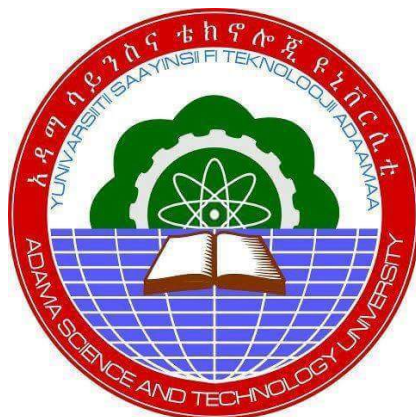


**MATHEMATICAL INVESTIGATIONS INTO BOUNDARY LAYER FLOWS OF
CASSON NANOFUIDS WITH HEAT AND MASS TRANSFER CHARACTERISTICS**



Kifle Adula Duguma

**A Dissertation Submitted to the Department of Applied Mathematics
School of Applied Natural Sciences**

**Presented in Fulfillment of the Requirement for the Degree of Doctor of Philosophy in Applied
Mathematics(Specialization in Computational Fluid Dynamics)**

**Office of Graduate Studies
Adama Science and Technology University**

**January 2024
Adama, Ethiopia**

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Declaration

I declare that "**Mathematical Investigations into Boundary Layer Flows of Casson Nanofluids with Heat and Mass Transfer Characteristics**" is my own work, that it has not been submitted before for any degree or examination at any other university, and that all sources I have used or quoted have been indicated and acknowledged by complete references.

Mr. Kifle Adula Duguma
Name of Student

Signature

Date

Recommendation

We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled "**Mathematical Investigations into Boundary Layer Flows of Casson Nanofluids with Heat and Mass Transfer Characteristics**" prepared under our guidance by Kifle Adula submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics. Therefore, we recommend the submission of the dissertation to the department for further review and defense.

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Approval Page

We, hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**Mathematical Investigations into Boundary Layer Flows of Casson Nanofluids with Heat and Mass Transfer Characteristics**" and developed by Kifle Adula Duguma.

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We, the undersigned, members of the Board of Examiners of the dissertation open defense by Kifle Adula Duguma have read and evaluated the dissertation entitled "**Mathematical Investigations into Boundary Layer Flows of Casson Nanofluids with Heat and Mass Transfer Characteristics**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics.

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Finally, approval and acceptance of the dissertation is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the candidate's Department Graduate Council (DGC) and School Graduate Committee (SGC).

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Dedication

It is dedicated to my family and parents.

Abstract

Mounting temperatures in electronic devices during operation may damage sensitive internal components if too much thermal energy is accumulated inside the system. The advent of innovative ultrahigh-performance thermal management technologies known as nanofluid has provided a veritable platform to improve system performance and reliability by removing the high heat generated in engineering and industrial devices. Hence, due to its numerous significance, the study of boundary layer Casson nanofluid flow with mass and heat transfer has garnered vast interest in engineering applications. Therefore, this dissertation examined the combined applications of boundary layer Casson nanofluid flow through a non-Darcian porous medium in the process of mass and heat transfer enhancement. Taking this into consideration, theoretically computational analysis has been done on Casson fluid flow problems such as thermophoresis and Brownian motion effects on boundary layer Casson nanofluid flow with mass and heat transfer subjected to injection/suction and chemical reactions, modeling of heat enhancement characteristics of Cu-H₂O-Casson nanofluid flow subjected to injection/suction, and stagnation point flow of CoFe₂O₄/TiO₂-H₂O-Casson nanofluid past a convectively heated permeable slippery shrinking/stretching sheet in a Darcy-Forchheimer porous medium in the presence of viscous dissipation. By applying suitable similarity transformations, the governing nonlinear partial differential equations (PDEs) are converted into a system of nonlinear ordinary differential equations (ODEs) and then tackled numerically using the shooting method with the Runge-Kutta-Fehlberg fourth-fifth order (RKF45) integration procedure embedded in the MAPLE software package. In all problems, dual solutions are demonstrated, and using hydrodynamic temporal stability analysis, their properties are investigated. The impacts of the embedded governing parameters on temperature, velocity, and concentration profiles, as well as skin friction coefficient, the Nusselt and Sherwood numbers are demonstrated in detail through graphs, charts, and tables. The stability analysis revealed that only the upper branch solutions are stable and physically realizable, whereas the lower branch (second) solutions are unstable and physically unachievable. Furthermore, the smallest positive eigenvalue is numerically displayed for the upper branch solutions.

Keywords: Casson nanofluid; Dual Solutions; Shrinking/Stretching sheet; Convective heating; Temporal stability analysis; Darcy-Forchheimer porous medium; Viscous dissipation; Stagnation point flow.

List of Publications

From this PhD dissertation three articles listed below are published, in peer reviewed journals and indexed in Scopus to the scientific community in the field of applied mathematics and engineering journal.

1. Duguma, K.A., Makinde, O.D., and Enyadene, L.G. (2023). Stability analysis of dual solutions of convective flow of casson nanofluid past a shrinking/stretching slippery sheet with thermophoresis and brownian motion in porous media. *Journal of Mathematics*, 2023. <https://doi.org/10.1155/2023/5954860>
2. Duguma, K.A., Makinde, O.D., and Enyadene, L.G. (2023). Dual Solutions and Stability Analysis of Cu-H₂O-Casson Nanofluid Convection past a Heated Stretching/Shrinking Slippery Sheet in a Porous Medium. *Computational and Mathematical Methods*, 2023. <https://doi.org/10.1155/2023/6671523>
3. Duguma, K.A., Makinde, O.D., and Enyadene, L.G. (2023). Stagnation Point Flow of CoFe₂O₄/TiO₂-H₂O-Casson Nanofluid Past a Slippery Stretching/Shrinking Cylindrical Surface in a Darcy-Forchheimer Porous Medium. *Journal of Engineering*, 2023. <https://doi.org/10.1155/2023/8238703>

Contents

Declaration	i
Recommendation	ii
Approval Page	iii
Acknowledgements	iv
Dedication	v
Abstract	vi
List of Publications	vii
Table of Contents	xi
List of Tables	xii
List of Figures	xv
Nomenclatures	xvi
1 Introduction	1
1.1 Background of the Study	1
1.1.1 Casson Fluid Flow and Its Applications	1
1.1.2 Nanofluid and its applications	5
1.1.3 Heat and Mass Transfers	10
1.1.4 Chemical Reaction	14
1.1.5 Flow with Viscous Dissipation	14
1.1.6 Flows with Slip Boundary Conditions	15
1.1.7 Flows with Suction/Injection	16
1.1.8 Porous Medium and Its Applications	16
1.1.9 Reynolds Number and Its Significance	17
1.2 Statement of the Problem	18
1.3 Objectives of the Study	19
1.3.1 General Objective of the Study	19

1.3.2	Specific Objectives of the Study	19
1.4	Significance of the Study	20
1.5	Scope of the Study	21
1.6	Organization of the Dissertation	21
2	Literature Review	22
2.1	Boundary Layer Flow of Casson Nanofluids	22
2.2	Boundary Layer Flow with Heat and Mass Transfer	24
2.3	Boundary Layer Flow in a Porous Medium	28
2.4	Stagnation Point Casson Nanofluids Flow in a PM	31
3	Research Methodology	37
3.1	Governing Equations	37
3.1.1	The Continuity Equation	37
3.1.2	The Momentum Equation (Navier-Stokes Equation)	38
3.1.3	The Energy Equation	42
3.1.4	The Species Concentration Equation	48
3.2	Computational Methods of Solution	49
3.2.1	The Shooting Technique	49
3.2.2	Generalized Newton Raphson Technique	50
3.2.3	Runge-Kutta-Fehlberg Technique (RKF45)	51
4	Stability Analysis of Dual Solutions of Convective Flow of Casson Nanofluid Past a Shrinking/Stretching Slippery Sheet in Porous Medium	54
4.1	Introduction	54
4.2	Formulation of Mathematical Model	54
4.3	Similarity Transformation	58
4.4	Physical Quantities of Engineering Interest	62
4.5	Stability Analysis	63
4.6	Procedures for Numerical Method	69
4.7	Results and Discussion	70
4.7.1	Velocity Profiles with Change of Parameters	71
4.7.2	Temperature Profiles with change of Parameters	73

4.7.3	Concentration Profiles with change of Parameters	75
4.7.4	The Skin Friction, Local Nusselt and Sherwood Numbers	77
4.7.5	Existence of Dual Solutions	81
5	Dual Solutions and Stability Analysis of Cu-H₂O-Casson Nanofluid Convection Past a Heated Stretching/Shrinking Slippery Sheet in a Porous Medium	87
5.1	Introduction	87
5.2	Mathematical Model Formulation	87
5.3	Similarity Transformation	89
5.4	Physical Quantities of Engineering Interest	92
5.5	Stability Analysis	93
5.6	Procedures for Numerical Methods	97
5.7	Results and Discussion	98
5.7.1	Existence of Dual Solutions	99
5.7.2	The Skin Friction and Rate of Heat Transfer	102
5.7.3	Heat Transfer Enhancement	104
5.7.4	Velocity Profile	106
5.7.5	Temperature Profile	107
6	Stagnation Point Flow of CoFe₂O₄/TiO₂-H₂O-Casson Nanofluid Past a Slippery Stretching/Shrinking Cylindrical Surface in a Darcy-Forchheimer Porous Medium	111
6.1	Introduction	111
6.2	Mathematical Model Formulation	111
6.3	Similarity Transformation	113
6.4	Engineering Interest Parameters	116
6.5	Stability Analysis	117
6.6	Numerical Procedures	121
6.7	Results and Discussion	122
6.7.1	Existence of Dual Solutions	122
6.7.2	Rate of Heat Transfer	125
6.7.3	Velocity Profile	129
6.7.4	Temperature Profile	133
6.7.5	Heat Transfer Enhancement	136

7	Summary, Conclusion and Recommendation	140
7.1	Summary	140
7.2	Conclusion	141
7.3	Recommendation and Suggestions for Future Research	145
7.3.1	Recommendation	145
7.3.2	Suggestions for Future Research	145
	References	146
	Appendix	172

List of Tables

1.1.1 Thermophysical properties of H ₂ O, Casson fluid, some nanoparticles at $T = 300K$	10
4.7.1 Computations showing comparison of numerical values of drag force (skin friction) for various values of λ with β , $Da \rightarrow \infty, Fr = \delta = S = 0$	70
4.7.2 Computations showing the effects of different parameters on dimensionless shear stress/local skin friction $(1 + \frac{1}{\beta})f''(0)$, heat transfer rate/local Nusselt number $-\theta'(0)$ and mass transfer rate/local Sherwood number $-\phi'(0)$	82
4.7.3 Numerical results of critical shrinking parameter (λ_c), upper and lower branch solutions of skin friction ($\sqrt{Re_x}C_f$) for different values of different parameters	85
4.7.4 Numerical results of critical shrinking parameter (λ_c), upper and lower branch of the least eigenvalues for varying values of different parameters	86
5.7.1 Comparison of values of skin friction $\sqrt{Re_x}C_f$ for varying values of stretching/shrinking parameter λ with $\phi = 0, \beta \rightarrow \infty, Da \rightarrow \infty, Fr = S = \delta = 0$	99
5.7.2 The computation showing critical shrinking parameter, skin friction and the smallest eigenvalues for both upper and lower solutions for varying values of ϕ and λ when $\beta = 10, Da = Fr = S = \delta = 0.1$ for Cu-H ₂ O nanofluid	103
5.7.3 The computation showing impact of parameters variation on critical shrinking parameter, skin friction and the smallest eigenvalues for both upper and lower solutions for Cu-H ₂ O nanofluid when $\phi = 0.1$ is fixed	103
6.7.1 Comparison of values of the dimensionless stream function (f) and temperature (θ) at $\eta = 2$ for varying values of the free stream Reynolds number Re when $\phi = Fr = \lambda = \delta = Ec = 0, Pr = 1.0, \beta \rightarrow \infty, Da \rightarrow \infty$ and $Bi \rightarrow \infty$	123
6.7.2 Comparison of values of skin friction $Re^{-\frac{1}{2}}f''(1)$ for varying values of the free stream Reynolds number Re , when $\beta \rightarrow \infty, Da \rightarrow \infty, Fr = \phi = \lambda = \delta = 0$	123
6.7.3 The computational results of critical shrinking parameter and skin friction for varying values of parameters for both upper branch (UB) and lower branch (LB) solutions	127
6.7.4 The computational results for critical shrinking parameter and the smallest eigenvalues ε of both upper branch (UB) and lower branch (LB) solutions	127
6.7.5 The computational results for velocity profile, skin friction, temperature profile, and the Nusselt number for varying values of parameters	132

List of Figures

1.1.1 Flow curve of a shear-thinning material (Mezger, 2015)	3
1.1.2 Some non-Newtonian and Casson Fluids.	4
1.1.3 Nanofluids Synthesis, (Sezer et al., 2019)	6
1.1.4 Brownian Motion on a Sphere, (Price and Williams, 1983)	7
1.1.5 Motion of Colloidal Particles, (Perrin, 1909)	7
1.1.6 The particle temperature effect on the thermophoresis (Su et al., 2022a)	7
1.1.7 Modes of Heat Transfer (Naomi Watts, 2021)	11
1.1.8 Natural, Forced and Mixed Convection (Cengel and Ghajar, 2015)	12
1.1.9 Mass Transport: Description of Diffusion Mechanism (Conidi et al., 2020)	13
1.1.10 Types of Slip Velocity: Modified from (Panigrahi and Asfer, 2008)	15
1.1.11 (a) Perforated Plate Acting as Porous Media (b) Darcy-Forchheimer Plate Flow Embedded in a Cylinder (Miguel, 2019)	17
1.1.12 Laminar-Transition-Turbulent Flow Reynolds Number (Halila et al., 2015)	18
3.1.1 Control Volume Physical Model	37
4.2.1 The Casson nanofluid flow physical model	55
4.7.1 Effects of: (a) Fr and β (b) δ and Da on the flow velocity profile.	72
4.7.2 Effects of: (a) λ and S on the flow velocity profile (b) Fr and β on the flow temperature profile.	72
4.7.3 Effects of: (a) δ and Da (b) λ and S on the flow temperature profile.	75
4.7.4 Effects of:(a) Pr and Ec (b) Bi and Le on the temperature profile.	75
4.7.5 Effects of: (a) Nt and R on the temperature (b) Fr and β on the flow concentration profiles.	77
4.7.6 Effects of: (a) δ and Da (b) λ and S on the flow concentration profile.	78
4.7.7 Effects of:(a) Pr and Ec (b) Bi and Le on the concentration profile.	78
4.7.8 Effects of: (a) Nt and R (b) Nb on the concentration profile	79
4.7.9 (a) Skin friction against Fr for varying values of β and Da . (b) Skin friction against S for varying values of δ and λ	79
4.7.10 (a) Nusselt number against Fr for varying values of Nt , Da , and β . (b) Nusselt number against S for varying values of δ and λ	80
4.7.11 Nusselt number against Pr for varying values of Ec and Bi	80

4.7.12 (a) Sherwood number against R for varying values of Le and Nb . (b) Sherwood number against Ec for varying values of Nb and Nt	81
4.7.13 Skin friction against λ for different values of (a) δ . (b) β	83
4.7.14 Skin friction against λ for different values of (a) Da . (b) Fr	84
4.7.15 Skin friction against λ for different values of S	84
4.7.16 Local Nusselt number against λ for different values of (a) Ec . (b) Bi	85
5.2.1 Physical flow diagram of the model.	88
5.7.1 Skin friction against λ for varying values of: (a) ϕ . (b) β	101
5.7.2 Skin friction against λ for varying values of: (a) Da . (b) Fr	101
5.7.3 Skin friction against λ for varying values of: (a) S . (b) δ	102
5.7.4 Nusselt number against λ for varying values of: (a) Ec . (b) Bi	102
5.7.5 Skin friction against λ for varying values of: (a) ϕ , δ , and Da (b) Fr , S , and β	105
5.7.6 Nusselt number against: (a) λ for varying values of ϕ , Da , δ , and β . (b) Bi for varying values of Ec , Fr , and S	105
5.7.7 Rate of heat transfer enhancement with ϕ for fixed values $Pr = 6.2$, $\beta = 10$, $Da = Fr = S = \delta = Ec = Bi = 0.1$	106
5.7.8 Velocity profile against η for varying values of: (a) λ and β (b) Da and Fr	107
5.7.9 Velocity profile against η for varying values of: (a) ϕ and S (b) δ	108
5.7.10 Temperature profile against η for varying values of: (a) λ and β (b) Da and Fr	110
5.7.11 Temperature profile against η for varying values of: (a) ϕ and S (b) Ec and Bi	110
5.7.12 Temperature profile against η for varying values of δ	110
6.2.1 Schematic diagram of a stationary cylinder with radial stagnation flow and Casson nanofluid in porous media	112
6.7.1 Skin friction against λ for varying values of: (a) ϕ . (b) β	125
6.7.2 Skin friction against λ for varying values of: (a) Da . (b) Fr	125
6.7.3 Skin friction against λ for varying values of: (a) Re . (b) δ	126
6.7.4 Skin friction against: (a) ϕ with varying β and δ (b) Da with varying Fr , and Re	126
6.7.5 Nusselt number against λ with varying (a) ϕ (b) β	129
6.7.6 Nusselt number against λ with varying (a) Da (b) Fr	129
6.7.7 Nusselt number against λ with varying (a) Re (b) δ	130
6.7.8 Nusselt number against λ with varying (a) Pr (b) Ec	130

6.7.9	Nusselt number against (a) λ with varying Bi . (b) ϕ with varying β and δ .	131
6.7.10	Nusselt number against (a) Fr with varying Da and Re (b) Ec with varying Bi and λ .	131
6.7.11	Velocity profile against η for varying values of: (a) ϕ (b) β .	133
6.7.12	Velocity profile against η for varying values of: (a) Da . (b) Fr .	134
6.7.13	Velocity profile against η for varying values of: (a) Re . (b) δ .	134
6.7.14	Velocity profile against η for varying values of λ .	135
6.7.15	Temperature profile against η for varying values of (a) ϕ (b) β .	136
6.7.16	Temperature profile against η for varying values of: (a) Da (b) Fr .	137
6.7.17	Temperature profile against η for varying values of: (a) Re (b) δ .	137
6.7.18	Temperature profile against η for varying values of: (a) Pr (b) Ec .	138
6.7.19	Temperature profile against η for varying values of: (a) Bi (b) λ .	138
6.7.20	Percentage heat transfer enhancement for varying values of ϕ .	139

Nomenclatures

a, b, c, n	Real constants	S	Suction/injection parameter
Bi	Biot number	T	Temperature of the fluid
C_f	Coefficient of the skin friction	T_f	Local fluid temperature
C_p	Specific heat at constant pressure	T_∞	Fluid ambient temperature
Da	Darcy number	u, v	Velocities along x, y coordinates
D_B	Brownian diffusion Coefficient	U_∞	Fluid free stream velocity
D_T	Thermophoretic diffusion Coefficient	V_0	Constant suction/injection velocity
e_{ij}	The rate of strain tensor	z, r	Axial and radial coordinates, respectively
Ec	Eckert number	Greek symbols	
f	Dimensionless stream function	β	Casson parameter/factor
F	Forchheimer drag force coefficient	γ	Chemical reaction rate
Fr	Forchheimer parameter	δ	Velocity slip parameter
h_f	Convective heat transfer coefficient	ε	Eigenvalue parameter
k_1	Porous medium permeability	η	Similarity variable
k_f	Base fluid thermal conductivity	θ	Dimensionless temperature
k_s	Nanoparticles' thermal conductivity	λ	Velocity ratio parameter
k_{nf}	Effective thermal conductivity	μ_B	Plastic dynamic viscosity
L	Slip length coefficient	μ_f	Dynamic viscosity of the base fluid
Le	Lewis number	μ_{nf}	Effective dynamic viscosity
$\mathbf{n}$	Normal unit vector	ν_f	Base fluids' Kinematic viscosity
N_b	Brownian movement parameter	π	$(i, j)^{th}$ component of deformation rate
N_t	Thermophoresis parameter	π_c	Critical value of π
Nu_x	Local Nusselt number	ρ_f	Density of the base fluid
Pr	Prandtl number	ρ_s	Density of the solid nanoparticle
p_y	Yield stress of the fluid	ρ_{nf}	Effective density of the nanofluid
$\mathbf{q}$	velocity of the nanofluid	$(\rho C_p)_{nf}$	Nanofluid effective heat capacity
q_w	Heat flux	τ	Non-dimensional time variable
q_s	Mass(specie) flux	τ_{ij}	Components of stress tensor
Re	Reynolds number	τ_w	Wall skin friction
R	Chemical reaction parameter	ϕ	Nanoparticles volume fraction

Chapter 1

Introduction

1.1 Background of the Study

Fluid is any substance which is fit for streaming effectively when shear pressure is connected on it. The fluid comprises of two sub-types: viscous (real) and inviscid (ideal) fluids ([Barnea and Taitel, 1993](#)). Viscous fluid theory plays an imperative role in actual life due to its incredible applications in science and technology. The fluids having no(zero) viscosity are called in-viscid fluids. Fluid dynamics is the main branch of fluid mechanics, which like the study of any other branches of science needs mathematical analysis as well as experimentation ([Eckert, 2007](#)). The analytical approach helps us finding the solution of certain idealized and simplified problems and understands the unity behind apparently dissimilar phenomena. The field is very vast and has given birth to other subjects like meteorology, Newtonian flows, non-Newtonian flows, gas dynamics, computational fluid dynamics, magnetohydrodynamics (MHD), etc. Computational fluid dynamics (CFD) is a field that uses numerical analysis and data structures to analyze and solve problems that involve fluid flows ([Chung, 2002](#)), applied to a wide range of research and engineering problems in many fields of study and industries, including aerodynamics and aerospace analysis, hypersonics, weather simulation, natural science and environmental engineering, industrial system design and analysis, biological engineering, fluid flows with heat and mass transfer, engine and combustion analysis, and visual effects for film and games.

1.1.1 Casson Fluid Flow and Its Applications

Newtonian fluids are fluids that gratify the Newton's viscosity law ($\tau_{xy} \propto \frac{\partial u}{\partial y}$), which is mathematically written as

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}. \quad (1.1.1)$$

where τ_{xy} is shear stress, μ is dynamic viscosity, and $\frac{\partial u}{\partial y} := \dot{\gamma}$ is shear strain rate or velocity gradient ([Zhang et al., 2022](#)). Examples: mineral oil, water, alcohol, gasoline, etc. are Newtonian fluids, because they continue to exemplify fluid properties no matter how fast they are mixed or stirred.

In non-Newtonian fluid, shear stresses and rates of strain (velocity gradient) are not directly related ([Nakamura and Sawada, 1988](#)). That is, non-Newtonian fluid is a kind of fluid that its shear stress τ is not always keeps a linear relationship with the shear strain rate. In non-Newtonian fluids, viscosity, the gradual deformation by shear or tensile stresses, can change when under force to either more liquid or more solid. For example: ketchup becomes runnier when shaken and is thus a non-Newtonian fluid. Such fluids cannot be portrayed because of nonexistence of single

constitutive connection among stress and rate of strain (Pramanik, 2014). Since no single non-Newtonian model predicts every one of the properties of non-Newtonian fluid along these lines examinations proposed different non-Newtonian fluid models. In this case, mathematically can be written as (Shenoy, 1994):

$$\tau_{xy} \propto \left(\frac{\partial u}{\partial y} \right)^n, \quad n \neq 1. \quad (1.1.2)$$

so that

$$\tau_{xy} = j \left(\frac{\partial u}{\partial y} \right)^n, \quad n \neq 1. \quad (1.1.3)$$

where n and j represents the index of flow behaviour and consistency, respectively. Here, if j is constant and $n = 1$, the flow becomes Newtonian.

Based on the special rheological properties (rheology-the science of flow and deformation of matter and describes the interrelation between force, deformation and time), non-Newtonian fluids classified according to whether its viscosity function varies with the duration of the shear τ or not (Qiye and Jingan, 2002; Sochi, 2010) as: (a) time-independent or steady non-Newtonian fluid, sometimes called the generalized Newtonian fluid, -its viscosity is only related to the shear strain rate $\dot{\gamma}$. These include, Bingham, nonlinear Bingham, pseudoplastic, and dilatant fluids. (b) time-dependent or unsteady non-Newtonian fluids - viscosity is not only related to $\dot{\gamma}$ but also related to duration of the shear τ . These type include, thixotropic (liquids or gases whose viscosity decreases upon applying stress over a known time period), rheopexic (the longer the fluid undergoes shearing force, the higher its viscosity), and viscoelastic fluids (fluids that exhibit an elastic solid and a viscous liquid under different circumstances). Under a given shear strain rate $\dot{\gamma}$, the apparent viscosity of thixotropic fluid decreases with the time, the apparent viscosity of rheopexic fluid increases with the time and the most important characteristic of viscoelastic fluid is the strain recovery which will happen as the external force is removed (Qiye and Jingan, 2002; Sochi, 2010).

Because of their special flow characteristics non-Newtonian fluids have applications in the following fields: nuclear waste disposal, chemical catalytic reactors, geothermal energy production, ground water hydrology, transpiration cooling, petroleum reservoirs, mechanical seal, rocket propellant development, biomedical applications (nano-drug delivery, cancer therapeutics, cryopreservation), magnetorheological finishing and electrorheological polishing, plastic polymer, boring/drilling mud, optical fibers, paper generation, hot moving, metal turning, and cooling of metallic plates in a cooling shower, etc. (Peng et al., 2014).

Casson fluid is a subclass of non-Newtonian fluids introduced by Casson (Casson, 1959). Casson fluid is a shear-thinning fluid that is assumed to have an infinite viscosity at zero shear rate, yield stress (force needed to change from elastic to plastic deformation) below which no flow occurs and

zero viscosity at an infinite shear rate (Dash et al., 1996; Ali et al., 2020). This behaviour is sometimes incorrectly termed thixotropy because the equilibrium flow curve of a thixotropic material is often shear-thinning. However, unlike shear-thinning behaviour, thixotropy is time-dependent. Casson fluid behaves like elastic solids, and, for this kind of fluid, a yield shear stress exists in the constitutive equation. Figure 1.1.1 shows the curves of: (a) *shear rate*($\dot{\gamma}$) versus *shear stress*(τ) and (b) *shear rate*($\dot{\gamma}$) versus *viscosity*(μ) for non-Newtonian shear thinning fluids.

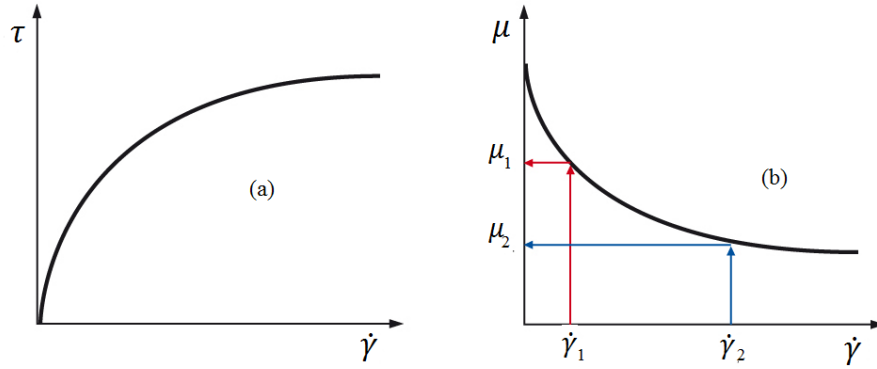


Figure 1.1.1: Flow curve of a shear-thinning material (Mezger, 2015)

The Casson fluid model discovered by (Casson, 1959), for the prediction of the flow behavior of pigment-oil suspensions. The shear stress τ in this model obeys the relation:

$$\sqrt{\tau} = \sqrt{p_y} + \sqrt{\mu \dot{\gamma}}. \quad (1.1.4)$$

where p_y the Casson yield stress (the value of stress at a point on a stress-strain curve (yield point) that indicates the limit of elastic behavior and the beginning of plastic behavior), $\dot{\gamma}$ the shear rate and μ the Casson viscosity at infinite shear rate. Casson yield stress is a significant factor in many industrial processes such as: pumping, spreading and coating. Below the yield point, a material will deform elastically and will return to its original shape when the applied stress is removed. Examples of Casson fluid are: jelly, tomato sauce, honey, soup, and concentrated fruit juices, human blood (consisting of a core region of suspension of all the erythrocytes), chocolate, etc. (Sankar and Lee, 2011). Figures 1.1.2 are some non-Newtonian and Casson nanofluids (Pramanik, 2014; Alqarni et al., 2022).

The Casson fluids flows have various applications in sustenance handling or food processing (tomato sauce, honey, soup, jelly and concentrated fruit juices, etc), metallurgy, manufacturing of pharmaceutical products, drilling (penetrating) operations, bio-engineering/bio-designing operations (such as synovial fluids), manufacturing of industrial products (like china clay (Tamoor et al., 2017), paints, synthetic lubricants). Their sources from left to right then down are: <https://www.>

scottarmsdentalpractice.com/types-of-toothpaste/, <https://theviewfromgreatisland.com/how-to-make-vegan-mayonnaise/>, <https://physicsworld.com/a/the-physics-of-blood-spatter/>, <https://www.cookist.com/homemade-ketchup/>, <https://www.somewhatsimple.com/wp-content/uploads/2021/11/cornstarch-water-ooblek-experiment.jpg>, https://www-konga-com-res.cloudinary.com/w_auto,f_auto,fl_lossy,dpr_auto,q_auto/media/catalog/product/Q/H/62064_1522697066.jpg, <https://thecraftathomefamily.com/wp-content/uploads/2022/03/Yogurt-Silly-Putty-Featured.jpg>, <https://labpedia.net/wp-content/uploads/2020/01/Synovial-fluid-and-cavity.jpg>, <https://www.inspiredtaste.net/27956/easy-tomato-soup-recipe/>, <https://www.bbcgoodfood.com/recipes/collection/jelly-recipes/amp>, <https://www.fruitsmart.com/blog/the-power-of-juice-concentrates/>, <https://an-naseehatraders.com/>.



Toothpaste



Mayonnaise



Human Blood



Ketchup



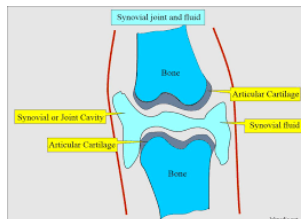
Cornstarch in Water



Bentonite Clay



Silly Putty



Synovial Fluid



Tomato Soup/Sauce



Jelly Recipes



Concentrated Fruit Juices



Honey

Figure 1.1.2: Some non-Newtonian and Casson Fluids.

1.1.2 Nanofluid and its applications

Nanofluids are a new class of fluids engineered by dispersing nanometer-sized materials (nanoparticles, nanofibers, nanotubes, nanowires, nanosheet, or droplets) in base fluids. In other words, nanofluids are nanoscale colloidal suspensions containing condensed nanomaterials. The term nanofluid has been first proposed by [Choi and Eastman \(1995\)](#) to indicate engineered colloids (suspensions or dispersions) consist of nanoparticles dispersed in a base fluid, and according ([Yu and Xie, 2012](#)) it designates a colloidal suspension containing nanoscale particles with any base fluid that doesn't dissolve the particles hosted by it. Nanomaterials (nanoparticles) are used in a variety of manufacturing processes, products and healthcare including: paints, filters, insulation, lubricant additives, etc. For example, in healthcare nanozymes are nanomaterials with enzyme-like characteristics.

The most commonly used nanoparticles for nanofluids formulation are single element (Cu, Fe, Ag, Al, Au, Si, etc), single element oxide (CuO, Cu₂O, Al₂O₃, TiO₂, MgO, Fe₃O₄, ZnO, etc), alloys (Cu-Zn, Fe-Ni, Ag-Cu, etc), multi-element oxides (CuZnFe₄O₄, NiFe₂O₄, ZnFe₂O₄, etc), metal carbides (SiC, B₄C, ZrC, etc), metal nitrides (SiN, TiN, AlN, etc), carbon materials (graphite, carbon nanotubes, diamond, multi-walled carbon nanotubes (MWCNT), single-walled carbon nanotubes (SWCNT), hydrogen exfoliated graphene (HEG), graphene oxide(GO), graphene nanoplatelets (GNPs) and/or combination of them (MgO-MWCNT, MWCNT-Fe₃O₄, CuO-HEG, Ag-HEG, MWCNT-HEG, etc), and the commonly used base-fluids for nanofluid formulation are: water, ethanol, ethylene glycol(EG), oil, ethylene oxide, glycerol, kerosene, toluene, etc. ([Gupta et al., 2017](#); [Sundar et al., 2017](#)).

Nanofluid are used in: nanomaterial processing, microelectronic cooling, combustion chambers, sterilization process, environmental pollutants, fluidized beds, automatic transmission fluid, conveying of powdered materials, petroleum industries, crude oil purification, engine coolant, and polymer coating, etc. Figure 1.1.3 demonstrates nanofluids synthesis (nanoparticles + base fluid + dispersant = stable nanofluid with enhanced thermal properties).

Nanofluids can be classified into two main categories: single material nanofluids, first studied by ([Choi and Eastman, 1995](#)) and hybrid nanofluids, first studied experimentally by ([Jana et al., 2007](#)).

The metallic particles change the heat conduction characteristics and transport properties of the base fluids. Nanoparticles provide a high surface area for heat transfer because of their special features, such as lower density, excellent chemical and physical stability, etc. For instance, copper

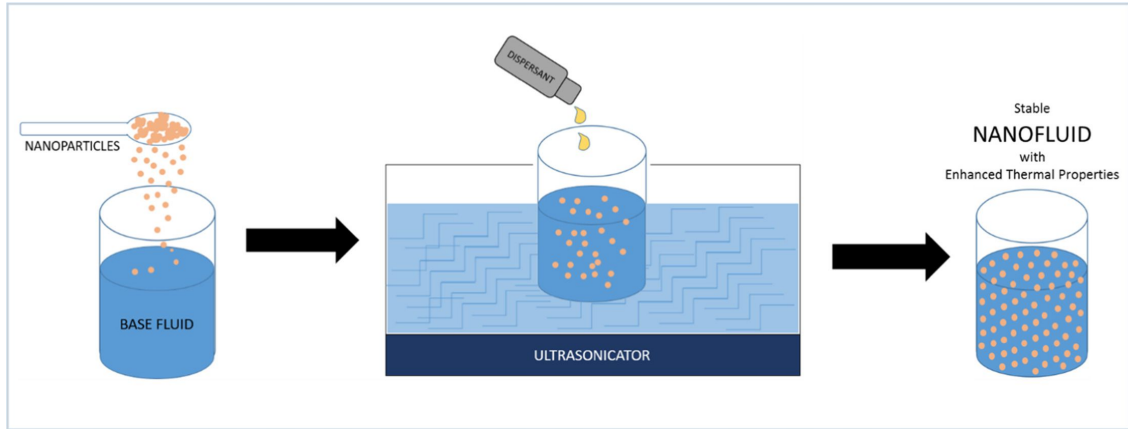


Figure 1.1.3: Nanofluids Synthesis, (Sezer et al., 2019)

(Cu) nanoparticles are relatively stable in terms of both physical and chemical properties with high surface-area-to-volume ratio and conducts electricity and heat very well, it has key applications in industrial machinery (heat exchangers) and used as an anti-microbial, anti-biotic, and anti-fungal agent when added to coatings, plastics, polymers, and textiles (Crisan et al., 2021). Moreover, due to its hard magnetic material with high coercivity and good mechanical stabilities at higher temperature and their great chemical and physical stability, cobalt ferrite CoFe_2O_4 is the most suitable for several applications in: audio, videotape, generator, etc. (Kazemi et al., 2018). Furthermore, because of its high thermal conductivity made TiO_2 nanoparticles for use as enhancements in heat transfer rate and has great applications in the areas of: paints and coatings, cooling of radiators and electronic devices, nucleate pool boiling, heat exchangers, preparation of sunscreen, catalysts, etc, (Ali et al., 2017).

The enhanced thermal properties of nanofluids are the main features of the nanofluids. Recently, nanofluid has gained significant attention due to its great potential for enhancing the thermal conductivity of the base fluids with a remarkable improvement in the convective heat transfer coefficient. There are two major nanofluids convective transport models: the Buongiorno's two-phase (non-homogeneous) model and Tiwari and Das single-phase (homogeneous).

Buongiorno (2006) modeled nanofluids convective heat transport based on nanoparticle-fluid relative velocity. Nanofluids act more like a fluid essentially two-phase fluids containing some common solid-liquids behaviors compared to other solid-liquid mixtures with larger sized particles. In nanofluids, it is recognized that nanoparticles do not follow the fluid streamlines passively. In fact, there are some reasons that induce a slip velocity between the nanoparticles and the base fluid. Movements of nanoparticles have significant impact on rheological and thermophysical properties of the nanofluids. Thus, the nanoparticles absolute velocity is the sum of both the base fluid veloc-

ity and a slip (relative) velocity. According to Buongiorno, there are seven slip mechanisms that creates a relative velocity between the base fluid and the nanoparticles viz. inertia, thermophoresis, Brownian diffusion, Magnus effect, diffusiophoresis, gravity, and fluid drainage. The main slip mechanisms in nanofluids are thermophoresis and Brownian diffusion due to the small size of nanoparticles.

Brownian diffusion is due to random drifting of suspended nanoparticles in the base fluid which originates from continuous collisions among the nanoparticles and liquid molecules. Figures 1.1.4 and 1.1.5 show the Brownian motion of a particle on a sphere and the tracings of the motion of colloidal three particles. The fluid's overall linear and angular momenta remain null over time. The kinetic energies of the molecular Brownian motions, together with those of molecular rotations and vibrations, sum up to the caloric component of a fluid's internal energy. Thus, Brownian motion of nanoparticles at the molecular and nanoscale level is a key mechanism governing the thermal behavior of nanofluids.

Thermophoresis is the diffusion of particles from hotter to colder region under the influence/presence of temperature gradient (in opposite direction) making a non-uniform nanoparticle distribution, which is not due to thermal convection currents (Su et al., 2022b). Figure 1.1.6 illustrates the particle temperature effect on the thermophoresis. The thermophoretic force has a number of practical applications, particularly in separation industry since different particle types move differently under the force of the temperature gradient. Example: thermophoresis is one of the methods used to separate different polymer particles in field flow fractionation. Therefore, investigating the nanoparticles motion is critical for evaluating the performance of nanoparticles inclusion to the base fluid as a heat transfer medium (Lok et al., 2006). Tiwari and Das (2007) developed a theory

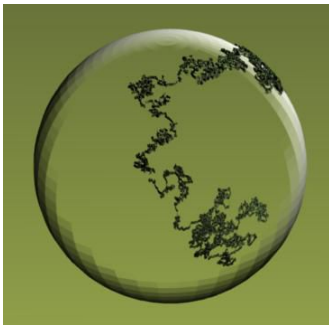


Figure 1.1.4: Brownian Motion on a Sphere, (Price and Williams, 1983)

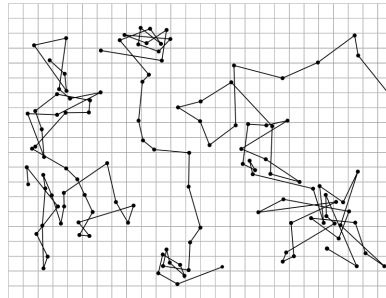


Figure 1.1.5: Motion of Colloidal Particles, (Perrin, 1909)

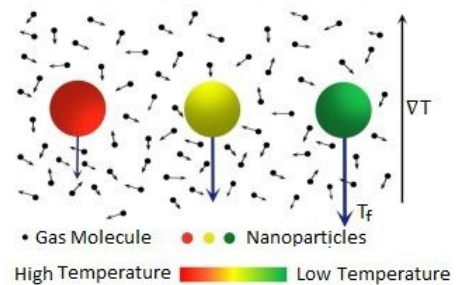


Figure 1.1.6: The particle temperature effect on the thermophoresis (Su et al., 2022a)

that rejects the impacts of thermophoretic forces and Brownian motion unlike Buongiorno's model.

In their assumption of the nanofluid flow is the nanoparticles are distributed within the base fluid homogeneously and of single phase. Accordingly, the nanofluid is a conventional homogeneous single-phase fluid which possesses the effective thermophysical properties of both components and their respective concentrations. Therefore, the equations of conservation (mass, momentum and energy) applied for single-phased fluids are extended and used for nanofluids directly.

The parameters influencing nanofluids effective thermal conductivity are: nanoparticles material thermal conductivity, nanoparticle(shape, size and concentration/percentage), basefluid type, nanofluid temperature, acidity (PH), clustering, additives. Nanofluids have better and remarkable thermophysical properties (such as thermal conductivity, thermal diffusivity, viscosity, and coefficients of convective heat transfer compared to those of base fluids (Scherer and Figueiredo Neto, 2005), and signify great potential applications in many fields, and hence nanofluids can also be known as, 'next generation heat transfer fluid'.

The *nanoparticles volumetric fraction*, ϕ defined as the ratio (Pak and Cho, 1998):

$$\phi = \frac{V_s}{V_s + V_f} \approx \frac{V_s}{V_f}. \quad (1.1.5)$$

where V_s and V_f are the volume of nanoparticles and base-fluid volume, respectively.

The nanofluid *effective viscosity*, μ_{nf} is a measure of the tendency of the suspension to resist the flow or can be defined as the ratio of the shear stress to the shear rate, and formulated as (Brinkman, 1952)

$$\mu_{nf} = \mu_f(1 - \phi)^{-2.5}. \quad (1.1.6)$$

where μ_f is base fluid viscosity.

The nanofluid *effective density*, ρ_{nf} of a nanofluid is considered to be a combined property of both basefluid and nanoparticles, it can be calculated theoretically using its basefluid density and nanoparticle density (Pak and Cho, 1998):

$$\rho_{nf} = (1 - \phi)\rho_f + \phi\rho_s. \quad (1.1.7)$$

where ρ_f , and ρ_s are the density of the base fluid, and density of the solid nanoparticle, respectively.

Thermal conductivity of a material is a measure of its ability to conduct heat. The nanofluid *effective thermal conductivity*, k_{nf} is defined as (Wasp et al., 1977)

$$k_{nf} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}k_f. \quad (1.1.8)$$

where k_f , and k_s are the base fluid thermal conductivity, and nanoparticles thermal conductivity, respectively. Note that:

$$k_{nf} = \frac{k_s + (n-1)k_f - (n-1)\phi(k_f - k_s)}{k_s + (n-1)k_f + \phi(k_f - k_s)}k_f.$$

$n = \frac{3}{\Psi}$, where Ψ , called the 'sphericity', is defined as the ratio of the surface area of the sphere to that of the particle for the same volume. For spherical particles $\Psi = 1$ and for the cylinders $\Psi = 0.5$. For instance, the copper particle is spherical in shape so that $n = 3$, Hamilton and Crosser ([Hamilton and Crosser, 1962](#)).

A good nanofluid is characterized by high thermal conductivity and low viscosity which was explained by ([Choi and Eastman, 1995](#)) and the main reason why nanofluid possess enhanced thermal properties is because of the suspended nanoparticles increase the effective thermal conductivity of the fluid.

The nanofluids *effective specific heat*, $(C_p)_{nf}$ is the amount of heat needed to increase the temperature of a gram of nanofluid by a degree Celsius, formulated as ([Pak and Cho, 1998](#)):

$$(C_p)_{nf} = (1 - \phi)(C_p)_f + \phi(C_p)_s. \quad (1.1.9)$$

where $(C_p)_f$ and $(C_p)_s$ are the specific heat capacity of the basefluid and the specific heat capacity of the nanoparticles, respectively.

Thermal diffusivity measures the ability of a material to conduct thermal energy with respect to its ability to store thermal energy. The nanofluids *effective thermal diffusivity*, α_{nf} is theoretically can be calculated using the formula ([Eggers and Kabelac, 2016](#)):

$$\alpha_{nf} = \frac{k_{nf}}{\rho_{nf}(C_p)_{nf}}. \quad (1.1.10)$$

The nanofluids (the synthesized nanoparticle-liquid suspension) *effective heat capacity*, $(\rho C_p)_{nf}$ is expressed as ([Xuan and Roetzel, 2000](#)):

$$(\rho C_p)_{nf} = (1 - \phi)(\rho C_p)_f + \phi(\rho C_p)_s. \quad (1.1.11)$$

It is important to emphasize that $(\rho C_p)_{nf}$ which accounts for the sensible heat of the nanoparticles as they move homogeneously with the fluid.

The *Prandtl number*, Pr is the quantity that relates momentum diffusivity to thermal diffusivity, mathematically formulated as ([Favre-Marinet and Tardu, 2013](#)):

$$Pr = \frac{\mu C_p}{k}. \quad (1.1.12)$$

The thermophysical properties of the base fluid H₂O, Casson fluid, and some nanoparticles at $T = 300K$ are given in Table 1.1.1 ([Tshivhi and Makinde, 2021](#); [Shaw et al., 2020](#)). In this dissertation, both the single-phase and two-phase nanofluids models with stated arguments are focused to investigate the hydrodynamic, thermal, and solutal characteristics of nanofluids boundary layer flow.

Table 1.1.1: Thermophysical properties of H₂O, Casson fluid, some nanoparticles at $T = 300K$

Properties	H ₂ O	Casson fluid	Cu	CoFe ₂ O ₄	TiO ₂
Density, ρ [$kg.m^{-3}$]	997.1	1060	8933	4907	4250
Thermal Conductivity, k [$W.m^{-1}.K^{-1}$]	0.613	0.505	401	3.7	8.9568
Specific heat, C_p [$J.kg^{-1}.K$]	4179	3490	385	700	686.2

1.1.3 Heat and Mass Transfers

The temperature difference in the fluid flow occurs due to layers/thicknesses of fluid and/or variation between boundary of the flow and its environment, which resulted in the transfers of a quantity from the system of higher to lower temperature called heat (Kreith et al., 2011). The transfer of heat is the fundamental process transportation of energy which is a base for thermal power generation such as in engineering fields and manufacturing industry in: energy extraction from atomic piles, power stations design, aerodynamic heating, high power motors cooling, high speed aircraft, heat exchangers using liquid metal coolant, food and chemical plants, vehicles atmospheric re-entry, heat utilization stored in the subterranean layer of the earth, etc. (Dessie and Kishan, 2014) and (Kumar et al., 2018). There are four modes of heat transfer (thermal energy transport) (Naomi Watts, 2021), Figure 1.1.7:

- i. **Conduction or Diffusion** - the transfer of energy between objects that are in physical contact. Thermal conductivity is the property of a material to conduct heat and evaluated primarily in terms of Fourier's Law for heat conduction, following (Grattan-Guinness, 2005; Condorcet and Heaviside, 2008):

$$\vec{q}_d = -k\nabla T. \quad (1.1.13)$$

where $\vec{q}_d$ is heat transferred vector due to conduction, $k > 0$ is the thermal conductivity, and ∇T is the temperature gradient.

- ii. **Radiation** - the transfer of energy by the electromagnetic emission (photons) due to the changes in the electronic configurations of the molecules or atoms (Holman, 2009). The radiation heat transfer phenomenon is defined by the Stefan-Boltzmann law as:

$$q_r = \epsilon \sigma^* T_w^4. \quad (1.1.14)$$

where q_r is heat flux due to the thermal radiation, σ^* is the Stefan-Boltzmann constant, ϵ is the surface's emissivity with $0 \leq \epsilon \leq 1$, and T_w is the surface wall temperature.

- iii. **Convection** - the mechanism of heat(energy) transfer between an object and its environment, due to a macroscopic fluid motion. In other words, convective heat transfer is the transfer of heat between two bodies by currents of moving fluid, and it is a process that is essentially

the transfer of heat via mass transfer (Burmeister, 1993). The average temperature is a reference for evaluating properties related to convective heat transfer. Heat transfer rate due to convection heat stated by Newton's cooling law as:

$$\vec{q}_c = h(T_w - T_\infty). \quad (1.1.15)$$

where $\vec{q}_c$ is the quantity of heat transferred vector due to convection, T_∞ is the free stream temperature of the fluid, h is the coefficient of convection heat transfer, and T_w is the surface wall temperature.

- iv. **Advection** - the transport mechanism of a fluid from one location to another, and is dependent on motion and momentum of that fluid. Accordingly, it stated as (Zhao, 2021):

$$\vec{q}_a = \rho C_p \vec{U} (T_w - T_\infty). \quad (1.1.16)$$

where $\vec{q}_a$ is the quantity of heat transferred vector due to advection, $\vec{U}$ is the bulk velocity vector, T_∞ is the free stream temperature of the fluid, and T_w is the surface wall temperature.

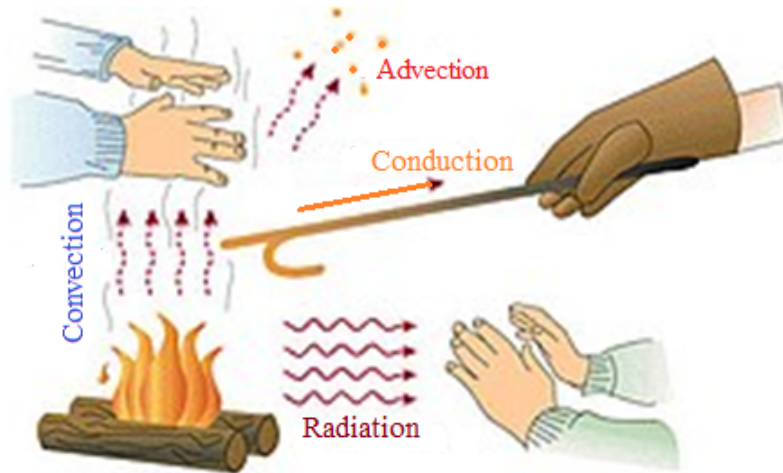


Figure 1.1.7: Modes of Heat Transfer (Naomi Watts, 2021)

(<https://commons.wikimedia.org/wiki/File:Heat-transmittance-means2.jpg>)

Convection refer to cumulative transport and advection refers to the transport due to bulk fluid. Convection (convective heat transfer) is usually the dominant form of heat transfer in fluids (Mahapatra and Nandy, 2013). Convection is classified as:

- a. **Free or Natural Convection** is a type of fluid flow/motion of a liquid (such as water) or a gas (such as air), where the fluid motion is not generated by any external source but by some parts of the fluid being heavier than other parts, which is due to differences in temperature or concentration of dissolved matter in the fluid. The driving force for natural convection is

gravity, so that the buoyant force causes denser parts of the fluid to move downwards and less dense part to move upwards. Thus, the hotter volume transfers heat towards the cooler volume of that fluid (Bergman et al., 2011b). Generally, the densities of most fluids decrease with increase in temperature.

- b. **Forced Convection** is a mechanism, or type of transport in which fluid motion is generated by an external source such as a pump, fan, suction device, etc. Some of the most practical applications of forced convection are: Heating, Ventilation, and Air Conditioning (HVAC) systems, cooling of electronic devices, industrial heat exchangers, automotive radiators, aerospace applications, drying processes, power generation, solar collectors, etc. (Joshi and Makayama, 2003; UKEssays, 2018; Hazizi, 2023).
- c. **Mixed Convection** occurs when natural and forced convection mechanisms act together to transfer heat, in other words, situations where both pressure forces and buoyant forces interact (Sun et al., 2012). The contribution of both mechanisms heat transfer is largely determined by the fluid's flow, temperature, geometry, and orientation and nature of the fluid. Mixed convection is mostly found in very-high-power-output devices such as in: nuclear reactor technology, some aspects of electronic cooling and the design of water filled thermal energy storage tanks (Bejan, 2013). Figure 1.1.8(c) demonstrates natural, forced, and mixed convection heat transfer (Cengel and Ghajar, 2015; Hasan, 2018).

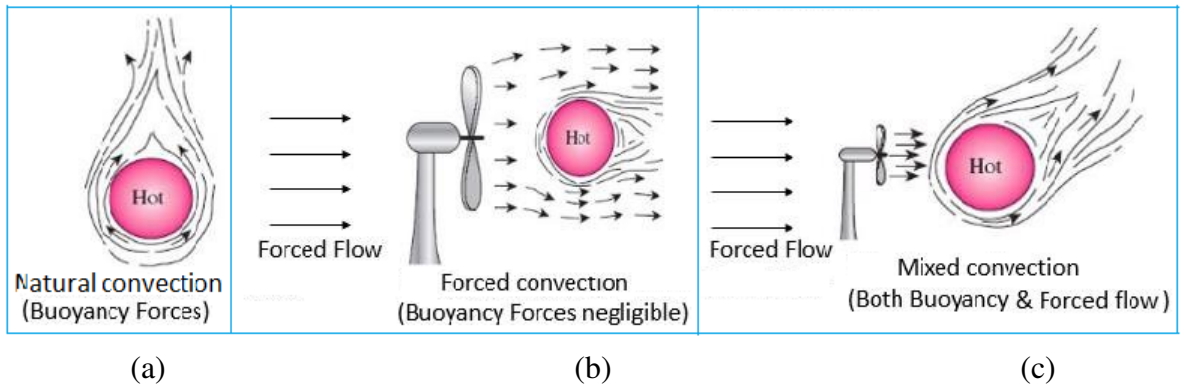


Figure 1.1.8: Natural, Forced and Mixed Convection (Cengel and Ghajar, 2015)

The thermal transport by convection from a flow's surface to a moving fluid is called heat transfer rate (Nusselt number Nu) is defined as:

$$Nu = \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = \frac{h}{k}L. \quad (1.1.17)$$

where $h \equiv \frac{k}{L}Nu$ is the coefficient of convective heat transfer, k is thermal conductivity of the fluid, and L is the characteristic length.

Mass transfer, transport of mass within a fluid-fluid interface or a fluid-solid interface, can be defined as the net movement of mass from one location to another (Baehr and Stephan, 2011). Simply, mass transfer is mass in transit as the result of a species concentration, temperature or pressure gradients difference, where the most common is due to a concentration gradient, in a mixture (fluid or solid material). Mass transfer involves diffusive and convective transport of chemical species within physical systems. For instance, mass transfer occurs naturally in absorption, evaporation (water from a pond to the atmosphere), drying, precipitation, membrane filtration and distillation processes. Figure 1.1.9 displays particles diffusion through a membrane.

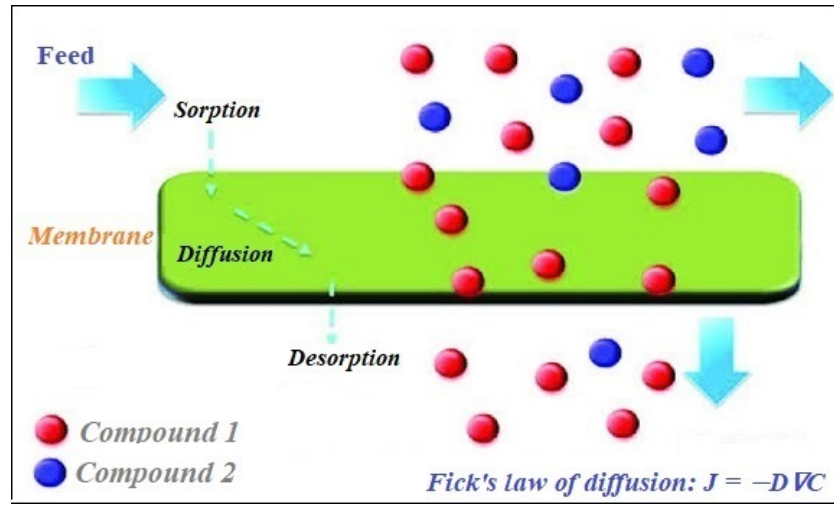


Figure 1.1.9: Mass Transport: Description of Diffusion Mechanism (Conidi et al., 2020)

Mass transfer has great applications in the processes such as the purification of blood in the kidneys and liver, distillation of alcohol, industrial processes (such as separation of chemical components in distillation columns, absorbers such as scrubbers or stripping or activated carbon beds, liquid-liquid extraction) (Baehr and Stephan, 2011). The mass diffusion equation derived from Fick's law (analogous to Fourier's law of heat diffusion) is given by (Bergman et al., 2011a):

$$\mathbf{J} = -D \nabla C. \quad (1.1.18)$$

where $\mathbf{J}$ is the molar flux vector, D is the coefficient of mass diffusivity (which depends on the species type and the medium of diffusion), and ∇C is the concentration gradient.

The convective mass transfer rate is defined by the ratio of the convective mass transfer to the diffusive mass transport rate (also known as the Sherwood number Sh or mass transfer Nusselt number) is given as:

$$Sh = \frac{h}{D} L. \quad (1.1.19)$$

where $h \equiv \frac{D}{L} Sh$ is the coefficient of convective mass transfer, D is mass diffusivity, and L is the characteristic length.

1.1.4 Chemical Reaction

Chemical reaction is defined as a process in which one or more substances (called the reactants) are converted to one or more different substances (called the product) (Kotz et al., 2014). Chemical reactions are categorized as homogeneous (reactants and products are in the same phase) and heterogeneous (reactants are in two or more phases) processes, depending on their occurrence (an interface or as a single phase). The generation of species in a homogeneous reaction is the same as an internal source of heat generation, whereas heterogeneous reaction can be assumed as a boundary condition same as the constant heat transfer condition and enables faster in heat transfer processes (Davis and Davis, 2012).

The reaction rate is the speed at which a chemical reaction takes place, and hence proportional to the growth in the concentration of a product per unit time and to the decay in the concentration of a reactant per unit time (McMurry et al., 2020). Chemical reactions are inseparable from transfer of mass due to the diminishing reactants move to the location at which reaction takes place while the rising products must move away. Chemical reactions affect a fluid flow properties since their occurrence either uses energy (i.e., endothermic reactions absorb in energy and the temperature of the surroundings reduces) or dissipates energy (i.e., exothermic reactions transmit energy to the surroundings and the temperature of the surrounding rises) in the fluid (Jan et al., 2019).

Therefore, boundary layer flow of nanofluid in the presence of chemical reaction with heat and mass transfer problems are of numerous applications in many processes (Sehra et al., 2021). To generalize, the study of heat and mass transfer with chemical reaction got due attention for their applications in the areas such as in nuclear reactor cores, modeling of fire and combustion, semiconductor wafers and electronic chips, air and water pollution control, fibrous insulation, atmospheric flows, and many other chemical engineering problems (Najib et al., 2014).

1.1.5 Flow with Viscous Dissipation

Viscous dissipation is the irreversible process by means of which the work done by a fluid flow on adjacent layers (mechanical energy) because of the action of shear force (viscous) is transformed into heat (internal energy) (Morini, 2008; Rashad et al., 2014). The temperature distribution and heat transfer in the fluid flow regime is highly influenced by viscous dissipation and considered as energy source which has impact on the rate of heat transfer (Ibrahim and Tulu, 2019). More discussions about viscous dissipation are considered in the articles (Kumar et al., 2018; Lund et al., 2020a; Joshi et al., 2022). Viscous dissipation has great importance in forced convection in various fluid flow such as annular duct, porous medium, parallel-plate microchannel, and hence, currently

attracted researchers in the area of flow with viscous dissipation past porous media (Ahmed et al., 2021; Chen and Tso, 2011; Loh et al., 2022).

1.1.6 Flows with Slip Boundary Conditions

A wall boundary condition is a "non-slip" condition which assumes the velocity at the wall boundary is 0 no matter what wall roughness value is applied, whereas the "slip" boundary (slip/symmetry boundary) permits the fluid flow to move along the wall boundary, which mean that the velocity at slip/symmetry boundary is different from 0 (Rahmati et al., 2018; Rapp, 2022). In other words, the slip wall boundary condition is the fluid flow where viscous effects at the wall are negligible and/or the mesh size under consideration is much bigger than the boundary layer thickness (so that the boundary layer effects is insignificant), and thus the slip boundary is the proper boundary condition for symmetry flow surfaces. The no-slip boundary condition is extensively used since it simplifies complex situations with some limitations, and if the surfaces are sufficiently smooth where the fluid cannot be in contact with the wall it is called slip boundary condition (Sehra et al., 2021). Figure 1.1.10 shows different slip conditions.

In many applied problems in Engineering and Science, boundary conditions play an essential role so that a meaningful solution is gained. There are situations with no slip conditions which lead to unrealistic property such as in the spreading of a liquid on a solid substrates, corner flow and the polymer melts extrusion from a capillary tube (Thompson and Troian, 1997), and hence no slip condition must be replaced by slip boundary condition when fluid flows about microfluidic and nanofluidic (Asfer and Panigrahi, 2015; Nguyen et al., 2019). The slip flow region for the fluid problem is very significant in the era of modern sciences and technology (Jan et al., 2019). Generally, in the fluid flow problems slip conditions has vital role in chemical sciences/engineering and industries (Najib et al., 2014).

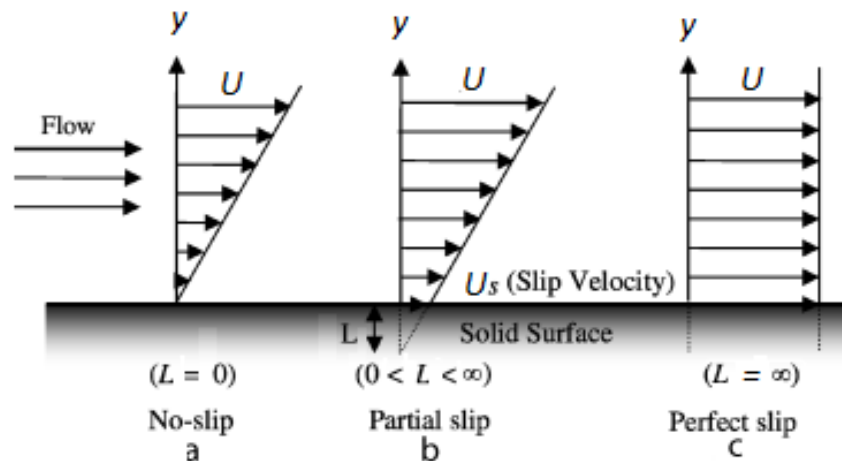


Figure 1.1.10: Types of Slip Velocity: Modified from (Panigrahi and Asfer, 2008)

1.1.7 Flows with Suction/Injection

Suction or injection on the boundary layer control plays a vital role in the areas of space sciences, aerodynamics, and extensive engineering applications with mass transfer and chemical reactions (Uwanta and Hamza, 2014). Suction is one of the techniques of boundary layer control, which have the aim of reducing drag/friction on bodies in an external fluid flow or of minimizing losses of energy in channels (Epifanov, 2011), and this method was first suggested by (Prandtl, 1904) as one of the ways of preventing or "delaying" boundary layer separation. Moreover, it was investigated that the suction of the boundary layer on the stretched surface of flow delays the backflow while in injection enhances the strength of the reverse (or velocity overshoot) fluid flow (Ali, 1996).

1.1.8 Porous Medium and Its Applications

A porous medium/material is a material containing voids/pores, such that the skeletal portion (usually a solid, but structures like foams are often analyzed) of the material is called "frame" or "matrix" and the pores are typically filled with a fluid. Porosity which is the ratio of the void space to the total volume is characterized by a porous medium, and hence the quality of the fluid to flow through a porous medium is determined by the porosity or permeability of the material (Heinemann and Mittermeir, 2013; Ganji and Kachapi, 2015; Nield and Bejan, 2017). Due to interconnection between the pores it is possible for the flow of one or more fluids through the media. For instance, wood, sandstone, limestone, biological tissues, beach sand, and human lung are natural media, whereas composite materials, ceramics, and metallic foams with high porosity, etc., are industrial media (Nazari et al., 2019).

Fluid flow in the porous medium is influenced by various forces, and its essence is to take in energy and produce fluid through the wellbore. Thus, the relationship between energy and rate of flow becomes the most important problem in fluid flow mechanics past porous media (Heinemann and Mittermeir, 2013). Darcy's law relates the velocity and the pressure gradient/drop in an unbounded porous medium, and hence for the viscous fluid flows at low speed past a porous medium the pressure drop due to the drag force is directly proportional to the velocity of the fluid and therefore, mathematically given by:

$$\nabla P = -\frac{\mu_f L}{k_1 A} \mathbf{U}. \quad (1.1.20)$$

where $\mathbf{U}$ is velocity vector, L is the characteristic length, A is the characteristic cross-sectional area, k_1 is the permeability of the porous medium (determined experimentally), and μ_f is the dynamic viscosity of the fluid. If A is unity, Eq. (1.1.20) is reduced to $\nabla P = -\frac{\mu_f}{k_1} \mathbf{U}$, which is standard Darcy's law. The equation $\mathbf{U} = -\frac{k_1}{\mu_f} \nabla P$ is scalar for isotropic porous medium (i.e., when the two permeabilities (absolute/effective and relative) are equal) (Nield et al., 2006; Heinemann

and Mittermeir, 2013).

Taking what Darcy's law missed, that is, the case when inertial resistance forces are dominant, the drag force and the surface drag becomes comparable, which is resulted in the discovery of non-Darcy formula called Darcy-Forchheimer by adding a modification term that affects the pressure drop. According to (Joseph et al., 1982) the pressure gradient in the flow due to the porous medium, ∇P is given by

$$\nabla P = -\frac{\mu_f}{k_1} \mathbf{U} - \frac{C_F \rho_f}{\sqrt{k_1}} |\mathbf{U}| \mathbf{U}. \quad (1.1.21)$$

where k_1 is permeability of the porous medium, $C_F = \frac{11}{20} \left(1 - \frac{11}{20} \frac{d}{D_e}\right)$ is a dimensionless form-drag constant, d is the diameter of spheres of the porous medium, and $D_e = \frac{2wh}{w+h}$ is the equivalent diameter of the bed (defined in terms of the height h and width w of the bed) (Nield et al., 2006). Figure 1.1.11 illustrates some of Darcy-Forchheimer porous media flows.

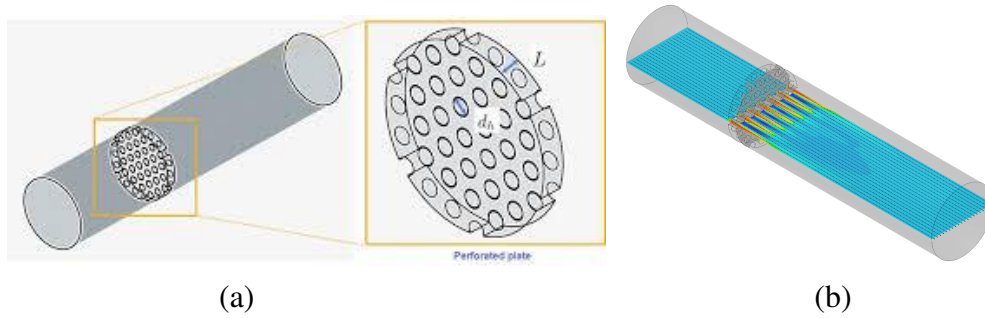


Figure 1.1.11: (a) Perforated Plate Acting as Porous Media (b) Darcy-Forchheimer Plate Flow Embedded in a Cylinder (Miguel, 2019)

Areas that the technology requires porous media as a tool are in: soil science (the soil contains and transports water and nutrients to plants), hydrology (a water bearing and sealing layer), chemical engineering (applied as filter or catalyst bed), petroleum engineering (reservoir rock stores crude oil and natural gas), according to (Heinemann and Mittermeir, 2013), and seepage of water in river beds, migration of pollutants into the soil and aquifers, mass and heat transport in packed bed reactor columns, purification and filtration processes, drying of porous materials in textile industries, the saturation of porous materials by chemicals, etc. (Kausar et al., 2019; Tadesse et al., 2021a).

1.1.9 Reynolds Number and Its Significance

For fluid flows in the porous medium, a Reynolds number Re is defined as the ratio of inertia force to viscous force, that is essential for CFD since it is a fundamental block used to estimate the transition from the laminar to turbulent boundary layer fluid flows (Heinemann and Mittermeir, 2013) (see Figure 1.1.12).

Laminar flow is sheet-like with low velocity in sheet surface or tube, where any material point

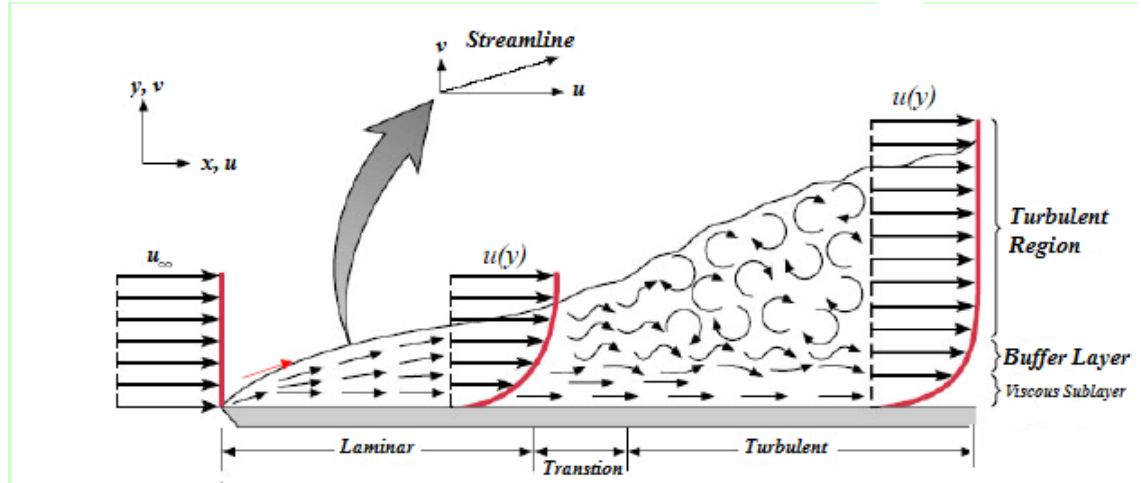


Figure 1.1.12: Laminar-Transition-Turbulent Flow Reynolds Number (Halila et al., 2015)

moves smoothly in a straight line that is parallel to the axis of the surface of flow or tube for which $Re < 2000$); turbulent flow is eddy current like flow for which $Re > 4000$; and transition flow for $2000 < Re < 4000$. Based on (Xue et al., 2021), the dimensionless Reynolds number in porous media is given by:

$$Re = \frac{1}{1750} \left(\frac{\mathbf{q} \rho_f \sqrt{k_1}}{\mu_f \phi^{1.5}} \right) = \frac{\mathbf{q} L}{\nu_f}. \quad (1.1.22)$$

where $\mathbf{q} = \frac{\mathbf{U}}{\phi} = -\frac{k_1}{\mu_f \phi} \nabla P$ is the real flow velocity, $\nu_f = \frac{\mu_f}{\rho_f}$ the kinematic viscosity of the fluid, $\phi = \frac{V_p}{V_p + V_s}$ is the dimensionless porosity, V_p is the volume of the void/pore, V_s is the volume of the solid material, and $\mathbf{U}$ is Darcy velocity, $L = \frac{1}{1750} \frac{\sqrt{k_1}}{\phi^{1.5}}$ is the characteristic length.

Reynolds number gets its applications in fluid mechanics, such as in the calculation on friction factor and drag in internal and external flows, that lends value to hydraulic study, aerofoil design, wing tunnel testing, climate science, atomized fuel droplets, HVAC (Heating, Ventilation, and Air Conditioning). Darcy flow is in the laminar flow where $Re < 0.2$ or $Re < 0.3$. For air flowing through an airplane $Re = 5,000,000$ and a seagull's wing $Re = 100,000$, a circular pipe $Re = 2320$, blood flowing in the brain $Re = 100$, and water flowing in the reservoirs $Re < 0.2$ (Heinemann and Mittermeir, 2013).

1.2 Statement of the Problem

Theoretical investigation of boundary layer flow of non-Newtonian nanofluids with heat and mass transfer characteristics past a surface embedded in a porous medium have attracted special attention of researchers in recent time due to their wide range of applications in industry and technology. Casson fluid is a non-Newtonian fluid which exhibits some interesting characteristics. It behaves like an elastic solid when a small amount of stress is applied on it but after a certain value of stress applied on it, it starts behaving like a fluid. Casson nanofluids have great significance in biomechanics and polymer industries. In the fields of material processing, engineering processing, food

processing, oil reservoir engineering, agricultural water dispersal, chemical industry, bioengineering and in many others, Casson nanofluids have great importance. In the world-wide industry of polymer processing, the effects of dispersed nanoparticles on typical characteristics of polymer melts and solutions have constituted a challenging research problem in industries. Moreover, a lot of research are needed in order to understand the behavior of a Casson nanofluids, when it flows past a stretching or contracting heated surface in a porous medium as this process bears a wide range of applications in industries such as glass blasting, paper cooling and textile industry, etc.

Mathematically speaking, this study examines the stability of dual solutions arising from non-linear models describing the boundary flow of Casson nanofluids with heat and mass transfer past stretching and shrinking surface. This is necessary in order to determine and predict the physically realistic solution of such flow with heat and mass transfer scenario under experimental investigation for industrial and engineering applications. Therefore, filling the gap in the work of (Titiloye et al., 2019; Tadesse et al., 2021a,b), this dissertation considers the boundary layer flow of Casson nanofluid flow subjected to a slippery surface of different shapes in a non-Darcian porous medium. In particular, the following problems were investigated:

1. Stability analysis of dual solutions of convective flow of Casson nanofluid past a shrinking/stretching slippery sheet with thermophoresis and Brownian motion in porous media.
2. Dual solutions and stability analysis of Cu-H₂O-Casson nanofluid convection past a heated stretching/shrinking slippery sheet in a porous medium.
3. Stagnation point flow of CoFe₂O₄/TiO₂-H₂O-Casson nanofluid past a slippery stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium.

1.3 Objectives of the Study

1.3.1 General Objective of the Study

The general objective of this dissertation is to investigate mathematically boundary layer Casson nanofluid flow with mass and heat transfer properties.

1.3.2 Specific Objectives of the Study

The specific objectives of this dissertation are to:

- i. formulate a mathematical model for thermophoresis and Brownian motion effects on Casson nanofluid flows over convectively heated permeable stretching/shrinking slippery sheet in a

Darcy-Forchheimer porous medium.

- ii. develop a mathematical model for heat transfer enhancement of Cu-H₂O-Casson nanofluid flows past convectively heated permeable stretching/shrinking slippery sheet in a Darcy-Forchheimer porous medium.
- iii. establish a mathematical model of stagnation point flow of CoFe₂O₄/TiO₂-H₂O-Casson nanofluid past a slippery stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium.
- iv. analyze the existence of dual/multiple solutions and perform the temporal stability analysis for the basic Casson fluid flows.
- v. investigate the effects of various embedded parameters on the concentration, temperature, and velocity profiles, and moreover on the coefficient of skin friction, the rate of heat transfer and the rate of mass transfer.

1.4 Significance of the Study

The study is very important for the following reasons:

- Assist to analyze the mass and heat flow properties in operation of heat-exchange apparatus, food processing and storage, geophysical systems, electro-chemistry, fibrous insulation, metallurgy, the design of pebble bed nuclear reactors, underground disposal of nuclear or non-nuclear waste, microelectronics cooling, etc. so that products with better quality and specifications are obtained.
- Helps to understand heat transfer enhancement as a result of added nanoparticles: Cu-H₂O, CoFe₂O₄-H₂O, and TiO₂-H₂O into the Casson fluids.
- Used as input for many industrial designing process such as geothermal energy extractors, nuclear waste disposal, heat exchangers, cooling of nuclear reactors, microelectronics cooling systems, radiators, etc.
- Helpful for further study on boundary layer flow of Casson nanofluid with mass and heat transfer properties and establish a base for further investigations on both single-and two-phase flows.

1.5 Scope of the Study

The current study is focused on laminar, steady, two-dimensional boundary layer Casson nanofluid flow with convective mass and heat transfer of a viscous and an incompressible fluid flow past a stretching/shrinking horizontal flat and cylindrical surfaces embedded in a porous media. The parameters such as Casson factor, velocity slip, velocity ratio, permeability, thermal and solutal convection, viscous and porous dissipation, chemical reaction, thermophoresis, Brownian motion, and suction/injection are included in the discussion of the boundary layer characteristics. For porous media under consideration, the governing PDEs are approximated by Darcy and non-Darcy flow models. The similarity transformations are used to transform PDEs with boundary conditions. The numerical scheme, the Runge-Kutta-Fehlberg fourth-fifth (RKF45) order method, along with the shooting technique, is used to solve the similarity equations. The study investigates applications in boundary layer control such as in reducing the drag in aerodynamics, polymer industry, cooling systems, polymeric materials processing, solar power technology, nuclear reactors, etc., where the detailed applications for each problem are provided in the relevant chapters.

1.6 Organization of the Dissertation

The chapters of this dissertation are organized as follows. Chapter 1 deals with the general introduction regarding this dissertation. Chapter 2 discussed previous and up-to-dated related literature, whereas Chapter 3 discussed in detail the derivation of Casson fluid flow governing equations by considering, conservation laws and relevant mathematical techniques used for solving boundary value problems. Chapter 4 deals with stability analysis of dual solutions of convective flow of Casson nanofluid past a permeable shrinking/stretching slippery sheet embedded though porous media in the presence of thermophoresis, Brownian motion, viscous dissipation, and chemical reactions. Moreover, analysis of dual solutions and heat transfer enhancement of Cu-H₂O-Casson nanofluid convection past a heated permeable stretching/shrinking slippery sheet in a porous medium elaborated in Chapter 5. Furthermore, Chapter 6 considered stagnation point flow of CoFe₂O₄/TiO₂-H₂O-Casson nanofluid past a slippery stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium and flow properties are discussed for shrinking surface in detail. The last, Chapter 7, presents the main results and conclusions recommendations, and future perspectives of this research work.

Chapter 2

Literature Review

2.1 Boundary Layer Flow of Casson Nanofluids

The concept of boundary layer (BL) fluid flow (the boundary layer is the microscopic or thin region of fluid flow in which flow is influenced by the friction between the solid surface and the fluid) and heat transfer over or towards a surface has been of great application in many manufacturing processes, such as glass-fiber production, metallurgical processes, wire drawing, glass blowing, melt-spinning and annealing and tinning of copper wires, the manufacturing of plastic and rubber sheets, and crystal growing. Moreover, due to its enormous applications in areas of industry and engineering such as damping and braking devices, printing technology, personal protective equipment, food products, and drag-reducing agents, the boundary layer flows of non-Newtonian fluids with heat and mass transfer properties due to stretching sheet surfaces have attracted researchers' attention ([Hayat et al., 2008](#)).

[Jawad et al. \(2018\)](#) presented a mathematical model for defining the fluid film flow of non-Newtonian nanofluids through malleable viscidness and thermal conductivity in the presence of a uniform magnetic field past a stretching sheet. Using HAM, they discovered that an increase in the value of the permeability parameter (reciprocal of Darcy number) caused a reduction in the permeability of porous medium. The boundary layer flow of non-Newtonian fluids presented by ([Layek et al., 2018](#)), and they showed that for increasing values of Prandtl number Pr and greater values of Eckert number Ec , increment in the temperature profile was observed. Again, ([Rizwana et al., 2020](#)) concentrated on the non-Newtonian fluid flow of Cu-H₂O nanofluid flow, concluding that the non-Newtonian parameter makes the fluid velocity faster while the temperature profile drops with the non-Newtonian parameter β and nanoparticle volume fraction ϕ and the system heats up from the impact of viscous dissipation.

Casson fluids are used to characterize non-Newtonian fluids with shear-thinning properties. The impacts of temperature-dependent viscosity and thermal conductivity of an incompressible MHD Casson fluid flow along a vertical porous surface in the presence of suction and viscous dissipation were investigated by ([Animasaun, 2015](#)), and ([Tamoor et al., 2017](#)) studied the MHD flow of Casson fluid past a stretching cylinder.

[Kumar et al. \(2019\)](#) considered the heat transport characteristics of non-Newtonian MHD Casson fluid flows due to stretching cylinders with heat generation and noticed that the influence

curvature parameters have a tendency to increase both the velocity and thermal profiles. They concluded that the temperature profile reduces as the Casson parameter increases. The thin-film Casson fluid three-dimensional flow past a steady rotating inclined plane in the presence of injection or suction was examined by (Saeed et al., 2019). They obtained that for increasing values of β , the fluid motion was reduced since the Casson fluid produces a resistance force. Moreover, (Rizwana et al., 2020) concentrated on the non-Newtonian Cu-H₂O- nanofluid flow. They concluded that an increase in β enhanced the fluid motion and decreased its boundary layer thickness, while the temperature profile dropped with β and ϕ . Further, (Mandal and Layek, 2021) investigated an incompressible, two-dimensional, and unsteady MHD electrically conducting a Casson fluid flow. They observed the overshoot in similarity temperature and velocity profiles due to the Casson fluid stress distribution and imposed partial slip at the boundary wall. Moreover, the flow and thermal boundary layer thickness were reduced for enhanced values of β .

The use of nanofluids, mixtures of nanoparticles, and base fluids has played a great role in heat transfer enhancement, cooling electronic devices, and various manufacturing industries and engineering processes due to the high development of thermal devices, (Tadesse et al., 2021a) discussed. The Casson nanofluid (composed of water-based nanoparticles Cu, Ag, Au, Al, Al₂O₃, and TiO₂) flow was analyzed by (Palaniammal and Saritha, 2017). It was revealed that the velocity profile decreased with the rising values of ϕ and β , whereas the velocity profile along the x -direction was reduced with ϕ and decreased with an increase in the values of β . Furthermore, (Zubair et al., 2019) investigated a steady three-dimensional Casson nanofluid flow through two parallel rotating and stretching plates. They found that the temperature profile was an enhancing function of ϕ , whereas the velocity profile along x -direction was reduced with β , and hence, the skin friction was enhanced with increasing β values. (Mahanta et al., 2019) studied the Casson nanofluid flow, and they observed that the temperature profile increased with β and the velocity profile dropped with β .

Titiloye et al. (2019) studied two-dimensional steady laminar boundary layer Casson nanofluid flow, and showed that increments in the values of β enhanced velocity and velocity gradient and reduced temperature and temperature gradient. (Shah et al., 2020) investigated electrically conducting Casson nanofluid flow. Accordingly, it was revealed that enhancing the values of β reduced the shear stress of the fluid, and as a result, it relaxed the fluid to move with higher velocity, whereas the surface drag force boosted up. Moreover, (Lanjwani et al., 2021) numerically investigated the MHD boundary layer flow of Ti₆Al₄V and Fe₃O₂-Casson-based nanofluid. They concluded that

the velocity profile was enhanced with ϕ , but the velocity profile and its boundary layer thicknesses were reduced for increasing values of β ; however, the temperature profile and its thermal boundary layer thicknesses were raised as ϕ and Bi values increased.

[Basha and Sivaraj \(2022\)](#) numerically explored Casson nanofluid flow. They demonstrated that as β increased, the velocity profile and the factor of skin friction at the surface intensified for the real solution and the temperature profile diminished. Moreover, ([Shatnawi et al., 2022](#)) made a mathematical analysis of Casson nanofluid flow over a vertical Riga plate. They obtained that as ϕ increased, the velocity declined due to the incremental viscosity, and as β increased, the heat transfer rate and drag force dropped due to an increment in the shear thinning, resulting in reduced skin friction and a raised Nusselt number. Further, ([Basha and Sivaraj, 2022](#)) numerically investigated the Casson nanofluid flow. From their analysis, they found that the drag force factor increased at the surface as the values of β augmented for the real solution. [Yousef et al. \(2022\)](#) investigated Casson-Williamson nanofluid flow. Their analysis revealed that increments in β values intensify the temperature profile. [Khan et al. \(2022\)](#) theoretically investigated how Casson nanofluids flow past a vertical Riga plate. They obtained that the velocity profile upsurged as the values of β increased and dropped as the values of ϕ increased; moreover, the drag force was reduced for enhancement in the values of ϕ , whereas the temperature profile dropped for enhancing values of ϕ . [Adeyeye et al. \(2022\)](#) presented the magnetized Casson nanomaterial flow and solved it using the block method, which is suitable for dealing with ODE systems of different orders. The numerical results showed that the velocity profile declined with rising β values. Moreover, it is noted that the rate of heat transfer increased with the enhancement in the thermal slip parameter, and a lower temperature profile due to a larger β was observed.

2.2 Boundary Layer Flow with Heat and Mass Transfer

The study of mass and heat transfer in the presence of chemical reactions has great practical importance to scientists and engineers because it almost appears universally in different physical fields such as engineering sciences, medical sciences, and different industrial processes. [Andersson et al. \(1994\)](#) investigated numerically the laminar boundary layer of mass and heat transfer problems for nanofluid flows on a linearly stretching sheet surface. They discovered that the principal impacts of a destructive chemical reaction were an increased mass transfer rate and a reduced thickness of the concentration boundary layer from the stretching sheet surface to the surrounding fluid flow, where these effects are less pronounced for higher-order reactions than for a first-order reaction.

[Animasaun \(2015\)](#) investigated theoretically the impacts of thermophoresis and some thermo-physical characteristics on the free convective mass and heat transfer of non-Darcian MHD flow with the n^{th} -order of chemical reaction. It was revealed that an increment in β values resulted in reduced concentration and temperature profiles; increasing Prandtl number Pr reduced the flow and also strongly depressed temperatures at the boundary layer; the thermophoretic parameter Nt had a decreasing effect on concentration profiles; and the temperature profiles intensified with an augmentation in the values of Nt .

[Palaniammal and Saritha \(2017\)](#) analyzed Casson nanofluid (composed of water-based nanoparticles Cu, Ag, Au, Al, Al_2O_3 , and TiO_2) flow with chemical reaction by solving analytically. It was revealed that the velocity profile was reduced by raising the values of S , ϕ and β . Moreover, the temperature profile was improved by increasing the values of ϕ and Ec , whereas it was reduced for enhancing the values of S , β , and furthermore, Pr , and the species concentration profile was raised with the enhancing values of β , ϕ , and diminished for increasing values of S , Sc and R .

[Mahanta et al. \(2019\)](#) studied the thermophoresis and Brownian motion effects on a non-linearly stretching sheet surface with Casson nanofluid focusing and emphasizing mass and heat transfer for the laminar steady Casson nanofluid with convective and velocity slip boundary conditions by using bvp4c of Matlab to solve the model equations. They observed that the temperature profile raised with β and the velocity profile dropped with β ; the temperature profile was enhanced with higher values of Nt and Brownian motion parameter Nb , whereas the concentration profile diminished with Nt and Nb . Moreover, the thermal boundary layer thickness was reduced with Pr and raised with higher values of Nb , whereas the concentration profiles got higher with the Lewis number Le .

[Khashi'ie et al. \(2020\)](#) investigated the heat transfer and the boundary layer flow characteristics of a hybrid Cu- Al_2O_3 /H $_2$ O nanofluid numerically by employing a bvp4c solver embedded in MATLAB software. The results showed that the velocity profile was inclined with the increment in Cu ϕ values, and the temperature profile escalated with increasing values of Ec and Cu ϕ . The increment in Ec didn't affect the boundary layer separation but reduced the rate of heat transfer.

[Joshi et al. \(2021\)](#) studied 3-dimensional free convective flow of magnetic hybrid nanofluid (SWCNT + Ag-H $_2$ O) past a bidirectional stretchable porous surface with heat generation and chemical reaction by using the shooting technique along with the Runge-Kutta-Fehlberg fourth- and fifth-order methods for computation. They showed that the mass transfer enhancement rate function for the chemical reaction was higher for the lower order than the higher order. Moreover,

as R intensified, the magnitude of surface drag forces (skin friction coefficient) increased.

Tshivhi and Makinde (2021) examined the effects of magnetic field on thermal boundary layer flow of a conducting nanofluid consisting of $\text{Fe}_3\text{O}_4\text{-H}_2\text{O}$, $\text{Al}_2\text{O}_3\text{-H}_2\text{O}$ and $\text{Cu-H}_2\text{O}$ over a convectively heated, stretching, fixed, shrinking, and slippery surface and tackled the problem using the shooting method with the Runge-Kutta-Fehlberg integration scheme. They revealed that $\text{Cu-H}_2\text{O}$ provided the highest rate of heat transfer enhancement, followed by $\text{Fe}_3\text{O}_4\text{-H}_2\text{O}$ and $\text{Al}_2\text{O}_3\text{-H}_2\text{O}$, and hence, it implied that $\text{Cu-H}_2\text{O}$ may be used as a better coolant compared to the two others. Moreover, the nanofluid velocity boundary layer thickness decreased with an increment in the surface slipperiness parameters δ and ϕ . Furthermore, the thermal boundary layer thickness was amplified with an increase in Eckert number Ec due to viscous dissipation, whereas an increment in ϕ , δ , and Bi reduced thermal boundary layer thickness because of sheet surface convective heating. Lastly, they also deduced that the sheet surface skin friction escalated with increased values of ϕ but diminished with increment in δ values, whereas the heat transfer rate was enhanced with rising ϕ , δ and Bi values due to sheet convective heating, while an increment in Ec values due to viscous dissipation lessened the heat transfer rate.

Tadesse et al. (2021b) investigated the MHD two-dimensional flow of an electrically conducting H_2O -based nanofluid by generating the numerical solutions using MAPLE software. They found that the thermal and velocity profiles dropped with suction/injection parameter S , and the drag force coefficient increased with both convective parameters (Biot number) Bi and Nb but reduced with increasing S values. Moreover, they revealed that the heat transfer rate decreased with Nb , Bi , and Nt values, and the mass transfer rate increased with Nt but decreased with Nb .

Varun Kumar et al. (2022) investigated the salient features of a Casson nanoliquid stream past a curved stretching sheet surface in the presence of Brownian motion and thermophoresis influenced by chemical reaction and magnetic field. They demonstrated that the velocity gradient was diminished for escalating values of β and improved for ascending values of the curvature parameter κ ($\equiv \sqrt{Re}$), but the temperature profile and rate of mass transfer decreased with rising κ and R , respectively. Moreover, the concentration gradient diminished for escalating values of Sherwood number Sc and chemical reaction parameter R , whereas raising values of Nt and Nb deteriorated the mass transfer rate and improved the thermal gradient of the fluid flow.

Verma et al. (2022) numerically analyzed the steady 2D boundary layer flows of graphene-water ($\text{Gr-H}_2\text{O}$) and graphene oxide-water ($\text{GO-H}_2\text{O}$) nanofluids past an exponentially shrinking sheet surface embedded inside a porous medium, which was analyzed and solved with RK4 by

using the shooting technique. The Gr-H₂O nanofluid delayed the boundary layer separation, produced a higher surface drag and coefficient of surface cooling rate, enhanced the velocity profile, and lowered the temperature profile when compared with the GO-H₂O nanofluid. Moreover, it was displayed that as ϕ increased, the boundary layer separation improved, surface drag declined when $S < 2.5$ and then boosted, the coefficient of surface cooling rate Nu declined, velocity profile dropped, and temperature profile boosted; as wall mass transfer parameter S increased, the surface drag uplifted, the coefficient of surface cooling rate Nu upsurged, velocity profile elevated, and temperature profile decayed; and as Pr values increased, the temperature profile decayed.

Basha and Sivaraj (2022) numerically explored the Casson nanofluid flow in the presence of suction/injection by applying the bvp4c MATLAB solver to solve the transformed equations. They demonstrated that as β increased, the velocity profile escalated and the temperature profile dropped; an increment in Nt and Nb enhanced the Casson nanofluid temperature profile; raising the values of β and S augmented the factor of skin friction at the surface for the real solution; and Brownian motion possessed the same critical point on the rates of mass and heat transfer.

Dey et al. (2022) explored unsteady and steady Casson fluid with solutal and thermal transfer flow by solving numerically utilizing the "MATLAB routine bvp4c scheme". It was observed that by enlarging the thermal transfer of the fluid, β controls the motion of the fluid flow. Moreover, under the suction environment, escalation in Pr enriched the rate of heat transfer of the fluid for the unsteady case, and the mass accumulation rate of the fluid was enhanced due to an increment in R for both steady and unsteady flow.

Tawade et al. (2022) explored the problem of laminar steady flow on a two-dimensional boundary layer using heat transfer of Casson nanofluid by means of the 4th-order Runge-Kutta method along with the shooting technique. They obtained that an increment in Nt values intensified the temperature and concentration profiles and reduced the Nusselt number, whereas an increment in Nb values raised the temperature profile and reduced both the concentration profile and the Nusselt number. Moreover, enhancing the Prandtl number Pr dropped the Nusselt number and hence raised the temperature profile.

Yousef et al. (2022) investigated Casson-Williamson nanofluid flow with the combined impact of viscous dissipation and slip velocity phenomena in the presence of thermal radiation, chemical reactions between nanoparticles, and heat absorption (generation) immersed in a saturated porous medium numerically depicted with the shooting method. Their analysis revealed that as Ec, R , and Nb increased, the local Sherwood number improved and the local Nusselt number reduced,

whereas increments in β and δ intensified the temperature profile. Moreover, as δ increased, the mass flux rate, heat transfer rate, and wall shear stress all diminished; an increment in β and δ improved the concentration profile, while an increase in Ec , R , and Nb values diminished it.

[Essam M et al. \(2023\)](#) studied the impacts of Brownian motion and thermal radiation on the mixed convection mass and heat transfer of nanofluid flow using the Runge-Kutta fourth-order method and the shooting approach. Accordingly, they showed that, due to nanofluid particle collision, an increment in Nb values upsurged the temperature profile, generating thermal energy, and when working with hot surfaces, a thermophoretic force pushed out the nanoparticle concentration away from the surface of flow. Moreover, the heat transfer rate Nu was dropped by 63.83% and 83.31% when the Nb values was improved for suction and injection, respectively, and an increment in Nt values reduced Nu by 50.48% and 65.08% for the cases of suction and injection, respectively.

[Shahzad et al. \(2023\)](#) analyzed a bioconvective micropolar nanofluid flow in the presence of gyrotactic microorganisms between two parallel disk surfaces with convective thermal boundary conditions. They discovered that as Bi and Nb increased, the thermal profile escalated throughout the central plane, and as Nt increased, the concentration profile was boosted throughout the central plane but declined as Sc was escalated. More detailed discussions were also found in the references ([Shah et al., 2020](#); [Joshi et al., 2021](#)).

2.3 Boundary Layer Flow in a Porous Medium

Studies of fluid flow and heat transfer in a porous medium (PM) have attracted the attention of searchers towards fluid flow in porous media for the past several decades because of the wide range of applications, such as water developments in geothermal supplies, drying technologies, geothermal operations, gas-cleaning filtration, petroleum industry, energy storage setups, building heat insulation-related materials, disposal reactors for nuclear waste, fossil fuel beds, etc.

There are different models to simulate porous media: the Darcy model, the non-Darcy mode, and non-equilibrium model. For low-velocity flow with weak porosity conditions, a pioneering semi-empirical equation was developed by Darcy. Later, ([Forchheimer, 1901](#)) prophesied a modified equation known as the Darcy-Forchheimer equation by introducing quadratic terms in the governing momentum equation of the fluid flow.

[Titiloye et al. \(2019\)](#) studied a two-dimensional steady laminar mixed convective boundary layer Casson nanofluid flow along a stretching/shrinking sheet with multiple slip boundary conditions in a non-Darcian porous medium, and concluded that increasing δ values enhanced the

velocity and reduced both temperature and nanoparticle concentration, increment in β and Da enhanced velocity and velocity gradient, whereas increasing Fr reduced both, increment in β reduced temperature and temperature gradient, whereas enhancing Ec accelerates both.

Tadesse et al. (2021b) investigated the MHD two-dimensional stagnation point flow of electrically conducting H_2O -based nanofluid towards a permeable shrinking/stretching in a porous medium by generating the numerical solutions using MAPLE software. They found that the thermal and velocity profiles diminished with Da and S , and the drag force coefficient increased with both Bi and Nb but decreased with increasing S . Moreover, they revealed that the heat transfer rate was lowered for the shrinking sheet compared to the stretching sheet and also decreased with Nb , Bi , and Nt , and also the mass transfer rate increased with Nt but decreased with Nb .

Verma et al. (2022) numerically analyzed the steady 2D boundary layer flows of graphene-water (Gr- H_2O) and graphene oxide-water (GO- H_2O) nanofluids past an exponentially shrinking sheet surface embedded inside a porous medium and solved them with RK4 by using the shooting technique. They showed that the Gr- H_2O nanofluid delayed the boundary layer separation, produced a higher surface drag and coefficient of surface cooling rate, enhanced the velocity profile, and lowered the temperature profile when compared with the GO- H_2O nanofluid. Moreover, it was displayed that as ϕ valued increased, the boundary layer separation improved, surface drag declined when $S < 2.5$ and then boosted, the coefficient of surface cooling rate Nu declined, the velocity profile dropped, and the temperature profile was boosted. As Da reduced, boundary layer separation was delayed, surface drag was uplifted, and the coefficient of surface cooling rate Nu upsurged, velocity profile elevated, and temperature profile decayed as wall mass transfer parameter S increased the surface drag and the coefficient of surface cooling rate Nu upsurged, velocity profile elevated, and temperature profile decayed; and as Pr values escalated, the temperature profile decayed.

Ganesh et al. (2018) studied nanofluid flow with second-order slip past a shrinking/stretching sheet surface embedded in a Darcy-Forchheimer porous medium in the presence of Ohmic and viscous dissipation impacts and solved numerically using the Runge-Kutta 4th order technique with the shooting method. In both shrinking and stretching sheet surfaces, the temperature profile is enhanced by viscous dissipation, and the thinning of the thermal boundary layer in the shrinking and stretching sheet upper branch solution occurs for increasing values of λ .

Rasool et al. (2020) discussed the heat and mass transfer characteristics of Casson-type MHD nanofluid flow through porous medium past non-linear stretching sheet surface by virtue of shoot-

ing technique with a fourth-order RK scheme. They deduced that as Da and Fr increased, the momentum and corresponding boundary layer thickness decreased, and the increment in β raised both the wall drag and the heat transfer rate, whereas an increase in δ increased skin friction and reduced the heat transfer rate.

[Das et al. \(2020\)](#) characterized the Darcy-Forchheimer flow of a magneto-couple stress fluid over an inclined, exponentially stretching flat surface. It was noted that the augmented values of Da and Fr diminished the stream function as well as the velocity profile, whereas the temperature profile increased with increasing Da .

[Bisht and Sharma \(2021\)](#) investigated the Darcy-Forchheimer flow of Sisko nanofluid with convective thermal boundary conditions and viscous dissipation, and they revealed that an increment in Da values enhanced, while an increment in Fr reduced the rate of heat transfer, and moreover, the values of Da control the skin friction.

[Tadesse et al. \(2021a\)](#) discussed the hydromagnetic stagnation point of magnetite ferrofluid flow of a convectively heated, shrinking/stretching permeable sheet surface, which was exposed to injection/suction in a Darcy-Forchheimer porous medium, where to generate the numerical solutions, they used MAPLE software. For the real solution, the coefficient of skin friction was intensified with an increment in the values of ϕ and $S > 0$, but was doomed for increasing Da , Fr , and $S < 0$ values; the velocity profile increased with the value of ϕ and $S > 0$ dropped with the values of Da . Moreover, as Da increased the momentum boundary, layer thickness raised, but it diminished with an increment in the values of ϕ , $S > 0$, and also the temperature and its boundary layer thickness were amplified with an enhancement in the values of ϕ and Ec but diminished with an increase in Da and Fr . Furthermore, they illustrated that the heat transfer rate increased with increasing ϕ and $S > 0$, whereas it decreased with Da and Fr .

[Shahzad et al. \(2023\)](#) analyzed the Darcy-Forchheimer medium effects in a bioconvective micropolar nanofluid flow in the presence of gyrotactic microorganisms between two parallel, coaxial, and radially stretching disk surfaces with convective thermal boundary conditions with the impacts of thermal radiation and heat conductivity flow. Their analysis revealed that for the enhancing Fr , the radial velocity diminished near the upper disk and raised near the lower disk; as Bi , and Nb increased, the thermal profile escalated throughout the central plane, and as Nt increased, the concentration profile was boosted throughout the central plane, but it declined as Sc values increased.

[Essam M et al. \(2023\)](#) studied the impacts of Brownian motion and thermal radiation on the mixed convection mass and heat transfer of nanofluid past a porous plate embedded in a Darcy-

Forchheimer flow using the Runge-Kutta fourth-order method and the shooting approach. Accordingly, they showed that as Da improved, the fluid velocity profile and boundary layer thickness diminished due to the intensification of fluid flow resistance.

[Lund et al. \(2022\)](#) analyzed the impact of Darcy-Forchheimer on the flow of 2D MHD nanofluid across an exponentially shrinking sheet surface in the presence of thermal slip conditions, joule heating, viscous dissipation, and first-order velocity solving them by the `bvp4c` technique in the MATLAB software package. It was revealed that, for the real solution, the skin friction coefficient enhanced and the heat transfer rate diminished with increasing ϕ values; the velocity profile was reduced for enhanced $\frac{1}{Da}$, δ , and ϕ values along the η -direction and decreased as the magnitude of Fr values intensified. More results regarding porous media flow are also found in the references ([Merkin, 1986](#); [Harris et al., 2009](#); [Animasaun, 2015](#); [Alizadeh et al., 2019, 2020](#); [Hong et al., 2020](#); [Joshi et al., 2021](#); [Singh et al., 2021](#); [Yousef et al., 2022](#)).

2.4 Stagnation Point Casson Nanofluids Flow in a PM

In fluid dynamics, a stagnation point is a point on the surface of a body submerged in a fluid flow where the fluid velocity is 0, so that stagnation point flows describe the flow or motion of a fluid in the immediate neighborhood of a stagnation point/region at the front of a blunt-nosed body, exists on all solid bodies moving in a fluid, ([Merkin et al., 2021](#)). The steady state heat transfer in an axisymmetric stagnation flow on a circular cylinder was analyzed ([Reddy Gorla, 1976](#)). [Gorla \(1993\)](#) investigated the boundary layer solutions for the axisymmetric stagnation mixed convection flow past a vertical cylinder, and ([Harris et al., 2009](#)) considered the steady mixed convection stagnation point boundary layer flow on an impermeable surface with slip. Moreover, ([Shatnawi et al., 2022](#)) made a mathematical analysis of the stagnation point flow of Casson nanofluid flow over a surface of a vertical Riga plate, one of the external agents used to limit the friction force and control the flow of fluid. The study of flow at the stagnation point (which means fluid flow over a solid surface's stagnation area) of nanofluids have many applications in plastic sheet extrusion, manufacturing, and industrial processing, aerodynamics, cooling and drying of paper products, etc. ([Tadesse et al., 2021a](#)).

Dual solutions represent different branches of solutions that are obtained under the same conditions by guessing some missing initial values, and these solutions are called upper branch, or physical/first solution, and lower branch, or non-physical/second solution ([Alsenafi and Ferdows, 2021](#)). Due to various physical flow structures and orientations, fluids flow problems involve dual or multiple branched solutions: ([Merkin, 1986](#); [Alsenafi and Ferdows, 2021](#)) investigated that dual

solution occurs due to reversed (opposing) flow (Weidman et al., 2006; Yao et al., 2019; Hamid et al., 2019) dual solution exists because of suction/blowing (Harris et al., 2009) discussed stagnation point flow and found that the dual solutions exist for assisting flows at the surface of the flow. The boundary layer flow is caused by the stretching or shrinking of surfaces and is practiced in manufacturing processes and industrial engineering such as polymer or metal extrusions, continuous glass casting, wire drawing, etc. (Waini et al., 2020). Temporal stability refers to the stability of a flow system depending on time. This can be done by taking a time-dependent problem for possible steady-state flows and confirming that only the first solution is stable and physically realizable since it possesses the smallest positive eigenvalue, whereas the second solution is unstable and cannot be realized (and will cause turbulence in the flow system).

Nandeppanavar and Shakunthala (2017) considered the impacts of a stagnation point on flow rate and heat transfer past a stretching cylinder in the presence of SWCNT and MWCNT. They found that the velocity and thermal boundary layer increased as the ϕ augmented. However, $\lambda > 0$ indicated the velocity accumulation of the potential flow in the neighborhood of the stagnation point and the decrement in the thermal boundary layer. Moreover, as the curvature and ϕ values increased the thermal boundary layer escalated.

Alizadeh et al. (2020) investigated mixed convection stagnation-point nanofluid flow heat transfer enhancement past a cylinder inserted in a porous medium. They demonstrated that an increment in ϕ results in the fattening of the thermal boundary layer and the enhancement of the heat transfer rate. Moreover, variation in convective heating has a considerable impact on the rate of heat transfer.

Poornima et al. (2022) mathematically studied heat transfer in boundary-layer stagnation flow past a stretching/shrinking cylinder. They found that the coefficients of drag force of the nanofluid flows over the stretching/shrinking cylinder and the heat transfer rate at the surface enhanced with an increase in Re of flows.

Shatnawi et al. (2022) made a mathematical analysis of the stagnation point flow of Casson nanofluid flow over a vertical Riga plate surface and solved it through the bvp4c technique using the software of Matlab packages. As ϕ increased, the velocity declined, which is due to the increment in the viscosity resulting in decreased velocity, and as β increased, the heat transfer rate and drag force diminished due to the increment in the shear thinning, which resulted in reduced skin friction and a raised Nusselt number.

Makinde (2018) theoretically investigated the stagnation point hydromagnetic flow of Fe_3O_4 -

H₂O past a convectively heated permeable shrinking/stretching sheet and revealed that dual solutions exist for a certain range of stretching/shrinking parameters ($\lambda_c < \lambda < 0$) and confirmed that the upper branch solution is temporally stable and physically realizable while the lower branch solution is not, and moreover, he confirmed the existence of a critical shrinking parameter value λ_c below which no real solution exists.

Mishra et al. (2020) performed numerically a steady MHD two-dimensional mass and heat transfer boundary layer flow of optically thick radiative, an incompressible, electrically conducting viscous fluid driven by past a permeable shrinking/stretching sheet surface in the presence of homogeneous chemical reactions, heat absorption, Joule, and viscous dissipations by implementing a shooting technique along with a fourth-order Runge-Kutta scheme. They discovered dual solutions that accounted for the shrinking sheet surface case. By performing the stability analysis, they confirmed that only the upper branch solution is stable and physically reliable, whereas the lower branch solution is unstable. Further, it was revealed that the critical shrinking parameter λ_c range got widened because of suction and magnetic field, for which the dual solution occurs, and for a physically reliable solution, the shear stress (drag force coefficient) at a shrinking/stretching sheet got augmented owing to magnetic field and suction.

Lund et al. (2020a) examined the rate of heat transfer at the steady MHD stagnation point Casson fluid flows over the stretching/shrinking surface in the presence of viscous dissipation and thermal radiation by applying the three-stage Labatto III-A method to demonstrate stability analysis with the use of a bvp4c solver in MATLAB. The transformed nonlinear ODEs were solved by using the shooting technique in the MAPLE software package. Double solutions occurred for the various ranges of λ and S , and using stability analysis, they confirmed that only the first branch solution was stable. On the shrinking surface, for the first branch solution, they demonstrated that the velocity and its boundary layer thickness diminished with increasing λ values. The thermal boundary thickness advanced with enhanced values of Ec , while the temperature profile dropped with rising values of Pr . Moreover, the skin friction coefficient was reduced as λ values were increased and improved for intensified S values.

Tadesse et al. (2021a) discussed the MHD stagnation point flow of a magnetite ferrofluid in a Darcy-Forchheimer porous medium. They confirmed the existence of a dual solution for the shrinking parameter, reaching the conclusion that the upper branch solution is hydrodynamically stable and is physically achievable, while the lower branch solution is physically unrealistic and unstable. Moreover, the nanoparticle volume fraction, suction/injection widened the range of λ

for which the solution exists, whereas increments in the porous medium and the porous medium inertia parameters narrowed it.

Tshivhi and Makinde (2021) examined the effects of magnetic fields on the thermal boundary layer flow of a conducting nanofluid consisting of $\text{Fe}_3\text{O}_4\text{-H}_2\text{O}$, $\text{Al}_2\text{O}_3\text{-H}_2\text{O}$, and $\text{Cu-H}_2\text{O}$ over a convectively heated, stretching, fixed, shrinking, and slippery surface. They tackled the problem using the shooting method with the Runge-Kutta-Fehlberg integration scheme. They revealed that in all three nanofluids, dual solutions exist for a certain interval/range of parameters ($\lambda_c < \lambda < 0$) in the case of flow over a shrinking sheet surface, and a critical value of the shrinking parameter value (λ_c) exists below which no real solution can be obtained, so that this critical value λ_c shows the extent to which the sheet surface can shrink during the cooling process. Moreover, using temporal stability analysis, they found the smallest eigenvalue that indicates only the upper branch solution is stable and physically realizable, while the lower branch solution is not. On the other hand, the nanofluid velocity boundary layer thickness decreased with an increment in δ and ϕ values. Furthermore, thermal boundary layer thickness amplified with an upsurging in Ec due to viscous dissipation, whereas an increment in ϕ , δ , and Bi reduced thermal boundary layer thickness because of sheet surface convective heating lessened it. Lastly, they also deduced that the sheet surface skin friction intensified with an increment in ϕ values but diminished with δ , whereas the heat transfer rate was enhanced with increment in ϕ values, δ and Bi , due to sheet convective heating, while an increment in Ec due to viscous dissipation lessened the heat transfer rate.

Lanjwani et al. (2021) numerically investigated the mixed convection magnetohydrodynamic (MHD) boundary layer flow and heat transfer behaviors of $\text{Ti}_6\text{Al}_4\text{V}$ and $\text{Fe}_3\text{O}_4\text{-Casson}$ -based nanofluid past an exponential shrinking/stretching surface using the shooting method in the Maple software package. As a result, they found triple solutions, which occurred at different ranges of S and λ in relation to both opposing and assisting flow conditions. Moreover, their work revealed that the skin friction rate decreased for $\lambda > 0$ and escalated for $\lambda < 0$ as S was enhanced for both $\text{Ti}_6\text{Al}_4\text{V}$ and $\text{Fe}_3\text{O}_4\text{-Casson}$ -based nanofluid flow; the rate of heat transfer diminished by rising in S for both Fe_3O_4 and $\text{Ti}_6\text{Al}_4\text{V-Casson}$ nanofluid flow; and $\text{Fe}_3\text{O}_4\text{-Casson}$ nanofluid shows a lower rate of skin friction when compared to $\text{Ti}_6\text{Al}_4\text{V-Casson}$ nanofluid, while the opposite was observed in rate of heat transfer. Further, they concluded that the velocity profile enhanced when λ and ϕ increased, but the velocity profile and its boundary layer thicknesses reduced for increasing values of β and S ; the temperature profile dropped for raising values of Pr , λ , and S , but temperature profile and its thermal boundary layer thicknesses raised as ϕ and Bi values increased.

Using Chebyshev Differential Quadrature, [Basha and Sivaraj \(2022\)](#) numerically investigated the dual solutions and stability analysis over the extending/contracting wedge and stagnation point for the Casson nanofluid flow. They found that the stagnation point has higher rates of heat transfer than the wedge, and the drag force factor increased at the surface as the values of β and S increased for the first solution.

[Dey et al. \(2022\)](#) explored the nature of dual solutions of unsteady and steady for the Casson fluid with solutal and thermal transfer flow over a permeable elongating sheet surface by solving numerically utilizing the "MATLAB routine bvp4c scheme". Due to time-dependent surface change, the flow and temperature properties over the sheet change too, and hence dual-type flow solutions came to exist. Stability scrutiny was implemented, and they also confirmed that the upper branch solution is the only visually achievable and stable solution. Moreover, by enlarging the thermal transfer of the fluid, β controls the motion of the fluid flow; under the suction environment, escalation in Pr enriches the rate of heat transfer of the fluid for the unsteady case; and the mass accumulation rate of the fluid is enhanced due to an increment in R for both steady and unsteady flow.

[Basha and Sivaraj \(2022\)](#) numerically explored the dual natures and stability performance of Casson nanofluid flow past a contracting/extending permeable stagnation point and wedge in the presence of suction/injection by applying the bvp4c MATLAB solver to solve the transformed equations. They demonstrated two solutions in a certain region when the stagnation point and wedge surface are contracted/extended, and using deep stability test the first branch solution was physically realizable and hence stable; as β increased, the velocity profile was raised for the first solution but dropped for the second solution, and the reverse was true for the temperature profile; an increment in Nt and Nb enhanced the Casson nanofluid temperature profile for both solution branches; raising the values of β and S augmented the factor of skin friction at the surface for the first solution; Brownian motion possessed the same critical point on the rates of mass and heat transfer; and higher rates of heat and mass transfer was observed for the stagnation point case rather than the wedge.

[Murad et al. \(2023\)](#) solved the heat transfer properties of Casson-Carreau fluid at stagnation point over a continuous moving plate surface using the developed collocation method which is based on B-spline functions. They concluded that as $\lambda < 0$ and β increased, the fluid flow velocity profile and rate of heat transfer were enhanced, whereas the temperature profile was reduced. Also, they deduced that for certain intervals of increasing $\lambda < 0$ the drag force enhanced. More stability

analyses on the occurrence of dual/multiple solutions due to stretching/shrinking surfaces were explained in the references ([Weidman et al., 2008](#); [Rohni et al., 2020](#); [Lund et al., 2020b, 2021](#); [RamReddy and Saran, 2022](#)).

From the aforementioned literature and to the best of our knowledge, no scientific work has been done to consider the collective effects of the embedded parameters (of our current works) on the Casson nanofluid flow with heat and mass transfer characteristics, and hence, more attention is needed for fluid flow problems, mass and heat transfer past a slippery stretching/shrinking surfaces embedded in a Darcy-Forchheimer porous medium in the presence of convective heating and porous dissipation.

Chapter 3

Research Methodology

3.1 Governing Equations

Based on the objective of this dissertation, that is, to determine of velocity, temperature, and species concentration distribution, the basic laws (conservation of mass, momentum, and energy) are demonstrated for the purpose of analysis of momentum, heat, and mass transfer of fluid flow.

The basic leading equations of the flow of viscous incompressible fluids comprises the continuity equation (conservation of mass), motion equation (conservation of momentum), energy equation (conservation of energy) and species concentration equation (conservation of species). These basic physical laws are expressed with mathematical equations in two forms: partial differential equations or integral equations.

Moreover, the governing partial differential equations for continuity, momentum, energy and chemical species concentration were formulated under the assumption that the flow is two-dimensional, incompressible, steady, laminar, and the surface is convectively heated, and the porous media is homogeneous, saturated, and isotropic.

3.1.1 The Continuity Equation

Consider a fixed finite control volume V of arbitrary shape of finite size bounded by a surface S . Let dS be a surface element of S and $\mathbf{n}$ the unit vector normal to it drawn outward as illustrated in Figure (3.1.1).

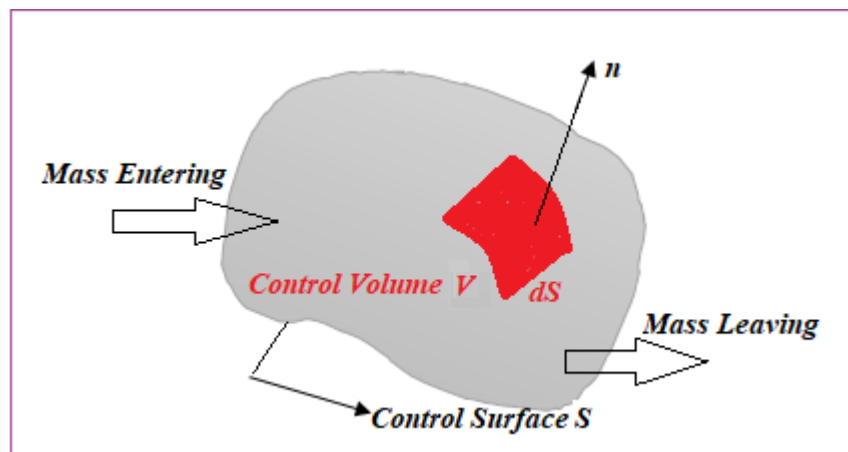


Figure 3.1.1: Control Volume Physical Model

Let the velocity of the nanofluid be $\mathbf{q} = (u, v, w)$ and mass density be ρ at a point (x, y, z) at a time t within a control volume V . Since $m = \rho V$, for small mass dm , then $dm = \rho dV$. Thus, the total mass within V is

$$m = \int_V dm = \int_V \rho dV. \quad (3.1.1)$$

According to the law of conservation of matter, the mass of a nanofluid in V is constant whenever there is no sources/sinks (no mass generation/destruction). In other words, the time rate of change in mass of a nanofluid in V is zero. Thus,

$$\frac{dm}{dt} = \frac{d}{dt} \int_V \rho dV = 0. \quad (3.1.2)$$

Here,

$$\begin{aligned} \frac{d}{dt} \int_V \rho dV &= \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho \mathbf{q} \cdot \mathbf{n} dS \quad (\text{Reynolds transport theorem} \equiv RTT) \\ &= \int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot (\rho \mathbf{q}) dV \quad (\text{Gauss divergence theorem} \equiv GDT) \\ &= \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \right) dV. \end{aligned}$$

Comparing this result with Eq. (3.1.2) gives

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \right) dV = 0. \quad (3.1.3)$$

Since the volume under consideration is arbitrary, (that is, regardless of the size, shape, or location of volume V),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0, \quad (3.1.4)$$

and this Eq. (3.1.4) is called continuity equation. Physically, this equation describes the movement of a fluid element, that is, when the fluid is in motion its mass is conserved.

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{q}) + \nabla \rho \cdot \mathbf{q} = 0, \quad [\because \nabla \cdot (\rho \mathbf{q}) = \rho(\nabla \cdot \mathbf{q}) + \nabla \rho \cdot \mathbf{q}]. \quad (3.1.5)$$

For the incompressible nanofluid under consideration $\frac{\partial \rho}{\partial t} = 0$, since ρ is constant in space and time and resulted in $\nabla \rho = 0$. Therefore, Eq. (3.1.5) is reduced to

$$\nabla \cdot \mathbf{q} = 0. \quad (3.1.6)$$

For two dimensional nanofluid flow, Eq. (3.1.6) can be re-written as (in Cartesian coordinate form)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (3.1.7)$$

and in cylindrical coordinates form $(x, y, z) \rightarrow (r, \theta, z)$ is

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{1}{r} \frac{\partial(v)}{\partial \theta} + \frac{\partial(w)}{\partial z} = 0.$$

Avoiding the angular dimension θ with $r \ll z$,

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{\partial(w)}{\partial z} = 0. \quad (3.1.8)$$

3.1.2 The Momentum Equation (Navier-Stokes Equation)

To derive the Navier-Stokes equations (which describe the momentum balance in the fluid flow) the Newton's second law of motion is applied, which states that "the total time rate of change of

linear momentum is conserved". In other words, total time rate of change of momentum of the fluid particles in the control volume is the same as the overall sum of forces acting on the fluid particles in the control volume. But the total time rate of change of momentum of the fluid particle in the control volume is given by

$$I_m = \frac{d}{dt} \int_V (\rho \mathbf{q}) dV,$$

and the total forces acting F_t on the fluid particle is

$$F_t = F_b + F_s,$$

where F_s is the total surface forces and F_b is the total body forces. Thus,

$$I_m = F_t \Rightarrow \frac{d}{dt} \int_V (\rho \mathbf{q}) dV = F_b + F_s. \quad (3.1.9)$$

But

$$\begin{aligned} \frac{d}{dt} \int_V (\rho \mathbf{q}) dV &= \int_V \frac{\partial(\rho \mathbf{q})}{\partial t} dV + \int_S (\rho \mathbf{q})(\mathbf{q} \cdot \mathbf{n}) dS \quad (\text{RTT}) \\ &= \int_V \frac{\partial(\rho \mathbf{q})}{\partial t} dV + \int_V \nabla \cdot (\rho \mathbf{q}) \mathbf{q} dV \quad (\text{GDT}) \\ &= \int_V \left(\frac{\partial(\rho \mathbf{q})}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \mathbf{q} \right) dV. \end{aligned}$$

Thus, Eq. (3.1.9) becomes

$$\int_V \left(\frac{\partial(\rho \mathbf{q})}{\partial t} + \nabla \cdot (\rho \mathbf{q}) \mathbf{q} \right) dV = F_b + F_s. \quad (3.1.10)$$

Using identity from vector calculus,

$$\nabla \cdot (\rho \mathbf{q}) \mathbf{q} = \mathbf{q} \cdot \nabla (\rho \mathbf{q}) + \rho \mathbf{q} (\nabla \cdot \mathbf{q}) = \rho (\mathbf{q} \cdot \nabla) \mathbf{q} + \rho \mathbf{q} (\nabla \cdot \mathbf{q}) = \rho (\mathbf{q} \cdot \nabla) \mathbf{q},$$

since from continuity equation $\nabla \cdot \mathbf{q} = 0$. Thus, Eq. (3.1.10) is simplified as

$$\int_V \left(\frac{\partial(\rho \mathbf{q})}{\partial t} + \rho (\mathbf{q} \cdot \nabla) \mathbf{q} \right) dV = \int_V \rho \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) dV = F_b + F_s. \quad (3.1.11)$$

To simplify the RHS of Eq. (3.1.11), the total surface force F_s (force per unit surface area) is

$$F_s = \int_S (\boldsymbol{\tau} + P_d) \cdot \mathbf{n} dS = \int_V \nabla \cdot (\boldsymbol{\tau} + P_d) dV, \quad (\text{GDT}) \quad (3.1.12)$$

where $\boldsymbol{\tau}$ and P_d are stress tensor (force per unit surface area) and the extra pressure stress tensor (drag resistance force due to the motion of the fluid particles towards pore spaces of the porous medium), respectively. For the current problem, avoiding the body force, and combining Eqs. (3.1.11) and (3.1.12)

$$\int_V \rho \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) dV = \int_V \nabla \cdot (\boldsymbol{\tau} + P_d) dV. \quad (3.1.13)$$

Since V is arbitrary,

$$\rho \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) = \nabla \cdot (\boldsymbol{\tau} + P_d) = \nabla \cdot \boldsymbol{\tau} + \nabla \cdot P_d. \quad (3.1.14)$$

which is known as the momentum equation (the Navier-Stokes equations) for incompressible fluid flow. In this case, the stress tensor τ is given by

$$\tau = -PI + \tau_{ij} \quad (3.1.15)$$

where P , I and τ_{ij} is hydrostatic pressure stress (the normal stress tensor), the identity matrix tensor and the Cauchy (shearing) stress tensor, respectively.

The rheological equations representing the basic form of incompressible Casson fluid flow with isotropic properties are given as (Animasaun, 2015; Subba Rao et al., 2015; Nakamura and Sawada, 1988):

$$\tau_{ij} = \begin{cases} 2 \left(\mu_C + \frac{p_y}{\sqrt{2\pi}} \right) \sigma_{ij} & \text{if } \pi > \pi_c \\ 2 \left(\mu_C + \frac{p_y}{\sqrt{2\pi_c}} \right) \sigma_{ij} & \text{if } \pi < \pi_c \end{cases} \quad (3.1.16)$$

where τ_{ij} , μ_C , p_y , π , σ_{ij} , and π_c represent the components of stress tensor, the non-Newtonian plastic dynamic viscosity of the Casson fluid, the fluid yield stress, $(i, j)^{th}$ rate of deformation component, strain tensor rate, and π 's critical value, respectively, which is computed according to model of the non-Newtonian fluid flow. Here

$$p_y \equiv \frac{\mu_C \sqrt{2\pi_c}}{\beta}, \quad \pi = \sigma_{ij} \sigma_{ij}, \quad \sigma_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

where $(u_1, u_2, u_3) \equiv (u, v, w)$ and $(x_1, x_2, x_3) \equiv (x, y, z)$ and β is called the non-Newtonian/Casson factor/parameter.

For Casson fluid flow $\pi > \pi_c$. The Casson dynamic viscosity is defined as: $\mu_f = \mu_C + \frac{p_y}{\sqrt{2\pi}}$. Thus on substitution, the kinematic viscosity becomes: $\nu_f = \frac{\mu_C}{\rho_f} \left(1 + \frac{1}{\beta} \right)$. Thus for the Casson fluid flow under consideration

$$\tau_{ij} = \mu_C \left(1 + \frac{1}{\beta} \right) \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]. \quad (3.1.17)$$

The Cauchy stress tensor τ_{ij} in two dimension is described in matrix as

$$\tau_{ij} = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} = \mu_C \left(1 + \frac{1}{\beta} \right) \begin{bmatrix} 2 \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2 \frac{\partial v}{\partial y} \end{bmatrix}. \quad (3.1.18)$$

Combining the continuity equation with the assumption from the problem under consideration with V_0 is the uniform (constant) suction/injection velocity at the permeable sheet surface in the y -direction so that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$, and hence Eq. (3.1.3) is reduced to

$$\tau_{ij} = \mu_C \left(1 + \frac{1}{\beta} \right) \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}. \quad (3.1.19)$$

Therefore,

$$\tau = \begin{bmatrix} -P & \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} \\ \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} & -P \end{bmatrix} \quad (3.1.20)$$

and hence

$$\nabla \cdot \boldsymbol{\tau} = \begin{bmatrix} \frac{\partial(-P)}{\partial x} + \frac{\partial}{\partial y} \left(\mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} \right) \\ \frac{\partial}{\partial x} \left(\mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} \right) + \frac{\partial(-P)}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial P}{\partial x} + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) \\ -\frac{\partial P}{\partial y} \end{bmatrix}.$$

That is,

$$\nabla \cdot \boldsymbol{\tau} = \begin{bmatrix} -\frac{\partial P}{\partial x} + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) \\ -\frac{\partial P}{\partial y} \end{bmatrix}. \quad (3.1.21)$$

Moreover, for the extra pressure P_E (Papanastasiou et al., 2021), $P_d \equiv -P_E I$ is given by

$$P_d = \begin{bmatrix} -P_E & 0 \\ 0 & -P_E \end{bmatrix} \Rightarrow \nabla \cdot P_d = \begin{bmatrix} -\frac{\partial P_E}{\partial x} + 0 \\ 0 - \frac{\partial P_E}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial P_E}{\partial x} \\ -\frac{\partial P_E}{\partial y} \end{bmatrix} \quad (3.1.22)$$

where $\frac{\partial P_E}{\partial y} = 0$ due to flow direction. Therefore,

$$\nabla \cdot (\boldsymbol{\tau} + P_d) = \begin{bmatrix} -\frac{\partial P}{\partial x} + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial P_E}{\partial x} \\ -\frac{\partial P}{\partial y} - \frac{\partial P_E}{\partial y} \end{bmatrix}.$$

Following Eq. (1.1.21) for the Casson fluid the Darcy-Forchheimer porous media flow model, the extra pressure drop ∇P is given by

$$\nabla P = -\frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) \mathbf{U} - \frac{F}{\sqrt{k_1}} |\mathbf{U}| \mathbf{U} \quad (3.1.23)$$

where $F \equiv \rho_f C_F$ is the coefficient of porous inertial resistance.

Note that for flow in one direction (i.e., one dimensional flow problem) when the extra pressure stress P_d is isotropic, then P_d may be considered as a scalar function or a tensor of rank 0. Therefore, for an isotropic porous medium considered under this study, i.e., when the pressure tensor is isotropic (its components are unchanged by rotation of the surface), the Darcy velocity $\mathbf{U} = -\frac{k_1}{\mu_f} \nabla P$ is scalar and hence $\nabla \cdot P_d \equiv -\frac{\partial P_E}{\partial x}$ in the x -direction is given by

$$\frac{\partial P_E}{\partial x} = -\frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\sqrt{k_1}} u^2. \quad (3.1.24)$$

For steady state flow under study, $\frac{\partial \mathbf{q}}{\partial t} = 0$ and hence the LHS of Eq. (3.1.14) is reduced to

$$\begin{aligned} \rho(\mathbf{q} \cdot \nabla) \mathbf{q} &\equiv \rho \left((u, v) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right) (u, v) = \rho \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) (u, v) \\ &= \rho \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) (u, v) = \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}, u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \\ &= \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}, 0 \right), \quad (\because v = V_0 \text{ is a constant}). \end{aligned} \quad (3.1.25)$$

Using Eqs. (3.1.2), (3.1.24) and (3.1.25) into Eq. (3.1.14), the x -momentum equation becomes

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial P}{\partial x} - \frac{\partial P_E}{\partial y} + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) - \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\sqrt{k_1}} u^2 \quad (3.1.26)$$

and the y-momentum equation becomes

$$\frac{\partial P}{\partial y} = \frac{\partial P_E}{\partial y} = 0. \quad (3.1.27)$$

In cylindrical coordinate form, $(x, y, z) \rightarrow (r, \theta, z)$, the momentum equation becomes,

$$\begin{aligned} \rho \left(u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} \right) = & -\frac{\partial P}{\partial r} + \mu_C \left(1 + \frac{1}{\beta} \right) \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(ru)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right. \\ & \left. - \frac{2}{r^2} \frac{\partial v}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} \right] - \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\sqrt{k_1}} u^2. \end{aligned}$$

Avoiding the angular dimension θ with $r \ll z$,

$$\rho \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu_C \left(1 + \frac{1}{\beta} \right) \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(ru)}{\partial r} \right) \right] - \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\sqrt{k_1}} u^2. \quad (3.1.28)$$

3.1.3 The Energy Equation

This equation is derived from the first law of thermodynamics (the energy conservation principle) which states that "The time rate of change of energy within the volume element dV is equals to the sum of the rate that energy is received by heat added/removed and work done in the interaction process." This law is formulated as,

$$\frac{dE}{dt} = \frac{dQ}{dt} + \frac{dW}{dt}, \quad (3.1.29)$$

where E , Q , and W are total energy (sum of internal energy and kinetic energy), amount of heat added/removed, and the work term (which could also absorb the potential energy), respectively.

The energy per unit mass is given by $e = \varepsilon_E + \frac{1}{2} \mathbf{q}^2 + gz$ (Incropera et al., 1996; Bergman et al., 2011a). By multiplying both sides with dm , we get $\rho e dV = \rho \varepsilon_E dV + \rho \frac{1}{2} \mathbf{q}^2 dV + \rho g z dV$, where $dm = \rho dV$ is the differential of nanofluid mass element of the control volume. Now, for a nanofluid under consideration the amount of the differential of the total energy $dE \equiv \rho e dV$ (in the absence of potential energy $\equiv \rho g z dV$ of the flow) within the volume element dV is given as;

$$dE = \rho \varepsilon_E dV + \frac{1}{2} \rho \mathbf{q}^2 dV = \rho \left(\varepsilon_E + \frac{1}{2} \mathbf{q}^2 \right) dV, \quad (3.1.30)$$

where g , ε_E , and $\frac{1}{2} \mathbf{q}^2$ are the gravitational acceleration, internal energy per unit mass, and kinetic energy of the unit mass, respectively. Integrating both sides of Eq. (3.1.30) gives

$$\begin{aligned}
E &= \int_V \rho \left(\varepsilon_E + \frac{1}{2} \mathbf{q}^2 \right) dV. \\
\Rightarrow \frac{dE}{dt} &= \frac{d}{dt} \int_V \rho \left(\varepsilon_E + \frac{1}{2} \mathbf{q}^2 \right) dV \\
&= \int_V \rho \left(\frac{\partial \varepsilon_E}{\partial t} + \frac{1}{2} \frac{\partial \mathbf{q}^2}{\partial t} \right) dV + \int_S \rho \left(\varepsilon_E + \frac{1}{2} \mathbf{q}^2 \right) (\mathbf{q} \cdot \mathbf{n}) dS, \quad (\text{RTT}) \\
&= \int_V \rho \left(\frac{\partial \varepsilon_E}{\partial t} + \frac{1}{2} \frac{\partial \mathbf{q}^2}{\partial t} \right) dV + \int_V \nabla \cdot \mathbf{q} \rho \left(\varepsilon_E + \frac{1}{2} \mathbf{q}^2 \right) dV, \quad (\text{GDT}) \\
&= \int_V \rho \left(\frac{\partial \varepsilon_E}{\partial t} + \frac{1}{2} \frac{\partial \mathbf{q}^2}{\partial t} \right) dV + \int_V \rho (\nabla \cdot \mathbf{q} \varepsilon_E) dV + \int_V \frac{1}{2} \rho \left[\nabla \cdot \mathbf{q} (\mathbf{q}^2) \right] dV \\
&= \int_V \rho \left(\frac{\partial \varepsilon_E}{\partial t} + \nabla \cdot \mathbf{q} \varepsilon_E \right) dV + \int_V \frac{1}{2} \rho \left[\frac{\partial \mathbf{q}^2}{\partial t} + \nabla \cdot \mathbf{q} (\mathbf{q}^2) \right] dV. \\
\Rightarrow \frac{dE}{dt} &= \int_V \rho \left(\frac{\partial \varepsilon_E}{\partial t} + \nabla \cdot \mathbf{q} \varepsilon_E \right) dV + \int_V \frac{1}{2} \rho \left[\frac{\partial \mathbf{q}^2}{\partial t} + \nabla \cdot \mathbf{q} (\mathbf{q}^2) \right] dV. \quad (3.1.31)
\end{aligned}$$

To simplify Eq. (3.1.31)

$$\frac{\partial \varepsilon_E}{\partial t} = C_p \frac{\partial T}{\partial t} \quad (\because \varepsilon_E = C_p T), \quad (3.1.32)$$

where C_p is specific heat of the nanofluid at constant pressure. But,

$$\begin{aligned}
\nabla \cdot \mathbf{q} \varepsilon_E &= \varepsilon_E (\nabla \cdot \mathbf{q}) + \mathbf{q} \cdot \nabla \varepsilon_E \\
&= \mathbf{q} \cdot \nabla \varepsilon_E \quad (\because \nabla \cdot \mathbf{q} = 0) \\
&= C_p (\mathbf{q} \cdot \nabla) T \quad (\because \varepsilon_E = C_p T) \\
\therefore, \nabla \cdot \mathbf{q} \varepsilon_E &= C_p (\mathbf{q} \cdot \nabla) T. \quad (3.1.33)
\end{aligned}$$

Since $\mathbf{q}^2$ is scalar

$$\frac{\partial \mathbf{q}^2}{\partial t} = 2\mathbf{q} \frac{\partial \mathbf{q}}{\partial t}. \quad (3.1.34)$$

Finally,

$$\begin{aligned}
\nabla \cdot \mathbf{q} (\mathbf{q}^2) &= \mathbf{q}^2 (\nabla \cdot \mathbf{q}) + \nabla \mathbf{q}^2 \cdot \mathbf{q} = \nabla \mathbf{q}^2 \cdot \mathbf{q} \quad (\because \nabla \cdot \mathbf{q} = 0) \\
&= \left(\frac{\partial \mathbf{q}^2}{\partial x}, \frac{\partial \mathbf{q}^2}{\partial y}, \frac{\partial \mathbf{q}^2}{\partial z} \right) \cdot (u, v, w) = \left(2\mathbf{q} \frac{\partial \mathbf{q}}{\partial x}, 2\mathbf{q} \frac{\partial \mathbf{q}}{\partial y}, 2\mathbf{q} \frac{\partial \mathbf{q}}{\partial z} \right) \cdot (u, v, w) \\
&= 2\mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial x}, \frac{\partial \mathbf{q}}{\partial y}, \frac{\partial \mathbf{q}}{\partial z} \right) \mathbf{q} = 2\mathbf{q} (\mathbf{q} \cdot \nabla) \mathbf{q} \\
\Rightarrow \nabla \cdot \mathbf{q} (\mathbf{q}^2) &= 2\mathbf{q} (\mathbf{q} \cdot \nabla) \mathbf{q}. \quad (3.1.35)
\end{aligned}$$

Re-substituting Eqs. (3.1.32)-(3.1.35) into Eq. (3.1.31)

$$\frac{dE}{dt} = \int_V \rho C_p \left[\frac{\partial T}{\partial t} + (\mathbf{q} \cdot \nabla) T \right] dV + \int_V \rho \mathbf{q} \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right] dV. \quad (3.1.36)$$

Work is done when an applied force acts on an object through a distance. Thus, work done on a volume element dV by the force vector $\mathbf{F}$ at the position vector $\mathbf{r}$ is given by

$$dW = \mathbf{F} \cdot d\mathbf{r}.$$

Taking derivative of both sides with respect to time

$$\frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot (u, v, w) = \mathbf{F} \cdot \mathbf{q}.$$

But the surface force is given by $F_s = \int_S (\boldsymbol{\tau} + P_d) \cdot \mathbf{n} dS$,

where $\boldsymbol{\tau}$ and P_d are stress tensor and drag resistance force (extra pressure stress), respectively.

Thus, the work done due to surface force (body force is zero) is given by

$$\begin{aligned} \frac{dW}{dt} &= \mathbf{q} \cdot \mathbf{F}_s = \int_S \mathbf{q}(\boldsymbol{\tau} + P_d) \cdot \mathbf{n} dS = \int_V \nabla \cdot [\mathbf{q}(\boldsymbol{\tau} + P_d)] dV \quad (\text{GDT}) \\ &= \int_V [\nabla \cdot (\mathbf{q}\boldsymbol{\tau}) + \nabla \cdot (\mathbf{q}P_d)] dV = \int_V \nabla \cdot (\mathbf{q}\boldsymbol{\tau}) dV. \quad [\because \nabla \cdot (\mathbf{q}P_d) = 0] \end{aligned}$$

Since the drag resistance force is perpendicular to motion of the nanofluid particles towards pore spaces of the porous medium, $\nabla \cdot (\mathbf{q}P_d) = 0$. Hence

$$\frac{dW}{dt} = \int_V \nabla \cdot (\mathbf{q}\boldsymbol{\tau}) dV. \quad (3.1.37)$$

Based on vector and tensor calculus, for a vector $\mathbf{v}$ and a tensor $\boldsymbol{\omega}$, $\nabla \cdot (\mathbf{v}\boldsymbol{\omega}) = \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\omega}) + \nabla \mathbf{v} : \boldsymbol{\omega}$.

Thus, $\nabla \cdot (\mathbf{q}\boldsymbol{\tau}) = \mathbf{q} \cdot (\nabla \cdot \boldsymbol{\tau}) + \nabla \mathbf{q} : \boldsymbol{\tau} = \mathbf{q}(\nabla \cdot \boldsymbol{\tau}) + \text{Trace}(\nabla \mathbf{q}\boldsymbol{\tau})$, because it is known that the double-dot product of two tensors (square matrices of the same order) A and B (such that $B = B^T$) is calculated as $A : B = \text{Trace}(AB)$, which is a scalar value. Therefore, Eq. (3.1.37) becomes

$$\begin{aligned} \frac{dW}{dt} &= \int_V [\mathbf{q}(\nabla \cdot \boldsymbol{\tau}) + \text{Trace}(\nabla \mathbf{q}\boldsymbol{\tau})] dV \\ &= \int_V [\mathbf{q}(\nabla \cdot \boldsymbol{\tau}) + \mathbf{q}(\nabla \cdot P_d) - \mathbf{q}(\nabla \cdot P_d) + \text{Trace}(\nabla \mathbf{q}\boldsymbol{\tau})] dV. \end{aligned} \quad (3.1.38)$$

Multiplying both sides of Eq. (3.1.14) with $\mathbf{q}$,

$$\rho \mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) = \mathbf{q}(\nabla \cdot \boldsymbol{\tau}) + \mathbf{q}(\nabla \cdot P_d). \quad (3.1.39)$$

Inserting Eq. (3.1.39) into Eq. (3.1.38) yields,

$$\frac{dW}{dt} = \int_V \left[\rho \mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) - \mathbf{q}(\nabla \cdot P_d) + \text{Trace}(\nabla \mathbf{q}\boldsymbol{\tau}) \right] dV. \quad (3.1.40)$$

To simplify $\text{Trace}(\nabla \mathbf{q}\boldsymbol{\tau})$, using Eq. (3.1.40),

$$\begin{aligned} \nabla \mathbf{q}\boldsymbol{\tau} &= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} -P & \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} \\ \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} & -P \end{bmatrix} \\ &= \begin{bmatrix} -P \frac{\partial u}{\partial x} + \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} & \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} - P \frac{\partial v}{\partial x} \\ -P \frac{\partial u}{\partial y} + \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} & \mu_C \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} - P \frac{\partial v}{\partial y} \end{bmatrix} \end{aligned}$$

Since $v = V_0$ is constant and $\frac{\partial u}{\partial x} = 0$ from continuity equation,

$$\nabla \mathbf{q}\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 \\ -P \frac{\partial u}{\partial y} & \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 \end{bmatrix}$$

$$\Rightarrow \text{Trace}(\nabla \mathbf{q} \boldsymbol{\tau}) = 0 + \mu_C \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2 = \mu_C \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2$$

The equation

$$\nabla \mathbf{q} : \boldsymbol{\tau} = \text{Trace}(\nabla \mathbf{q} \boldsymbol{\tau}) = \mu_C \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2 \quad (3.1.41)$$

is called the dissipation energy. It can be also noted that $\boldsymbol{\tau} \cdot \nabla \mathbf{q} = \nabla \mathbf{q} : \boldsymbol{\tau} = \text{Trace}(\nabla \mathbf{q} \boldsymbol{\tau})$. The dot product for two square matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the Frobenius inner product, and computed as $A \cdot B = \sum \sum a_{ij} b_{ij}$. Therefore, $\boldsymbol{\tau} \cdot \nabla \mathbf{q} = -P \frac{\partial u}{\partial x} + \mu_C \left(1 + \frac{1}{\beta}\right) \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + \mu_C \left(1 + \frac{1}{\beta}\right) \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} - P \frac{\partial v}{\partial y} = \mu_C \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2$.

Now, using Eqs. (3.1.22) and Eq. (3.1.24)

$$\begin{aligned} \mathbf{q}(\nabla \cdot P_d) &= (u, V_0) \begin{bmatrix} -\frac{\partial P_E}{\partial x} \\ -\frac{\partial P_E}{\partial y} \end{bmatrix} = -u \frac{\partial P_E}{\partial x} - V_0 \frac{\partial P_E}{\partial y} \\ &= u \left(-\frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta}\right) u - \frac{F}{\sqrt{k_1}} u^2 \right) + V_0(0) = -\frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta}\right) u^2 - \frac{F}{\sqrt{k_1}} u^3. \end{aligned}$$

Therefore, again by inserting Eqs. (3.1.41) and (3.1.3) into Eq. (3.1.40)

$$\frac{dW}{dt} = \int_V \left[\rho \mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) + \mu_C \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta}\right) u^2 + \frac{F}{\sqrt{k_1}} u^3 \right] dV. \quad (3.1.42)$$

To discuss the heat transfer term, let us start with the two special types of mass fluxes. According to (Buongiorno, 2006) in order to develop a realistic two-component model for transport phenomena in nanofluids, it is important to understand the mechanisms by which the nanoparticles can develop a slip velocity with respect to the base fluid. The two main slip mechanisms are Brownian and thermophoretic diffusions. Brownian motion is the random motion of nanoparticles due to the continuous collisions between the nanoparticles and the molecules of the basefluid, whereas thermophoresis is nanoparticle movement from warmer to colder region (in opposite direction of the temperature gradient), making a non-uniform nanoparticle volume fraction distribution. In nanofluids Buongiorno obtained four equations: two continuity equations for mass conservation, momentum, and heat energy transport experimentally. Under boundary layer approximations the momentum equation for nanofluid remained the same with that of base fluid. The Brownian motion and thermophoresis effects were exhibited through the energy and the species concentration equations only due to the presence of nanoparticles.

The nanoparticle mass flux due to Brownian diffusion $\mathbf{J}_{p,B}$ is given by

$$\mathbf{J}_{p,B} = -\rho_p D_B \nabla C, \quad D_B = \frac{k_B T}{3\pi\mu d_p} \quad (3.1.43)$$

where, T is the nanofluid temperature, k_B is the Boltzmann's constant and d_p is the nanoparticle diameter, ρ_p is density of the nanoparticle and C is concentration of nanoparticle, and D_B is Brow-

nian diffusion coefficient (which is given by the Einstein-Stokes's equation).

The nanoparticle mass flux due to thermophoretic effects $\mathbf{J}_{p,T}$ is given by

$$\mathbf{J}_{p,T} = -\rho_p D_T \frac{\nabla T}{T_\infty}, \quad D_T \equiv \bar{\beta} \frac{\mu}{\rho_p} C \quad (3.1.44)$$

where D_T is thermal diffusion coefficient and $\bar{\beta}$ is proportionality factor and given by $\bar{\beta} = 0.26 \frac{k_f}{2k_f + k_p}$ (k_f and k_p are thermal conductivity of the base fluid and the nanoparticle material, respectively).

Thus, the total nanoparticle mass flux (which is the sum of the nanoparticle mass fluxes due Brownian and thermophoresis diffusions) is given as

$$\mathbf{J}_p = -\rho_p \left(D_B \nabla C + D_T \frac{\nabla T}{T_\infty} \right). \quad (3.1.45)$$

Define the heat flux vector $\mathbf{H}$ which points in the direction of thermal energy flow and as the amount of heat energy per unit area having units of energy per area per time. Therefore, the net rate of heat transfer through the surface S is given as

$$\frac{dQ}{dt} = - \int_S (\mathbf{H} - \mathbf{h}_p \mathbf{J}_p) \cdot \mathbf{n} dS = - \int_S \mathbf{H} \cdot \mathbf{n} dS + \int_S \mathbf{h}_p \mathbf{J}_p \cdot \mathbf{n} dS, \quad (3.1.46)$$

where in the RHS, the first term is the contribution of heat conduction through base fluid and the second term is the heat flux due to nanoparticle diffusion, and $\mathbf{h}_p \equiv (C_p)_p T$ represents the specific enthalpy of the nanoparticle with T assumed to be constant over a given surface (isothermal wall).

Here, the minus sign shows heat flux transfers from higher to lower direction. Thus,

$$\int_S \mathbf{h}_p \mathbf{J}_p \cdot \mathbf{n} dS = \mathbf{h}_p \int_S \mathbf{J}_p \cdot \mathbf{n} dS. \quad (3.1.47)$$

Hence

$$\begin{aligned} \frac{dQ}{dt} &= - \int_S \mathbf{H} \cdot \mathbf{n} dS + \mathbf{h}_p \int_S \mathbf{J}_p \cdot \mathbf{n} dS = - \int_V \nabla \cdot \mathbf{H} dV + \mathbf{h}_p \int_V \nabla \cdot \mathbf{J}_p dV \quad (\text{GDT}) \\ &= - \int_V (\nabla \cdot \mathbf{H} - \mathbf{h}_p \nabla \cdot \mathbf{J}_p) dV. \end{aligned} \quad (3.1.48)$$

The sum of the Fourier heat flux (by conduction) and the heat flux by nanoparticle diffusion will give the heat flux vector $\mathbf{H}$. That is,

$$\mathbf{H} = -k_f \nabla T + \mathbf{h}_p \mathbf{J}_p \quad (3.1.49)$$

where T is the nanofluid temperature and k_f is the nanofluid thermal conductivity. Then using divergence property,

$$\nabla \cdot \mathbf{H} = -k_f \nabla^2 T + \nabla \mathbf{h}_p \cdot \mathbf{J}_p + \mathbf{h}_p \nabla \cdot \mathbf{J}_p \quad (3.1.50)$$

Inserting Eq. (3.1.50) into Eq. (3.1.48) gives

$$\frac{dQ}{dt} = - \int_V (-k_f \nabla^2 T + \nabla \mathbf{h}_p \cdot \mathbf{J}_p) dV. \quad (3.1.51)$$

Using Eq. (3.1.45) into Eq. (3.1.51)

$$\begin{aligned}
\frac{dQ}{dt} &= - \int_V \left[-k_f \nabla^2 T + \nabla((C_p)_p T) \cdot \left(-\rho_p \left(D_B \nabla C + D_T \frac{\nabla T}{T_\infty} \right) \right) \right] dV \\
&= \int_V \left[k_f \nabla^2 T + (\rho C_p)_p \left(D_B \nabla C \cdot \nabla T + D_T \frac{(\nabla T)^2}{T_\infty} \right) \right] dV.
\end{aligned} \tag{3.1.52}$$

Substituting Eqs. (3.1.36), (3.1.42) and (3.1.52) into Eq. (3.1.29)

$$\begin{aligned}
&\int_V \rho C_p \left[\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T \right] dV + \int_V \rho \mathbf{q} \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right] dV \\
&= \int_V \left[\rho \mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 \right. \\
&\quad \left. + \frac{F}{\sqrt{k_1}} u^3 \right] dV + \int_V \left[k_f \nabla^2 T + (\rho C_p)_p \left(D_B \nabla C \cdot \nabla T + D_T \frac{(\nabla T)^2}{T_\infty} \right) \right] dV.
\end{aligned} \tag{3.1.53}$$

As V was arbitrary chosen Eq. (3.1.53) one can re-written as

$$\begin{aligned}
&\rho C_p \left[\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T \right] + \rho \mathbf{q} \left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right] \\
&= \rho \mathbf{q} \left(\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla) \mathbf{q} \right) + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 \\
&\quad + \frac{F}{\sqrt{k_1}} u^3 + k_f \nabla^2 T + (\rho C_p)_p \left(D_B \nabla C \cdot \nabla T + D_T \frac{(\nabla T)^2}{T_\infty} \right).
\end{aligned} \tag{3.1.54}$$

On simplification Eq. (3.1.54) becomes

$$\begin{aligned}
\rho C_p \left[\frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T \right] &= k_f \nabla^2 T + (\rho C_p)_p \left(D_B \nabla C \cdot \nabla T + D_T \frac{(\nabla T)^2}{T_\infty} \right) \\
&\quad + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 + \frac{F}{\sqrt{k_1}} u^3.
\end{aligned} \tag{3.1.55}$$

For steady state flow $\frac{\partial T}{\partial t} = 0$ and hence in two dimensional case, Eq. (3.1.55) is expanded as

$$\begin{aligned}
\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) &= k_f \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu_C \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 \\
&\quad + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 + \frac{F}{\sqrt{k_1}} u^3 + (\rho C_p)_p D_B \left(\frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) \\
&\quad + (\rho C_p)_p \frac{D_T}{T_\infty} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right].
\end{aligned} \tag{3.1.56}$$

In cylindrical coordinate form, $(x, y, z) \rightarrow (r, \theta, z)$,

$$\begin{aligned}
\rho C_p \left(u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \theta} + w \frac{\partial T}{\partial z} \right) &= k_f \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] \\
&\quad + 2\mu_C \left(1 + \frac{1}{\beta} \right) \left[\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] \\
&\quad + \mu_C \left(1 + \frac{1}{\beta} \right) \left[\left(\frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{v}{r} \right)^2 \right)^2 \right] \\
&\quad + \frac{\mu_C}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 + \frac{F}{\sqrt{k_1}} u^3.
\end{aligned}$$

Avoiding the angular dimension θ with $r \ll z$

$$\rho C_p \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = k_f \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + 2\mu_c \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial r} \right)^2 + \frac{\mu_c}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 + \frac{F}{\sqrt{k_1}} u^3. \quad (3.1.57)$$

3.1.4 The Species Concentration Equation

This equation is formulated based on the law of mass balance. This principle can be applied to a differential control volume in a similar to the continuity relation. But here, if the conservation of a given species are considered, this relation should also include a term that accounts for the production or disappearance of species by chemical reaction within the volume.

Let begin with an arbitrary control volume V with a boundary S . The control volume is fixed in space, and fluid might move through it. Species may accumulate in V , it may travel in and out of V by bulk convection or by diffusion, and it may be created within V by homogeneous reactions. Therefore, the rate of change of the mass species in control volume V is then described by the following balance:

$$\left\{ \begin{array}{c} \text{net rate of} \\ \text{accumulation within} \\ \text{control volume} \end{array} \right\} = - \left\{ \begin{array}{c} \text{net rate of mass} \\ \text{convection into} \\ \text{control volume} \end{array} \right\} + \left\{ \begin{array}{c} \text{net rate of chemical} \\ \text{production within} \\ \text{the control volume} \end{array} \right\}$$

Then the total rate of change of the concentration species C within the control volume dV is

$$\begin{aligned} \frac{d}{dt} \int_V \rho_p C dV &\equiv \rho_p \left(\int_V \frac{\partial C}{\partial t} dV + \int_S C(\mathbf{q} \cdot \mathbf{n}) dS \right) \quad (\text{RTT}) \\ &= \rho_p \left(\int_V \frac{\partial C}{\partial t} dV + \int_V \nabla \cdot (C\mathbf{q}) dV \right). \quad (\text{GDT}) \end{aligned} \quad (3.1.58)$$

Again, the net rate of efflux of mass of the species into the control volume is given by

$$- \int_S \mathbf{J}_p \cdot \mathbf{n} dS \equiv - \int_V \nabla \cdot \mathbf{J}_p dV \quad (\text{GDT}) \quad (3.1.59)$$

where dS surface area element.

The molar reaction rate of chemical creation of species $\bar{r}$ (represents the rate at which species are produced or consumed by all homogeneous reactions, given in kg/m^3s) within the control volume is given as

$$\int_V \bar{r} dV. \quad (3.1.60)$$

Thus, the law of mass balance(material balance) equation becomes,

$$\frac{d}{dt} \int_V \rho_p C dV = - \int_V \nabla \cdot \mathbf{J}_p dV + \int_V \bar{r} dV. \quad (3.1.61)$$

And re-substituting with equivalences in Eq. (3.1.58) into Eq. (3.1.61) gives

$$\rho_p \left(\int_V \frac{\partial C}{\partial t} dV + \int_V \nabla \cdot (C\mathbf{q}) dV \right) = - \int_V \nabla \cdot \mathbf{J}_p dV + \int_V \bar{\mathbf{r}} dV. \quad (3.1.62)$$

Since $\nabla \cdot (C\mathbf{q}) = C(\nabla \cdot \mathbf{q}) + \mathbf{q} \cdot \nabla C = \mathbf{q} \cdot \nabla C$ (because $\nabla \cdot \mathbf{q} = 0$ from continuity equation), Eq. (3.1.62) is reduced to

$$\rho_p \left(\int_V \frac{\partial C}{\partial t} dV + \int_V (\mathbf{q} \cdot \nabla C) dV \right) = - \int_V \nabla \cdot \mathbf{J}_p dV + \int_V \bar{\mathbf{r}} dV. \quad (3.1.63)$$

The control volume was chosen arbitrarily, so removing the integral,

$$\rho_p \left(\frac{\partial C}{\partial t} + (\mathbf{q} \cdot \nabla C) \right) = - \nabla \cdot \mathbf{J}_p + \bar{\mathbf{r}}. \quad (3.1.64)$$

Inserting the total mass flux $\mathbf{J}_p$ from Eq. (3.1.45) for steady state flow Eq. (3.1.64) can be re-written as

$$\rho_p (\mathbf{q} \cdot \nabla C) = - \nabla \cdot \left(-\rho_p \left(D_B \nabla C + D_T \frac{\nabla T}{T_\infty} \right) \right) + \bar{\mathbf{r}} \quad (3.1.65)$$

which is simplified to

$$\rho_p (\mathbf{q} \cdot \nabla C) = \rho_p \left(D_B \nabla^2 C + D_T \frac{\nabla^2 T}{T_\infty} \right) + \bar{\mathbf{r}}. \quad (3.1.66)$$

For two dimensional Cartesian form Eq. (3.1.66) is equivalently expanded as

$$\rho_p \left(u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) = \rho_p \left[D_B \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \right] + \bar{\mathbf{r}} \quad (3.1.67)$$

where $\bar{\mathbf{r}} = \gamma \rho_p \Delta C$ such that γ the chemical reaction rate and ΔC concentration difference.

3.2 Computational Methods of Solution

In this dissertation, some mathematical methods were used to solve the transformed coupled non-linear boundary value ODEs. In most cases of the fluid dynamics/mechanics problems there is no exact solutions due to the highly coupled and non-linearity nature of the problems. In this subsection the numerical computations techniques to obtain the solution were explained. The general computational algorithms under this research were implemented using MAPLE symbolic package. The computational methods used for Solving BVPs are:

3.2.1 The Shooting Technique

The shooting method is an iterative algorithm that reestablishes the formulated BVPs to a corresponding initial value problem (IVPs) with its suitable initial conditions. The new converted problem needs the solution of the IVP with the initial conditions arbitrary selected to guess the

boundary conditions at the end points. This procedure is repeated again with a new set of initial conditions till the desired accuracy is obtained or a limit to the iteration is reached if these boundary conditions are not satisfied to the required accuracy.

To explain how to apply shooting technique, take the two point boundary value problem

$$\begin{aligned} & y'''(x) = f(x, y, y', y'') \\ \text{with } & \begin{cases} y(x_a) = l, \\ y(x_b) = m, \quad x \in [a, b] \\ y'(x_b) = n, \end{cases} \end{aligned} \quad (3.2.1)$$

In order to apply the shooting technique to tackle this problem, the given higher order ODE must be converted to equivalent first-order system ODEs. Therefore, by introducing new variables: $y(x) = f_1$, $y'(x) = f_2$, $y''(x) = f_3$, and assuming the missed initial conditions as $f_2(x_a) = p$, $f_3(x_a) = q$, where p and q are called the shooting parameters, the following system of IVPs are obtained:

$$\begin{aligned} f_1' &= f_2, & f_1(x_a) &= l, \\ f_2' &= f_3, & f_2(x_a) &= p, \\ f_3' &= f(x, f_1, f_2, f_3), & f_3(x_a) &= q, \end{aligned} \quad (3.2.2)$$

Now, the solution of the IVP in Eq. (3.2.2) should satisfy the following two mandatory conditions,

$$\begin{aligned} F_1(p, q) &= f_1(x_b; p, q) - m = 0, \\ F_2(p, q) &= f_2(x_b; p, q) - n = 0. \end{aligned} \quad (3.2.3)$$

To obtain the zeros (p, q) of this system a standardized root-finding technique called generalized Newton-Raphson iteration method is used.

3.2.2 Generalized Newton Raphson Technique

The Newton-Raphson Method is an iterative technique for root-finding to a real-valued function, and method has quadratic rate of convergence under appropriate conditions.

Consider a system of n nonlinear equations with n unknowns, as the one described below:

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \cdot \\ \cdot \\ \cdot \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases} \quad (3.2.4)$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, n$, are nonlinear functions. This non-linear system of equations in Eq. (3.2.4) can be rewritten in compacted form as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \quad (3.2.5)$$

where $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\mathbf{f} = [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})]^T$, $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$, and $\mathbf{0} = [0, 0, \dots, 0]^T$. Taking an initial guess for vector $\mathbf{x}$, say, $\mathbf{x}^{(0)}$, where $\mathbf{f}(\mathbf{x}^{(0)}) \neq \mathbf{0}$. Therefore, it is needed to find $\Delta\mathbf{x}^{(0)}$ so that $\mathbf{f}(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) = \mathbf{0}$. Using the first-order Taylor series expansion $\mathbf{f}(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)})$ is approximated as:

$$\mathbf{f}(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) \approx \mathbf{f}(\mathbf{x}^{(0)}) + \mathbf{J}_{\mathbf{c}}^{(0)} \Delta\mathbf{x}^{(0)}, \quad (3.2.6)$$

where $\mathbf{J}_{\mathbf{c}}$ is $n \times n$ Jacobian:

$$\mathbf{J}_{\mathbf{c}} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix} \quad (3.2.7)$$

From Eq. (3.2.6) since it is sought to have $\mathbf{f}(\mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)}) = \mathbf{0}$, $\Delta\mathbf{x}^{(0)}$ can be computed as

$$\Delta\mathbf{x}^{(0)} \approx -[\mathbf{J}_{\mathbf{c}}^{(0)}]^{-1} \mathbf{f}(\mathbf{x}^{(0)}) \quad (3.2.8)$$

Thus, vector $\mathbf{x}$ can be updated as:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \Delta\mathbf{x}^{(0)} \quad (3.2.9)$$

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - [\mathbf{J}_{\mathbf{c}}^{(i)}]^{-1} \mathbf{f}(\mathbf{x}^{(i)}), \quad (3.2.10)$$

where i is the iteration counter.

3.2.3 Runge-Kutta-Fehlberg Technique (RKF45)

This technique needs six function evaluations per step but it provides an automatic error estimate and at the same time produces better accuracy than the standard fourth-order method in the solution of an IVP. Fehlberg showed that four of these function values combined with one set of coefficients could be used to produce a fourth-order method while all six values combined with another set of coefficients could be used to produce a fifth-order method. Comparison of the values produced by the fourth-order and fifth-order methods then leads to an estimate of the error which can be used for step size control (Conte and De Boor, 2017). According to (Faires and Burden, 2012), if the two answers are in close agreement, the approximation is accepted, and if the two answers do not agree to a specified accuracy, the step size is reduced. Moreover, if the answers agree to more significant digits than required, the step size is increased.

Consider an initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y(t)) \quad (3.2.11)$$

where $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$, $f \in [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with an initial condition $y(0) = y_0$.

It is needed in a numerical approximation of the continuously differentiable solution $y(t)$ of the IVP Eq. (3.2.11) over the time interval $t \in [a, b]$. To this aim, subdivide the interval $[a, b]$ into N equal sub-intervals and select the mesh points t_j

$$t_j = a + jh, \quad j = 0, 1, \dots, N, \quad h = \frac{b-a}{N} \quad (3.2.12)$$

The value h is called a step size.

The family of explicit Runge-Kutta (RK) methods of the m 'th stage is given by

$$y(t_{n+1}) := y_{n+1} = y_n + \sum_{i=1}^m c_i k_i \quad (3.2.13)$$

where

$$\begin{aligned} k_1 &= hf(t_n, x_n), \\ k_2 &= hf(t_n + \alpha_2 h, y_n + h\beta_{21}k_1(t_n, x_n)), \\ k_3 &= hf(t_n + \alpha_3 h, y_n + h(\beta_{31}k_1(t_n, y_n) + \beta_{32}k_2(t_n, x_n))), \\ &\vdots \\ k_m &= hf(x_n + \alpha_m h, y_n + \sum_{j=1}^{m-1} \beta_{mj}k_j) \end{aligned} \quad (3.2.14)$$

To specify a particular method, it is needed to provide the integer m (the number of stages), and the coefficients α_i (for $i = 2, 3, \dots, m$), β_{ij} (for $1 \leq j < i \leq m$), and c_i (for $i = 1, 2, \dots, m$).

To describe briefly the approach taken by Fehlberg, assume that the integration of the equation $y' = f(t, y)$ up to a point t_n with a step size h , and it is required to find an estimate of $y(t)$ at $t = t_{n+1}$. One estimate will be given by the formula, taking $m = 6$.

$$y_{n+1} = y_n + \sum_{i=1}^6 c_i k_i \quad (3.2.15)$$

for certain coefficients c_i and a second estimate will be given by

$$\bar{y}_{n+1} = y_n + \sum_{i=1}^6 \bar{c}_i k_i \quad (3.2.16)$$

for another set of coefficients $\bar{c}_i$. The error estimate for step size control is then computed as follows:

$$\epsilon = |y_{k+1} - \bar{y}_{k+1}| = \sum_{i=1}^6 (c_i - \bar{c}_i) k_i \quad (3.2.17)$$

and it can be used as described earlier to estimate the proper step h for the next integration. The functions k_i are the same in both formulas and can be expressed in the form

$$k_i = hf(t_n + \alpha_i h, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j), \quad i = 1, \dots, 6 \quad (3.2.18)$$

There are many possible choices of the coefficients α_i and β_{ij} that will lead to Runge-Kutta methods of order 5 as Fehlberg proposed. Accordingly the following six values are obtained:

$$\begin{aligned} k_1 &= hf(t_n, y_n), \\ k_2 &= hf\left(t_n + \frac{1}{4}h, y_n + \frac{1}{4}k_1\right), \\ k_3 &= hf\left(t_n + \frac{3}{8}h, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right), \\ k_4 &= hf\left(t_n + \frac{12}{13}h, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right), \\ k_5 &= hf\left(t_n + h, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right), \\ k_6 &= hf\left(t_n + \frac{1}{2}h, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right), \end{aligned} \quad (3.2.19)$$

Then, an approximation to the solution of the IVP is made using a Runge-Kutta method of order 4 using the obtained k_i 's:

$$\bar{y}_{k+1} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5, \quad (3.2.20)$$

Again, a better value for the solution is determined using a Runge-Kutta method of order 5 inserting obtained k_i values:

$$y_{k+1} = y_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6, \quad (3.2.21)$$

An estimate of the error can be obtained by subtracting the two values obtained. If the error exceeds a specific threshold, the results can be recalculated using a smaller step size. The new optimal step size h_n can be determined by multiplying the scalar s times the old step size h_o (Mathews and Fink, 2004). The scalar is given by

$$s = \left(\frac{h_o \epsilon_{tol}}{2\epsilon} \right)^{\frac{1}{4}} \quad (3.2.22)$$

so that optimal step size becomes

$$h_n = sh_o \equiv h_o \left(\frac{h_o \epsilon_{tol}}{2\epsilon} \right)^{\frac{1}{4}} \quad (3.2.23)$$

where ϵ_{tol} is the expected error or prescribed tolerance.

Chapter 4

Stability Analysis of Dual Solutions of Convective Flow of Casson Nanofluid Past a Shrinking/Stretching Slippery Sheet in Porous Medium

4.1 Introduction

Thermally driven boundary layer flow of Casson nanofluid in a porous medium has several applications in chemical and mechanical engineering, such as in food processing and storage, geophysical systems, electro-chemistry, fibrous insulation, metallurgy, the design of pebble bed nuclear reactors, underground disposal of nuclear or non-nuclear waste, microelectronics cooling. A careful observation of the aforementioned literature reveals that no scientific work has been done to study the collective impacts of the embedded parameters under consideration on the Casson nanofluid flow with heat and mass transfer properties.

In this dissertation, the main goal of this chapter is to investigate the occurrence of a dual solution which is expected due to the shrinking surface, and temporal stability analysis must be performed to determine the stable and physically reliable solutions that buttress the theoretical relevance of the work, for the hydrodynamic boundary layer flow of Casson nanofluid past a slippery permeable stretching/shrinking sheet in a Darcy-Forchheimer porous media with the presence of viscous dissipation, chemical reaction, and convective heating. Particularly, the Brownian motion and thermophoresis effects on the entire flow, thermal, and solutal profiles were entertained thoroughly for their engineering cooling applications, for instance, in heat exchangers, cooling of nuclear reactors, electronic cooling systems (such as fat plates), radiators, etc.. Numerical results for the nanoparticle concentration, temperature, and velocity profiles are illustrated graphically, and coefficient of skin friction, mass, and heat transfer rates are discussed both graphically and quantitatively. The application of this study can be found in industrial and engineering cooling of convectively heated surfaces for efficient operation of the system using nanofluid.

4.2 Formulation of Mathematical Model

Consider the steady incompressible two-dimensional laminar boundary layer flow of a Casson nanofluid over a convectively heated permeable linearly shrinking/stretching sheet surface of velocity $U_w \equiv bx$, and the physical flow model is graphically displayed in Figure 4.2.1. The presence of thermophoresis, Brownian motion, viscous dissipation, chemical reaction, and slip conditions

are considered in mass and heat transfer models. The x -axis is taken parallel to the plate flow direction, and the y -axis is perpendicular to it. The velocity components parallel to the x and y axes are given by u and v , respectively. T_∞ , $U_\infty(x) \equiv ax$, C_∞ , and $T_f(x) \equiv T_\infty + nx^2$ represent ambient temperature, ambient velocity, ambient concentration (the ratio of the mass of the nanoparticles to the sum of masses of the base fluid and nanoparticles), and temperature of the fluid beneath the sheet surface, respectively. It is assumed that the surface of the plate is subjected to slip boundary conditions.

Note that the nanoparticles concentration boundary layer appears first, followed by thermal and momentum boundary layer as shown in the problem geometry Figure 4.2.1, because in two phase flow scenario, the nanoparticles tends to move close to the sheet surface due to its higher density as compared to the base fluid during the flow process, the thermal boundary layer may follow as a results of heat transfer enhancement capability of nanofluid and finally the momentum boundary layer resulting from the continuous motion of the nanofluid where in our assumption the Prandtl number Pr greater than unity ($Pr > 1$).

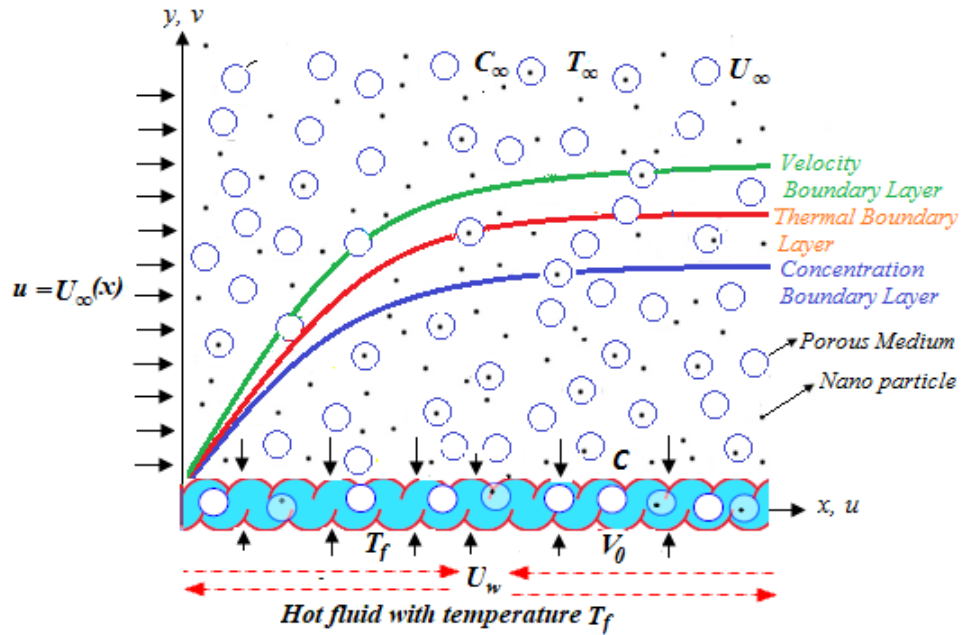


Figure 4.2.1: The Casson nanofluid flow physical model

Let $U = (u(x, y), v(x, y))$, $T = T(x, y)$, and $C = C(x, y)$ represent the velocity, temperature, and nanoparticle volume fraction of the Casson nanofluid flow under consideration, respectively. Moreover, following (Buongiorno, 2006; Titiloye et al., 2019; Oyelakin et al., 2016; Nield, 2000) and Eqs. (3.1.7), (3.1.26), (3.1.56), and (3.1.67), the boundary layer through Darcy Forchheimer porous media (which is homogeneous, saturated, and isotropic) with mass and heat transfer of

the Casson nanofluid equations are expressed:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= -\frac{\partial P}{\partial x} + \mu_B \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\mu_B}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\sqrt{k_1}} u^2, \\
(\rho C_p)_{nf} \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) &= k_f \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu_B \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\mu_B}{k_1} \left(1 + \frac{1}{\beta} \right) u^2 \\
&\quad + \frac{F}{\sqrt{\rho C_p k_1}} u^3 + (\rho C_p)_p D_B \left(\frac{\partial C}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + (\rho C_p)_p \frac{D_T}{T_\infty} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right], \\
\rho_p \left(u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} \right) &= \rho_p \left[D_B \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \right] + \bar{\mathbf{r}}.
\end{aligned}$$

Taking the standard boundary-layer approximation based on scale analysis, It is obtained that variations along the y-axis within the boundary layer is much faster than the variations along the x-axis.

i.e, $y \ll x$ and $v \ll u$ implies

$$\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}, \quad \frac{\partial T}{\partial x} \ll \frac{\partial T}{\partial y}, \quad \text{and} \quad \frac{\partial C}{\partial x} \ll \frac{\partial C}{\partial y}.$$

Following the above discussion, the governing equations are simplified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.2.1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) - \frac{\nu}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\rho \sqrt{k_1}} u^2 \tag{4.2.2}$$

$$\begin{aligned}
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{k_f}{\rho C_p} \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\nu}{C_p k_1} \left(1 + \frac{1}{\beta} \right) u^2 + \frac{F}{\rho C_p \sqrt{k_1}} u^3 \\
&\quad + \frac{(\rho C_p)_p}{(\rho C_p)_{nf}} \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right]
\end{aligned} \tag{4.2.3}$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} + \gamma \Delta C \tag{4.2.4}$$

where the change in concentration of species $\Delta C = C_\infty - C$, the dynamic viscosity $\nu = \frac{\mu_B}{\rho}$, $\mu_B \equiv \mu$, and the nanofluid density $\rho \equiv \rho_f$.

Here the momentum Eq. (4.2.2) is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} \right) - \frac{\nu}{k_1} \left(1 + \frac{1}{\beta} \right) u - \frac{F}{\rho \sqrt{k_1}} u^2 \tag{4.2.5}$$

To balance the momentum equation at free stream, (i.e. $y \rightarrow \infty$)

$$-\frac{1}{\rho} \frac{\partial P}{\partial x} = U_\infty \frac{\partial U_\infty}{\partial x} + \frac{\nu}{k_1} \left(1 + \frac{1}{\beta} \right) U_\infty + \frac{F}{\rho \sqrt{k_1}} U_\infty^2 \tag{4.2.6}$$

Thus, by redefining the momentum equation in such a way that it is balanced with the free stream flow, and hence, following (Makinde, 2011; Makinde and Mutuku, 2014; Tadesse et al., 2021a) Eq. (4.2.5) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{\partial U_\infty}{\partial x} + v \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} - \frac{1}{k_1} (u - U_\infty) \right) - \frac{F}{\rho \sqrt{k_1}} (u - U_\infty)^2 \quad (4.2.7)$$

Note that, the expression $(u - U_\infty)^2$ in the Eq. (4.2.7) is based on the free stream pressure gradient expression in Eq. (4.2.6) and the fact that both left and right side of model momentum Eq. (4.2.7) must be equal at free stream when $u = U_\infty$ and $v = 0$. Although using the expression $(u^2 - U_\infty^2)$ may also satisfy the free stream condition, however, the expression $(u - U_\infty)^2$ is most appropriate to use in this context since it better reflects the nonlinear effects and also satisfies the free stream conditions. More generally, as $u \rightarrow U_\infty$, $(u - U_\infty)^2 = u^2 - 2uU_\infty + U_\infty^2 \rightarrow u^2 - 2U_\infty^2 + U_\infty^2 = u^2 - U_\infty^2$. Similarly, the balanced energy equation is defined as

$$\begin{aligned} u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \bar{\tau} \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{v}{C_p} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 \\ + \frac{v}{C_p k_1} \left(1 + \frac{1}{\beta} \right) (u - U_\infty)^2 + \frac{F}{\rho C_p \sqrt{k_1}} (u - U_\infty)^3 \end{aligned} \quad (4.2.8)$$

Collecting these balanced equations, the governing equations for the nanofluid flow over a stretching/shrinking sheet are re-written as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4.2.9)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{\partial U_\infty}{\partial x} + v \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial y^2} - \frac{1}{k_1} (u - U_\infty) \right) - \frac{F}{\rho \sqrt{k_1}} (u - U_\infty)^2 \quad (4.2.10)$$

$$\begin{aligned} u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \bar{\tau} \left[D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{v}{C_p} \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial u}{\partial y} \right)^2 \\ + \frac{v}{C_p k_1} \left(1 + \frac{1}{\beta} \right) (u - U_\infty)^2 + \frac{F}{\rho C_p \sqrt{k_1}} (u - U_\infty)^3 \end{aligned} \quad (4.2.11)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty} \frac{\partial^2 T}{\partial y^2} - \gamma (C - C_\infty) \quad (4.2.12)$$

with the following boundary conditions, following (Tshivhi and Makinde, 2021; Shah et al., 2020; Tadesse et al., 2021b; Affy, 2017),

$$\left. \begin{aligned} u(x, 0) &= U_w(x) + \frac{\mu}{L} \left(1 + \frac{1}{\beta} \right) \frac{\partial u}{\partial y}, \quad v(x, 0) = V_0, \\ -k_f \frac{\partial T}{\partial y}(x, 0) &= h_f [T_f(x, 0) - T(x, 0)], \quad D_B \frac{\partial C}{\partial y}(x, 0) = \frac{D_T}{T_\infty} \frac{\partial T}{\partial y}(x, 0), \\ u(x, \infty) &\rightarrow U_\infty(x), \quad T(x, \infty) \rightarrow T_\infty, \quad C(x, \infty) \rightarrow C_\infty. \end{aligned} \right\} \quad (4.2.13)$$

where, ρ is the fluid density, k_1 is the porous media permeability, F is coefficient of Forchheimer's resistance/drag force, $\alpha = \frac{k_f}{(\rho C_p)_f}$ is thermal diffusivity (the ratio of heat conducted to heat stored) of the fluid, $\bar{\tau} = \frac{(\rho C_p)_p}{(\rho C_p)_{nf}}$ is the quotient of the effective heat capacity of the nanoparticle to that of the nanofluid, C_p is specific heat at a constant pressure of the fluid, D_B is coefficient of Brownian diffusion, D_T is coefficient thermophoretic diffusion, γ is chemical reaction rate, μ is fluid dynamic viscosity, L is coefficient slip length (which is the measure of an extrapolated distance across the wall where the tangential velocity component becomes zero, and computed by taking the product of density and velocity of the fluid), V_0 is the uniform suction/injection velocity at the permeable sheet surface, k_f is the fluid thermal conductivity, h_f is coefficient of convective heat transfer, T_f is convective fluid temperature, a is free stream linear flow rate, b is linear stretching/shrinking rate of the surface (for $b > 0$ stretching wall and for $b < 0$ shrinking wall).

4.3 Similarity Transformation

By taking the dimensionless variables, Eqs. (4.2.9)-(4.2.13) can be transferred to the non-dimensional form to get the similar solutions (Shah et al., 2020; Tadesse et al., 2021b; Mustafa and Khan, 2015):

$$\eta = \sqrt{\frac{a}{v}}y, \quad \psi(x,y) = x\sqrt{av}f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_\infty}. \quad (4.3.1)$$

where $\psi = x\sqrt{av}f(\eta)$ is the stream function such that (Tadesse et al., 2021b):

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{d\eta}{dy} = x\sqrt{av}f'(\eta) \cdot \sqrt{\frac{a}{v}} = axf'(\eta) = U_\infty f' \quad (4.3.2)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{av}f(\eta) = -\sqrt{av}f \quad (4.3.3)$$

And hence using the similarity transformation obtained above, the similarity transformation processes to ordinary differential equations, denoting the n^{th} derivative of f with respect to η , $f^{(n)}(\eta) \equiv f^{(n)}$, are as shown below,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{d\eta}{dx} = af'(\eta) = af', \quad (4.3.4)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta} \frac{d\eta}{dy} = -\sqrt{av}f'(\eta)\sqrt{\frac{a}{v}} = -af', \quad (4.3.5)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \frac{d\eta}{dy} = axf''(\eta)\sqrt{\frac{a}{v}} = U_\infty \sqrt{\frac{a}{v}}f'', \quad (4.3.6)$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(U_\infty \sqrt{\frac{a}{v}}f'' \right) = ax\sqrt{\frac{a}{v}}f'''(\eta)\sqrt{\frac{a}{v}} = \frac{aU_\infty}{v}f'''. \quad (4.3.7)$$

Again,

$$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \Rightarrow T = T_\infty + (T_f - T_\infty)\theta(\eta) \equiv \theta nx^2 + T_\infty, \quad \frac{\partial T}{\partial x} = 2\theta nx, \quad (4.3.8)$$

$$\frac{\partial T}{\partial y} = (T_f - T_\infty) \frac{\partial \theta(\eta)}{\partial y} = (T_f - T_\infty) \theta' \frac{\partial \eta}{\partial y} = nx^2 \sqrt{\frac{a}{v}} \theta', \quad (4.3.9)$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left((T_f - T_\infty) \theta' \sqrt{\frac{a}{v}} \right) = nx^2 \frac{a}{v} \theta''. \quad (4.3.10)$$

and finally,

$$\phi(\eta) = \frac{C - C_\infty}{C_\infty} \Rightarrow C = C_\infty + C_\infty \phi(\eta) \equiv C = C_\infty(1 + \phi(\eta)), \quad \frac{\partial C}{\partial x} = 0, \quad (4.3.11)$$

$$\frac{\partial C}{\partial y} = C_\infty \frac{\partial \phi}{\partial y}(\eta) = C_\infty \frac{\partial \phi}{\partial \eta} \frac{d\eta}{dy} = C_\infty \phi'(\eta) \sqrt{\frac{a}{v}} = C_\infty \sqrt{\frac{a}{v}} \phi', \quad (4.3.12)$$

and

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial C}{\partial y} \right) = \frac{\partial}{\partial y} \left(C_\infty \sqrt{\frac{a}{v}} \phi' \right) = C_\infty \sqrt{\frac{a}{v}} \phi'' \sqrt{\frac{a}{v}} = C_\infty \frac{a}{v} \phi''. \quad (4.3.13)$$

The Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = af' + (-af') = af' - af' = 0 \quad (4.3.14)$$

The Momentum Equation

Substituting Eqs. (4.3.2) - (4.3.7) into the x-momentum Eq. (4.2.10),

$$\begin{aligned} (U_\infty f')(af') + (-\sqrt{av}f)(U_\infty \sqrt{\frac{a}{v}} f'') &= U_\infty a + v \left(1 + \frac{1}{\beta} \right) \left(\frac{aU_\infty}{v} f''' \right. \\ &\quad \left. - \frac{1}{k_1} (U_\infty f' - U_\infty) \right) - \frac{F}{\rho \sqrt{k_1}} (U_\infty f' - U_\infty)^2 \end{aligned} \quad (4.3.15)$$

Factoring common terms

$$U_\infty a [(f')^2 - ff''] = U_\infty a + v \left(1 + \frac{1}{\beta} \right) \left(\frac{aU_\infty}{v} f''' - \frac{U_\infty}{k_1} (f' - 1) \right) - \frac{U_\infty^2 F}{\rho \sqrt{k_1}} (f' - 1)^2 \quad (4.3.16)$$

Dividing both sides by $U_\infty a$

$$(f')^2 - ff'' = 1 + v \left(1 + \frac{1}{\beta} \right) \left(\frac{1}{v} f''' - \frac{1}{ak_1} (f' - 1) \right) - \frac{U_\infty F}{a\rho \sqrt{k_1}} (f' - 1)^2 \quad (4.3.17)$$

Re-writing with some modification

$$\left(1 + \frac{1}{\beta} \right) \left(f''' - \frac{v}{ak_1} (f' - 1) \right) + ff'' - (f')^2 - \frac{xF}{\rho \sqrt{k_1}} (f' - 1)^2 + 1 = 0 \quad (4.3.18)$$

On substituting with parameters, finally

$$\left(1 + \frac{1}{\beta} \right) \left(f''' - \frac{1}{Da} (f' - 1) \right) + ff'' - (f')^2 - Fr(f' - 1)^2 + 1 = 0 \quad (4.3.19)$$

The Energy Equation

Substituting Eqs. (4.3.1) - (4.3.13) into the energy Eq. (4.2.11),

$$\begin{aligned}
& (U_\infty f')(2nx\theta) + (-\sqrt{av}f)(nx^2\theta'\sqrt{\frac{a}{v}}) = \alpha(nx^2\frac{a}{v}\theta'') \\
& + \bar{\tau} \left[D_B(C_\infty\sqrt{\frac{a}{v}}\phi')(nx^2\sqrt{\frac{a}{v}}\theta') + \frac{D_T}{T_\infty} \left(nx^2\sqrt{\frac{a}{v}}\theta' \right)^2 \right] + \frac{v}{C_p} \left(1 + \frac{1}{\beta} \right) \cdot \\
& \left(U_\infty\sqrt{\frac{a}{v}}f'' \right)^2 + \frac{v}{C_p k_1} \left(1 + \frac{1}{\beta} \right) (U_\infty f' - U_\infty)^2 + \frac{F}{\rho C_p \sqrt{k_1}} (U_\infty f' - U_\infty)^3
\end{aligned} \tag{4.3.20}$$

Collecting common factors

$$\begin{aligned}
& (U_\infty nx)(2f'\theta - f\theta') = \frac{\alpha}{v}(U_\infty nx)\theta'' \\
& + \bar{\tau} \left[D_B C_\infty \frac{(U_\infty nx)}{v} \phi' \theta' + \frac{D_T}{v T_\infty} (U_\infty n^2 x^3) (\theta')^2 \right] + \frac{U_\infty^2 a}{C_p} \left(1 + \frac{1}{\beta} \right) (f'')^2 \\
& + \frac{v U_\infty^2}{C_p k_1} \left(1 + \frac{1}{\beta} \right) (f' - 1)^2 + \frac{U_\infty^3 F}{\rho C_p \sqrt{k_1}} (f' - 1)^3
\end{aligned} \tag{4.3.21}$$

Dividing both sides by $(U_\infty nx)$

$$\begin{aligned}
2f'\theta - f\theta' &= \frac{\alpha}{v}\theta'' + \bar{\tau} \left[\frac{D_B C_\infty}{v} \phi' \theta' + \frac{D_T}{v T_\infty} (nx^2) (\theta')^2 \right] \\
& + \frac{a^2}{nC_p} \left(1 + \frac{1}{\beta} \right) (f'')^2 + \frac{av}{nC_p k_1} \left(1 + \frac{1}{\beta} \right) (f' - 1)^2 + \frac{a^2 x F}{n \rho C_p \sqrt{k_1}} (f' - 1)^3
\end{aligned} \tag{4.3.22}$$

Replacing terms with equivalent identities and arranging gives

$$\begin{aligned}
2f'\theta - f\theta' &= \frac{\alpha}{v}\theta'' + \left[\frac{\bar{\tau} D_B C_\infty}{v} \phi' \theta' + \frac{\bar{\tau} D_T (T_f - T_\infty)}{v T_\infty} (\theta')^2 \right] \\
& + \frac{a^2}{nC_p} \left[\left(1 + \frac{1}{\beta} \right) (f'')^2 + \frac{v}{a k_1} \left(1 + \frac{1}{\beta} \right) (f' - 1)^2 + \frac{x F}{\rho \sqrt{k_1}} (f' - 1)^3 \right]
\end{aligned} \tag{4.3.23}$$

Re-arranging

$$\begin{aligned}
& \frac{1}{Pr} \theta'' + f\theta' - 2f'\theta + (Nb\phi' + Nt\theta') \theta' \\
& + Ec \left[\left(1 + \frac{1}{\beta} \right) (f'')^2 + \frac{1}{Da} \left(1 + \frac{1}{\beta} \right) (f' - 1)^2 + Fr(f' - 1)^3 \right] = 0
\end{aligned} \tag{4.3.24}$$

where $\frac{a^2}{nC_p} = \frac{a^2 x^2}{nx^2 C_p} = \frac{U_\infty^2}{(T_f - T_\infty) C_p}$

The Concentration Equation

Substituting Eqs. (4.3.1) - (4.3.13) into the concentration Eq. (4.2.12),

$$(U_\infty f')(0) + (-\sqrt{av}f) \left(C_\infty \sqrt{\frac{a}{v}} \phi' \right) = D_B \left(C_\infty \frac{a}{v} \phi'' \right) + \frac{D_T}{T_\infty} \left((T_f - T_\infty) \theta'' \frac{a}{v} \right) - \gamma C_\infty \phi \tag{4.3.25}$$

Simplifying

$$-(C_\infty a) f \phi' = (C_\infty a) \frac{D_B}{v} \phi'' + \frac{a D_T}{v T_\infty} (T_f - T_\infty) \theta'' - \gamma C_\infty \phi \tag{4.3.26}$$

Dividing both sides by $(C_\infty a)$ gives

$$-f\phi' = \frac{D_B}{v} \phi'' + \frac{D_T (T_f - T_\infty)}{v C_\infty T_\infty} \theta'' - \frac{\gamma}{a} \phi \tag{4.3.27}$$

Multiplying both sides by $(\frac{v}{D_B})$ gives

$$-\frac{v}{D_B}f\phi' = \phi'' + \frac{D_T(T_f - T_\infty)}{D_B C_\infty T_\infty}\theta'' - \frac{v}{D_B}\frac{\gamma}{a}\phi \quad (4.3.28)$$

Re-arranging

$$\phi'' + \frac{D_T(T_f - T_\infty)}{D_B T_\infty C_\infty}\theta'' + \frac{v}{D_B}\left[f\phi' - \frac{\gamma}{a}\phi\right] = 0 \quad (4.3.29)$$

Substituting with identities and re-arranging

$$\phi'' + \frac{Nt}{Nb}\theta'' + PrLe(f\phi' - R\phi) = 0 \quad (4.3.30)$$

To transform the boundary conditions

When $\eta \rightarrow 0$

$$\begin{aligned} u = U_w + \frac{\mu}{L}\left(1 + \frac{1}{\beta}\right)\frac{\partial y}{\partial \eta} &\Rightarrow U_\infty f'(0) = bx + \frac{\mu}{L}\left(1 + \frac{1}{\beta}\right)\left(\sqrt{\frac{a}{v}}U_\infty\right)f''(0) \\ &\Rightarrow f'(0) = \frac{b}{a} + \frac{\mu}{L}\sqrt{\frac{a}{v}}\left(1 + \frac{1}{\beta}\right)f''(0). \quad \therefore, f'(0) = \lambda + \delta\left(1 + \frac{1}{\beta}\right)f''(0), \end{aligned}$$

$$v = V_0 \Rightarrow -\sqrt{av}f(0) = V_0 \Rightarrow f(0) = -\frac{V_0}{\sqrt{av}}. \quad \therefore, f(0) = S,$$

$$\begin{aligned} -k_f\frac{\partial T}{\partial y} = h_f(T_f - T) &\Rightarrow -k_f(T_f - T_\infty)\sqrt{\frac{a}{v}}\theta'(0) = h_f[T_f - (T_\infty(T_f - T_\infty)\theta(0))] \\ &\Rightarrow -k_f(T_f - T_\infty)\sqrt{\frac{a}{v}}\theta'(0) = h_f(T_f - T_\infty)(1 - \theta(0)) \Rightarrow \theta'(0) = \frac{h_f}{k_f}\sqrt{\frac{a}{v}}(\theta(0) - 1) \\ \therefore, \theta'(0) &= Bi[\theta(0) - 1]. \end{aligned}$$

and

$$\begin{aligned} D_B\frac{\partial C}{\partial y} = -\frac{D_T}{T_\infty}\frac{\partial T}{\partial y} &\Rightarrow D_B\sqrt{\frac{a}{v}}C_\infty\phi'(0) = -\frac{D_T}{T_\infty}(T_f - T_\infty)\sqrt{\frac{a}{v}}\theta'(0) \\ &\Rightarrow \phi'(0) = \frac{D_T(T_f - T_\infty)}{D_B T_\infty C_\infty}\theta'(0). \quad \therefore, \phi'(0) = \frac{Nt}{Nb}\theta'(0). \end{aligned}$$

When $\eta \rightarrow \infty$

$$u \rightarrow U_\infty \Rightarrow U_\infty f'(\infty) \rightarrow U_\infty \Rightarrow f'(\infty) \rightarrow 1. \quad \therefore, f'(\infty) = 1.$$

$$T \rightarrow T_\infty \Rightarrow (T_f - T_\infty)\theta(\infty) + T_\infty \rightarrow T_\infty \Rightarrow \theta(\infty) \rightarrow 0. \quad \therefore, \theta(\infty) = 0.$$

and

$$C \rightarrow C_\infty \Rightarrow C_\infty[\phi(0) + 1] \rightarrow C_\infty \Rightarrow \phi(0) + 1 \rightarrow 1, \quad \therefore, \phi(\infty) = 0.$$

Collecting, the transformed nonlinear ordinary differential equations gives

$$\left(1 + \frac{1}{\beta}\right)\left[f''' - \frac{1}{Da}(f' - 1)\right] + ff'' - (f')^2 - Fr(f' - 1)^2 + 1 = 0, \quad (4.3.31)$$

$$\begin{aligned} \frac{\theta''}{Pr} + f\theta' - 2f'\theta + (Nb\phi' + Nt\theta')\theta' + Ec\left(1 + \frac{1}{\beta}\right)\left[(f'')^2 + \frac{1}{Da}(f' - 1)^2\right] \\ + EcFr(f' - 1)^3 = 0, \end{aligned} \quad (4.3.32)$$

$$\phi'' + \frac{Nt}{Nb}\theta'' + PrLe(f\phi' - R\phi) = 0. \quad (4.3.33)$$

With the dimensionless form of the boundary conditions

$$\eta = 0 : \left. \begin{aligned} f(0) &= S, & f'(0) &= \lambda + \delta \left(1 + \frac{1}{\beta}\right) f''(0), \\ \theta'(0) &= Bi[\theta(0) - 1], & \phi'(0) &= \frac{N_t}{N_b} \theta'(0), \end{aligned} \right\} \quad (4.3.34)$$

$$\eta \rightarrow \infty : \quad f'(\infty) = 1, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0. \quad (4.3.35)$$

where η is the independent similarity variable, Fr is the Forchheimer (second order porous resistance) parameter which is the dimensionless (Joseph et al., 1982), λ is the shrinking/stretching (velocity ratio) parameter such that for stretching $\lambda > 0$ and for shrinking $\lambda < 0$, Ec is the Eckert number (viscous dissipation parameter), Le is the Lewis number, Da is the Darcy number (porous media parameter), N_b is the Brownian motion parameter, Pr is the Prandtl number, Bi is the Biot number (convective heating parameter), R is the chemical reaction parameter, S is the suction/injection parameter (such that for suction $S > 0$ or $-V_0$, and for injection $S < 0$ or $+V_0$), δ is the velocity slip (slipperiness) parameter, and N_t is the thermophoretic parameter.

$$\begin{aligned} Da &= \frac{ak_1}{v}, & Fr &= \frac{xF}{\rho\sqrt{k_1}}, & \lambda &= \frac{b}{a}, & Pr &= \frac{v}{\alpha}, \\ Ec &= \frac{U_\infty^2}{c_p(T_f - T_\infty)}, & N_t &= \frac{\bar{\tau}D_T(T_f - T_\infty)}{vT_\infty}, & N_b &= \frac{\bar{\tau}D_B C_\infty}{v}, \\ Le &= \frac{\alpha}{D_B}, & R &= \frac{\gamma}{a}, & S &= -\frac{V_0}{\sqrt{av}}, & \delta &= \frac{\mu}{L}\sqrt{\frac{a}{v}}, & Bi &= \frac{h_f}{k_f}\sqrt{\frac{v}{a}}. \end{aligned} \quad (4.3.36)$$

4.4 Physical Quantities of Engineering Interest

The dimensionless coefficient of skin friction which determines the friction that occurs at the boundary of the fluid flow and the surface over which it flows, the local Nusselt number which measures the heat flux through the surface, and the local Sherwood number measures the mass flux through the surface are interesting engineering physical quantities of this problem. For the Casson nanofluid flow, τ_w the wall shear stress, q_w the wall heat flux (due to the temperature gradient), and q_s the wall mass flux (due to the Brownian motion force) are given as

$$\begin{aligned} \tau_w &= \mu \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right) \Big|_{\eta=0} = \mu \left(1 + \frac{1}{\beta}\right) \left[U_\infty \sqrt{\frac{a}{v}} f''(0)\right], \\ q_w &= -k_f \left(\frac{\partial T}{\partial y}\right) \Big|_{\eta=0} = -k_f (T_f - T_\infty) \sqrt{\frac{a}{v}} \theta'(0), \\ q_s &= -D_B \left(\frac{\partial C}{\partial y}\right) \Big|_{\eta=0} = -D_B \sqrt{\frac{a}{v}} C_\infty \phi'(0). \end{aligned} \quad (4.4.1)$$

Thus, the physical quantities the drag force on the surface (coefficient of the local Skin friction), the rate of heat transfer (temerature gradient or the local Nusselt number) and the rate of mass

transfer (local Sherwood number) are defined as:

$$C_f \equiv \frac{\tau_w}{\rho U_\infty^2}, \quad Nu_x \equiv \frac{xq_w}{k_f(T_f - T_\infty)}, \quad Sh_x \equiv \frac{xq_s}{D_B C_\infty}. \quad (4.4.2)$$

To simplify this quantities

$$C_f = \frac{\tau_w}{\rho U_\infty^2} = \frac{\mu}{\rho U_\infty} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{a}{v}} f''(0) = \left(1 + \frac{1}{\beta}\right) \sqrt{\left(\frac{v^2 a}{v U_\infty^2}\right)} f''(0) \quad (4.4.3)$$

$$= \left(1 + \frac{1}{\beta}\right) \sqrt{\left(\frac{va}{U_\infty ax}\right)} f''(0) = \left(1 + \frac{1}{\beta}\right) \sqrt{\left(\frac{v}{U_\infty x}\right)} f''(0) = \left(1 + \frac{1}{\beta}\right) \frac{1}{\sqrt{Re_x}} f''(0),$$

$$Nu_x = \frac{xq_w}{k_f(T_f - T_\infty)} = \frac{x}{k_f(T_f - T_\infty)} \left[-k_f(T_f - T_\infty) \sqrt{\frac{a}{v}} \theta'(0) \right] = -\sqrt{\frac{ax^2}{v}} \theta'(0) \quad (4.4.4)$$

$$= -\sqrt{\frac{U_\infty x}{v}} \theta'(0) = -\sqrt{Re_x} \theta'(0),$$

and

$$Sh_x = \frac{xq_s}{D_B C_\infty} = \frac{x}{D_B C_\infty} \left[-D_B \sqrt{\frac{a}{v}} C_\infty \phi'(0) \right] = -x \sqrt{\frac{a}{v}} \phi'(0) = -\sqrt{\frac{ax^2}{v}} \phi'(0) \quad (4.4.5)$$

$$= -\sqrt{\frac{U_\infty x}{v}} \phi'(0) = -\sqrt{Re_x} \phi'(0).$$

Collecting these results

$$Re_x^{\frac{1}{2}} C_f = \left(1 + \frac{1}{\beta}\right) f''(0), \quad Re_x^{-\frac{1}{2}} Nu_x = -\theta'(0), \quad Re_x^{-\frac{1}{2}} Sh_x = -\phi'(0). \quad (4.4.6)$$

where Re_x denotes the local Reynolds number.

4.5 Stability Analysis

Since the problem under investigation involves shrinking/stretching surfaces, according to literature it could have dual/multiple solutions under various fluid flow conditions. Moreover, it is important to test the physical reliability of the solutions obtained. To do so, a linear temporal stability test that uses a time derivative is applied. To start, Eqs. (4.2.10)-(4.2.12) should be converted into time-dependent flow equation (unsteady form) (Merkin, 1986).

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + v \left(1 + \frac{1}{\beta}\right) \left[\frac{\partial^2 u}{\partial y^2} - \frac{1}{k_1} (u - U_\infty) \right] - \frac{F}{\rho \sqrt{k_1}} (u - U_\infty)^2 \quad (4.5.1)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[D_B \left(\frac{\partial C}{\partial y} \frac{\partial T}{\partial y} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial y} \right)^2 \right] + \left(\frac{v}{c_p} \right) \left(1 + \frac{1}{\beta} \right) \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{1}{k_1} \right) (u - U_\infty)^2 \right] + \frac{F}{\rho c_p \sqrt{k_1}} (u - U_\infty)^3 \quad (4.5.2)$$

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \left(\frac{\partial^2 C}{\partial y^2} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial^2 T}{\partial y^2} \right) - \gamma (C - C_\infty) \quad (4.5.3)$$

where t denotes the time.

A new similarity transformation is introduced:

$$\begin{aligned}\eta &= \sqrt{\frac{a}{v}}y, \quad u = axf'(\eta, \tau), \quad v = -\sqrt{av}f(\eta, \tau), \quad \Psi = x\sqrt{av}f(\eta, \tau), \\ \theta(\eta, \tau) &= \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi(\eta, \tau) = \frac{C - C_\infty}{C_\infty}, \quad \tau = at.\end{aligned}\tag{4.5.4}$$

where τ is the non-dimensional time variable.

In addition to what results from Eqs. (4.3.2)-(4.3.13),

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{\partial}{\partial \tau} \left(U_\infty \frac{\partial f}{\partial \eta}(\eta, \tau) \right) = U_\infty \frac{\partial^2 f}{\partial \tau \partial \eta}(\eta, \tau) \\ \frac{\partial T}{\partial \tau} &= \frac{\partial}{\partial \tau} \{ [T_\infty + (T_f - T_\infty) \theta(\eta, \tau)] \} = (T_f - T_\infty) \frac{\partial \theta}{\partial \tau}(\eta, \tau) \\ \frac{\partial C}{\partial \tau} &= \frac{\partial}{\partial \tau} \{ C_\infty [1 + \phi(\eta, \tau)] \} = C_\infty \frac{\partial \phi}{\partial \tau}(\eta, \tau)\end{aligned}\tag{4.5.5}$$

By using Eq. (4.5.4) and its derivations Eqs. (4.3.2)-(4.3.13) and Eq.(4.5.5), Eqs. (4.5.1) - (4.5.3) become

$$\begin{aligned}(1 + \frac{1}{\beta}) \left[\frac{\partial^3 f}{\partial \eta^3} - \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right) \right] + f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial}{\partial \tau} \left(\frac{\partial f}{\partial \eta} \right) - Fr \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 \\ - \left(\frac{\partial f}{\partial \eta} \right)^2 + 1 = 0,\end{aligned}\tag{4.5.6}$$

$$\begin{aligned}\frac{1}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} + \left(f + N_b \frac{\partial \phi}{\partial \eta} + N_t \frac{\partial \theta}{\partial \eta} \right) \frac{\partial \theta}{\partial \eta} - \frac{\partial \theta}{\partial \tau} - 2\theta \frac{\partial f}{\partial \eta} \\ + Ec(1 + \frac{1}{\beta}) \left[\left(\frac{\partial^2 f}{\partial \eta^2} \right)^2 + \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 \right] + EcFr \left(\frac{\partial f}{\partial \eta} - 1 \right)^3 = 0,\end{aligned}\tag{4.5.7}$$

$$\frac{\partial^2 \phi}{\partial \eta^2} + \frac{N_t}{N_b} \frac{\partial^2 \theta}{\partial \eta^2} + PrLe \left(f \frac{\partial \phi}{\partial \eta} - \frac{\partial \phi}{\partial \tau} - R\phi \right) = 0.\tag{4.5.8}$$

along with new boundary conditions:

$$\begin{aligned}f(0, \tau) &= S, \quad \frac{\partial f}{\partial \eta}(0, \tau) = \lambda + \delta(1 + \frac{1}{\beta}) \frac{\partial^2 f}{\partial \eta^2}(0, \tau), \\ \frac{\partial \theta}{\partial \eta}(0, \tau) &= Bi[\theta(0, \tau) - 1], \quad \frac{\partial \phi}{\partial \eta}(0, \tau) = \frac{N_t}{N_b} \frac{\partial \theta}{\partial \eta}(0, \tau), \\ \frac{\partial f}{\partial \eta}(\infty, \tau) &= 1, \quad \theta(\infty, \tau) = 0, \quad \phi(\infty, \tau) = 0.\end{aligned}\tag{4.5.9}$$

To test the stability of the steady flow solution $f(\eta) = f_0(\eta)$, $\theta(\eta) = \theta_0(\eta)$, and $\phi(\eta) = \phi_0(\eta)$ satisfying the set of boundary-value problem Eq. (4.5.9), can be written as

$$\begin{aligned}f(\eta, \tau) &= f_0(\eta) + e^{-\varepsilon\tau}F(\eta, \tau), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\varepsilon\tau}G(\eta, \tau), \\ \phi(\eta, \tau) &= \phi_0(\eta) + e^{-\varepsilon\tau}H(\eta, \tau).\end{aligned}\tag{4.5.10}$$

where $F(\eta, \tau)$, $G(\eta, \tau)$ and $H(\eta, \tau)$ are very small relative values of $f_0(\eta)$, $\theta_0(\eta)$, and $\phi_0(\eta)$ respectively; and ε is the unknown eigenvalues. Now the higher partial derivatives of Eq. (4.5.2),

$$\begin{aligned}
f(\eta, \tau) &= f_0(\eta) + e^{-\varepsilon\tau}F(\eta, \tau) \equiv f = f_0 + e^{-\varepsilon\tau}F \\
\Rightarrow \frac{\partial f}{\partial \eta}(\eta, \tau) &= \frac{\partial f_0}{\partial \eta}(\eta) + e^{-\varepsilon\tau} \frac{\partial F}{\partial \eta}(\eta, \tau) \equiv f' = f'_0 + e^{-\varepsilon\tau}F' \\
\Rightarrow \frac{\partial^2 f}{\partial \eta^2}(\eta, \tau) &= \frac{\partial^2 f_0}{\partial \eta^2}(\eta) + e^{-\varepsilon\tau} \frac{\partial^2 F}{\partial \eta^2}(\eta, \tau) \equiv f'' = f''_0 + e^{-\varepsilon\tau}F'' \\
\Rightarrow \frac{\partial^3 f}{\partial \eta^3}(\eta, \tau) &= \frac{\partial^3 f_0}{\partial \eta^3}(\eta) + e^{-\varepsilon\tau} \frac{\partial^3 F}{\partial \eta^3}(\eta, \tau) \equiv f''' = f'''_0 + e^{-\varepsilon\tau}F''' \\
\Rightarrow \frac{\partial}{\partial \tau} \left(\frac{\partial f}{\partial \eta}(\eta, \tau) \right) &= -\varepsilon e^{-\varepsilon\tau} \frac{\partial F}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial}{\partial \tau} \left(\frac{\partial F}{\partial \eta}(\eta, \tau) \right) \\
&\equiv \frac{\partial f'}{\partial \tau} = e^{-\varepsilon\tau} \left(-\varepsilon F' + \frac{\partial F'}{\partial \tau} \right)
\end{aligned} \tag{4.5.11}$$

Inserting the results from Eq. (4.5.11) into Eq. (4.5.1) gives

$$\begin{aligned}
(1 + \frac{1}{\beta}) \left[(f'''_0 + e^{-\varepsilon\tau}F''') - \frac{1}{Da} (f'_0 + e^{-\varepsilon\tau}F' - 1) \right] &+ (f_0 + e^{-\varepsilon\tau}F)(f''_0 + e^{-\varepsilon\tau}F'') \\
- Fr(f'_0 + e^{-\varepsilon\tau}F' - 1)^2 - (f'_0 + e^{-\varepsilon\tau}F')^2 &+ e^{\varepsilon\tau} \left(\varepsilon F' - \frac{\partial F'}{\partial \tau} \right) + 1 = 0
\end{aligned} \tag{4.5.12}$$

Re-arranging

$$\begin{aligned}
&\left\{ (1 + \frac{1}{\beta}) \left[f'''_0 - \frac{1}{Da}(f'_0 - 1) \right] + f_0 f''_0 - (f'_0)^2 - Fr(f'_0 - 1)^2 + 1 \right\} \\
&+ \left\{ (1 + \frac{1}{\beta}) \left[F''' - \frac{1}{Da}F' \right] + f_0 F'' - [2Fr(f'_0 - 1) + 2f'_0 - \varepsilon] F' + f''_0 F \right\} e^{-\varepsilon\tau} \\
&- \frac{\partial F'}{\partial \tau} e^{-\varepsilon\tau} + [FF'' - Fr(F')^2 - (F')^2] e^{-2\varepsilon\tau} = 0
\end{aligned} \tag{4.5.13}$$

Since $f \rightarrow f_0$ the first row of Eq. (4.5.13) goes to zero and then dividing both sides of the rest equation with $e^{-\varepsilon\tau}$ reduces it to

$$\begin{aligned}
&\left\{ (1 + \frac{1}{\beta}) \left[F''' - \frac{1}{Da}F' \right] + f_0 F'' - [2Fr(f'_0 - 1) + 2f'_0 - \varepsilon] F' + f''_0 F \right\} \\
&- \frac{\partial F'}{\partial \tau} + [FF'' - Fr - (F')^2] e^{-\varepsilon\tau} = 0
\end{aligned} \tag{4.5.14}$$

For large values of τ , $e^{-\varepsilon\tau} \rightarrow 0$ and hence $[FF'' - Fr - (F')^2] e^{-\varepsilon\tau} = 0$ so that Eq. (4.5.14) reduced to

$$(1 + \frac{1}{\beta}) \left[F''' - \frac{1}{Da}F' \right] + f_0 F'' - [2Fr(f'_0 - 1) + 2f'_0 - \varepsilon] F' + f''_0 F - \frac{\partial F'}{\partial \tau} = 0 \tag{4.5.15}$$

Again,

$$\begin{aligned}
\theta(\eta, \tau) &= \theta_0(\eta) + e^{-\varepsilon\tau}G(\eta, \tau) \equiv \theta = \theta_0 + e^{-\varepsilon\tau}G \\
\Rightarrow \frac{\partial \theta}{\partial \eta}(\eta, \tau) &= \frac{\partial \theta_0}{\partial \eta}(\eta) + e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(\eta, \tau) \equiv \theta' = \theta'_0 + e^{-\varepsilon\tau}G' \\
\Rightarrow \frac{\partial^2 \theta}{\partial \eta^2}(\eta, \tau) &= \frac{\partial^2 \theta_0}{\partial \eta^2}(\eta) + e^{-\varepsilon\tau} \frac{\partial^2 G}{\partial \eta^2}(\eta, \tau) \equiv \theta'' = \theta''_0 + e^{-\varepsilon\tau}G'' \\
\Rightarrow \frac{\partial \theta}{\partial \tau}(\eta, \tau) &= -\varepsilon e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial}{\partial \tau} \left(\frac{\partial G}{\partial \eta}(\eta, \tau) \right) \equiv \frac{\partial \theta}{\partial \tau} = e^{-\varepsilon\tau} \left(-\varepsilon G + \frac{\partial G}{\partial \tau} \right)
\end{aligned} \tag{4.5.16}$$

$$\begin{aligned}
\phi(\eta, \tau) &= \phi_0(\eta) + e^{-\varepsilon\tau}H(\eta, \tau) \equiv \phi = \phi_0 + e^{-\varepsilon\tau}H \\
\Rightarrow \frac{\partial\phi}{\partial\eta}(\eta, \tau) &= \frac{\partial\phi_0}{\partial\eta}(\eta) + e^{-\varepsilon\tau}\frac{\partial H}{\partial\eta}(\eta, \tau) \equiv \phi' = \phi'_0 + e^{-\varepsilon\tau}H' \\
\Rightarrow \frac{\partial^2\phi}{\partial\eta^2}(\eta, \tau) &= \frac{\partial^2\phi_0}{\partial\eta^2}(\eta) + e^{-\varepsilon\tau}\frac{\partial^2 H}{\partial\eta^2}(\eta, \tau) \equiv \phi'' = \phi''_0 + e^{-\varepsilon\tau}H'' \\
\Rightarrow \frac{\partial\phi}{\partial\tau}(\eta, \tau) &= -\varepsilon e^{-\varepsilon\tau}\frac{\partial H}{\partial\eta} + e^{-\varepsilon\tau}\frac{\partial}{\partial\tau}\left(\frac{\partial H}{\partial\eta}(\eta, \tau)\right) \equiv \frac{\partial\phi}{\partial\tau} = e^{-\varepsilon\tau}\left(-\varepsilon H + \frac{\partial H}{\partial\tau}\right)
\end{aligned} \tag{4.5.17}$$

Inserting the results from Eqs. (4.5.11), (4.5.16), and (4.5.17) into Eq. (4.5.2)

$$\begin{aligned}
&\frac{1}{Pr}(\theta''_0 + e^{-\varepsilon\tau}G'') + [f_0 + e^{-\varepsilon\tau}F + N_b(\phi'_0 + e^{-\varepsilon\tau}H') + N_t(\theta'_0 + e^{-\varepsilon\tau}G')] \cdot \\
&(\theta'_0 + e^{-\varepsilon\tau}G') - \left(-\varepsilon e^{-\varepsilon\tau}G' + \varepsilon\frac{\partial G'}{\partial\tau}\right) - \theta(f'_0 + e^{-\varepsilon\tau}F') \\
&+ Ec\left(1 + \frac{1}{\beta}\right) \left[(f''_0 + e^{-\varepsilon\tau}F'')^2 + \frac{1}{Da}(f''_0 + e^{-\varepsilon\tau}F' - 1)^2\right] \\
&+ EcFr(f'_0 + e^{-\varepsilon\tau}F' - 1)^3 + e^{-\varepsilon\tau}\left(\varepsilon G - \frac{\partial G}{\partial\tau}\right) = 0
\end{aligned} \tag{4.5.18}$$

Again re-arranging

$$\begin{aligned}
&\left\{\frac{\theta''_0}{Pr} + (f_0 + N_t\theta'_0 + N_b\phi'_0)\theta'_0 - 2f'_0\theta_0 + Ec\left[\left(1 + \frac{1}{\beta}\right) \cdot \right. \right. \\
&\left. \left. \left[(f''_0)^2 + \frac{1}{Da}(f'_0 - 1)^2\right] + Fr(f'_0 - 1)^3\right]\right\} + \\
&\left\{\frac{1}{Pr}G'' + (f_0 + Nb\phi_0 + 2Nt\theta'_0)G' - \frac{\partial G}{\partial\tau} + (\varepsilon - f'_0)G - \frac{\partial G}{\partial\tau} + Ec\left(1 + \frac{1}{\beta}\right) \cdot \right. \\
&\left. \left[2f''_0F'' + \frac{2}{Da}Ec(f'_0 - 1)\right]F' + 3EcFr(f'_0 - 1)^2F' + (F + NbH')\theta'_0\right\}e^{-\varepsilon\tau} \\
&\left\{Nt(G')^2 - G + Ec\left(1 + \frac{1}{\beta}\right)(F'')^2\frac{1}{Da}(F')^2 + 3(f'_0 - 1)(F')^2 + \right. \\
&\left. e^{-\varepsilon\tau}(F')^3\right\}e^{-2\varepsilon\tau} = 0
\end{aligned} \tag{4.5.19}$$

From Eq. (4.5.19), since $f \rightarrow f_0$, $\theta \rightarrow \theta_0$ and $\phi \rightarrow \phi_0$ the expression in the first braces goes to zero.

Dividing both sides of the rest by $e^{-\varepsilon\tau}$ reduces the equation to

$$\begin{aligned}
&\left\{\frac{1}{Pr}G'' + (f_0 + Nb\phi_0 + 2Nt\theta'_0)G' - \frac{\partial G}{\partial\tau} + (\varepsilon - f'_0)G - \frac{\partial G}{\partial\tau} + Ec\left(1 + \frac{1}{\beta}\right) \cdot \right. \\
&\left. \left[2f''_0F'' + \frac{2}{Da}Ec(f'_0 - 1)\right]F' + 3EcFr(f'_0 - 1)^2F' + (F + NbH')\theta'_0\right\} \\
&\left\{Nt(G')^2 - G + Ec\left(1 + \frac{1}{\beta}\right)(F'')^2\frac{1}{Da}(F')^2 + 3(f'_0 - 1)(F')^2 + \right. \\
&\left. e^{-\varepsilon\tau}(F')^3\right\}e^{-\varepsilon\tau} = 0
\end{aligned} \tag{4.5.20}$$

and for large values of τ , $e^{-\varepsilon\tau} \rightarrow 0$ and hence Eq. (4.5.20) becomes

$$\begin{aligned}
&\frac{1}{Pr}G'' + (f_0 + Nb\phi_0 + 2Nt\theta'_0)G' + (\varepsilon - f'_0)G - \frac{\partial G}{\partial\tau} + Ec\left(1 + \frac{1}{\beta}\right) \cdot \\
&\left[2f''_0F'' + \frac{2}{Da}Ec(f'_0 - 1)\right]F' + 3EcFr(f'_0 - 1)^2F' + (F + NbH')\theta'_0 - \frac{\partial G}{\partial\tau} = 0
\end{aligned} \tag{4.5.21}$$

Lastly, plugging the results from Eqs. (4.5.11), (4.5.16), and (4.5.17) into Eq. (4.5.3) gives

$$\begin{aligned} & (\phi_0'' + e^{-\varepsilon\tau}H'') + PrLe \left[(f_0 + e^{-\varepsilon\tau}F)(\phi_0' + e^{-\varepsilon\tau}H') + \frac{N_t}{N_b}(\theta_0'' + e^{-\varepsilon\tau}G'') \right] \\ & - PrLe(\phi_0' + e^{-\varepsilon\tau}H') + PrLe e^{\varepsilon\tau} \left(\varepsilon H - \frac{\partial H}{\partial \tau} \right) - R(\phi_0 + e^{-\varepsilon\tau}H)\phi = 0 \end{aligned} \quad (4.5.22)$$

Re-arranging

$$\begin{aligned} & \left[\phi'' + \frac{N_t}{N_b} \theta'' + PrLe(f_0\phi_0' - R\phi) \right] - PrLe \frac{\partial H}{\partial \tau} + \\ & \left[H'' + \frac{N_t}{N_b} G'' + PrLe(f_0H' + (\varepsilon - R)H + \phi_0'F) \right] e^{-\varepsilon\tau} + PrLe FH' e^{-2\varepsilon\tau} = 0 \end{aligned} \quad (4.5.23)$$

Following the above assumptions for this equation, and finally reduced to

$$H'' + \frac{N_t}{N_b} G'' + PrLe(f_0H' + (\varepsilon - R)H + \phi_0'F) - PrLe \frac{\partial H}{\partial \tau} = 0 \quad (4.5.24)$$

To transform the new boundary conditions:

When $\eta \rightarrow 0$

$$f(0, \tau) = S \Rightarrow f(0) + e^{-\varepsilon\tau}F(0, \tau) = S \Rightarrow S + e^{-\varepsilon\tau}F(0, \tau) = S. \quad \therefore, F(0) = 0.$$

$$\begin{aligned} \frac{\partial f}{\partial \eta}(0, \tau) &= \lambda + \delta(1 + \frac{1}{\beta}) \frac{\partial^2 f}{\partial \eta^2}(0, \tau) \\ \Rightarrow \frac{\partial f_0(0)}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial F(0, \tau)}{\partial \eta} &= \lambda + \delta(1 + \frac{1}{\beta}) \left(\frac{\partial^2 f_0}{\partial \eta^2} + e^{-\varepsilon\tau} \frac{\partial^2 F_0(0, \tau)}{\partial \eta^2} \right) \\ \Rightarrow \frac{\partial f_0(0)}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial F(0, \tau)}{\partial \eta} &= \left[\lambda + \delta(1 + \frac{1}{\beta}) \frac{\partial^2 f_0}{\partial \eta^2} \right] + \delta(1 + \frac{1}{\beta}) e^{-\varepsilon\tau} \frac{\partial^2 F_0(0, \tau)}{\partial \eta^2} \\ \Rightarrow e^{-\varepsilon\tau} \frac{\partial F(0, \tau)}{\partial \eta} &= \delta(1 + \frac{1}{\beta}) e^{-\varepsilon\tau} \frac{\partial^2 F_0(0, \tau)}{\partial \eta^2} \Rightarrow \frac{\partial F(0, \tau)}{\partial \eta} = \delta(1 + \frac{1}{\beta}) \frac{\partial^2 F_0(0, \tau)}{\partial \eta^2}. \end{aligned}$$

$$\therefore, F'(0) = \delta(1 + \frac{1}{\beta}) F''(0)$$

$$\frac{\partial \theta}{\partial \eta}(0, \tau) = Bi[\theta(0, \tau) - 1] \Rightarrow \frac{\partial \theta_0}{\partial \eta}(0) + e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau) = Bi[\theta_0(0) + e^{-\varepsilon\tau}G(0, \tau) - 1]$$

$$\Rightarrow \frac{\partial \theta_0}{\partial \eta}(0) + e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau) = \cancel{Bi[\theta_0(0) - 1]} + e^{-\varepsilon\tau} BiG(0, \tau)$$

$$\Rightarrow e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau) = e^{-\varepsilon\tau} BiG(0, \tau) \quad \therefore, G'(0) = BiG(0)$$

$$\frac{\partial \phi}{\partial \eta}(0, \tau) = \frac{N_t}{N_b} \frac{\partial \theta}{\partial \eta}(0, \tau)$$

$$\Rightarrow \frac{\partial \phi_0}{\partial \eta}(0) + e^{-\varepsilon\tau} \frac{\partial H}{\partial \eta}(0, \tau) = \frac{N_t}{N_b} \left(\frac{\partial \theta_0}{\partial \eta}(0) + e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau) \right)$$

$$\Rightarrow \frac{\partial \phi_0}{\partial \eta}(0) + e^{-\varepsilon\tau} \frac{\partial H}{\partial \eta}(0, \tau) = \frac{N_t}{N_b} \frac{\partial \theta_0}{\partial \eta}(0) + \frac{N_t}{N_b} e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau)$$

$$\Rightarrow e^{-\varepsilon\tau} \frac{\partial H}{\partial \eta}(0, \tau) = \frac{N_t}{N_b} e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(0, \tau) \quad \therefore, H'(0) = \frac{N_t}{N_b} G(0)$$

When $\eta \rightarrow \infty$

$$\frac{\partial f}{\partial \eta}(\infty, \tau) \rightarrow 1 \Rightarrow \frac{\partial f_0(\infty)}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial F(\infty, \tau)}{\partial \eta} \rightarrow 1 \quad \therefore, F'(\infty) = 0$$

$$\theta(\infty, \tau) \rightarrow 0 \Rightarrow \cancel{\theta_0(\infty)} + e^{-\varepsilon\tau} G(\infty, \tau) \rightarrow \emptyset \quad \therefore, G(\infty) = 0$$

$$\phi(\infty, \tau) \rightarrow 0 \Rightarrow \cancel{\phi_0(\infty)} + e^{-\varepsilon\tau} H(\infty, \tau) \rightarrow \emptyset \quad \therefore, H(\infty) = 0$$

Collecting the results from Eqs. (4.5.15), (4.5.22), and (4.5.24) the following equations are obtained:

$$(1 + \frac{1}{\beta}) \left[F''' - \frac{1}{Da} F' \right] + f_0 F'' - [2Fr(f'_0 - 1) + 2f'_0 - \varepsilon] F' + f_0'' F - \frac{\partial F'}{\partial \tau} = 0 \quad (4.5.25)$$

$$\begin{aligned} \frac{1}{Pr} G'' + (f_0 + Nb\phi_0 + 2Nt\theta'_0) G' + (\varepsilon - f'_0) G - \frac{\partial G}{\partial \tau} + Ec(1 + \frac{1}{\beta}) \cdot \\ \left[2f_0'' F'' + \frac{2}{Da} Ec(f'_0 - 1) \right] F' + 3EcFr(f'_0 - 1)^2 F' + (F + NbH') \theta'_0 - \frac{\partial G}{\partial \tau} = 0 \end{aligned} \quad (4.5.26)$$

$$H'' + \frac{Nt}{Nb} G'' + PrLe [f_0 H' + (\varepsilon - R)H + \phi'_0 F] - PrLe \frac{\partial H}{\partial \tau} = 0 \quad (4.5.27)$$

with boundary conditions

$$\begin{aligned} F(0) = 0, \quad F'(0) = \delta(1 + \frac{1}{\beta}) F''(0), \quad G'(0) = BiG(0), \\ H'(0) = \frac{Nt}{Nb} G(0), F'(\infty) = 0, \quad G(\infty) = 0, \quad H(\infty) = 0. \end{aligned} \quad (4.5.28)$$

Based on (Weidman et al., 2006), the initial decay/drop or growth/rise of the solution of Eqs. (4.5.10) can be determined by setting $\tau = 0$ so that $H = H_0(\eta)$, $G = G_0(\eta)$, and $F = F_0(\eta)$ in Eqs. (4.5.25)-(4.5.27). On solving the eigenvalue problem in Eqs. (4.5.25)-(4.5.27), an infinite set of eigenvalues, say, $\varepsilon < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots$ are obtained. From these values, taking the least eigenvalue, ε . The least eigenvalue ε is negative implying that disturbance within the flow grows and hence is unstable and physically not attainable. If the least eigenvalue ε is positive, it shows that the solution is stable and physically realizable. The linearized eigenvalue equations are given by:

$$(1 + \frac{1}{\beta}) \left[F_0''' - \frac{1}{Da} F_0' \right] + f_0 F_0'' - [2Fr(f'_0 - 1) + 2f'_0 - \varepsilon] F_0' + f_0'' F_0 = 0 \quad (4.5.29)$$

$$\begin{aligned} \frac{1}{Pr} G_0'' + (f_0 + Nb\phi'_0 + 2Nt\theta'_0) G_0' + (\varepsilon - f'_0) G_0 + Ec(1 + \frac{1}{\beta}) \left[2f_0'' F_0'' + \frac{2}{Da} (f'_0 - 1) F_0' \right] \\ + 3EcFr(f'_0 - 1)^2 F_0' + (F_0 + NbH_0) \theta'_0 = 0 \end{aligned} \quad (4.5.30)$$

$$H_0'' + \frac{Nt}{Nb} G_0'' + PrLe (f_0 H_0' + (\varepsilon - R)H_0 + \phi'_0 F_0) = 0 \quad (4.5.31)$$

subject to boundary condition:

$$F_0(0) = 0, \quad F_0' = \delta(1 + \frac{1}{\beta}) F_0'', \quad G_0'(0) = BiG_0(0), \quad H_0' = \frac{Nt}{Nb} G_0'(0), \quad (4.5.32)$$

$$F_0''(0) = 1, \quad F_0'(\infty) \rightarrow 0, \quad G(\infty) \rightarrow 0, \quad H(\infty) \rightarrow 0.$$

Since according to (Harris et al., 2009), the stability of the Eqs. (4.5.29)-(4.5.31) of steady-

state flow solution is regulated by the least eigenvalue ε , the range/interval of possible eigenvalues could be obtained by relaxing a suitable boundary condition: $H_0(\eta)$, $G_0(\eta)$, and $F'_0(\eta)$. For this reason, $F'_0(\infty) \rightarrow 0$ can be relaxed into $F''_0(0) = 1$ for this problem.

4.6 Procedures for Numerical Method

To analyze the coupled Eqs. (4.3.31)-(4.3.33) subjected to the boundary conditions in Eq. (4.3.34) for the flow model, the technique called fourth-fifth-order Runge-Kutta-Fehlberg for various numerical values of the governing parameters is applied. This method is used by many researchers in solving convective boundary layer flow problems because it is more accurate, efficient, and adaptive with an error tolerance of order 5. To use this numerical solver, it is begun with converting the coupled boundary value problem to a system of first-order ODEs by defining new variables, let

$$f = f_1, \quad f' = f_2, \quad f'' = f_3, \quad \theta = f_4, \quad \theta' = f_5, \quad \phi = f_6, \quad \phi' = f_7 \quad (4.6.1)$$

Then, the following are obtained

$$\begin{aligned} f'_1 &= f_2 \\ f'_2 &= f_3 \\ f'_3 &= 1/Da(f_2 - 1) + (1 + 1/\beta)^{-1}[f_2^2 - f_1 f_3 + Fr(f_2 - 1)^2 - 1] \\ f'_4 &= f_5 \\ f'_5 &= -Pr\{(f_1 + N_b f_2 + N_t f_5)f_5 - 2f_2 f_4 \\ &\quad + Ec(1 + 1/\beta)[f_3^2 + 1/Da(f_2 - 1)^2] + EcFr(f_2 - 1)^3\} \\ f'_6 &= f_7 \\ f'_7 &= PrLe(Rf_6 - f_1 f_7) + PrN_t/N_b\{(f_1 + N_b f_2 + N_t f_5)f_5 - 2f_2 f_4 \\ &\quad + Ec(1 + 1/\beta)[f_3^2 + 1/Da(f_2 - 1)^2] + EcFr(f_2 - 1)^3\} \end{aligned} \quad (4.6.2)$$

whereas, ' represents the derivatives f, θ and ϕ with respect to η . Here, the initial conditions are given by

$$\begin{bmatrix} f_1(0) \\ f_2(0) \\ f_3(0) \\ f_4(0) \\ f_5(0) \\ f_6(0) \\ f_7(0) \end{bmatrix} = \begin{bmatrix} S \\ \lambda + \delta(1 + \frac{1}{\beta})q \\ q \\ r \\ Bi(r - 1) \\ t \\ Bi\frac{N_t}{N_b}(r - 1) \end{bmatrix} \quad (4.6.3)$$

The initial guess for the missing values q, r , and t (that is, $f''(0)$, $\theta'(0)$, and $\phi'(0)$) are selected systematically and computed by shooting method till the boundary conditions at $\eta \rightarrow \infty$ are attained, and then the IVP in Eqs. (4.6.2) and (4.6.3) are solved using fourth-fifth-order Runge-Kutta-Fehlberg method.

4.7 Results and Discussion

To approximate the numerical value of η_∞ , it is begun by taking estimate value and solve the problem consisting of Eqs. (4.3.31)-(4.3.33). This procedure is reiterated by taking larger value of η_∞ till the difference of consecutive values of the lacking boundary conditions for $\eta = 0$ differs only with required significant figures. Then, the last numerical value η_∞ is assumed to be the approximate value of η_∞ for the specific set of embedded parameters. For this study, $\eta_\infty = 11$ is considered, and hence $f'(\infty) = 1, \theta(\infty) = 0$, and $\phi(\infty) = 0$ become $f'(11) = 1, \theta(11) = 0$, and $\phi(11) = 0$. Therefore, the choice of $\eta_\infty = 11$ confirms that all the numerical solutions approached correctly the asymptotic values. For this purpose, MAPLE software packages are used for obtaining the computational solutions of the transformed problem. For each computation, the step size used is $\Delta\eta = 0.001$, with an average CPU time elapsed of 5.5 seconds.

The validity and convergence of the solutions are compared with previous results in Table 4.7.1. For our case, numerical results were validated with the literature (Wang, 2008; Nazar et al., 2011), and it is observed that there is an excellent agreement with the published works. Numerical solutions were calculated in the ranges $0.1 \leq \beta \leq 0.3$, $1 \leq Da \leq 16$, $0.05 \leq Fr \leq 1.5$, $0.05 \leq Pr \leq 8.2$, $0.1 \leq Ec \leq 0.3$, $0.1 \leq Nt \leq 0.3$, $0.1 \leq Nb \leq 0.3$, $0.05 \leq Le \leq 5$, $0.05 \leq R \leq 0.8$, $-0.2 \leq S \leq 0.5$, $-0.25 \leq \lambda \leq 1$, $0.1 \leq \delta \leq 0.2$ and $0.1 \leq Bi \leq 0.17$.

Table 4.7.1: Computations showing comparison of numerical values of drag force (skin friction) for various values of λ with $\beta, Da \rightarrow \infty, Fr = \delta = S = 0$.

λ	Wang (2008)	Nazar et al. (2011)	Present result
-0.25	1.40224	1.40224	1.402240810
0	1.232588	1.232588	1.232587650
0.2	1.05113	1.05113	1.051132999
0.5	0.71329	0.71330	0.713294956
1	0	0	0

The combined impacts of nanofluids parameters (thermophoretic and Brownian motion parameters) with Casson parameter, porous medium, viscous dissipation, slip conditions, convective heating on the non-dimensionalized concentration, temperature, velocity, Sherwood number, Nusselt

number, and skin friction over a permeable shrinking/stretching sheet were analyzed with numerical computation.

4.7.1 Velocity Profiles with Change of Parameters

Figure 4.7.1-4.7.2(a) illustrates the impacts of Forchheimer (porous medium inertia) parameter Fr , Casson parameter β , the velocity slip parameter δ , the porous media parameter (Darcy number) Da , the stretching/shrinking parameter λ and injection/suction parameter S on the velocity profile $f'(\eta)$ of the Casson nanofluids flow. These figures show that the flow velocity profile increases with the increasing value of similarity transformation η .

Figure 4.7.1(a) demonstrates the effects of Casson factor/parameter β and Forchheimer parameter Fr (which indicates the ratio of pressure gradient caused by solid-fluid interactions to that of inertia and viscous resistances) on the velocity of the flow. This figure depicts the influence of Casson factor β on the fluid velocity and it can be seen that as the values of β increase, the velocity decreases near the sheet surface and then dominates. An increment in Casson factor β resulted in the reduction of the fluid viscosity (which is due to applied stress) which ultimately enhances the flow velocity. The velocity profile escalates when β is increased, which leads to a thinner boundary layer flow. In other words, due to the inverse proportionality of the Casson factor to plastic viscosity, the flow of the Casson nanofluid gets escalated. The Newtonian fluid case happened as β gets higher ($\beta \rightarrow \infty$), and it is observed that the velocity of Newtonian fluid is greater than the Casson nanofluid with a diminished boundary layer. Moreover, it is observed that increasing Fr decelerates the Casson nanofluid flow velocity profile, which is because of the existence of an inertial impact that drags the fluid flow backward.

Figure 4.7.1(b) demonstrated the effects of Darcy number Da (the relative effect of the permeability of the media versus its cross-sectional area) and the velocity slip (slipperiness) parameter δ on the velocity profile. The fluid velocity profile is increasing with the slipperiness parameter δ with decreasing momentum boundary layer thickness. Perhaps, this happened because when slipperiness occurs, the fluid particles run away from the sheet surface and this leads to the diminution of shear force that in turn promotes the Casson nanofluid flow. The decrement in the velocity profile in response to enhanced Da in the flow of Casson nanofluids due to the porous medium established and built-in is observed in this figure. The existence of more and more porous matrix provides an increasingly resistant force that produces increasingly delayed flow and boosts the shrinking of the boundary layer.

Figure 4.7.2(a) shows the impacts of shrinking/stretching parameter λ and suction/injection

parameter S on the velocity profile. This figure reveals that when the stretching/shrinking parameter λ increases the momentum boundary layer thickness and the velocity enhances. In reality, for the shrinking case, the velocity could be reduced, however, this hindering factor is dominated by the slipperiness effect. The increment in the injection/suction parameter S augments the velocity of the nanofluid as depicted in the same figure. The imposition of sheet surface fluid suction for this problem affects increasing the Casson nanofluid flow. It is crucial to note that the velocity profile of the fluid flow decreases with increasing values of injection ($S < 0$), however, here the velocity slip dominates the injection effect due to the high temperature of the surface.

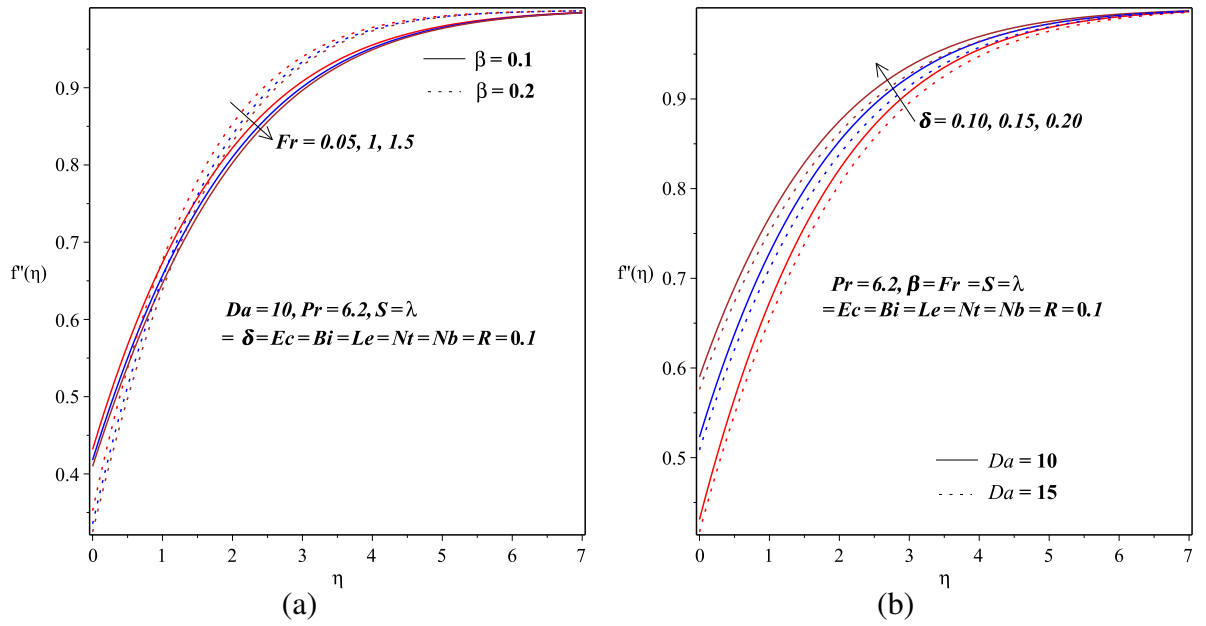


Figure 4.7.1: Effects of: (a) Fr and β (b) δ and Da on the flow velocity profile.

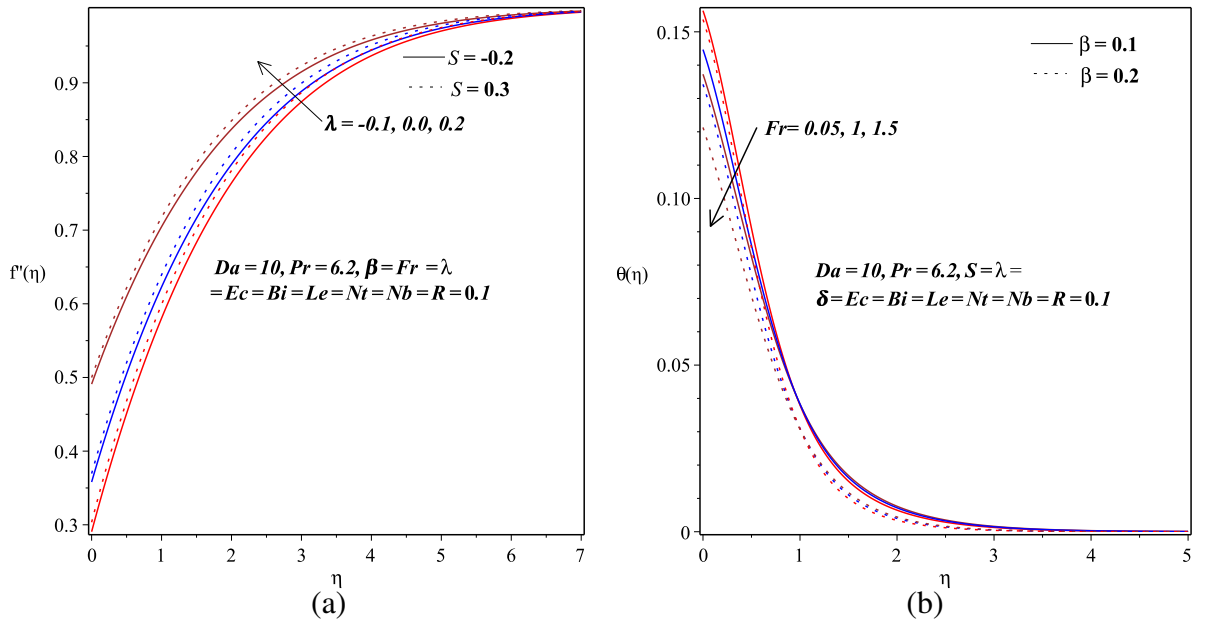


Figure 4.7.2: Effects of: (a) λ and S on the flow velocity profile (b) Fr and β on the flow temperature profile.

4.7.2 Temperature Profiles with change of Parameters

Figures 4.7.2(b)-4.7.5(a) illustrate the impacts of various embedded parametric factors such as the Casson, slipperiness, Forchheimer, shrinking/stretching, injection/suction, and thermophoresis parameters, and Eckert, Darcy, Lewis, Prandtl, and Biot numbers on the temperature profile and its corresponding boundary layer thickness.

From Figure 4.7.2(b) it is seen that the increment in the values of Forchheimer parameter Fr drops the temperature profile near the boundary of the sheet surface (up to $\eta = 0.96$), and then interchanges. Near the sheet surface, this happened since the porous medium and its inertial resistance give additional resistivity to the flow of the nanofluid mechanism, which forces the fluid flow at a low velocity and results in reduced a temperature, and gradually starts to become stronger as η increases (move away from the sheet surface). From the same plot, the temperature of the nanofluid flow and its thermal boundary layer thickness decreases with the enhancement of β values. Physically, when β values become higher the strength of the yield stress of the Casson fluid is weakened and hence promotes the plastic dynamic viscosity and hence resulting in a diminishing of the thickness of the thermal boundary layer of the flow temperature profile. Note that the thermal boundary layer thickness of the Casson nanofluid is greater than that of the Newtonian type of fluid flow. The values of β get higher($\beta \rightarrow \infty$) showing stronger interactions and molecular motion within the Casson nanofluid behaving as a Newtonian Casson fluid which ultimately enhances the Casson nanofluid temperature profile. Therefore, decreasing the stress improves the fluid flow and drops the Casson nanofluid flow heat transfer.

Figure 4.7.3(a) demonstrates that the temperature profile drops with an increment in the values of the velocity slip parameter δ . The temperature distribution gets decreased because of increment in the values of Da within a few layers of fluid flow from the sheet surface ($\eta = 1.081$) as observed from the same figure. Note that the existence of a porous matrix increases a resistant force to flow of the fluid that hinders the flow velocity and hence elevates the fluid temperature profile and thickness of the thermal boundary layer at the sheet surface. Importantly, for $\eta > 1.081$ the temperature field lines precede the opposite course because of an increment in the values of Da .

From Figure 4.7.3(b), it is depicted that enhancing λ and S results in the diminution of the temperature of the thermal boundary layer thickness of the fluid. That is, with the involvement of the suction effect, the thermal boundary layer thickness is reduced, enhancing the temperature gradient on the surface of the sheet and causing warmer fluid away from the boundary layer, which is agreeable with the temperature profile of the nanofluid.

The impacts of Prandtl number Pr (which is the quotient of kinematic viscosity (momentum diffusivity) to thermal diffusivity) and Eckert number Ec (which indicates kinetic energy to heat dissipation potential ratio) were explained using Figure 4.7.4(a). From the figure, it can be seen that as Prandtl number Pr increases, the thermal boundary layer thickness rises with an increase in nanofluid temperature at the sheet surface due to the dominant viscous effects at sheet surface with increasing Pr . However, within the boundary layer region, the trend is opposite with a decrease in thermal boundary layer thickness and fluid temperature due to dominant effects of thermal diffusivity. Again, from this figure enhancement of the fluid temperature profile and the thermal boundary layer thickness was observed due to the augmenting values of Ec . In reality, Ec transforms the kinetic energy into internal energy in opposition to the stresses of the viscous fluid. Because of that, the increment in viscous dissipation inclines to boost the fluid flow thermal boundary layer thickness and the temperature profile. That is, the thermal boundary layer thickness increases gradually due to an escalation in dissipation increases the flow's thermal conductivity which upgrades the thermal boundary layers.

In Figure 4.7.4(b) the impacts of Biot number Bi and Lewis number Le are demonstrated. Physically, the Biot number is the ratio of internal thermal resistance at the surface of the object to the boundary layer thermal resistance. It was observed that an increment in the convective heat transfer parameter Bi results in thermal convection boosting at the wall and accelerates the fluid's temperature profile. That is, intensifying the values of Bi reveals increment in the temperature fields and the thermal boundary layer thickness. Again from the figure, it can be seen that an increment in Le resulted in a reduction of the temperature profile. Lewis number Le (the ratio of thermal diffusivity to specie/mass diffusivity ratio) is used for characterizing the fluid flows where there are convective mass and heat transfers (simultaneously). Both the temperature profile and thickness of the thermal boundary layer decrease for augmenting values of the Lewis number Le because of thermal diffusivity.

Figure 4.7.5(a) demonstrates the effects of thermophoresis Nt and the chemical reaction parameter R on the temperature field. The temperature of the fluid augments with the thermophoresis parameter Nt as detailed in this figure. Since the thermophoresis process accelerates the nanoparticle movement from hotter to colder positions, accordingly, heat moves rapidly from the hotter sheet surface to the fluid flow, and thus, it enhances the temperature of the Casson nanofluid. The presented figure validates the theory defined above, for given values of Nt a rising the thermal boundary layer thickness and the temperature field are demonstrated. The same figure displays

the temperature profile for various values of the chemical reaction parameter R , and it is seen that increasing the values of R produces a slight reduction of the temperature profile and thermal boundary layer thickness.

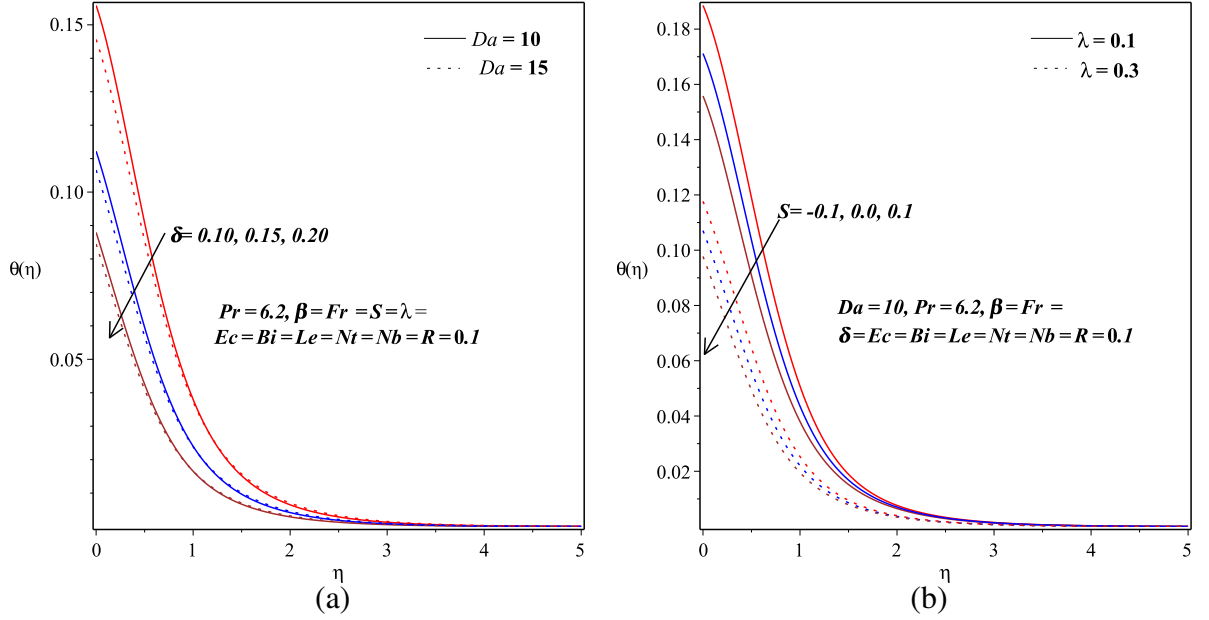


Figure 4.7.3: Effects of: (a) δ and Da (b) λ and S on the flow temperature profile.

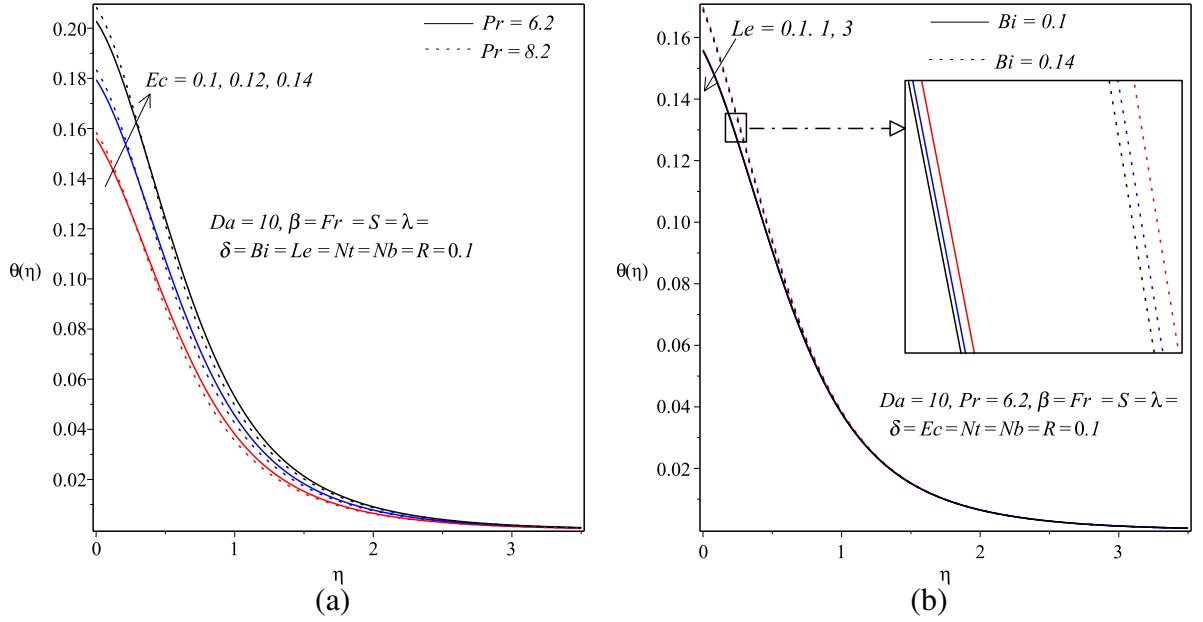


Figure 4.7.4: Effects of: (a) Pr and Ec (b) Bi and Le on the temperature profile.

4.7.3 Concentration Profiles with change of Parameters

Figures 4.7.5(b)-4.7.8 demonstrate the impacts of different embedded parameters: the Casson, slipperiness, Forchheimer, shrinking/stretching, Brownian motion, injection/suction, and thermophoresis parameters, and Eckert, Darcy, Lewis, Prandtl, and Biot numbers on the concentration field $\phi(\eta)$ and the corresponding solutal boundary layer thickness.

Figure 4.7.5(b) illustrates that as the Forchheimer and Casson parameters increase both the concentration field and its boundary layer thickness decrease near the surface of the sheet and then decrease in the free stream region. Again, the concentration profile increases as the values of the velocity slip parameter and Darcy number rise, which is due to the diminution of the accessible area in the boundary layer of the fluid flow, due to the shrinking of the surface, as presented in Figure 4.7.6(a). In reality, the higher values of Da designate the porousness of the porous medium. Thus, as values of Da enhanced, the strength of permeability of the porous medium augments, which concludes the escalating conduct in rate of mass transportation.

Nearer to the surface of flow, increasing the values of the stretching parameter λ increases the concentration profile and its boundary layer thickness and reverses in the free stream. Increment in the values of injection parameter S resulted in the enhancement of both the concentration field and the corresponding boundary layer thickness nearer to the sheet surface and then switches, as displayed in Figure 4.7.6(b).

Figure 4.7.7(a) shows that an improvement in Prandtl number Pr leads to a reduced concentration profile and its boundary layer thickness. The concentration profile is the decreasing function of the dissipation parameter (Eckert number Ec) up to $\eta = 1.23$ and then reverses, as it is observed from the same figure. The increasing Ec rises the internal energy, which accordingly increases the temperature field of the fluid flows and results in a dropping of the concentration profile. It is noted that the Eckert number Ec is usually used in nuclear systems engineering for compressible fluid flow at advanced speeds.

Figure 4.7.7(b) shows the changes in the concentration profile corresponding to different values of Biot number Bi , and it explains that the concentration field and related the boundary layer thickness are enhanced as the values of Biot number Bi increases. That is, since the distribution of nanoparticle concentration is driven by the temperature profile, one foresees that an increasing Biot number Bi results in a deeper penetration of the concentration profile. For increasing the Lewis number Le , the concentration field and its boundary layer thickness decrease.

Figure 4.7.8(a) portrays the effects of the thermophoretic parameter Nt on concentration fields. The concentration fields are increasing functions of Nt . This property is observed due to the higher Nt moves the nanoparticles of the fluid from a hotter to a colder region, which produces an increment in the concentration profile of the fluid flow. Hence, an augmenting effect of the thermophoresis parameter on the concentration field is observed here. The impacts of the destructive reaction parameter R on the concentration profile is displayed in the same figure. It is noted that

$R > 0$ implies the presence of a chemical reaction (i.e., the destructive consequence of the reaction). Here, it is observed that enhancing $R > 0$ reduces the concentration field and the associated boundary layer thickness.

Figure 4.7.8(b) exhibits the influence of Brownian motion on the concentration field. Brownian motion is responsible for a dropping in the concentration profile. The higher Brownian motion parameter Nb heats up the thickness of concentration boundary layer, which repels the fluid nanoparticles from the stretching/shrinking sheet surface. Therefore, the greater Nb resulted in a falling of the concentration field.

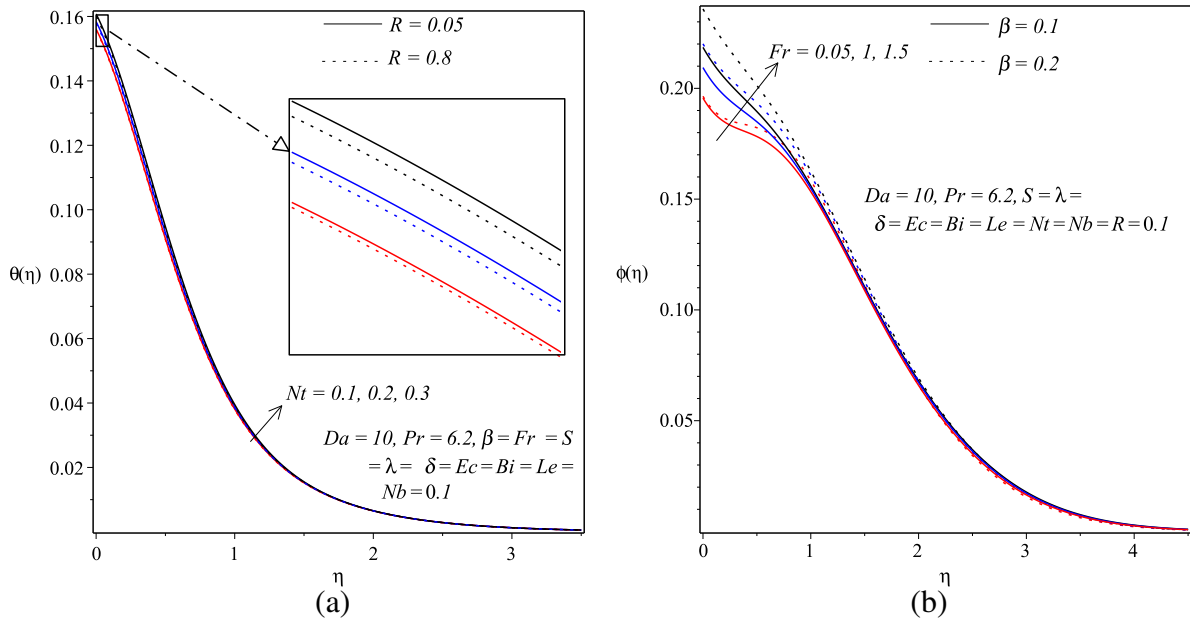


Figure 4.7.5: Effects of: (a) Nt and R on the temperature (b) Fr and β on the flow concentration profiles.

4.7.4 The Skin Friction, Local Nusselt and Sherwood Numbers

The effects of different parameters on the skin friction (a resistant force exerted on an object moving in a fluid), the local Nusselt number (convective to conductive heat transfer across a boundary), and the local Sherwood number (convective mass transfer to the rate of diffusive mass transport) are discussed in Figures 4.7.9-4.7.12 and Table 4.7.2.

Figures 4.7.9 and Table 4.7.2 demonstrate the variation of skin friction with different parameters with a stretching parameter $\lambda = 0.2$, and it is observed that increments in the values of the suction/injection parameter enhance the skin friction, whereas increasing the values of the Casson, Forchheimer, shrinking/stretching, velocity slip parameters, and Darcy number reduce the skin friction. Perhaps the reduction in the coefficient of skin friction for the mentioned parameters is due to the slipperiness of the flow surface.

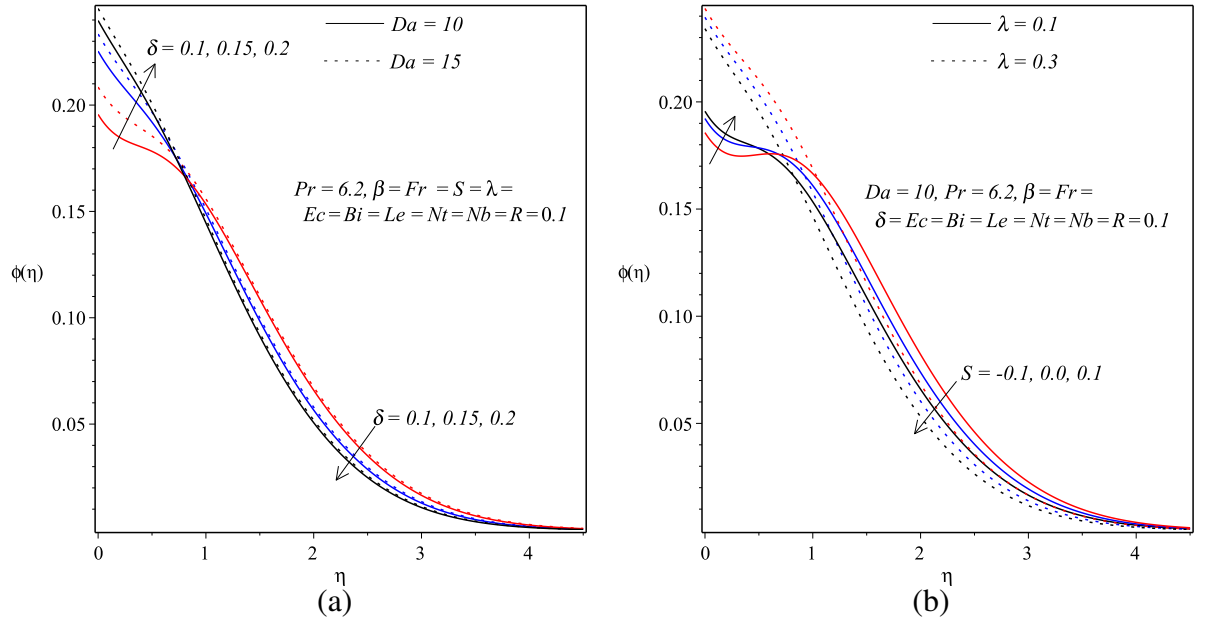


Figure 4.7.6: Effects of: (a) δ and Da (b) λ and S on the flow concentration profile.

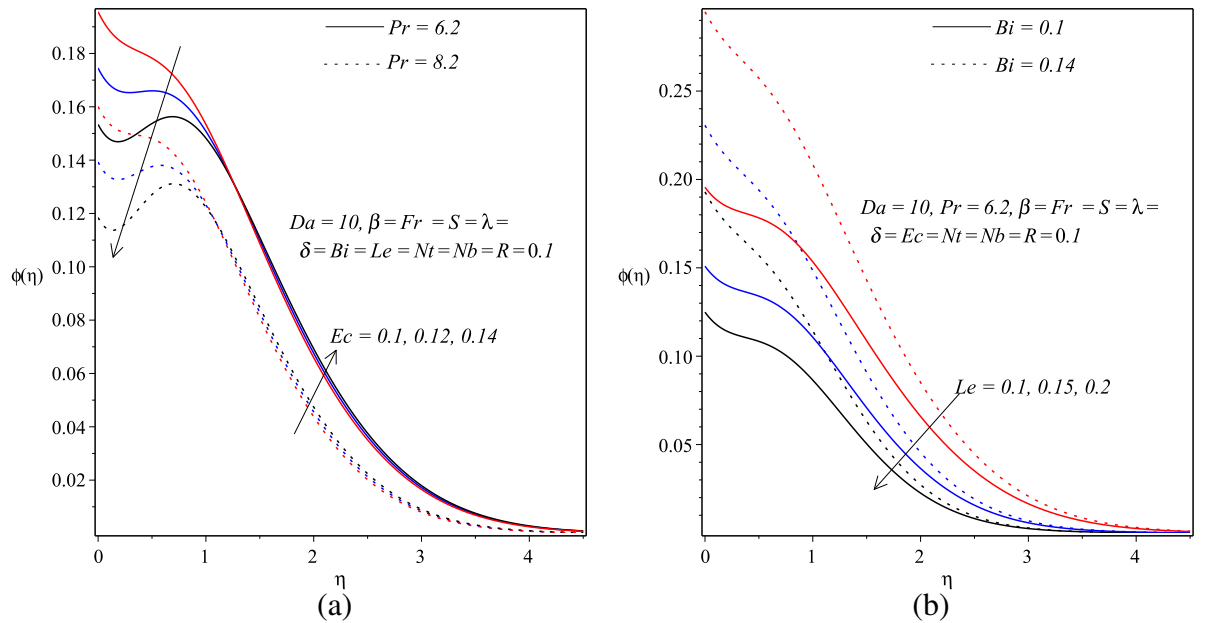


Figure 4.7.7: Effects of: (a) Pr and Ec (b) Bi and Le on the concentration profile.

The effects of Casson factor β , Lewis number Le , Darcy number Da , Forchheimer parameter Fr , Prandtl number Pr , suction/injection parameter S , Eckert number Ec , stretching/shrinking parameter λ , velocity slip parameter δ , convective heating parameter Bi , and thermophoresis parameter Nt on the local Nusselt number (heat transfer rate) are presented in Figures 4.7.10-4.7.11 and Table 4.7.2. It can be seen that augmentation in the values of the Casson, Forchheimer, slipperiness, stretching, chemical reaction, and suction parameters, and Darcy, Prandtl, Lewis, and Biot numbers increase the Nusselt number (heat transfer rate), however, increasing the values of

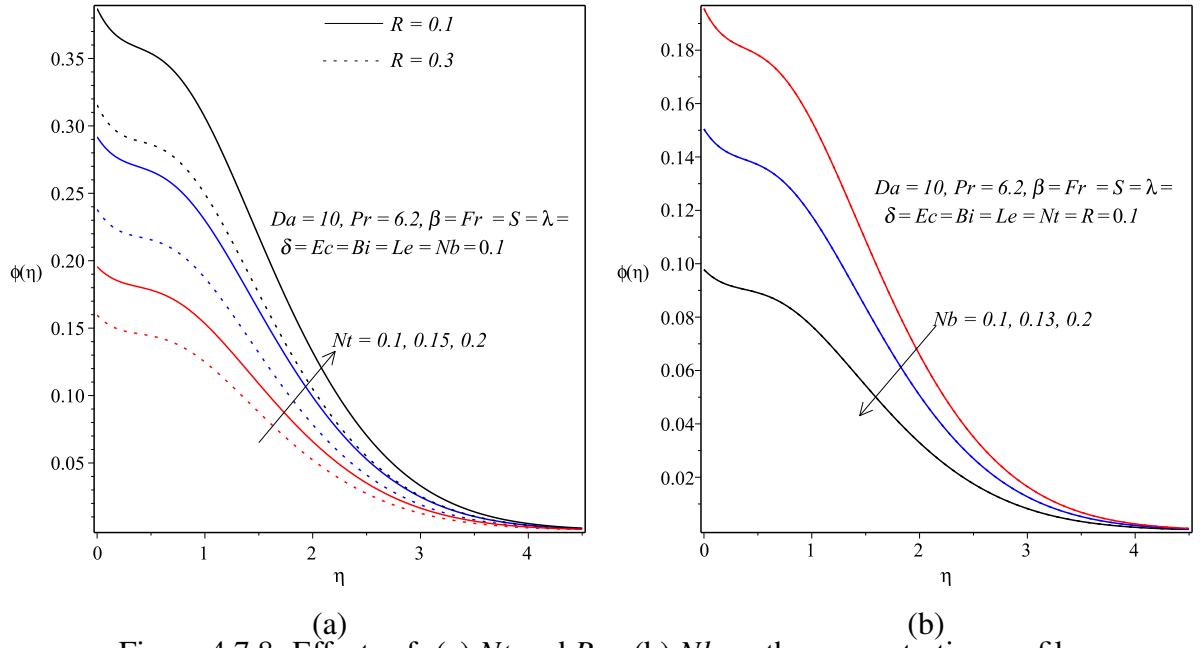


Figure 4.7.8: Effects of: (a) Nt and R (b) Nb on the concentration profile

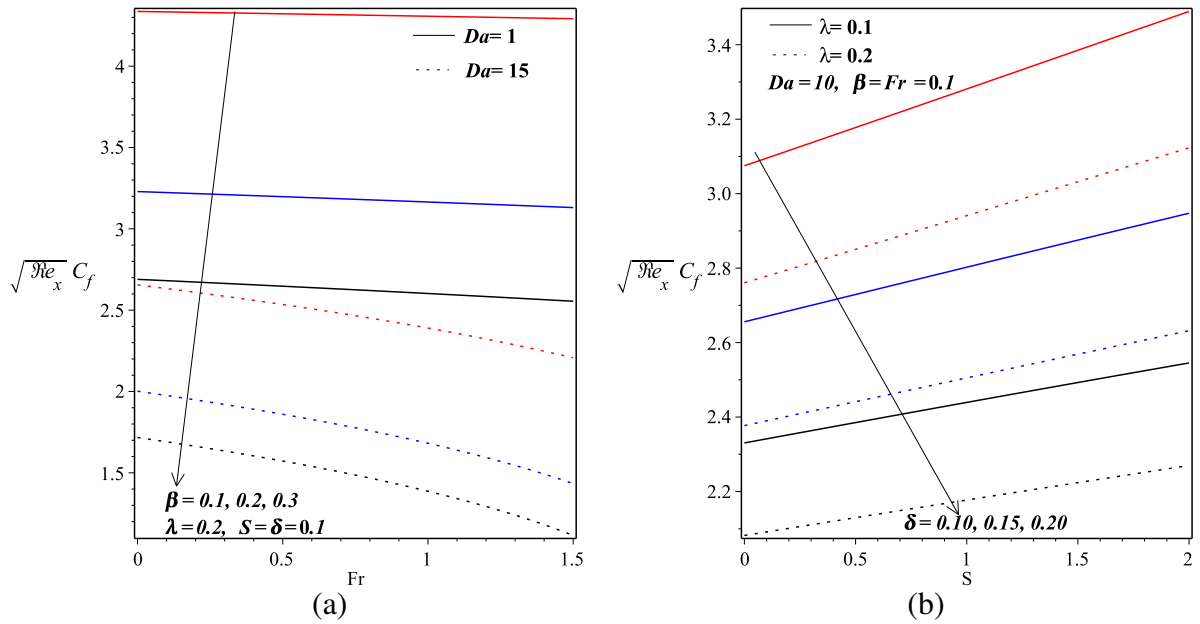


Figure 4.7.9: (a) Skin friction against Fr for varying values of β and Da . (b) Skin friction against S for varying values of δ and λ .

the thermophoresis parameter and Eckert number diminish the rate of heat transfer of the flow. The local Nusselt number drops with an increment in the values of β , which leads to the conclusion that the heat transfer rate at the sheet surface is higher in Casson nanofluid when compared to Newtonian fluid flow. From Table 4.7.2, it can be noticed that for increasing values of R , the heat transfer rate escalates. In reality, for sturdier chemical reactions, a destructive consequence is observed, that decays the chemical substance. Thus, for augmented R , the concentration gradient (the local Sherwood number) steps up. From Figure 4.7.10(b)

In computation, it is obtained that the rate of mass transfer (local Sherwood number) is equal to

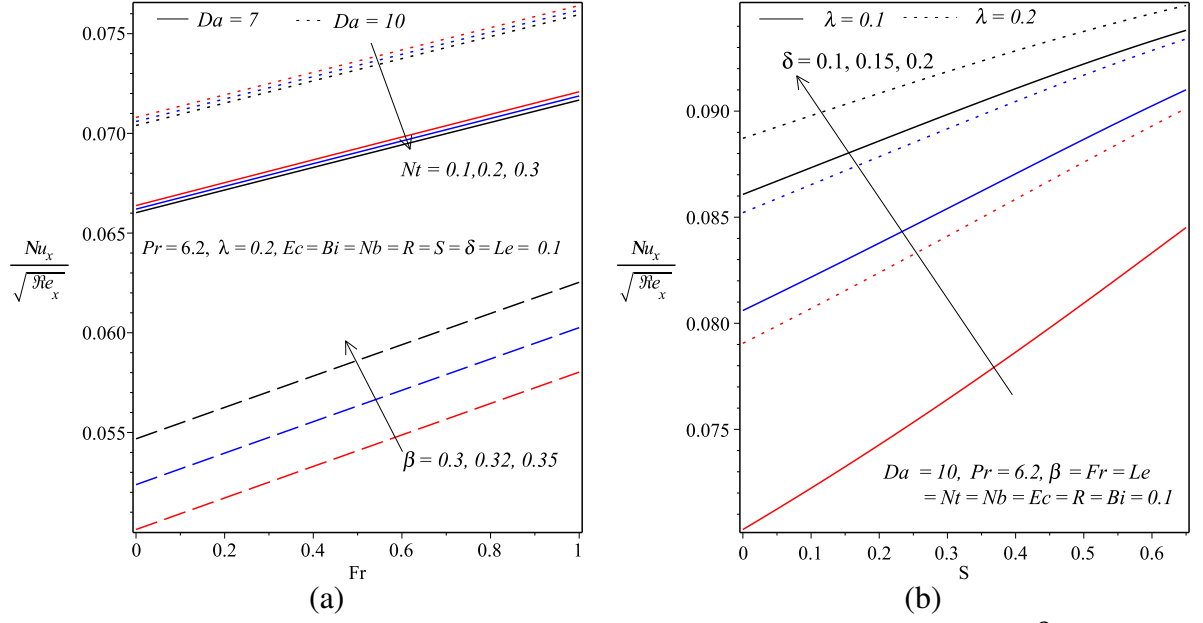


Figure 4.7.10: (a) Nusselt number against Fr for varying values of Nt , Da , and β . (b) Nusselt number against S for varying values of δ and λ .

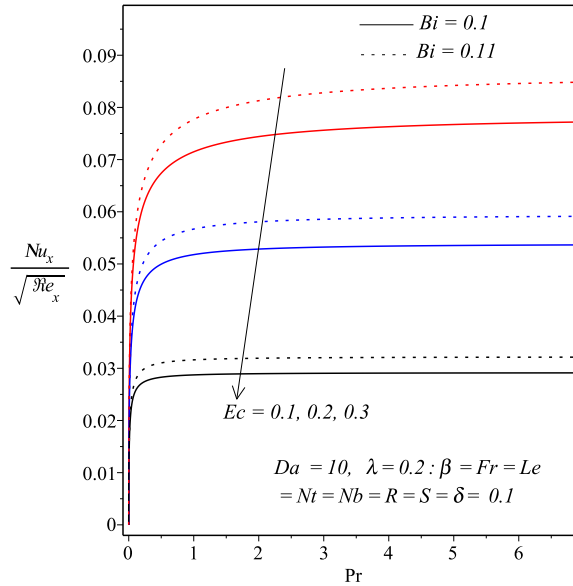


Figure 4.7.11: Nusselt number against Pr for varying values of Ec and Bi .

the parametric multiple of the heat transfer rate (temperature gradient or the local Nusselt number). That is, $\frac{Sh_x}{\sqrt{Re_x}} = \frac{Nt}{Nb} \frac{Nu_x}{\sqrt{Re_x}}$ and hence it is possible to derive a conclusion for the mass transfer rate based on the plots for the Nusselt number and Table 4.7.2 without plotting for the Sherwood number. Therefore, the Sherwood number (mass transfer rate) increases with raising values of the suction, Casson, Forchheimer, stretching, thermophoresis, chemical reaction, and velocity slip parameters, and Darcy, Prandtl, Lewis, and Biot numbers, and decreases with enhancing the values of Brownian motion parameter and Eckert number.

Note that the rate of mass transfer is an increasing function of the thermophoresis parameter Nt , which implies that as the nanoparticles are repelled from the stretching/shrinking surface due to thermophoresis processes, the rising Nt values accelerates the diffusion of nanoparticles into the boundary layer, increasing the mass transfer rate, Figure 4.7.12.

Moreover, from Figure 4.7.12 and Table 4.7.2, it is revealed that an increment in the values of the Brownian motion parameter Nb drops the mass transfer rate. As a consequence, the Brownian motion tends to pull the nanoparticles toward the stretching/shrinking surface, and furthermore, increasing the Brownian motion parameter Nb intensifies the nanoparticles' diffusion into the boundary layer, decelerating the mass transfer rate.

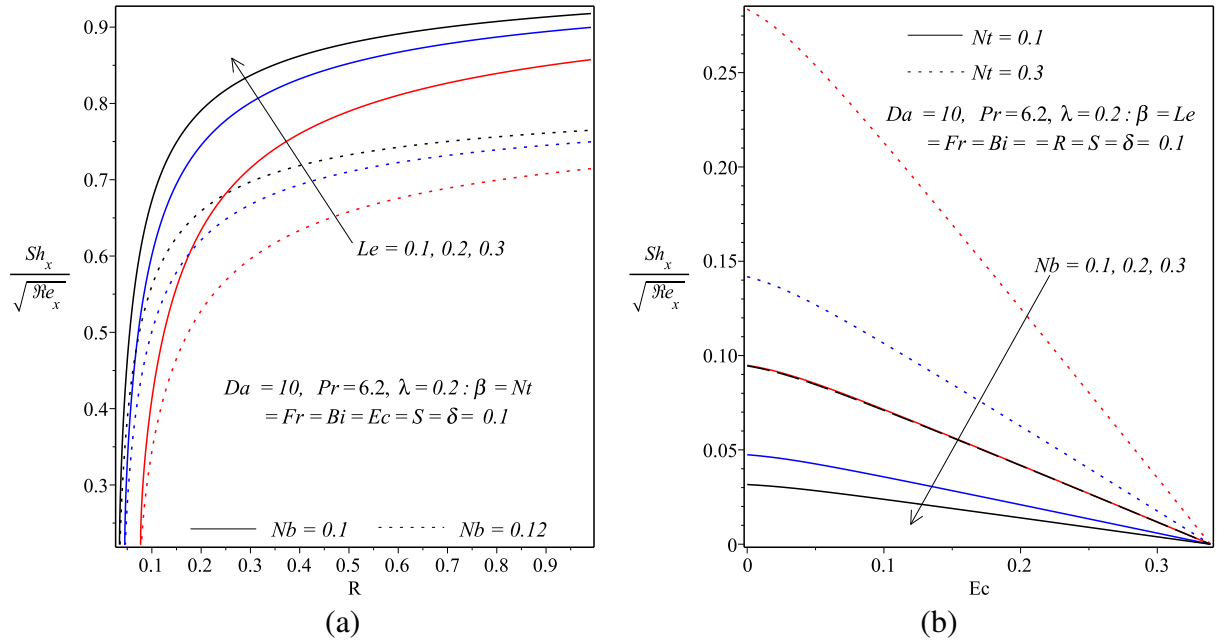


Figure 4.7.12: (a) Sherwood number against R for varying values of Le and Nb . (b) Sherwood number against Ec for varying values of Nb and Nt .

Table 4.7.2 (for $Pr = 6.2, Da = 10, \lambda = 0.2, \beta = Fr = S = Nt = \delta = Ec = R = Bi = Le = Nb = 0.1$ gives the reference point values $(1 + \frac{1}{\beta})f''(0) = 2.962800997, -\theta'(0) = 0.0876258570$ and $-\phi'(0) = 0.0876258570$) presents the results obtained by numerical investigations of the combined effects of different embedded factors on the skin friction, rates of mass and heat transfers from fluid flows over a convectively heated permeable shrinking/stretching slippery sheet surface in a porous media. This table best agrees with the discussion under this subtopic.

4.7.5 Existence of Dual Solutions

For a shrinking parameter $\lambda < 0$, it is possible to obtain dual/multiple solutions for the boundary layer flow equations. These dual solutions occur in the interval $\lambda_c < \lambda < 0$ and for $\lambda < \lambda_c < 0$,

Table 4.7.2: Computations showing the effects of different parameters on dimensionless shear stress/local skin friction $(1 + \frac{1}{\beta})f''(0)$, heat transfer rate/local Nusselt number $-\theta'(0)$ and mass transfer rate/local Sherwood number $-\phi'(0)$.

Parameters	Varying values	$(1 + \frac{1}{\beta})f''(0)$	$-\theta'(0)$	$-\phi'(0)$
		2.962800997	0.0876258570	0.0876258570
β	0.2	2.260998919	0.0878870051	0.0878870051
	0.3	1.950192405	0.0881960358	0.0881960358
Da	5	3.255265012	0.0859136542	0.0859136542
	1	4.374282429	0.0815482233	0.0815482233
Fr	0.2	2.952523448	0.0877004313	0.0877004313
	0.3	2.942101222	0.0877764640	0.0877764640
S	0.3	2.998454219	0.0895800667	0.0895800667
	0.5	3.034102359	0.0910737847	0.0910737847
λ	0.1	3.309018370	0.0843988472	0.0843988472
	-0.1	3.982559810	0.0752642915	0.0752642915
δ	0.15	2.518460891	0.0907763749	0.0907763749
	0.20	2.185817697	0.0925699557	0.0925699557
Pr	4	2.962800997	0.0875736381	0.0875736381
	1	2.962800997	0.0859760767	0.0859760767
Ec	0.2	2.962800997	0.0789620250	0.0789620250
	0.3	2.962800997	0.0703085810	0.0703085810
Bi	0.14	2.962800997	0.1207806956	0.1207806956
	0.17	2.962800997	0.1449574092	0.1449574092
Nt	0.2	2.962800997	0.0874397590	0.1748795180
	0.3	2.962800997	0.0872501123	0.2617503370
Le	0.2	2.962800997	0.0876312362	0.0876312362
	0.3	2.962800997	0.0876353478	0.0876353478
Nb	0.2	2.962800997	0.0876258570	0.0438129285
	0.3	2.962800997	0.0876258570	0.0292086190
R	0.2	2.962800997	0.0876297218	0.0876297218
	0.3	2.962800997	0.0876329204	0.0876329204

no solution exists, where λ_c is the critical minimum value of λ for which a unique solution of the equations exists. In other words, when the λ_c is attained, the sheet surface will stop shrinking, and the solution converges only $\lambda_c < \lambda < 0$ for the shrinking sheet surface. The value of λ_c depends on the values of the other embedded flow factors in the governing model equations. Thus, within the interval of shrinking parameter (λ) values, only one realistic solution exists. Several studies explained that only the upper branch (first) solution for the case of shrinking sheets surface is physically realistic and stable, whereas the lower branch solution diverges with small perturbations or disturbances.

Figures 4.7.12(b)-4.7.15 demonstrate the impacts of various parameters (velocity slip parameter δ , Casson parameter β , Darcy number Da , Forchheimer parameter Fr , suction parameter S ,

Eckert Ec , and Biot numbers) on the coefficient of local skin friction. From the displayed plots, Figures 4.7.12(b)-4.7.15 it is seen that increasing the values of the suction parameter enhances the skin friction for the first solution because suction at the boundary of the sheet surface slows down the fluid flow and rises the shear stress, whereas increments in the Casson factor, Darcy number, Forchheimer, and velocity slip parameters reduce the coefficient of skin friction in the shrinking sheet surface case.

The coefficient of surface drag force (skin friction) is observed to drop upon improving the Casson factor β , which means that less applied force is sufficient to move the Casson nanofluid past the sheet surface for higher β . In other words, rising in $\beta \rightarrow \infty$ implies that the fluid misses its non-Newtonian properties and behaves as Newtonian fluids, and thus its velocity augments, due to the dropping in the shear stress. Again, from these figures, it is depicted that augmenting the suction and velocity slip parameters resulted in the widening of solution interval (increment in $|\lambda_c|$). However, the diminishing behavior in $|\lambda_c|$ is observed for increasing the values of the Casson parameter, Darcy number, and Forchheimer parameter. These graphically displayed results agree sufficiently with the computed numerical results presented in Tables 4.7.2 and 4.7.3.

From Figure 4.7.16, it is demonstrated that intensifying the values of Biot and Eckert numbers increases the heat transfer rate for the shrinking parameter λ . The interval of the solution is increases with the Eckert number and decreases with the Biot number. Again, these observed results sufficiently agree with the computed results presented in Table 4.7.2.

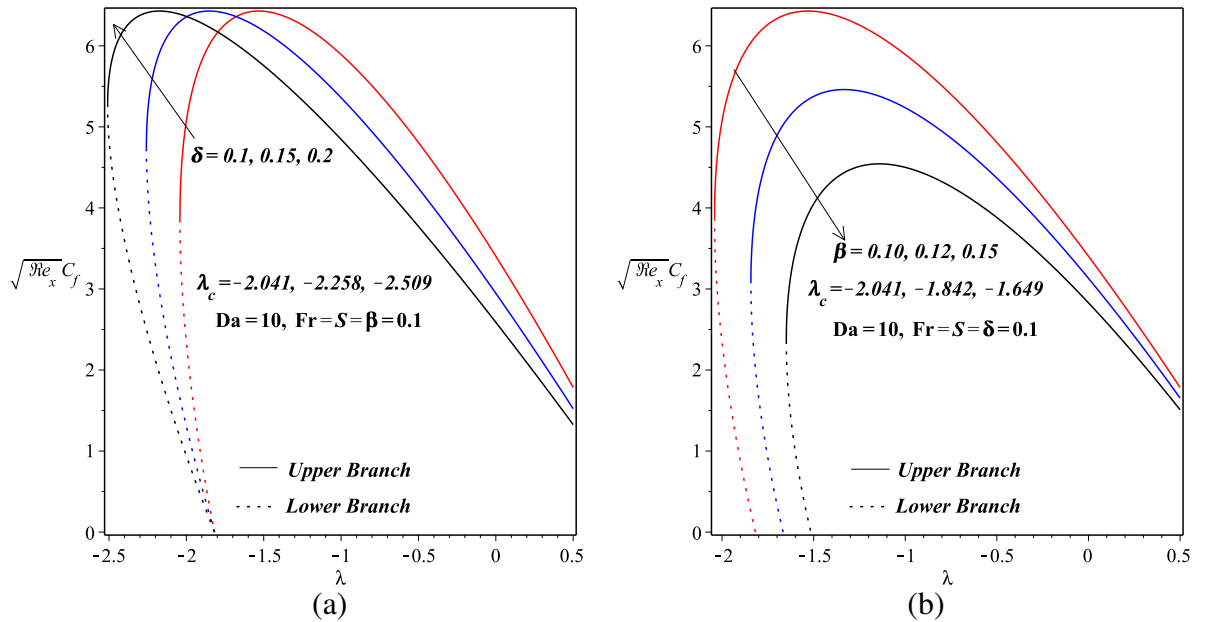


Figure 4.7.13: Skin friction against λ for different values of (a) δ . (b) β .

The tabulated results show the dual solution occurs for some range of λ . To determine the stable

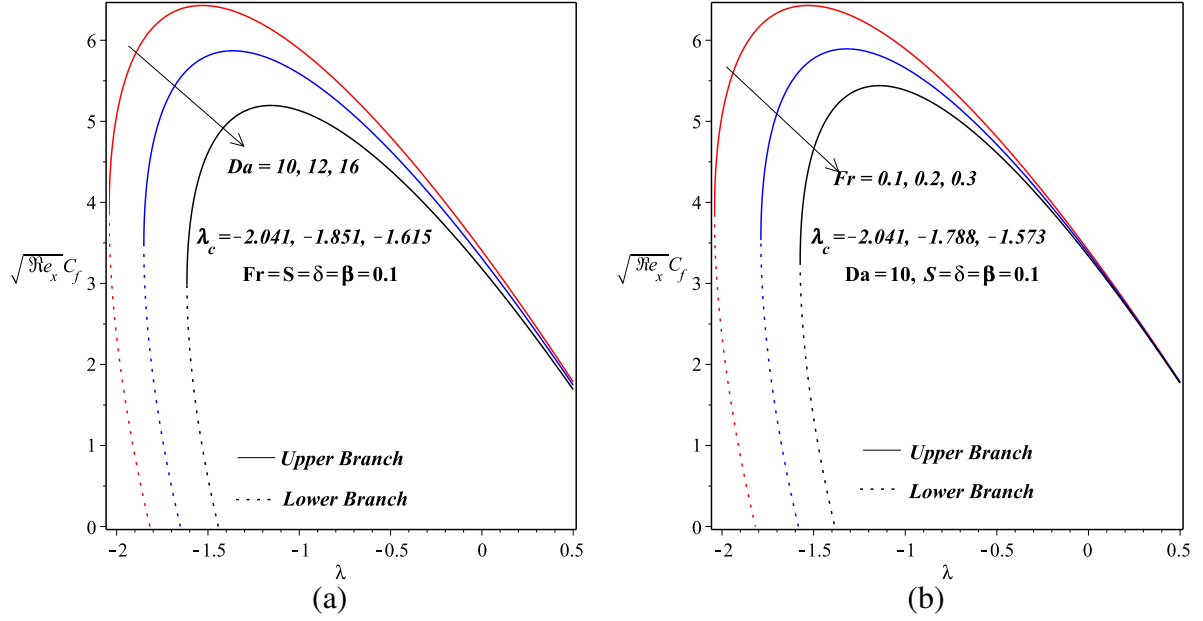


Figure 4.7.14: Skin friction against λ for different values of (a) Da . (b) Fr .

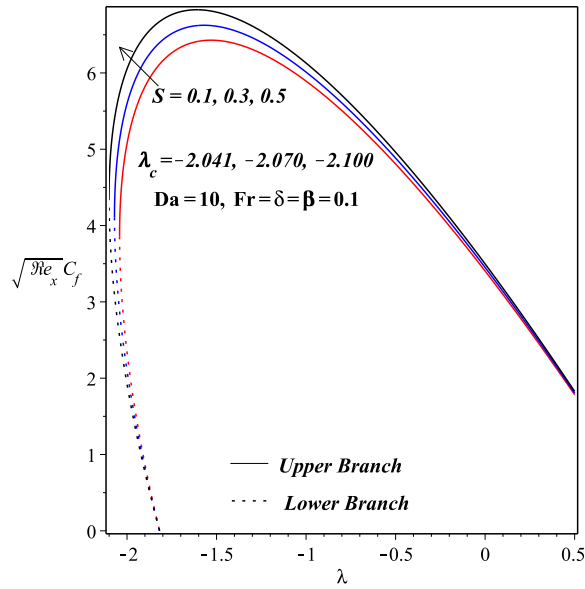


Figure 4.7.15: Skin friction against λ for different values of S .

solution the temporal stability analysis was performed for the problem under investigation for the case of shrinking surfaces. As displayed in Table 4.7.4, for various values λ the least eigenvalue ε is numerically calculated for the temporary change of small disturbances/perturbations for the basic steady fluid flow by varying the numerical values of the Casson, slipperiness, and injection/suction parameters, the Darcy number, and the Forchheimer parameter. From this table, since the value of the least eigenvalue ε is positive for the shrinking sheet surface, the first solution is stable temporally and therefore, physically achievable. Clearly, the negative value of ε revealed that the lower branch (second) solution is unstable and physically not realistic. Note that, $\varepsilon > 0$

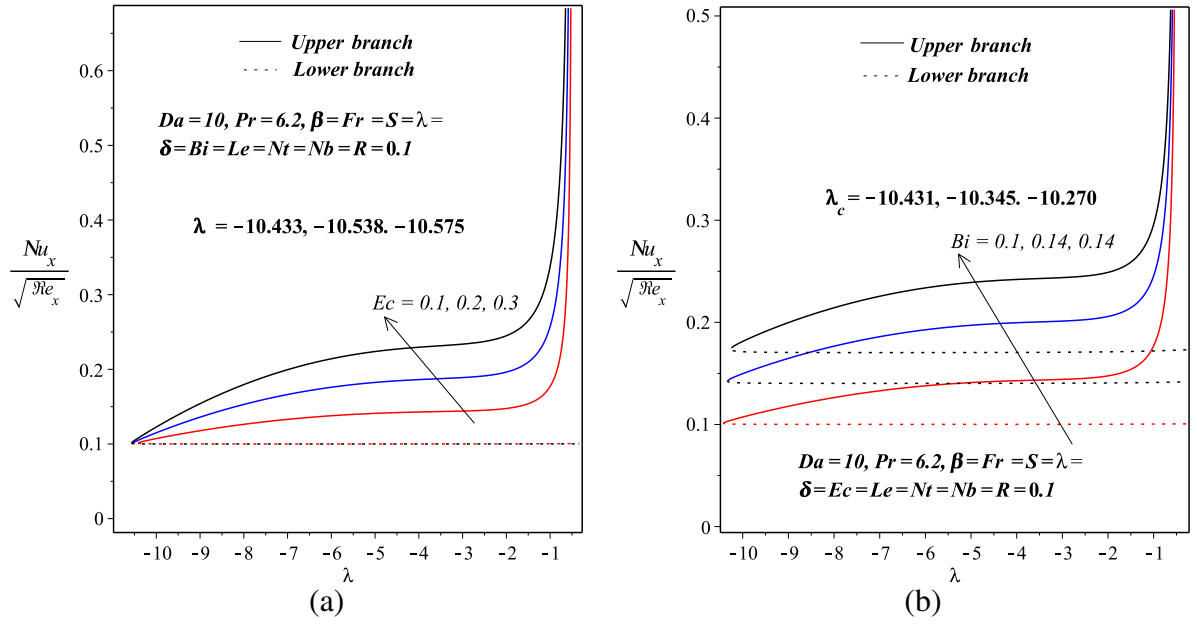


Figure 4.7.16: Local Nusselt number against λ for different values of (a) Ec . (b) Bi .

demonstrates the small disturbance rate declines for the first solution, whereas $\varepsilon < 0$ shows the increment of the disturbances for the second solution.

Table 4.7.3: Numerical results of critical shrinking parameter (λ_c), upper and lower branch solutions of skin friction ($\sqrt{Re_x}C_f$) for different values of different parameters.

Da	Fr	S	δ	β	λ_c	λ	$\sqrt{Re_x}C_f$	
							Lower B.	Upper B.
10	0.1	0.1	0.1	0.1	-2.041	-2.032	3.1631566	4.483433
12	0.1	0.1	0.1	0.1	-1.851	-1.843	2.8443656	2.8443656
16	0.1	0.1	0.1	0.1	-1.615	-1.608	2.4538616	3.478158
10	0.2	0.1	0.1	0.1	-1.788	-1.780	2.9253387	4.088929
10	0.3	0.1	0.1	0.1	-1.573	-1.567	2.7345476	3.746372
10	0.1	0.3	0.1	0.1	-2.070	-2.061	3.4360885	4.708393
10	0.1	0.5	0.1	0.1	-2.100	-2.096	3.8958574	4.786905
10	0.1	0.1	0.15	0.1	-2.258	-2.256	4.4737190	4.979667
10	0.1	0.1	0.2	0.10	-2.509	-2.508	5.1112185	5.382166
10	0.1	0.1	0.1	0.12	-1.842	-1.828	2.2706655	3.775570
10	0.1	0.1	0.1	0.15	-1.649	-1.628	1.4800467	3.118625

Table 4.7.4: Numerical results of critical shrinking parameter (λ_c), upper and lower branch of the least eigenvalues for varying values of different parameters.

Da	Fr	S	δ	β	λ_c	λ	ε	
							Lower B.	Upper B.
10	0.1	0.1	0.1	0.1	-2.041	-2.032	-0.59955535	0.0529634
12	0.1	0.1	0.1	0.1	-1.851	-1.843	-0.57292914	0.0292364
16	0.1	0.1	0.1	0.1	-1.615	-1.608	-0.54119359	0.0111859
10	0.2	0.1	0.1	0.1	-1.788	-1.780	-0.58594192	0.0735937
10	0.3	0.1	0.1	0.1	-1.573	-1.567	-0.55361026	0.0920778
10	0.1	0.3	0.1	0.1	-2.070	-2.061	-0.55231504	0.0778383
10	0.1	0.5	0.1	0.1	-2.100	-2.096	-0.42219478	0.0195818
10	0.1	0.1	0.15	0.1	-2.258	-2.256	-0.22597910	0.0683772
10	0.1	0.1	0.2	0.10	-2.509	-2.508	-0.05875678	0.1269797
10	0.1	0.1	0.1	0.12	-1.842	-1.828	-0.73701377	0.0777459
10	0.1	0.1	0.1	0.15	-1.649	-1.628	-0.88991622	0.1045761

Chapter 5

Dual Solutions and Stability Analysis of Cu-H₂O-Casson Nanofluid Convection Past a Heated Stretching/Shrinking Slippery Sheet in a Porous Medium

5.1 Introduction

A thorough observation of the aforementioned above-cited reviews and works reveals that no scientific analysis has been done to study the mutual effects of slipperiness of the surface, nanoparticles volume fraction, thermal and viscous dissipation, porous media and stretching/shrinking surface on the rate of heat transfer enhancement in a flow into boundary layer of Cu-H₂O Casson nanofluids. Filling the gap in the work of ([Tadesse et al., 2021a](#)) (in which they consider MHD Newtonian flow), this chapter of the dissertation considers non-Newtonian flow subjected to a slippery surface.

Since in this type of fluid flow problem, dual solutions possibly exist due to stretching/shrinking surface, the main goal of this chapter is to do stability analysis to determine physically reliable and stable solutions and enhancement of heat transfer as a result of the addition of the Cu-H₂O nanoparticle into Casson fluids for the coupled transformed ordinary differential equations. Moreover, the numerical results of the coefficient of skin friction, rates of heat transfer, and velocity and temperature profiles are demonstrated and discussed both graphically and quantitatively. Thermally driven boundary layer flow of Casson nanofluid in a porous medium has several applications in chemical and mechanical engineering, e.g. food processing and storage, geophysical systems, electro-chemistry, fibrous insulation, metallurgy, the design of pebble(rock) bed nuclear reactors, underground disposal of nuclear or non-nuclear waste, microelectronics cooling. Casson nanofluid is more helpful for cooling and friction reducing agents compared to Newtonian type nanofluid flow.

5.2 Mathematical Model Formulation

Consider a two-dimensional, viscous, homogeneous, steady, isotropic, laminar, incompressible boundary layer flow of a Cu-H₂O Casson nanofluid over a linearly stretching/shrinking sheet within a Darcy-Forchheimer porous medium. The flow is subjected to suction/injection of constant velocity V_0 perpendicular to the sheet surface. The shrinking/stretching sheet moves with velocity $U_w = bx$ and the flow is subjected to slip condition. The surface below the shrinking/stretching sheet surface is heated convectively by a hot fluid having an initial temperature T_f that gives a

The above Eqs. (5.2.2)-(5.2.4) represent the governing equations of the nanofluid flow when $\beta \rightarrow \infty$ and the Casson fluid flow when $\beta \neq \infty$ and $\phi = 0$.

5.3 Similarity Transformation

Eqs. (5.2.2)-(5.2.3) with the boundary (5.2.4) can be transformed to the dimensionless form by introducing non-dimensional variables stated below, to obtain the similarity solutions (Makinde, 2018; Tadesse et al., 2021a):

$$\eta = \sqrt{\frac{a}{v_f}}y, \quad \psi(x,y) = x\sqrt{av_f}f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}. \quad (5.3.1)$$

where the stream function $\psi = x\sqrt{av_f}f(\eta)$ is written as a velocity component as:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{d\eta}{dy} = x\sqrt{av_f}f'(\eta) \cdot \sqrt{\frac{a}{v_f}} = axf'(\eta) = U_\infty f', \quad (5.3.2)$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{av_f}f(\eta) = -\sqrt{av_f}f. \quad (5.3.3)$$

Following the same procedure as in Chapter 4, the following results are obtained:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{d\eta}{dx} = af'(\eta) = af', \quad (5.3.4)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta} \frac{d\eta}{dy} = -\sqrt{av_f}f'(\eta) \sqrt{\frac{a}{v_f}} = -af', \quad (5.3.5)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \frac{d\eta}{dy} = axf''(\eta) \sqrt{\frac{a}{v_f}} = U_\infty \sqrt{\frac{a}{v_f}} f'', \quad (5.3.6)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(U_\infty \sqrt{\frac{a}{v_f}} f'' \right) = ax \sqrt{\frac{a}{v_f}} f'''(\eta) \sqrt{\frac{a}{v_f}} = \frac{aU_\infty}{v_f} f'''. \quad (5.3.7)$$

and,

$$\theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \Rightarrow T = T_\infty + (T_f - T_\infty)\theta(\eta) \equiv \theta nx^2 + T_\infty, \quad \frac{\partial T}{\partial x} = 2\theta nx, \quad (5.3.8)$$

$$\frac{\partial T}{\partial y} = (T_f - T_\infty) \frac{\partial \theta(\eta)}{\partial y} = (T_f - T_\infty) \theta' \frac{\partial \eta}{\partial y} = nx^2 \sqrt{\frac{a}{v_f}} \theta', \quad (5.3.9)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left((T_f - T_\infty) \theta' \sqrt{\frac{a}{v_f}} \right) = nx^2 \frac{a}{v_f} \theta''. \quad (5.3.10)$$

From Eqs. (1.1.6)-(1.1.8) and (1.1.11) the following are obtained:

$$\begin{aligned} A_1 &= \frac{\mu_{nf}}{\mu_f} = (1 - \phi)^{-2.5} \Rightarrow \mu_{nf} = A_1 \mu_f, \quad A_2 = \frac{\rho_{nf}}{\rho_f} = 1 - \phi + \phi \frac{\rho_s}{\rho_f} \Rightarrow \rho_{nf} = A_2 \rho_f, \\ A_3 &= \frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} \Rightarrow k_{nf} = A_3 k_f, \\ A_4 &= \frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = 1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \Rightarrow (\rho C_p)_{nf} = A_4 (\rho C_p)_f. \end{aligned} \quad (5.3.11)$$

The Continuity equation

The equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = af' + (-af') = af' - af' = 0, \quad \text{which satisfies stream function } \psi \quad (5.3.12)$$

The Momentum equation

Substituting Eqs. (5.3.2-5.3.7) and (5.3.11) into Eq. (5.2.2) gives

$$\begin{aligned} (U_\infty f')(af') + (-\sqrt{av_f}f) \left(U_\infty \sqrt{\frac{a}{v_f}} f'' \right) &= U_\infty a + \left(\frac{A_1 \mu_f}{A_2 \rho_f} \right) \left(1 + \frac{1}{\beta} \right) \left(\frac{a U_\infty}{v_f} f''' \right) \\ - \left(\frac{A_1 \mu_f}{A_2 \rho_f k_1} \right) \left(1 + \frac{1}{\beta} \right) (U_\infty f' - U_\infty) &- \frac{F}{A_2 \rho_f \sqrt{k_1}} (U_\infty f' - U_\infty)^2 \end{aligned} \quad (5.3.13)$$

Re-arranging,

$$\begin{aligned} (U_\infty a)((f')^2 - ff'') &= U_\infty a + v_f \frac{A_1}{A_2} \left(1 + \frac{1}{\beta} \right) \left(\frac{U_\infty a}{v_f} f''' \right) \\ - U_\infty \frac{A_1}{A_2} \frac{v_f}{k_1} \left(1 + \frac{1}{\beta} \right) (f' - 1) &- \frac{U_\infty^2 F}{A_2 \rho_f \sqrt{k_1}} (f' - 1)^2 \end{aligned} \quad (5.3.14)$$

Dividing both sides by $U_\infty a$

$$(f')^2 - ff'' = 1 + \frac{A_1}{A_2} \left(1 + \frac{1}{\beta} \right) f''' - \frac{A_1}{A_2} \frac{v_f}{ak_1} \left(1 + \frac{1}{\beta} \right) (f' - 1) - \frac{x F}{A_2 \rho_f \sqrt{k_1}} (f' - 1)^2 \quad (5.3.15)$$

Re-substituting with non dimensional parameters and re-arranging more gives

$$A_1 \left(1 + \frac{1}{\beta} \right) \left[f''' - \frac{1}{Da} (f' - 1) \right] + A_2 (ff'' - (f')^2 + 1) - Fr (f' - 1)^2 = 0 \quad (5.3.16)$$

where,

$$Da = \frac{ak_1}{v_f}, \quad Fr = \frac{x F}{\rho_f \sqrt{k_1}}. \quad (5.3.17)$$

The Energy equation

Substituting Eqs. (5.3.2)-(5.3.11) into the energy Eq. (5.2.3) gives

$$\begin{aligned} (U_\infty f')(2\theta nx) + (-\sqrt{av_f}f)(nx^2 \sqrt{\frac{a}{v_f}} \theta') &= \frac{A_3 k_f}{A_4 (\rho C_p)_f} \left(nx^2 \frac{a}{v_f} \theta'' \right) \\ + \frac{A_1 \mu_f}{A_4 (\rho C_p)_f} \left(1 + \frac{1}{\beta} \right) (U_\infty \sqrt{\frac{a}{v_f}} f'')^2 &+ \frac{A_1 \mu_f}{(A_4 (\rho C_p)_f k_1)} \left(1 + \frac{1}{\beta} \right) (U_\infty f' - U_\infty)^2 \\ + \frac{F}{(A_4 (\rho C_p)_f \sqrt{k_1})} (U_\infty f' - U_\infty)^3 \end{aligned} \quad (5.3.18)$$

Re-arranging the terms gives

$$\begin{aligned} (U_\infty nx f')(2\theta) - (U_\infty nx) f \theta' &= (U_\infty nx) \frac{A_3 k_f}{A_4 v_f (\rho C_p)_f} \theta'' \\ + \frac{U_\infty^2 a A_1}{A_4 (C_p)_f} \left(1 + \frac{1}{\beta} \right) (f'')^2 &+ \frac{U_\infty^2 A_1 v_f}{A_4 (C_p)_f k_1} \left(1 + \frac{1}{\beta} \right) (f' - 1)^2 \\ + \frac{U_\infty^3 F}{A_4 (\rho C_p)_f \sqrt{k_1}} (f' - 1)^3. \end{aligned} \quad (5.3.19)$$

Dividing both sides by $U_\infty nx$ gives

$$2f'\theta - f\theta' = \frac{A_3}{A_4} \frac{k_f}{v_f(\rho C_p)_f} \theta'' + \frac{A_1 U_\infty^2}{A_4 (C_p)_f n x^2} \left(1 + \frac{1}{\beta}\right) (f'')^2 + \frac{A_1 U_\infty^2}{A_4 (C_p)_f n x^2} \frac{v_f}{a k_1} \left(1 + \frac{1}{\beta}\right) (f' - 1)^2 + \frac{U_\infty^2}{A_4 (C_p)_f n x^2} \frac{x F}{\rho_f \sqrt{k_1}} (f' - 1)^3. \quad (5.3.20)$$

Introducing non-dimensional parameters re-arranging more gives

$$A_3 \frac{\theta''}{Pr} + A_4 (f\theta' - 2f'\theta) + A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[(f'')^2 + \frac{1}{Da} (f' - 1)^2 \right] + Ec Fr (f' - 1)^3 = 0, \quad (5.3.21)$$

where

$$Pr = \frac{v_f(\rho C_p)_f}{k_f}, \quad Ec = \frac{U_\infty^2}{(C_p)_f n x^2} = \frac{U_\infty^2}{(C_p)_f (T_f - T_\infty)}. \quad (5.3.22)$$

The boundary conditions are derived as

When $\eta \rightarrow 0$

$$\begin{aligned} u = U_w + \frac{\mu_f}{L} \left(1 + \frac{1}{\beta}\right) \frac{\partial y}{\partial \eta} &\Rightarrow U_\infty f'(0) = bx + \frac{\mu_f}{L} \left(1 + \frac{1}{\beta}\right) \left(\sqrt{\frac{a}{v_f}} U_\infty\right) f''(0) \\ \Rightarrow f'(0) = \frac{b}{a} + \frac{\mu_f}{L} \sqrt{\frac{a}{v_f}} \left(1 + \frac{1}{\beta}\right) f''(0). &\therefore, f'(0) = \lambda + \delta \left(1 + \frac{1}{\beta}\right) f''(0). \\ v = V_0 \Rightarrow -\sqrt{a v} f(0) = V_0 \Rightarrow f(0) = -\frac{V_0}{\sqrt{a v}}. &\therefore, f(0) = S. \end{aligned}$$

and

$$\begin{aligned} -k_f \frac{\partial T}{\partial y} = h_f (T_f - T) &\Rightarrow -k_f (T_f - T_\infty) \sqrt{\frac{a}{v_f}} \theta'(0) = h_f [T_f - (T_\infty (T_f - T_\infty) \theta(0))] \\ \Rightarrow -k_f (T_f - T_\infty) \sqrt{\frac{a}{v_f}} \theta'(0) = h_f (T_f - T_\infty) (1 - \theta(0)) &\Rightarrow \theta'(0) = \frac{h_f}{k_f} \sqrt{\frac{a}{v_f}} (\theta(0) - 1). \\ \therefore, \theta'(0) = Bi [\theta(0) - 1]. \end{aligned}$$

When $\eta \rightarrow \infty$

$$u \rightarrow U_\infty \Rightarrow U_\infty f'(\infty) \rightarrow U_\infty \Rightarrow f'(\infty) \rightarrow 1. \therefore, f'(\infty) = 1.$$

and

$$T \rightarrow T_\infty \Rightarrow (T_f - T_\infty) \theta(\infty) + T_\infty \rightarrow T_\infty \Rightarrow \theta(\infty) \rightarrow 0. \therefore, \theta(\infty) = 0.$$

Collecting the results above, the following nonlinear ordinary differential equations are obtained:

$$A_1 \left(1 + \frac{1}{\beta}\right) \left[f''' - \frac{1}{Da} (f' - 1) \right] + A_2 [f f'' - (f')^2 + 1] - Fr (f' - 1)^2 = 0, \quad (5.3.23)$$

$$A_3 \frac{\theta''}{Pr} + A_4 (f\theta' - 2f'\theta) + A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[(f'')^2 + \frac{1}{Da} (f' - 1)^2 \right] + Ec Fr (f' - 1)^3 = 0, \quad (5.3.24)$$

with the boundary conditions,

$$f(0) = S, f'(0) = \lambda + \delta \left(1 + \frac{1}{\beta}\right) f''(0), \theta'(0) = Bi [\theta(0) - 1], f'(\infty) = 1, \theta(\infty) = 0. \quad (5.3.25)$$

where the parameters, the dimensionless numbers and the variables given as below:

$$\begin{aligned} Da = \frac{a k_1}{v_f}, \quad A_1 = (1 - \phi)^{-2.5}, \quad A_2 = 1 - \phi + \phi \frac{\rho_s}{\rho_f}, \quad Fr = \frac{x F}{\rho_f \sqrt{k_1}}, \\ A_3 = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}, \quad Pr = \frac{v_f(\rho C_p)_f}{k_f}, \quad A_4 = 1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \\ Ec = \frac{U_\infty^2}{(C_p)_f (T_f - T_\infty)}, \quad \delta = \frac{\mu_f}{L} \sqrt{\frac{a}{v_f}}, \quad S = -\frac{V_0}{\sqrt{a v_f}}, \quad Bi = \frac{h_f}{k_f} \sqrt{\frac{a}{v_f}}, \quad \lambda = \frac{b}{a} \end{aligned} \quad (5.3.26)$$

where η is the similarity variable, λ is the velocity ratio (stretching/shrinking) parameter where $\lambda > 0$ for stretching and $\lambda < 0$ for shrinking of the sheet, Da is the Darcy number (porous media parameter), Fr is the Forchheimer (second order porous resistance) parameter, Pr is the Prandtl number, Ec is the Eckert number, S is the constant mass flux parameter where $S > 0$ for suction and $S < 0$ for injection of the fluid, δ is the velocity slip parameter and Bi is the Biot number (convective parameter).

5.4 Physical Quantities of Engineering Interest

The skin friction coefficient, and the local Nusselt number are engineering interest quantities of this problem. The wall shear stress τ_w , and the wall heat flux q_w are defined as:

$$\begin{aligned}\tau_w &= \mu_f \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right) \Big|_{\eta=0} = \mu_f \left(1 + \frac{1}{\beta}\right) \left[U_\infty \sqrt{\frac{a}{v_f}} f''(0)\right], \\ q_w &= -k_f \left(\frac{\partial T}{\partial y}\right) \Big|_{\eta=0} = -k_f (T_f - T_\infty) \sqrt{\frac{a}{v_f}} \theta'(0)\end{aligned}\quad (5.4.1)$$

Thus, coefficient of the local skin friction, and the heat transfer rate (the local Nusselt number) are defined as:

$$C_f \equiv \frac{\tau_w}{\rho U_\infty^2}, \quad Nu_x \equiv \frac{x q_w}{k_f (T_f - T_\infty)}.\quad (5.4.2)$$

To simplify these quantities

$$\begin{aligned}C_f &= \frac{\tau_w}{\rho_f U_\infty^2} = \frac{\mu_f}{\rho_f U_\infty} \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{a}{v_f}} f''(0) = \left(1 + \frac{1}{\beta}\right) \sqrt{\left(\frac{v_f a}{U_\infty^2}\right)} f''(0) \\ &= \left(1 + \frac{1}{\beta}\right) \sqrt{\left(\frac{v_f}{U_\infty x}\right)} f''(0) = \left(1 + \frac{1}{\beta}\right) \sqrt{\frac{1}{Re_x}} f''(0) = \left(1 + \frac{1}{\beta}\right) \frac{1}{\sqrt{Re_x}} f''(0).\end{aligned}\quad (5.4.3)$$

and

$$\begin{aligned}Nu_x &= \frac{x q_w}{k_f (T_f - T_\infty)} = \frac{x}{k_f (T_f - T_\infty)} \left[-k_f (T_f - T_\infty) \sqrt{\frac{a}{v_f}} \theta'(0)\right] = -\sqrt{\frac{ax^2}{v_f}} \theta'(0) \\ &= -\sqrt{\frac{U_\infty x}{v_f}} \theta'(0) = -\sqrt{Re_x} \theta'(0).\end{aligned}\quad (5.4.4)$$

Collecting these results

$$Re_x^{\frac{1}{2}} C_f = \left(1 + \frac{1}{\beta}\right) f''(0), \quad Re_x^{-\frac{1}{2}} Nu_x = -\theta'(0).\quad (5.4.5)$$

where Re_x denotes the local Reynolds number.

To compute the heat transfer enhancement (HTE) of the nanoparticles, the formula below is used (Makinde, 2018; Tadesse et al., 2021a).

$$HTE = \frac{\frac{Nu}{\sqrt{Re_x}}(\phi \neq 0) - \frac{Nu}{\sqrt{Re_x}}(\phi = 0)}{\frac{Nu}{\sqrt{Re_x}}(\phi = 0)} * 100\quad (5.4.6)$$

Where $Re_x = \frac{U_\infty x}{\nu_f}$ is the local Reynolds number.

5.5 Stability Analysis

Fluid flow models of such kind could have dual solutions based on the physical parameters within the problem from the numerical result obtained. Therefore, by making stability analysis, the solution which is stable and physically practicable is determined. Mathematically this analysis is performed to validate the real solution among all the others. To do this, the same procedures are followed, it has been done in Chapter 4. Thus, to employ the stability analysis, Eqs. (5.2.2) and (5.2.3) should be rewritten in unsteady (time dependent) case, according to (Merkin, 1986). Thus,

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} \\ - \frac{\mu_{nf}}{\rho_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty) - \frac{F}{\rho_{nf} \sqrt{k_1}} (u - U_\infty)^2 \end{aligned} \quad (5.5.1)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{nf}}{(\rho C_p)_{nf}} \frac{\partial^2 T}{\partial y^2} + \frac{\mu_{nf}}{(\rho C_p)_{nf}} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial y}\right)^2 \\ + \frac{\mu_{nf}}{(\rho C_p)_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty)^2 + \frac{F}{(\rho C_p)_{nf} \sqrt{k_1}} (u - U_\infty)^3 \end{aligned} \quad (5.5.2)$$

where t denotes the time.

Introducing a new similarity transformation,

$$\begin{aligned} \eta = \sqrt{\frac{a}{\nu_f}} y, \quad \Psi = x \sqrt{a \nu_f} f(\eta, \tau), \quad \theta(\eta, \tau) = \frac{T - T_\infty}{T_f - T_\infty}, \\ u = a x f'(\eta, \tau), \quad v = -\sqrt{a \nu_f} f(\eta, \tau), \quad \tau = a t. \end{aligned} \quad (5.5.3)$$

It is noted that

$$\begin{aligned} \frac{\partial u}{\partial \tau} = \frac{\partial}{\partial \tau} \left(U_\infty \frac{\partial f}{\partial \eta}(\eta, \tau) \right) = U_\infty \frac{\partial^2 f}{\partial \tau \partial \eta}(\eta, \tau), \\ \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial \tau} [T_\infty + (T_f - T_\infty) \theta(\eta, \tau)] = (T_f - T_\infty) \frac{\partial \theta}{\partial \tau}(\eta, \tau). \end{aligned} \quad (5.5.4)$$

Now using Eqs. (5.5.3) and (5.5.4) into Eqs. (5.5.1) and (5.5.2) give

$$\begin{aligned} \left(1 + \frac{1}{\beta}\right) A_1 \left[\frac{\partial^3 f}{\partial \eta^3} - \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right) \right] + A_2 \left[f \frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta} \right)^2 - \frac{\partial^2 f}{\partial \tau \partial \eta} + 1 \right] \\ - Fr \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 = 0 \end{aligned} \quad (5.5.5)$$

$$\begin{aligned} \frac{A_3}{Pr} \frac{\partial^2 \theta}{\partial \eta^2} + A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[\left(\frac{\partial^2 f}{\partial \eta^2} \right)^2 + \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 \right] + A_4 \left(f \frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta} \theta \right) \\ A_4 \frac{\partial \theta}{\partial \tau} + Ec Fr \left(\frac{\partial f}{\partial \eta} - 1 \right)^3 = 0 \end{aligned} \quad (5.5.6)$$

with the boundary conditions

$$\begin{aligned} f(0, \tau) = S, \quad \frac{\partial f}{\partial \eta}(0, \tau) &= \lambda + \delta \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 f}{\partial \eta^2}(0, \tau), \\ \frac{\partial \theta}{\partial \eta}(0, \tau) &= Bi[\theta(0, \tau) - 1], \quad \frac{\partial f}{\partial \eta}(\infty, \tau) \rightarrow 1, \quad \theta(\infty, \tau) \rightarrow 0. \end{aligned} \quad (5.5.7)$$

Based on (Harris et al., 2009), in order to test the stability of solutions of $f(\eta) = f_0(\eta)$ and $\theta(\eta) = \theta_0(\eta)$ that satisfy boundary value problems Eqs. (5.3.23)-(5.3.25) by introducing perturbed functions:

$$\begin{aligned} f(\eta, \tau) &= f_0(\eta) + e^{-\varepsilon\tau} F(\eta, \tau) \\ \theta(\eta, \tau) &= \theta_0(\eta) + e^{-\varepsilon\tau} G(\eta, \tau) \end{aligned} \quad (5.5.8)$$

where ε is an unknown eigenvalue parameter (a small disturbance of growth or decay) that provides an infinite set of the eigenvalues $\varepsilon < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots$ and $F(\eta, \tau)$ and $G(\eta, \tau)$ are small relative to $f_0(\eta)$ and $\theta_0(\eta)$, respectively. Now the higher partial derivatives of Eq.(5.5.8) becomes

$$\begin{aligned} f(\eta, \tau) &= f_0(\eta) + e^{-\varepsilon\tau} F(\eta, \tau) \equiv f = f_0 + e^{-\varepsilon\tau} F \\ \Rightarrow \frac{\partial f}{\partial \eta}(\eta, \tau) &= \frac{\partial f_0}{\partial \eta}(\eta) + e^{-\varepsilon\tau} \frac{\partial F}{\partial \eta}(\eta, \tau) \equiv f' = f'_0 + e^{-\varepsilon\tau} F' \\ \Rightarrow \frac{\partial^2 f}{\partial \eta^2}(\eta, \tau) &= \frac{\partial^2 f_0}{\partial \eta^2}(\eta) + e^{-\varepsilon\tau} \frac{\partial^2 F}{\partial \eta^2}(\eta, \tau) \equiv f'' = f''_0 + e^{-\varepsilon\tau} F'' \\ \Rightarrow \frac{\partial^3 f}{\partial \eta^3}(\eta, \tau) &= \frac{\partial^3 f_0}{\partial \eta^3}(\eta) + e^{-\varepsilon\tau} \frac{\partial^3 F}{\partial \eta^3}(\eta, \tau) \equiv f''' = f'''_0 + e^{-\varepsilon\tau} F''' \\ \Rightarrow \frac{\partial}{\partial \tau} \left(\frac{\partial f}{\partial \eta}(\eta, \tau) \right) &= -\varepsilon e^{-\varepsilon\tau} \frac{\partial F}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial}{\partial \tau} \left(\frac{\partial F}{\partial \eta}(\eta, \tau) \right) \equiv \frac{\partial f}{\partial \tau} = e^{-\varepsilon\tau} \left(-\varepsilon F' + \frac{\partial F'}{\partial \tau} \right) \end{aligned} \quad (5.5.9)$$

and

$$\begin{aligned} \theta(\eta, \tau) &= \theta_0(\eta) + e^{-\varepsilon\tau} G(\eta, \tau) \equiv \theta = \theta_0 + e^{-\varepsilon\tau} G \\ \Rightarrow \frac{\partial \theta}{\partial \eta}(\eta, \tau) &= \frac{\partial \theta_0}{\partial \eta}(\eta) + e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta}(\eta, \tau) \equiv \theta' = \theta'_0 + e^{-\varepsilon\tau} G' \\ \Rightarrow \frac{\partial^2 \theta}{\partial \eta^2}(\eta, \tau) &= \frac{\partial^2 \theta_0}{\partial \eta^2}(\eta) + e^{-\varepsilon\tau} \frac{\partial^2 G}{\partial \eta^2}(\eta, \tau) \equiv \theta'' = \theta''_0 + e^{-\varepsilon\tau} G'' \\ \Rightarrow \frac{\partial \theta}{\partial \tau}(\eta, \tau) &= -\varepsilon e^{-\varepsilon\tau} \frac{\partial G}{\partial \eta} + e^{-\varepsilon\tau} \frac{\partial}{\partial \tau} \left(\frac{\partial G}{\partial \eta}(\eta, \tau) \right) \equiv \frac{\partial \theta}{\partial \tau} = e^{-\varepsilon\tau} \left(-\varepsilon G + \frac{\partial G}{\partial \tau} \right) \end{aligned} \quad (5.5.10)$$

Inserting the results from Eq. (5.5.8) into Eq. (5.5.5) gives

$$\begin{aligned} (1 + \frac{1}{\beta}) A_1 [(f_0''' + e^{-\varepsilon\tau} F''') - \frac{1}{Da} (f_0' + e^{-\varepsilon\tau} F' - 1)] + A_2 [(f_0 + e^{-\varepsilon\tau} F) \cdot \\ (f_0' + e^{-\varepsilon\tau} F') - (f_0' + e^{-\varepsilon\tau} F')^2 - e^{-\varepsilon\tau} (-\varepsilon F' + \frac{\partial F'}{\partial \tau}) + 1] \\ - Fr (f_0' + e^{-\varepsilon\tau} F' - 1)^2 = 0 \end{aligned} \quad (5.5.11)$$

Re-arranging the equation,

$$\begin{aligned} (1 + \frac{1}{\beta}) A_1 \left[f_0''' - \frac{1}{Da} (f_0' - 1) \right] + A_2 \left[f_0 f_0'' - (f_0')^2 + 1 \right] - Fr (f_0' - 1)^2 \\ + \left\{ (1 + \frac{1}{\beta}) A_1 F''' + A_2 f_0 F'' + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} (1 + \frac{1}{\beta}) + 2(A_2 + Fr) f_0' \right) \right] F' \right. \\ \left. - A_2 \frac{\partial F'}{\partial \tau} + A_2 f_0'' F - A_2 \frac{\partial F'}{\partial \tau} \right\} e^{-\varepsilon\tau} + \{ A_2 F F'' - A_2 (F')^2 - Fr (F')^2 \} e^{-2\varepsilon\tau} = 0 \end{aligned} \quad (5.5.12)$$

Since $f \rightarrow f_0$ the first row of (5.5.12) becomes zero and then dividing both sides of the rest equation with $e^{-\varepsilon\tau}$ gives

$$\begin{aligned} & \left(1 + \frac{1}{\beta}\right) A_1 F''' + A_2 f_0 F'' + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) \right] F' \\ & - A_2 \frac{\partial F'}{\partial \tau} + A_2 f_0'' F - A_2 \frac{\partial F'}{\partial \tau} + \{A_2 F F'' - A_2 (F')^2 - Fr (F')^2\} e^{-\varepsilon\tau} = 0 \end{aligned} \quad (5.5.13)$$

For large values of τ , $e^{-\varepsilon\tau} \rightarrow 0$ and hence $\{A_2 F F'' - A_2 (F')^2 - Fr (F')^2\} e^{-\varepsilon\tau} = 0$ so that (5.5.13) becomes

$$\begin{aligned} & \left(1 + \frac{1}{\beta}\right) A_1 F''' + A_2 f_0 F'' + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) \right] F' \\ & - A_2 \frac{\partial F'}{\partial \tau} + A_2 f_0'' F - A_2 \frac{\partial F'}{\partial \tau} = 0 \end{aligned} \quad (5.5.14)$$

Again using Eqs. (5.5.9) and (5.5.10) into Eq. (5.5.6)

$$\begin{aligned} & \frac{A_3}{Pr} (\theta_0'' + e^{-\varepsilon\tau} G'') + A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[(f_0'' + e^{-\varepsilon\tau} F'')^2 + \frac{1}{Da} (f_0' + e^{-\varepsilon\tau} F')^2 \right] \\ & + A_4 [(f_0 + e^{-\varepsilon\tau} F) (\theta_0' + e^{-\varepsilon\tau} G') - (f_0' + e^{-\varepsilon\tau} F') (\theta_0 + e^{-\varepsilon\tau} G)] \\ & - e^{-\varepsilon\tau} (-\varepsilon G + \frac{\partial G}{\partial \tau}) + Ec Fr (f_0' + e^{-\varepsilon\tau} F' - 1)^3 = 0 \end{aligned} \quad (5.5.15)$$

Arranging Eq. (5.5.15) gives

$$\begin{aligned} & \left\{ \frac{A_3}{Pr} \theta_0'' + A_4 (f_0 \theta_0' - 2f_0' \theta_0) + A_1 Ec \left(1 + \frac{1}{\beta}\right) [(f_0'')^2 + \frac{1}{Da} (f_0' - 1)^2] + Ec Fr (f_0' - 1)^3 \right\} \\ & \left\{ \frac{A_3}{Pr} G'' + A_4 f_0 G' + A_4 (\varepsilon - 2f_0') G + 2A_1 Ec \left(1 + \frac{1}{\beta}\right) f_0' F'' + \left[\frac{2A_1 Ec}{Da} \left(1 + \frac{1}{\beta}\right) \cdot \right. \right. \\ & \left. \left. (f_0' - 1) + 3Ec Fr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F - A_4 \frac{\partial G}{\partial \tau} \right\} e^{-\varepsilon\tau} \\ & \left\{ A_4 (F G' - 2F' G) + A_1 Ec \left(1 + \frac{1}{\beta}\right) [2(F'')^2 + \frac{1}{Da} (F')^2] \right. \\ & \left. + 3E Fr (f_0' - 1) (F')^2 + (Ec Fr (F')^3) e^{-\varepsilon\tau} \right\} e^{-2\varepsilon\tau} = 0 \end{aligned} \quad (5.5.16)$$

From (5.5.16), since $f \rightarrow f_0$ and $\theta \rightarrow \theta_0$ the expression in the first braces goes to zero. Dividing both sides of the rest by $e^{-\varepsilon\tau}$ reduces the equation to

$$\begin{aligned} & \left\{ \frac{A_3}{Pr} G'' + A_4 f_0 G' + A_4 (\varepsilon - 2f_0') G + 2A_1 Ec \left(1 + \frac{1}{\beta}\right) f_0' F'' + \left[\frac{2A_1 Ec}{Da} \left(1 + \frac{1}{\beta}\right) \cdot \right. \right. \\ & \left. \left. (f_0' - 1) + 3Ec Fr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F - A_4 \frac{\partial G}{\partial \tau} \right\} \\ & \left\{ A_4 (F G' - 2F' G) + A_1 Ec \left(1 + \frac{1}{\beta}\right) [2(F'')^2 + \frac{1}{Da} (F')^2] \right. \\ & \left. + 3E Fr (f_0' - 1) (F')^2 + (Ec Fr (F')^3) e^{-\varepsilon\tau} \right\} e^{-\varepsilon\tau} = 0 \end{aligned} \quad (5.5.17)$$

Since for large values of τ , $e^{-\varepsilon\tau} \rightarrow 0$, Eq. (5.5.17) becomes

$$\begin{aligned} & \frac{A_3}{Pr} G'' + A_4 f_0 G' + A_4 (\varepsilon - 2f_0') G + 2A_1 Ec \left(1 + \frac{1}{\beta}\right) f_0' F'' + \left[\frac{2A_1 Ec}{Da} \left(1 + \frac{1}{\beta}\right) \cdot \right. \\ & \left. (f_0' - 1) + 3Ec Fr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F - A_4 \frac{\partial G}{\partial \tau} = 0 \end{aligned} \quad (5.5.18)$$

Therefore, collecting the reduced equation becomes

$$(1 + \frac{1}{\beta}) A_1 F''' + A_2 f_0 F'' + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} (1 + \frac{1}{\beta}) + 2(A_2 + Fr) f_0' \right) \right] F' - A_2 \frac{\partial F'}{\partial \tau} + A_2 f_0'' F - A_2 \frac{\partial F'}{\partial \tau} = 0 \quad (5.5.19)$$

$$\frac{A_3}{Pr} G'' + A_4 f_0 G' + A_4 (\varepsilon - 2f_0') G + 2A_1 Ec (1 + \frac{1}{\beta}) f_0' F'' + \left[\frac{2A_1 Ec}{Da} (1 + \frac{1}{\beta}) \cdot (f_0' - 1) + 3EcFr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F - A_4 \frac{\partial G}{\partial \tau} = 0 \quad (5.5.20)$$

subjected to the boundary conditions:

$$F(0, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(0, \tau) = \delta(1 + \frac{1}{\beta}) \frac{\partial^2 F}{\partial \eta^2}(0, \tau), \quad \frac{\partial G}{\partial \eta}(0, \tau) = Bi G(0, \tau), \quad \frac{\partial F}{\partial \eta}(\infty, \tau) \rightarrow 0, \quad G(\infty, \tau) \rightarrow 0. \quad (5.5.21)$$

Following (Weidman et al., 2006), the initial growth or decay of the solution (5.5.8) can be identified by obtaining the steady state solution setting $\tau = 0$ so that $F = F_0(\eta)$ and $G = G_0(\eta)$ in Eqs. (5.5.19) and (5.5.20), where $0 < F_0(\eta) \ll 1$ and $0 < G_0(\eta) \ll 1$. The stability of the solution depends upon the sign of the smallest eigenvalue ε . If the value of ε_1 is positive, that shows the flow is stable and there is an initial decay. Conversely, if the value of ε is negative, that shows the flow is unstable and illustrates an initial growth of disturbance. The linearized Eigenvalue problem is given by

$$(1 + \frac{1}{\beta}) A_1 F_0''' + A_2 f_0 F_0'' + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} (1 + \frac{1}{\beta}) + 2(A_2 + Fr) f_0' \right) \right] F_0' + A_2 f_0'' F_0 = 0 \quad (5.5.22)$$

$$\frac{A_3}{Pr} G_0'' + A_4 f_0 G_0' + A_4 (\varepsilon - 2f_0') G_0 + 2A_1 Ec (1 + \frac{1}{\beta}) f_0' F_0'' + \left[\frac{2A_1 Ec}{Da} (1 + \frac{1}{\beta}) (f_0' - 1) + 3EcFr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F_0' + A_4 \theta_0' F_0 = 0 \quad (5.5.23)$$

subjecting to boundary conditions:

$$F_0(0) = 0, \quad F_0'(0) = \delta(1 + \frac{1}{\beta}) F_0''(0), \quad G_0'(0) = Bi G_0(0), \quad F_0'(\infty) \rightarrow 0, \quad G_0(\infty) \rightarrow 0 \quad (5.5.24)$$

To obtain the smallest eigenvalues, there is a need to relax one of the boundary conditions in the form of the initial condition as suggested by (Harris et al., 2009). In this problem, $F_0'(\infty) \rightarrow 0$ has been relaxed in the initial form as $F_0''(0) = 1$. The smallest negative eigenvalues point out the initial development of the disturbance and the solution of the flow is unstable. On the other hand, the smallest positive related eigenvalue value shows the fluid flow is stable and physically realizable. Thus, the modified boundary conditions (5.5.24) become

$$F_0(0) = 0, \quad F_0'(0) = \delta(1 + \frac{1}{\beta}) F_0''(0), \quad G_0'(0) = Bi G_0(0), \quad F_0''(0) = 1 \quad (5.5.25)$$

$$F_0'(\infty) \rightarrow 0, \quad G_0(\infty) \rightarrow 0$$

5.6 Procedures for Numerical Methods

To solve Eqs. (5.3.23) and (5.3.24) along with the boundary condition (5.3.25) numerically, the shooting method along with fourth-fifth order Runge-Kutta-Fehlberg technique embedded in the Maple software is used. In order to apply the solver, first the system is reduced as a set of equivalent ordinary differential equations of first order using the substitutions $y(1) = f$ and $y(4) = \theta$ as below:

$$\begin{aligned}
 y(1)' &= f' = y(2), \\
 y(2)' &= f'' = y(3), \\
 y(3)' &= f''' = \frac{1}{Da}(y(2) - 1) - \frac{1}{(1 + \frac{1}{\beta})} \left\{ \frac{A_2}{A_1} [y(1)y(3) - (y(2))^2 + 1] \right. \\
 &\quad \left. - \frac{Fr}{A_1} (y(2) - 1)^2 \right\}, \\
 y(4)' &= \theta' = y(5), \\
 y(5)' &= \theta'' = -\frac{Pr}{A_3} \{ A_4 [y(1)y(5) - 2y(2)y(4)] + A_1 Ec(1 + \frac{1}{\beta}) [(y(3))^2 \\
 &\quad + \frac{1}{Da} (y(2) - 1)^2] + Ec Fr (y(2) - 1)^3 \}.
 \end{aligned} \tag{5.6.1}$$

For the boundary conditions,

$$\begin{aligned}
 ya(1) &= S, \quad ya(2) = \lambda + \delta(1 + \frac{1}{\beta})k_1, \quad ya(3) = k_1, \quad yb(2) = 1, \\
 ya(4) &= k_2, \quad ya(5) = Bi[k_2 - 1], \quad yb(4) = 0.
 \end{aligned} \tag{5.6.2}$$

For the case of stability analysis Eqs. (5.5.22) and (5.5.23) with the boundary conditions (5.5.24), the same procedures are followed as in Chapter 4 to convert into first order ordinary differential equations. New substitutions are introduced to: $y(1) = F_0$, $y(4) = G_0$, $z(1) = f_0$ and $z(4) = \theta_0$:

$$\begin{aligned}
 y(1)' &= F_0' = y(2), \\
 y(2)' &= F_0'' = y(3), \\
 y(3)' &= F_0''' = \frac{1}{A_1(1 + \frac{1}{\beta})} \left\{ \left[\frac{A_1}{Da} (1 + \frac{1}{\beta}) + 2(A_2 + Fr)z(2) - (A_2\epsilon + 2Fr) \right] y(2) \right. \\
 &\quad \left. - A_2 [z(1)y(3) + y(1)z(3)] \right\}, \\
 y(4)' &= G_0' = y(5), \\
 y(5)' &= G_0''' = -\frac{Pr}{A_3} \{ A_4 [[z(1)y(5) + z(5)y(1) + \epsilon y(4)] - 2[y(2)z(4) + z(2)y(4)]] \\
 &\quad + 2A_1 Ec(1 + \frac{1}{\beta})y(3)z(3) + [\frac{2A_1 Ec}{Da} (1 + \frac{1}{\beta})(z(2) - 1) \\
 &\quad + 3Ec Fr(z(2) - 1)^2]y(2) \}.
 \end{aligned} \tag{5.6.3}$$

For the boundary conditions,

$$ya(1) = 0, \quad ya(2) = \delta(1 + \frac{1}{\beta})k_1, \quad ya(3) = k_1, \quad yb(2) = 0, \quad (5.6.4)$$

$$ya(4) = k_2, \quad ya(5) = Bi k_2, \quad ya(3) = 1, \quad b(4) = 0.$$

and

$$za(1) = S, \quad za(2) = \lambda + \delta(1 + \frac{1}{\beta})l_1, \quad za(3) = l_1, \quad zb(2) = 1, \quad (5.6.5)$$

$$a(4) = l_1, \quad za(5) = Bi[l_2 - 1], \quad zb(4) = 0.$$

To determine the unknown initial conditions k_1, k_2, l_1 and l_2 , they are shot for arbitrary slope so that the solution of the system of ODEs satisfies the boundary conditions at ∞ and its accuracy is checked by comparing the calculated quantities with the provided end points. After obtaining these values, the fourth-fifth order Runge-Kutta-Fehlberg technique are applied to solve a system of first-order ODEs in Eq. (5.6.1) with boundary conditions (5.6.2) and determine ε from Eqs. (5.5.22)-(5.5.24). To get the dual solutions, different initial approximates for the values of k_1, k_2 are considered, where all profiles asymptotically satisfy the ∞ boundary conditions.

5.7 Results and Discussion

In this study, the results of the combined effects of velocity ratio (stretching/shrinking) parameter λ , Casson parameter(factor) β , Darcy number Da , Forchheimer (porous medium inertia) parameter Fr , suction/injection parameter S , velocity slip (slipperiness) parameter δ , Prandtl number Pr , Eckert number Ec , Biot number Bi and the nanoparticles volume fraction ϕ on the fluid flow temperature and velocity profiles are demonstrated using plots and computations are made for these embedded parameters. The range of parameters considered in the study are: $0.0 \leq \phi \leq 0.1$, $0.1 \leq \beta \leq 10$, $0.1 \leq Da \leq 20$, $0.1 \leq Fr \leq 5$, $0.02 \leq S \leq 0.5$, $0.1 \leq \delta \leq 0.5$, $0.1 \leq Ec \leq 0.3$, $0.05 \leq Bi \leq 0.15$. Since the single phase nanofluid flow model under investigation is a water base with Copper nanoparticles, the effective Prandtl number of working nanofluid is that of pure water (i.e. $Pr = 6.2$), (Shaw et al., 2020; Upreti et al., 2022), whereas the universal values of parameters used in the present study: $\beta = 10$, $Pr = 6.2$, $\phi = Da = Fr = S = \delta = Ec = Bi = 0.1$. In addition, the occurrence of two solutions for certain ranges of parameter variations are demonstrated for the coefficient of skin friction C_f and Nusselt number (rate of heat transfer) Nu_x in graphs and/or tables for diverse numerical quantities of parameters. The existence of dual solutions due to shrinking surface for certain ranges of parameter variations are shown for the skin friction coefficient using graphs and tables, and also the variations of Nusselt numbers are demonstrated in graphs for different values of change of parameters.

The governing nonlinear ordinary differential equations: Eqs. (5.3.23) and (5.3.24) with the

boundary conditions (5.3.25) are solved numerically using shooting techniques built-in Maple2018 software. The MAPLE solver has been used widely by many researchers to solve the boundary value problem (bvp). To validate this method, the computational results obtained with that of the coefficient of skin friction of preceding works given in (Nazar et al., 2011; Jumana et al., 2020; Tadesse et al., 2021a), as presented in Table 5.7.1 below are compared. It is observed that there is a nice agreement and hence the method is proper for tackling problems of the current study. The velocity and temperature profiles as well as skin friction and Nusselt number are graphically presented.

Table 5.7.1: Comparison of values of skin friction $\sqrt{Re_x}C_f$ for varying values of stretching/shrinking parameter λ with $\phi = 0, \beta \rightarrow \infty, Da \rightarrow \infty, Fr = S = \delta = 0$

λ	Nazar et al. (2011)	Jumana et al. (2020)	Tadesse et al. (2021a)	Present Result
-0.25	1.40224	-	1.402241	1.402240800
0	1.232588	1.232587	1.232588	1.232587800
0.2	1.05113	1.051129	1.051130	1.051129994
0.5	0.71329	0.713295	0.713295	0.713294956
1	0	0.000000	0	0

5.7.1 Existence of Dual Solutions

In Figures 5.7.1-5.7.3, the impacts of different values of the involving parameters on the skin friction for shrinking parameter λ are presented. These figures demonstrate that dual solutions exist with upper (solid curve) and lower (dotted curve) branches for $\lambda > \lambda_c$ and no real solution for $\lambda < \lambda_c$. For shrinking parameter λ , the critical value λ_c is the quantity where the upper and lower branch solutions meet each other. In other words, physically λ_c shows the extent to which the sheet surface can shrink while processing. From these figures it is observed that $|\lambda_c|$ enhances with S and δ , whereas it decreases with increasing values of ϕ , β , Da , and Fr . That means the interval of the shrinking parameter λ for which the similarity solution exists widens as suction and slipperiness of the surface parameters increase, whereas it diminishes as the Casson factor, nanoparticles volume fraction, porous medium and porous medium inertia parameters increase. On the other side of this critical value λ_c , no similar solutions exist because of the boundary layer separation from the surface of the sheet and it is not possible to get the solution using the boundary layer approximations.

In Figures 5.7.1(a), it is seen that for the upper branch solution for the shrinking parameter λ , the value of the skin friction coefficient (surface drag force) lessened with an increment in nanoparticles volume fraction ϕ and reversed as approaching the critical value of the shrinking parameter. In reality, the rising nanoparticle volume fraction means that the nanoparticles and the base fluid

collide with each other, which improves the motion of the nanofluid; as a result, the thickness of the momentum boundary layer diminishes and increases the surface drag force. This result agrees with the computed and tabulated values in Table 5.7.2. For $\lambda = 1$, it is always obtained $f''(0) = 0$ for all values of ϕ since the fluid velocity is the same as velocity of the stretching/shrinking surface of the sheet.

In Figure 5.7.1(b), it is seen that the skin friction coefficient diminishes as the Casson factor β increases for the upper branch solution. This is due to an increment in β , which means that the fluid loses its non-Newtonian behavior and acts like a Newtonian type and hence its velocity enhances, resulting in the reduction of shear stress. Figures 5.7.2(a) and 5.7.2(b) illustrate the coefficient of skin friction reduces by intensifying values of porous medium parameter Da and porous medium inertia parameter Fr . As demonstrated in Figure 5.7.3(a), the coefficient of skin friction augments with increasing values of suction parameter S for the upper branch solution, since suction at the boundary surface retards flow of the fluid and increases the coefficient of skin friction. Figure 5.7.3(b) also demonstrates the skin friction for the shrinking case, and it is observed that for the upper branch solution, the surface drag force diminished with an increment in slipperiness parameter δ , and reversed as it approached the critical value of the shrinking parameter. All these results from 5.7.1(b)-5.7.3(b) best agree with the numerical results in Table 5.7.3.

Figures 5.7.4 demonstrates the presence of a dual solution using the graph of local Nusselt number Nu_x , by taking particular numerical values of viscous dissipation parameter Ec and convective heating parameter Bi , to observe their impact on the process of heat transfer and the intervals of dual solution existence, for the shrinking sheet. It is noted from the model that the energy and momentum equations are coupled and hence the Nusselt number characterizes a dual solution for $\lambda_c < \lambda < 0$ in the case of a shrinking sheet surface as displayed for particular governing parameters in figures. In 5.7.4 (a), it is observed that as viscous dissipation parameter Ec increases, the local Nusselt number enhances for the upper branch solution and drops for lower branch solutions, where the upper branch solution is the one which is stable and physically realizable while the lower branch solution is not. Thus, minimum viscous dissipation (minimum Ec) is an acceptable sign for the rate of heat transfer enhancement of the nanofluid as the literature reveals. The interval of solution widens as the viscous dissipation Ec escalates. Again, 5.7.4(b) reveals that as convective heating parameter Bi increases, the local Nusselt number upsurges for both the upper and lower branch solutions for shrinking sheet surface. That means, the coefficient of heat transfer caused by the hot fluid beneath the sheet is directly associated with the convective heating parameter. How-

ever, unlike the viscous dissipation parameter case, the interval of existence of the dual solution gets narrowed for increasing convective heating parameters. From the computational results of

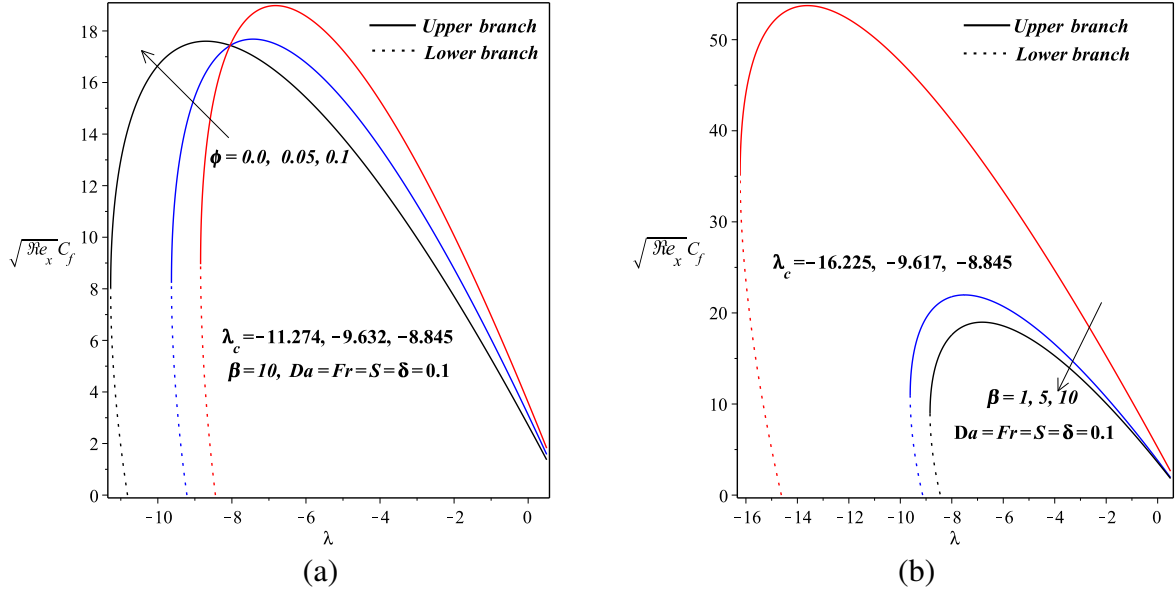


Figure 5.7.1: Skin friction against λ for varying values of: (a) ϕ . (b) β .

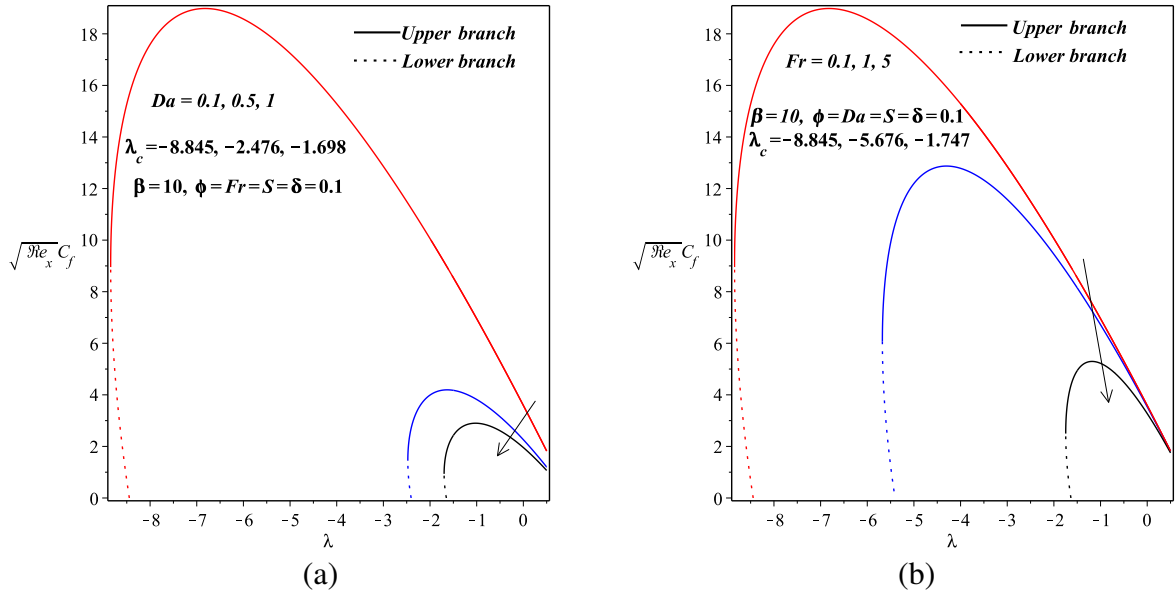


Figure 5.7.2: Skin friction against λ for varying values of: (a) Da . (b) Fr .

this problem, the dual solution exists for some interval of λ . Temporal stability analysis is made to determine stable solutions among different solutions appear due to a shrinking sheet surface. As detailed in Table 5.7.2, for different values of Cu-H₂O nanoparticle volume fraction ϕ and velocity ratio λ the smallest eigenvalue ε is calculated for the temporary change of small perturbations regarding the basic steady flow, where the values of $\beta = 10, Da = Fr = S = \delta = 0.1$ are fixed. Similar calculations were done for fixed $\phi = 0.1$ as presented in Table 5.7.3. It is noted that since the value of the smallest eigenvalue ε is positive, the upper branch solution for shrinking sheet

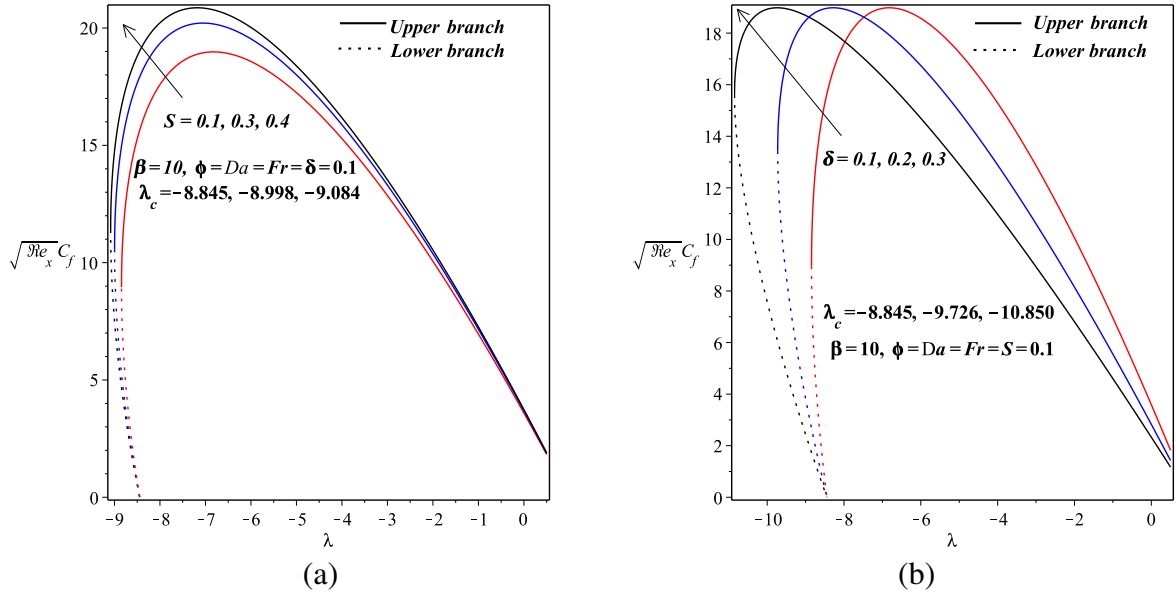


Figure 5.7.3: Skin friction against λ for varying values of: (a) S . (b) δ .

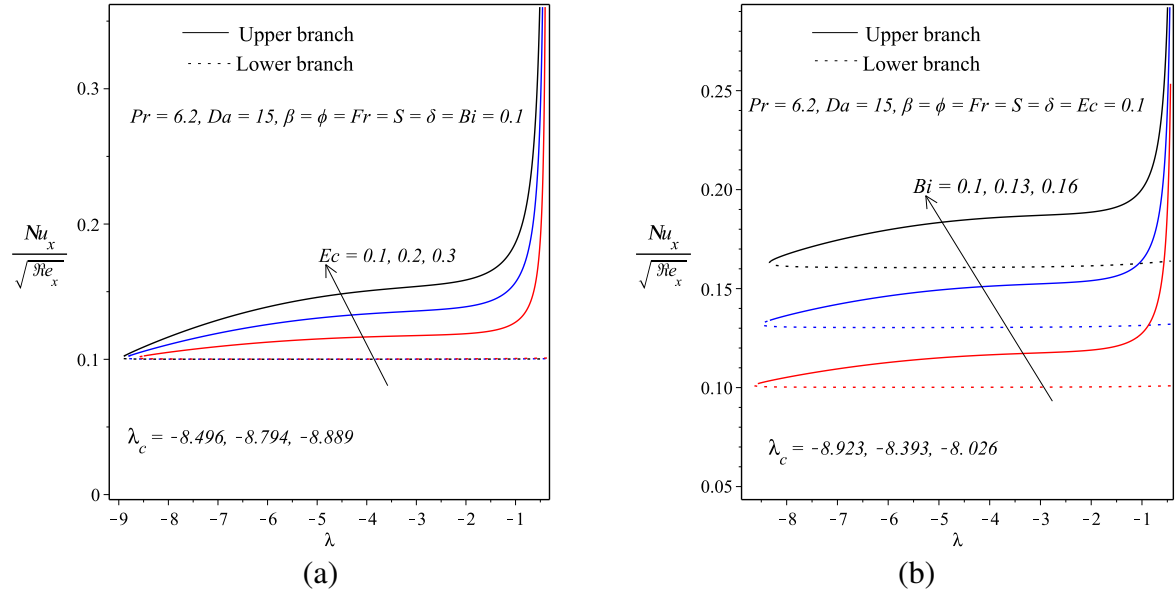


Figure 5.7.4: Nusselt number against λ for varying values of: (a) Ec . (b) Bi .

surface is hydrodynamically temporally stable, and hence, physically realizable. Clearly, for the lower branch solution, the negative value of ε revealed that it is unstable and physically unrealistic. In addition, $\varepsilon > 0$ demonstrates the rate of declination of small disturbance in the upper branch solution, whereas $\varepsilon < 0$ or the lower branch solution shows the increment of the disturbances.

5.7.2 The Skin Friction and Rate of Heat Transfer

Figure 5.7.5 demonstrate the effects of different embedded parameters on the skin friction. Accordingly, from from these figures, it can be seen that as suction and nanoparticles volume fraction increases the skin friction enhances. However, this figure shows that, for increment in the values of Casson parameter β , porous media parameter Da , porous media inertia parameter Fr , slipperiness

Table 5.7.2: The computation showing critical shrinking parameter, skin friction and the smallest eigenvalues for both upper and lower solutions for varying values of ϕ and λ when $\beta = 10, Da = Fr = S = \delta = 0.1$ for Cu-H₂O nanofluid.

Varying Parameters			$\sqrt{Re_x}C_f$		ε	
ϕ	λ_c	λ	Lower Branch	Upper Branch	Lower Branch	Upper Branch
0	-11.274	-11.245	6.1226218	9.7873344	-1.8948508	0.2557403
		-11.250	6.2977932	9.6332684	-1.7985628	0.1584437
		-11.255	6.4913428	9.4608324	-1.6915341	0.0504055
0.05	-9.632	-9.610	6.4645936	9.8859367	-1.4190913	0.2465134
		-9.614	6.6359242	9.7343401	-1.3409342	0.1672097
		-9.617	6.7765913	9.6084782	-1.2764123	0.1018277
0.1	-8.845	-8.821	6.8970069	10.8728661	-1.4272733	0.2188636
		-8.826	7.1259643	10.6726304	-1.3390939	0.1290218
		-8.831	7.3851530	10.4421769	-1.2384452	0.0267103

Table 5.7.3: The computation showing impact of parameters variation on critical shrinking parameter, skin friction and the smallest eigenvalues for both upper and lower solutions for Cu-H₂O nanofluid when $\phi = 0.1$ is fixed.

β	Da	Fr	S	δ	λ_c	λ	$\sqrt{Re_x}C_f$		ε	
							Lower B.	Upper B.	Lower B.	Upper B.
10	0.1	0.1	0.1	0.1	-8.845	-8.821	6.897006	10.872865	-1.427273	0.218863
5	0.1	0.1	0.1	0.1	-9.617	-9.601	9.095182	12.600703	-1.252935	0.108050
1	0.1	0.1	0.1	0.1	-16.225	-16.211	32.431338	37.553862	-1.044620	0.396915
10	0.5	0.1	0.1	0.1	-2.476	-2.440	0.428272	2.368089	-1.375845	0.027543
10	1	0.1	0.1	0.1	-1.698	-1.650	0.004414	1.765601	-1.444670	0.097651
10	0.1	1	0.1	0.1	-5.676	-5.550	2.035769	9.389230	-4.281724	1.600540
10	0.1	5	0.1	0.1	-1.747	-1.700	0.929175	3.800993	-0.491905	8.418994
10	0.1	0.1	0.3	0.1	-8.998	-8.971	8.431013	12.530511	-1.368711	0.331972
10	0.1	0.1	0.4	0.1	-9.084	-9.070	9.872623	12.811560	-1.087406	0.131775
10	0.1	0.1	0.1	0.2	-9.726	-9.712	12.336980	14.416381	-0.757477	0.349037
10	0.1	0.1	0.1	0.3	-10.850	-10.810	14.224548	16.739973	-0.870792	0.845219

parameter δ , and stretching parameter λ , the skin friction diminishes.

Figures 5.7.6 illustrate the impacts of different parameters on the heat transfer rate (Nusselt number Nu). Figure 5.7.6(a) reveals that the intensification in the values of the Casson factor β , porous media parameter Da , slipperiness parameter δ , nanoparticles volume fraction ϕ , and stretching parameter λ resulted in the enhancement of the local Nusselt number for the stretching sheet case.

An increment in the Casson factor β escalates the fluid motion due to slipperiness and drops the thermal profile, resulting in enhanced heat transfer rate, whereas as the porous media parameter increases, both the fluid motion and the thermal profile diminish, which also enhances the Nusselt number (rate of heat transfer) at the stretching sheets surface.

An increment in the values of nanoparticle volume fraction ϕ resulted in an improved the Nusselt number Nu , which in reality means that the rate of heat transfer at the surface of the sheet enhances as nanoparticle volume fraction increases. In other words, the addition of the Cu-H₂O nanoparticles volume fraction produces an increment in the temperature gradient at the sheet surface and results in the rate of heat transfer from the surface of the sheet to the fluid. Again, it can be concluded that the heat transfer properties of the Casson fluid becomes better because of the addition of the Cu-H₂O nanoparticles into it. The same figure also revealed that rising in the values of stretching parameter λ resulted in the ascending of the Nusselt number Nu , which means as stretching of the surface increases, the heat transfer at the surface improves. Moreover, the stepping-up in the values of slip parameter δ enhances the Nusselt number Nu , and hence, the heat transfer rate escalates, which is due to improvement in slipperiness of the sheet surface reducing adhesion of the fluid to it, as observed from the same figure.

Figure 5.7.6(b) explains the impact of porous media inertia parameter Fr , suction parameter S , Eckert number Ec , and Biot number Bi on the Nusselt number for stretching sheet surfaces. It is demonstrated that for all these parameters the heat transfer rate enhances. An increment in suction parameter S boosts the processes of heat transfer rate from the stretching sheet surface, which is due to an increase in the suction rate that upsurges the Cu-H₂O nanoparticles accumulation on the stretching sheet surface that resulted in an enhancement in heat transfer rate (Nusselt number Nu) of the fluid. In other words, the minimum temperature gradient at the sheet surface, which is closely linked with an inverse relation to the heat transfer rate from the surface of the sheet. This shows that higher suction parameter S resulted in a considerably higher heat transfer rate. This figure also illustrates the impact of viscous dissipation on the management of heat transfer from the surface of the sheet.

It is observed that the viscous dissipation parameter Ec intensifies the heat transfer rate falls, which is due to the slipperiness of the surface and the addition of nanoparticles, and moreover, the flow motion increases, and hence the temperature gradient drops, which, in turn, results in an enhanced rate of heat transfer. Due to convective heating of the sheet surface, it is seen that the Nusselt number (heat transfer rate) rises as convective parameter Bi augments, as seen from the same figure.

5.7.3 Heat Transfer Enhancement

The chart in Figure 5.7.7 demonstrates how fast heat transfer is enhanced for the increasing values of Cu-H₂O nanoparticle for the Cu-H₂O-Casson nanofluid flow. For the shrinking sheet, tak-

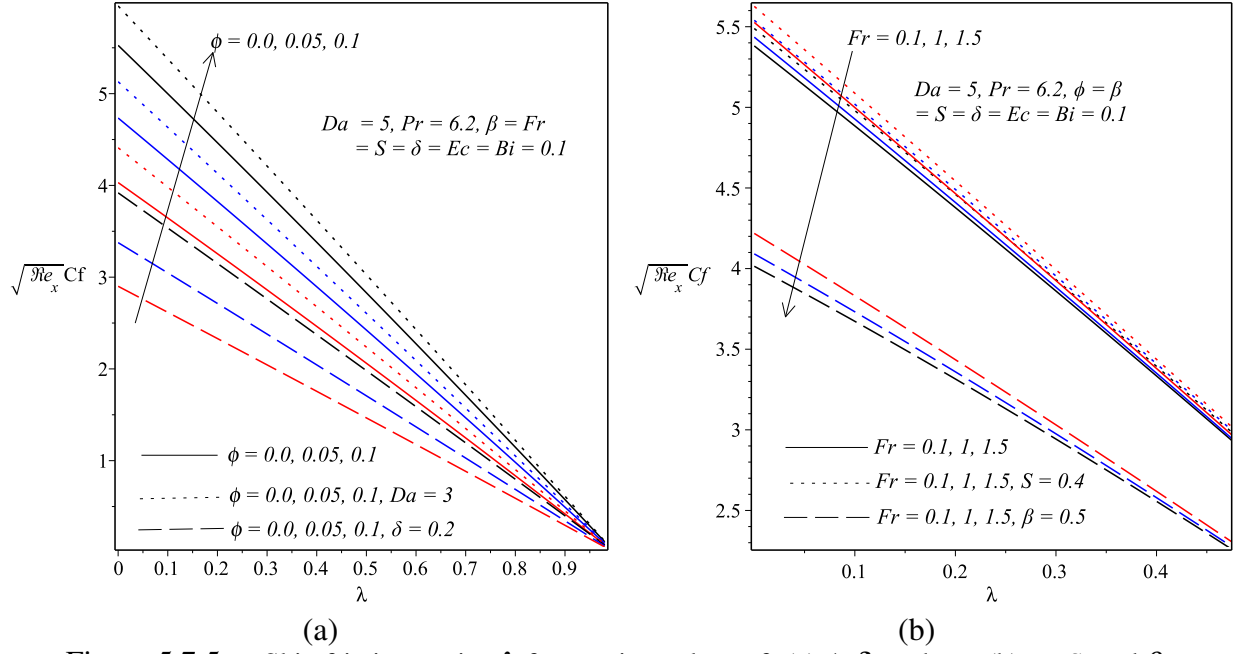


Figure 5.7.5: Skin friction against λ for varying values of: (a) ϕ, δ , and Da (b) Fr, S , and β .

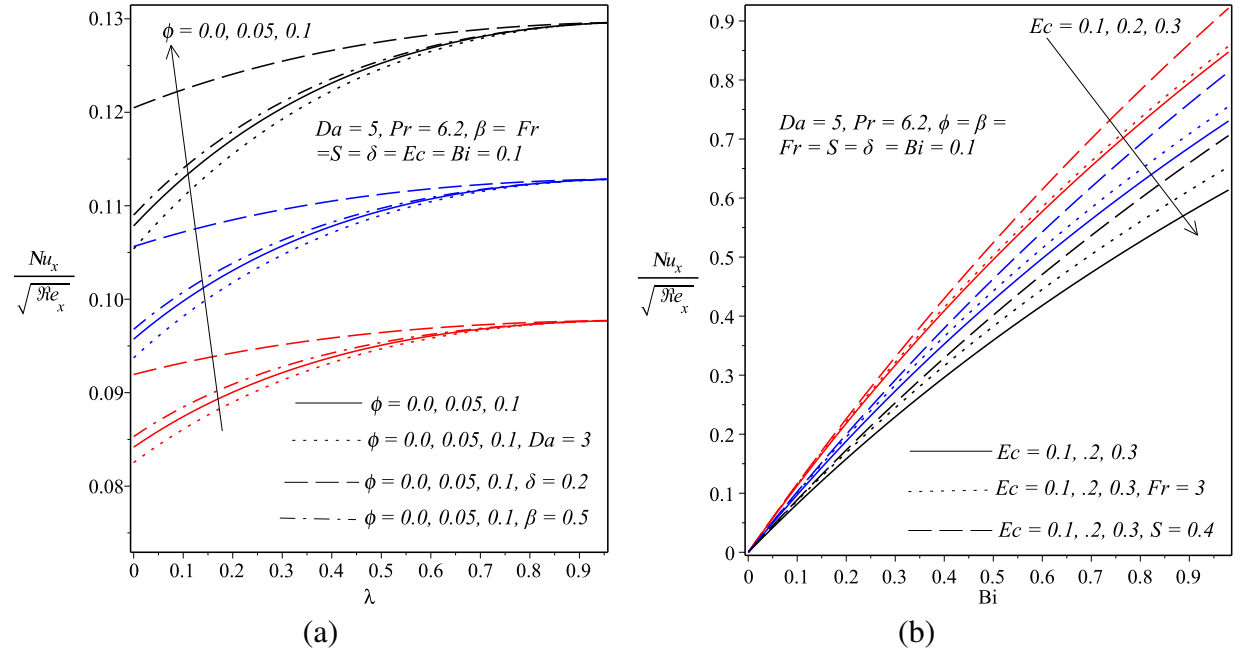


Figure 5.7.6: Nusselt number against: (a) λ for varying values of ϕ, Da, δ , and β . (b) Bi for varying values of Ec, Fr , and S .

ing 10% of the nanoparticles volume fraction, the heat transfer rate is boosted up to 114.5% for Cu-H₂O. Generally, the heat transfer enhancement rate drops from the shrinking sheet towards the stretching sheet surface as observed from the chart. Moreover, the rate of heat transfer enhancement upsurge for working nanofluids with increasing Cu-H₂O nanoparticle volume fraction. Studies revealed that Cu-H₂O nanoparticles serve as a better coolant for industrial and engineering usage when compared to other water-based nanoparticles.

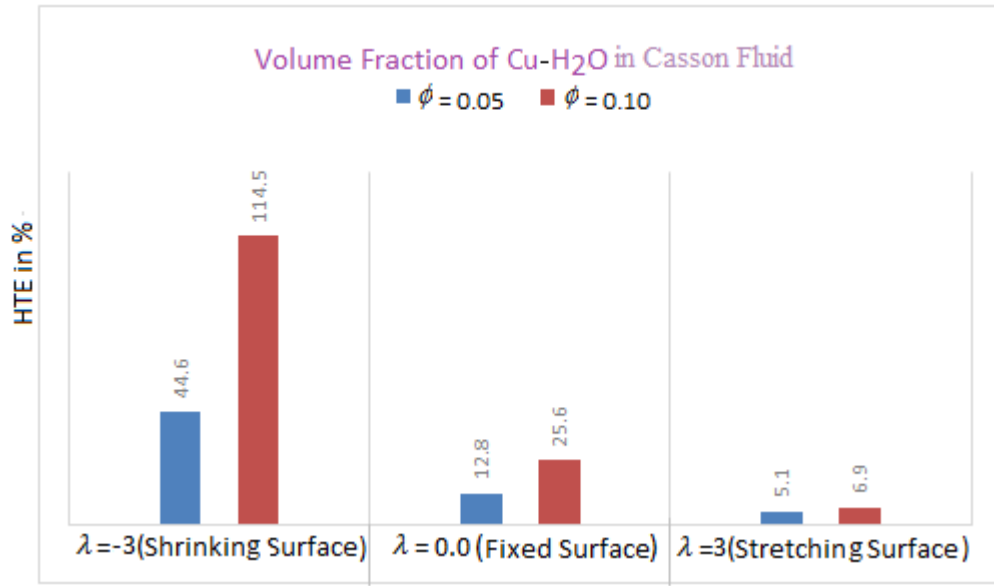


Figure 5.7.7: Rate of heat transfer enhancement with ϕ for fixed values $Pr = 6.2, \beta = 10, Da = Fr = S = \delta = Ec = Bi = 0.1$.

5.7.4 Velocity Profile

Figures 5.7.8 and 5.7.9 present the impacts of different values of velocity ratio (stretching/shrinking) parameter λ , Casson parameter(factor) β , Darcy number (porous media) Da , Forchheimer (porous medium inertia) parameter Fr , suction/injection parameter S , velocity slip (slipperiness) parameter δ , and nanoparticles volume fraction ϕ on fluid velocity profile for the upper branch solution (for the stable upper branch solution only).

Figure 5.7.8(a) displays the effect of Casson factor β on the velocity profile. It is revealed that initially, the velocity profile and the flow boundary layer thickness decrease with increasing β , but after a certain distance from the stretching sheet surface, the velocity profile starts to enhance with increasing β . It is observable that the fluid velocity is minimal near the sheet surface but it tends to be high as far away from the sheet surface. The main cause for getting such a result is that the augmented Casson factor reduces the yield stress (fluid behaves like a Newtonian one when Casson factor becomes very large) which in turn restraints the velocity of the Cu-H₂O-Casson nanofluid. Here, for all Casson factor β , $f'(\eta)$ is an increasing function of η . It is also important to note that when the values of β indefinitely inflate ($\beta \rightarrow \infty$), the non-Newtonian property of the Casson fluid goes, and it acts like a Newtonian fluid flow, and hence, the flow boundary layer thickness of Cu-H₂O-Casson nanofluid drops, and the velocity profile increases far from the surface of the sheet. This also happened due to the plastic viscosity of the Casson fluid, which is inversely proportional to the Casson factor. From the same figure, it is observed that an increment in the

velocity ratio parameter λ resulted in the enhancement of the flow velocity profile and its boundary layer thickness.

In Figure 5.7.8(b), it is observed that diminutive velocity profiles $f'(\eta)$ in response to increment in porous media parameter Da on the boundary layer flow of Cu-H₂O-Casson nanofluid. This is due to the presence of more and more porous matrix producing more and more resistant forces that contribute to more and more retarded flow and shrinking of the corresponding boundary layer thickness. This figure again describes how the porous medium inertia parameter Fr engulfs the velocity profile $f'(\eta)$ within the flow regime. Thus, the increment in porous media inertial resistance looms the velocity profile and the related reduced boundary layer thickness.

Figure 5.7.9(a) depicts that intensifying the values of Cu-H₂O volume fraction ϕ increases the velocity of the flow. That is, the higher ϕ augments the velocity profile due to the spherical-shape of Cu nanoparticle. Physically, the higher ϕ enhances the motion of the spherical-shaped Cu nanoparticles which diminishes the boundary layer thickness, and improves the velocity profile of Cu-H₂O-Casson nanofluid flow. The same figure also shows that improving the value of suction S augments the velocity of the flow, which is because suction is dominated by slipperiness of the surface and reduces the drag on the surface to control the separation of the boundary layer. As the slipperiness parameter δ rises the velocity profile escalates, whereas its boundary layer thickness diminishes as demonstrated in Figure 5.7.9(b).

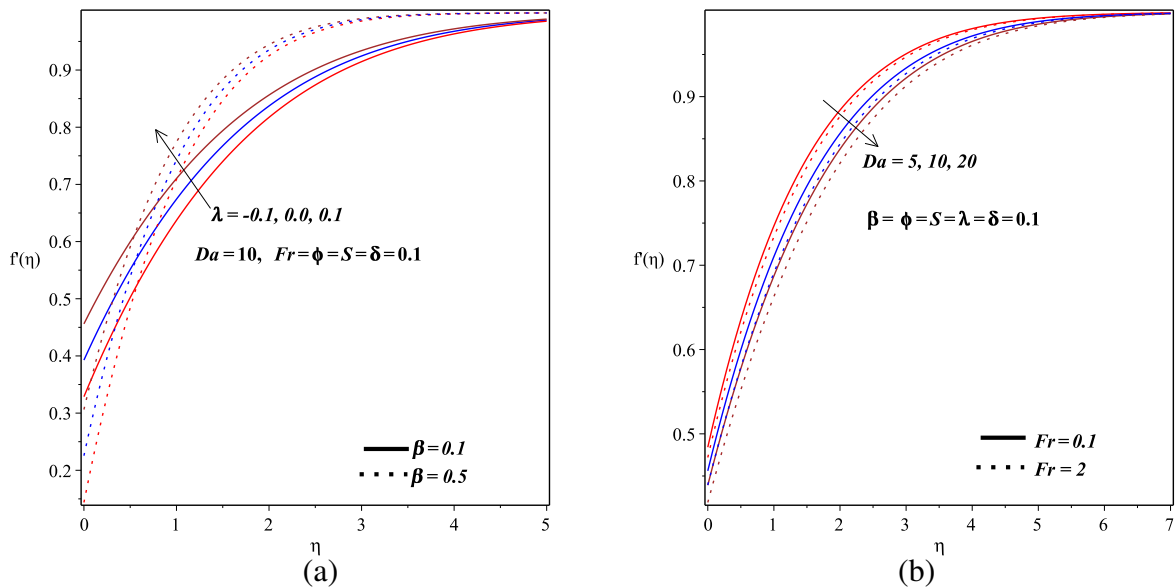


Figure 5.7.8: Velocity profile against η for varying values of: (a) λ and β (b) Da and Fr .

5.7.5 Temperature Profile

The effects of different values of the parameters: velocity ratio (stretching/shrinking) parameter λ , Casson factor β , Darcy number Da , Forchheimer (porous medium inertia) parameter Fr , suc-

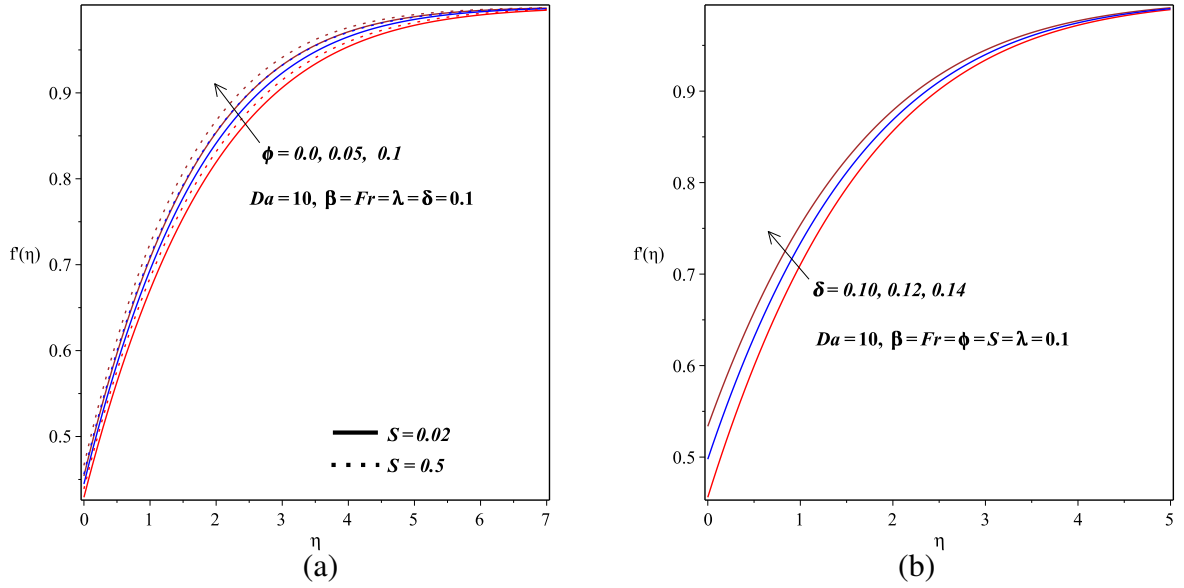


Figure 5.7.9: Velocity profile against η for varying values of: (a) ϕ and S (b) δ .

tion/injection parameter S , velocity slip (slipperiness) parameter δ , Prandtl number Pr , Eckert number Ec , Biot number Bi and nanoparticles volume fraction ϕ on fluid temperature profile for the upper branch solution were demonstrated in Figures 5.7.10-5.7.12(a).

In Figure 5.7.10(a), it is observed that the temperature profiles are belittled for enhancing values of the Casson factor β . As the Casson factor values get higher, the yield stress strength of Cu-H₂O-Casson nanofluid weakens, which, in turn, enhances the plastic dynamic viscosity, and therefore, the thickness of the thermal boundary layer diminishes, for raising values of the Casson factor β . Here, it is noted that the Cu-H₂O-Casson nanofluids thermal boundary layer thickness is greater than that of the Newtonian fluid. It is known that as the values of the Casson factor get higher ($\beta \rightarrow \infty$), the Cu-H₂O-Casson nanofluids act as a Newtonian fluid. Therefore, reducing stress escalates the fluid flow and drops its heat transfer. The same figure also illustrates that an increment in the velocity ratio λ resulted in a diminishing of the temperature profile of the Cu-H₂O-Casson nanofluid flow, which shows that stretching of the surface tends to squeeze the thermal boundary layer of the Cu-H₂O-Casson nanofluid flow.

The temperature profile due to the porous medium parameter Da is demonstrated in 5.7.10(b). The temperature profile falls due to an increment in Da within a few boundary layers of fluid from the surface of the sheet ($\eta < 1.13$), and beyond ($\eta > 1.13$) the temperature profile switches trend due to an increment in Da . This figure also displays the impact of the nonlinear porous medium inertia parameter Fr on the fluid temperature profile. It is observed that an increment in the porous medium inertia parameter produces the falling fluid temperature profile and its thermal boundary layer thickness.

By looking at Figure 5.7.11(a), to observe how the temperature profile behaves in dealing with different Cu-H₂O nanoparticle volume fraction ϕ , and it is seen that as ϕ increases, the temperature profile $\theta(\eta)$ and its thermal boundary layer also increase. Since the Cu nanoparticle has high thermal conductivity, physically, it means increasing nanoparticle volume fraction ϕ upsurges the temperature profile and heat transfer rate. That is, due to enhanced thermal conductivity, Cu-H₂O nanoparticles heighten the thermal enhancement of the Casson nanofluid past the permeable surface. The same figure reveals that augmented fluid suction parameter S produces both a diminutive temperature profile and its thermal boundary layer thickness. As a result, when impacts of viscous dissipation overrun, for augmenting suction parameter S , the thermal boundary layer thickness diminishes. The distribution of the temperature field gets more consistent within the boundary layer of flow because of the impact of suction.

Figure 5.7.11(b) illustrates the impact of viscous dissipation (Eckert number Ec) on temperature profile. There is an indication that an increment in the viscous dissipation parameter is highly beneficial to the growth of temperature profile in the flow region. It is known that viscosity absorbs kinetic energy from the fluid motion and changes it into internal energy that improves heating the fluid flow, hence increasing the temperature profile. Moreover, the same figure demonstrates that augmenting the convective heating (increment in Biot number Bi) enhances the temperature profile of Cu-H₂O-Casson nanofluid flow. That means, the coefficient of heat transfer caused by the hot fluid beneath the sheet is directly associated with the convective heating parameter Bi . That is, behavior of temperature profiles due to convective heating is higher near the boundary layer, implicating that the fluid temperature adjacent to the sheet surface surpasses that of the free stream fluid flow. Thus, the increasing convective heating parameter Bi resulted in an increment in the convective temperature profile of the fluid flow. In Figure 5.7.12(a), it is observed that increasing sheet surface slipperiness δ reduces both the temperature profile and the thermal boundary layer thickness of the Cu-H₂O-Casson nanofluid flow.

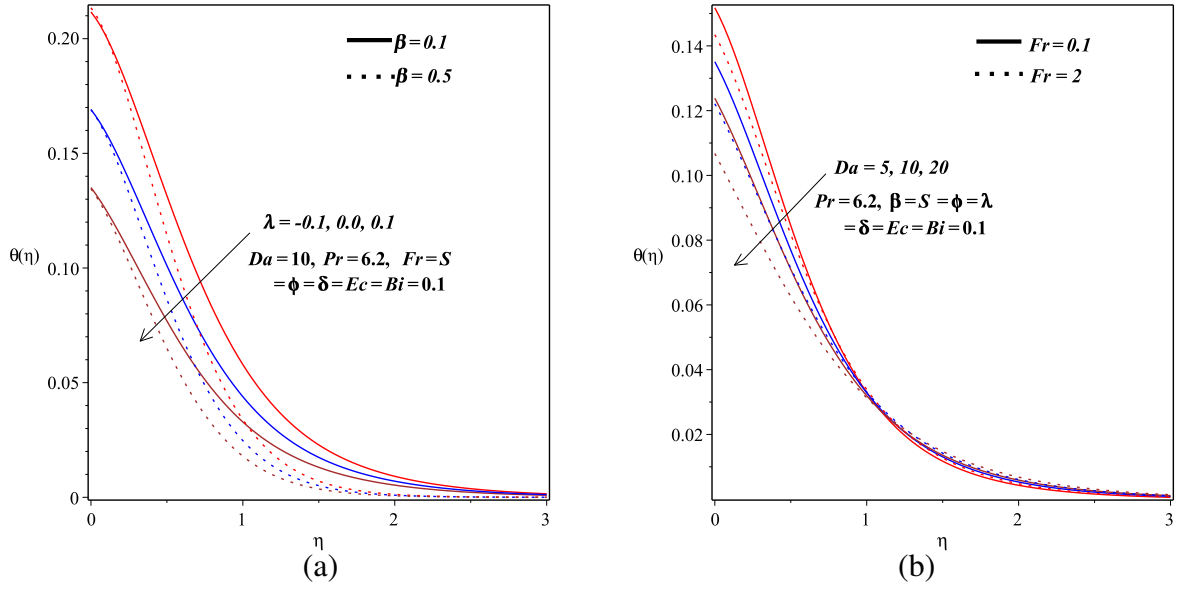


Figure 5.7.10: Temperature profile against η for varying values of: (a) λ and β (b) Da and Fr .

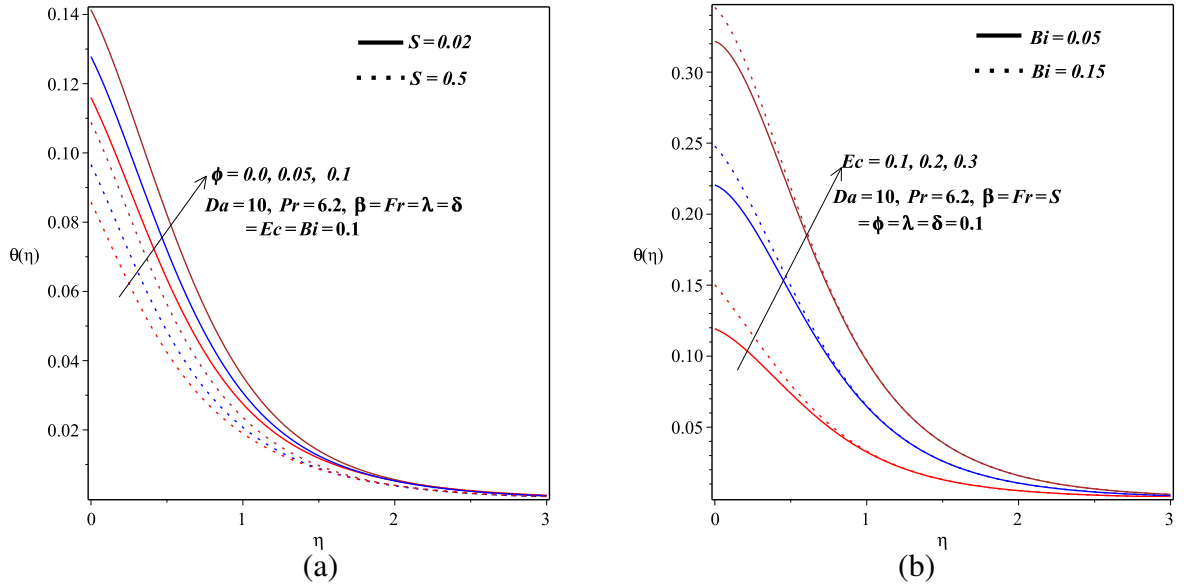


Figure 5.7.11: Temperature profile against η for varying values of: (a) ϕ and S (b) Ec and Bi .

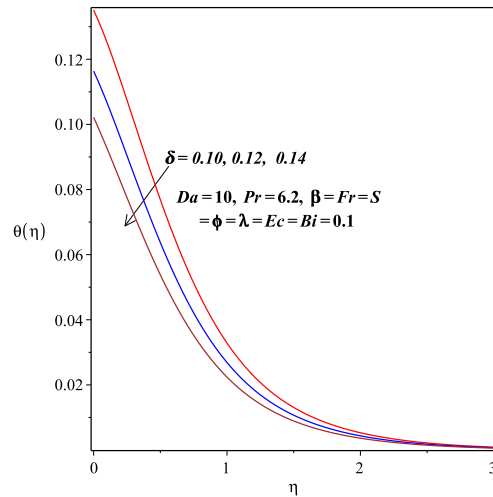


Figure 5.7.12: Temperature profile against η for varying values of δ .

Chapter 6

Stagnation Point Flow of $\text{CoFe}_2\text{O}_4/\text{TiO}_2\text{-H}_2\text{O}$ -Casson Nanofluid Past a Slippery Stretching/Shrinking Cylindrical Surface in a Darcy-Forchheimer Porous Medium

6.1 Introduction

The collective effects of the all embedded parameters under consideration on the complex Casson nanofluid stagnation point flow with heat transfer characteristics is investigated in this chapter. The main goal of this chapter is to investigate the existence of dual solutions, which is expected due to shrinking surfaces, and apply stability analysis to determine the stable and physically reliable solutions that buttress the theoretical relevance of the work for the hydrodynamic Casson nanofluid flow past a stretching/shrinking slippery surface in a Darcy-Forchheimer porous medium with the presence of viscous dissipation and convective heating using $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ and $\text{TiO}_2\text{-H}_2\text{O}$ as nanoparticles in comparison, extending the previous works/articles ([Duguma et al., 2023a,b](#)) with considering cylindrical geometry of flow surface, where the novelty is tested nearer to the critical shrinking parameter $|\lambda_c|$.

Moreover, this study brings significant input for the field of chemical and mechanical engineering sciences. Particularly, nanofluids are widely employed in different cooling systems in engineering and industries for effective heat removal from electronics. For the modeled boundary layer PDEs which were transformed into similar ODEs with their corresponding boundary conditions, the numerical results of similar: velocity and thermal profiles, skin friction coefficient, and rates of heat transfer and enhancement were discussed both graphically and quantitatively using the shooting technique with bvp solver embedded in Maple software packages.

6.2 Mathematical Model Formulation

Consider a laminar, steady, viscous, incompressible two-dimensional stagnation point flow of $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ (Cobalt Ferrite-Water which is magnetic nanoparticle) and $\text{TiO}_2\text{-H}_2\text{O}$ (Titanium Dioxide-Water) Casson nanofluid towards a horizontal linearly stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium with surface velocity $U_w = 2bz$ and free stream stagnation point velocity $U_\infty = 2cz$. The flow physical model presented in Figure 6.2.1, modified from model used by ([Alizadeh et al., 2019](#)), is a convectively heated horizontal cylinder embedded

in porous media. A cylindrical coordinate system is used, where the axial length of the cylinder (z -axis) is considered in the direction of the stretching/shrinking surface and the dimension (r -axis) is the radial change of the cylinder. The free stream temperature of the fluid flow is taken as T_∞ and the convectively heated temperature of the stretching/shrinking surface of the cylinder is $T_f = T_\infty + nz^2$.

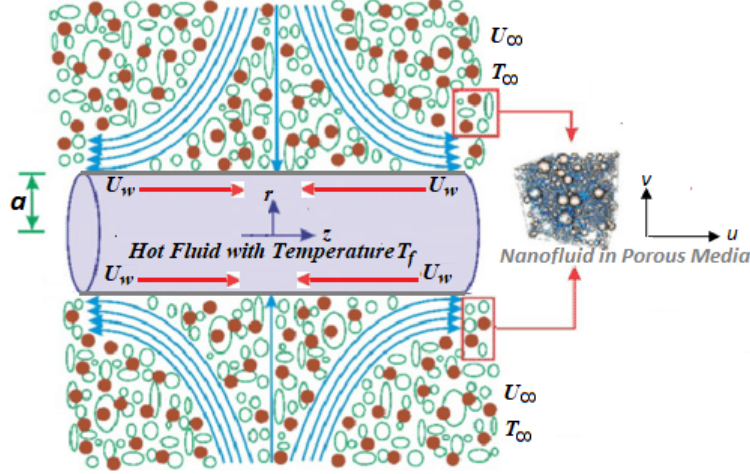


Figure 6.2.1: Schematic diagram of a stationary cylinder with radial stagnation flow and Casson nanofluid in porous media

This problem is an extension of the problems in Chapters 4 and 5 except the changing from Cartesian into cylindrical coordinate system. Following Eqs. (3.1.8), (3.1.28) and (3.1.57) for steady incompressible non-Newtonian (Casson fluid) flow, due to physical diagram orientation switching the role of r and z , with Tiwari-Das convective transport model equations, the Darcy-Forchheimer flow model of flows in porous media, and the governing equations of continuity, the balanced momentum and energy equations are given by:

$$\frac{\partial(ru)}{\partial z} + \frac{\partial(rv)}{\partial r} = 0 \quad (6.2.1)$$

$$u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} = U_\infty \frac{dU_\infty}{dz} + \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) - \frac{\mu_{nf}}{\rho_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty) - \frac{F}{\rho_{nf} \sqrt{k_1}} (u - U_\infty)^2 \quad (6.2.2)$$

$$u \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial r} = \frac{k_{nf}}{(\rho c_p)_{nf}} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}\right) + \frac{\mu_{nf}}{(\rho c_p)_{nf}} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial r}\right)^2 + \frac{\mu_{nf}}{(\rho c_p)_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty)^2 + \frac{F}{(\rho c_p)_{nf} \sqrt{k_1}} (u - U_\infty)^3 \quad (6.2.3)$$

subjected to boundary conditions given by,

$$u(z, a) = U_w(z) + \frac{\mu_f}{L} \left(1 + \frac{1}{\beta}\right) \frac{\partial u}{\partial r}, \quad -k_f \frac{\partial T}{\partial r}(z, a) = h_f [T_f(z) - T(z, a)], \quad (6.2.4)$$

$$v(z, a) = 0, \quad u(z, \infty) \rightarrow U_\infty(z), \quad T(z, \infty) \rightarrow T_\infty.$$

where, u , v , μ_{nf} , β , ρ_{nf} , k_1 , F , k_{nf} , $(\rho C_p)_{nf}$, C_p , μ_f , L , k_f , h_f , a , b , c , and n are the z direction velocity, r direction velocity, effective dynamic viscosity, non-Newtonian/Casson parameter, effective density, permeability of the porous medium, Forchheimer drag force coefficient, effective thermal conductivity, effective heat capacity, specific heat at constant pressure, dynamic viscosity, slip length coefficient, thermal conductivity, convective heat transfer coefficient, radius of the cylinder, constant of strain rate at the cylinder surface, constant of free stream strain rate of the nanoparticles, and real constant (Km^{-2}) of the Casson nanofluid flow, respectively.

6.3 Similarity Transformation

The governing Eqs. (6.2.2) to (6.2.3) are transformed into the dimensionless form by introducing non-dimensional variables defined as below, to obtain the similar solutions (Singh et al., 2021; Waini et al., 2022):

$$\eta = \left(\frac{r}{a}\right)^2, \quad \psi = \frac{a^2 U_\infty}{2} f(\eta) = a^2 c z f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}. \quad (6.3.1)$$

where the stream function is related to a velocity component (Papanastasiou et al., 2021) as:

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} = 2c z f' = U_\infty f', \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial z} = -\frac{a^2 c}{r} f = -\frac{c a}{\sqrt{\eta}} f = -\frac{U_\infty a^2}{2 z r} f. \quad (6.3.2)$$

Thus, from Eqs. (6.3.1) and (6.3.2),

$$\begin{aligned} \frac{\partial u}{\partial r} &= 2c z f'' \cdot \frac{2r}{a^2} = \frac{2U_\infty f'' r}{a^2}, \quad \frac{\partial^2 u}{\partial r^2} = \frac{2 \cdot 2c z c}{a^2} \left[r f''' \left(\frac{2r}{a^2} \right) + f'' \right] = \frac{2U_\infty}{a^2} (2\eta f''' + f''), \\ \frac{\partial u}{\partial z} &= 2c f' = \frac{U_\infty}{z} f', \quad \frac{\partial U_\infty}{\partial z} = 2c = \frac{2c z}{z} = \frac{U_\infty}{z}, \\ \frac{\partial T}{\partial r} &= n z^2 \theta' \cdot \frac{2r}{a^2} = \frac{2n z^2 r}{a^2} \theta' = \frac{2r(T_f - T_\infty)}{a^2} \theta', \\ \frac{\partial^2 T}{\partial r^2} &= \frac{2n z^2}{a^2} \left[r \theta'' \left(\frac{2r}{a^2} + \theta' \right) \right] = \frac{2(T_f - T_\infty)}{a^2} (2\eta \theta'' + \theta'), \quad \frac{\partial T}{\partial z} = 2n z \theta = \frac{2(T_f - T_\infty)}{z} \theta. \end{aligned} \quad (6.3.3)$$

and we define

$$\begin{aligned} A_1 &= \frac{\mu_{nf}}{\mu_f} = (1 - \phi)^{-2.5} \Rightarrow \mu_{nf} = A_1 \mu_f, \\ A_2 &= \frac{\rho_{nf}}{\rho_f} = 1 - \phi + \phi \frac{\rho_s}{\rho_f} \Rightarrow \rho_{nf} = A_2 \rho_f, \\ A_3 &= \frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)} \Rightarrow k_{nf} = A_3 k_f, \\ A_4 &= \frac{(\rho C_p)_{nf}}{(\rho C_p)_f} = 1 - \phi + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f} \Rightarrow (\rho C_p)_{nf} = A_4 (\rho C_p)_f. \end{aligned} \quad (6.3.4)$$

The Continuity equation

Using Eq. (6.3.3) into the equation of continuity,

$$\begin{aligned} \frac{\partial(r \cdot 2czf')}{\partial z} + \frac{\partial(r \cdot \frac{ca}{\sqrt{\eta}}f)}{\partial r} &= 0 \Rightarrow 2crf' - \frac{\partial}{\partial r} \left[ca^2 f \left(\left(\frac{r}{a} \right)^2 \right) \right] = 0 \\ \Rightarrow 2crf' - 2crf' &= 0 \end{aligned} \quad (6.3.5)$$

which satisfies the chosen stream function.

The Momentum Equation

Using Eqs. (6.3.2)-(6.3.4) into Eq. (3.1.2),

$$\begin{aligned} (U_\infty f') \left(\frac{U_\infty}{z} f' \right) + \left(-\frac{U_\infty a^2}{2zr} f \right) \left(2 \frac{U_\infty f'' r}{a^2} \right) &= U_\infty \cdot \frac{U_\infty}{z} + \frac{A_1 \mu_f}{A_2 \rho_f} \left(1 + \frac{1}{\beta} \right) \cdot \\ \left[\frac{2U_\infty}{a^2} (2\eta f''' + f'') + \frac{1}{r} \frac{2U_\infty f'' r}{a^2} \right] - \frac{A_1 \mu_f}{A_2 \rho_f k_1} \left(1 + \frac{1}{\beta} \right) (U_\infty f' - U_\infty) &- \frac{F}{A_2 \rho_f \sqrt{k_1}} \cdot \\ (U_\infty f' - U_\infty)^2 \end{aligned} \quad (6.3.6)$$

Simplifying

$$\begin{aligned} \frac{U_\infty^2}{z} (f')^2 - \frac{U_\infty^2}{z} f f'' &= \frac{U_\infty^2}{z} + \frac{A_1}{A_2} v_f \left(1 + \frac{1}{\beta} \right) \frac{2U_\infty}{a^2} (2\eta f''' + 2f'') \\ - U_\infty \frac{A_1}{A_2 k_1} v_f \left(1 + \frac{1}{\beta} \right) (f' - 1) &- U_\infty^2 \frac{F}{A_2 \rho_f \sqrt{k_1}} (f' - 1)^2 \end{aligned} \quad (6.3.7)$$

Multiplying both sides by $\frac{A_2 z}{U_\infty}$ and arranging more gives

$$\begin{aligned} A_1 \left(1 + \frac{1}{\beta} \right) \left[\frac{2v_f}{ca^2} (\eta f''' + f'') - \frac{v_f}{2ck_1} (f' - 1) \right] &+ A_2 (ff'' - (f')^2 + 1) \\ - \frac{zF}{\rho_f \sqrt{k_1}} (f' - 1)^2 &= 0 \end{aligned} \quad (6.3.8)$$

Substituting with non dimensional parameters gives

$$A_1 \left(1 + \frac{1}{\beta} \right) \left[\frac{1}{Re} (\eta f''' + f'') - \frac{1}{Da} (f' - 1) \right] + A_2 (ff'' - (f')^2 + 1) - Fr(f' - 1)^2 = 0 \quad (6.3.9)$$

where

$$Re = \frac{ca^2}{2v_f}, \quad Da = \frac{2ck_1}{v_f} \quad \text{and} \quad Fr = \frac{zF}{\rho_f \sqrt{k_1}}.$$

The Energy Equation

Using Eqs. (6.3.2)-(6.3.4) into Eq. (6.2.3),

$$\begin{aligned} (U_\infty f') \left(\frac{2(T_f - T_\infty)}{z} \theta \right) + \left(-\frac{U_\infty a^2}{2zr} f \right) \left(\frac{2r(T_f - T_\infty)}{a^2} \theta' \right) &= \\ \frac{A_3 k_f}{A_4 (\rho C_p)_f} \left[\frac{2(T_f - T_\infty)}{a^2} (2\eta \theta'' + \theta') + \frac{1}{r} \frac{2r(T_f - T_\infty)}{a^2} \theta' \right] &+ \frac{A_1 \mu_f}{(A_4 (\rho C_p)_f)} \left(1 + \frac{1}{\beta} \right) \left(\frac{2U_\infty f'' r}{a^2} \right)^2 + \frac{A_1 \mu_f}{A_4 (\rho C_p)_f k_1} \left(1 + \frac{1}{\beta} \right) \cdot \\ (U_\infty f' - U_\infty)^2 + \frac{F}{A_4 (\rho C_p)_f \sqrt{k_1}} (U_\infty f' - U_\infty)^3 \end{aligned} \quad (6.3.10)$$

Re-arranging and simplifying gives

$$\begin{aligned} \frac{U_\infty(T_f - T_\infty)}{z} (2f'\theta - f\theta') &= \frac{A_3}{A_4} \frac{k_f}{(\rho C_p)_f} \frac{4(T_f - T_\infty)}{a^2} (\eta\theta'' + \theta') + \frac{A_1}{A_4} \frac{v_f}{(C_p)_f} \left(1 + \frac{1}{\beta}\right). \\ \frac{4U_\infty^2 r^2}{a^4} (f'')^2 + \frac{A_1}{A_4} \frac{v_f}{(C_p)_f k_1} U_\infty^2 \left(1 + \frac{1}{\beta}\right) (f' - 1)^2 &+ \frac{U_\infty^3 F}{A_4(\rho C_p)_f \sqrt{k_1}} (f' - 1)^3 \end{aligned} \quad (6.3.11)$$

Multiplying both sides by $\frac{z}{U_\infty(T_f - T_\infty)}$ gives

$$\begin{aligned} 2f'\theta - f\theta' &= \frac{A_3}{A_4} \frac{k_f}{(\rho C_p)_f} \frac{4}{a^2 U_\infty} (\eta\theta'' + \theta') + \frac{A_1}{A_4} \frac{v_f}{(C_p)_f} \left(1 + \frac{1}{\beta}\right) \frac{4zU_\infty r^2}{a^4(T_f - T_\infty)} (f'')^2 \\ &+ \frac{A_1}{A_4} \frac{zv_f}{(C_p)_f k_1} \frac{U_\infty}{T_f - T_\infty} \left(1 + \frac{1}{\beta}\right) (f' - 1)^2 + \frac{U_\infty^2}{T_f - T_\infty} \frac{zF}{A_4(\rho C_p)_f \sqrt{k_1}} (f' - 1)^3 \end{aligned} \quad (6.3.12)$$

Simplification and arranging more

$$\begin{aligned} 2f'\theta - f\theta' &= \frac{A_3}{A_4} \frac{k_f}{v_f(\rho C_p)_f} \frac{2v_f}{a^2 c} (\eta\theta'' + \theta') + \frac{A_1}{A_4} \frac{2v_f}{a^2 c} \frac{U_\infty^2}{(C_p)_f(T_f - T_\infty)} \\ &\left(1 + \frac{1}{\beta}\right) \left(\frac{r}{a}\right)^2 (f'')^2 + \frac{A_1}{A_4} \frac{v_f}{2ck_1} \frac{U_\infty^2}{A_4(C_p)_f(T_f - T_\infty)} \left(1 + \frac{1}{\beta}\right) (f' - 1)^2 \\ &+ \frac{U_\infty^2}{(C_p)_f(T_f - T_\infty)} \frac{zF}{(\rho_f \sqrt{k_1})} (f' - 1)^3 \end{aligned} \quad (6.3.13)$$

Substituting with dimensionless parameters

$$\begin{aligned} \frac{A_3}{Pr} (\eta\theta'' + \theta') + A_4 Re(f\theta' - 2f'\theta) + 4A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[\eta f''^2 + \frac{Re}{Da} (f' - 1)^2\right] \\ + 4Re Ec Fr (f' - 1)^3 = 0 \end{aligned} \quad (6.3.14)$$

The boundary conditions are:

When $r = a$, that is, at $\eta = 1$,

$$\begin{aligned} u = u_w + \frac{\mu_f}{L} \left(1 + \frac{1}{\beta}\right) \frac{\partial u}{\partial r} &\Rightarrow U_\infty f' = 2bz + \frac{\mu_f}{L} \left(1 + \frac{1}{\beta}\right) \left(\frac{2U_\infty f'' r}{a^2}\right) \\ \Rightarrow f'(1) = \frac{b}{c} + \frac{\mu_f}{L} \frac{2r}{a^2} \left(1 + \frac{1}{\beta}\right) f''(1) &\Rightarrow f'(1) = \lambda + \delta \left(1 + \frac{1}{\beta}\right) f''(1), \end{aligned} \quad (6.3.15)$$

$$v = 0 \Rightarrow -ca^2 f(1) = 0 \Rightarrow f(1) = 0, \quad (6.3.16)$$

$$\begin{aligned} -k_f \frac{\partial T}{\partial r} &= h_f [T_f(z) - T(z, a)] \\ \Rightarrow -k_f \frac{2r(T_f - T_\infty)}{a^2} \theta'(1) &= h_f \{T_f - [T_\infty + (T_f - T_\infty)\theta(1)]\} \\ \Rightarrow -k_f \frac{2r(T_f - T_\infty)}{a^2} \theta'(1) &= h_f (T_f - T_\infty) [1 - \theta(1)] \\ \Rightarrow -k_f \frac{2r}{a^2} \theta'(1) &= h_f [1 - \theta(1)] \\ \Rightarrow \theta'(1) &= -\frac{a^2 h_f}{2rk_f} \sqrt{\frac{a}{v_f}} (\theta(1) - 1) \Rightarrow \theta'(1) = Bi[\theta(1) - 1]. \end{aligned} \quad (6.3.17)$$

When $r \rightarrow \infty$, that is, at $\eta \rightarrow \infty$,

$$\begin{aligned} u(z, \infty) \rightarrow U_\infty(z) &\Rightarrow 2czf'(\infty) \rightarrow 2cz \Rightarrow f'(\infty) \rightarrow 1, \\ T(z, \infty) \rightarrow T_\infty &\Rightarrow (nz^2)\theta(\infty) + T_\infty \rightarrow T_\infty \Rightarrow \theta(\infty) \rightarrow 0. \end{aligned} \quad (6.3.18)$$

Collecting the above results, give the nonlinear ordinary differential equations below:

$$\frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \left[\eta f''' + f'' + \frac{Re}{Da} (1 - f') \right] + [A_2 (ff'' - f'^2 + 1) - Fr(1 - f')^2] = 0 \quad (6.3.19)$$

$$\begin{aligned} \frac{A_3}{Pr} (\eta \theta'' + \theta') + A_4 Re (f \theta' - 2 f' \theta) + 4 A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[\eta f'^2 + \frac{Re}{Da} (f' - 1)^2 \right] \\ + 4 Re Ec Fr (f' - 1)^3 = 0 \end{aligned} \quad (6.3.20)$$

along with the dimensionless boundary conditions

$$f(1) = 0, f'(1) = \lambda + \delta \left(1 + \frac{1}{\beta}\right) f''(1), \theta'(1) = -Bi[1 - \theta(1)], f'(\infty) \rightarrow 1, \theta(\infty) \rightarrow 0 \quad (6.3.21)$$

where η , Da , Fr , λ , Re , Pr , δ , Ec , and Bi are the similarity variable, Darcy number (porous media parameter), Forchheimer (second order porous resistance) parameter, velocity ratio (stretching/shrinking) parameter (where $\lambda > 0$ for stretching and $\lambda < 0$ for shrinking of the sheet), free stream Reynolds number (note that $Re \equiv \frac{1}{4\gamma^2}$, where $\gamma = \sqrt{\frac{z v_f}{U_\infty a^2}}$ is the curvature of the cylindrical surface (Gorla, 1993; Alizadeh et al., 2020), Prandtl number, velocity slip parameter, Eckert number, and Biot number (convective parameter), respectively. These mentioned parameters are defined as follows:

$$\begin{aligned} Da = \frac{2ck_1}{v_f}, \quad Fr = \frac{zF}{\rho_f \sqrt{k_1}}, \quad Ec = \frac{U_\infty^2}{(C_p)_f (T_f - T_\infty)}, \quad \lambda = \frac{b}{c}, \\ Bi = \frac{a^2 h_f}{2rk_f}, \quad Re = \frac{a^2 c}{2v_f}, \quad Pr = \frac{v_f (\rho C_p)_f}{k_f}, \quad \delta = \frac{2r\mu_f}{a^2 L}, \end{aligned} \quad (6.3.22)$$

6.4 Engineering Interest Parameters

As discussed in Chapters 3 and 4 the important physical quantities of engineering interest in this problem are the dimensionless skin friction coefficient and the local Nusselt number. The wall skin friction τ_w and heat flux q_w are computed as:

$$\begin{aligned} \tau_w = \mu_{nf} \left(1 + \frac{1}{\beta}\right) \frac{\partial u}{\partial r} \Big|_{r=a} = A_1 \mu_f \left(1 + \frac{1}{\beta}\right) \frac{2r U_\infty f''}{a^2} \Big|_{r=a} = A_1 \mu_f \left(1 + \frac{1}{\beta}\right) \left(\frac{2U_\infty f''(1)}{a} \right), \\ q_w = -k_{nf} \frac{\partial T}{\partial r} \Big|_{r=a} = -A_3 k_f \left(\frac{2r(T_f - T_\infty)}{a^2} \theta'(\eta) \right) \Big|_{r=a} = -A_3 k_f \frac{2(T_f - T_\infty)}{a} \theta'(1). \end{aligned} \quad (6.4.1)$$

But the coefficient of skin friction C_f and the local Nusselt number Nu_z are calculated as:

$$\begin{aligned}
C_f &= \frac{\tau_w}{\rho_f U_\infty^2} = A_1 \frac{\mu_f}{\rho_f} \left(1 + \frac{1}{\beta}\right) \left(\frac{2U_\infty f''(1)}{aU_\infty^2}\right) \\
&= A_1 v_f \left(1 + \frac{1}{\beta}\right) \left(\frac{2f''(1)}{aU_\infty}\right) = A_1 v_f \left(1 + \frac{1}{\beta}\right) \left(\frac{2v_f \cdot a}{a^2 c \cdot 2z}\right) f''(1) \\
&= A_1 v_f \left(1 + \frac{1}{\beta}\right) \left(\frac{a}{2z Re}\right) f''(1) = \frac{A_1}{2} \left(1 + \frac{1}{\beta}\right) f''(1) \\
\therefore, \frac{z}{a} Re C_f &= \frac{A_1}{2} \left(1 + \frac{1}{\beta}\right) f''(1) \\
Nu_z &= \frac{z q_w}{k_f (T_f - T_\infty)} = -z A_3 \frac{k_f}{k_f} \left(\frac{2(T_f - T_\infty)}{a(T_f - T_\infty)}\right) \theta'(1) = -2A_3 \frac{z}{a} \theta'(1) \\
\therefore, \frac{a}{z} Nu_z &= -2A_3 \theta'(1)
\end{aligned} \tag{6.4.2}$$

where $Re = \frac{a^2 c}{2v_f}$ represents the Reynolds number (Gorla, 1993).

The heat transfer enhancement (HTE) of the CoFe₂O₄-H₂O and TiO₂-H₂O nanoparticles are computed using the formula, (Makinde, 2018; Tadesse et al., 2021a).

$$HTE = \frac{Nu_z(\phi \neq 0) - Nu_z(\phi = 0)}{Nu_z(\phi = 0)} * 100$$

6.5 Stability Analysis

Stability analysis helps to validate the real solution among different solution could appear while solving the problem. That is, the stability analysis is important to test the reliability of the non-unique similarity solutions. To do this, the same procedure is followed as it has been done in Chapter 4 and 5. Thus, the unsteady state form of Eqs. (6.2.2) and (6.2.3) is given by

$$\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} &= U_\infty \frac{dU_\infty}{dz} + \frac{\mu_{nf}}{\rho_{nf}} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) \\
&\quad - \frac{\mu_{nf}}{\rho_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty) - \frac{F}{\rho_{nf} \sqrt{k_1}} (u - U_\infty)^2
\end{aligned} \tag{6.5.1}$$

$$\begin{aligned}
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial r} &= \frac{k_{nf}}{(\rho c_p)_{nf}} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}\right) + \frac{\mu_{nf}}{(\rho c_p)_{nf}} \left(1 + \frac{1}{\beta}\right) \left(\frac{\partial u}{\partial r}\right)^2 \\
&\quad + \frac{\mu_{nf}}{(\rho C_p)_{nf} k_1} \left(1 + \frac{1}{\beta}\right) (u - U_\infty)^2 + \frac{F}{(\rho C_p)_{nf} \sqrt{k_1}} (u - U_\infty)^3
\end{aligned} \tag{6.5.2}$$

where t denotes the time.

Again, the stability analysis required to introduce of new similarity transformations as mentioned in

$$\begin{aligned}
\eta &= \left(\frac{r}{a}\right)^2, \quad \psi = \frac{aU_\infty}{2} f(\eta, \tau), \quad \theta(\eta, \tau) = \frac{T - T_\infty}{T_f - T_\infty}, \quad \tau = at, \\
u &= U_\infty f'(\eta, \tau), \quad v = -\frac{ca}{\sqrt{\eta}} f(\eta, \tau) = -\frac{U_\infty a^2}{2zr} f(\eta, \tau).
\end{aligned} \tag{6.5.3}$$

and hence

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{\partial}{\partial \tau} \left(U_{\infty} \frac{\partial f}{\partial \eta}(\eta, \tau) \right) = U_{\infty} \frac{\partial^2 f}{\partial \tau \partial \eta}(\eta, \tau), \\ \frac{\partial T}{\partial \tau} &= \frac{\partial}{\partial \tau} \{ [T_{\infty} + (T_f - T_{\infty}) \theta(\eta, \tau)] \} = (T_f - T_{\infty}) \frac{\partial \theta}{\partial \tau}(\eta, \tau).\end{aligned}$$

Inserting Eqs. (6.5.3) and the above equations into Eqs. (6.5.1) and (6.5.2),

$$\begin{aligned}\frac{A_1}{Re} \left(1 + \frac{1}{\beta} \right) \left[\eta \frac{\partial^3 f}{\partial \eta^3} + \frac{\partial^2 f}{\partial \eta^2} + \frac{Re}{Da} \left(1 - \frac{\partial f}{\partial \eta} \right) \right] + A_2 \left(f \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f^2}{\partial \eta} + 1 \right) \\ - A_2 \left[\frac{\partial^2 f}{\partial \tau \partial \eta} - Fr \left(1 - \frac{\partial f}{\partial \eta} \right)^2 \right] = 0\end{aligned}\quad (6.5.4)$$

$$\begin{aligned}\frac{A_3}{RePr} \left(\eta \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial \theta}{\partial \eta} \right) + A_4 \left(f \frac{\partial \theta}{\partial \eta} - 2 \frac{\partial f}{\partial \eta} \theta - \frac{\partial \theta}{\partial \tau} \right) + A_1 Ec \left(1 + \frac{1}{\beta} \right) \cdot \\ \left[\frac{\eta}{Re} \frac{\partial^2 f}{\partial \eta^2} + \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 \right] + Ec Fr \left(\frac{\partial f}{\partial \eta} - 1 \right)^3 = 0.\end{aligned}\quad (6.5.5)$$

with the time dependent boundary conditions

$$\begin{aligned}f(1, \tau) = 0, \quad \frac{\partial f}{\partial \eta}(1, \tau) = \lambda + \delta \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 f}{\partial \eta^2}(1, \tau), \\ \frac{\partial \theta}{\partial \eta}(1, \tau) = -Bi[1 - \theta(1, \tau)], \quad \frac{\partial f}{\partial \eta}(\infty, \tau) \rightarrow 1, \quad \theta(\infty, \tau) \rightarrow 0.\end{aligned}\quad (6.5.6)$$

To test the stability of the steady flow solution of $f(\eta) = f_0(\eta)$ and $\theta(\eta) = \theta_0(\eta)$ that satisfy boundary value problems in Eqs. (6.5.4)-(6.5.6) perturbed functions are introduced [Harris et al. \(2009\)](#).

$$\begin{aligned}f(\eta, \tau) &= f_0(\eta) + e^{-\varepsilon \tau} F(\eta, \tau), \\ \theta(\eta, \tau) &= \theta_0(\eta) + e^{-\varepsilon \tau} G(\eta, \tau).\end{aligned}\quad (6.5.7)$$

where ε is an unknown eigenvalue parameter (a small disturbance of growth or decay) that provides an infinite set of the eigenvalues $\varepsilon < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots$, and $F(\eta, \tau)$ and $G(\eta, \tau)$ are small relative to $f_0(\eta)$ and $\theta_0(\eta)$, respectively. Since the perturbation equations are similar to that of Chapter 5, inserting the results from Eq. (5.5.9) into Eq. (6.5.4),

$$\begin{aligned}\frac{A_1}{Re} \left(1 + \frac{1}{\beta} \right) \left[\eta f_0''' + e^{-\varepsilon \tau} F''' + f_0'' + e^{-\varepsilon \tau} F'' + \frac{Re}{Da} (1 - f_0' + e^{-\varepsilon \tau} F') \right] \\ + A_2 \left[(f_0 + e^{-\varepsilon \tau} F) (f_0'' + e^{-\varepsilon \tau} F'') - (f_0' + e^{-\varepsilon \tau} F')^2 + 1 \right] \\ - A_2 \left[e^{-\varepsilon \tau} \left(-\varepsilon F' + \frac{\partial F'}{\partial \tau} \right) - Fr (1 - f_0' + e^{-\varepsilon \tau} F')^2 \right] = 0\end{aligned}\quad (6.5.8)$$

By re-arranging Eq. (6.5.8), it becomes

$$\begin{aligned}
& \left\{ \frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \left[\eta f_0''' + f_0'' + \frac{Re}{Da} (1 - f_0') \right] + [A_2 (f_0 f_0'' - f_0'^2 + 1) - Fr (1 - f_0')^2] \right\} \\
& \left\{ \frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \eta F''' + \left[\frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) + A_2 f_0 \right] F'' - A_2 \frac{\partial F'}{\partial \tau} \right. \\
& \left. + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) F' \right] + A_2 f_0'' F \right\} e^{-\varepsilon \tau} \\
& \{A_2 (FF'' - (F')^2 - Fr(F')^2)\} e^{-2\varepsilon \tau} = 0
\end{aligned} \tag{6.5.9}$$

Since $f \rightarrow f_0$ the expression in the first braces of Eq. (6.5.9) becomes zero and then dividing both sides of the rest equation with $e^{-\varepsilon \tau}$ gives

$$\begin{aligned}
& \left\{ \frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \eta F''' + \left[\frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) + A_2 f_0 \right] F'' - A_2 \frac{\partial F'}{\partial \tau} \right. \\
& \left. + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) F' \right] + A_2 f_0'' F \right\} \\
& \{A_2 (FF'' - (F')^2 - Fr(F')^2)\} e^{-\varepsilon \tau} = 0
\end{aligned} \tag{6.5.10}$$

For large values of τ , $e^{-\varepsilon \tau} \rightarrow 0$ and hence $\{A_2 (FF'' - (F')^2 - Fr(F')^2)\} e^{-\varepsilon \tau} = 0$ so that Eq. (6.5.10) becomes

$$\begin{aligned}
& \frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \eta F''' + \left[\frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) + A_2 f_0 \right] F'' - A_2 \frac{\partial F'}{\partial \tau} \\
& + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) F' \right] + A_2 f_0'' F = 0
\end{aligned} \tag{6.5.11}$$

Again using Eq. (5.5.10) into Eq. (6.5.5)

$$\begin{aligned}
& \frac{A_3}{RePr} (\eta \theta_0'' + e^{-\varepsilon \tau} G'' + \theta_0' + e^{-\varepsilon \tau} G') + A_4 [(f_0 + e^{-\varepsilon \tau} F) (\theta_0' + e^{-\varepsilon \tau} G') \\
& - 2 (f_0' + e^{-\varepsilon \tau} F') (\theta_0 + e^{-\varepsilon \tau} G) - \frac{\partial \theta}{\partial \tau}] + 4A_1 Ec \left(1 + \frac{1}{\beta}\right) \cdot \\
& \left[\frac{\eta}{Re} (f_0'' + e^{-\varepsilon \tau} F'') + \frac{1}{Da} \left(\frac{\partial f}{\partial \eta} - 1 \right)^2 \right] + 4Ec Fr (f_0' + e^{-\varepsilon \tau} F' - 1)^3 = 0
\end{aligned} \tag{6.5.12}$$

Arranging Eq. (6.5.12) gives

$$\begin{aligned}
& \left\{ \frac{A_3}{RePr} (\eta \theta_0'' + \theta_0') + A_4 (f \theta_0' - 2f' \theta_0) + 4A_1 Ec \left(1 + \frac{1}{\beta}\right) \left[\frac{\eta}{Re} f_0''^2 + \frac{1}{Da} (f_0' - 1)^2 \right] \right. \\
& \left. + 4Ec Fr (f_0' - 1)^3 \right\} + \left\{ \frac{A_3}{RePr} \eta G'' + \left(\frac{A_3}{RePr} + A_4 f_0 \right) G' + A_4 (\varepsilon - 2f_0') G \right. \\
& \left. + 8A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta}\right) f_0'' F'' + \left[\frac{8A_1 Ec}{Da} \left(1 + \frac{1}{\beta}\right) (f_0' - 1) + 12Ec Fr (f_0' - 1)^2 \right. \right. \\
& \left. \left. - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F + A_4 \left(\frac{\partial \theta_0}{\partial \tau} - \frac{\partial G}{\partial \tau} \right) \right\} e^{-\varepsilon \tau} \\
& \{A_4 (FG' - 2F'G) + A_1 \frac{4Ec}{Re} \left(1 + \frac{1}{\beta}\right) \eta [2(F'')^2 + \frac{1}{Da} (F')^2] + 12Ec Fr (f_0' - 1) (F')^2 \\
& + Ec Fr (F')^3\} e^{-2\varepsilon \tau} = 0
\end{aligned} \tag{6.5.13}$$

From Eq. (6.5.13), since $f \rightarrow f_0$ and $\theta \rightarrow \theta_0$ the expression in the first braces goes to zero.

Dividing both sides of the rest by $e^{-\varepsilon\tau}$ reduces the equation to

$$\begin{aligned} & \frac{A_3}{RePr} \eta G'' + \left(\frac{A_3}{RePr} + A_4 f_0 \right) G' + A_4 (\varepsilon - 2f'_0) G \\ & + 8A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta} \right) f_0'' F'' + \left[\frac{8A_1 Ec}{Da} \left(1 + \frac{1}{\beta} \right) (f'_0 - 1) + 12EcFr (f'_0 - 1)^2 \right. \\ & \left. - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F + A_4 \left(\frac{\partial \theta_0}{\partial \tau} - \frac{\partial G}{\partial \tau} \right) \\ & \{ A_4 (FG' - 2F'G) + 4A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta} \right) \eta [2(F'')^2 + \frac{1}{Da} (F')^2] + 12E c F r (f'_0 - 1) (F')^2 \\ & + 4E c F r (F')^3 e^{-\varepsilon\tau} \} e^{-\varepsilon\tau} = 0 \end{aligned} \quad (6.5.14)$$

since for large values of τ , $e^{-\varepsilon\tau} \rightarrow 0$ the expression in the last braces goes to zero, and hence, Eq. (6.5.14) becomes

$$\begin{aligned} & \frac{A_3}{RePr} \eta G'' + \left(\frac{A_3}{RePr} + A_4 f_0 \right) G' + A_4 (\varepsilon - 2f'_0) G + 8A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta} \right) f_0'' F'' \\ & + \left[\frac{8A_1 Ec}{Da} \left(1 + \frac{1}{\beta} \right) (f'_0 - 1) + 12EcFr (f'_0 - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F \\ & + A_4 \left(\frac{\partial \theta_0}{\partial \tau} - \frac{\partial G}{\partial \tau} \right) = 0 \end{aligned} \quad (6.5.15)$$

Therefore, collecting the reduced equation give

$$\begin{aligned} & \frac{A_1}{Re} \left(1 + \frac{1}{\beta} \right) \eta F''' + \left[\frac{A_1}{Re} \left(1 + \frac{1}{\beta} \right) + A_2 f_0 \right] F'' - A_2 \frac{\partial F'}{\partial \tau} \\ & + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta} \right) + 2(A_2 + Fr) f'_0 \right) F' \right] + A_2 f_0'' F = 0 \end{aligned} \quad (6.5.16)$$

$$\begin{aligned} & \frac{A_3}{RePr} \eta G'' + \left(\frac{A_3}{RePr} + A_4 f_0 \right) G' + A_4 (\varepsilon - 2f'_0) G + 8A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta} \right) f_0'' F'' \\ & + \left[\frac{8A_1 Ec}{Da} \left(1 + \frac{1}{\beta} \right) (f'_0 - 1) + 12EcFr (f'_0 - 1)^2 - 2A_4 \theta_0 \right] F' + A_4 \theta_0' F \\ & + A_4 \left(\frac{\partial \theta_0}{\partial \tau} - \frac{\partial G}{\partial \tau} \right) = 0 \end{aligned} \quad (6.5.17)$$

and the corresponding boundary conditions are:

$$\begin{aligned} & F(1, \tau) = 0, \quad \frac{\partial F}{\partial \eta}(1, \tau) = \delta \left(1 + \frac{1}{\beta} \right) \frac{\partial^2 F}{\partial \eta^2}(1, \tau), \quad \frac{\partial G}{\partial \eta}(1, \tau) = Bi G(1, \tau), \\ & \frac{\partial F}{\partial \eta}(\infty, \tau) \rightarrow 0, \quad G(\infty, \tau) \rightarrow 0. \end{aligned} \quad (6.5.18)$$

Following (Weidman et al., 2006), the initial growth or decay of the solution (6.5.7) can be identified by obtaining the steady state solution setting $\tau = 0$ so that $F = F_0(\eta)$ and $G = G_0(\eta)$ in Eqs. (6.5.16) and (6.5.17), where $0 < F_0(\eta) \ll 1$ and $0 < G_0(\eta) \ll 1$. The stability of solution depends upon sign of smallest eigenvalue ε . If value of ε_1 is positive that show the flow is a stable and there is the initial decay. Conversely, if value of ε is negative that shows the flow is the unstable and

illustrate an initial growth of disturbance. The linearized Eigenvalue problem are given by

$$\begin{aligned} & \frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) \eta F_0''' + \left[\frac{A_1}{Re} \left(1 + \frac{1}{\beta}\right) + A_2 f_0 \right] F_0'' \\ & + \left[A_2 \varepsilon + 2Fr - \left(\frac{A_1}{Da} \left(1 + \frac{1}{\beta}\right) + 2(A_2 + Fr) f_0' \right) F_0' \right] + A_2 f_0'' F_0 = 0 \end{aligned} \quad (6.5.19)$$

$$\begin{aligned} & \frac{A_3}{RePr} \eta G_0'' + \left(\frac{A_3}{RePr} + A_4 f_0 \right) G_0' + A_4 (\varepsilon - 2f_0') G_0 + 8A_1 \frac{Ec}{Re} \left(1 + \frac{1}{\beta}\right) f_0'' F_0'' \\ & + \left[\frac{8A_1 Ec}{Da} \left(1 + \frac{1}{\beta}\right) (f_0' - 1) + 12EcFr (f_0' - 1)^2 - 2A_4 \theta_0 \right] F_0' + A_4 \theta_0' F_0 = 0 \end{aligned} \quad (6.5.20)$$

subjected to boundary conditions:

$$F_0(1) = 0, \quad F_0'(1) = \delta \left(1 + \frac{1}{\beta}\right) F_0''(1), \quad G_0'(1) = Bi \, G_0(1), \quad F_0'(\infty) \rightarrow 0, \quad G_0(\infty) \rightarrow 0. \quad (6.5.21)$$

To obtain stability analysis of the solutions, Eq. (6.5.19) and Eq. (6.5.20) of linear eigenvalue problem along the boundary conditions in Eq. (6.5.21) have been solved by using bvp solver functions in the MAPLE software and obtained the smallest eigenvalue ε . To obtain smallest eigenvalues, there is a need to relax one of the boundary conditions in form of the initial condition as suggested by (Harris et al., 2009). In this problem, $F_0'(\infty) \rightarrow 0$ has been relaxed in the initial form as $F_0''(0) = 1$. The smallest negative eigenvalues point out initial development of the disturbance and the solution of the flow is unstable. On other hand, the smallest positive related eigenvalue value shows the fluid flow is stable and physical realizable. Thus, the modified boundary conditions (6.5.21) become

$$\begin{aligned} F_0(1) = 0, \quad F_0'(1) = \delta \left(1 + \frac{1}{\beta}\right) F_0''(1), \quad G_0'(1) = Bi \, G_0(1), \quad F_0''(0) = 1, \\ F_0'(\infty) \rightarrow 0, \quad G_0(\infty) \rightarrow 0 \end{aligned} \quad (6.5.22)$$

6.6 Numerical Procedures

The procedures are similar to that of Chapters 4 and 5.

- Convert the boundary value problem into initial value problem.
- Choose the suitable finite value of η .
- Shooting technique is used to guess missing initial conditions by an iterative process until boundary conditions are satisfied.
- This process was repeated until the results correct up to the desired accuracy are obtained.

6.7 Results and Discussion

In this chapter, the cumulative effects of the velocity ratio (stretching/shrinking) parameter λ , free stream Reynolds number Re , Casson parameter (factor) β , porous media parameter (Darcy number) Da , porous media inertial resistance parameter (Forchheimer parameter) Fr , velocity slip (slipperiness) parameter δ , Prandtl number Pr , viscous dissipation parameter (Eckert number) Ec , convective heating parameter (Biot number), Bi and nanoparticle volume fractions ϕ , on the fluid velocity and temperature profiles, drag force, and heat transfer rate are illustrated graphically in a chart, and numerically computed results are presented in tables. The range of parameters used in this article are: $0.0 \leq \phi \leq 0.1$, $0.1 \leq \beta \leq 10$, $1 \leq Da \leq 10$, $0.1 \leq Fr \leq 0.3$, $0.5 \leq Re \leq 1$, $0.1 \leq \delta \leq 0.15$, $2.2 \leq Pr \leq 6.2$, $0.1 \leq Ec \leq 0.3$, $0.1 \leq Bi \leq 0.14$. Moreover, the appearance of dual solutions for some intervals of varying parameter values is explained for the coefficients skin friction (surface drag force) $\frac{A_1}{2}(1 + 1/\beta)f''(1)$ and Nusselt number (heat transfer rate) $-2A_3\theta'(1)$ in plots and/or tables for various numerical quantities of the involving parameters.

For the governing systems of highly nonlinear ordinary differential Eqs. (6.3.19)-(6.3.20) that cannot be solved analytically, they are numerically tackled using the shooting technique with Maple 2018 (which is coded with a finite difference fourth order accuracy level and solves the boundary value problem - bvp), subjected to initial and far field boundary conditions (6.3.21). To validate this method, the computational results of: i) dimensionless stream function (f) and dimensionless temperature (θ) at $\eta = 2$ (Reddy Gorla, 1976; Alizadeh et al., 2020) presented in Table 6.7.1, and ii) the coefficient of skin friction Reddy Gorla (1976); Wang (1974) presented in Table 6.7.2, are compared. It is seen from the tabulated results that the comparison reveals a better agreement for each value of the dimensionless stream function, dimensionless temperature, and coefficient of skin friction. Therefore, the authors, are guaranteed that the method and results obtained under this study are all valid and acceptable. The velocity, temperature, local Nusselt number, and local skin friction profiles are illustrated graphically and presented in tables.

6.7.1 Existence of Dual Solutions

For the shrinking parameter λ the influences of varying embedded parameters on the skin friction are presented in Figures 6.7.1-6.7.4 for $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ (CF) nanoparticle case only, and this result is compared with that of $\text{TiO}_2\text{-H}_2\text{O}$ (TD) nanoparticle case using Table 6.7.3. Accordingly, it is revealed that dual solutions exist: the upper (solid curve) and lower (dotted curve) solution branches exist for $\lambda > \lambda_c$, and no real solution exists for $\lambda < \lambda_c$. That means, λ_c is the shrinking parameter

Table 6.7.1: Comparison of values of the dimensionless stream function (f) and temperature (θ) at $\eta = 2$ for varying values of the free stream Reynolds number Re when , $\phi = Fr = \lambda = \delta = Ec = 0$, $Pr = 1.0$, $\beta \rightarrow \infty$, $Da \rightarrow \infty$ and $Bi \rightarrow \infty$.

	$f(2)$			$\theta(2)$		
Re	Reddy Gorla (1976)	Alizadeh et al. (2020)	Present Result	Reddy Gorla (1976)	Alizadeh et al. (2020)	Present Result
0.01	0.12075	0.12051	0.1207572	0.84549	0.84557	0.8455064
0.1	0.22652	0.22659	0.2265285	0.73715	0.73701	0.7371582
1	0.46647	0.46683	0.4664705	0.46070	0.46045	0.4606932
10	0.78731	0.78725	0.7873119	0.02970	0.02983	0.0297425

Table 6.7.2: Comparison of values of skin friction $Re^{-\frac{1}{2}} f''(1)$ for varying values of the free stream Reynolds number Re , when $\beta \rightarrow \infty$, $Da \rightarrow \infty$, $Fr = \phi = \lambda = \delta = 0$

Re	Wang (1974)	Reddy Gorla (1976)	Present Result
0.2	1.7577	1.75770	1.75771210
1	1.484185	1.484185	1.48418510
10	1.31643	1.316427	1.31643081
100	-	1.259642	1.25964253

value for the upper and lower solutions, which physically demonstrates the extent to which the surface is able to shrink while in processing. Moreover, Figures 6.7.1-6.7.4, showed that the skin friction is a decreasing function of the shrinking parameter λ for all the parameters involved in the upper branch solutions except nearer to λ_c , whereas it is purely a decreasing function for the lower branch solutions. $|\lambda_c|$ (the range of λ for which the similarity solution appears) increases with the slipperiness parameter δ and higher values of the nanoparticle volume fraction parameter ϕ , whereas it decreases with increasing values of the Casson factor β , the porous media parameter Da , porous media inertia resistance parameter Fr , and Reynolds number Re for the upper branch solutions. Beyond this critical value λ_c , no similarity solutions exist due to the boundary layer separation from the surface, which leads us to the difficulty of using boundary layer approximations to solve the problem. Furthermore, from these figures it is observed that the upper and lower branch solutions for the coefficient of skin friction are in opposite trend except for increment in values of ϕ and δ . Comparing the graphical results with Table 6.7.3, it is revealed that the critical value λ_c is wider for TD with respect to CF nanoparticles.

Figures 6.7.1(a) and 6.7.4(a) demonstrate that for the shrinking parameter λ , the coefficient of skin friction gets enhanced as the nanoparticle volume fraction ϕ increases for the upper branch solution, which could be due to the high coercivity CF. Physically, the augmentation in nanoparticle volume fraction implies that the base fluid and the nanoparticles collision raises the motion of the

nanofluid, resulting in a diminishing of the momentum boundary layer thickness and increasing the drag force at the surface. It is noted that in Figure 6.7.4, 6.7.9(b) and 6.7.10 the thinner and bold curves represent upper and lower branch solutions, respectively.

The coefficient of skin friction falls as the Casson parameter β values increases, for the upper branch solution in Figures 6.7.1(b) and 6.7.4(a). The coefficient of skin friction is observed to drop upon improving the Casson factor β , which means that less applied force is required to move the Casson nanofluid past the surface for higher values of the Casson factor β . In other words, raising in $\beta \rightarrow \infty$ implies that the fluid misses its non-Newtonian properties and behaves like a Newtonian type, and hence its velocity augments due to the reduction in the shear stress.

Figures 6.7.2(a) and 6.7.4(b) reveal that the skin friction decreases as the porous media parameter (Darcy number Da) gets higher for the upper branch solution. Again, the skin friction drops as the porous media inertial parameter Fr increases for the upper branch solution as illustrated in Figures 6.7.2(b) and 6.7.4(b).

As demonstrated in Figures 6.7.3(a) and Figure 6.7.4(b) the coefficient of skin friction drops with an increment in the values of the Reynolds number Re for the upper branch solution. The skin friction (surface drag force) diminishes with an increment in the values of the slipperiness parameter δ as illustrated in Figures 6.7.3(b) and 6.7.4(b) for the upper branch solution.

Generally, the above results are observed nearer to the critical shrinking parameter λ_c values as illustrated by the plots, the computed results from Tables 6.7.3 and 6.7.4 support what is discussed above. From Table 6.7.5, far from λ_c (say at $\lambda = -0.1$), it is revealed that coefficient of the skin friction increases only with rising values of ϕ and Re but drops with other parameters. Moreover, it is observable that the skin friction (surface drag force) coefficient is higher for CF compared to TD near the critical λ_c , as it is observed from Table 6.7.4.

From the numerical results of this problem, the dual solution exists for some interval of λ . The stability analysis is made to determine stable solutions within different solutions that arise due to shrinking cylindrical surfaces. As detailed in Table 6.7.4, for varying values of involving parameters, the smallest eigenvalue ε is calculated for the temporary change of small disturbances/perturbations regarding the basic steady flow, with respect to the fixed values of $Da = 10, Re = 0.5, \phi = \beta = Fr = \delta = 0.1$. From the table, it is observed that corresponding to the upper branch solutions, the smallest eigenvalue ε obtained is positive for shrinking surfaces, implicating that the upper branch solution is hydrodynamically temporally stable and therefore physically realizable. Clearly, for the lower branch solution, the negative value of ε revealed that it is unsta-

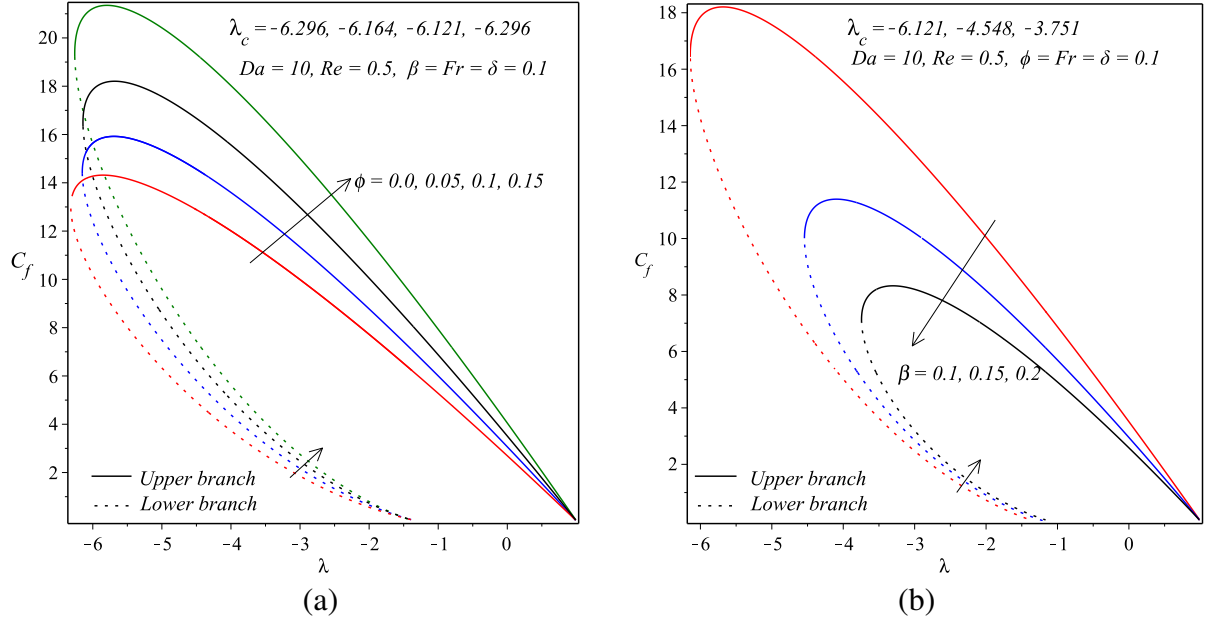


Figure 6.7.1: Skin friction against λ for varying values of: (a) ϕ . (b) β .

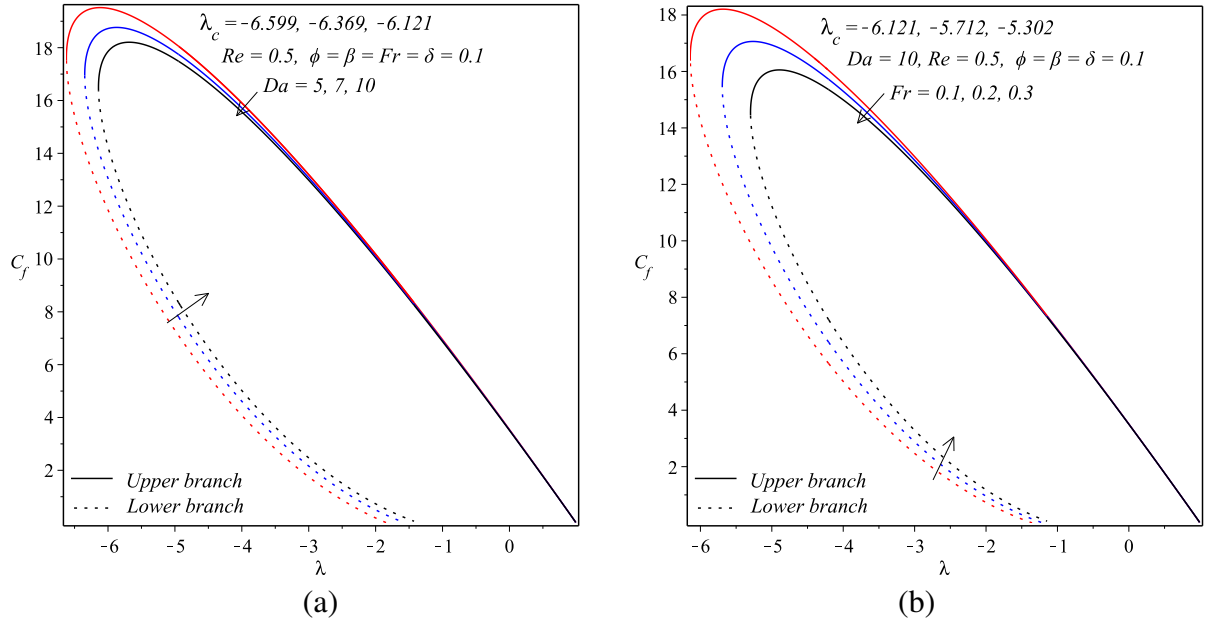


Figure 6.7.2: Skin friction against λ for varying values of: (a) Da . (b) Fr .

ble and physically unachievable. In addition, $\varepsilon > 0$ demonstrates the rate of declination of small disturbances on the upper branch solution, whereas $\varepsilon < 0$ for the lower branch solution shows the enhancement of the disturbances.

6.7.2 Rate of Heat Transfer

Figures 6.7.5-6.7.10 and Table 6.7.4 explain the effects of all the parameters under discussion on the heat transfer rate (local Nusselt number Nu). It is noted from the model that the energy and momentum equations are coupled, and hence the Nusselt number characterizes a dual solution for $\lambda_c < \lambda < 0$ for the case of shrinking surfaces. Moreover, it is revealed that dual solutions

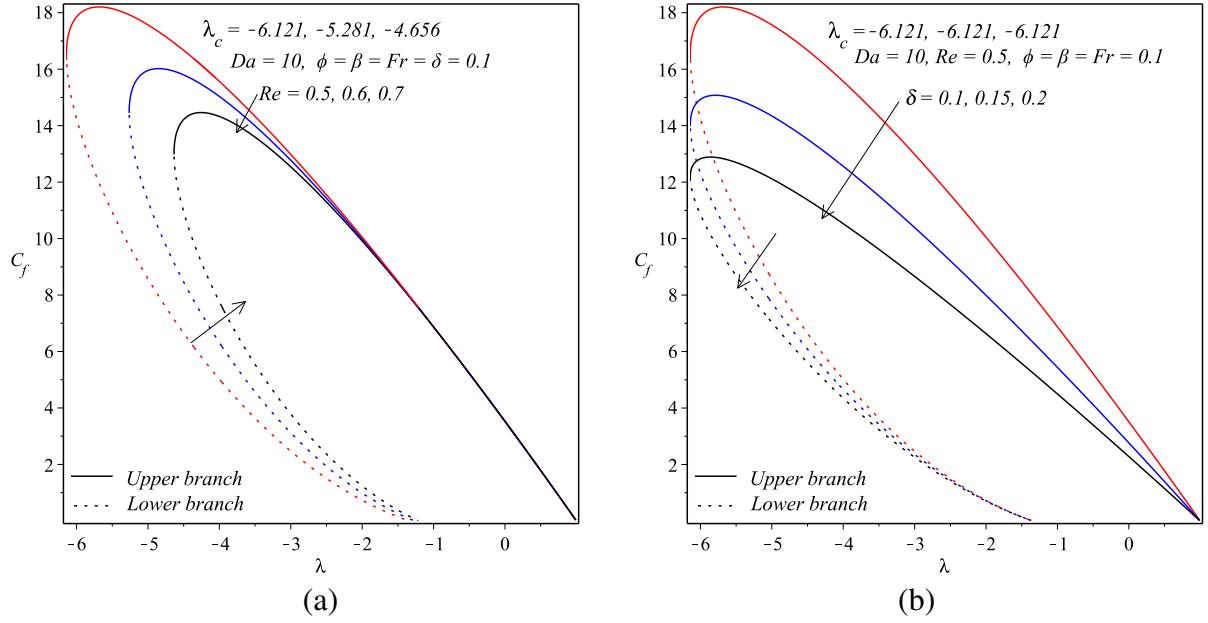


Figure 6.7.3: Skin friction against λ for varying values of: (a) Re . (b) δ .

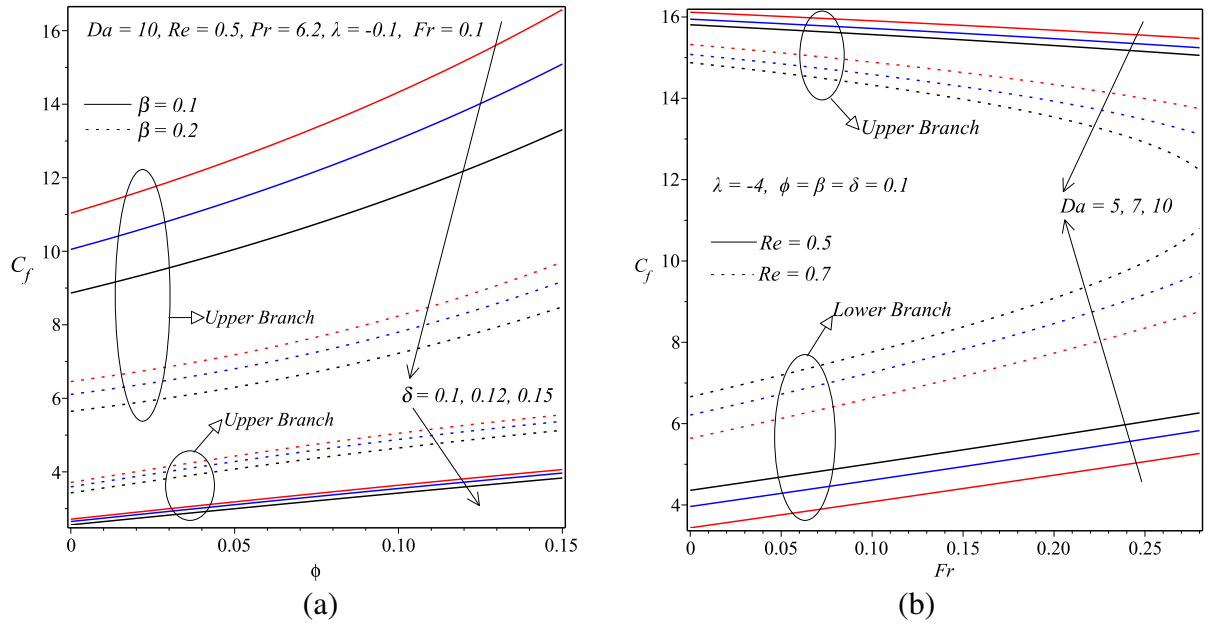


Figure 6.7.4: Skin friction against: (a) ϕ with varying β and δ (b) Da with varying Fr , and Re .

exist and that the upper branch solutions (represented by the solid curve) for the Nusselt number are an increasing function of the shrinking parameter λ , whereas the opposite trend is observed for the lower branch solutions. The critical shrinking parameter $|\lambda_c|$ for the solution decreases for increasing values of nanoparticle volume fraction, porous media inertia parameter, and Biot numbers, whereas for the Casson, Reynolds, slipperiness, and velocity ratio parameters, Darcy, Prandtl, and Eckert numbers, it gets widened for the upper branch solutions.

It is observable that rising values of nanoparticle volume fraction ϕ resulted in an ascending heat transfer rate (Nusselt number Nu) for the upper branch solution, as seen in Figures 6.7.5(a)

Table 6.7.3: The computational results of critical shrinking parameter and skin friction for varying values of parameters for both upper branch (UB) and lower branch (LB) solutions.

Values		CoFe ₂ O ₄ -H ₂ O nanoparticle case				TiO ₂ -H ₂ O nanoparticle case			
		Shrinking Parameter		Skin Friction, $\frac{z}{a}ReC_f$		Shrinking Parameter		Skin Friction, $\frac{z}{a}ReC_f$	
		λ_c	λ	LB	UB	λ_c	λ	LB	UB
ϕ	0.0	-6.296	-6.291	7.804424	13.41113	-6.296	-6.291	7.441272	13.41113
	0.05	-6.163	-6.160	8.751245	14.71521	-6.291	-6.287	8.589124	14.79627
Da	5	-6.599	-6.594	9.824713	18.23317	-6.858	-6.853	10.88821	18.54962
	7	-6.369	-6.364	9.973547	17.17354	-6.597	-6.594	10.77125	17.79559
		-6.121	-6.117	10.08922	17.06109	-6.358	-6.353	10.398941	17.39894
β	0.15	-4.548	-4.544	5.432811	10.35182	-4.733	-4.730	6.055124	10.41189
	0.2	-3.751	-3.747	4.017591	7.557561	-3.887	-3.883	4.002217	7.609331
Fr	0.2	-5.712	-5.708	8.421154	15.75589	-5.913	-5.910	10.00817	16.10817
	0.3	-5.302	-5.298	7.798566	14.99881	-5.490	-5.486	8.507728	15.19526
Re	0.6	-5.281	-5.278	8.672531	14.76315	-5.467	-5.462	8.009925	15.30241
	0.7	-4.656	-4.652	7.121718	13.30115	-4.822	-4.817	7.553292	13.63655
δ	0.15	-6.121	-6.117	8.500941	14.44912	-6.358	-6.353	8.522848	14.71149
	0.2	-6.121	-6.117	7.298143	12.41321	-6.358	-6.353	7.811162	12.72238

Table 6.7.4: The computational results for critical shrinking parameter and the smallest eigenvalues ε of both upper branch (UB) and lower branch (LB) solutions.

Values		CoFe ₂ O ₄ -H ₂ O nanoparticle case				TiO ₂ -H ₂ O nanoparticle case			
		Shrinking Parameter		Eigenvalue ε		Shrinking Parameter		Eigenvalue ε	
		λ_c	λ	LB	UB	λ_c	λ	LB	UB
ϕ	0.0	-6.296	-6.291	-0.433091	4.128013	-6.337	-6.333	-0.889731	4.241711
	0.05	-6.163	-6.160	-0.454080	3.850957	-6.291	-6.287	-0.725491	3.715033
Da	5	-6.599	-6.594	-0.573897	4.644802	-6.858	-6.853	-0.048141	4.671653
	7	-6.369	-6.364	-0.533424	3.956016	-6.597	-6.594	-0.170224	4.156063
		-6.121	-6.117	-0.335706	3.909736	-6.358	-6.353	-0.365095	3.937407
β	0.15	-6.121	-6.117	-0.335706	3.909736	-6.358	-6.353	-0.365095	3.937407
	0.2	-3.751	-3.747	-0.171095	1.350239	-3.887	-3.883	-0.319463	2.738349
Fr	0.2	-5.712	-5.708	-1.096991	3.821874	-5.913	-5.910	-0.067453	3.898320
	0.3	-5.302	-5.298	-1.030511	4.037972	-5.490	-5.486	-0.663476	4.042332
Re	0.6	-5.281	-5.278	-0.309304	3.469464	-5.467	-5.462	-0.991754	3.650911
	0.7	-4.656	-4.652	-0.637060	3.288788	-4.822	-4.817	-0.478861	3.375309
δ	0.15	-6.121	-6.117	-0.773934	3.939137	-6.358	-6.353	-1.050795	3.932585
	0.2	-6.121	-6.117	-1.068120	3.973537	-6.358	-6.353	-0.811844	4.019688

and 6.7.9(b) nearer to the critical shrinking parameter λ_c .

Figures 6.7.5(b) and 6.7.9(b) reveal that the augmentation in the values of the Casson factor β produces an intensified heat transfer rate nearer to the critical shrinking parameter values for the upper branch solution. Physically, an increment in the Casson factor augments the fluid motion and the thermal profile, resulting in an increase in heat transfer rate. As $\beta \rightarrow \infty$ (the Newtonian fluid) the heat transfer rate is highly reduced compared to the Casson fluid.

Figures 6.7.6 and 6.7.10(a) demonstrate that the intensification in the values of porous media parameter Da resulted in the rising of the heat transfer rate, whereas for increasing porous inertia resistance parameter Fr the heat transfer rate drops for the upper branch solutions nearer to the

critical shrinking parameter λ_c . For the upper branch solution, nearer to the critical shrinking parameter λ_c , increment in the Reynolds number Re drops the convective heat transfer rate, as demonstrated in Figures 6.7.7a and 6.7.10a. Physically, the added nanoparticles increases viscosity Figure 6.7.1(a) producing dominated inertial forces nearer to λ_c . Moreover, the retarded flow rate at this region lead to the overcoming of conductive heat transfer to the convective one due to the better thermal conductivity of the used nanoparticles than the normal base fluid, and therefore, weak temperature gradient (Nusselt number) is observed resulted in reduced heat transfer rate of the Casson Nanofluid flow.

Stepping up in the values of the slip parameter δ enhances the convective heat transfer rate (Nusselt number Nu) in the upper branch solution as seen from Figures 6.7.7b and 6.7.9b, nearer to the critical shrinking parameter λ_c values. Physically, heat transfer rate increases as a result of an improvement in the slipperiness of the cylindrical surface, reducing adhesion of the nanofluid to it.

Figure 6.7.8a shows that as the Prandtl number Pr increases, the convective heat transfer rate gets higher in the upper branch solution nearer to the critical shrinking parameter λ_c values. Physically, this is due to an increment in Pr , which signifies that the momentum diffuses more quickly and the velocity boundary layer is less thick than the thermal boundary layer of the fluid, meaning more heat transfer by convection is dominant in the fluid flow.

Figures 6.7.8(b) and 6.7.10(b) illustrate as the viscous dissipation parameter Ec increases, local Nusselt number (heat transfer rate) also increases nearer to the critical shrinking λ_c values for the upper branch solution. Figure 6.7.9(a) and 6.7.10(b) demonstrate the heat transfer rate (local Nusselt number Nu_z) against the Biot number Bi , and as it can be observed, the heat transfer rate Nu_z is decreasing as the convective heating parameter (Biot number Bi) increases for the upper branch solution nearer to the critical shrinking λ_c values. Physically, it means that the coefficient of heat transfer caused by the hot fluid beneath the sheet is directly associated with the convective heating parameter. Moreover, from Figure 6.7.10(b) it is demonstrated that the heat transfer rate (Nu) escalates as the values of the shrinking parameter λ increases for the upper branch solution.

Generally, it is observed from the graphs and Table 6.7.5 that the heat transfer rate (the Nusselt number) is more pronounced for TD compared to CF nanoparticles. Moreover, nearer to λ_c , heat transfer rate (Nusselt number) increases for rising values of β , Pr , and Ec but decreases for higher values of Fr and Bi , however the opposite trend holds far from λ_c (say at $\lambda = -0.08$).

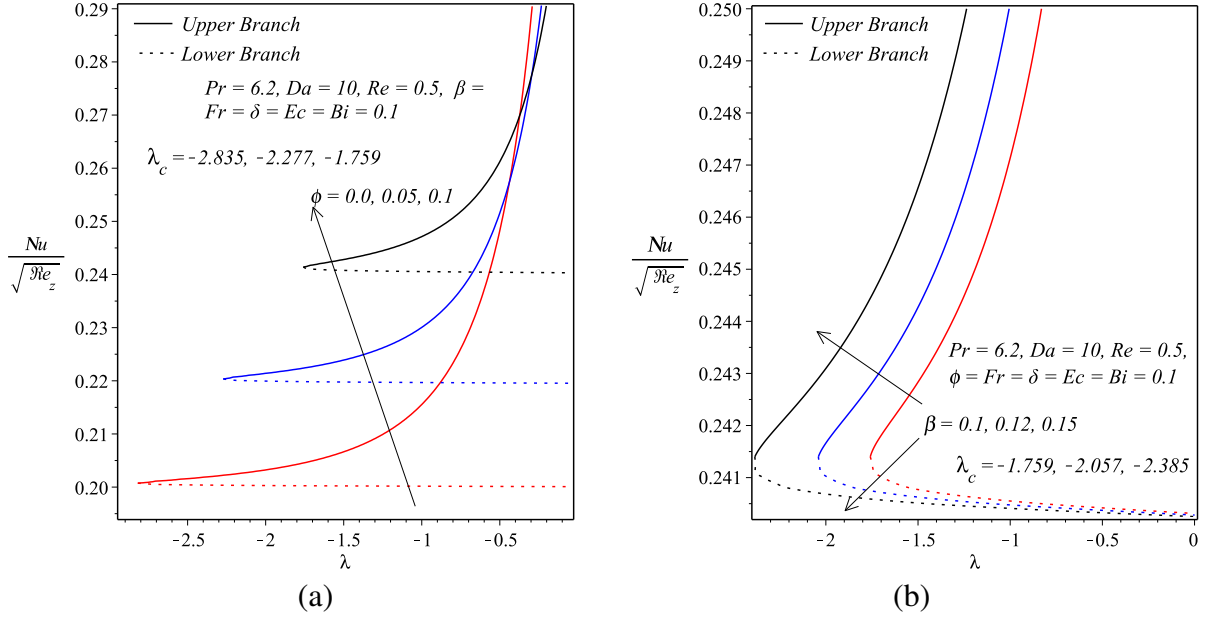


Figure 6.7.5: Nusselt number against λ with varying (a) ϕ (b) β .

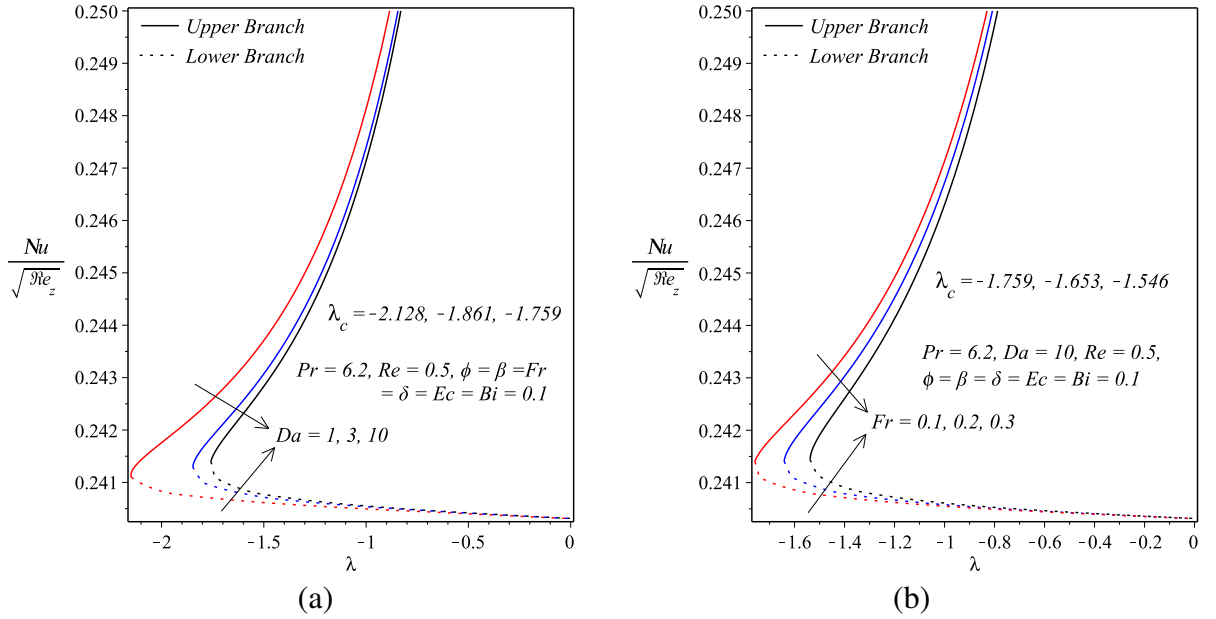


Figure 6.7.6: Nusselt number against λ with varying (a) Da (b) Fr .

6.7.3 Velocity Profile

Figures 6.7.11-6.7.14 demonstrate the effects of different values of the embedded parameters on the fluid flow velocity profile in the case of CF nanoparticles for the UB (which is the only stable and physically realizable one) and LB (unstable and cannot be realized) solutions nearer to the critical shrinking λ_c values. In all plots, it is demonstrated that the upper and lower branch solutions are in opposite trend. Moreover, the velocity profile is an increasing function of η , whereas for all rising values of ϕ, β, Da, Fr, Re , and δ the velocity profile is decreasing with raising of its momentum boundary layer thicknesses nearer to $|\lambda_c|$ in the upper branch solutions. Furthermore, Fr, EC and

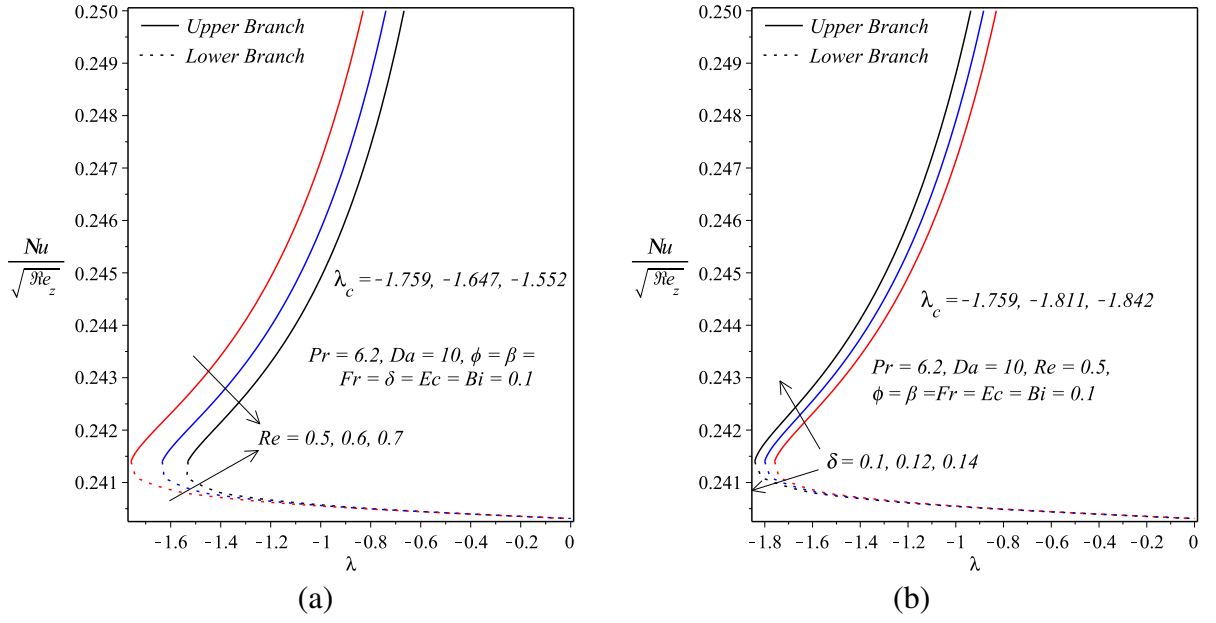


Figure 6.7.7: Nusselt number against λ with varying (a) Re (b) δ .

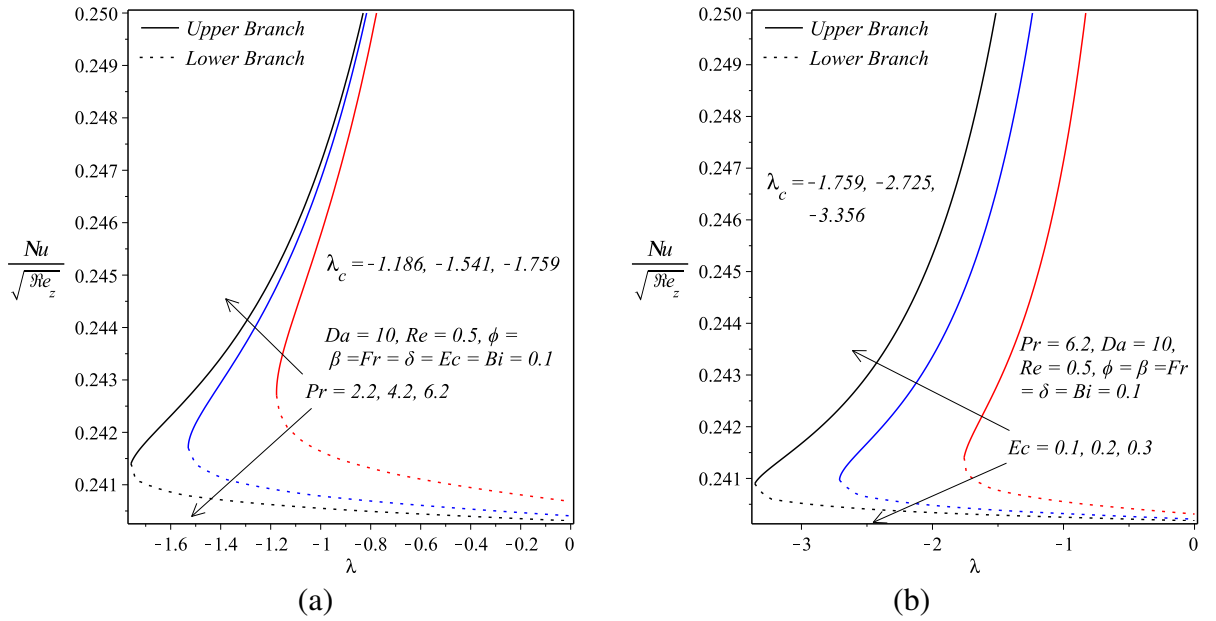
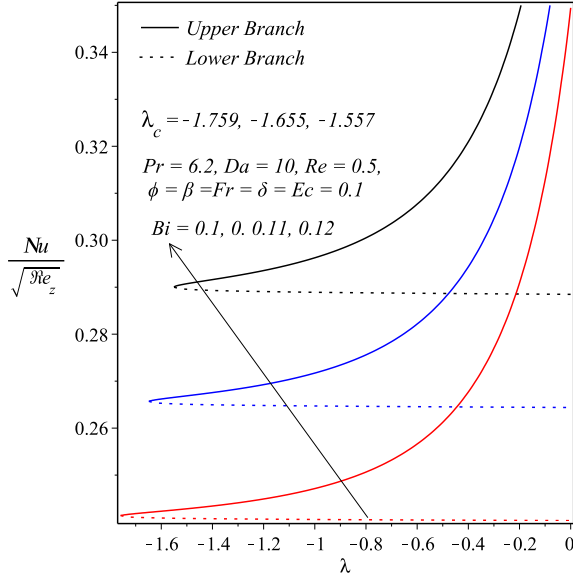


Figure 6.7.8: Nusselt number against λ with varying (a) Pr (b) Ec .

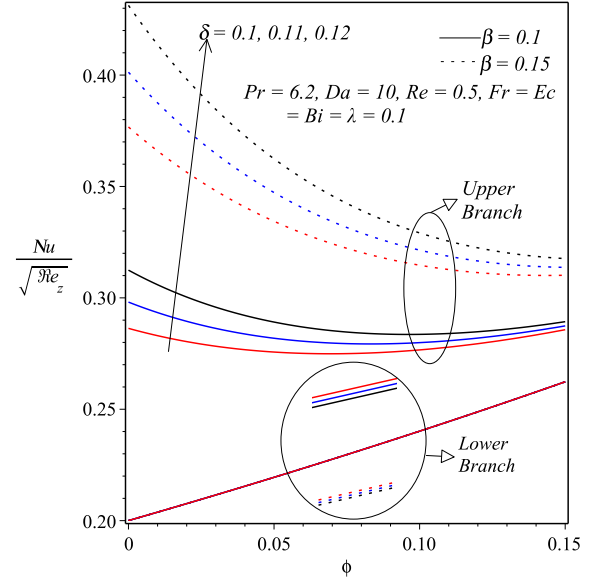
Bi do not have observable effect on the velocity profile and coefficient of the skin friction as Table 6.7.5, and no plot is demonstrated in this regard.

The influence of CF nanoparticle volume fraction ϕ on the flow velocity profile are displayed in Figure 6.7.11(a), and it is observed that the velocity profile drops and the flow boundary layer thickness increases with enhancing nanoparticle volume fraction ϕ for the upper branch solution nearer to $|\lambda_c|$, and the lower branch solution drops.

Figure 6.7.11(b) depicted the influence of the Casson factor β on the fluid velocity profile, and it is observed that the velocity profile decreases with rising boundary layer thickness as the values

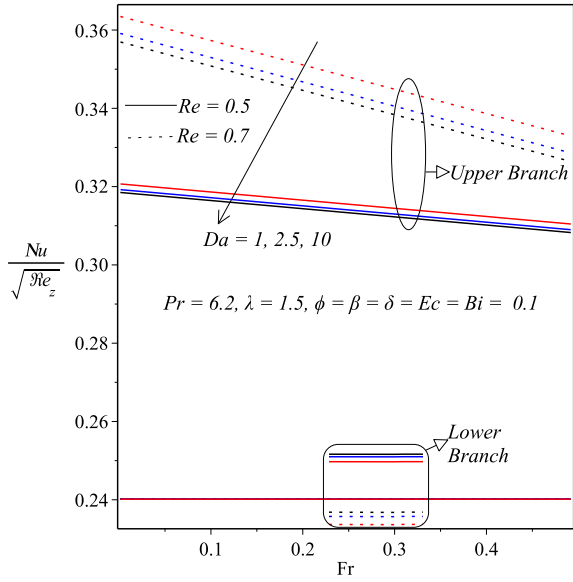


(a)

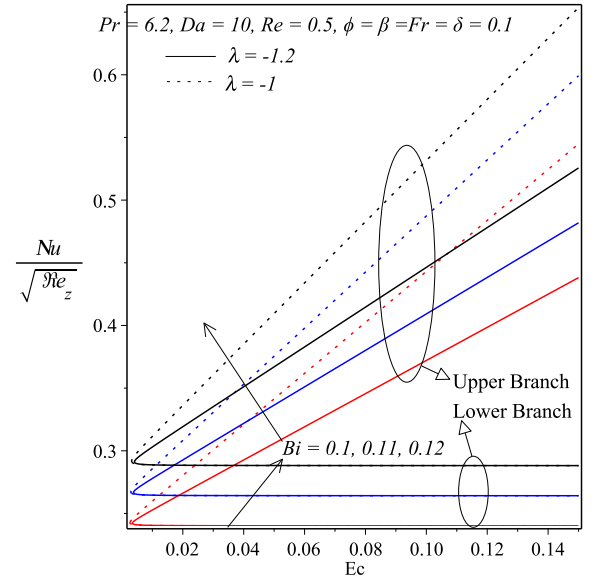


(b)

Figure 6.7.9: Nusselt number against (a) λ with varying Bi . (b) ϕ with varying β and δ .



(a)



(b)

Figure 6.7.10: Nusselt number against (a) Fr with varying Da and Re (b) Ec with varying Bi and λ .

of β augment for both the upper branch solutions. That is, the velocity of Casson nanofluid flow gets diminished when β increased, which might be due to the inverse proportionality of the Casson factor to plastic viscosity. The Newtonian fluid case happened as β got higher ($\beta \rightarrow \infty$), and it is observed that the velocity of the Newtonian fluid is lower than that of the Casson nanofluid with widened boundary layer thicknesses.

The diminishing velocity profiles and increasing boundary layer thickness are observed for the rising porous media parameter Da which could be due to the influence of the application of

Table 6.7.5: The computational results for velocity profile, skin friction, temperature profile, and the Nusselt number for varying values of parameters.

Parameters		CoFe ₂ O ₄ -H ₂ O Nanoparticle				TiO ₂ -H ₂ O Nanoparticle			
Name	Values	$f'(1)$	$\frac{z}{a}ReC_f$	$\theta(1)$	$\frac{a}{z}Nu_z$	$f'(1)$	$\frac{z}{a}ReC_f$	$\theta(1)$	$\frac{a}{z}Nu_z$
ϕ	0.0	0.3443	2.1214179	0.7885156	0.0422969	0.3443	2.1214179	0.7885156	0.0422969
	0.05	0.3470	2.4273017	0.8733792	0.0277817	0.3456	2.4190332	0.8690478	0.0295468
Pr	2.2	0.3480	2.7850414	0.6941179	0.0734467	0.3455	2.7685602	0.6854795	0.0797462
	4.2	0.3480	2.7850414	0.8611957	0.0333289	0.3455	2.7685602	0.8517387	0.0375914
Ec	0.08	0.3480	2.7850414	0.7882636	0.0508410	0.3455	2.7685602	0.7809155	0.0555485
	0.09	0.3480	2.7850414	0.8805976	0.0286702	0.3455	2.7685602	0.8721518	0.0324157
		0.3480	2.7850414	0.9729317	0.0064995	0.3455	2.7685602	0.9633880	0.0092829
β	0.15	0.2755	2.3130649	0.9932522	0.0016203	0.2727	2.2949169	0.9799335	0.0050878
	0.2	0.2316	2.0275647	0.9866240	0.0032118	0.2287	2.0088823	0.9708014	0.0074033
	1	0.0881	1.0935478	0.8090054	0.0458605	0.0856	1.0774762	0.7871929	0.0539569
Da	12	0.3432	2.7538371	0.9515903	0.0116239	0.3406	2.7367457	0.9419657	0.0147145
	15	0.3382	2.7213940	0.9289476	0.0170607	0.3355	2.7036478	0.9192120	0.0204837
Fr	0.2	0.3473	2.7800302	0.9688303	0.0074843	0.3447	2.7634303	0.9592648	0.0103283
	0.3	0.3465	2.7749759	0.9646787	0.0084812	0.3439	2.7582552	0.9550899	0.0113869
Re	0.6	0.3632	2.8840611	0.8886610	0.0267341	0.3605	2.8663237	0.8807833	0.0302272
	0.7	0.3769	2.9729610	0.8217108	0.0428098	0.3740	2.9542409	0.8151097	0.0468786
δ	0.12	0.3972	2.5874533	0.8004842	0.0479066	0.3945	2.5731069	0.7940280	0.0522239
	0.14	0.4396	2.4151383	0.6721195	0.0787288	0.4369	2.4025556	0.6677003	0.0842540
λ	0	0.3975	2.5862853	0.7995491	0.0481311	0.3951	2.5708519	0.7922350	0.0526785
	0.08	0.4467	2.3862249	0.6523665	0.0834718	0.4445	2.3718715	0.6468595	0.0895382
Bi	0.11	0.3480	2.7850414	0.9730653	0.0071142	0.3455	2.7685602	0.9635739	0.0101594
	0.12	0.3480	2.7850414	0.9731975	0.0077228	0.3455	2.7685602	0.9637579	0.0110270

the nanoparticles into the Casson fluid and porous matrix resistance force, which will retard flow as demonstrated in Figure 6.7.12(a) within the flow regimes of the Casson nanofluid for both the upper branch solutions. Similarly, from Figure 6.7.12(b) it is observed that the dropping of velocity profiles for rising in porous media inertial resistance parameter Fr for the upper branch solutions with increasing boundary layer thickness.

The impacts of Reynolds number Re (which measures the ratio of momentum/inertia to viscous forces) on the velocity profile are illustrated in Figure 6.7.13(a) and it is seen that as Re increases, the velocity profile drops nearer to the critical shrinking parameter value λ_c as with rising boundary layer thickness for the upper branch solutions, however, the velocity profile upsurges for lower branch solution.

Figure 6.7.13(b) shows that as the slipperiness parameter δ increases, the velocity profile decreases with increasing boundary layer thickness for both the upper and lower branch solutions. Moreover, Figure 6.7.14 shows that as the values of the velocity ratio parameter λ increases, the velocity profile also increases, and its boundary layer thickness diminishes in the upper branch solution.

To generalize, it is observable from the graphs under this subsection and Table 6.7.5 the veloc-

ity profile is higher for CF compared to TD in the case of the upper branch solution at the surface of flow. Moreover, for the shrinking surface case, unlike nearer to λ_c where the velocity profile decreases with all parameters, except for λ , Table 6.7.5 (where the universal results on the 5th row is computed for $Da = 10, Re = 0.5, \lambda = -0.08, Pr = 6.2, \phi = \beta = Fr = \delta = Ec = Bi = 0.1$ for both $CoFe_2O_4-H_2O$ and TiO_2-H_2O nanoparticles) revealed that far from λ_c (say at $\lambda = -0.08$), the velocity profile increases for rising values of ϕ, Da, Re, δ and λ for the upper branch solution. Furthermore, fluctuating values of Pr, Ec and Bi showed an unobservant effects on the velocity profile and its boundary layer thickness, and the skin friction coefficient, perhaps due to dominance in the thermal diffusivity, energy, and resistance for convection at the surface of the casson nanofluid in the upper branch solutions.

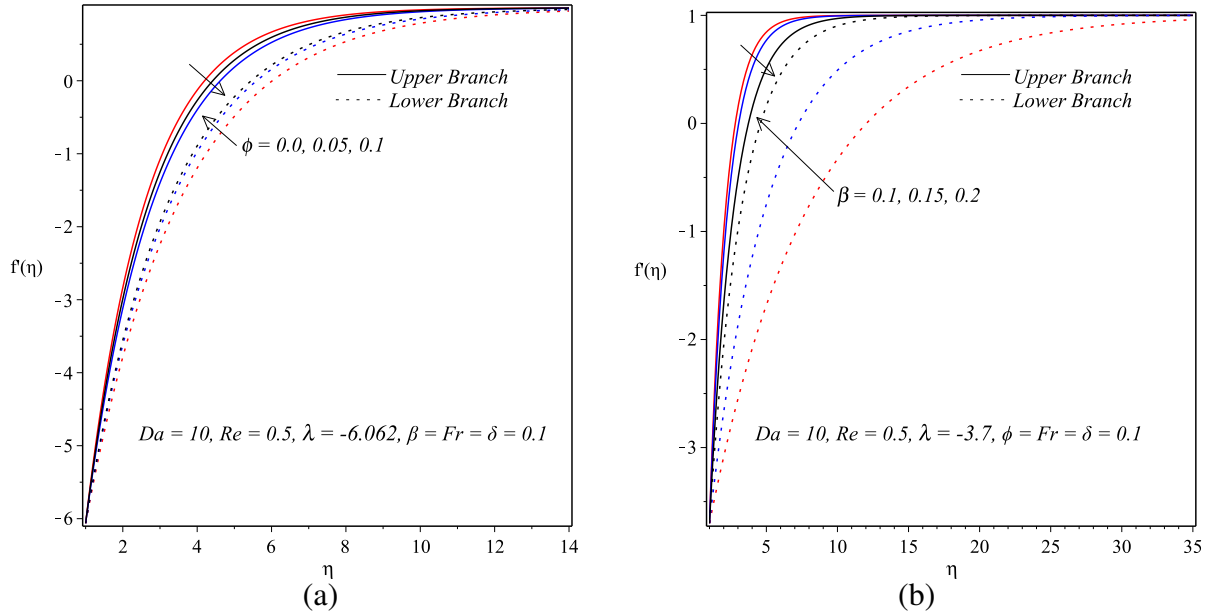


Figure 6.7.11: Velocity profile against η for varying values of: (a) ϕ (b) β .

6.7.4 Temperature Profile

The overall impacts of variation in the values of the involving parameters on the fluid temperature profile and related boundary layer thickness for the shrinking parameter λ are demonstrated in Figures 6.7.15-6.7.19. It is observed from the plots that the temperature profile is a decreasing function of η for all parameters under consideration nearer to λ_c . The upper and lower branch solution are in opposite trend except for the parameters ϕ and Bi .

The temperature profile behavior in dealing with different values of nanoparticle volume fraction ϕ is illustrated in Figure 6.7.15(a), and it is observed that the temperature profile and its boundary layer thickness diminishes for increasing ϕ in both upper and lower branch solutions. Physically, increasing the nanoparticle volume fraction ϕ raises the thermal conductivity of the

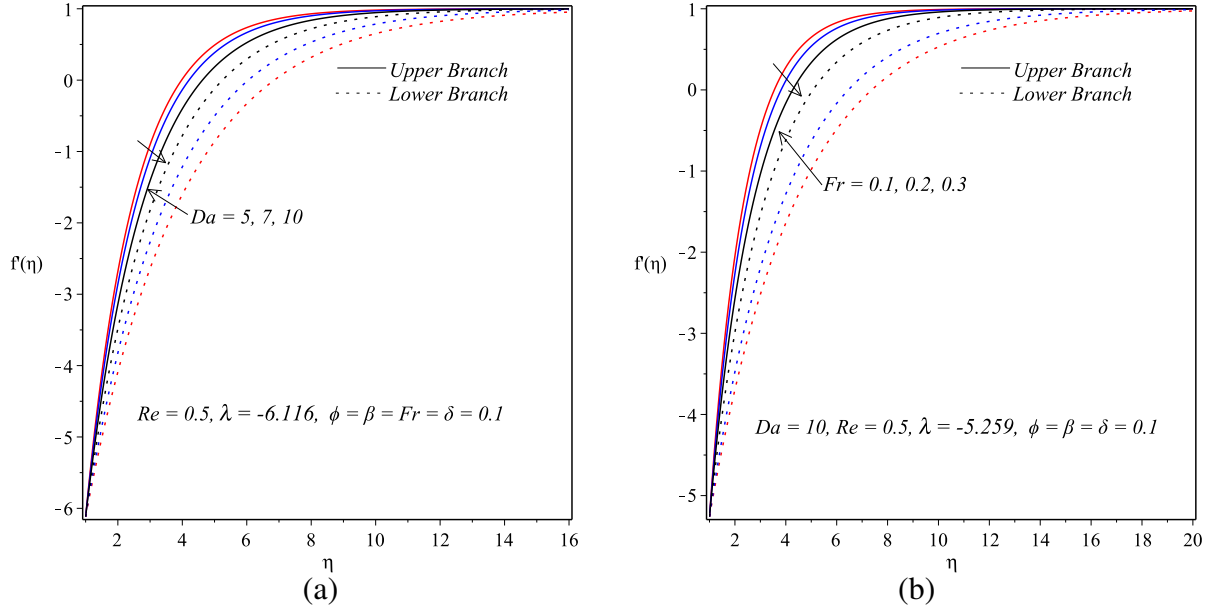


Figure 6.7.12: Velocity profile against η for varying values of: (a) Da . (b) Fr .

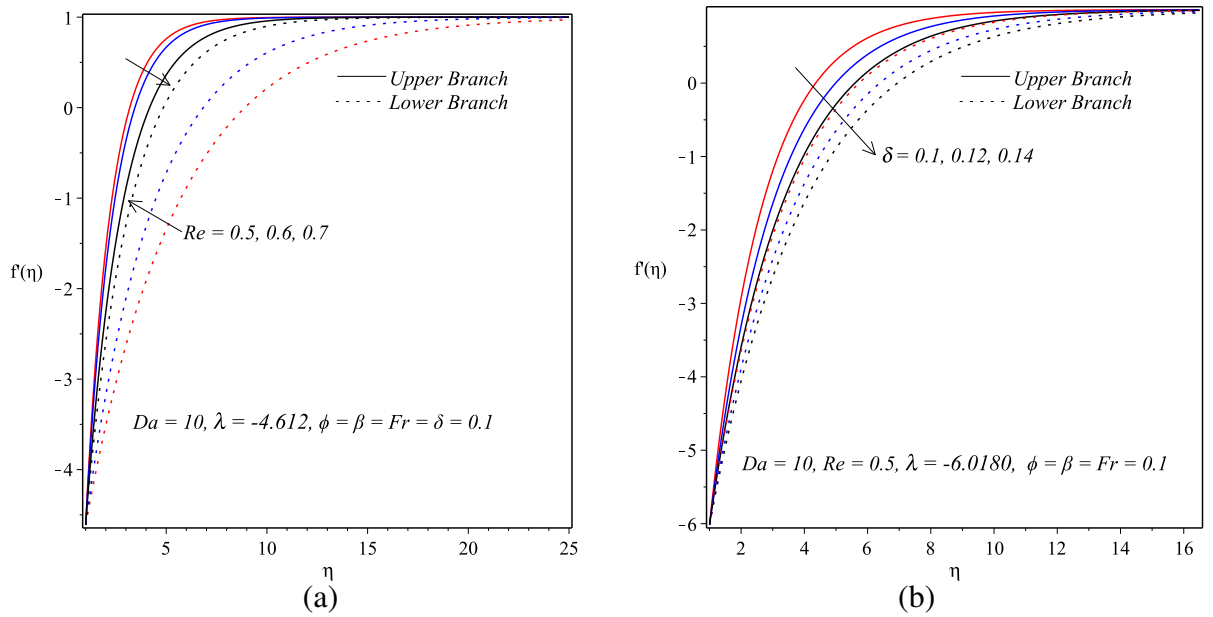


Figure 6.7.13: Velocity profile against η for varying values of: (a) Re . (b) δ .

fluid, and hence, intensify the temperature gradient of the Casson nanofluid, and drops the thermal behavior of Casson nanofluid flow.

From Figure 6.7.15(b), it is observed that the temperature profile and the corresponding boundary layer thickness diminishes with increasing values of the Casson factor β for the upper branch solution. Physically, when the Casson factor values become higher, the strength of the yield stress of the Casson fluid is weakened, enhancing the plastic dynamic viscosity and therefore diminishing the thickness of the thermal boundary layer of the flow temperature profile. Note that the Casson nanofluid's thermal boundary layer thickness is higher than that of the Newtonian fluid. The values

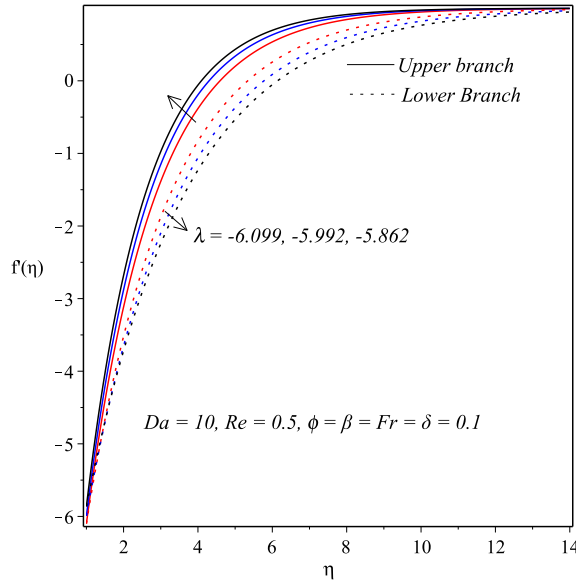


Figure 6.7.14: Velocity profile against η for varying values of λ .

of the Casson parameter get higher ($\beta \rightarrow \infty$) shows weaker interactions of molecular motion within the Casson nanofluid behaving as a Newtonian Casson fluid, which ultimately reduces the Casson nanofluid's temperature profile.

Figure 6.7.16(a) showed that the temperature profile and its boundary layer thickness raises against the increment in Da within the flow regime for the upper branch solution. Physically, the increment in porous matrix reduces the temperature gradient and hence enhances the thermal properties of the Casson nanofluid flow. Moreover, from Figure 6.7.16(b) it is observed that the temperature profile and its boundary layer thickness increase against the increment in the values of Fr for the upper branch solution.

Figure 6.7.17(a) revealed that rising values of the Reynolds number Re resulted in a intensification of the temperature profile and its thermal boundary layer thickness for the upper branch solutions. Physically, it indicates that as the Reynolds number Re raises the temperature gradient drops due to dominance in thermal conductivity, and hence, the temperature of the fluid starts to rise. In Figure 6.7.17(b) it is demonstrated that an increment in the slipperiness parameter δ resulted in the dropping of the temperature profile and its thermal boundary layer thickness for the upper branch solution. Physically, the slipperiness of the surface facilitates the temperature gradient, resulting in a reduction of the temperature profile and a lessening of the corresponding thermal boundary layer thickness of the Casson nanofluid flow.

The effect of the Prandtl number Pr on the thermal properties of the Casson nanofluid flows is revealed, and it is observed that as Pr upsurges, the temperature and its boundary layer thickness diminishes for the upper branch solutions, as presented in Figure 6.7.18(a). Figure 6.7.18(b)

illustrates the influence of the viscous dissipation parameter (Eckert number Ec) on the thermal profile. The rising of both the temperature profile and the boundary layer thickness are observed at the expense of enhancing the viscous dissipation parameter Ec for the upper branch solution. It is known that the kinetic energy is absorbed by viscosity from the fluid motion and converted into internal energy that improves the heating of the fluid flow, and thus increasing both the temperature profile and its boundary layer thickness.

Figure 6.7.19(a) depicts that increasing the convective heating parameter Bi drops the temperature profile and the related boundary layer thickness of Casson nanofluid flow for the upper branch solution. Physically, for $Bi < 1$ the temperature on the surface and beneath the surface (inside the cylinder) will be approximately similar, and/or the surface is a good conductor of heat, so the temperature is uniformly distributed throughout the surface of the cylinder. As the velocity ratio λ increases, the temperature profile drops and the thermal boundary layer diminishes for the upper branch solutions as depicted in Figure 6.7.19(b).

Generalizing based on the plots under discussion and Table 6.7.5, the temperature profile is higher for CF compared to TD for the upper branch solutions. Moreover, nearer to λ_c , temperature profile and its thermal boundary layer decrease for increasing values of β, Pr, Ec , and Bi , however the opposite trend holds far from λ_c (say at $\lambda = -0.08$).

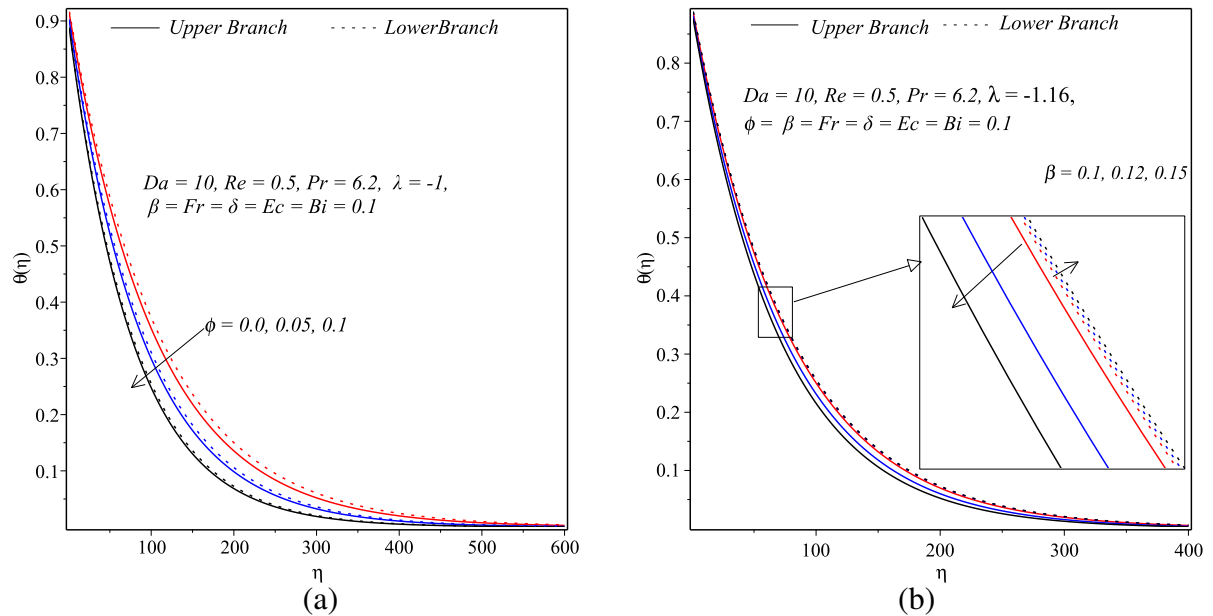


Figure 6.7.15: Temperature profile against η for varying values of (a) ϕ (b) β .

6.7.5 Heat Transfer Enhancement

The chart in Figure 6.7.20 demonstrates how rate heat transfer is enhanced for the increasing values of the nanoparticles' Hydrated Cobalt Ferrite ($\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$) and/or Hydrated Titanium Dioxide

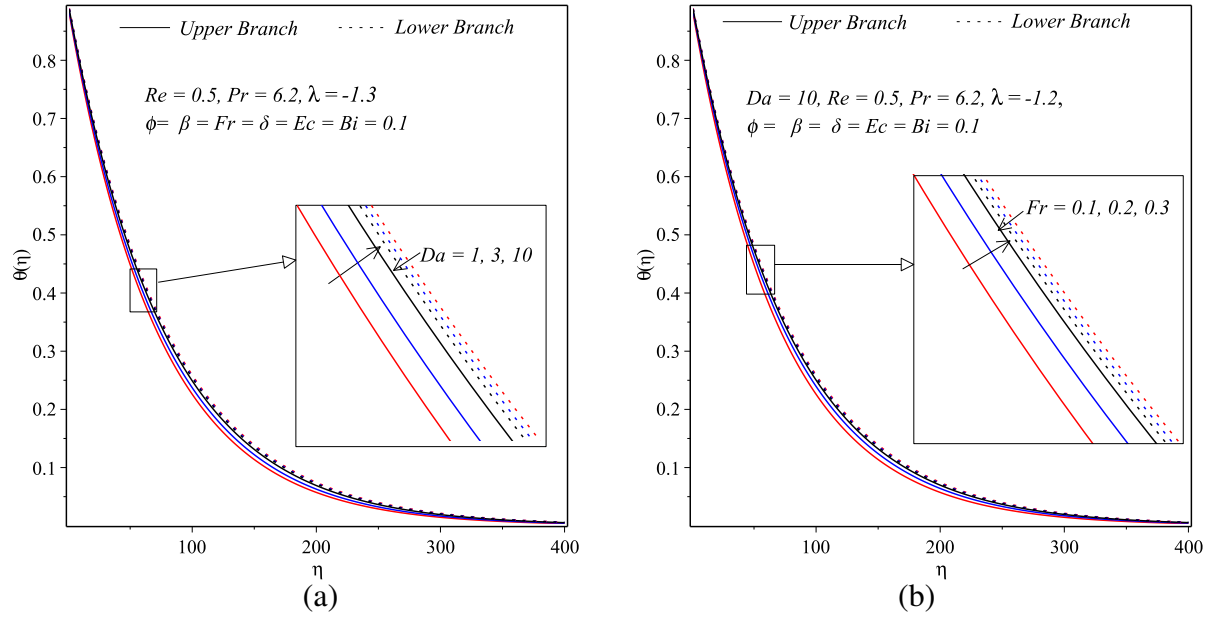


Figure 6.7.16: Temperature profile against η for varying values of: (a) Da (b) Fr .

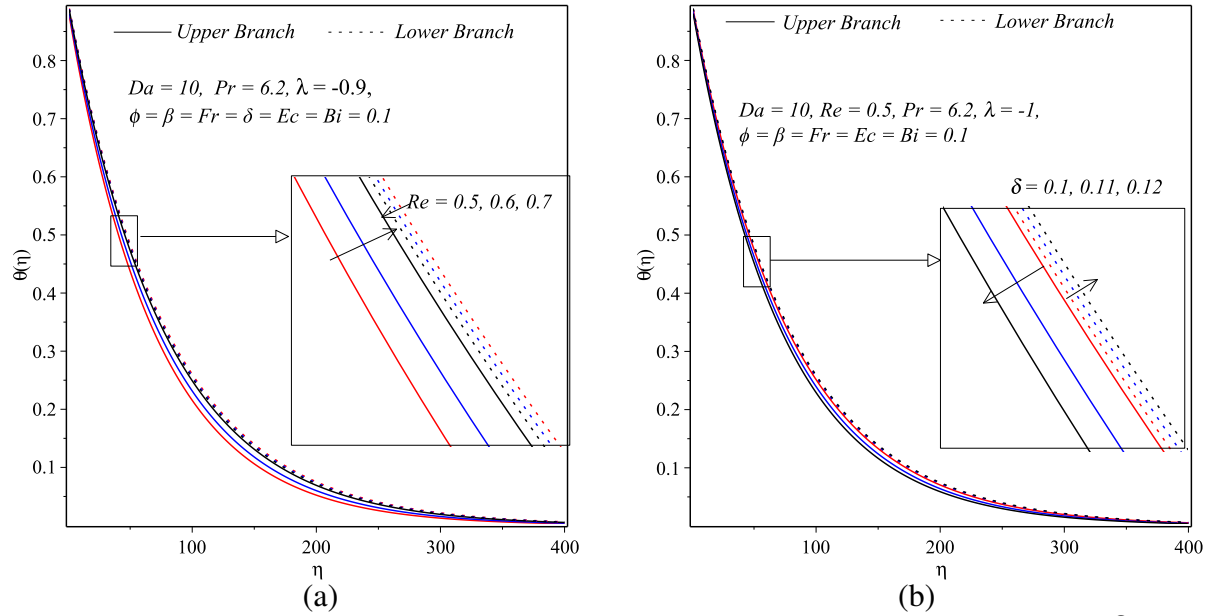


Figure 6.7.17: Temperature profile against η for varying values of: (a) Re (b) δ .

($\text{TiO}_2\text{-H}_2\text{O}$) volume fraction ϕ in the Casson fluid for the shrinking, fixed, and stretching surfaces (where $\beta = 5, Pr = 6.2, Da = 10, Re = 0.5, Fr = \delta = Ec = Bi = 0.1$). The heat transfer enhancement is more pronounced for $\text{TiO}_2\text{-H}_2\text{O}$ nanoparticles compared to that of $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ nanoparticles, which could be due to the larger thermal conductivity of TD with respect to CF. Moreover, the rate of heat transfer enhancement is higher for the working Casson nanofluids with increasing the values of ϕ for a higher shrinking parameter compared to other cases. Studies revealed that nanoparticles serve as better coolants for industrial and engineering usage when compared to base fluids; for instance, TD can be used as a better enhancer of heat transfer rate and cooling of radiators and electronic devices, etc.

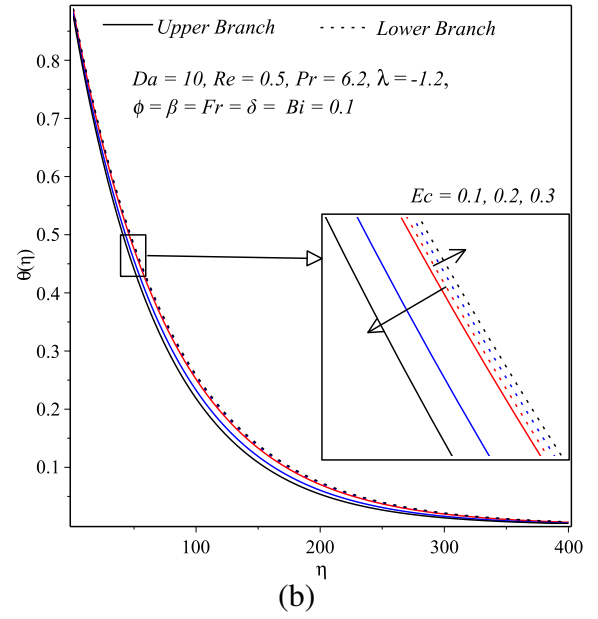
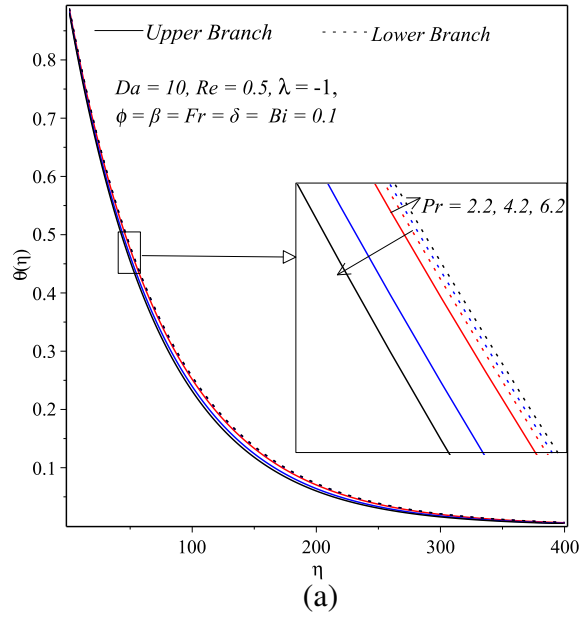


Figure 6.7.18: Temperature profile against η for varying values of: (a) Pr (b) Ec .

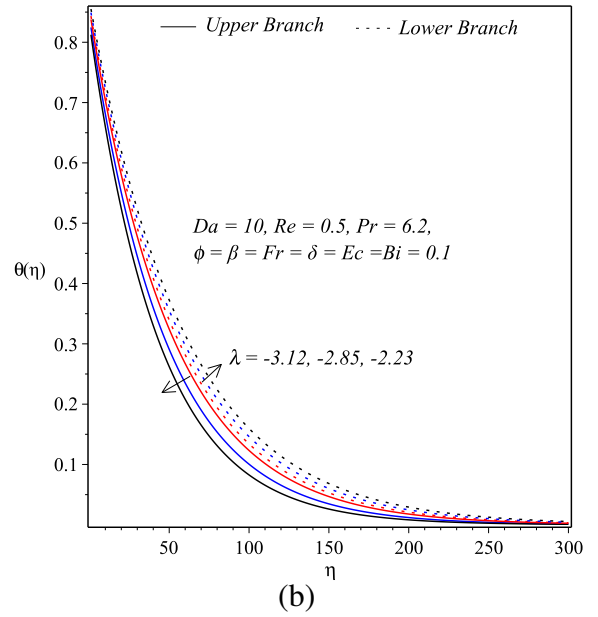
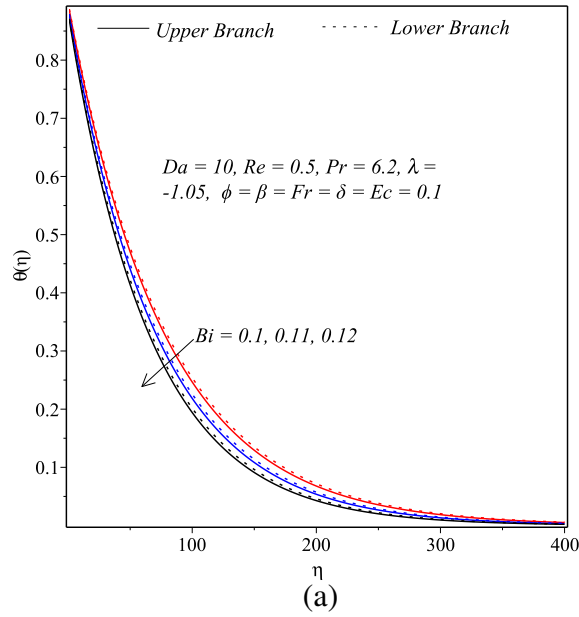


Figure 6.7.19: Temperature profile against η for varying values of: (a) Bi (b) λ .

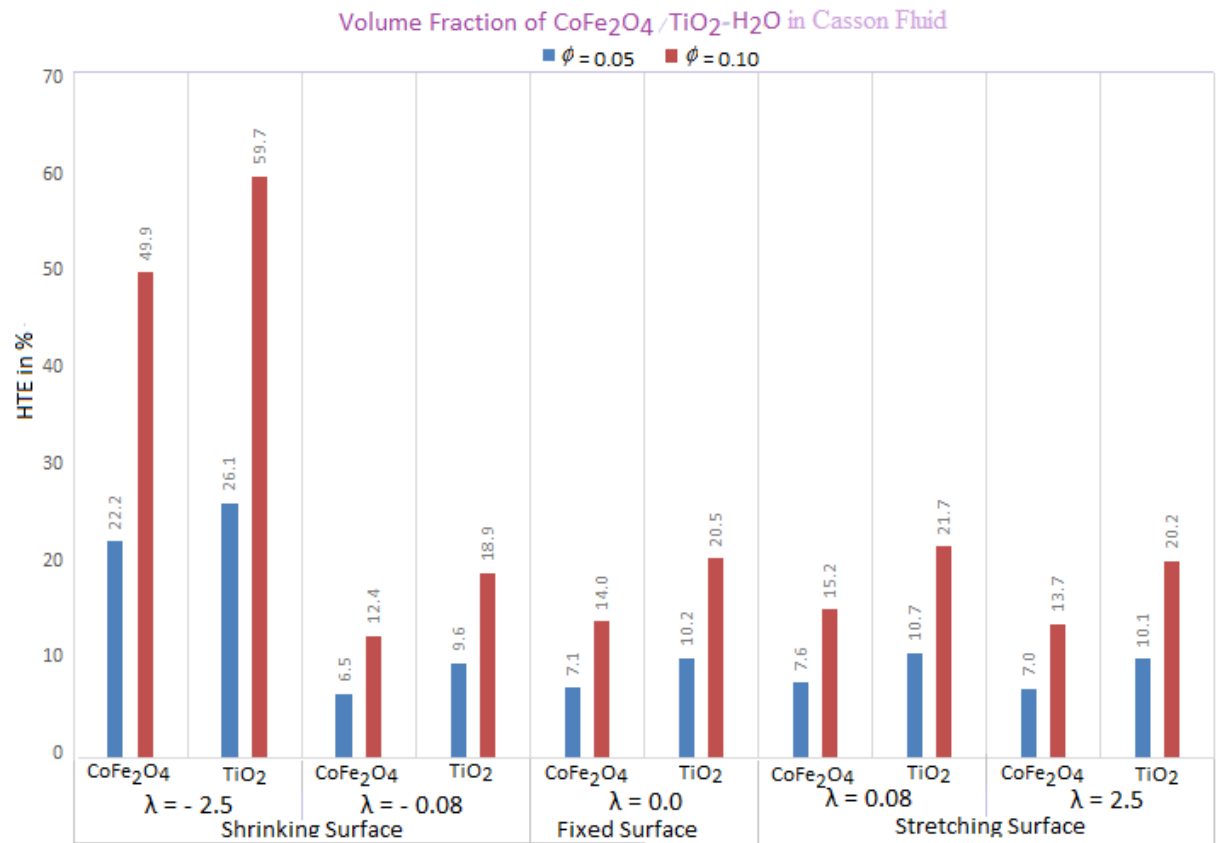


Figure 6.7.20: Percentage heat transfer enhancement for varying values of ϕ .

Chapter 7

Summary, Conclusion and Recommendation

7.1 Summary

Considering the immense applications of the boundary layer flow theory in the field of fluid dynamics, the purpose of this research is to investigate mathematical models of the Casson nanofluid flows and to analyze the mass and heat transfer characteristics of this fluid in a boundary layer regime. The general objective of this dissertation was obtained via the specific objectives: to formulate a mathematical model to analyze the effects of thermophoresis and Brownian motion on Casson nanofluid flows over convectively heated permeable stretching/shrinking slippery sheet in a Darcy-Forchheimer porous medium; to develop a model for heat transfer enhancement of Cu-H₂O-Casson nanofluid flows past convectively heated permeable stretching/shrinking slippery sheet in a Darcy-Forchheimer porous medium; to establish a mathematical model of stagnation point flow of CoFe₂O₄/TiO₂-H₂O-Casson nanofluid past a slippery stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium; to analyze the existence of dual/multiple solutions and perform the temporal stability analysis for the basic Casson nanofluid flows; to investigate the effects of various embedded parameters on the concentration, temperature, and velocity profiles, and moreover on the coefficient of skin friction, mass transfer rate, and heat transfer rate and enhancement. The mathematical models used here are primarily based on the continuity, Navier-Stokes, energy, mass concentration, and Casson rheological equations, and solved numerically so that for consistency the obtained results were compared, and presented in graphs and charts, whereas the computed results were tabulated and quantitatively discussed.

Chapter 1 focuses on the background of the study with a general introduction, basic definitions and formulas of physical properties in fluid flow, motivation of the study, statement of the problem, objectives, significance, scope of the study and organization of the dissertation. Again, chapter 2 detailed studies on boundary layer theory in general and particularly about boundary layer Casson nanofluid flow past a permeable slippery stretching/shrinking sheet and boundary layer flow in a non-Darcian porous medium. In chapter 3, derivations of fluid flow the general governing equations based on the laws of conservation and the computational methods for solving boundary value problems used are discussed.

In chapter 4 effects of thermophoresis and Brownian motion on flows of Casson nanofluid over a shrinking/stretching slippery sheet in the porous media are studied. The impacts of some

embedded parameters on fluid temperature profile, velocity profile, species concentration profile, wall skin friction, wall heat transfer, and wall mass transfer are demonstrated and the existence of dual real solutions for a specific range of shrinking parameters λ is observed. Further, besides the impacts of slipperiness of flow surface, convective heating, and viscous dissipation, the effects of porous media are considered.

Modeling heat enhancement of Cu-H₂O-Casson nanofluid past a heated stretching/shrinking slippery sheet in a porous medium is investigated in chapter 5. The impacts of flow governing parameters on Casson nanofluid flow and heat transfer behaviors are analyzed by using a Tiwari and Das single-phase flow model. Moreover, hydrodynamic stability analysis is done to identify solutions that are hydrodynamically stable and physically achievable.

Applying Tiwari and Das single-phase flow model on stagnation point flow of CoFe₂O₄-H₂O- and TiO₂-H₂O-Casson nanofluid past a slippery stretching/shrinking cylindrical surface in a Darcy-Forchheimer porous medium the collective impacts of the nanoparticle volume fraction, slip condition, viscous dissipation, Casson factor, porous medium, porous medium inertia parameters, and convective heating on the dimensionless fluid velocity and temperature profiles, wall skin friction, and wall heat transfer transfer behaviors are investigated in chapter 6. Furthermore, temporal stability analysis is carried out numerically to identify intervals of occurrence of stable and physically reliable solutions. The effects of some governing parameters' variation on the intervals of occurrence of dual solutions are presented in graphs, chart and tables.

This dissertation analyzed the existence of dual solutions and the numerical solutions to the problems with the impacts of varying values of some embedded parameters. The solutions obtained to the problems are all theoretical, and these solutions are obtained by applying the fourth-fifth order Runge-Kutta-Fehlberg technique along with the shooting technique numerically, and MAPLE software is used to generate the numerical solutions.

7.2 Conclusion

In this dissertation, a laminar steady two-dimensional boundary layer flow of Casson nanofluid with mass and heat transfer behaviors in non-Darcian porous medium problem is investigated. These problems are mathematically formulated using the Buongiorno convective transport flow, and Tiwari and Das nanofluids flow models. From the computed and illustrated numerical results obtained and the fluid flow model analysis in the chapters 4, 5 and 6, the findings are itemized as follows:

- For the shrinking sheet, dual solutions exist for a specific range $\lambda_c < \lambda < 0$, and the value of $|\lambda_c|$ increases only for the increment in the values of the suction and slipperiness parameters. Also, increasing the values of Eckert and Biot numbers augments the heat transfer rate (Nusselt number), but the value of $|\lambda_c|$ increases only for the Ec .
- For the shrinking sheet surface, the value of the least eigenvalue ε is positive, which implies that among the dual solutions, only the first solution is temporally stable and therefore physically achievable, while the second solution diverges.
- The velocity profile increases with slipperiness, velocity ratio, and mass suction/injection parameters and decreases with Darcy number and Forchheimer parameter.
- The escalation in Casson, porous media, porous media resistance, slipperiness, velocity ratio, and suction/ injection parameters reduces the temperature and increases the concentration profiles of the fluid flow.
- As the thermophoresis and convective parameters increase, the temperature and concentration profiles intensify, however, both diminish with an escalating rate of chemical reaction and Lewis number. An increase in Brownian motion resulted in a dropping of the concentration of the nanoparticle, and had an insignificant effect on temperature.
- As the values of the suction parameter increase, the skin friction increases, whereas escalating the values of the velocity ratio, slipperiness, Casson, porous media, and porous media resistance parameters reduces the skin friction coefficient.
- The heat transfer rate (the Nusselt number) increases with all embedded parameters except for thermophoresis parameter and Eckert number. An increment in the Brownian motion has no significant effect on heat transfer rate.
- The Sherwood number (mass transfer rate) increases with all embedded parameters except for Brownian motion parameter and Eckert numbers.
- The temperature profile, thermal boundary layer thickness, and skin friction diminish, whereas the concentration profile, mass and heat transfer rates increase for the Newtonian fluid ($\beta \rightarrow \infty$) compared to the Casson nanofluid.
- There is a critical value of shrinking parameter λ_c below which no real, dual, or unique solutions exist. The critical value $|\lambda_c|$ widens with increment in Cu-H₂O nanoparticle volume

fraction, suction, slipperiness of surface, and viscous dissipation parameters, whereas it diminishes with increase in Casson factor, Darcy number, and Forchheimer parameter.

- Temporal stability analysis provided the smallest eigenvalue ε , which revealed that only the upper branch solution is stable and physically realizable, whereas the lower branch solution is unstable and not realistic.
- The skin friction (wall shear stress) at the sheet surface increases as nanoparticle volume fraction and suction parameters escalate, but reduces with increment in Casson factor, slipperiness of the sheet, Darcy number and Forchheimer parameter.
- The velocity profile augments with the increment in nanoparticle volume fraction, velocity ratio, and suction and slipperiness parameters. However, it decreases with increasing values of the Casson factor, and porous medium and porous medium inertia parameters for the upper branch solution.
- Widening of the momentum boundary layer thickness is observed with enhancement in porous medium and porous medium inertia parameters, whereas it diminishes with increment in velocity ratio, suction, slipperiness of surface, Casson factor, and nanoparticle volume fraction parameters.
- The temperature profile and the thermal boundary layer thickness increase with incremental in nanoparticle volume fraction, Casson factor, viscous dissipation, and convective heating, whereas they decrease with increment in the velocity ratio, suction, slipperiness, porous medium, and porous medium inertia parameters.
- The rate of heat transfer intensifies with the increment in the Casson factor, porous medium, porous medium inertia, suction, velocity ratio, slipperiness, nanoparticle volume fraction, and convective heating parameters, but drops with rising in viscous dissipation parameter.
- As $\beta \rightarrow \infty$, the coefficient of skin friction, temperature profile, thermal and momentum boundary layer thicknesses reduce, whereas the velocity and rate of heat transfer escalates.
- Increment of Cu-H₂O nanoparticles volume fraction into the Casson fluid, boosts the heat transfer enhancement rate higher for the shrinking sheet surface.
- There is a critical value of shrinking parameter λ_c that determines the interval of solutions, such that the critical value $|\lambda_c|$ widens only for higher values of nanoparticle volume fraction and slipperiness parameters, and increment in $|\lambda_c|$ is higher for TiO₂-H₂O than CoFe₂O₄-H₂O.

- The skin friction (drag force) coefficient of flow escalates only as the nanoparticle volume fraction parameter increase and reduce with an increment in the values of other parameters for the upper branch solutions.
- For shrinking surface, nearer to the critical shrinking parameter λ_c , the coefficient of skin friction increasingly overshoot as λ increases for all parameters, resulted in reduced velocity profile with increasing momentum boundary layer thickness for all parameters in the upper branch solutions.
- For the shrinking surface, there is unobservant effects of the Prandtl, Eckert, and Biot numbers on flow velocity profile and coefficient of the skin friction.
- The coefficient of skin friction (drag force) and the temperature profile are a decreasing function of the shrinking parameter λ , however the reverse is true for velocity profile and rate of heat transfer in the upper branch solutions.
- For the upper branch solutions, the rate of heat transfer drops for increasing values of nanoparticle volume fraction parameter, Forchheimer parameter, and Biot number nearer to λ_c and upsurges for the other parameters.
- Increasing the amount of nanoparticle volume fraction in the Casson fluid, boosts the heat transfer enhancement rate, which is higher for $\text{TiO}_2\text{-H}_2\text{O}$ than $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$.
- The temporal stability analysis determined the smallest eigenvalue ε which revealed that only the upper branch solution is stable and physically realizable, whereas the lower branch solution is unstable and not realistic for the flow problem.
- For the upper branch solutions, the velocity profile drops with all parameters nearer to λ_c , and enhances with increment in the values of the nanoparticle volume fraction parameter, slipperiness parameter, Darcy number, and the Reynolds number far from λ_c for the shrinking surface.
- For the upper branch solutions, the temperature profile and thermal boundary layer thickness increase only with enhancing values of nanoparticle volume fractions and porous inertia resistance parameters nearer to λ_c , and increase the nanoparticle volume fraction parameter, Casson factor, Prandtl number, Eckert number, and Biot number far from λ_c .

- As $\beta \rightarrow \infty$, $|\lambda_c|$ surface drag force coefficient, velocity profile, and temperature profile with its boundary layer thickness diminish, whereas the heat transfer rate and momentum boundary layer increase for the upper branch solutions nearer to λ_c .
- The critical shrinking parameter $|\lambda_c|$ and heat transfer rate and enhancement are higher for $\text{TiO}_2\text{-H}_2\text{O}$ nanoparticles compared to $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ nanoparticles, whereas the skin friction coefficient, velocity, and temperature profiles are higher for $\text{CoFe}_2\text{O}_4\text{-H}_2\text{O}$ nanoparticles relative to $\text{TiO}_2\text{-H}_2\text{O}$ for the upper branch solutions.

7.3 Recommendation and Suggestions for Future Research

7.3.1 Recommendation

The results obtained in this dissertation have a lot of applications in various industrial processes and engineering, particularly in designing of heat transfer systems. Therefore, by combining the theoretical findings of this study with experimental investigations, can be used as a platform for improving the existing design of systems performance and reliability in the process of cooling electronic devices using Casson nanofluid flow past a surfaces filled with porous media.

7.3.2 Suggestions for Future Research

Due to the advancement in nanotechnology in the areas of industrial processing and engineering the significance of nanofluid flow problems of the type under discussion studies can be extended with different perspectives as follows.

- The Casson nanofluid with mass and heat transfer does not include the impacts of various significant factors such as: different factors influencing the nanoparticles, hybrid nanoparticles, etc., and hence can be used for the future perspectives.
- Other non-Newtonian nanofluids flow models: Power Law, Bird-Carreau, Cross-Power, Herschel-Bulkley,... (both two-phase and single-phase flow models) can be considered.
- The flow geometry discussed here are only plate and cylindrical surface, and hence, it is possible to consider cone, spheres, disc and wedges. etc. with different dimensional flow.
- Various nanoparticles which (MWCNT, SWCNT, Ag, CuO, etc.) can be studied.

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Appendix

Maple Codes for Solving Problem in Chapter 5

#Dual Solutions and Stability Analysis of Cu-H₂O-Casson Nanofluid Convection Past a Heated Stretching/Shrinking Slippery Sheet in a Porous Medium.

#D(F)(y) represents velocity profile

#T(y) represents temperature profile

#Copper cps=385, hs=8933, ks=401, as:=59.6*10⁽⁶⁾

#Water cpf=4179, hf=997.1, kf=0.613, af:=5.5*10⁽⁻⁶⁾ r=0..0.1,

#b := β , d1 := δ , T := θ , L := λ , r := ϕ , y := η , u := $f'(\eta)$

with(plots):

#Copper

cps:=385:hs:=8933:ks:=401:r:=0:

#Water

cpf:=4179:hf:=997.1:kf:=0.613:

b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.02:Pr:=6.2:Ec:=0.1:Bi:=0.1:

A1:=((1-r)^(-2.5)):

A2:=(1-r+r*hs/hf):

A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):

A4:=(1-r+r*(hs*cps)/(hf*cpf)):

fcns:={F(y),T(y)}:

sys:=(1+1/b)*A1*(diff(F(y),y\$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y\$2)-diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,

A3*diff(T(y),y\$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+

A1*Pr*Ec*(1+1/b)*(diff(F(y),y\$2)*diff(F(y),y\$2)+(diff(F(y),y)-1)*

(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*

(diff(F(y),y)-1)=0\$:

p1:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),

D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):

p1u:=odeplot(p1,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"], style=line, color=black):

p1t:=odeplot(p1,[y,T(y)],0..10,numpoints=100,labels=["y","T"], style=line, color=black):

with(plots):

#Copper

cps:=385:hs:=8933:ks:=401:r:=0.05:

#Water

cpf:=4179:hf:=997.1:kf:=0.613:

```

b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.02:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(-2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:={F(y),T(y)}:
sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-
diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,
A3*diff(T(y),y$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+
A1*Pr*Ec*(1+1/b)*(diff(F(y),y$2)*diff(F(y),y$2)+(diff(F(y),y)-1)*
(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*
(diff(F(y),y)-1)=0$:
p2:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),
D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
p2u:=odeplot(p2,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"],style=line,
color=red):
p2t:=odeplot(p2,[y,T(y)],0..10,numpoints=100,labels=["y","T"],style=line,
color=red):
-----
with(plots):
#Copper
cps:=385:hs:=8933:ks:=401:r:=0.1:
#Water
cpf:=4179:hf:=997.1:kf:=0.613:
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.02:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(-2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:={F(y),T(y)}:
sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-
diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,
A3*diff(T(y),y$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+
A1*Pr*Ec*(1+1/b)*(diff(F(y),y$2)*diff(F(y),y$2)+(diff(F(y),y)-1)*
(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*
(diff(F(y),y)-1)=0$:
p3:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),
D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
p3u:=odeplot(p3,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"],style=line,

```

```

color=blue):
p3t:=odeplot(p3,[y,T(y)],0..10,numpoints=100,labels=["y","T"],style=line,
color=blue):
-----

with(plots):
#Copper
cps:=385:hs:=8933:ks:=401:r:=0:
#Water
cpf:=4179:hf:=997.1:kf:=0.613:
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.5:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(-2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:={F(y),T(y)}:
sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-
diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,
A3*diff(T(y),y$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+
A1*Pr*Ec*(1+1/b)*(diff(F(y),y$2)*diff(F(y),y$2)+(diff(F(y),y)-1)*
(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*
(diff(F(y),y)-1)=0$:
p4:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),
D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
p4u:=odeplot(p4,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"],style=dot,
color=black):
p4t:=odeplot(p4,[y,T(y)],0..10,numpoints=100,labels=["y","T"],style=dot,
color=black):
-----

with(plots):
#Copper
cps:=385:hs:=8933:ks:=401:r:=0.05:
#Water
cpf:=4179:hf:=997.1:kf:=0.613:
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.5:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(-2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:={F(y),T(y)}:

```

```

sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-
diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,
A3*diff(T(y),y$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+
A1*Pr*Ec*(1+1/b)*(diff(F(y),y$2)*diff(F(y),y$2)+(diff(F(y),y)-1)*
(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*
(diff(F(y),y)-1)=0$:
p5:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),
D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
p5u:=odeplot(p5,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"],style=dot,
color=red):
p5t:=odeplot(p5,[y,T(y)],0..10,numpoints=100,labels=["y","T"],style=dot,
color=red):
-----

with(plots):
#Copper
cps:=385:hs:=8933:ks:=401:r:=0.1:
#Water
cpf:=4179:hf:=997.1:kf:=0.613:
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.5:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(-2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:={F(y),T(y)}:
sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-
diff(F(y),y)*diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0,
A3*diff(T(y),y$2)+A4*Pr*(F(y)*diff(T(y),y)-2*diff(F(y),y)*T(y))+
A1*Pr*Ec*(1+1/b)*(diff(F(y),y$2)*diff(F(y),y$2)+(diff(F(y),y)-1)*
(diff(F(y),y)-1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*
(diff(F(y),y)-1)=0$:
p6:= dsolve({sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),
D(F)(10)=1,T(10)=0},fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
p6u:=odeplot(p6,[y,D(F)(y)],0..10,numpoints=100,labels=["y","u"],style=dot,
color=blue):
p6t:=odeplot(p6,[y,T(y)],0..10,numpoints=100,labels=["y","T"],style=dot,
color=blue):

plots[display]({p1u,p2u,p3u,p4u,p5u,p6u});
plots[display]({p1t,p2t,p3t,p4t,p5t,p6t});-----

```

Skin friction for dual solutions

with(plots):

#Copper

hs:=8933:

#Water

hf:=997.1:

b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0:

A1:=((1-r)^(-2.5)): A2:=(1-r+r*hs/hf):

i:=1:

R:=array(1..352):SK:=array(1..352):

for L from -11.274 by 0.1 to 0.5 do

$$\begin{aligned} FF := & A1^3 * A2 * Da * L^5 * b^2 + A1^3 * Da * Fr * L^5 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b^2 * d1 + 3 * A1^2 * Da * Fr * G1 * \\ & L^4 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b^2 * d1^3 + \\ & Da * Fr * G1^3 * L^2 * b^2 * d1^3 + A1^3 * A2 * Da * L^5 * b - 3 * A1^3 * A2 * Da * L^4 * b^2 + A1^3 * Da * Fr * L^5 * b - 5 * A1^3 * Da * \\ & Fr * L^4 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b * d1 - A1^2 * A2 * Da * G1 * L^3 * S * b^2 - 8 * A1^2 * A2 * Da * G1 * L^3 * b^2 * d1 + \\ & 3 * A1^2 * Da * Fr * G1 * L^4 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b * d1^2 - A1 * A2 * \\ & Da * G1^2 * L^2 * S * b^2 * d1 - 7 * A1 * A2 * Da * G1^2 * L^2 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b * d1^2 - 9 * A1 * Da * Fr * \\ & G1^2 * L^2 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b * d1^3 - 2 * A2 * Da * G1^3 * L * b^2 * d1^3 + Da * Fr * G1^3 * L^2 * b * d1^3 - 2 * Da * \\ & Fr * G1^3 * L * b^2 * d1^3 + A1^4 * L^4 * b^2 - 3 * A1^3 * A2 * Da * L^4 * b + 2 * A1^3 * A2 * Da * L^3 * b^2 - 5 * A1^3 * Da * Fr * L^4 * b + \\ & 10 * A1^3 * Da * Fr * L^3 * b^2 + 2 * A1^3 * G1 * L^3 * b^2 * d1 - 8 * A1^2 * A2 * Da * G1 * L^3 * b * d1 + 3 * A1^2 * A2 * Da * G1 * L^2 * \\ & S * b^2 + 6 * A1^2 * A2 * Da * G1 * L^2 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b * d1 + 18 * A1^2 * Da * Fr * G1 * L^2 * b^2 * d1 + \\ & A1^2 * G1^2 * L^2 * b^2 * d1^2 - 7 * A1 * A2 * Da * G1^2 * L^2 * b * d1^2 + 2 * A1 * A2 * Da * G1^2 * L * S * b^2 * d1 + 5 * A1 * A2 * Da * \\ & G1^2 * L * b^2 * d1^2 - 9 * A1 * Da * Fr * G1^2 * L^2 * b * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b^2 * d1^2 - 2 * A2 * Da * G1^3 * L * b * \\ & d1^3 + A2 * Da * G1^3 * b^2 * d1^3 - 2 * Da * Fr * G1^3 * L * b * d1^3 + Da * Fr * G1^3 * b^2 * d1^3 + 2 * A1^4 * L^4 * b - 4 * A1^4 * L^3 * \\ & b^2 + 2 * A1^3 * A2 * Da * L^3 * b + 2 * A1^3 * A2 * Da * L^2 * b^2 + 10 * A1^3 * Da * Fr * L^3 * b - 10 * A1^3 * Da * Fr * L^2 * b^2 + 4 * \\ & A1^3 * G1 * L^3 * b * d1 - 6 * A1^3 * G1 * L^2 * b^2 * d1 + 6 * A1^2 * A2 * Da * G1 * L^2 * b * d1 - 3 * A1^2 * A2 * Da * G1 * L * S * \\ & b^2 + 18 * A1^2 * Da * Fr * G1 * L^2 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L * b^2 * d1 - A1^2 * Da * G1^2 * L^2 * b^2 + 2 * A1^2 * G1^2 * \\ & L^2 * b * d1^2 - 2 * A1^2 * G1^2 * L * b^2 * d1^2 + 5 * A1 * A2 * Da * G1^2 * L * b * d1^2 - A1 * A2 * Da * G1^2 * S * b^2 * d1 - A1 * A2 * \\ & Da * G1^2 * b^2 * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b^2 * d1^2 + A2 * Da * G1^3 * b * d1^3 + Da * \\ & Fr * G1^3 * b * d1^3 + A1^4 * L^4 - 8 * A1^4 * L^3 * b + 6 * A1^4 * L^2 * b^2 + 2 * A1^3 * A2 * Da * L^2 * b - 3 * A1^3 * A2 * Da * L * b^2 - \\ & 10 * A1^3 * Da * Fr * L^2 * b + 5 * A1^3 * Da * Fr * L * b^2 + 2 * A1^3 * G1 * L^3 * d1 - 12 * A1^3 * G1 * L^2 * b * d1 + 6 * A1^3 * G1 * \\ & L * b^2 * d1 + A1^2 * A2 * Da * G1 * S * b^2 - A1^2 * A2 * Da * G1 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L * b * d1 + 3 * A1^2 * Da * \\ & Fr * G1 * b^2 * d1 + 2 * A1^2 * Da * G1^2 * L * b^2 + A1^2 * G1^2 * L^2 * d1^2 - 4 * A1^2 * G1^2 * L * b * d1^2 + A1^2 * G1^2 * b^2 * d1^2 - \\ & A1 * A2 * Da * G1^2 * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b * d1^2 - 4 * A1^4 * L^3 + 12 * A1^4 * L^2 * b - 4 * A1^4 * L * b^2 - 3 * A1^3 * \\ & A2 * Da * L * b + A1^3 * A2 * Da * b^2 + 5 * A1^3 * Da * Fr * L * b - A1^3 * Da * Fr * b^2 - 6 * A1^3 * G1 * L^2 * d1 + 12 * A1^3 * G1 * \\ & L * b * d1 - 2 * A1^3 * G1 * b^2 * d1 - A1^2 * A2 * Da * G1 * b * d1 + 3 * A1^2 * Da * Fr * G1 * b * d1 - A1^2 * Da * G1^2 * b^2 - 2 * \\ & A1^2 * G1^2 * L * d1^2 + 2 * A1^2 * G1^2 * b * d1^2 + 6 * A1^4 * L^2 - 8 * A1^4 * L * b + A1^4 * b^2 + A1^3 * A2 * Da * b - A1^3 * Da * Fr * \\ & b + 6 * A1^3 * G1 * L * d1 - 4 * A1^3 * G1 * b * d1 + A1^2 * G1^2 * d1^2 - 4 * A1^4 * L + 2 * A1^4 * b - 2 * A1^3 * G1 * d1 + A1^4 = 0 : \end{aligned}$$

```

FF1:=fsolve(FF,G1):
SK[i]:=FF1[2]:
R[i]:=L:
i:=i+1:
od:
L:='L':
points:=seq([R[j],SK[j]],j=1..300):
pcf1:=pointplot(points,labels=["L","Cf"],style=line,thickness=3,colour=black):
with(plots):
#Copper
hs:=8933:
#Water
hf:=997.1:
b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0:
A1:=((1-r)^(-2.5)):
A2:=(1-r*r*hs/hf):
i:=1:
R:=array(1..352):SK:=array(1..352):
for L from -11.274 by 0.005 to -9 do
FF:=A1^3*A2*Da*L^5*b^2+A1^3*Da*Fr*L^5*b^2+3*A1^2*A2*Da*G1*L^4*b^2*d1+3*A1^2*Da*Fr*G1*
L^4*b^2*d1+3*A1*A2*Da*G1^2*L^3*b^2*d1^2+3*A1*Da*Fr*G1^2*L^3*b^2*d1^2+A2*Da*G1^3*L^2*b^2*d1^3+
Da*Fr*G1^3*L^2*b^2*d1^3+A1^3*A2*Da*L^5*b-3*A1^3*A2*Da*L^4*b^2+A1^3*Da*Fr*L^5*b-5*A1^3*Da*
Fr*L^4*b^2+3*A1^2*A2*Da*G1*L^4*b*d1-A1^2*A2*Da*G1*L^3*S*b^2-8*A1^2*A2*Da*G1*L^3*b^2*d1+
3*A1^2*Da*Fr*G1*L^4*b*d1-12*A1^2*Da*Fr*G1*L^3*b^2*d1+3*A1*A2*Da*G1^2*L^3*b*d1^2-A1*A2*
Da*G1^2*L^2*S*b^2*d1-7*A1*A2*Da*G1^2*L^2*b^2*d1^2+3*A1*Da*Fr*G1^2*L^3*b*d1^2-9*A1*Da*Fr*
G1^2*L^2*b^2*d1^2+A2*Da*G1^3*L^2*b*d1^3-2*A2*Da*G1^3*L*b^2*d1^3+Da*Fr*G1^3*L^2*b*d1^3-2*Da*
Fr*G1^3*L*b^2*d1^3+A1^4*L^4*b^2-3*A1^3*A2*Da*L^4*b+2*A1^3*A2*Da*L^3*b^2-5*A1^3*Da*Fr*L^4*b+
10*A1^3*Da*Fr*L^3*b^2+2*A1^3*G1*L^3*b^2*d1-8*A1^2*A2*Da*G1*L^3*b*d1+3*A1^2*A2*Da*G1*L^2*
S*b^2+6*A1^2*A2*Da*G1*L^2*b^2*d1-12*A1^2*Da*Fr*G1*L^3*b*d1+18*A1^2*Da*Fr*G1*L^2*b^2*d1+
A1^2*G1^2*L^2*b^2*d1^2-7*A1*A2*Da*G1^2*L^2*b*d1^2+2*A1*A2*Da*G1^2*L*S*b^2*d1+5*A1*A2*Da*
G1^2*L*b^2*d1^2-9*A1*Da*Fr*G1^2*L^2*b*d1^2+9*A1*Da*Fr*G1^2*L*b^2*d1^2-2*A2*Da*G1^3*L*b*
d1^3+A2*Da*G1^3*b^2*d1^3-2*Da*Fr*G1^3*L*b*d1^3+Da*Fr*G1^3*b^2*d1^3+2*A1^4*L^4*b-4*A1^4*L^3*
b^2+2*A1^3*A2*Da*L^3*b+2*A1^3*A2*Da*L^2*b^2+10*A1^3*Da*Fr*L^3*b-10*A1^3*Da*Fr*L^2*b^2+4*
A1^3*G1*L^3*b*d1-6*A1^3*G1*L^2*b^2*d1+6*A1^2*A2*Da*G1*L^2*b*d1-3*A1^2*A2*Da*G1*L*S*
b^2+18*A1^2*Da*Fr*G1*L^2*b*d1-12*A1^2*Da*Fr*G1*L*b^2*d1-A1^2*Da*G1^2*L^2*b^2+2*A1^2*G1^2*
L^2*b*d1^2-2*A1^2*G1^2*L*b^2*d1^2+5*A1*A2*Da*G1^2*L*b*d1^2-A1*A2*Da*G1^2*S*b^2*d1-A1*A2*
Da*G1^2*b^2*d1^2+9*A1*Da*Fr*G1^2*L*b*d1^2-3*A1*Da*Fr*G1^2*b^2*d1^2+A2*Da*G1^3*b*d1^3+Da*
Fr*G1^3*b*d1^3+A1^4*L^4-8*A1^4*L^3*b+6*A1^4*L^2*b^2+2*A1^3*A2*Da*L^2*b-3*A1^3*A2*Da*L*b^2-
10*A1^3*Da*Fr*L^2*b+5*A1^3*Da*Fr*L*b^2+2*A1^3*G1*L^3*d1-12*A1^3*G1*L^2*b*d1+6*A1^3*G1*

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L*b^2*d1+A1^2*A2*Da*G1*S*b^2-A1^2*A2*Da*G1*b^2*d1-12*A1^2*Da*Fr*G1*L*b*d1+3*A1^2*Da*
Fr*G1*b^2*d1+2*A1^2*Da*G1^2*L*b^2+A1^2*G1^2*L^2*d1^2-4*A1^2*G1^2*L*b*d1^2+A1^2*G1^2*b^2*d1^2-
A1*A2*Da*G1^2*b*d1^2-3*A1*Da*Fr*G1^2*b*d1^2-4*A1^4*L^3+12*A1^4*L^2*b-4*A1^4*L*b^2-3*A1^3*
A2*Da*L*b+A1^3*A2*Da*b^2+5*A1^3*Da*Fr*L*b-A1^3*Da*Fr*b^2-6*A1^3*G1*L^2*d1+12*A1^3*G1*
L*b*d1-2*A1^3*G1*b^2*d1-A1^2*A2*Da*G1*b*d1+3*A1^2*Da*Fr*G1*b*d1-A1^2*Da*G1^2*b^2-2*
A1^2*G1^2*L*d1^2+2*A1^2*G1^2*b*d1^2+6*A1^4*L^2-8*A1^4*L*b+A1^4*b^2+A1^3*A2*Da*b-A1^3*Da*Fr*
b+6*A1^3*G1*L*d1-4*A1^3*G1*b*d1+A1^2*G1^2*d1^2-4*A1^4*L+2*A1^4*b-2*A1^3*G1*d1+A1^4=0:
FF1:=fsolve(FF,G1):
SK[i]:=FF1[1]:
R[i]:=L:
i:=i+1:
od:
L:='L':
points:=seq([R[j],SK[j]],j=1..300):
pcfl1:=pointplot(points,labels=["L","Cf"]
,style=line,linestyle=dot,thickness=3,colour=black):
with(plots):
#Copper
hs:=8933:
#Water
hf:=997.1:
b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0.05:
A1:=((1-r)^(2.5)):
A2:=(1-r*hs/hf):
i:=1:
R:=array(1..352):SK:=array(1..352):
for L from -9.632 by 0.1 to 0.5 do
FF:=A1^3*A2*Da*L^5*b^2+A1^3*Da*Fr*L^5*b^2+3*A1^2*A2*Da*G1*L^4*b^2*d1+3*A1^2*Da*Fr*G1*
L^4*b^2*d1+3*A1*A2*Da*G1^2*L^3*b^2*d1^2+3*A1*Da*Fr*G1^2*L^3*b^2*d1^2+A2*Da*G1^3*L^2*b^2*d1^3+
Da*Fr*G1^3*L^2*b^2*d1^3+A1^3*A2*Da*L^5*b-3*A1^3*A2*Da*L^4*b^2+A1^3*Da*Fr*L^5*b-5*A1^3*Da*
Fr*L^4*b^2+3*A1^2*A2*Da*G1*L^4*b*d1-A1^2*A2*Da*G1*L^3*S*b^2-8*A1^2*A2*Da*G1*L^3*b^2*d1+
3*A1^2*Da*Fr*G1*L^4*b*d1-12*A1^2*Da*Fr*G1*L^3*b^2*d1+3*A1*A2*Da*G1^2*L^3*b*d1^2-A1*A2*
Da*G1^2*L^2*S*b^2*d1-7*A1*A2*Da*G1^2*L^2*b^2*d1^2+3*A1*Da*Fr*G1^2*L^3*b*d1^2-9*A1*Da*Fr*
G1^2*L^2*b^2*d1^2+A2*Da*G1^3*L^2*b*d1^3-2*A2*Da*G1^3*L*b^2*d1^3+Da*Fr*G1^3*L^2*b*d1^3-2*Da*
Fr*G1^3*L*b^2*d1^3+A1^4*L^4*b^2-3*A1^3*A2*Da*L^4*b+2*A1^3*A2*Da*L^3*b^2-5*A1^3*Da*Fr*L^4*b+
10*A1^3*Da*Fr*L^3*b^2+2*A1^3*G1*L^3*b^2*d1-8*A1^2*A2*Da*G1*L^3*b*d1+3*A1^2*A2*Da*G1*L^2*
S*b^2+6*A1^2*A2*Da*G1*L^2*b^2*d1-12*A1^2*Da*Fr*G1*L^3*b*d1+18*A1^2*Da*Fr*G1*L^2*b^2*d1+
A1^2*G1^2*L^2*b^2*d1^2-7*A1*A2*Da*G1^2*L^2*b*d1^2+2*A1*A2*Da*G1^2*L*S*b^2*d1+5*A1*A2*Da*
G1^2*L*b^2*d1^2-9*A1*Da*Fr*G1^2*L^2*b*d1^2+9*A1*Da*Fr*G1^2*L*b^2*d1^2-2*A2*Da*G1^3*L*b*

```

$$\begin{aligned}
& d1^3 + A2 * Da * G1^3 * b^2 * d1^3 - 2 * Da * Fr * G1^3 * L * b * d1^3 + Da * Fr * G1^3 * b^2 * d1^3 + 2 * A1^4 * L^4 * b - 4 * A1^4 * L^3 * \\
& b^2 + 2 * A1^3 * A2 * Da * L^3 * b + 2 * A1^3 * A2 * Da * L^2 * b^2 + 10 * A1^3 * Da * Fr * L^3 * b - 10 * A1^3 * Da * Fr * L^2 * b^2 + 4 * \\
& A1^3 * G1 * L^3 * b * d1 - 6 * A1^3 * G1 * L^2 * b^2 * d1 + 6 * A1^2 * A2 * Da * G1 * L^2 * b * d1 - 3 * A1^2 * A2 * Da * G1 * L * S * \\
& b^2 + 18 * A1^2 * Da * Fr * G1 * L^2 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L * b^2 * d1 - A1^2 * Da * G1^2 * L^2 * b^2 + 2 * A1^2 * G1^2 * \\
& L^2 * b * d1^2 - 2 * A1^2 * G1^2 * L * b^2 * d1^2 + 5 * A1 * A2 * Da * G1^2 * L * b * d1^2 - A1 * A2 * Da * G1^2 * S * b^2 * d1 - A1 * A2 * \\
& Da * G1^2 * b^2 * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b^2 * d1^2 + A2 * Da * G1^3 * b * d1^3 + Da * \\
& Fr * G1^3 * b * d1^3 + A1^4 * L^4 - 8 * A1^4 * L^3 * b + 6 * A1^4 * L^2 * b^2 + 2 * A1^3 * A2 * Da * L^2 * b - 3 * A1^3 * A2 * Da * L * b^2 - \\
& 10 * A1^3 * Da * Fr * L^2 * b + 5 * A1^3 * Da * Fr * L * b^2 + 2 * A1^3 * G1 * L^3 * d1 - 12 * A1^3 * G1 * L^2 * b * d1 + 6 * A1^3 * G1 * \\
& L * b^2 * d1 + A1^2 * A2 * Da * G1 * S * b^2 - A1^2 * A2 * Da * G1 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L * b * d1 + 3 * A1^2 * Da * \\
& Fr * G1 * b^2 * d1 + 2 * A1^2 * Da * G1^2 * L * b^2 + A1^2 * G1^2 * L^2 * d1^2 - 4 * A1^2 * G1^2 * L * b * d1^2 + A1^2 * G1^2 * b^2 * d1^2 - \\
& A1 * A2 * Da * G1^2 * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b * d1^2 - 4 * A1^4 * L^3 + 12 * A1^4 * L^2 * b - 4 * A1^4 * L * b^2 - 3 * A1^3 * \\
& A2 * Da * L * b + A1^3 * A2 * Da * b^2 + 5 * A1^3 * Da * Fr * L * b - A1^3 * Da * Fr * b^2 - 6 * A1^3 * G1 * L^2 * d1 + 12 * A1^3 * G1 * \\
& L * b * d1 - 2 * A1^3 * G1 * b^2 * d1 - A1^2 * A2 * Da * G1 * b * d1 + 3 * A1^2 * Da * Fr * G1 * b * d1 - A1^2 * Da * G1^2 * b^2 - 2 * \\
& A1^2 * G1^2 * L * d1^2 + 2 * A1^2 * G1^2 * b * d1^2 + 6 * A1^4 * L^2 - 8 * A1^4 * L * b + A1^4 * b^2 + A1^3 * A2 * Da * b - A1^3 * Da * Fr * \\
& b + 6 * A1^3 * G1 * L * d1 - 4 * A1^3 * G1 * b * d1 + A1^2 * G1^2 * d1^2 - 4 * A1^4 * L + 2 * A1^4 * b - 2 * A1^3 * G1 * d1 + A1^4 = 0 :
\end{aligned}$$

FF1:=fsolve(FF,G1):

SK[i]:=FF1[2]:

R[i]:=L:

i:=i+1:

od:

L:='L': points:=seq([R[j],SK[j]],j=1..300):

pcf2:=pointplot(points,labels=["L","Cf"],style=line,

thickness=3,colour=red):

with(plots):

#Copper

hs:=8933:

#Water

hf:=997.1:

b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0.05:

A1:=((1-r)^(2.5)):

A2:=(1-r+r*hs/hf):

i:=1:

R:=array(1..352):SK:=array(1..352):

for L from -9.632 by 0.005 to -9 do

$$\begin{aligned}
FF := & A1^3 * A2 * Da * L^5 * b^2 + A1^3 * Da * Fr * L^5 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b^2 * d1 + 3 * A1^2 * Da * Fr * G1 * \\
& L^4 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b^2 * d1^3 + \\
& Da * Fr * G1^3 * L^2 * b^2 * d1^3 + A1^3 * A2 * Da * L^5 * b - 3 * A1^3 * A2 * Da * L^4 * b^2 + A1^3 * Da * Fr * L^5 * b - 5 * A1^3 * Da * \\
& Fr * L^4 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b * d1 - A1^2 * A2 * Da * G1 * L^3 * S * b^2 - 8 * A1^2 * A2 * Da * G1 * L^3 * b^2 * d1 + \\
& 3 * A1^2 * Da * Fr * G1 * L^4 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b * d1^2 - A1 * A2 *
\end{aligned}$$

$$\begin{aligned}
& Da * G1^2 * L^2 * S * b^2 * d1 - 7 * A1 * A2 * Da * G1^2 * L^2 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b * d1^2 - 9 * A1 * Da * Fr * \\
& G1^2 * L^2 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b * d1^3 - 2 * A2 * Da * G1^3 * L * b^2 * d1^3 + Da * Fr * G1^3 * L^2 * b * d1^3 - 2 * Da * \\
& Fr * G1^3 * L * b^2 * d1^3 + A1^4 * L^4 * b^2 - 3 * A1^3 * A2 * Da * L^4 * b + 2 * A1^3 * A2 * Da * L^3 * b^2 - 5 * A1^3 * Da * Fr * L^4 * b + \\
& 10 * A1^3 * Da * Fr * L^3 * b^2 + 2 * A1^3 * G1 * L^3 * b^2 * d1 - 8 * A1^2 * A2 * Da * G1 * L^3 * b * d1 + 3 * A1^2 * A2 * Da * G1 * L^2 * \\
& S * b^2 + 6 * A1^2 * A2 * Da * G1 * L^2 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b * d1 + 18 * A1^2 * Da * Fr * G1 * L^2 * b^2 * d1 + \\
& A1^2 * G1^2 * L^2 * b^2 * d1^2 - 7 * A1 * A2 * Da * G1^2 * L^2 * b * d1^2 + 2 * A1 * A2 * Da * G1^2 * L * S * b^2 * d1 + 5 * A1 * A2 * Da * \\
& G1^2 * L * b^2 * d1^2 - 9 * A1 * Da * Fr * G1^2 * L^2 * b * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b^2 * d1^2 - 2 * A2 * Da * G1^3 * L * b * \\
& d1^3 + A2 * Da * G1^3 * b^2 * d1^3 - 2 * Da * Fr * G1^3 * L * b * d1^3 + Da * Fr * G1^3 * b^2 * d1^3 + 2 * A1^4 * L^4 * b - 4 * A1^4 * L^3 * \\
& b^2 + 2 * A1^3 * A2 * Da * L^3 * b + 2 * A1^3 * A2 * Da * L^2 * b^2 + 10 * A1^3 * Da * Fr * L^3 * b - 10 * A1^3 * Da * Fr * L^2 * b^2 + 4 * \\
& A1^3 * G1 * L^3 * b * d1 - 6 * A1^3 * G1 * L^2 * b^2 * d1 + 6 * A1^2 * A2 * Da * G1 * L^2 * b * d1 - 3 * A1^2 * A2 * Da * G1 * L * S * \\
& b^2 + 18 * A1^2 * Da * Fr * G1 * L^2 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L * b^2 * d1 - A1^2 * Da * G1^2 * L^2 * b^2 + 2 * A1^2 * G1^2 * \\
& L^2 * b * d1^2 - 2 * A1^2 * G1^2 * L * b^2 * d1^2 + 5 * A1 * A2 * Da * G1^2 * L * b * d1^2 - A1 * A2 * Da * G1^2 * S * b^2 * d1 - A1 * A2 * \\
& Da * G1^2 * b^2 * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b^2 * d1^2 + A2 * Da * G1^3 * b * d1^3 + Da * \\
& Fr * G1^3 * b * d1^3 + A1^4 * L^4 - 8 * A1^4 * L^3 * b + 6 * A1^4 * L^2 * b^2 + 2 * A1^3 * A2 * Da * L^2 * b - 3 * A1^3 * A2 * Da * L * b^2 - \\
& 10 * A1^3 * Da * Fr * L^2 * b + 5 * A1^3 * Da * Fr * L * b^2 + 2 * A1^3 * G1 * L^3 * d1 - 12 * A1^3 * G1 * L^2 * b * d1 + 6 * A1^3 * G1 * \\
& L * b^2 * d1 + A1^2 * A2 * Da * G1 * S * b^2 - A1^2 * A2 * Da * G1 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L * b * d1 + 3 * A1^2 * Da * \\
& Fr * G1 * b^2 * d1 + 2 * A1^2 * Da * G1^2 * L * b^2 + A1^2 * G1^2 * L^2 * d1^2 - 4 * A1^2 * G1^2 * L * b * d1^2 + A1^2 * G1^2 * b^2 * d1^2 - \\
& A1 * A2 * Da * G1^2 * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b * d1^2 - 4 * A1^4 * L^3 + 12 * A1^4 * L^2 * b - 4 * A1^4 * L * b^2 - 3 * A1^3 * \\
& A2 * Da * L * b + A1^3 * A2 * Da * b^2 + 5 * A1^3 * Da * Fr * L * b - A1^3 * Da * Fr * b^2 - 6 * A1^3 * G1 * L^2 * d1 + 12 * A1^3 * G1 * \\
& L * b * d1 - 2 * A1^3 * G1 * b^2 * d1 - A1^2 * A2 * Da * G1 * b * d1 + 3 * A1^2 * Da * Fr * G1 * b * d1 - A1^2 * Da * G1^2 * b^2 - 2 * \\
& A1^2 * G1^2 * L * d1^2 + 2 * A1^2 * G1^2 * b * d1^2 + 6 * A1^4 * L^2 - 8 * A1^4 * L * b + A1^4 * b^2 + A1^3 * A2 * Da * b - A1^3 * Da * Fr * \\
& b + 6 * A1^3 * G1 * L * d1 - 4 * A1^3 * G1 * b * d1 + A1^2 * G1^2 * d1^2 - 4 * A1^4 * L + 2 * A1^4 * b - 2 * A1^3 * G1 * d1 + A1^4 = 0:
\end{aligned}$$

FF1:=fsolve(FF,G1):

SK[i]:=FF1[1]:

R[i]:=L:

i:=i+1:

od:

L:='L':

points:=seq([R[j],SK[j]],j=1..300):

pcf21:=pointplot(points,labels=["L","Cf"],style=line,

linestyle=dot,thickness=3,colour=red):

with(plots):

#Copper

hs:=8933:

#Water

hf:=997.1:

b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0.1:

A1:=((1-r)^(-2.5)):

A2:=(1-r+r*hs/hf):

i:=1:

R:=array(1..352):SK:=array(1..352):

for L from -8.845 by 0.1 to 0.5 do

$$\begin{aligned} FF := & A1^3 * A2 * Da * L^5 * b^2 + A1^3 * Da * Fr * L^5 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b^2 * d1 + 3 * A1^2 * Da * Fr * G1 * \\ & L^4 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b^2 * d1^3 + \\ & Da * Fr * G1^3 * L^2 * b^2 * d1^3 + A1^3 * A2 * Da * L^5 * b - 3 * A1^3 * A2 * Da * L^4 * b^2 + A1^3 * Da * Fr * L^5 * b - 5 * A1^3 * Da * \\ & Fr * L^4 * b^2 + 3 * A1^2 * A2 * Da * G1 * L^4 * b * d1 - A1^2 * A2 * Da * G1 * L^3 * S * b^2 - 8 * A1^2 * A2 * Da * G1 * L^3 * b^2 * d1 + \\ & 3 * A1^2 * Da * Fr * G1 * L^4 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b^2 * d1 + 3 * A1 * A2 * Da * G1^2 * L^3 * b * d1^2 - A1 * A2 * \\ & Da * G1^2 * L^2 * S * b^2 * d1 - 7 * A1 * A2 * Da * G1^2 * L^2 * b^2 * d1^2 + 3 * A1 * Da * Fr * G1^2 * L^3 * b * d1^2 - 9 * A1 * Da * Fr * \\ & G1^2 * L^2 * b^2 * d1^2 + A2 * Da * G1^3 * L^2 * b * d1^3 - 2 * A2 * Da * G1^3 * L * b^2 * d1^3 + Da * Fr * G1^3 * L^2 * b * d1^3 - 2 * Da * \\ & Fr * G1^3 * L * b^2 * d1^3 + A1^4 * L^4 * b^2 - 3 * A1^3 * A2 * Da * L^4 * b + 2 * A1^3 * A2 * Da * L^3 * b^2 - 5 * A1^3 * Da * Fr * L^4 * b + \\ & 10 * A1^3 * Da * Fr * L^3 * b^2 + 2 * A1^3 * G1 * L^3 * b^2 * d1 - 8 * A1^2 * A2 * Da * G1 * L^3 * b * d1 + 3 * A1^2 * A2 * Da * G1 * L^2 * \\ & S * b^2 + 6 * A1^2 * A2 * Da * G1 * L^2 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L^3 * b * d1 + 18 * A1^2 * Da * Fr * G1 * L^2 * b^2 * d1 + \\ & A1^2 * G1^2 * L^2 * b^2 * d1^2 - 7 * A1 * A2 * Da * G1^2 * L^2 * b * d1^2 + 2 * A1 * A2 * Da * G1^2 * L * S * b^2 * d1 + 5 * A1 * A2 * Da * \\ & G1^2 * L * b^2 * d1^2 - 9 * A1 * Da * Fr * G1^2 * L^2 * b * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b^2 * d1^2 - 2 * A2 * Da * G1^3 * L * b * \\ & d1^3 + A2 * Da * G1^3 * b^2 * d1^3 - 2 * Da * Fr * G1^3 * L * b * d1^3 + Da * Fr * G1^3 * b^2 * d1^3 + 2 * A1^4 * L^4 * b - 4 * A1^4 * L^3 * \\ & b^2 + 2 * A1^3 * A2 * Da * L^3 * b + 2 * A1^3 * A2 * Da * L^2 * b^2 + 10 * A1^3 * Da * Fr * L^3 * b - 10 * A1^3 * Da * Fr * L^2 * b^2 + 4 * \\ & A1^3 * G1 * L^3 * b * d1 - 6 * A1^3 * G1 * L^2 * b^2 * d1 + 6 * A1^2 * A2 * Da * G1 * L^2 * b * d1 - 3 * A1^2 * A2 * Da * G1 * L * S * \\ & b^2 + 18 * A1^2 * Da * Fr * G1 * L^2 * b * d1 - 12 * A1^2 * Da * Fr * G1 * L * b^2 * d1 - A1^2 * Da * G1^2 * L^2 * b^2 + 2 * A1^2 * G1^2 * \\ & L^2 * b * d1^2 - 2 * A1^2 * G1^2 * L * b^2 * d1^2 + 5 * A1 * A2 * Da * G1^2 * L * b * d1^2 - A1 * A2 * Da * G1^2 * S * b^2 * d1 - A1 * A2 * \\ & Da * G1^2 * b^2 * d1^2 + 9 * A1 * Da * Fr * G1^2 * L * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b^2 * d1^2 + A2 * Da * G1^3 * b * d1^3 + Da * \\ & Fr * G1^3 * b * d1^3 + A1^4 * L^4 - 8 * A1^4 * L^3 * b + 6 * A1^4 * L^2 * b^2 + 2 * A1^3 * A2 * Da * L^2 * b - 3 * A1^3 * A2 * Da * L * b^2 - \\ & 10 * A1^3 * Da * Fr * L^2 * b + 5 * A1^3 * Da * Fr * L * b^2 + 2 * A1^3 * G1 * L^3 * d1 - 12 * A1^3 * G1 * L^2 * b * d1 + 6 * A1^3 * G1 * \\ & L * b^2 * d1 + A1^2 * A2 * Da * G1 * S * b^2 - A1^2 * A2 * Da * G1 * b^2 * d1 - 12 * A1^2 * Da * Fr * G1 * L * b * d1 + 3 * A1^2 * Da * \\ & Fr * G1 * b^2 * d1 + 2 * A1^2 * Da * G1^2 * L * b^2 + A1^2 * G1^2 * L^2 * d1^2 - 4 * A1^2 * G1^2 * L * b * d1^2 + A1^2 * G1^2 * b^2 * d1^2 - \\ & A1 * A2 * Da * G1^2 * b * d1^2 - 3 * A1 * Da * Fr * G1^2 * b * d1^2 - 4 * A1^4 * L^3 + 12 * A1^4 * L^2 * b - 4 * A1^4 * L * b^2 - 3 * A1^3 * \\ & A2 * Da * L * b + A1^3 * A2 * Da * b^2 + 5 * A1^3 * Da * Fr * L * b - A1^3 * Da * Fr * b^2 - 6 * A1^3 * G1 * L^2 * d1 + 12 * A1^3 * G1 * \\ & L * b * d1 - 2 * A1^3 * G1 * b^2 * d1 - A1^2 * A2 * Da * G1 * b * d1 + 3 * A1^2 * Da * Fr * G1 * b * d1 - A1^2 * Da * G1^2 * b^2 - 2 * \\ & A1^2 * G1^2 * L * d1^2 + 2 * A1^2 * G1^2 * b * d1^2 + 6 * A1^4 * L^2 - 8 * A1^4 * L * b + A1^4 * b^2 + A1^3 * A2 * Da * b - A1^3 * Da * Fr * \\ & b + 6 * A1^3 * G1 * L * d1 - 4 * A1^3 * G1 * b * d1 + A1^2 * G1^2 * d1^2 - 4 * A1^4 * L + 2 * A1^4 * b - 2 * A1^3 * G1 * d1 + A1^4 = 0: \end{aligned}$$

FF1:=fsolve(FF,G1):

SK[i]:=FF1[2]:

R[i]:=L:

i:=i+1:

od:

L:='L':

points:=seq([R[j],SK[j]],j=1..300):

pcf3:=pointplot(points,labels=["L","CF"],style=line,

thickness=3,colour=blue):

```

with(plots):
#Copper
hs:=8933:
#Water
hf:=997.1:
b:=10:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:r:=0.1:
A1:=((1-r)^( - 2.5)):
A2:=(1-r+r*hs/hf):
i:=1:
R:=array(1..352):SK:=array(1..352):
for L from -8.845 by 0.005 to -8.445 do
FF := A1^3*A2*Da*L^5*b^2 + A1^3*Da*Fr*L^5*b^2 + 3*A1^2*A2*Da*G1*L^4*b^2*d1 + 3*A1^2*Da*Fr*G1*
L^4*b^2*d1 + 3*A1*A2*Da*G1^2*L^3*b^2*d1^2 + 3*A1*Da*Fr*G1^2*L^3*b^2*d1^2 + A2*Da*G1^3*L^2*b^2*d1^3 +
Da*Fr*G1^3*L^2*b^2*d1^3 + A1^3*A2*Da*L^5*b - 3*A1^3*A2*Da*L^4*b^2 + A1^3*Da*Fr*L^5*b - 5*A1^3*Da*
Fr*L^4*b^2 + 3*A1^2*A2*Da*G1*L^4*b*d1 - A1^2*A2*Da*G1*L^3*S*b^2 - 8*A1^2*A2*Da*G1*L^3*b^2*d1 +
3*A1^2*Da*Fr*G1*L^4*b*d1 - 12*A1^2*Da*Fr*G1*L^3*b^2*d1 + 3*A1*A2*Da*G1^2*L^3*b*d1^2 - A1*A2*
Da*G1^2*L^2*S*b^2*d1 - 7*A1*A2*Da*G1^2*L^2*b^2*d1^2 + 3*A1*Da*Fr*G1^2*L^3*b*d1^2 - 9*A1*Da*Fr*
G1^2*L^2*b^2*d1^2 + A2*Da*G1^3*L^2*b*d1^3 - 2*A2*Da*G1^3*L*b^2*d1^3 + Da*Fr*G1^3*L^2*b*d1^3 - 2*Da*
Fr*G1^3*L*b^2*d1^3 + A1^4*L^4*b^2 - 3*A1^3*A2*Da*L^4*b + 2*A1^3*A2*Da*L^3*b^2 - 5*A1^3*Da*Fr*L^4*b +
10*A1^3*Da*Fr*L^3*b^2 + 2*A1^3*G1*L^3*b^2*d1 - 8*A1^2*A2*Da*G1*L^3*b*d1 + 3*A1^2*A2*Da*G1*L^2*
S*b^2 + 6*A1^2*A2*Da*G1*L^2*b^2*d1 - 12*A1^2*Da*Fr*G1*L^3*b*d1 + 18*A1^2*Da*Fr*G1*L^2*b^2*d1 +
A1^2*G1^2*L^2*b^2*d1^2 - 7*A1*A2*Da*G1^2*L^2*b*d1^2 + 2*A1*A2*Da*G1^2*L*S*b^2*d1 + 5*A1*A2*Da*
G1^2*L*b^2*d1^2 - 9*A1*Da*Fr*G1^2*L^2*b*d1^2 + 9*A1*Da*Fr*G1^2*L*b^2*d1^2 - 2*A2*Da*G1^3*L*b*
d1^3 + A2*Da*G1^3*b^2*d1^3 - 2*Da*Fr*G1^3*L*b*d1^3 + Da*Fr*G1^3*b^2*d1^3 + 2*A1^4*L^4*b - 4*A1^4*L^3*
b^2 + 2*A1^3*A2*Da*L^3*b + 2*A1^3*A2*Da*L^2*b^2 + 10*A1^3*Da*Fr*L^3*b - 10*A1^3*Da*Fr*L^2*b^2 + 4*
A1^3*G1*L^3*b*d1 - 6*A1^3*G1*L^2*b^2*d1 + 6*A1^2*A2*Da*G1*L^2*b*d1 - 3*A1^2*A2*Da*G1*L*S*
b^2 + 18*A1^2*Da*Fr*G1*L^2*b*d1 - 12*A1^2*Da*Fr*G1*L*b^2*d1 - A1^2*Da*G1^2*L^2*b^2 + 2*A1^2*G1^2*
L^2*b*d1^2 - 2*A1^2*G1^2*L*b^2*d1^2 + 5*A1*A2*Da*G1^2*L*b*d1^2 - A1*A2*Da*G1^2*S*b^2*d1 - A1*A2*
Da*G1^2*b^2*d1^2 + 9*A1*Da*Fr*G1^2*L*b*d1^2 - 3*A1*Da*Fr*G1^2*b^2*d1^2 + A2*Da*G1^3*b*d1^3 + Da*
Fr*G1^3*b*d1^3 + A1^4*L^4 - 8*A1^4*L^3*b + 6*A1^4*L^2*b^2 + 2*A1^3*A2*Da*L^2*b - 3*A1^3*A2*Da*L*b^2 -
10*A1^3*Da*Fr*L^2*b + 5*A1^3*Da*Fr*L*b^2 + 2*A1^3*G1*L^3*d1 - 12*A1^3*G1*L^2*b*d1 + 6*A1^3*G1*
L*b^2*d1 + A1^2*A2*Da*G1*S*b^2 - A1^2*A2*Da*G1*b^2*d1 - 12*A1^2*Da*Fr*G1*L*b*d1 + 3*A1^2*Da*
Fr*G1*b^2*d1 + 2*A1^2*Da*G1^2*L*b^2 + A1^2*G1^2*L^2*d1^2 - 4*A1^2*G1^2*L*b*d1^2 + A1^2*G1^2*b^2*d1^2 -
A1*A2*Da*G1^2*b*d1^2 - 3*A1*Da*Fr*G1^2*b*d1^2 - 4*A1^4*L^3 + 12*A1^4*L^2*b - 4*A1^4*L*b^2 - 3*A1^3*
A2*Da*L*b + A1^3*A2*Da*b^2 + 5*A1^3*Da*Fr*L*b - A1^3*Da*Fr*b^2 - 6*A1^3*G1*L^2*d1 + 12*A1^3*G1*
L*b*d1 - 2*A1^3*G1*b^2*d1 - A1^2*A2*Da*G1*b*d1 + 3*A1^2*Da*Fr*G1*b*d1 - A1^2*Da*G1^2*b^2 - 2*
A1^2*G1^2*L*d1^2 + 2*A1^2*G1^2*b*d1^2 + 6*A1^4*L^2 - 8*A1^4*L*b + A1^4*b^2 + A1^3*A2*Da*b - A1^3*Da*Fr*
b + 6*A1^3*G1*L*d1 - 4*A1^3*G1*b*d1 + A1^2*G1^2*d1^2 - 4*A1^4*L + 2*A1^4*b - 2*A1^3*G1*d1 + A1^4 = 0:
FF1:=fsolve(FF,G1):

```

```

SK[i]:=FF1[1]:
R[i]:=L:
i:=i+1:
od:
L:='L':
points:=seq([R[j],SK[j]],j=1..300):
pcf31:=pointplot(points,labels=["L","Cf"],style=line,linestyle=dot,thickness=3,colour=blue):
plots[display](pcf1,pcf11,pcf2,pcf21,pcf3,pcf31);
-----

```

Skin Friction and Heat Transfer Rate

```

-----
with(plots):
R:=array(1..352):T1:=array(1..352):
#Copper
cps:=385: hs:=8933: ks:=401: r:=0.1:
#Water
cpf:=4179: hf:=997.1: kf:=0.613:
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:S:=0.1:Pr:=6.2:Ec:=0.1:Bi:=0.1:
A1:=((1-r)^(2.5)):
A2:=(1-r+r*hs/hf):
A3:=(ks+2*kf-2*r*(kf-ks))/(ks+2*kf+r*(kf-ks)):
A4:=(1-r+r*(hs*cps)/(hf*cpf)):
fcns:=F(y),T(y):
i:=1:
for L from -1 by 0.02 to 1 do
sys:=(1+1/b)*A1*(diff(F(y),y$3)-(diff(F(y),y)-1)/Da)+A2*(F(y)*diff(F(y),y$2)-diff(F(y),y)*
diff(F(y),y)+1)-Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0, A3*diff(T(y),y2)+A4*Pr*(F(y)*diff(T(y),y)-
2*diff(F(y),y)*T(y))+A1*Pr*Ec*(1+1/b)*(diff(F(y),y2)*diff(F(y),y2)+(diff(F(y),y)-1)*(diff(F(y),y)-
1)/Da)+Pr*Ec*Fr*(diff(F(y),y)-1)*(diff(F(y),y)-1)*(diff(F(y),y)-1)=0:
p1:= dsolve(sys,F(0)=S,D(F)(0)=L+d1*(1+1/b)*D(D(F))(0),D(T)(0)=-Bi*(1-T(0)),D(F)(10)=1,T(10)=0,
fcns,type=numeric,method=bvp[midrich],abserr=1e-10):
R[i]:=L:
#Skin friction
T1[i]:=A1*(1+1/b)*rhs(p1(0)[4]):
#Nusselt Number T2[i]:=-A3*rhs(p1(0)[6]):
i:=i+1:
od:
L:='L':
points1:=seq([R[j],T1[j]],j=1..100):
points2:=seq([R[j],T2[j]],j=1..100):

```

```
ps1:=pointplot(points1,labels=["r","Cf"],style=line,color=black):
pn1:=pointplot(points2,labels=["r","Nu"],style=line,color=black):
```

Computing the Smallest Eigenvalues

```
with(plots):
#Copper
hs:=8933:
#Water
hf:=997.1:
#Parameter values and nanofluids properties
b:=0.1:d1:=0.1:Fr:=0.1:Da:=0.1:L:=0.1:S:=0.1:r:=0.1:G1:=6.297793:
A1:=((1-r)(-2.5)):
A2:=(1-r+r*hs/hf):
m:=G1/((A1*(1-L)-G1*d1)*(1+1/b)):
#Exact solution for basic flow
F1:=S+y+(1-L)*(exp(-m*y)-1)/(m*(1+d1*m*(1+1/b))):
fcns:=H(y):
#Solving eigenvalue problem
sys:=(1+1/b)*A1*diff(H(y),y$3)+A2*F1*diff(H(y),y$2)+(A2*x-2*A2*diff(F1,y)-(1+1/b)*(A1/Da)-
2*Fr*(diff(F1,y)-1))*diff(H(y),y)+A2*diff(F1,y$2)*H(y)=0:
p1:= dsolve(sys,H(0)=0,D(H)(0)=d1*(1+1/b)*D(D(H))(0),D(D(H))(0)=1,D(H)(10)=0,fcns,type=numeric,
method=bvp[midrich],abserr=1e-10):
```