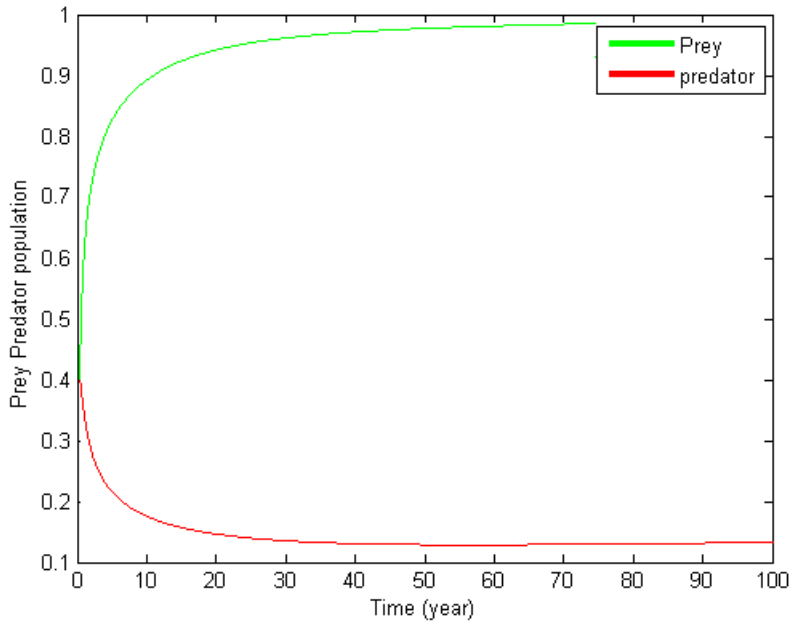

Figure 4.2: For $u(0 = 0.4), v(0) = 0.5\alpha = 0.85, \beta = 0.2, \lambda = 0.8, \omega = 0.2$ shows u approaches 1 and v approaches 0 in finite time

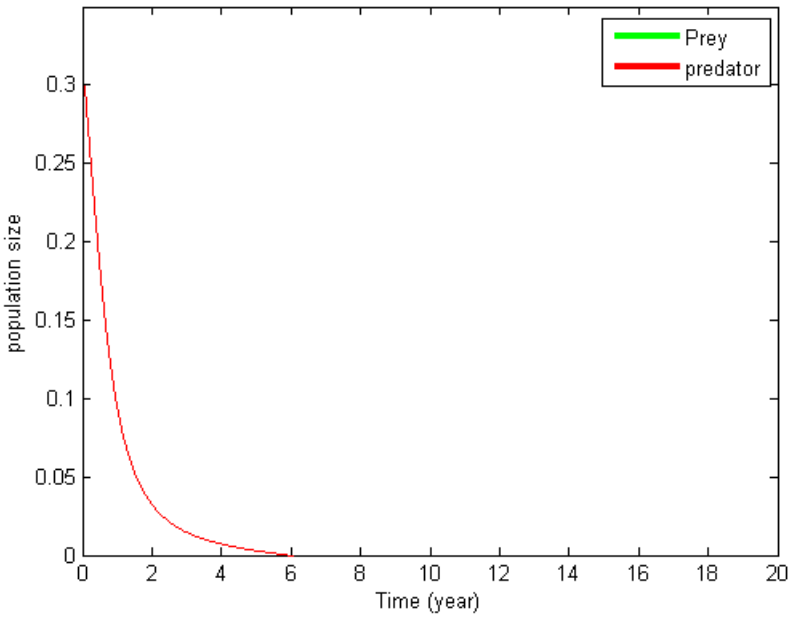

Figure 4.3: consider the values $\alpha = 0.25, \beta = 0.85, \lambda = 1, \omega = 0.4$. For this choice of parameter values the unique axial equilibrium point is stable.The time plot and the phase diagram illustrate the result and predators' extinction

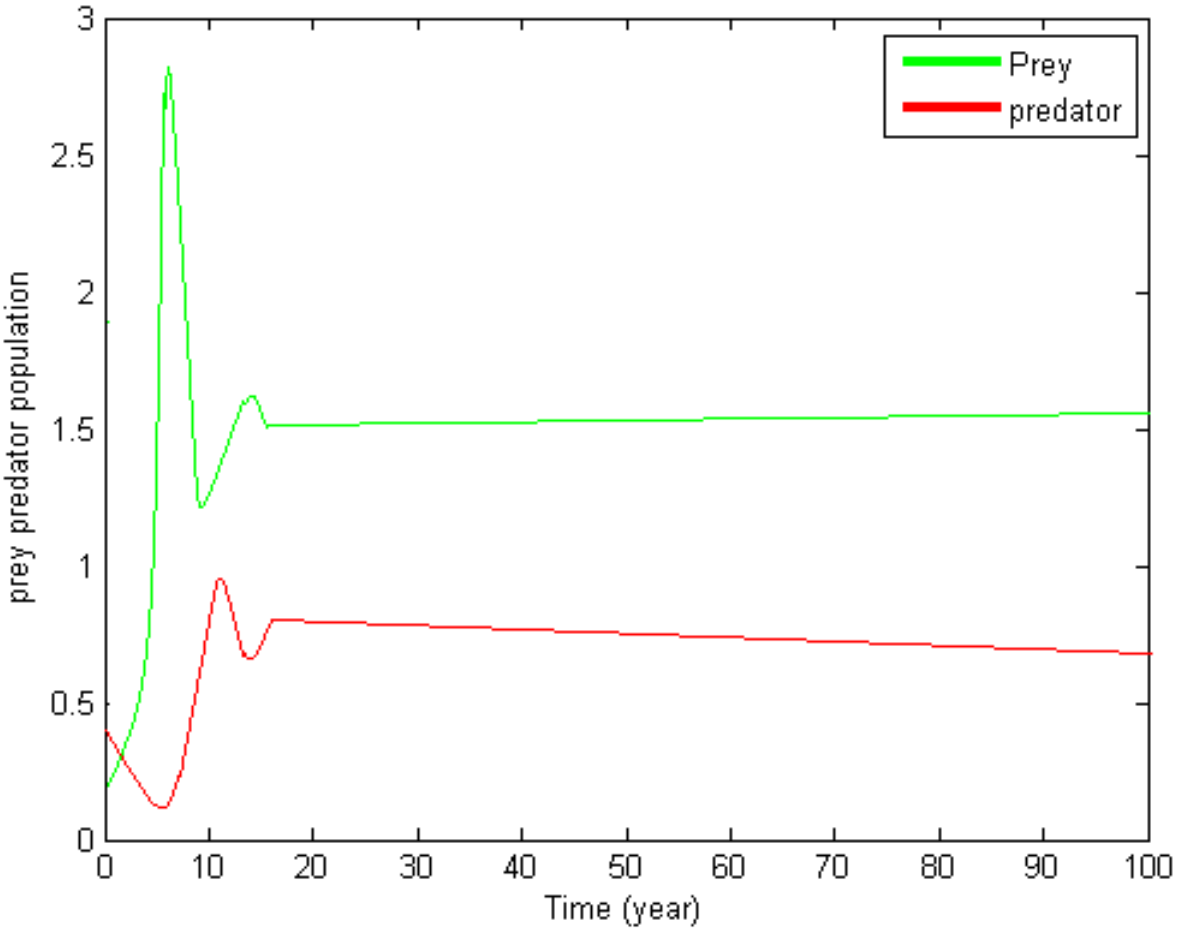

Figure 4.4: Here $u(0 = 0.1), v(0) = 0.35\alpha = 0.25, \beta = 0.85, \lambda = 1, \omega = 0.46u, v$ approach their equilibrium values $u^* = 1.5257732, v^* = 0.61855617$.

Chapter 5

Discussion and Conclusion

5.1 Discussion

In this paper a model that describes the prey-predator with prey refuge in Holling type II Functional response has been proposed and studied analytically as well as numerically. The effect of each parameter on the dynamical behavior of system is studied numerically and the trajectories of the system are drown in the typical figures. According to these figures, which represent the solution of system for the data given, The System has no periodic dynamics rather than the fact that the system approaches asymptotically to one of their equilibrium points depending on the set of parameter data and the stability conditions that are satisfied. The existence of an equilibrium point within the first quadrant is conditional refuge size, because if $\alpha \to \frac{(1+\omega)(\beta-\lambda)}{\beta}$, collapses to the point $(u, v) = (1, 0)$.

From Theorem (3.3.5), We can find the relationship between the area that is trimmed and the refuge functions so that it does not change the quantity of preys in the ecosystem $\frac{c}{ab} + \delta = \frac{c}{b}$ (Because,$\gamma = \frac{1}{a}$) Therefore $\delta = (1-\gamma)\frac{c}{b}$

From Theorem (3.3.6), $\frac{c}{b} = \frac{c}{ab} + \frac{c}{ab(1-\eta)} = (1-\gamma)\frac{c}{b}$ (Because,$\gamma = \frac{1}{a}$) $\Leftrightarrow \eta = \frac{a-2}{a-1}$

Therefore,$\eta = \frac{a-2}{a-1}$ with the same conditions it can be a relationship between δ of $X_r = \delta$ and η of $X_r = \eta X$, obtaining that $\delta = \frac{\gamma\eta}{b(1-\eta)}$

Theorem (3.3.7) considers that the relation between the size of the refuge and the carrying capacity $ab(1 - \frac{\delta}{K+\eta})K - c > 0)$.

This theorem observes that the area portion that is when clipping the ecosystem appears, if the equilibrium point inside the first quadrant can change its stability.

5.2 Conclusion

This paper presents a prey-predator model with Holling type II functional response incorporating prey refuge. Refuge can be considered as, areas in which the predator is not successfully controlling the prey and important for the biological control of a predator. The main focus of this paper was to introduce mathematical models of biological systems and techniques for their analysis.Moreover, we established some new results such as the existence of stable or unstable equilibrium points under suitable values of parameters in the models. Two species can coexist in the case of stable condition; otherwise they might be extinct in the case of unstable condition. The preys react fleeing the refuge due to the presence of a certain amount of predator.This work was shown in Theorem (3.4.4), the assertion that regular use refuge stabilizes predator-prey interaction depends on the size of the refuge. We show that this way of modeling the refuge function limits cycles are obtained, which indicates periodic solutions. It is important to note that the parameter of maximum physical capacity of refuge in influences in the number of equilibrium points.

When the area is modified from A to γA the equilibrium point to the interior of the first quadrant comes closer to the origin. We can find the relationship between the clipped area and the refuge function so that it does not change the quantity of preys in the equilibrium point in the quadrant with biological sense It concludes by analyzing the dynamics of predator prey Lotka-Volterra, that modifying the A area at where they live species by construction activities of civil works. The equilibrium point to the interior of the first quadrant comes closer to the origin in the same proportion the intervened area. The conditions for the stabilities are similar to that of the original system, We can find the relationship between the area that is intervened and the refuge functions so that it does not change the quantity of preys in the ecosystem. We observe that a refuge takes good advantage for the prey. It is observed that the refuge function proposed presents the best option to preserve the species, since it relates the size of the refuge, the carrying capacity and the land fraction that can take off and keep the original stability conditions without the interference.

5.3 Recommendation

The knowledge of the impact of prey refuge use by a fraction of prey-population is relevant in the context of bio-economic and conservation management, because it helps in regulating the harvesting activity in the ecosystem and management of reserves or non-take zones; also it is essential for conservation of endangered species creating protected areas (reserves) for preserve them . For this reason, other models for interaction between species must be considered.

Appendix A

Mathematical Preliminaries

A.1 Basic definition and results

In this section, we provide the basic mathematical definitions and results, which are crucial for a better understanding of this thesis work.

Definition A.1.1. Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2 . Then the equilibrium point $(0, 0)$ can be:

1. If $\lambda_1 < \lambda_2 < 0$ then, $(0, 0)$ is asymptotically stable and we refer to it as a *sink*.
2. If $0 < \lambda_1 < \lambda_2$ then, $(0, 0)$ is unstable and we refer to it as a *source*.
3. If $\lambda_1 < 0 < \lambda_2$ then, $(0, 0)$ is unstable and we refer to it as a *saddle*.
4. If $\lambda_1 = -ib, \lambda_2 = +ib$,then $(0, 0)$ is stable and we refer to it as a *center* or *orbit*.
5. If $\lambda_1 = -a - ib, \lambda_2 = -a + ib$ for $a > 0$, then $(0, 0)$ is asymptotically stable and we refer to it as a*stable spiral*.
6. If $\lambda_1 = a - ib, \lambda_2 = a + ib$ for $a > 0$, then $(0, 0)$ is unstable and we refer to it as an *unstable spiral*.
7. If $\lambda_1 = \lambda_2 < 0$,and the single eigenvalue has two linearly independent eigenvectors then, $(0, 0)$ is asymptotically stable and we refer to it as a *stable proper node*.
8. If $\lambda_1 = \lambda_2 < 0$, and the single eigenvalue does not have two linearly independent eigenvectors then $(0, 0)$ is asymptotically stable and we refer to it as a *stable improper node*.
9. If $\lambda_1 = \lambda_2 > 0$,and the single eigenvalue has two linearly independent eigenvectors then, $(0, 0)$ is unstable and we refer to it as a *unstable proper node*.
10. If $\lambda_1 = \lambda_2 > 0$, and the single eigenvalue does not have two linearly independent eigenvectors then $(0, 0)$ is asymptotically stable and we refer to it as a *unstable improper node*.

Consider the nonlinear autonomous dynamical system

$$\frac{dx}{dt} = f(x) \tag{A.1}$$

where $x \in \Re^n$

Definition A.1.2. A point $x^* \in R^n$ is an equilibrium point of (A.1) if $f(x^*) = 0$. An equilibrium point x^* is said to be hyperbolic if none of the eigenvalues of the Jacobian matrix $Df(x^*)$ of $f(x)$, calculated at the equilibrium point x , has real part equal to zero.

Definition A.1.3. Saddle-Node Equilibrium Point[18]: A non-hyperbolic equilibrium point $p \in R^n$ of (A.1) is called a saddle-node equilibrium point if the following conditions are satisfied:
(i) $Dxf(p)$ has a unique simple null eigenvalue and none of the other eigenvalues have real part equal to zero.
(ii) $w(D_x^2 f(p)(v, v)) \neq 0$, with v as the right eigenvector and w the left eigenvector associated with the null eigenvalue.

Definition A.1.4. A function $f(x)$ is said to be locally Lipschitz on a domain $\Omega \subset R^n$ if each point of Ω has a neighborhood Ω_0 such that f satisfies

$\|f(x) - f(y)\| \leq L\|x - y\|$
for all $x, y \epsilon \Omega_0$ f is globally Lipschitz if it is locally Lipschitz for every neighborhood Ω_0 of R^n .

Definition A.1.5. Suppose x_0 is an equilibrium point of the function F, then

1. x_0 is an attracting equilibrium point of F , if $|F(x_0)| < 1$
2. x_0 is a repelling equilibrium point of F,if $|F(x_0)| > 1$
3. x_0 is a neutral equilibrium point of F,if $|F(x_0)| = 1$

Attractive and repellent fixed points are sometimes know as sinks and sources, respectively. Fixed points that are neutral can behave in various ways. They can be weakly attracting, weakly repelling or neither attracting nor repelling.

Theorem A.1.1. ***stable and unstable equilibrium.*** *An equilibrium point (x^*, y^*) of the differential equation is stable if all the eigenvalues of J, the Jacobian evaluated at (x^*, y^*) have negative real parts. The equilibrium point is unstable if at least one of the eigenvalues has a positive real part.*
As a summary,
***Asymptotically stable**. A critical point is asymptotically stable if all eigenvalues of the jacobian matrix J are negative, or have negative real parts.*
***Unstable** A critical point is if at least one eigenvalue of the jacobian matrix J is positive, or has positive real part.*
***Stable (or neutrally stable)** Each trajectory move about the critical point within a finite range of distance.*

Definition A.1.6. Predation[12] is the consumption of one organism (prey) by another (predator)

Definition A.1.7. Invariant set . A set of points $A \in E$ is **invariant under** f if $x \in A \Rightarrow O(x) \in A$. (Can also define forward and backward invariant sets in the obvious way).

Definition A.1.8. Trace of marix A The sum of the diagonal entries of an $n \times n$ matrix A is called the trace of matrix A , denoted by T(A).

Definition A.1.9. limit cycle A limit cycle is an isolated closed trajectory.

A.2 Equilibria and Stability

Definition A.2.1. Equilibrium Point [18] An equilibrium point (fixed point, stationary point, rest point, singularity, critical point, steady state) of the system (A.1) is a point $x_* \epsilon R^n$ such that

$$f(x_*) = 0$$

i.e., a solution which does not change in time.
Stability properties characterize how a system behaves if its state is initiated close to, but not precisely at a given equilibrium point.

Definition A.2.2. Stable [18] A critical point $x_*$is stable provided that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$\|x_0 - x_*\| < \delta \Longrightarrow \|x(t) - x_*\| < \varepsilon$ for $t > 0$.

The critical point x_* is called unstable if it is not stable

Definition A.2.3. Asymptotically Stable[18] A critical point x_* is asymptotically sta ble if there exists $\delta > 0$ such that

$\|x_0 - x_*\| < \delta \Longrightarrow \lim_{\longrightarrow \infty} x(t) = x_*$.

A.3 The Routh-Hurwitz Criteria

Theorem A.3.1. *Given a polynomial* $p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + ...a_{n-1}\lambda + a_n$ *where the coefficients* a_i *are real constants,* $i = 1, 2, 3...n$ *define the n Hurwitz matrices using the coefficients of the characteristic polynomial:*
$H_1 = \begin{bmatrix} a_1 \end{bmatrix}$

$$,H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},$$

$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \cdots$ *and*

$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \cdots & \cdots & a_n \end{bmatrix}$ *where* $a_j = 0$ *for* $j > n$ *. All the roots of polynomial* $p(\lambda)$ *are negative or have negative real parts iff. the determinants of all Hurwitz matrices are positive:*

$detH_j > 0$,$j = 1, 2, 3...n$

for $n = 1$,*then* $detH_1 = a_1 > 0$

for $n = 2$ *then* $detH_2 = H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} = a_1 a_2 > 0$ *or* $a_1 > 0$ *and* $a_2 > 0$ *and*

for $n = 3$ *then* $detH_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = a_1 > 0, a_3 > 0$ *and* $a_1 a_2 - a_3 > 0$ *polynomials of degree* $1, 2$ *and* 3*, the Routh Hurwitz Criteria are summarized as follows:*

$n = 1$ *then* $a_1 > 0$

$n = 2$ *then* $a_1 > 0$ *and* $a_2 > 0$

$n = 3$ *then* $a_1 > 0$, $a_2 > 0$ *and* $a_1 a_2 - a_3 > 0$

A.4 Linear Systems and Linearizion

A.4.1 Linear Systems

Consider a linear systems

$$\frac{dx}{dt} = AX \tag{A.2}$$

,$x \epsilon R^n$, where $A$is a matrix of order n. Then the solutions of $(A.3)$ is given by

$x(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} x_i$, $c_i's$ are arbitrary constants

where $\lambda_i's$ are the eigenvalues of A and $x_i's$ are the corresponding eigenvectors. The stability of the solutions of a linear systems is determined by the sign of the eigen values of its coefficient matrix. Particularly, we have the following characterization theorem for a linear systems

Theorem A.4.1. *Let A be a 2×2 matrix and consider the system*

$\dot{x} = Ax,$

where $x \in R^2$, then stability is equivalent to

$traceA < 0 and\ detA > 0$

A.4.2 Linearization

To determine the behavior near a critical point, we will linearize the non-linear system around the critical point and use our knowledge of linear systems. The linearization of the non-linear system (A.1) around the equilibrium point x_* is

$$\frac{dx}{dt} = Jx \tag{A.3}$$

where J is the Jacobian matrix evaluated at x_* , i.e,

$$J := J(x_*) = \begin{bmatrix} \frac{df_1}{dx_1}(x_*) & \frac{df_1}{dx_2}(x*) & \cdots & \frac{df_1}{dx_n}(x_*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_n}{dx_1}(x_*) & \frac{df_n}{dx_2}(x_*) & \cdots & \frac{df_n}{dx_n}(x_*) \end{bmatrix}$$

The local behavior of the solutions near a critical point is described by the following theorems

Theorem A.4.2. ***Poincar - Lyapunov*** *Suppose x_* is a critical point of the non-linear system $(A.1)$, and suppose the $real(\lambda)$, the real part of the eigenvalues of J (the lin- earization) are negative. Then the critical point is locally asymptotically stable.*

Theorem A.4.3. *Suppose x_* is a critical point, and the real part of at least one eigen value of J is positive. Then the critical point is unstable*

Theorem A.4.4. ***The Poincare-Bendixson Theorem*** *For a system of two first order ordinary differential equations, consider a closed bounded region D. Suppose a positive half path, H, lies entirely within D. Then one of the following is true:*
1. H is a closed trajectory, e.g. a limit cycle;
2. H asymptotically tends to a closed trajectory, e.g. a limit cycle;
3. H terminates on a stationary point.
Therefore, if D does not have a stationary point then there must be a limit cycle.

A.4.3 Hartman-Grobman Theorem

Definition A.4.1. Hyperbolic Equilibrium Point A critical point is called hyperbolic if the real part of the eigenvalues of the Jacobian matrix J are nonzero. The next theorem is a guarantee that the non-linear system inherits the behavior of the linearized system.

Theorem A.4.5. ***Hartman-Grobman.*** *Suppose that x_* is a hyperbolic critical pointof system the nonlinear system $(A.1)$. Then, in a neighborhood of x_* , the system $(A.1)$ and its corresponding linearization are equivalent; i.e., there is a homeomorphism h that maps trajectories in $(A.1)$ near x_* onto trajectories in $(A.4)$.*

A.5 Lyapunov Theory

A powerful method for analyzing the stability of an equilibrium point is based on the use of Lyapunov functions [39].

Definition A.5.1. A function V : $R^n \Longrightarrow R$ is said to be a positive-definite function if

$(i) V(x) > 0$ for all$x \neq 0$,

$(ii) V(x) = 0$ if and only if $x = 0$.

Definition A.5.2. Let x_* be an equilibrium point of $(A.1)$ and let $V : U \Longrightarrow R$ be a C^1 function defined on some neighbourhood U of x_*such that
(i) V is positive-definite,
$(ii)\dot{V}(x) \leq 0$ in $U \setminus x$.
Any function,V , that satisfies the Conditions (i) and (ii) above is called a Lyapunov Function

Definition A.5.3. Invariant Set A set M is an invariant set with respect to $(A.1)$ if
$x(0) \in M \Rightarrow x(t) \in M, \forall t \in R$

Definition A.5.4. Positively Invariant A set M is a positively invariant set if
$x(0) \in M \Rightarrow x(t) \in M, \forall t \geq 0$

A.6 Some famous functional responses

In this section, we describe some famous functional responses which are crucial for a better understanding of this thesis work. As referred in [12],there are three types of Functional responses described as follows.
Holling Type I functional response The Holling Type I functional response is
$g_1(N) = AN, A > 0$ and the logistic ODE with proportional or constant rate harvesting is $\frac{dN}{dt} = rN(1 - \frac{N}{K}) - ANP$. For predators with a Type I functional response, the rate of prey consumption increases linearly with the prey population size. If the number of prey quadruple, the predators will eat four times as much per day. Predators with such unlimited appetites are rarely found in nature.
Holling Type II functional response.The Holling Type II functional response is
$g_2(N) = \frac{AN}{B+N}$, where $A, B > 0$ and the logistic ODE with Holling Type II response is $\frac{dN}{dt} = rN(1 - \frac{N}{K}) - \frac{AN}{B+N}P$. For predators with a Type II functional response, the rate of

prey consumption increases with the prey popula- tion size, but saturates at some maximum level A. This functional response seems to be the most common and is well documented in empirical studies . Other names for this functional response are the Monod response or Michaelis-Merten response. This response is characteristic of organisms that require non-trivial amounts of time to capture and ingest their prey. Holling gave a simple mechanistic explanation of this functional response. Predation involves two tasks: searching for prey and consuming the prey (chasing, killing, eating, and digesting).At low prey densities, predators spend most of their time searching for prey, and at high prey densities, predators spend most of their time on handling prey. Transforming from prey density to prey population gives the desired functional form.

Holling Type III functional response The Holling Type III functional response is $g_3(N) = \frac{AN^2}{(B^2+N^2}$), where $A, B > 0$ and the logistic ODE with Holling Type III response is $\frac{dN}{dt} = rN(1-\frac{N}{K}) - \frac{AN^2}{B^2+N^2}P$. The parameter B is known as the switching value, and when $N = B$ the value of the Type III predation function is $A/2$, exactly one-half its maximum.This response is characteristic of predators that below a certain prey density threshold, do not eat much of the prey, but when the prey density is above the threshold, their feeding rate increases as the prey population increases, but eventually levels off to an asymptote.This response captures two feeding effects. As for the Type II response, at high prey densities, predators spend most of their time on prey handling. Also, since $g_3^{'}(N) = 0$, at low prey densities, predators have a difficult time finding prey. This could occur for several reasons. Predators may lose their search image of the prey. There could be a small number of refuges which hide the small number of prey from predators. At low prey density, predators may consume alternate prey. Depending on the parameters, there can be either two, three, or four equilibrium points. The condition $g_3^{'}(N) = 0$ ensures that the equilibrium point $N = 0$ is always unstable. Thus the prey population never goes extinct

A.7 Bifurcation

Bifurcation is the qualitative changes in the dynamics of a system, which occurs when the system parameters are changed. The parameter values where bifurcation occurs are called bifurcation values (bifurcation point). A standard definition of bifurcation at a point is given below.

Definition A.7.1. let

$$\frac{dx}{dt} = f(x, \mu), x \in R, x \in R \tag{A.4}$$

be a one-parameter family of one-dimensional ODEs. An equilibrium solution of $(A.4)$ given by $(x, \mu) = (0, 0)$ is said to undergo bifurcation at $\mu = 0$ if the flow for μ near zero and x

near zero is not qualitatively the same as the flow near $x = 0$ at $\mu = 0$. There are several types of bifurcation which includes;
Saddle-node bifurcation : which occurs when fixed points exist and are destroyed by changing the values of some pa rameters.
Transcritical bifurcation: which corresponds to the case where the fixed points change stability with the change of the values of some parameter.
Pitchfork bifurcation: where equilibrium points appear and disappear in symmetrical pairs.
Hopf bifurcation: which happens when a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the imaginary axis of the complex plane.

A.8 Matlab Codes

Matlab codes or ode45 codes for figure 4.1.

```
function dc=prey predator(t,c)
    dc=zeros(2,1);
    alpha=0.85;
    Omega=0.4;
    labda=0.5;
    beta=0.2;
    dc(1)=c(1)*[(1-c(1))*(c(1)+Omega)-(c(1)+Omega-alpha)*c(2)];
    dc(2)=(2)*[beta*(c(1)+Omega-alpha)*c(1)-labda*(c(1)+Omega)];
    t=[0 40];
    y0=[0.4, 3.5];
    [T,C1] = ode45(@prey predator,t,y0);
    plot(T,C1(:,2),'g','LineWidth',3)
    xlabel('Time(year)')
    ylabel('Prey Predator population ')
    hold on
    t=[0 40];
    y0=[0.5,3.5];
    [T,C] = ode45(@prey predator,t,y0);
    plot(T,C(:,2),'r','LineWidth',3)
    plot(T,C1(:,2),'g',T,C(:,2),'r','LineWidth',3)
    legend('Prey','predator')
    axis([0 40 0 3.5])
    xlabel('Time (year)')
    ylabel('Prey Predator population ')
```

Matlab codes or ode45 codes for figure 4.2.

```
function dc=prey predator(t,c)
```

```
dc=zeros(2,1);
alpha=0.85;
Omega=0.65;
labda=0.5;
beta=0.25;
dc(1)=c(1)*[(1-c(1))*(c(1)+Omega)-(c(1)+Omega-alpha)*c(2)];
dc(2)=(2)*[beta*(c(1)+Omega-alpha)*c(1)-labda*(c(1)+Omega)];
t=[0 100];
y0=[0.4, 1.2];
[T,C1] = ode45(@prey predator,t,y0);
plot(T,C1(:,2),'g','LineWidth',3)
xlabel('Time(year)')
ylabel('Prey Predator population ')
hold on
t=[0 100];
y0=[0.5,1.2];
[T,C] = ode45(@prey predator,t,y0);
plot(T,C(:,2),'r','LineWidth',3)
plot(T,C1(:,2),'g',T,C(:,2),'r','LineWidth',3)
legend('Prey','predator')
axis([0 100 0 1])
xlabel('Time (year)')
ylabel('Prey Predator population ')
```

Matlab codes or ode45 codes for figure 4.3.

```
function dc=prey predator(t,c)
dc=zeros(2,1);
alpha=0.5;
Omega=0.4;
labda=0.35;
beta=0.82;
dc(1)=c(1)*[(1-c(1))*(c(1)+Omega)-(c(1)+Omega-alpha)*c(2)];
dc(2)=(2)*[beta*(c(1)+Omega-alpha)*c(1)-labda*(c(1)+Omega)];
t=[0 20];
y0=[0.2, 0.55];
[T,C1] = ode45(@prey predator,t,y0);
plot(T,C1(:,2),'g','LineWidth',3)
xlabel('Time(year)')
ylabel(' population size ')
hold on
t=[0 20];
```

```
y0=[0.1,0.55];
[T,C] = ode45(@prey predator,t,y0);
plot(T,C(:,2),'r','LineWidth',3)
plot(T,C1(:,2),'g',T,C(:,2),'r','LineWidth',3)
legend('Prey','predator')
axis([0 20 0 0.35])
xlabel('Time (year)')
ylabel(' population size ')
```

Matlab codes or ode45 codes for figure 4.4.

```
function dc=prey predator(t,c)
dc=zeros(2,1);
alpha=0.25;
Omega=0.6;
labda=1;
beta=0.85;
dc(1)=c(1)*[(1-c(1))*(c(1)+Omega)-(c(1)+Omega-alpha)*c(2)];
dc(2)=(2)*[beta*(c(1)+Omega-alpha)*c(1)-labda*(c(1)+Omega)];
t=[0 100];
y0=[0.35, 3];
[T,C1] = ode45(@prey predator,t,y0);
plot(T,C1(:,2),'g','LineWidth',3)
xlabel('Time(year)')
ylabel(' prey predator population ')
hold on
t=[0 100];
y0=[0.53,3];
[T,C] = ode45(@prey predator,t,y0);
plot(T,C(:,2),'r','LineWidth',3)
plot(T,C1(:,2),'g',T,C(:,2),'r','LineWidth',3)
legend('Prey','predator')
axis([0 100 0 3])
xlabel('Time (year)')
ylabel(' prey predator population ')
```

Bibliography

[1] A sih, (1987). Prey refuges and predator-prey stability, Theoretical Population Biology: Theoretical Population Biology, 31(1987), 1-12

[2] Collings, J.B (1995).Bifurcation and stability analysis of a temperature-dependent mite predator prey interaction model incorporating a prey refuge.
Bull. Math. Biol. 57(1),63- 76

[3] C. S. Holling, (1965). The functional response of predators to prey density and its role in mimicry and population regulation, Mem. Entomolog. Soc. Can. 45, 3-60

[4] E. Almanza-Vasquez, E. Gonzalez-Olivares, B. Gonzalez-Yanez . Dynam-ics of Lotka-Volterra predator-prey model considering saturated refuge for prey, BIOMAT 2011 International Symposium on Mathematical and Computational Biology, World Scientific Co. Pte. Ltd., Singapore (2012), 62-72.

[5] E. C. Pielou, (1977). Mathematical Ecology, A wiley-Interscience Publication, John Wiley and Sons, New York

[6] E.GonzalezOlivers and R.Ramos-Jilberto(2003). Dynamic consequences of prey refuges in a simple model system more prey, fewer predators and enhanced stability, Ecological Modeling,166(2003),no. 1-2, 135-146.

[7] E.Saez and E.Gonzalez-Olivares(1999). Dynamics of a predator-prey model, SIAM J. April.Math., 59 (1999), no.5, 1867-1878.

[8] F. Courchamp, L. Berec and J. Gascoigne (2008). Allee effects in Ecology and Conservation, Oxford University Press.

[9] Ginzburg, L.R., (1988). Assuming reproduction to be a function of consumption raises doubts about some popular predator prey models. J. Anim. Ecol. 67, 325-327.

[10] Harrison, G.W., (1979). Global stability of predatorprey interactions.
J. Math. Biol. 8,39-171.

[11] Hassell, M.P. (1978). The Dynamics of Arthropod PredatorPrey Systems. Princeton University Press,Prince-ton, NJ

[12] Holling, C.S. (1970). Some characteristics of simple types of predation and parasitism. Can. Entomo l91,385395

[13] : Hoy, M.A.: Almonds (California). In: Helle, W., Sabelis, M.W. (eds.), (1985)...Spider Mites: Their Biology, Natural Enemies and Control. World Crop Pest, vol. 1B, pp. 229 310.Elsevier, Amsterdam

[14] Huang, Y., Chen, F., Zhongs, L.(2006). Stability analysis of preypredator model with Holling type III response function incorporating a prey refuge. J. Appl. Math. Comput. 182, 672-683

[15] J. D. Murray, (1993). Mathematical Biology, Springer-Verlag, New York

[16] J.Maynard-smith(1974). Models in Ecology, University press Cambridge
Use the refuge to protect the food chain

[17] J.N.Mc Nair(1986) The effects of refuges on predator-prey interactions: reconsideration, Theoretical population Biology, 29(1986), no.1, 38-63

[18] J. Sotomayor(1973), Generic bifurcations of dynamical systems. In Dynamical Systems,, 549-560.

[19] Lotka, A. J. (1925). Elements of physical biology.Williams and Wilkins. Baltimore, Maryland, USA

[20] May, R.M. (1974). Stability and Complexity in Model Ecosystems.Princeton University Press, Princeton, p. 261.

[21] M. Kot, (2001).Elements of Mathematical Ecology,Cambridge University Press, Cambridge

[22] Murdoch, W.W., Oaten, A. (1975).Predation and populationstability. Adv. Ecol. Res. 9, 2-132.

[23] P.D.N.Srinivasu and I.L. Gayatri(2005). Influence of prey reserve capacity on predator-prey dynamics, Ecological Modeling, 181(2005), 191-202.

[24] P. T. Abrams and C. J.(1996). Walters, Invulnerable prey and the paradox of enrichment, Ecology, 77 (4), (1996), 1125-1133

[25] P. Turchin (2003). Complex population Dynamics: A Theoretical /Empirical synthesis, University Press, Princeton

[26] P. Turchin(2013). Complex population dynamics, A theoretical/empirical synthesis, Monographs in Population Biology 35, Princeton University Press

[27] R.J Taylor(1984). Predation, Chapman and Hall, NewYork

[28] R. M. May(2001).Stability and complexity in model ecosystems, Princeton University Press

[29] Rosenzweig, M.L.(1971) S.R.Paradox of enrichment: destabilization of exploitation ecosystem in ecological time. Science 171, 385-387

[30] Ruxton, G.D. (1995). Short term refuge use and stability of predatorprey models.Theor.Popul. Biol. 47, 1-17.

[31] Scheffer, M., de Boer, R.J., (1995):Implications of spatial heterogeneity for the paradox of enrichment. Ecology 76 (7), 2270-2277.

[32] V.krivan(1998). Effects of optimal anti-predator behavior of prey on predator-prey dynamics:The role of refuges, Theoretical population Biology, 53(1998), 131-142.

[33] Volterra, V. (1927). Variations and fluctuations in the numbers of coexisting animal species.

[34] Yodzis, P.(1989).Introduction to Theoretical Ecology. Harper and Row, New York, p. 384.