ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
SCHOOL OF APPLIED NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS

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Adane Kebede Tare

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTERS OF SCIENCE IN APPLIED MATHEMATICS

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Abstract

In this thesis research work we analyzed the impact of time delay on the dynamics of a predator-prey model with in the growth rate of the prey equation and constant rate of harvesting on both populations. We identified conditions for stable co-existence of the system. The study showed the existence of stable equilibrium point for the model with harvesting maximum profit. The result also showed that the time delay can induce a globally asymptotically stable equilibrium point in the positive quadrant can occur for the model with and without harvesting. Further,the result implies the predator and prey populations can live in coexistence.

Keywords : Predator-Prey, Time Delay, Jacobian Matrix, Eigenvalues, Effort of Harvesting, Profit.

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Chapter 1

Introduction

Ecosystems are characterized by the interaction between different species and natural environment. In the natural world species do not exist alone; it is of more ecological significance to study the persistence(permanent)-extinction threshold of each population in systems of two or more interacting species subjected to predator-prey systems. From the point of view of human needs, the exploitation of biological resources, the management of renewable resources, and the harvesting of populations are commonly human purpose of achieving the economic interest in fishery, forestry, and wildlife management, Chen et al(2013). Harvest management has been used to control increasing population and to meet the public demands for recreation, animal damage control or commercial harvesting. Many populations are managed under the assumption the population will continue to increase until it approaches the limits of the available resources to support it.

1.1 Historical background

1.1.1 Malthusian growth

Malthusian growth: a population increases exponentially as long as resources are unlimited. Defining $p(t)$ to be the population abundance at time t this growth can be described by the differential equation with growth rate r. Malthus (1798)

$$\frac{dp}{dt} = rp(t) \tag{1.1.1}$$

The equation tells us that a population grows in proportion to its current size and $p(t) = ce^{rt}$ is a solution to this equation, where c is an arbitrary constant. Indeed, if we differentiate p(t), we obtain $\frac{dp(t)}{dt} = rce^{rt} = rp(t)$. The population at the time t = 0 is $p(0) = p_0$, p(t) = P_0e^{rt} . The differential equation $\frac{dp}{dt} = rp(t), p(0) = p_0$ is an initial condition. Since the solution to equation(1.1.1) is $P(t) = ce^{kt}$, and we say that the population grows exponentially. Not all populations grow exponentially; a bacteria culture in a petri dish would grow unbounded and soon be much larger than the size of the laboratory. But limited resources (e.g., space, food, essential nutrients) can stop population growth introduced empirically by Verhulst (1838) in what is called today **logistic growth**,

$$\frac{dp}{dt} = rp(t)(1 - \frac{p}{k}) \tag{1.1.2}$$

The carrying capacity k defines some population abundance limit beyond which the population growth rate becomes negative, while below it, this growth rate is positive. Thus, k is some equilibrium value towards which the population abundance tends to converge. The logistic equation (1.1.2) has a trivial equilibrium (x = 0 , i.e. the population dies out) and a nontrivial equilibrium x = k (the population has reached the environment capacity). We can see that the population p goes to infinity for $r > 0$ and p goes to 0 for $r < 0$. The case $r > 0$ is biologically impossible since every population is restricted by the carrying capacity of its environment.

However, in population dynamics, the population size shows often an oscillatory behavior. A very famous example is given by Nicholson's(1957) data relative to cultures of the sheep blowfly Lucilia cuprina solutions to the logistic equation (1.1.2) do not show any oscillatory behavior. But if one includes a constant delay in (1.1.2), the result is the following equation

$$\frac{dp}{dt} = rp(t)(1 - \frac{p(t-\tau)}{k}) \tag{1.1.3}$$

where r and k are as described before and the delay $\tau > 0$ is a constant

1.1.2 The Logistic Model with Harvesting

Harvesting is a technique used for biological resource exploitation. However, overexploitation is the most responsible factor to the persistence of harvested stocks and causes not only economic loss but also changes the ecosystem structure and functioning, Chakraborty(2012). If we allow harvesting at a constant rate E, the logistic growth model becomes

$$\frac{dp}{dt} = rp(1 - \frac{p}{k}) - E \tag{1.1.4}$$

In this case $\frac{rk}{4}$ is the maximum sustainable yield.

1.1.3 Delay Models

Delay is a general concept that can represent different phenomena such as the time it takes for the primogeniture to reach maturity or the finite gestation period of a species. Mathematical delays are input in model to correct the classical logistic model, which assumes that the growth rate of a population at time t is determined by the number of individual at that time. The delay in the model means when predator consumes prey, they use the time to reproduce the next generation. Of course, ecological delays are complex and the mathematical representation is often a simplification of reality. Delay is frequently used in a predator-prey model to represent the ecological process more accurately. The general observation is that the time delay makes a stable equilibrium unstable and harvesting, on the other hand, helps to regain the stability, Freedaman(1987).

1.2 Mathematical preliminary

Initial value problems (IVPs) for ODEs like

$$\begin{aligned} x'(t) &= f(t, x(t)), t \geq t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{1.2.5}$$

$f : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, are used to model many physical, biological in which x represents some quantity which evolves in time or economical phenomena, in several cases, in order

to make a mathematical model closer to the real phenomenon, one can change the right-hand side of (1.2.5) in an ODE, the evolution at time t depends on the current state at time t and possibly on t. So an ODE does cannot directly account for such things as one way to overcome this limitation: delay differential equations (DDE) and prefer a delay equation approach:

$$\begin{aligned} x'(t) &= f(t, x(t), x(t-\tau_1), ..., x(t-\tau_n)), t \geq t_0, \\ x(t) &= \phi(t), \quad t \leq t_0, \end{aligned} \tag{1.2.6}$$

with $f : \mathbb{R} \times \mathbb{R}^m \times (\mathbb{R}^m)^n \longrightarrow \mathbb{R}^m$. According to the complexity of the problem, the delays τ_i, which are always nonnegative, may be constant values (constant delay problem), or functions of t (time-dependent delay problem), or functions of t and y itself (state-dependent delay problem). The function ϕ in (1.2.6) is called the history of the solution and it is at least depend in [r, t_0] , where

$$r = \quad \overset{min}{1 \leq i \leq n} \left\{ \begin{array}{c} min(t-\tau_0) \\ t \geq t_0 \end{array} \right\},$$

For constant delay, the lower bound r can be determined a priori. Suppose $f(x_1^*, x_2^*, ..., x_n^*) =$ 0, n = 1,2,3,... is a steady state of system (1.2.6).

Definition 1.1. The steady state x = x^* of system (1.2.6) is said to be absolutely stable (i.e., asymptotically stable independent of the delays) if it is asymptotically stable for all delays $\tau_j \geq 0 (1 \leq j \leq m)$. x = x^* is said to be conditionally stable (i.e., asymptotically stable depending on the delays) if it is asymptotically stable for τ_j (1$\leq j \leq$m) in some intervals, but not necessarily for all delays $\tau_j \geq 0 (1 \leq j \leq m)$.Ruan(2009)

The linearized system of (1.2.6) at x = x^* has the form:

$$x'(t) = A_0 x(t) + \sum_{j=1}^{m} A_j x(t-\tau_j), \tag{1.2.7}$$

where x $\in \mathbb{R}^n$, each $A_j (0 \leq j \leq m)$ is an n $\times$ n constant matrix. Thus, the steady state $x = x^*$ of system (1.2.6) is absolutely stable (conditionally stable) if the trivial solution of the linearized system (1.2.7) is absolutely stable (conditionally stable). Note that we

are dealing with local stability of system (1.2.6). The characteristic equation associated with system (1.2.7) takes the transcendental polynomial equation form:

$$det\left[\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\tau\lambda}\right] = 0 \tag{1.2.8}$$

delayed differential equation (DDE) has the general form (1.2.6). In this study, we consider a delay in the specific growth rate of prey to incorporate the effect of density dependence feedback mechanism which takes τ units of time to respond to changes in the prey population(P) or gestation period.

$$\frac{dp}{dt} = rp(t)(1 - \frac{p(t-\tau)}{k})$$

where r and k are as described before and the delay $\tau > 0$ is a constant. See model(1.1.2)

1.3 Systems dynamics

Mathematical modeling and analysis of multiple species ecological problems was first done by Volterra (1927). Volterra had been introduced to an ecological problem that in the years after the first World War, the proportion of the predatory fishes caught in the Upper Adriatic Sea was found to be considerably higher than in the years before the war, whereas the proportion of prey fishes was down. In order to come out with an explanation to this ecological problem, Volterra formulated and analyzed a system of ordinary differential equations which is represented as below:

$$\begin{aligned} \frac{dx}{dt} &= x(a - by) \\ \frac{dy}{dt} &= y(-c + dx), \end{aligned} \tag{1.3.9}$$

where a, b, c and d were positive constants; x and y were the densities of the prey fish and predator fish respectively. This system of differential equations was also studied independently by Lotka (1925) in the context of chemical kinetics and is now known as the Lotka-Volterra model.

In real world ecosystems prey populations do not grow exponentially in the absence of predator, but rather their size is eventually limited by the absence of resources. To account for this, the logestic model has a population regulation term with Holling type I functional response is given by

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{k}) - axy, \\ \frac{dy}{dt} &= bxy - cy. \end{aligned} \tag{1.3.10}$$

Here, x(t) and y(t) represent the density of prey and predator populations at time t, respectively. Parameters r, k, a, c and b are positive constants that denote the intrinsic reproduction rate, carrying capacity for the prey, capture rate of the prey, death rate of the predator in the absence of the prey and the growth yield of the predator, respectively. Modeling is a frequently evolving process, to gain a deep understanding of the mathematical aspects of the problem and to yield non trivial biological insights, one must carefully construct biologically meaningful and mathematically tractable population models, Kuang(2002). System is a collection of several equations with several unknowns. The general form for a mathematical model that describes the dynamics between any two species, having prey-predator type of interaction, has the following structure

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y), \end{aligned} \tag{1.3.11}$$

by studying the phase portrait in the phase plane(Brauer 2011)

The Gause type predator-prey model with Holling type I functional response is given by

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{k}) - mxy \quad , \\ \frac{dy}{dt} &= -dy + msxy. \end{aligned}$$

Here, $x(t)$ and $y(t)$ represent the density of prey and predator populations at time t, respectively. Parameters r, k, m, d and s are positive constants that denote the intrinsic reproduction rate, carrying capacity for the prey, capture rate of the prey, death rate of the predator in the absence of the prey and the growth yield of the predator, respectively.

Freedman (1980) came up with a generalized prey-predator model represented by the system of differential equations below:

$$\frac{dx}{dt} = xg(x) - yf(x),$$
$$\frac{dy}{dt} = y[-e + p(x)].$$

where x and y are the densities of the prey and predator respectively, g(x) is the growth rate of the prey in absence of the predators, f (x) is the functional response of the predators with respect to prey x, and p(x) is the numerical response of the predator. In most cases, p(x) is a product of a constant and f (x). The functions g(x), f (x) and p(x) are continuous and differentiable and have the following specific properties

- g(x) = $\alpha > 0$, $\frac{dg}{dx} < 0$ for all x ≥ 0 and there exists a real number k called the carrying capacity such that g(k) = 0.
- f (0) = 0, $\frac{df}{dx} > 0$ for all x ≥ 0.
- p (0) = 0, $\frac{dp}{dx} > 0$ for all x ≥ 0 and $lim_{x\to\infty}p(x) = p_\infty \leq \infty$

A system dynamics model is a mathematical object that deserves mathematical analysis. Any system dynamics model is a mathematical object known as dynamical system,

that can be formulated as, Stuart(1996).

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, ..., x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, ..., x_n) \\
&\cdot \\
&\cdot \\
&\cdot \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, ..., x_n)
\end{aligned} \tag{1.3.12}$$

where $f_i, \forall i = 1, 2, 3, ..., n; x \in \mathbb{R}^n$, f_i is a possibly nonlinear function. Normally, the function f is highly involved as it represents the whole model, but, at least conceptually, a model can be thought as a mathematical object of the form (1.3.12).

Definition 1.2. A point $E^* = (x^*, y^*)$ is said to be an equilibrium point (or steady state point) of system (1.3.11) if it satisfies the following equation f(x^*, y^*) = 0 and g(x^*, y^*) = 0

Linearization

Linearization is only an approximation of the non-linear system. Linearization only tells us how solutions behave near the equilibrium point. A solution curve might behave quite differently if it is far away from the equilibrium solution. We now give a small perturbation to the system about the steady state (E^*), and mathematically this means we put $x = X + x^*$ and $y = Y + y^*$. This implies

$$\frac{dX}{dt} = f(x^* + X, y^* + Y) = f(x^*, y^*) + Xf_x(x^*, y^*) + Yf_y(x^*, y^*) + ...$$

higher order terms (by Taylor series expansion of two variables) Similarly,

$$\frac{dY}{dt} = g(x^* + X, y^* + Y) = g(x^*, y^*) + Xg_x(x^*, y^*) + Yg_y(x^*, y^*) + ...$$

higher order terms where $f_x(x^*,y^*)$ is evaluated at the steady state (x^*, y^*). Since by definition, f$(x^*, y^*) = 0$, g$(x^*, y^*) = 0$, by neglecting second and higher order terms,Banerjee(2014).

we get

$$\begin{aligned} \frac{dX}{dt} &= f_x(x^*, y^*)X + f_y(x^*, y^*)Y \\ \frac{dY}{dt} &= g_x(x^*, y^*)X + g_y(x^*, y^*)Y. \end{aligned} \tag{1.3.13}$$

The Jacobian matrix of model (3.3.2) takes the form

$$A = J(E_i) = \begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}$$

The nature of the curve can be determined by the characteristic polynomial of the matrix J. Thus, det(A-Iλ) = 0. λ^2 - $tr(J)\lambda$ + det(J) = 0. Let $tr(J) = \psi$ and det (J) = $\Delta \Rightarrow \lambda^2$ - $\psi\lambda + \Delta = 0 \Rightarrow \lambda_{1,2} = \dfrac{\psi \pm \sqrt{\psi^2 - 4\Delta}}{2}$ where ψ is the trace of the Jacobian matrix, Δ is the determinant of Jacobian matrix and λ_1, λ_2 are the eigenvalues.

Definition 1.3. Let (x^*, y^*) be an equilibrium point of system (1.3.11). Then (x^*, y^*) is called a hyperbolic equilibrium point, Allen(2007) if non of the eigenvalues of $J = \partial f(x^*, y^*)$ have zero real part, otherwise it is a non-hyperbolic equilibrium point. According to this arguments equilibrium point (x^*, y^*) of the system (1.3.12) can be classified as i.**A saddle point** If $\psi < 0$, then the eigenvalues have opposite signs, that is, $\lambda_1 < 0 < \lambda_2$. Since the discriminant $\psi^2 - 4\Delta$ is positive, the eigenvalues are real. The equilibrium point is called a saddle.
ii. **Stable node** If $\Delta > 0$ and $\psi^2 - 4\Delta \geq 0$, then the equilibrium point is a stable node when $\psi < 0$, since both eigenvalues are real and negative.
iii. **Unstable node** If $\Delta > 0$ and $\psi^2 - 4\Delta \geq 0$, then the equilibrium point is unstable when $\psi > 0$, since both eigenvalues are real and negative.
iv. **Stable focus** If $\Delta > 0$ and $\psi^2 - 4\Delta < 0$, then the equilibrium point is a stable focus when $\psi < 0$, since the eigenvalues are complex whose real parts are negative.

1.3.1 Routh-Hurwitz conditions

For a system of autonomous ODEs, the Routh-Hurwitz conditions provide a necessary and sufficient condition for all the eigenvalues of the Jacobian matrix of an equilibrium point to have negative real parts. It follows from the method of linearization that such an equilibrium point is locally attracting. The Gause type predator-prey model with Holling

type I functional response is given by

$$\begin{aligned} \frac{dx}{dt} &= x(a - by) \\ \frac{dy}{dt} &= y(-c + dx), \end{aligned} \tag{1.3.14}$$

Definition 1.4. Routh-Hurwitz criterion, May(1973): Consider the following polynomial of degree n , which is defined by: $p_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n = 0$
Let $D_1 = a_1$,

$$D_2 = det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix},$$

$$D_k = det \begin{bmatrix} a_1 & a_3 & a_5 & ... & a_{2k-1} \\ 1 & a_2 & a_4 & ... & a_{2k-2} \\ 0 & a_1 & a_3 & ... & a_{2k-3} \\ : & 1 & a_2 & ... & a_{2k-4} \\ : & : & : & ... & : \\ 0 & 0 & 0 & ... & a_k \end{bmatrix}$$

where $a_i = 0$ if $i > n$. Then the roots of $P_n(\lambda)$ have negative real part if and only if $D_k > 0$ for all k = 1 , 2 ,..., n . Now, by applying this criterion when n = 2 , we have $\lambda^2 + a_1\lambda + a_2 = 0$ and hence $D_1 = a_1$
Thus n = 2, the necessary and sufficient conditions for having roots with negative real part are $a_1 > 0$ and $a_2 > 0$
For n = 3, we have $p_3(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ and hence $D_1 = a_1$

$$D_2 = det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix} = a_1a_2 - a_3$$

$$D_3 = det \begin{bmatrix} a_1 & a_3 & 0 \\ 1 & a_2 & 0 \\ 0 & a_1 & a_3 \end{bmatrix} = (a_1a_2 - a_3)a_3$$

Thus, for n = 3 , the necessary and sufficient conditions for having roots with negative real part are $a_1 > 0, a_3 > 0$ and $a_1a_2 - a_3 > 0$

Definition 1.5. The system (1.3.12) is globally asymptotically stable. That is,

i if the system has a positive equilibrium state E^*, then this equilibrium state is unique and globally asymptotically stable (in $\mathbb{R}^2_+$);

ii. if the system has no positive equilibrium state, then there is a predator-free equilibrium state $E_0 = (x_0, 0)$, which is globally asymptotically stable.

Theorem 1.1 (Lyapunov)**.** *Suppose there is a function* $\mathbb{R}^n \to \mathbb{R}$

i Defined and continuously differentiable on the set $E = B(x^*, \varepsilon)$*;*

ii $V(x^*) = 0$*; and*

iii $V(x) \geq 0$ *if* $x \neq x^*$ *then the equilibrium* x^* *is*

1. *Stable if V'(x) $\leq$ 0 for all $x \in E$.*
2. *Asymptotically stable if V'(x) $<$ 0 for all $x \in E$, except possibly at x^* itself.*
3. *Unstable if V'(x) $\geq$ 0 for all $x \in E$, except possibly at x^**

1.4 Statement of the problem

Interaction between food and its eater is the basic rule of nature. Understanding the dynamical relationship between predator and prey is the central goal of ecology. Depending upon the behavior of populations, more suitable 'functional response' has been developed as a quantification of the relative responsiveness of the predation rate to change in prey density at various populations of prey. The classical Lotka-Volterra model has been modified incorporating Verhulst or Logistic growth to take into account the fact that the resource is limited, and also to avoid the structural unstable nature of the model. In this thesis , we will try to study a mathematical model which clearly describes the existing reality between predator-prey systems to generate stable ecological system. Hence, the study will raise the following leading research questions:

1 Under what condition a mathematical model efficiently describes the dynamics of a delayed predator-prey system with constant rate of harvesting.
2 Is the system both locally and globally stable?
3 What are the ecological interpretations of the solution of the governing equation?

1.5 Objective of the study

1.5.1 General Objectives

The general objective of this study is investigate the dynamics of a delayed predator-prey system with constant rate of harvesting.

1.5.2 Specific Objectives

The specific objective of the study is to:

- describe a mathematical model of the dynamics of a delayed predator-prey system with constant rate of harvesting.
- to examine the effects of both harvesting effort and delay on the model dynamics.
- to analyze the predator-prey model with constant rate of harvesting in both populations.
- study local and global stability of the system with harvesting.
- examine the ecological implications of the analysis.

1.6 Significance of the study

The finding of this study is essential for the following reasons.

1 Help policy makers in making possible decisions to regulate ecological balance of predator-prey population.
2 useful for the mathematicians who are interested in the ecology fields because this research will give us more understanding in predator-prey model with or without harvesting and the effects on the stability of the population.
3 Suggest the possible intervention mechanisms.
4 Initiate other researchers to investigate further analysis.
5 Give information about co-existence system

1.7 Delimitation of the study

The present study will be limited to the analysis of a delayed predator-prey system with constant rate of harvesting. Secondary data will be used to validate the proposed model.

1.8 Research design and methodology

We describe a delayed predator-prey system with constant rate of harvesting using non-linear differential equations. Both local and global stability is discussed. Ecological implications of the analytical findings is presented critically.

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Chapter 2

Review literature

In the real world, the biosphere is an important zone for biological activities that are mainly responsible for the changes in ecology and environment and the growth rate of different species mainly depend on ecology, carrying capacity of environment etc. As a consequence growth rate of the prey pieces is an important matter for the predator- prey interaction model. During the last few decades, there has been great interest in the construction and study of models for the population dynamics. In mathematical biology the highest priority given in establishing conditions for the stability of ecosystems and for stable coexistence of interacting populations. Ecologists serve the real-field data to form mathematical models, and mathematicians study the various characteristics of the models that return the biologists to gain insight into complex ecosystems. Predator-prey systems are basic differential equation models for describing the interactions between two species with a pair of positive-negative feedbacks. While the predator-prey model of systems of two ordinary differential equations has relatively simple dynamics due to the powerful Poincare-Bendixon theory, the models of more species, or the models with spatial structure, delay effect, nonlocal effect are mathematically much more challenging, Lotka(1925), Davidson et al(2004), Braza (2012), Holling (1965).

Feng (2007) considered a differential equation system with diffusion and time delays which

models the dynamics of predator-prey interactions within three biological species. The results obtained by him explicitly present the effects of all the environmental data (growth rates and interaction rates) on the ultimate bounds of the three biological species and numerical simulations of the model were given to demonstrate the pattern of dynamics (extinction, persistence, and permanence) in the ecological model. The effect of constant rate harvesting on the dynamics of predator-prey systems has been investigated by Dai and Tang (1998), Myerscough et al.(1992) and Xiao and Ruan (1999), and they obtained very rich and interesting dynamical behaviours. Nakaoka et al.(2006) examined a Lotka-Volterra predator-prey system with two delays. They concluded that a positive equilibrium of the system is globally asymptotically stable for small delays and the system exhibits some chaotic behavior when time delay becomes large. They analytically determined critical values of time delay through which the system undergoes a Hopf bifurcation. The dynamics of a predator-prey system, where predator population has two stages, juvenile and adult with harvesting were modelled using a system of delay differential equation by Kar (2007). He concluded that both the delay and harvesting effort may play a significant role on the stability of the system.

Kar (2003) studied a prey-predator system with delay, Holling Type II functional response and harvesting of the prey. The study showed that as harvesting effort increased, the predators population decreased as expected. More importantly, if the harvesting effort was above a critical value which was determined in the study, the dynamics of the system changed from limit cycle to global asymptotic stability. This showed that harvest ing of the prey alone indirectly affected the population density of the predators and also played a crucial role in stabilizing the dynamics of the prey-predator systems. The delay term was included to ensure that only mature prey were harvested.

Fay and Greeff (2006) proposed a model to study the dynamics of the lion-wildebeest-zebra interaction in Kruger National Park (South Africa) in which the lion predated on the wildebeests and zebras. Starting with a simple model, they showed that by carefully incorporating in the simple model, terms to represent plausible biological aspects such as logistic growth with mutualism among wildebeests and zebras, type II functional response for the lions, seasonal calving of zebras, cropping of lions, obtained a model which comes close to fitting experimental data available. Their approach showed how well thought mathematical concepts should be incorporated in a model.

In terms of predator-prey systems with constant yield harvesting, Huang et al.(2013) systematically studied the dynamical properties of a predator-prey model of Holling and Leslie type with nonzero constant yield prey harvesting. They have shown that the harvested model can exhibit richer dynamics compared to the model with no harvesting, such as appearance of numerous kinds of bifurcations for the model, including saddle-node bifurcation, Hopf bifurcation, repelling and attracting Bogdanov-Takens bifurcations of co-dimensions 2 and 3.

In terms of constant-effort harvesting, a ratio-dependent predator-prey model in which the prey is continuously being harvested at a linear function rate was studied by Xiao et al. (2009). They proved that the system has different behaviors for various parameter values. Particularly, there exist areas of coexistence in which both populations become extinct, and areas of conditional coexistence depending on the initial values. Makinde (2007) developed an algorithm to approach the solution of the ratio-dependent predator-prey system with constant-effort harvesting. In addition to these work mentioned above, the effect of constant-effort harvesting in predator-prey models has been studied in, Rojas-Palma,et al.(2012).

Kar and Pahari (2007) studied the effect of harvesting and time delay on the dynamics of the generalized Gause type predator-prey models. Hoeckstra and Bergh (2005) focused on optimal harvesting of prey in a predator-prey ecosystem. They found the conditions for the existence of the predators when the predators and humans compete for prey.

May et al.(1979) proposed the following model to describe the interaction of predators and their prey subjected to various harvesting regimes:

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{k}) - axy - H_1, \\ \frac{dy}{dt} &= bxy - cy - H_2. \end{aligned} \tag{2.0.1}$$

where x(t) and y(t) represent the prey and predators densities at time t, respectively; r and k describe the intrinsic growth rate and the carrying capacity of the prey in the absence of predators, respectively; a is the maximum value at which per capita reduction rate of the prey x can attain; r is the intrinsic growth rate of predators; bx takes on the role of a prey-dependent carrying capacity for predators and b is a measure of the quality of the food for predators. H_1 and H_2 describe the effect of harvesting on the prey and predators, respectively.

Yafia (2007) established an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions through considering a model with one delay. A model in which the revenue is generated from fishing and the growth of fish depends upon the plankton which in turn grows logistically is developed by Dhar et al. (2008). They further formulated the model with delay in digestion of plankton by fish and found the threshold value of conversional parameter for Hopf-bifurcation. Kar and Matsuda (2006), and Kar and Pahari (2007) discussed the predator prey model with time delay and analyzed the effects of time delay on model dynamics such as the time delay may change the stability of equilibrium points and even cause a switching of stabilities.

From the above literature survey it is very relevant to point out that no attempt has been made to study the dynamics of a predator-prey system in the presence of an alternative resource regarding maximize Profit. Also we have considered a gestation delay of prey population and harvesting of both species regarding maximize profit. As the global population increase at an exponential rate, we put our environment under dangerous pressure. Therefore, having the tools to avoid over-exploiting the ecosystem is a small but important step in keeping the global ecosystem in a viable state.

Chapter 3

Model Analysis

3.1 Introduction

In this chapter, we present model description, formulation and analysis. The formulation of model consists of the construction of the equations based on the intended assumptions. Terms representing logistic growth of the prey species in absence of the predator are included in the prey equations. Constant rate harvesting of both predator and prey is incorporated in the model. The model has two non linear autonomous ordinary differential equations describing how the population densities of the two species would vary with time. The analysis includes the calculation and determination of the stability of equilibrium points. The findings consist of an interpretation of the analytical outcomes.

3.2 Assumptions, variables and parameters of the Predator Prey Model

3.2.1 The assumptions

Some general assumptions are considered in the case of interaction of prey-predator systems:

i. In the absence of the predators, the prey population grows logistically.
ii. Both the prey and predator populations are under harvesting.
iii. The rate of predation upon the prey is proportional to the encounters of predators and prey.
iv. The predator population increases, due to predation upon preys.
v. Predators will die out exponentially in the absence of preys.
vi. The species live in an ecosystem where external factors such as droughts, fires, epidemics are stable or have a similar effect on the interacting species.
vii. The carrying capacity (k) means the maximum population size of the prey. It depends on the amount of resources available in the ecosystem, the size of the population and the amount of resources for each individual's consumption.

3.2.2 The Variables

The following variables are used in the model:

- x(t) - the population of the prey at time t. It is assumed that the prey population grows logistically to its carrying capacity k with intrinsic growth rate r. The prey population gathered in a group to form a herd to save themselves from predator population.
- y(t) - the population of the predator at time t. For simplicity let x(t) and y(t) be represented as x and y.

3.2.3 The parameters

The following are the parameters used in the model:

1. r is per capita intrinsic growth rate for prey x.
2. k is environmental carrying capacity for prey x.

3. a denote the rate of consumption of prey by predator.
4. b denote conversion of prey consumed into predator reproduction rate.
5. c denote natural mortality rate of predator y.
6. E_x denote the rate of harvesting for the populations x .
7. E_y denote the rate of harvesting for the populations y.
8. τ denote the gestation period of prey.

3.2.4 The model equations

From the model description, assumptions, definition of variables and parameters in section (4.2), the equations to represent the dynamics of the preys and predators in ecosystem are formulated as,

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t)(1 - \frac{x(t-\tau)}{k}) - ax(t)y(t) - E_x \\ \frac{dy(t)}{dt} &= bx(t)y(t) - cy(t) - E_y \end{aligned} \tag{3.2.1}$$

with initial conditions $x(\theta)$, $y(\theta) \geq 0$, $\theta \in [-\tau, 0)$ and $x(0) > 0, y(0) > 0$. Here, E_x and E_y are the rates of harvesting for the prey and predator population respectively.

Now, the conservative resources are important such that the focused attention on management of harvesting in predator-prey system has become an interesting topic in mathematical bio-economic research. The population with harvesting is related to the renewable resource management. The exploitation of harvesting population species is used in fishery, forestry and wildlife management. The population dynamics with harvesting is related to the optimal management of renewable resources. The goals of management are to make the population increase or decrease to harvest the population for a continuing yield. The harvesting process as fishery and hunting includes searching for food or sport. Thus, the harvesting is a controller of the density of population. The functional form of harvest is generally considered using the phrase catch-per-unit-effort (CPUE) hypothesis, Clark (1990) to describe an assumption that catch per unit effort is proportional to the

stock level. $E_x = E_1 q_1 x$ and $E_y = E_2 q_2 y$, where E_1 and E_2 are, respectively, the harvesting effort used to harvest from prey and predator population, q_1 and q_2 are, respectively, the catchability coefficient of prey and predator population. Here in this model we assume $q_1 = q_2 = 1$. If we put in a good effort (labors, tools) then the harvesting rate increases. However, the rate of harvesting is limited by a carrying capacity. So, to manage the population dynamics in ecology, we consider the predator and prey population model (3.5.6) and non-selective harvesting of both species. For a nonlinear delay system, there are two types of stabilities: absolute stability (independent of the delay) and conditional stability (depending on the delay). Here we consider four cases separately with and without time delay and with and without harvesting.

Case 1. For $\tau = 0$, $E_x = E_y = 0$ the system (3.2.1). Such a system is called Predator - Prey population Model.

Case 2. For $\tau = 0$ the system (3.2.1), such a system is called predator-prey population with constant rate of harvesting model.

Case 3. For $E_x = E_y = 0$ the system (3.2.1), such a system is called a delayed predator-prey population Model.

Case 4. Model equation is called a delayed predator-prey population Model with harvesting. Now we will discuss each case as follows.

3.3 A Predator-Prey population Model

For $\tau = 0$, $E_x = E_y = 0$ the system (3.2.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= rx(1 - \frac{x}{k}) - axy, \\ \frac{dy}{dt} &= bxy - cy. \end{aligned} \tag{3.3.2}$$

where r, a, b, c, k, are positive constants. Such model is called usual predator prey model.

3.3.1 Existence of equilibrium points of the system

In this section, we establish conditions for the existence of the equilibrium points of the system. we consider the prey nullcline and predator nullcline of this system(3.3.2), which are given by:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) - axy \quad = 0,$$
$$\frac{dy}{dt} = bxy - cy = 0.$$

It is obvious that the equilibria are the intersections of these nullclines. Then solving the system for x and y we have three equilibrium points $E_0 = (0, 0)$, $E_1 = (k, 0)$ and $E^* = (x^*, y^*) = (\frac{c}{b}, \frac{r(kb - c)}{abk})$. We assume that kb - c > 0 in order to get a positive equilibrium point. The existence of $E_0 = (0, 0)$ is trivial. This solution effectively represents the extinction of both species. If both populations are at 0, then they will continue to be so indefinitely. The axial equilibrium $E_1 = (k, 0)$ exists . Thus, in the absence of predator y, the density of prey x will increase or decrease until it reaches the carrying capacity k.

3.3.2 Stability of the equilibrium points

In this section we discuss the stability of different equilibrium points of the system (3.2.1), by computing the variational matrix at various equilibrium points and using the Routh-Hurwitz criterion, May(1973). An equilibrium point is stable if a small change in the initial conditions gives a point which tends toward the equilibrium as the independent variable tends towards positive infinity. An equilibrium point is unstable if a small change in the initial conditions gives a point which veers away from the equilibrium as the independent variable tends towards positive infinity. The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues evaluated at each equilibrium point. An equilibrium point is asymptotically stable if all eigenvalues of the Jacobian matrix have negative real parts and it is unstable if at least one eigenvalue has positive real part. The Jacobian matrix of model (3.3.2) takes the form

$$J(E_i) = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix}$$

which gives

$$J(E_i) = \begin{bmatrix} r - \frac{2rx}{k} - ay & -ax \\ by & bx - c. \end{bmatrix}$$

i. Trivial equilibrium $E_0(0,0)$. (i.e. the population dies out). The Jacobian matrix evaluated at E_0 gives,

$$J(E_0) = A = \begin{bmatrix} r & 0 \\ 0 & -c \end{bmatrix}, det(A - \lambda I) = 0 \Rightarrow f(\lambda) = \begin{vmatrix} r - \lambda & 0 \\ 0 & -c - \lambda \end{vmatrix}$$

characteristic equation of the Jacobian matrix J at E_0 is $f(\lambda) = \lambda^2 + (r - c)\lambda - rc$. The eigenvalues of $J(E_0)$ are r and -c with opposite sign and so $E_0(0,0)$ is saddle.

ii. At the axial equilibrium $(E_1) = (k, 0)$, (the population has reached the environment capacity). We have

$$J(E_1) = A = \begin{bmatrix} -r & -ak \\ 0 & bk - c \end{bmatrix}$$

We have, $\det(A - \lambda I) = 0$

$$f(\lambda) = \begin{vmatrix} -r - \lambda & -ak \\ 0 & bk - c - \lambda. \end{vmatrix}$$

The characteristic equation of the Jacobian matrix J at E_1 is $f(\lambda) = (\lambda)^2 + (c + r - bk)\lambda + (rc - rbk)$. We assume that $c + r - bk > 0$ and $(rc - rbk) > 0$ by Routh-Hurwitz criteria in order to get both eigenvalues have negative real parts. When c+r < bk we will get positive eigenvalues. Thus, the equilibrium point E_1 is unstable.

iii. At the interior equilibrium point E^* , we have

$$J(E^*) = A = \begin{bmatrix} \frac{rc}{bk} & -\frac{ac}{b} \\ \\ \frac{rbk - rc}{ak} & 0. \end{bmatrix}$$

$$0 = det(A - \lambda I)$$

$$f(\lambda) = \begin{vmatrix} -\frac{rc}{bk} - \lambda & -\frac{ac}{b} \\ \frac{rbk - rc}{ak} & 0 - \lambda. \end{vmatrix}$$

$$f(\lambda) = \lambda^2 + \frac{rc}{bk}\lambda + \frac{cr}{bk}(kb - c).$$

Let $m = \frac{rc}{bk}$ and n $= \frac{c}{bk}(rbk - rc)$ the eigenvalues of the Jacobian matrix are $\lambda_{1,2} = \frac{-m \pm \sqrt{m^2 - 4n}}{2}$. Since m and n are both positive, then both eigenvalues have negative real parts. It means that the equilibrium point E^* is locally asymptotically stable. Furthermore, since $kb - c > 0$ then the equilibrium point is also globally asymptotically stable. see Ho and Ou (2002).

3.4 Predator - prey population model and constant rate of harvesting

Consider the model of prey-predator system (3.2.1) with no time delay, we need to analyze the stability of the equilibrium point of this model.

The model without time delay is

$$\begin{aligned} \frac{dx}{dt} &= rx(1 - \frac{x}{k}) - axy - E_x, \\ \frac{dy}{dt} &= bxy - cy - E_y. \end{aligned} \tag{3.4.3}$$

where r, a, b, c, k, E_x and E_y are positive constants. The constants E_x and E_y denote the constant rate of harvesting from the populations x and y respectively.

By substituting $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ then we have the relations

$$\begin{aligned} rx(1 - \frac{x}{k}) - axy &= E_x \\ bxy - cy &= E_y. \end{aligned} \tag{3.4.4}$$

From (3.4.4) we have $y = \frac{rx - \frac{rx^2}{k} - E_x}{ax}$ which follows that $r^2 - 4\frac{r}{k}E_x$ should be positive in order to get the equilibrium point in the positive quadrant. Hence we have to assume

that $E_x > \frac{rk}{4}$ since E_y is positive, then from (3.4.4) we should assume that $E_x > \frac{rk}{4}$. $\frac{rb}{k}x^3 - (rb + \frac{rc}{k})x^2 + (E_x b + E_y a + rc)x - E_x c < 0$, for some positive $x > \frac{c}{b}$. Let the two equilibrium points be $E_1 = (x_1, y_1)$ and $E_2 = (x_2, y_2)$. The equilibrium point E_1 is possible to be asymptotically stable, while the equilibrium point E_2 is not stable, it is a saddle point. To analyze the stability of the equilibrium point the Jacobian matrix of the model is

$$J(E_i) = \begin{bmatrix} r - \frac{2rx}{k} - ay & -ax \\ by & bx - c. \end{bmatrix}$$

The characteristic equation of the Jacobian matrix at this point by letting

$f = r - \frac{2rx_1}{k} - ay_1$, $g = ax_1$, $h = by_1$ and $z = bx_1 - c$ is

$$f(\lambda) = \lambda^2 - (f + z)\lambda + fz + gh \tag{3.4.5}$$

Then the equilibrium point E is asymptotically stable when $fz + gh > 0$ and $f + z < 0$.

3.5 A delayed predator prey population model

In this section we consider a time delay into the predator-prey model. It is inherently assumed that all the metabolic energy a predator obtains through its food used for logistic growth which ultimately enhance the predator population. Here the predator population consumes the prey population at a constant rate , but the reproduction of predators after predating the prey population is not instantaneous thus it will be incorporated by some time lag required for of preys. This gestation period of preys is represented by τ. Starting from Hutchinsons delay logistic model May (1974) has proposed the following system

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t)(1 - \frac{x(t-\tau)}{k}) - ax(t)y(t) \\ \frac{dy(t)}{dt} &= bx(t)y(t) - cy(t), \end{aligned} \tag{3.5.6}$$

Where r, k, τ, a, c and b are positive constants. Model (3.5.6) contains a single discrete delay. The term $(1 - \frac{x(t-\tau)}{k})$ in model (3.5.6) denotes a density dependent feedback mechanism which takes τ units of time to respond to changes in the population density.

3.6 A delayed predator-prey population model with constant rate of harvesting

We consider that the predator and prey populations are harvested by the same gear of fishing, then it is reasonable to assume that the efforts of harvesting are the same, i.e, $E_x = E_y = E$. From model (3.2.1)

$$\begin{aligned}
\frac{dx(t)}{dt} &= rx(t)(1-\frac{x(t-\tau)}{k}) - ax(t)y(t) - Ex(t)\\
\frac{dy(t)}{dt} &= bx(t)y(t) - cy(t) - E(t)\\
\frac{dx(t)}{dt} &= (r-E)x(t) - rx(t)\frac{(r-E)x(t-\tau)}{(r-E)k} - ax(t)y(t)\\
\frac{dy(t)}{dt} &= bx(t)y(t) - (c+E)y(t)\\
\frac{dx(t)}{dt} &= r_1x(t)(1-\frac{x(t-\tau)}{k_1}) - ax(t)y(t)\\
\frac{dy(t)}{dt} &= bx(t)y(t) - c_1y(t),
\end{aligned} \tag{3.6.7}$$

Where $r_1 = r - E$, $k_1 = \dfrac{(r-E)k}{r}$, and $c_1 = c + E$.

We assume that $r > E$. This assumption is made to guarantee the intrinsic growth of prey population is greater than the effort of harvesting. The constants E_x and E_y denote the constant rate of harvesting from the populations x and y respectively. By substituting $\frac{dx(t)}{dt} = 0$ and $\frac{dy(t)}{dt} = 0$. Clearly, $E_0(0,0)$ is the trivial equilibrium point of the system (3.2.1) and all other possible equilibrium points of the system (3.2.1), considered in the first quadrant of x-y plane, are as follows:

$$\begin{aligned}
E_1 &= (x,0) \quad where, x = k_1\\
E_2 &= (0,y) \quad where, y = 0\\
E^* &= (x^*,y^*) = (\frac{c_1}{b}, \frac{r_1(k_1b-c_1)}{abk_1})
\end{aligned} \tag{3.6.8}$$

We consider the following system of DDEs equation

$$\begin{aligned}
x'(t) &= f(t,x(t),x(t-\tau)), t \geq t_0,\\
x(t) &= \phi(t), \quad t \leq t_0,
\end{aligned} \tag{3.6.9}$$

If it does not possess a periodic solution at $\tau = 0$, Hopf bifurcation at an equilibrium point x = 0 is investigated for the existence of a periodic solution. Chung, et al.(2002) Linearization of (3.6.9) produces the system

$$x'(t) = cx - dx(t-\tau) \tag{3.6.10}$$

where c and d are the Jacobians given by $c = f_X(x = x_0) \quad and \quad d = f_{X(t-\tau)}(x = x_0)$ The solution of the characteristic equation of (3.6) is nontrivial if and only if $\mid \lambda I - c - de^{-\tau\lambda} \mid = 0$. To have a positive equilibrium point we assume that $k_1 b - c_1 > 0$, that is; $0 < E < \dfrac{r_1(k_1 b - c_1)}{(kb + r)}$. Under this assumption the equilibrium point E^* is in the first quadrant. In order to understand the locally asymptotically stability of the equilibrium point E^* in the model with time delay, we analyze the associated linearization model with perturbation. From model (3.2.1)

$$\begin{aligned}
&\frac{dx(t)}{dt} = r_1 x(t)(1 - \frac{x_\tau}{k_1}) - ax(t)y(t) \\
&Let, \frac{dx}{dt} = f(x, x_\tau, y), x(t) = x, x(t-\tau) = x_\tau \\
&f(x, x_\tau, y) = r_1 x(1 - \frac{x_\tau}{k_1}) - axy \\
&\Rightarrow x'(t) = cx - dx_\tau + ey
\end{aligned}$$

where c and d are the Jacobians given by

$$c = \frac{\partial f(x, x_\tau, y)}{\partial x} \mid_{(x,y)=(x^*,y^*)}, \quad d = \frac{\partial f(x, x_\tau, y)}{\partial x_\tau} \mid_{(x,y)=(x^*,y^*)} and \quad e = \frac{\partial f(x, x_\tau, y)}{\partial y} \mid_{(x,y)=(x^*,y^*)}$$

Let substituting $u(t) = x(t) - x^*$ and $v(t) = y(t) - y^*$ into model (3.2.1) and neglecting the product terms to get the linearized model; similarly for $\dfrac{dy(t)}{dt} = bx(t)y(t) - c_1 y(t)$, we get the system

$$\begin{aligned}
\frac{du(t)}{dt} &= -\frac{r_1}{k_1} x^* u(t-\tau) - ax^* v(t) \\
\frac{dv(t)}{dt} &= by^* u(t).
\end{aligned}$$

Analyzing the local stability of the equilibrium point E^* in the model with time delay is equivalent to analyzing the stability of zero equilibrium point in the linearized model.

$$J(E_i) = \begin{bmatrix} -\dfrac{r_1}{k_1} x e^{-\lambda\tau} & ax \\ by & 0 \end{bmatrix}$$

$det(A - \lambda I) = 0$

$$J(E^*) = A = \begin{vmatrix} -\frac{r_1}{k_1}x^*e^{-\lambda\tau} - \lambda & ax^* \\ by^* & 0 - \lambda \end{vmatrix} = 0$$

From the linearized model we have the characteristic equation

$$\lambda^2 + \lambda p_1 e^{-\lambda\tau} + q_1 = 0, \tag{3.6.11}$$

where $p_1 = \frac{r_1}{k_1}x^*$ and $q_1 = abx^*y^*$. For $\tau = 0$ the characteristic equation becomes

$$\lambda^2 + \lambda p_1 + q_1 = 0, \tag{3.6.12}$$

which has the roots

$$\lambda = \frac{-p_1 \pm \sqrt{p_1^2 - 4q_1}}{2}. \tag{3.6.13}$$

Since p_1 and q_1 are both positive, the characteristic equation (3.6.12) has negative roots.

Theorem 3.1. *Let $\tau = 0$ and; $0 < E < \frac{r(kb-c)}{(kb+r)}$. Then the equilibrium point for model (3.6.7) is asymptotically stable.*

Proof. From the condition $0 < E < \frac{r(kb-c)}{(kb+r)}$ we get the equilibrium point E^* is in the positive quadrant. Since $\tau = 0$, then the characteristic equation as stated in equation (3.6.12) has negative eigenvalues. We conclude that the equilibrium point E^* is asymptotically stable. We know that the equilibrium point E^* of model (3.3.2) is globally asymptotically stable. Since the equilibrium point E^* of model (3.6.7) is similar with the equilibrium point E^* of model (3.3.2), and then we conclude that the equilibrium point E^* of model (3.6.7) is also globally asymptotically stable. □

Time-delay is an important and obvious factor in biological systems. Now for $\tau \neq 0$, if $\lambda = i\omega, \omega > 0$, is a root of the characteristic equation (3.6.11), then we have

$$\begin{aligned} &-\omega^2 + ip_1\omega e^{-i\omega\tau} + q_1 = 0, \\ &-\omega^2 + ip_1\omega\cos(\omega\tau) + p_1\omega\sin(\omega\tau) + q_1 = 0. \end{aligned}$$

Separating the real and imaginary parts, we have

$$\begin{aligned} -\omega^2 + p_1\omega\sin(\omega\tau) + q_1 = 0, \\ p_1\omega\cos(\omega\tau) = 0. \end{aligned} \tag{3.6.14}$$

Squaring both sides gives

$$p_1^2\omega^2\sin^2(\omega\tau) = \omega^4 - 2q_1\omega^2 + q_1^2,$$
$$p_1^2\omega^2\cos^2(\omega\tau) = 0.$$

Adding both equations and regrouping by powers of , we obtain the following fourth degree polynomial

$$\omega^4 - (p_1^2 + 2q_1)\omega^2 + q_1^2 = 0 \tag{3.6.15}$$

From which we obtain

$$\omega_\pm^2 = \frac{1}{2}\{(p_1^2 + 2q_1) \pm \sqrt{p_1^4 + 4p_1^2q_1}\}. \tag{3.6.16}$$

From (3.6.16) we can see that there are two positive solutions of ω^2. We can now find the values of $\tau_j^\pm$, by substituting ω^2 into equations (3.6.14) and solving for τ. We obtain

$$\tau_k^+ = \frac{\pi}{2\omega_+} + \frac{2k\pi}{\omega_+}, \quad \tau_k^- = \frac{3\pi}{2\omega_-} + \frac{2k\pi}{\omega_-}, \quad k = 0, 1, 2, ... \tag{3.6.17}$$

Theorem 3.2. *Let $kb - c > 0$, $0 < E < \dfrac{r(kb-c)}{(kb+r)}$ and $\tau_k^\pm$ be defined in equation (3.6.17). Then there exists a positive integer m such that there are m switches from stability to instability and to stability. In other words, when $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup ... \cup (\tau_{m-1}^-, \tau_m^+)$, the equilibrium point E^* for model (3.6.7) is stable and when $\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup ... \cup (\tau_{m-1}^+, \tau_{m-1}^-)$, the equilibrium point E^* is unstable. Therefore, there are Hopf bifurcations at E^* for $\tau = \tau_k^\pm, \quad k = 0, 1, 2, ...$*

Proof. From (3.6.13) we know that the equilibrium point E^* is stable for $\tau = 0$. Then to prove the theorem we need only to verify the transvers ability conditions, see Cushing (1977). Transversality conditions are optimality conditions often used along with Euler equations to characterize the optimal paths (plans, programs, trajectories,etc) of dynamic economic models. we want to show occurrence of hope bifurcation.

$$\frac{d(Re\lambda)}{d\tau}|_{\tau=\tau_k^+} > 0 \quad and \quad \frac{d(Re\lambda)}{d\tau}|_{\tau=\tau_k^-} < 0.$$

Differentiating the equation 3.6.11 with respect to τ we obtain

$$2\lambda\frac{d\lambda}{d\tau} + p_1e^{-\lambda\tau}\frac{d\lambda}{d\tau} + \lambda p_1e^{-\lambda\tau}(-\tau\frac{d\lambda}{d\tau} - \lambda) = 0$$
$$(2\lambda + (1 - \lambda\tau)p_1e^{-\lambda\tau})\frac{d\lambda}{d\tau} = \lambda^2p_1e^{-\lambda\tau}.$$

For convenience, we study $(\frac{d\lambda}{d\tau})^{-1}$ instead of $\frac{d\lambda}{d\tau}$. Then we have

$$\begin{aligned}(\frac{d\lambda}{d\tau})^{-1} &= \frac{2\lambda e^{\lambda\tau} + p_1(1-\lambda\tau)}{\lambda^2 p_1} \\ &= \frac{2\lambda e^{\lambda\tau} + p_1}{\lambda^2 p_1} - \frac{\tau}{\lambda}\end{aligned}$$

From the characteristic equation (3.6.11) we know that $e^{\lambda\tau} = \dfrac{-\lambda p_1}{\lambda^2 + q_1}$.

Then we have $(\frac{d\lambda}{d\tau})^{-1} = \dfrac{-\lambda^2 + q_1}{\lambda^2(\lambda^2 + q_1)} - \dfrac{\tau}{\lambda}$. Therefore

$$\begin{aligned}sign(\frac{d(Re\lambda)}{d\tau})_{\lambda=i\omega} &= sign(Re(\frac{d\lambda}{d\tau})^{-1})_{\lambda=i\omega} \\ &= sign(Re[\frac{-1}{\lambda^2+q_1}]_{\lambda=i\omega} + Re[\frac{q_1}{\lambda^4+\lambda^2 q_1}]_{\lambda=i\omega}) \\ &= sign(\frac{-1}{-\omega^2+q_1} + \frac{q_1}{\omega^4-\omega^2 q_1}) = sign(\frac{\omega^4-q_1^2}{\omega^2(\omega^2-q_1^2)}) \\ &= sign(\omega^4 - q_1^2).\end{aligned}$$

From equation(3.6.15) we know that $\omega^4 - q_1^2 = 2\omega^4 - (p_1^2 + 2q_1)\omega^2$. Then we have

$$sign(\frac{d(Re\lambda)}{d\tau})_{\lambda=i\omega} = sign(2\omega^4 - (p_1^2+2q_1)\omega^2) = sign(2\omega^2 - (p_1^2+2q_1)).$$

By substituting the expression for $\omega_\pm^2$, it is easy to see that the sign is positive for ω_+^2 and the sign is negative for ω_-^2. Therefore, crossing from left to right with increasing τ occurs for values of τ corresponding to ω_+ and crossing from right to left occurs for values of τ corresponding to ω_- . From (3.6.16) and the last result, we can verify that the transversality conditions are satisfied. Therefore $\tau_k^\pm$ are bifurcation values. □

Theorem 3.3. *Given $\tau > 0$. The equilibrium point E^* for model (3.6.7) is stable if the effort of harvesting satisfies $E \in E_b \cap E_s$, where $E_b = (0, \dfrac{r(kb-c)}{(kb+r)})$ and E_s is a set of the solution of the inequality*

$$f(E) = \frac{\pi^4}{4\tau^4} - \frac{\pi^2}{\tau^2}p_1^2 - \frac{2\pi^2}{\tau^2}q_1 + 4q_1^2 > 0.$$

Proof. In order the equilibrium point E^* exists in the positive quadrant we assume that $0 < E < \dfrac{r(kb-c)}{(kb+r)}$. From Theorem 2, we know that the equilibrium point E^* is stable when $\tau \in [0, \tau_0^+)$. From equation (3.6.16) we have $\omega_+ = [\frac{1}{2}\{(p_1^2+2q_1) + \sqrt{p_1^4+4p_1^2q_1}\}]^{\frac{1}{2}}$. Since the time delay (τ) is given the we have to set a condition such that $\tau < \dfrac{\pi}{2\omega_+}$ or $\omega_+ < \dfrac{\pi}{2\tau}$. Then we have

$$[\frac{1}{2}\{(p_1^2+2q_1) + \sqrt{p_1^4+4p_1^2q_1}\}]^{\frac{1}{2}} < \frac{\pi}{2\tau},$$

$$\frac{1}{2}\{(p_1^2+2q_1)+\sqrt{p_1^4+4p_1^2q_1}\}<\frac{\pi^2}{4\tau^2}$$

$$(p_1^2+2q_1)+\sqrt{p_1^4+4p_1^2q_1}<\frac{\pi^2}{2\tau^2},$$

$$\sqrt{p_1^4+4p_1^2q_1}<\frac{\pi^2}{2\tau^2}-(p_1^2+2q_1),$$

$$p_1^4+4p_1^2q_1<(\frac{\pi^2}{2\tau^2}-(p_1^2+2q_1))^2,$$

$$p_1^4+4p_1^2q_1<\frac{\pi^4}{4\tau^4}-\frac{\pi^2}{\tau^2}(p_1^2+2q_1)+(p_1^2+2q_1)^2.$$

After simplifying we get

$$\frac{\pi^4}{4\tau^4}-\frac{\pi^2}{\tau^2}p_1^2-\frac{2\pi^2}{\tau^2}q_1+4q_1^2>0.$$

Since both p_1 and q_1 depends on E then the last inequality can be solved in terms of E. Further we conclude that the equilibrium point E^* is stable when the effort satisfies $E\in E_b\cap E_s$. □

3.7 Profit Analysis

Now we relate the stable equilibrium point $E^*=(x^*,y^*)=(\frac{c_1}{b},\frac{r_1(k_1b-c_1)}{abk_1})$ to the maximum profit problem. We define total cost, $TC=c_f+c_vE$ is social cost and harvesting cost; total revenue, $TR=p_xx^*E+p_yy^*E$. Hence we have the profit function $\pi=p_xx^*E+p_yy^*E-c_f-c_vE$. Substituting $x^*=\frac{c_1}{b}$ and $y^*=\frac{r_1(k_1b-c_1)}{abk_1}$ to obtain $\pi=\frac{-A_1}{abk}E^2+\frac{B_1}{abk}E-C_f$
which has the critical point $E_c=\frac{B_1}{2A_1}$ Where $A_1=p_ykb+p_yr-ap_xk$ which is assumed to be positive and $B_1=ap_xkc-ac_vkb-p_ycr+p_ykbr$. Under this assumption, the critical point $E_c=\frac{B_1}{2A_1}$ minimizes the profit function. From the assumption $0<E_c<\frac{r(kb-c)}{(kb+r)}$, then we need a condition so that the critical point $E_c\in(0,\frac{r(kb-c)}{(kb+r)})$, i.e., $0<\frac{B_1}{2A_1}<\frac{r(kb-c)}{(kb+r)}$. In the case of Critical point $E_c>\frac{r(kb-c)}{(kb+r)}$, the profit function becomes maximum at $E_c=\frac{r(kb-c)}{(kb+r)}$. However, this situation leads to extinction of the population y, since $y^*=0$.

Chapter 4

Conclusion

The main focus of this thesis was to investigate the effect of a time delay and constant rate of harvesting on the predator prey system. The Lotka-Volterra predator-prey equation is a critical model in ecology. We have seen throughout this thesis that incorporating a harvesting constant in a model and varying it can change the stability of a system; although it does not seem to be a general rule. From the mathematical analysis change in stability is not supposed to happen when altering E as long as it respects the conditions and is below the critical harvesting rate (Ec). We have also seen that adding delays into a model can change its stability. We have shown that adding a delay in the logistic growth term of the prey equation created m switches from stability to instability as τ increases. This means that the system can go back to stability when unstable.

In this thesis, we established some new results in linearizing method of delayed differential equations. We have stated and proved several results giving criterion for the existence of equilibrium points, stability and bifurcations for the model system. In the model without time delay and harvesting, the positive equilibrium occurs when $kb - c > 0$ and it is globally asymptotically stable. This means that the predator and prey populations can live in coexistence. For the equilibrium point and also exists a critical value of the rate that maximizes the profit function. This means that under suitable values of the parameters, time delay and effort of harvesting, the predator and prey populations can remain

in existence and give greatest profit. This is valuable from the point of view of ecology.

4.1 Further development of the model

Our thesis is useful for future research work regarding a delayed predator-prey model with constant rate of harvesting. Further investigation can be carried out on this study especially regarding how to estimate parameter values used in the model from field data. Qualitative analysis on how to find the co-existence equilibrium point and conditions for local and global stability can be done following a different approach. Hopf bifurcation can be carried out on the model. Numerical simulation is left for further investigation.

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