Analysis of Boundary Domain-Integral Equations for Variable-Coefficient Dirichlet BVP in 2D

By

Abera Fekadu

A Thesis Submitted to
The program of Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the Degree of Master's of Science in Applied Mathematics(Specialization in Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

Adama, Ethiopia
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APPROVAL PAGE

This is certify that the thesis prepared by Abera Fekadu entitled "Analysis of Boundary Domain-Integral Equations for variable coefficient Dirichlet BVP in 2D" submitted in partial fulfillment of the requirement for the degree of Masters of Science in Mathematics with the regulation of the university and meets the accepted standards with respect to originality.

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Abstract

In this research, the Dirichlet boundary value problem for the second order "stationary heat transfer" elliptic partial differential equation with variable coefficient is considered in 2D. Using an appropriate parametrix function, this problem is reduced to some direct segregated systems of Boundary Domain Integral Equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain for the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is analysed and the invertibility of the BDIE operators is proved.

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Notation

Here, the notations used in this research are introduced and some definitions are given as follows.

$C_c^\infty(\Omega)$ - is the class of all infinitely differentiable functions on Ω with compact support.

$\Gamma := \partial\Omega$ - the boundary of the domain Ω.

$\triangle a$ Laplacian of a , $\triangle a = \nabla.\nabla a$.

Γh - boundary of Ωh.

$C(\Omega)$ - is the class of all continuous functions on Ω.

φ , Ψ - trial functions .

$C_0(\Omega)$ - is the class of all continuous functions on Ω that vanishes at boundary (for bounded sets) or ∞.

$W^{k,p}(\Omega)$ - is the class of all functions in $L^p(\Omega)$ whose Distributional Derivative up to order k are also in $L^p(\Omega)$.

$C^k(\Omega)$ - the Class of all $k - times$ continuously differentiable functions on Ω.

$\overline{\Omega}$ - closure of the Ω.

$C^\infty(\bar{\Omega})$ = Class of $C^\infty(\Omega)$ function such that all its derivatives can be extended continuously to $\bar{\Omega}$

Ω - the space domain.

$\mathbb{R}^n$ - denotes n - dimensional Euclidean space.

$H_0^k(\Omega)$ - is the space $W_0^{k,2}(\Omega)$.

∇^2 - Laplace operator.

$H^k(\Omega)$ - is the space $W^{k,2}(\Omega)$.

$D(\Omega)$ - is the class of all infinitely differentiable functions on Ω with compact support endowed with inductive limit topology.

ΓD - boundary parts with Dirichlet data.

∇a - Gradient of a.

i, j, l - indices.

Chapter 1

Introduction

1.1 Back ground of the study

Partial Differential Equations (PDEs) with variable coefficients often arise in Mathematical modeling of in homogeneous media (e.g. functionally graded materials or materials with damage induced in homogeneity) in solid mechanics, electro magnectics thermo-conductivity; fluid flows through porous media and other areas of physics and engineering[2, 6].

The Boundary Integral Equation Method (BIEM) is a well established tool for solution of Boundary Value Problems (BVPs) with constant coefficients. The main ingredient for reducing a BVP for a PDE to a BIE is a fundamental solution to the original PDE. However, it is generally not available in an analytical form for PDEs with variable coefficients[2]. Nevertheless, for a rather wide class of variable-coefficients PDEs it is possible to use instead an explicit parametric (Levi function) associated with a fundamental solution of corresponding frozen-coefficient PDEs and reduce BVPs for such PDEs in interior domains to systems of Boundary-Domain Integral Equations (BDIEs) for further numerical solution[3].

The Boundary Integral Equation (BIE) method (boundary element method, elastic potential method) has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility (at least for some problems with constant coefficients) of reducing a Boundary Value Problem (BVP) for a linear partial differential equation in a domain to an integral equation on the domain boundary, that is, to diminish the problem dimensionality by one. It leads to a diminution of the linear algebraic equations system, which results from discretization, and allows to obtain numerical solutions using small computer resources[10].

The main thing necessary for the reduction of a BVP to a BDIE is a fundamental solution to the original partial differential equation, available in an analytical form. After the fundamental solution is used in the corresponding Green formulae, one can reduce the problem to a boundary integral equation. However, such a fundamental solution is generally not available if the coefficients of the original BVP are not constant[1, 6, 10]. The BVPs of heat transfer with variable heat conductivity coefficients and the BVPs of elastic shells particularly belong to this category. One can use, in this case, a parametrix function, which is usually available, instead of the fundamental solution in the Green formulae. Parametric correctly describes the main part of the fundamental solution but is not required to satisfy the original differential equations apart from the singular point,[3].

1.2 Statment of the problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by infinitely differentiable curve $\partial\Omega$. We shall drive and investigate Boundary Domain Integral Equation(BDIE), for the following Dirichlet BVP Find a function $u \in H^1(\Omega)$ satisfying:

$$Au(x) := \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i}(a(x)\frac{\partial u(x)}{\partial x_i}) = f(x) \quad in \quad \Omega$$

$$\gamma^+ u(x) = \varphi_0 \quad on \quad \partial\Omega$$

Where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$ are given functions.

This research work will answer the following questions.

- What is the necessary condition for the solvability of this Dirichlet BVP ?
- Is the Boundary Value Problem uniquely solvable ?
- How to transform this Dirichlet BVP to BDIE systems ?
- Is the original BVP equivalent to the obtained BDIE systems ?
- Are the BDIE operators for the Dirichlet BVP invertible?

1.3 Objective of the study

1.3.1 General objective of the study

The main objective of this study will be to analyse the boundary domain integral equations for variable coefficient dirichlet boundary value problem in two dimension.

1.3.2 Specific objective of the study

The specific objective of this study will be as following

- To investigate solvability of the Dirichlet BVP.
- To show the Dirichlet BVPs with variable coefficient in interior domain can be reduced to some systems of BDIEs.
- To investigate invertibility of the boundary-domain integral operators obtained in the appropriate sobolev space.
- To show the equivalence of the original Boundary Value Problem and the obtained Boundary Domain Integral Equations.
- To investigate the invertibility of the Boundary Domain Integral operators of Dirichlet BVP.

1.4 Significance of the study

The study of this problem will have the following importance: The study of this research is highly important to understand the uniqueness, existence, and invertibility of Boundary Domain Integral Equation (BDIE) for variable coefficient Dirichlet BVP. It also improves understanding of the Boundary Domain Integral Equations (BDIE) so that reducing and analysing the given Dirichlet BVPs for variable coefficients. Also, initiates other researchs to under take further extension and important to understand analysis of BDIE for Variable coefficient dirichlet BVPs in 2D. It also uses as references for those who needs to conduct research on this area.

Chapter 2

Review of Literature

Many researchers give different analysis of Boundary-Domain Integral Equation for elliptic partial differential equation with variable coefficient subjected to some boundary condition. In this research we review some of this analysis developed by different researchers to study the analysis of boundary domain integral equations that are directly related to the objectives of the study. Reduction of the BVP with arbitrarily variable coefficients to boundary domain integral equation systems is usually not possible, since the fundamental solution necessarily for such reduction is generally not available in an analytical form (except for some special dependence of the coefficients on coordinates)[6]. It is possible to reduce a Boundary Value Problem (BVP) to some systems of boundary domain integral equations where the Dirichlet ,Neumann and mixed problems for some partial differential equations were reduced to BDIEs systems by using a parametric(Lev function).

In [1], an explicit parametric and the parametric-based third Greens identity has been used to reduce the mixed boundary value problem for the heat transfer partial differential equation with a variable coefficient to four different versions of direct segregated BDIES. The obtained BDIE systems contain integral operators defined on the domain under consideration as well as potential-type and pseudo differential operators defined on open sub manifolds of the boundary. It has been shown that the BDIE systems were equivalent to the original mixed BVP, and the operators of the BDIE systems were invertible in appropriate sobolev spaces.

In [2], the Localized boundary domain integral equation system associated with the Dirichlet boundary value problem for a scalar Laplace PDE with variable coefficient were formulated and analyzed. Mapping and jump properties of surface and volume integral potentials based on a localized parametric and constituting the LBDIE systems were studied in a scale of sobolev(Bessel potential) spaces. The main results established in the paper were the LBDIEs equivalence to the original variable-coefficient BVPs and the invertibility of the LBDIE operators in the corresponding sobolev spaces. Analysis of direct segregated boundary domain integral equations for variable-coefficient corresponding to mixed BVPs in exterior domains was presented in [3].

In [3], direct segregated systems of boundary domain integral equations were formulated for the mixed (Dirichlet-Neumann) BVPs for a scalar second order divergent elliptic PDE with a variable coefficient in an exterior three dimensional domain. The BDIE system equivalence to the original BVPs and the fredholm properties and the invertibility of the corresponding boundary domain integral operators were analysed in weighted sobolev spaces suitable for infinite domains. This analysis was based on the corresponding properties of the BVPs in weighted soboev spaces that were proved as well.

Analysis of segregated BDIEs based on a parametric and associated with the Dirichlet boundary value problems for a linear stationary diffusion partial differential equation with a variable coefficient were presented in [6]. Equivalence of the BDIEs to the original BVP, BDIE, solvability solution uniqueness or non-uniqueness,and invertibility of the BDIE operators are analyzed in sobolev spaces.It has been shown that the BDIE operators for the Dirichlet BVP were not invertible. The analysis of BDIEs for the second order stationary heat transfer elliptic partial differential equation with variable- coefficient Neumann BVP in $2D$ was presented in [12].

In [12], the interior Neumann problem for variable coefficient PDE in two-dimensional domain,where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\Omega)$ has been considered. The BVP has been reduced to two systems of BDIEs and their equivalence to the original BVP has been shown. The invertibility of the associated operators in the corresponding Bessel potential spaces has been also proved. In this research study, the BDIE formulations and analysis for the second order stationary heat transfer elliptic partial differential equation with variable coefficients based on Levi function parametric-based third Greens identity is considered in [12] and extended to the treatment of Dirichlet BVPs in two-dimensional setting.

Chapter 3

Preliminaries

3.1 Basic notations and spaces

In this section we will give some concepts and definition that are important in this research thesis. We denote a multi-index by $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and a point in the plane by $x = (x_1, x_2) \in \mathbb{R}^2$. The length of the multi-index is defined by $|\alpha| = \alpha_1 + \alpha_2$ and its factorial is defined by $\alpha! = \alpha_1!\alpha_2!$. The multi-index notation for the power x and derivative operator is denoted respectively as:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} = \frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \partial_2^{\alpha_2}}.$$

Let Ω be an open set in $\mathbb{R}^2$, then the set of all infinitely differentiable function on Ω with compact support, is denoted by $D(\Omega)$ or $C^\infty(\Omega)$. The function space $D^{'}(\Omega)$ consists of all continuous linear functionals over $D(\Omega)$. And we denote by $S(\mathbb{R}^2)$ the Schwartz space consisting of functions that together with all their derivatives, decay faster than any power of $|x|$ as $|x|$ goes to ∞. That means functions $\varphi \in C^\infty(\mathbb{R}^2)$ such that

$$\sup_{x \in \mathbb{R}^2} |x^\alpha (D^\beta \varphi)(x)| < \infty,$$

for all multi-indices α and β. The function spaces $S^{'}(\mathbb{R}^2)$ denotes the collection of all continuous linear functionals over $S(\mathbb{R}^2)$. Let $k \in \mathbb{N}_0$. The sobolev space $H^k(\Omega) = W_2^k(\Omega)$

consists of the functions $u \in L^2(\Omega)$ such that $D^\alpha u \in L_2(\Omega)$ for any multi-index $|\alpha| < k$,

$$H^k(\Omega) = \{u \in L_2(\Omega) : D^\alpha u \in L_2(\Omega), \quad for \quad |\alpha| \leq k.\}$$

Here the Derivatives $D^\alpha u$ are understood in the sense of Distributions. The space $H^k(\Omega)$ is a Hilbert space with the inner products given by:

$$\langle u, v\rangle_{H^k(\Omega)} = \sum_{|\alpha| \leqslant k} \langle D^\alpha u(x), D^\alpha v(x)\rangle_{L_2(\Omega)},$$

and the corresponding norm is given by

$$||u||^2_{H^k(\Omega)} = \sum_{|\alpha| \leq k} ||D^\alpha u||_{L_2(\Omega)}.$$

We denote that $H^0(\Omega) = L_2(\Omega)$. In particular, for $k = 1$, the space $H^1(\Omega)$ expressed as follow :

$$H^1(\Omega) = \{u \in L_2(\Omega); \frac{\partial u}{\partial x_i} \in L_2(\Omega), i = 1, 2\}.$$

And inner product in $H^1(\Omega)$is given by:

$$\begin{aligned} \langle u, v\rangle_{H^1(\Omega)} &= \sum_{|\alpha| \leq 1} \int_\Omega \overline{D^\alpha u(x)} D^\alpha v(x) dx \\ &= \int_\Omega \overline{u(x)} v(x) dx + \sum_{i=1}^{2} \int_\Omega \overline{\frac{\partial u}{\partial x_i}} \frac{\partial v}{\partial x_i} dx \\ &= \int_\Omega \overline{u(x)} v(x) dx + \int_\Omega \overline{\nabla u} \nabla v dx. \end{aligned}$$

While the norm is given by

$$||u||^2_{H^1(\Omega)} = ||u||^2_{L_2(\Omega)} + ||\nabla u||^2_{L_2(\Omega)}.$$

Definition 3.1. ($\mathbb{C}^k(\Omega)$ space, [15]). Let $\Omega \subseteq \mathbb{R}^2$ be some open subset and assume $k \in \mathbb{N}_0$. Then $\mathbb{C}^k(\Omega)$ is the space of functions which are bounded and k times continuously differentiable in Ω[see eg.on 15]. In particular, for $u \in \mathbb{C}^k(\Omega)$ the norm

$$||u||_{\mathbb{C}^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|,$$

is finite. Correspondingly, $\mathbb{C}^\infty(\Omega)$ is the space of functions which are bounded and infinitely continuously differentiable. For a function $u(x)$ defined for $x \in \Omega$ we denote

$$\sup u = \overline{\{x \in \Omega : u(x) \neq 0\}}$$

to be the support of the function u. Then,

$$\mathbb{C}^{\infty}(\Omega) = \{\mathbb{C}^{\infty}(\Omega) : \sup u \subset \Omega\}$$

is the space of $\mathbb{C}^{\infty}(\Omega)$ functions with compact support[15].

Definition 3.2. (The spaces $L^p(\Omega)$, [16]). We denote by $L^p(\Omega), (1 \leq p < \infty)$, is the space of equivalence classes of Lebesgue measurable functions u on the open subset $\Omega \in \mathbb{R}^n$ such that $|u|^p$ is integrable on Ω. We recall that two Lebesgue measurable functions u and v on Ω are said to be equivalent if they are equal almost all every where in Ω, i.e, $u(x) = v(x)$ for all x out side a set of Lebesgue measure zero,see [16].
The space $L^p(\Omega)$ is a Banach space with the norm,

$$||u||_{L^p(\Omega)} = (\int_{\Omega} |u(x)|^p dx)^{\frac{1}{p}}$$

In particular, for $p = 2$ we have the space of all square integrable functions $L^2(\Omega)$ which is also a Hilbert space with the inner product

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} u(x)v(\bar{x})dx \quad forall \quad u, v \in L^2(\Omega),$$

3.2 Test functions and Distributions

Definition 3.3. (Test functions, [18]). Let Ω be an open subset of $\mathbb{R}^n$. Then for $\phi : \Omega \to \mathbb{C}$ the support of ϕ, written supp ϕ, is defined as the closure in Ω of the set of $x \in \Omega$ for which $\phi(x) \neq 0$. The class of test functions $\mathcal{D}(\Omega)$ consists of all functions $\phi(x)$ defined in Ω, vanishing outside a bounded subset of Ω that stays away from the boundary of Ω and such that all partial derivatives of all orders of ϕ are continuous.

$$\mathcal{D}(\Omega) = \{\phi \in C^{\infty}(\Omega); \quad supp \quad of \quad \phi \quad is \quad compact \quad in \quad \Omega\}.$$

The set $\mathcal{D}(\Omega)$ also denoted by $C_0^{\infty}(\Omega)$.

Definition 3.4. (Distributions, [18]). A distribution or generalized function is a linear mapping $\phi \mapsto \langle u, \phi \rangle$ from $\mathcal{D}(\Omega)$ on to $\mathbb{R}$, which is continuous in the following sense: If $\phi_n \mapsto \phi$ in $\mathcal{D}(\Omega)$, then $\langle u, \phi_n \rangle \mapsto \langle u, \phi \rangle$. The set of all distributions is called $\mathcal{D}'(\Omega)$. In other words, $u : \mathcal{D}(\Omega) \mapsto \mathbb{R}$, where the values of u acting on φ denoted by

$$u : \varphi \mapsto u(\varphi) = u\varphi = \langle u, \varphi \rangle,$$

satisfies the following linearity and continuity criteria
1. For any $a, b \in \mathbb{C}$ and $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$,

$$\langle u, a\varphi_1 + b\varphi_2 \rangle = a\langle u, \varphi_1 \rangle + b\langle u, \varphi_2 \rangle.$$

2. $\langle u, \varphi_n \rangle \to \langle u, \varphi \rangle$ for $n \to \infty$ in $\mathbb{C}$, whenever $\varphi_n \to \varphi$ $\mathcal{D}(\Omega)$.

Remark 3.1. A continuous linear functional on $\mathcal{D}$ is called a distribution, or generalized function.

Example 3.1. The simplest example of a distribution is the functional generated by a locally integrable function, $f(x) \in L_1^{loc}(\Omega)$ and is given by

$$\langle f, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx, \forall \varphi \in \mathcal{D}(\Omega)$$

Proof. i. First, we show that the integral exists. Since $\phi \in \mathcal{D}(\Omega)$ has bounded support, contained in a rectangle Ω say, the integral is $\int_\Omega f(x)\varphi(x)dx$, which exists since the product of an integrable function and a continuous function is an integrable function. Hence f is a functional on $\mathcal{D}$.

(i). show linearity.

Let $a, b \in \mathbb{C}$, and let $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned} &\langle u, (a\varphi_1 + b\varphi_2)\rangle \\ &= \int_\Omega u(a\varphi_1 + b\varphi_2)dx \\ &= \int_\Omega ua\varphi_1 + ub\varphi_2 dx \\ &= \int_\Omega ua\varphi_1 dx + \int_\Omega ub\varphi_2 dx \\ &= a\int_\Omega u\varphi_1 dx + b\int_\Omega u\varphi_2 dx \\ &= a\langle u, \varphi_1\rangle + b\langle u, \varphi_2\rangle \end{aligned}$$

(ii). Continuity.

Let $\varphi_n \to \varphi$ in $\mathcal{D}$. Then there exist K a compact set, such that $supp\varphi_n \subset K$, $\forall n$ and $supp\varphi \subset K$. We have

$$|\langle f, \varphi_n\rangle - \langle f, \varphi\rangle| = |\int_K f(x)[\varphi_n - \varphi]dx|$$

$$\leq (max|\varphi_n - \varphi|)\int_K |f(x)|dx \to 0 as n \to \infty$$

because $\varphi_n \to \varphi$ uniformly. hence

$$\langle f, \varphi_n\rangle \to \langle f, \varphi\rangle.$$

So f is a continuous functional, that is, a distribution. □

Definition 3.5. (Locally integrable function, [15]).

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ is locally integrable if $\int_\Omega |f(x)|dx$ exists and

$$\int_\Omega |f(x)|dx < \infty.$$

Remark 3.2. Any continuous function is locally integrable; so is any piece wise continuous function.

Definition 3.6. (Fundamental solutional, [17]). Every parametrix function $P(x, y)$ which is a solution of equation $A_x P = 0$ is called a fundamental solution of the equation $Au = 0$.

Theorem 3.1. ***(Uniqueness theorem, [14]).*** *The solution of the Dirichlet problem, if it exists, is unique.*

Proof. Let ψ_1, ψ_2 be two solutions of the Dirichlet problems, then ψ_1, ψ_2 satisfy

$$\Delta\psi_1(x) + a^2\psi_1(x) = 0 \quad in \quad \Omega \quad suchthat \quad \psi_1 = f \quad on \quad \Gamma$$

$$\Delta\psi_2(x) + a^2\psi_2(x) = 0 \quad in \quad \Omega \quad suchthat \quad \psi_2 = f \quad on \quad \Gamma.$$

Since ψ_1 and ψ_2 are harmonic in Ω, $(\psi_1 - \psi_2)$ is also harmonic in Ω.

$$i.e. \quad \psi_1 - \psi_2 = 0 \quad on \quad \Gamma = \partial\Omega,$$

then implies

$$\psi_1 = \psi_2$$

at all interior point of Ω. Therefore the solution is unique. □

Definition 3.7. (Compactness, [9, 15]). Suppose that a set X is equipped with a metric $|.,.|_X$. An open cover of a subset $W \subset X$ is a family of open subsets of X whose union contains W. We say that W is compact if every open cover of W has a finite sub-cover. Every compact set is closed.

Definition 3.8. (Fredholm Operator, [9]). A bounded linear operator: $B : X \to Y$ is Fredholm if,
(1). range(B) is closed in Y,
(2). ker(B) and the factor space $Y_{/range(B)}$ are finite dimensional.
The index of B is then defined by

$$Index(B) = dim(ker(B)) - dim(Y_{/range(B)}).$$

Definition 3.9. (Adjoint Operator, [9]). Consider a linear map $A : X \to Y$. If X and Y are each equipped with a conjugation, then we define the adjoint operator $A^* : Y^* \to X^*$ by

$$(A^*w, u) = (w, Au) \quad for \quad w \in Y^*, u \in X.$$

3.3 Trace Theorem , Inverse Trace Theorem and sobolev space

3.3.1 Trace Theorem

Definition 3.10. (Trace Theorem, [9]). Trace of f on $\partial\Omega$ means that the restrictions of f to $\partial\Omega$. We recall the trace is a restriction of a function to the boundary and the interior trace operator is a continuous linear mapping from $H^s(\Omega)$ into $H^{s-\frac{1}{2}}(\partial\Omega)$.

$$i.e. \quad \gamma^+ : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega).$$

Trace Theorem states that:- Let $\Omega \subset \mathbb{R}^2$ be a $C^{k-1,1}$ - domain. For $\frac{1}{2} < s \leqslant k$, the interior Trace operator

$$\gamma_D : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma),$$

where $\gamma_D v = v/\Gamma$ is bounded. Then there exists $C_T > 0$ such that

$$||\gamma_D v||_{H^{s-\frac{1}{2}}(\Gamma)} \leqslant C_T ||V||_{H^s(\Omega)}, \qquad \forall \gamma \in H^s(\Omega).$$

3.3.2 Inverse Trace Theorem

Definition 3.11. (Inverse Trace Theorem, [9]). The Trace Operator, $\gamma_D : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$ has a continuous right inverse operator $\varepsilon : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$, satisfying:

$$(\gamma^+ o \varepsilon)(w) = w \qquad \forall w \in H^{s-\frac{1}{2}}(\Gamma).$$

There exists a constant $C_1 > 0$ such that

$$||\varepsilon w||_{H^s(\Omega)} \leq C_1 ||W||_{H^{s-\frac{1}{2}}(\Gamma)},$$

for all $W \in H^{s-\frac{1}{2}}(\Gamma)$. The trace theorem enables us to define unambiguously $\gamma^+ u$ or $u \in \partial\Omega$, provided that u is smooth enough to be in $H^m(\Omega)$, i.e. for any $u \in H^m(\Omega)$, we have $\partial^\alpha u \in H^1(\Omega)$ for $|\alpha| \leq m-1$. Thus by the trace theorem the values of $\partial^\alpha u$ on the boundary is well-defined and belongs to $L^p(\Omega)$ which implies $\gamma^+ \partial^\alpha u \in L^p(\Omega)$.

3.3.3 Sobolev Space

Definition 3.12. (Sobolev Space, [15]). Let Ω be a non-empty open subset of $\mathbb{R}^2$, m be a non-negative integer and $1 \leq p \leq \infty$.

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \forall \alpha \in N^n \cup 0, |\alpha| \leq m\}.$$

The sobolev space $H^1(\Omega)$ is the set of all $u \in L_2(\Omega)$ such that all the first partial derivatives $\dfrac{\partial u}{\partial x_i}$ belongs to $L_2(\Omega)$.

$$i.e. \quad H^1(\Omega) = \{u \in L_2(\Omega) : \nabla u \in L_2(\Omega)\}.$$

In the definition the derivatives are defined in distributional sense.

$$i.e. \quad \langle \partial x_j u, \varphi \rangle = -\langle u, \partial x_j \varphi \rangle.$$

In many applied situations , the Dirichlet integral $\int_\Omega |\nabla u|^2$ represents an energy. The functions in $H^1(\Omega)$ are therefore associated with configurations having finite energy.

Definition 3.13. (Bessel potential space, [9]). For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^2)$, the Sobolev space of order s on $\mathbb{R}^2$ by

$$H^s(\mathbb{R}^2) = \{u \in S^*(\mathcal{R}^2) : \mathcal{J}^s u \in L^2(\mathbb{R}^2)\},$$

and equip this space with the inner product

$$(u, v)_{H^s(\mathbb{R}^2)} = (\mathcal{J}u, \mathcal{J}v)$$

and the induced norm

$$||u||_{H^s(\mathbb{R}^2)} = \sqrt{(u, u)_{H^s(\mathbb{R}^2)}} = ||\mathcal{J}^s u||_{L_2(\mathbb{R}^2)}.$$

The space $H^{-s}(\mathbb{R}^2)$ is the dual space of $H^s(\mathbb{R}^2)$ equipped with the norm

$$||u||_{H^{-s}(\mathbb{R}^2)} = \sup_{0 \neq v \in H^s(\mathbb{R}^2)} \frac{|\langle uv \rangle|}{||v||_{H^s(\mathbb{R}^2)}} \qquad for \qquad u \in^{-s} (\mathbb{R}^2).$$

Chapter 4

Boundary Domain Integral Equation of Variable Coefficient Dirichlet BVP

In this Research, the Dirichlet boundary value problem for the second order stationary heat transfer elliptic partial differential equation with variable coefficient is considered in $2D$. Using an appropriate parametrix function, the problems are reduced. Consequently ,we need to set conditions on the domain for the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs.

The main thing necessary for the reduction of a BVP to a BIE is a fundamental solution to the original partial differential equation. After the fundamental solution is used in the corresponding Green formula, one can reduce the problem to a boundary integral equation. However, such a fundamental solution is generally not available if the coefficients of the original BVP are not constant. One can use, in this case, a parametrix function, which is usually available, instead of the fundamental solution in the Green formulae. Parametrix correctly describes the main part of the fundamental solution but is not required to satisfy the original differential equations apart from the singular point. However, in order for the reduction to be enabled, a fundamental solution for the PDE is necessary. Even though fundamental solutions are known for many equations with constant coefficients, they are not generally available in an explicit form for partial differential operators with variable coefficients.

4.1 Formulations of the Dirichlet Boundary Value Problem

4.1.1 The First Green's Identity

If $h \in C_0^1(\bar{\Omega})$, then

$$\int_\Omega \frac{\partial}{\partial x_i} h(x)dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) ds_x, (i = 1, 2) \tag{4.1}$$

where,

$$\gamma^+ h(x) = \lim_{y\in\Omega \to x \in \partial\Omega} h(y), \qquad for \qquad x \in \partial\Omega \tag{4.2}$$

is the interior boundary trace of $h(x)$. Using the results for density and trace theorem in [9] we can show that the integral relation (4.1) holds for any $h \in H^1(\Omega)$. Indeed , since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ then we have a sequence $h_n \in C^1(\bar{\Omega})$ such that $(||h_n - h||)_{H^1(\Omega)} \to 0$ as $n \to \infty$. Since $h_n \in C^1(\bar{\Omega})$, the integral relation (3) implies :

$$\int_\Omega \frac{\partial}{\partial x_i} h_n dx = \int_{\partial\Omega} \gamma^+ h_n n_i ds_x, (i = 1, 2) \tag{4.3}$$

Taking the limit as $n \to \infty$ in the equation (4.3)

$$\begin{aligned} |\int_\Omega \frac{\partial h_n}{\partial x_i} dx - \int_\Omega \frac{\partial h}{\partial x_i} dx| &\le \int_\Omega (|\frac{\partial h_n}{\partial x_i} - \frac{\partial h}{\partial x_i}|) dx \\ &\le (||h_n - h||)_{H^1(\Omega)} \to 0. \end{aligned}$$

Hence,

$$\int_\Omega \frac{\partial h_n}{\partial x_i} dx - \int_\Omega \frac{\partial h}{\partial x_i} dx \to 0.$$

For the right side of equation (4.1),

$$|\int_{\partial\Omega}\gamma^+ h_n n_i ds_x - \int_{\partial\Omega}\gamma^+ h n_i ds_x| \leq \int_{\partial\Omega}|\gamma^+(h_n - h)|n_i ds_x$$
$$\leq (||\gamma^+(h_n - h)||)_{L_2(\partial\Omega)},$$

the continuity of the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ and hence to $L^2(\partial\Omega)$ implies, $(||\gamma^+(h_n - h)||)_{L^2(\partial\Omega)} \leq (c||h_n - h||)_{L^2(\partial\Omega)} \to 0$ as $n \to \infty$. Hence,

$$\int_{\partial\Omega}\gamma^+ h_n n_i ds_x - \int_{\partial\Omega}\gamma^+ h n_i ds_x \to 0 \quad as \quad n \to \infty.$$

Now, for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, if we put $h(x) = a(x)\dfrac{\partial u(x)}{\partial x_j}v(x)$ and applying the Gauss Ostrogradski, we obtain

$$\int_{\Omega}\frac{\partial}{\partial x_j}(a(x)\frac{\partial u(x)}{\partial x_j}v(x))dx = \int_{\partial\Omega} n_j\gamma(a(x)\frac{\partial u(x)}{\partial x_j}v(x))ds_x.$$

summing over j and noting the relation :

$$\frac{\partial}{\partial x_j}(a(x)\frac{\partial u}{\partial x_j}v(x)) = \frac{\partial}{\partial x_j}(a(x)\frac{\partial u}{\partial x_j})v(x) + a(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j},$$

we obtain the following Green's first formula

$$\varepsilon(u, v) = -\int_{\Omega}(Au)(x)v(x)dx + \int_{\partial\Omega}T^+u(x)\gamma^+v(x)ds_x,$$

where,

$$\varepsilon(u, v) = \sum_{i=1}^{2}\int_{\Omega}a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx,$$

is the symmetric bi-linear form, and

$$T^+u(x) = \lim_{y\in\Omega\to x\in\partial\Omega}[\sum_{i=1}^{2}n_i(x)a(y)\frac{\partial}{\partial y_i}u(y)],$$

for $x \in \partial\Omega$, is the interior co-normal derivative.

Remark 4.1. For $v \in D(\Omega)$, $\gamma^+ v = 0$. If $u \in H^1(\Omega)$, then we can define Au as a distribution on Ω by :

$$(Au, v) = -\varepsilon(u, v) \qquad for \qquad v \in D(\Omega).$$

The subspace $H^{1,0}(\Omega; A)$ is defined as :

$$H^{1,0}(\Omega; A) = \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\},$$

with the norm

$$||g||^2_{H^{1,0}(\Omega;A)} = ||g||^2_{H^1(\Omega)} + (||Ag||)^2_{L_2(\Omega)}.$$

Definition 4.1. For $u \in H^{1,0}(\Omega; A)$ we can define the canonical (co-normal) derivative $T^+u \in H^{\frac{-1}{2}}(\partial\Omega)$ with the help of Green's formula (see eg. [5, 13] and the references there in).

$$\langle T^+u, w\rangle = \int_\Omega [(\gamma^+_{-1}w)Au + E(u, \gamma^+_{-1}w)]dx, \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega) \tag{4.4}$$

Where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the continuous interior trace operator γ^+, which maps $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, while $E(u, v) = \nabla u(x)a(x)\nabla v(x)$, and $\langle ., .\rangle_{\partial\Omega}$ denote the duality brackets between the spaces $H^{\frac{-1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ scalar product. The operator

$$T^+ : H^{1,0}(\Omega; A) \to H^{\frac{-1}{2}}(\partial\Omega)$$

is continuous and gives the continuous extension on $H^{1,0}(\Omega; A)$ of the classical co-normal derivative operator. For $u \in H^s(\Omega)$, $s > \frac{3}{2}$, then the canonical co-normal derivative defined by equation (4.4) coincides with the classical one , defined in the trace sense,

$$i.e. \qquad T^+u(x) = a(x)\gamma^+(\frac{\partial u(x)}{\partial n(x)}),$$

where $n(x)$ is the exterior unit normal vector.

Remark 4.2. The first Green's formula also holds for $u \in H^{1,0}(\Omega; A)$ and $v \in H^1(\Omega)$ as in [13], i.e

$$\int_\Omega E(u, v)dx = -\langle Au, v\rangle_\Omega + \langle T^+u, \gamma^+v\rangle_{\partial\Omega}. \tag{4.5}$$

4.1.2 The Second Green's Formula

In the first Green's formula by interchanging the role of u and v and subtracting the result, we obtain the Green's second formula. For $u, v \in H^{1,0}(\Omega; A)$, the second Green's formula is:

$$\int_\Omega (vAu - uAv)dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{4.6}$$

so we shall drive and investigate the BDIE systems for the following Dirichlet boundary value problem. Given the functions $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, and $f \in L_2(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying, $Au = f$ in Ω and $\gamma^+ u = \varphi_0$ on $\partial\Omega$.

Let Ω be a domain in $\mathbb{R}^2$ bounded. Let Ω^+ be a bounded by simple closed infinitely differentiable curve $\partial\Omega$, the set of all infinitely differentiable function on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals $\mathcal{D}(\Omega)$. For $s \in \mathbb{R}$ we denote by $H^s(\mathbb{R}^2)$ the Bessel potential space. Note that the space $H^1(\mathbb{R}^2)$ coincides with the sobolev space $W^{\frac{1}{2}}(\mathbb{R})$ with equivalent norm and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. For any non empty open set $\Omega \in \mathbb{R}^n$ we define,

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = \bigcup \Omega\}$$

for some $\bigcup \in H^s(\mathbb{R}^n)$. The space $\hat{H}^s(\Omega)$ is defined to be the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^n)$. For $s \in (\frac{-1}{2}, \frac{1}{2})$, $H^s(\Omega)$ can be identified with $\hat{H}^s(\Omega)$, then we shall consider the scalar elliptic partial differential equation

$$Au(x) = \sum_{1}^{2} \frac{\partial}{\partial x_i}(a(x)\frac{\partial u(x)}{\partial x_i}) = f(x) \qquad in \qquad \Omega \tag{4.7}$$

with $a(x) \in C^\infty(\mathbb{R}^2), a(x) > 0$. For a given functions $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$, we shall consider the Dirichlet boundary value problem for the function $u \in H^1(\Omega)$,

$$Au(x) = f \qquad in \qquad \Omega \tag{4.8}$$

$$\gamma^+ u = \varphi_0 \qquad on \qquad \partial\Omega \tag{4.9}$$

Here the above equation are understood in the distributional and trace sense respectively. In application the BVP in the above two solution may be describe a stationary heat transfer boundary value problem in isotropic in homogeneous two dimensional body Ω, where $u(x)$is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\varphi_0(x)$ is a known temperature on the boundary. Define a subspace as in[5, 7, 13]

$$H^{1,0}(\Omega, A) = \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\},$$

endowed with the norm

$$||g||^2 H^{1,0}(\Omega, A) = ||g||^2 L_2(\Omega),$$

for $u \in H^{1,0}(\Omega, A)$ then we can define the (canonical) co normal derivative $T^+u \in H^{\frac{-1}{2}}(\partial\Omega)$ in the weak form (see, eg. [5, 13] and the reference there in).

$$\langle T^+, u\rangle = \int_\Omega [(\gamma^+, \omega)Au + E(u, \gamma^+_{-1}\omega)]dx, \ \forall \omega \in H^{\frac{1}{2}}(\partial\Omega) \tag{4.10}$$

where

$$\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$$

is a continuous right inverse of the continuous right trace operator

$$\gamma^+ : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

while

$$E(u, v) = a(x)\nabla u(x).\nabla u(x)$$

is the symmetric bi-linear form. For $u \in H^s(\Omega)$, $s > \dfrac{3}{2}$ the canonical co normal derivative defined by equation above coincides with the classical one defined in the trace sense, that means

$$T^+u = an.\gamma^+\nabla u,$$

where $n(x)$ is the exterior unit normal vector.

Remark 4.3. The first Greens identity holds for any $H^{1,0}(\Omega; A)$ and $v \in H^1(\Omega)$([5, 13]). That means,

$$\int_\Omega E(u, v)dx = -\langle Au, v\rangle_\Omega + \langle T^+u, \gamma^+v\rangle_{\partial\Omega},$$

and the second Green identity holds for any $u, v \in H^{1,0}(\Omega; A)$. That means that

$$\int_\Omega (vAu - uAv)dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}$$

4.2 Parametric-based potential operators

A function $P(x, y)$ is a parametric function for the operator A, if

$$A_x P(x, y) = \delta(x - y) + R(x, y),$$

where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak singularity at $x = y$. In particular, see, eg. on [10], the function

$$P(x, y) = \frac{1}{2\pi a(y)} log|x - y|,$$

where $x, y \in \mathbb{R}^2$ is a parametrix for the operator A and the corresponding remainder is

$$R(x, y) = \sum_{1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \qquad where \qquad x, y \in \mathbb{R}^2.$$

If $a(x) = 1$ then A becomes the Laplace operator, $\triangle$ and the parametrix $p(x, y)$ becomes its fundamental solution, $P_{\triangle}(x, y)$. If $u \in H^{1,0}(\Omega; A)$ then from the second Greens identity, we have the following parametrix-based third Greens identity for $y \in \Omega$, [10],

$$u(y) = \int_{\partial\Omega} [\gamma^+ u(x) T_x^+ P(x, y) - P(x, y) T^+ u(x)] dx - \int_{\Omega} R(x, y) u(x) dx + \int_{\Omega} P(x, y) f(x) dx, \qquad y \in \Omega. \tag{4.11}$$

Note that the direct substitution of $v(x)$ by $P(x, y)$ in the second Green identity is not possible as it has singularity at $x = y$. This difficulty is avoided by replacing Ω by $\Omega \backslash B(y, \epsilon)$, where $B(y, \epsilon)$ is a disc of radius ϵ centered at y; taking the limit $\epsilon \to 0$ we then arrive at equation (4.11),e.g. [14]. The parametric-based logarithmic and remainder potential operators are defined, similar to [1] in the three dimensional case, as

$$\mathcal{P} g(y) = \int_{\Omega} p(x, y) g(x) dx,$$

$$\mathcal{R} g(y) = \int_{\Omega} R(x, y) g(x) dx.$$

The single and double layer potential operators corresponding to the parametrix $P(x, y)$ are defined for y is not belongs to $\partial\Omega$ as

$$V g(y) = -\int_{\partial\Omega} P(x, y) g(x) ds_x,$$

$$Wg(y) = -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x.$$

The following boundary integral (Pseudo-differential) operators

are also defined for $y \in \partial\Omega$,

$$Vg(y) = -\int_{\partial\Omega} P(x,y)g(x)ds_x,$$

$$Wg(y) := -\int_{\partial\Omega} T_x^+ P(x,y)g(x)ds_x,$$

$$W'g(y) := -\int_{\partial\Omega} T_Y^+ P(x,y)g(x)ds_x,$$

$$L^+g(y) := T_y^+ Wg(y).$$

Let $P_\triangle$,$V_\triangle$,$W_\triangle$,$\mathcal{V}_\triangle$,$\mathcal{W}_\triangle$ and $\mathcal{L}_\triangle$ denote the potentials corresponding to the operator $A =\triangle$. Then the following relations hold, [1] :

$$Pg = \frac{1}{a}P \triangle g, \qquad Rg = \frac{-1}{a(y)}\sum_1^2 \partial_i P \triangle [g\partial_i a] \tag{4.12}$$

$$Vg = \frac{1}{a}V \triangle g, \qquad Wg = \frac{1}{a}W \triangle (ag) \tag{4.13}$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\triangle g, \qquad \mathcal{W}g = \frac{1}{a}\mathcal{W}_\triangle (ag) \tag{4.14}$$

$$\mathcal{W}'g = \mathcal{W}' \triangle g + [a\frac{\partial}{\partial_n}(\frac{1}{a})\mathcal{V} \triangle g], \tag{4.15}$$

$$L^+g = L_\triangle^+(ag) + [a\frac{\partial}{\partial_n}(\frac{1}{a})]W_\triangle^+(ag). \tag{4.16}$$

Theorem 4.1. *Let $s \in \mathbb{R}$, then the following operators are continuous:*

$$V : H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega),$$

$$W : H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega),$$

$$V : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega),$$

$$W, W' : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega),$$

$$L^+ : H^s(\partial\Omega) \to H^{s-1}(\partial\Omega).$$

Proof. We have the corresponding mappings for the corresponding constant coefficient operators. (see eg.[5,9,15]). Equation (4.13), (4.14), (4.15), (4.16) also implies that the theorem claim. ☐

Theorem 4.2. *Let $u \in H^{\frac{-1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$, then the following jump relation hold on $\partial\Omega$.[9]*

$$\gamma^+ V u(y) = V u(y), \tag{4.17}$$

$$\gamma^+ W v(y) = \frac{-1}{2} v(y) + W v(y), \tag{4.18}$$

$$T^+ V u(y) = \frac{1}{2} u(y) + W' u(y), \tag{4.19}$$

For the constant coefficient case, this theorem is well known. Then taking into account the relations (4.17), (4.18) and (4.19), we can prove the theorem for the variable positive coefficient $a \in C^\infty(\mathbb{R}^2)$ as well.

Theorem 4.3. *Let Ω be a bounded open domain domain in $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. Then the following operators are continuous:*

$$P : \tilde{H}^s(\Omega) \to H^{s+2}(\Omega), \qquad s \in \mathbb{R} \tag{4.20}$$

$$: H^s(\Omega) \to H^{s+2}(\Omega), \qquad s > \frac{-1}{2} \tag{4.21}$$

$$R : \tilde{H}^s(\Omega) \to H^{s+1}(\Omega), \qquad s \in \mathbb{R} \tag{4.22}$$

$$: H^s(\Omega) \to H^{s+1}(\Omega), \qquad s > \frac{-1}{2} \tag{4.23}$$

$$\gamma^+ P : \tilde{s}(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > \frac{-3}{2} \tag{4.24}$$

$$: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \qquad s > \frac{-1}{2} \tag{4.25}$$

$$\gamma^+ R : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > \frac{-1}{2} \tag{4.26}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > \frac{-1}{2} \tag{4.27}$$

$$T^+ P : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > \frac{-1}{2} \tag{4.28}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \qquad s > \frac{-1}{2} \tag{4.29}$$

$$T^+ R : \tilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2} \tag{4.30}$$

$$: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \qquad s > \frac{1}{2} \tag{4.31}$$

Proof. The operator $P_\triangle$ (as in [2, 9, 12, 15]) is a homogeneous pseudo-differential operator of order -2 on $\mathbb{R}^2$ then mapping $P_\triangle : H^s_{comp}(\mathbb{R}^2) \to H^{s+2}_{loc}(\mathbb{R}^2)$ continuously for any $s \in \mathbb{R}$. Hence the application of trace theorem along with the relations (4.12), the operators (4.20),(4.22),(4.24),(4.26), (4.28),(4.30) are continuous. For $s \in (\frac{-1}{2}, \frac{1}{2})$, $\tilde{H}^s(\Omega)$ is identified with $H^s(\Omega)$, and equation (4.31) directly follows from equation (4.20). To prove the case $s \in (\frac{1}{2}, \frac{3}{2})$, we implement the Gauss divergence theorem and the fact that

$$\frac{\partial}{\partial x_j} log|x-y| = -\frac{\partial}{\partial y_j} log|x-y|,$$

and obtain:

$$\frac{\partial}{\partial y_j}(P \vartriangle g)(y) = -\frac{1}{2\pi}\int_\Omega g(x)\frac{\partial}{\partial x_j} log|x-y|dx,$$

$$= \frac{1}{2\pi}\int_\Omega log|x-y|\frac{\partial}{\partial x_j} g(x)dx - \frac{1}{2\pi}\int_{\partial\Omega} log|x-y|n_j\gamma^+ g(x)ds_x.$$

$$= P \vartriangle (\partial_j g)(y) + V \vartriangle (n_j\gamma^+ g)(y). \tag{4.32}$$

Now for $s \in (\frac{1}{2}, \frac{3}{2})$, since

$$\partial_j : H^s(\Omega) \to H^{s-1}(\Omega)$$

is continuous, we have

$$P \vartriangle \partial_j : H^s(\Omega) \to H^{s+1}(\Omega)$$

is continuous,and from the trace theorem

$$\gamma^+ g \in H^{s-\frac{1}{2}}(\partial\Omega)$$

and the properties of the single layer potential operator, we conclude that

$$\nabla P_\triangle : H^s(\Omega) \to H^{s+1}(\Omega)$$

is continuous. This implies that

$$P_\triangle : H^s(\Omega) \to H^{s+2}(\Omega)$$

is continuous, which along with the relation $Pg = \frac{1}{a}P_\triangle$ leads to the continuity of operator of equation (4.21) for $s \in (\frac{1}{2}, \frac{3}{2})$. Further, with the help of these results and the relation (4.32),we can verify by induction that the operator (4.21) is continuous for $s \in (k-\frac{1}{2}, k+\frac{1}{2})$, where k is an arbitrary non-negative integer.For the values $s = k+\frac{1}{2}$, the continuity of the operator (4.21) the follows due to the complex interpolation properties of Bessel potential spaces. The trace theorem will give the continuity of proof for the operators (4.24) and (4.25). We can follow the same procedure to prove the claim of the theorem concerning the operator R. The continuity of the operators $(4.28)-(4.31)$ follows if we remark that for the chosen s the co-normal derivative can be understood in the classical sense. (see, e.g. [9],Theorem 3.27) □

Corollary 4.4. *Let $s \in \mathbb{R}$. Then the following operators are compact*

$$V : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{4.33}$$

$$W : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{4.34}$$

$$W' : H^s(\partial\Omega) \to H^s(\partial\Omega) \tag{4.35}$$

Corollary 4.5. *The following operators are compact for any $s > \frac{1}{2}$*

$$R : H^s(\Omega) \to H^s(\Omega), \tag{4.36}$$

$$\gamma^+ R : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \tag{4.37}$$

$$T^+ R : H^s(\Omega) \to H^{s-\frac{3}{2}}(\partial\Omega). \tag{4.38}$$

4.3 Logarithmic potentials

The parametrix-based logarithmic and remainder potential operators are defined, similar to [2] and [3] in three dimensional case as,

$$\mathcal{P}g(y) = \int_\Omega P(x,y)g(x)dx,$$
$$\mathcal{R}g(y) = \int_\Omega R(x,y)g(x)dx$$

Let $P_\triangle$ denotes the fundamental solution of the Laplace operator $\triangle$, and the corresponding logarithmic potential by, $\mathcal{P}_\triangle$. Then

$$\mathcal{P}g = \frac{1}{a}\mathcal{P}_\triangle g$$

$$\mathcal{R}g = \frac{-1}{a(y)} \sum_{i=1}^{2} \partial_i \mathcal{P}_\triangle[g(\partial_i a)].$$

To see this,

$$\mathcal{R}g(y) = \int_\Omega R(x,y)g(x)dx$$
$$= \int_\Omega \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x-y|^2} \frac{\partial a(x)}{\partial x_i} g(x)dx$$
$$= \frac{1}{a(y)} \sum_{i=1}^{2} \int_\Omega \frac{x_i - y_i}{2\pi|x-y|^2} \frac{\partial a(x)}{\partial x_i} g(x)dx$$

$$= \frac{-1}{a(y)} \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \int_{\Omega} \frac{\log|x-y|}{2\pi} \frac{\partial a(x)}{\partial x_i} g(x)dx$$

$$= \frac{-1}{a(y)} \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \mathcal{P}_{\triangle}(\frac{\partial a(x)}{\partial x_i} g(x))$$

Remark 4.4. The Logarithmic potential,

$$\mathcal{P}_{\triangle} u(x) = \int_{\mathbb{R}^2} P_{\triangle}(x-y)u(y)dy$$

is a homogeneous pseudo-differential operator of degree -2.

From the definition of fundamental solution, $\triangle\ P_{\triangle}(x,y) = \delta(x-y)$.
If we take the Fourier transform in x both side we have,

$$-|\xi|^2 \mathcal{F} P_{\triangle}(.,y) = \mathcal{F}\delta(.,y)$$

$$= (2\pi)^{-1} e^{-iy\xi}.$$

This implies that for $\xi \neq 0$,

$$P_{\triangle}(x,y) = -(2\pi)^{-1} \mathcal{F}^{-1}(e^{-iy\xi} \frac{1}{|\xi|^2})$$

Putting this in

$$\mathcal{P}_{\triangle} u(x) = \int_{\mathbb{R}^2} P_{\triangle}(x-y)u(y)dy$$

we get,

$$\mathcal{P}u(x) = \int_{\mathbb{R}^n} -(2\pi)^{-1} \mathcal{F}^{-1}(\frac{e^{-iy\xi}}{|\xi|^2})u(y)dy,$$

$$= -\int_{\mathbb{R}^n} (2\pi)^{-2} [\int_{\Omega} e^{ix\xi} e^{-iy\xi} \frac{1}{|\xi|^2} d\xi] u(y)dy$$

$$= -\int_{\mathbb{R}^2} (2\pi)^{-2} [\int_{\Omega} e^{-iy\xi} u(y)dy] e^{ix\xi} \frac{1}{|\xi|^2} d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\xi} \frac{-1}{|\xi|^2} \hat{u}(\xi) d\xi$$

Therefore, $\mathcal{P}u(x) = \mathcal{F}_{\xi}^{-1} \to x(\frac{-1}{|\xi|^2}\hat{u}(\xi))$, hence it is homogeneous pseudo-differential operator with symbol $\delta(\xi) = \frac{-1}{|\xi|^2}$.

4.4 Layer potentials

The single and double layer potential operators corresponding to the parametrix $P(x, y)$ are defined for y is not belongs to $\partial\Omega$ as,

$$Vg(y) = \int_{\partial\Omega} P(x, y)g(x)ds_x,$$

$$Wg(y) = -\int_{\partial\Omega} T_x P(x, y)g(x)ds_x,$$

Where g is some scalar density function. The following boundary integral (pseudo-differential) operators are are also defined for $y \in \partial\Omega$,

$$\mathcal{V}g(y) = -\int_{\partial\Omega} P(x, y)g(x)ds_x,$$

$$\mathcal{W}g(y) = -\int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x,$$

$$\mathcal{W}'g(y) = -\int_{\partial\Omega} T_y^+ P(x, y)g(x)ds_x$$

$$\mathcal{L}g(y) = T_y^+ Wg(y).$$

4.5 Invertibility of the single layer potential operator

It is well known (see, [4],remark 1.42(ii), [15], proof of Theorem 6.22) for some $2D$ domain the kernel of the operator $\mathcal{V}_\Delta$ is non zero, which by equation (4.14) also implies that $ker(\mathcal{V}) \neq 0$ for the same domains. In order have the invertibility for the single layer potential operator in $2D$, we then define the following subspace of the space $H^{\frac{-1}{2}}(\partial\Omega)$, ([15], see eq. (6.30))

$$H_*^{\frac{-1}{2}}(\partial\Omega) = \{\phi \in H^{\frac{-1}{2}}(\partial\Omega) : \langle\phi, 1\rangle_{\partial\Omega} = 0\}.$$

Where the norm in

$$H_*^{\frac{-1}{2}}(\partial\Omega)$$

is the induced by the norm in $H^{\frac{-1}{2}}(\partial\Omega)$.

Theorem 4.6. *If $\psi \in H_*^{\frac{-1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof. The theorem holds for the operator $\mathcal{V}_\Delta$ (see, e.g. ([9], corollary 8.11(ii)), which by equation (4.14) implies it for the operator $\mathcal{V}$ as well. □

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^2$ have a diameter, $diam(\Omega) \leq 1$. Then the single layer potential*

$$\mathcal{V} : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

is invertible.

Proof. By ([15], Theorem 6.23), for $diam(\Omega) < 1$ the operator $\mathcal{V}_\Delta : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is $H^{\frac{-1}{2}}(\partial\Omega)$ - elliptic and since it is also bounded. Then by the first relation in (4.15) the invertibility of the operator,

$$\mathcal{V}_\Delta : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$$

also follows. □

4.6 The Third Green Identity

For $u \in H^{1,0}(A;\Omega)$, let us write the third Green's identity (4.11) using the surface and volume potential operator notations,

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}Au \quad in \quad \Omega \tag{4.39}$$

Applying the Trace operator to this equation and using the jump relations from Theorem 4.2, we have

$$\frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}Au \quad on \quad \partial\Omega \tag{4.40}$$

Similarly applying the co-normal derivative operator to the equation (4.39), and using again the jump relation ,we obtain

$$\frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}\gamma^+u = T^+\mathcal{P}Au \quad on \quad \partial\Omega \tag{4.41}$$

For some functions f, ψ, ϕ let us consider a more general indirect integral relation associated with equation (4.39).

$$u + \mathcal{R}u - V\psi + W\phi = \mathcal{P}f \quad in \quad \Omega. \tag{4.42}$$

Lemma 4.8. *Let* $u \in H^1(\Omega), f \in L_2(\Omega), \psi \in H^{\frac{-1}{2}}(\partial\Omega)$ *and* $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ *satisfy equation* (4.42). *Then* u *belongs to* $H^{1,0}(\Omega; A)$ *and is a solution of partial differential equation*
$Au = f \qquad in \qquad \Omega$ *and*
$V(\psi - T^+u)(y) - W(\phi - \gamma^+u)(y) = 0, \qquad y \in \Omega$

Lemma 4.9. (i). *Let either* $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)$ *and* $diam(\Omega) < 1$ *,or* $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)_*$.
If $V\psi^* = 0$ *in* Ω *,then* $\psi^* = 0 \qquad on \qquad \partial\Omega$.
(ii) *Let* $\phi^* \in H^{\frac{1}{2}}$. *If* $W\phi^* = 0$, *then* $\phi^* = 0 \qquad on \qquad \partial\Omega$.

Proof. (i). Taking the Trace of equation in Lemma 2(i) on $\partial\Omega$, by the jump relation (4.18), we have

$$\mathcal{V}\psi^*(y) = 0 \qquad on \qquad \partial\Omega.$$

If $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)$ and $diam(\Omega) < 1$, the the result follows from the invertibility of the single layer potential given by above theorem.
On the other hand if $\psi^* \in H^{\frac{-1}{2}}(\partial\Omega)_*$, then the result follows.
(ii). Let us take the trace of equation in Lemma 2(ii) on $\partial\Omega$ and use the jump relation (4.19) to obtain,

$$-\frac{1}{2}\phi^* + \mathcal{W}\phi^* = 0 \qquad on \qquad \partial\Omega.$$

Multiplying this equation by $a(y)$, denoting

$$\hat{\phi}^* = a\phi^*$$

and using the second relation in (4.14), we obtain equation

$$\frac{-1}{2}\hat{\phi}^* + \mathcal{W} \vartriangle \hat{\phi}^* = 0 \qquad on \qquad \partial\Omega.$$

It is well known that this equation has only the trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_\vartriangle$, $[7, Theorem 3.1]$ since $a(y)$ not equal to zero, then the result follows. □

4.7 Boundary-Domain Integral Equations (BDIEs)

To reduce the variable-coefficient Dirichlet Boundary Value Problem $Au = f$, and $\gamma^+u = \varphi_0$ to a segregated boundary-domain integral equation system, let us denote the unknown co-normal derivative as;

$$\psi = T^+u(x)H^{\frac{-1}{2}}(\partial\Omega)$$

and will further consider ψ as formally independent on u. Assuming that the function u satisfies partial differential equation (PDE), $Au = f$, by the substituting the Dirichlet condition into the third Green identity of equation

$$u + \mathcal{R}u - VT^{+} + W\gamma^{+}u = \mathcal{P}Au \quad in \quad \Omega$$

and either in to its trace of equation

$$\frac{1}{2}\gamma^{+}u + \gamma^{+}\mathcal{R}u - VT^{+}u + W\gamma^{+}u = \gamma^{+}\mathcal{P}Au \quad on \quad \partial\Omega$$

or in to its co-normal derivative of the equation

$$\frac{1}{2}T^{+}u + T^{+}\mathcal{R}u - W'T^{+}u + L^{+}\gamma^{+}u = T^{+}\mathcal{P}Au \quad on \quad \partial\Omega,$$

we can reduce the boundary value problem (bvp) (4.8) – (4.9) to two different systems of Boundary-Domain Integral Equations for the unknown function

$$u \in H^{1,0}(\Omega; A) \quad and \quad \psi = T^{+}u \in H^{\frac{-1}{2}}(\partial\Omega).$$

Boundary-Domain Integral Equation System $(D1)$ obtained from the following two equations is as follows:

$$u + Ru - VT^{+}u + W\gamma^{+}u = \mathcal{P}Au \quad in \quad \Omega$$

and

$$\frac{1}{2}\gamma^{+}u + \gamma^{+}Ru - VT^{+}u + W\gamma^{+}u = \gamma^{+}\mathcal{P}Au \quad on \quad \partial\Omega$$

is

$$u + Ru - V\psi = F_0 \quad in \quad \Omega$$

$$\gamma^{+}Ru - V\psi = \gamma^{+}F_0 - \varphi_0,$$

where

$$F = \mathcal{P}f - W\varphi_0 \quad in \quad \Omega \tag{4.43}$$

The system can be written in the matrix form as

$$\mathcal{A}^{1}\mathcal{U} = \mathcal{F} \quad where \quad \mathcal{U} = [u, \psi]^{T} \in H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega)$$

and

$$\mathcal{A}^1 = \begin{pmatrix} I + \mathcal{R} & -V \\ \gamma^+\mathcal{R} & -\mathcal{V} \end{pmatrix},$$

and

$$\mathcal{F}^1 = \begin{pmatrix} F_0 \\ \gamma^+F_0 - \varphi_0 \end{pmatrix}$$

From the mapping properties of Win above Theorem 1 and $\mathcal{P}$ in Theorem 3, we get the inclusion $F_0 \in H^{1,0}(\Omega; A)$, the Trace theorem implies

$$\gamma^+F_0 \in H^{\frac{1}{2}}(\partial\Omega).$$

Therefore

$$\mathcal{F}^1 \in H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega).$$

Due to the mapping properties of the operators involved in $\mathcal{A}^1$, the operator,

$$\mathcal{A}^1 : H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega)$$

is bounded.

Boundary Domain Integral Equation (BDIE) system $(D2)$ obtained from the equations

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}Au \quad in \quad \Omega,$$

and

$$\frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u = T^+\mathcal{P}Au \quad on \quad \partial\Omega$$

is

$$u + \mathcal{R}u - V\psi = F_0 \quad in \quad (\Omega),$$

$$\frac{1}{2}\psi + T^+\mathcal{R}u - \mathcal{W}'\psi = T^+F_0 \quad on \quad \partial\Omega,$$

where F_0 is given by the equation

$$F_0 = \mathcal{P}f - W\varphi_0 \quad in \quad \Omega.$$

In matrix form it can be written as

$$\mathcal{A}^2\mathcal{U} = \mathcal{F}^2,$$

where

$$\mathcal{A}^2 = \begin{pmatrix} I+\mathcal{R} & -V \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' \end{pmatrix},$$

$$\mathcal{F}^2 = \begin{pmatrix} F_0 \\ T^+F_0 \end{pmatrix}$$

Note that the operator;

$$\mathcal{A}^2 : H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega), \quad is \quad bounded.$$

4.8 Equivalence and Invertibility Theorems

In the following theorem we shall see the equivalence of the original Dirichlet Boundary Value Problem to the Boundary Domain Integral equation systems.

Theorem 4.10. *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$.*
(i). If some $u \in H^1(\Omega)$ solves the Boundary Value Problem (BVP) $(4.8)-(4.9)$, then the pair (u, ψ), where

$$\psi = T^+(u) \in H^{\frac{-1}{2}}(\partial\Omega) \tag{4.44}$$

solves the boundary domain integral equation systems $(D1)$ and $(D2)$.
(ii). If a pair

$$(u, \psi) \in H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

solves Boundary Domain Integral equation $(BDIE)$ system $(D1)$,and diam$(\Omega) < 1$, then u solves Boundary domain Integral Equation $(BDIE)$ $(D2)$ and Boundary Value Problem (BVP) $(4.8)-(4.9)$, this solution is unique, ψ satisfies the equation

$$\psi = T^+u \in H^{\frac{-1}{2}}(\partial\Omega).$$

(iii). If a pair

$$(u, \psi)H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

solves Boundary Domain Integral Equation $(BDIE)$ *system* $(D2)$*, the* u *solves the Boundary Domain Integral Equation* $(BDIE)$ *system* $(D2)$ *and Boundary Value Problem* (BVP) (4.8) *and* (4.9) *this solution is unique, and* ψ *satisfies the equation*

$$\psi = T^+u \in H^{\frac{-1}{2}}(\partial\Omega).$$

Proof. (i). Let $u \in H^1(\Omega)$ be solution of the Boundary Value Problem (BVP), (4.8) and (4.9). Since $f \in L_2(\Omega)$, we have that $u \in H^{1,0}(\Omega; A)$. Setting ψ by equation

$$\psi = T^+u \in H^{\frac{-1}{2}}(\partial\Omega)$$

and recalling how Boundary Domain Integral Equation $(BDIE)$ system $(D1)$ and $(D2)$ were constructed, and we obtain that (u, ψ) solves them. Let now a pair

$$(u, \psi) \in H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

solves the system $(D1)$ or $(D2)$. Due to the first equations in the Boundary Domain Integral Equation (BDIE) systems, the hypotheses of Lemma 1 are satisfied implying that u belongs to $H^{1,0}(\Omega; A)$ and solves partial differential the equation $\gamma^+u = \varphi_0$ in Ω, while the following equation also holds

$$V(\psi - T^+u)(y) - W(\varphi - \gamma^+u)(y) = 0, \qquad y \quad \in \quad \Omega. \tag{4.45}$$

(ii). Let

$$(u, \psi) \in H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

solve the system $(D1)$.
Taking the trace of the first equation in $(D1)$ and subtracting the second equation from it, we get $\gamma^+u = \varphi_0$ on $\partial\Omega$.
Thus the Dirichlet Boundary Condition is satisfied, then we have

$$V(\psi - T^+u)(y) = 0, y \in \Omega$$

Lemma $2(i)$ then implies $\psi = T^+u$.
(iii). Let now

$$(u, \psi) \in H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

solves the system $(D2)$.
Taking the co normal derivative of the first equation in $(D2)$ and subtracting the second equation from it, we get

$$\psi = T^+u \quad on \quad \partial\Omega.$$

Then inserting this in equation (4.44) gives

$$W(\varphi_0 - \gamma^+u)(y), \qquad y \in \Omega.$$

And Lemma $2(ii)$ implies

$$\varphi_0 = \gamma^+u \quad on \quad \partial\Omega.$$

The uniqueness of the Boundary Domain Integral Equation ($BDIE$) system solutions follows form the fact that the corresponding homogeneous Boundary Domain Integral Equation systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then paragraph (ii) and (iii) above imply that the homogeneous Boundary Domain Integral Equation systems also have only the trivial solutions. □

Theorem 4.11. *If $diam(\Omega) < 1$, the the following operators are invertible,*

$$\mathcal{A}^1 : H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega), \tag{4.46}$$

$$\mathcal{A}^1 : H^{1,0}(\Omega;A)XH^{\frac{-1}{2}}(\partial\Omega) \to H^{1,0}(\Omega,A)XH^{\frac{1}{2}}(\partial\Omega). \tag{4.47}$$

Proof. Theorem (4.44)(ii) implies that operators (4.46) and (4.47) are injective.
Let us denote

$$\mathcal{A}_0^1 = \begin{pmatrix} I & -V \\ 0 & \mathcal{V} \end{pmatrix}.$$

Then

$$\mathcal{A} : H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega)$$

is bounded.
It is invertible due to its triangular structure and invertibility of its diagonal operators

$$I : H^1(\Omega) \to H^1(\Omega)$$

and

$$-\mathcal{V} : H^{\frac{-1}{2}}(\partial\Omega \to H^{\frac{1}{2}}(\partial\Omega).$$

then the operator

$$\mathcal{A}^1 - \mathcal{A}_0^1 = \begin{pmatrix} R & 0 \\ \gamma^+ R & 0 \end{pmatrix}$$

$$H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega)$$

is compact, implying that operator (4.46) is a Fredholm operator with zero index, then the injectivety of the operator (4.46) implies its invertibility . To proof invertibility of operator (4.47) , we remark that for any

$$\mathcal{F} \in H^{1,0}(\Omega;A)XH^{\frac{1}{2}}(\partial\Omega),$$

a solution of equation

$$\mathcal{A}^1\mathcal{U} = \mathcal{F}^1,$$

can be written as

$$\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1,$$

where

$$(\mathcal{A}^1)^{-1} : H^1(\Omega)XH^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

is the continuous inverse to the operator (4.46). But due to Lemma 1 the first equation of system $(D1)$ implies that

$$\mathcal{U} = (\mathcal{A}^1)^{-1}\mathcal{F}^1 \in H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega),$$

and moreover the operator

$$(\mathcal{A}^1)^{-1} : H^{1,0}(\Omega; A)XH^{\frac{1}{2}} \to H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\Omega)$$

is continuous, which implies invertibility of the operator (4.47). The following similar assertion for the operator $\mathcal{A}^2$ holds without a limitation on the diameter of Ω. $\square$

Theorem 4.12. *The following operators are invertible.*

$$\mathcal{A}^2 : H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega), \tag{4.48}$$

$$\mathcal{A}^2 : H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega) \to H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega). \tag{4.49}$$

Proof. Theorem 4.14(iii) implies that the operator (4.48) and (4.49) are injective.
Let us denote

$$\mathcal{A}^2 = \begin{pmatrix} I & -V \\ 0 & -\frac{1}{2}I \end{pmatrix}.$$

Then

$$\mathcal{A}^2_0 : H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators

$$I : H^1(\Omega) \to H^1(\Omega)$$

and

$$I : H^{\frac{-1}{2}}(\partial\Omega) \to H^{\frac{-1}{2}}(\partial\Omega).$$

then the operator,

$$\mathcal{A}^2 - \mathcal{A}^2_0 = \begin{pmatrix} R & 0 \\ T^+R & -\mathcal{W} \end{pmatrix}$$

$$H^{1,0}(\Omega; A)XH^{\frac{-1}{2}}(\partial\Omega) \to H^1(\Omega)XH^{\frac{-1}{2}}(\partial\Omega)$$

is compact. This implies that operator (4.48) is a Fredholm operator with zero index and then injectivity of the operator (4.48) implies its invertibility. $\square$

Chapter 5

Conclusion

In this research, we have considered the interior Dirichlet problem for variable coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$.

The BVP was reduced to two systems of Boundary-Domain Integral Equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding sobolev spaces was also proved. In a similar way one can consider also the $2D$ versions of the BDIEs for the Neumann problem, mixed problem in interior and exterior domains, united BDIEs as well as the localized BDIEs, which were analyzed for $3D$ case. The Boundary Domain Integral Equations systems are in general uniquely solvable and invertible.

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