Quantum Analysis of Three-Level Cascade Laser with the Same and Different Frequencies Coupled to Vacuum Reservoir

by

AREGA WAKJIRA

A Thesis Submitted to

the Department of Applied Physics Program

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement

for the Degree of Masters in Physics(Quantum Optics)

Office of Graduate Studies

Adama Science and Technology University

Adama , Ethiopia

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Approved by the Examination Committee

External Examiner ____________________- ______________-

Internal Examiner ____________________- ______________

Advisor ____________________- ______________-

Chairman ____________________- ______________

DECLARATION

I hereby declare that this M.sc thesis is my original work and has not been presented for a degree in any other universities, and all sources of material used for the thesis have been duly acknowledged.

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Signature:—————————————

This Msc thesis has been submitted for examination with my approval as university advisor supervisor.

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Signature:—————————————

Date of submission:—————————————

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ABSTRACT

We obtained the equation of evolution of the expectation values of the cavity operators for a cascade three-level laser with the same and different frequencies by applying the pertinent master equation. Applying the steady state solutions of the expectation values, we studied the squeezing, photon statistics and entanglement of the cavity and output modes. The results obtained show that the light generated by this system is squeezed and entangled and, also the squeezing inside and outside the cavity is related with the entanglement. Furthermore, our calculation show that the mean photon number, the variance of the photon number, the degree of squeezing and entanglement of the cavity and output modes increase for large value of A, small value of η(if more atoms are initially prepared at the bottom level). Moreover, we found that as κ increase from 0 to 1, squeezing and entanglement of light inside the cavity is decrease whereas increase at outside.

Contents

1

INTRODUCTION

1.1 Back Ground of Study

Light has played a special role in our attempts to understand nature both classically and quantum mechanically. Classically light consists of waves with well-defined amplitude and phase such is not the case when we treat the field quantum mechanically. Scientists gained technology measure with great precision, strange phenomena was observed and this leads to the birth of quantum physics. Recent development of technology greatly improved our ability to control individual quantum system. This brings quantum technology beyond research to control of the concrete industrial programs. Quantum optics is a field of quantum physics that deals mainly with quantum properties of the light generated by various quantum optical system, such as LASER and MASER with their effect on the dynamics of atoms. MASER(Microwave Amplification by Stimulated Emission of Radiation) is the way that the coherent electromagnetic radiation can be generated in the radio frequency range. LASER(Light Amplification by Stimulated Emission of Radiation) is consists of a set of atoms interacting a resonant electromagnetic radiation in side a cavity with a single-port mirror. A laser operating well above threshold generates coherent light. The quantum properties of light are largely determined by the state of the light mode and the well known quantum state are: the number state, the coherent state, the chaotic state and the squeezed state.

Squeezing is one of the interesting non-classical features of light that has been attracting attention and studied by many authors[1-22]. In squeezed light the noise in one quadrature is below the vacuum or coherent level at the expense of enhanced fluctuation in the other quadrature, with product of the uncertainties in the two quadrature's satisfying the uncertainty relation. Squeezed light can be generated as a result of quantum optical process. Such as sub harmonic generation, second harmonic generation and three-level laser under certain conditions. A three-level cascade laser is one of the quantum optical systems that can produced squeezed light under certain condition. Squeezing light has potential applications in low-noise communication, weak signal detection, and optical amplification measurement.

Entanglement is another interesting non-classical features of light in which the density operator for state cannot be expressed as a product of density operators of the constituents. It has been drawing attention and studied by many authors[23-26]. Entangled photons are expected to have a potential applications in quantum memory and long distance quantum communications, since low frequency and narrow line width of the light can enhance efficient coupling between photons and atomic memories in quantum net work. Researchers in quantum entanglement was initiated by a 1935 paper[24], by Albert Einstein, Boris Podolsky and Nathan Rosen describing the EPR paradox and several papers by Erwin Schrodinger [26] shortly. Recently much attention has been paid to the generation and detection of continuous variable entanglement as it might be easier to manipulate than discrete counter parts, quantum bits in order to perform quantum information processing. On the other hand, the efficiency of the quantum information processing highly depends on the degree of entanglement. Hence, it is desirable to generate strongly entangled continuous variable state.

Three-level cascade laser is one of the most interesting and well characterized optical devices in quantum optics. In three-level cascade laser, three-level atoms in a cascade configuration are injected into a cavity coupled to a vacuum reservoir via a single-port mirror. The injected atoms may initially be prepared in a coherent superposition of the top and bottom levels. The set of energy levels of an atom consists of an infinite number of discrete levels corresponding to the bound states of the electron [22]. For a three-level atom, out of these set of energy levels only three-levels interact with electromagnetic radiation. When the three-level atom interacts with radiation, it under goes a transition from top to bottom level via the intermediate level by emitting two photons.

1.2 Statement of the Problem

In previous time, several authors have studied the quantum analysis of light with the same and different frequencies generated by a three-level cascade laser alone using different methods. In this study, we study the quantum analysis of light with the same and different frequencies produced by three-level cascade laser coupled with vacuum reservoir together by using steady state solution.

1.3 Objectives of the Study

1.3.1 General Objectives of the Study

The general objective of this thesis is to analyze the photon statistics, squeezing and entanglement properties of light inside and outside a cavity generated by three-level cascade laser with the same and different frequencies coupled with vacuum reservoir by applying the steady state solutions of cavity operators.

1.3.2 Specific Objectives of the Study

The specific objectives of this thesis are:

- ▶ to determine the master equation of cavity coupled to vacuum reservoir and then for system under consideration, to obtain the steady state solution of the expectation values of the cavity mode operator associated with the normal ordering.
- ▶ to calculate, applying the steady state solutions of the expectation values of the cavity mode variables, the quadrature variance of both cavity modes and output modes, the mean and the variance of photon number of the same and different frequencies created by a three-level cascade laser coupled to vacuum reservoir.
- ▶ to analyze, applying the Dual etal criteria[25], the entanglement of the cavity and output modes with the same and different frequencies.

2

REVIEW OF LITERATURE

Quantum optics deals mainly with quantum properties of the light generated by various optical systems and with the effect on the dynamics of atoms. The quantum properties of light are largely determined by the state of the light mode and the well known the quantum state; the coherent state, the chaotic state and the squeezed state. Squeezing light can be generated by quantum optical processes such as parametric oscillation(subharmonic generation), second harmonic generation and three level laser under certain condition.

A parametric oscillator is one of the most interesting and well characterized optical device in quantum optics. In this device a pump photon interacts with a non linear crystal inside a cavity and is down converted in two highly correlated photons. If these photons have the same frequency and the device is called a degenerate parametric oscillator, otherwise it is called a non degenerate parametric oscillator.

In second harmonic generation the fundamental mode interacts with a nonlinear crystal and is up converted into the second harmonic mode. It so happens that the second harmonic mode is in squeezed state. The fundamental mode is which is initially in a coherent state also gets squeezed as time progresses.

The three-level laser systems are V-type, λ-type and cascade type. In V-type, the three atoms are, $\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$. In this system the energy levels of $\mid a\rangle$, $\mid b\rangle$ are higher than that of $\mid c\rangle$. In λ-type atoms, $\mid a\rangle$ is at the highest energy level, while $\mid b\rangle$ and $\mid c\rangle$ are at lower level. Cascade type is defined as a quantum optical system in which three-level atoms in a cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected at a constant rate into a cavity coupled to a vacuum reservoir. These atoms are removed from the cavity after some time, which is long enough for the atoms to decay spontaneously to levels other than the middle or bottom. In this system the top, intermediate, and bottom levels of a three-level cascade atom can be conveniently denoted by $\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$, respectively. The transitions between levels $\mid a\rangle$ and $\mid b\rangle$ and between levels $\mid b\rangle$ and $\mid c\rangle$ is taken to be electric-dipole allowed, with direct transition between levels $\mid a\rangle$ and $\mid c\rangle$ is taken to be dipole forbidden. When a three-level cascade atom decays from the top level to the bottom level via the intermediate level, two photons of frequency ω_1 and ω_2 are emitted. If the two photons have the same frequency, the three-level atom is called degenerate otherwise non-degenerate. A three level atoms makes a transition from the top to the bottom level. Then it emits a photon and decays to the intermediate level. We certainly expect several three-level atoms to be involved in this process. The photons emitted from the top constitute one light mode and the other emitted from the intermediate level form another type. Due to the fact that these two photons shared the same intermediate level and they are said to have coherence which is responsible to the squeezing and entanglement exposition of the same and different frequencies of light modes. Three-level lasers in which a considerable role is played by the coherent superposition of the top and bottom levels of the injected atoms have been studied by different authors [9-14], [16-19] and [23]. These studies show that three-level lasers can generate light in a squeezed state under certain conditions.

Fesseha[9] has studied the squeezing properties of light by a degenerate three-level laser coupled to a vacuum reservoir. He has found that the squeezing increases with the linear gain coefficient and the perfect squeezing can be achieved for large values A and for small values of η.

Tesfa[10] has studied the squeezing properties of the cavity modes produced by a non degenerate three-level laser applying the solutions of stochastic differential equations. He has found that the two-mode cavity radiation exhibits squeezing if the atoms are initially prepared with more atoms in the bottom level than in the upper level, and the degree of squeezing increases with linear gain coefficient.

Friew[11] has studied the squeezing and statistical properties of the light generated by non-degenerate three-level laser whose cavity contains two parametric amplifiers coupled to an ordinary vacuum reservoir. He has found that parametric amplifiers increase the degree of squeezing.

Mesfin[12] has studied coherently prepared degenerate and non-degenerate three-level laser in the linear and adiabatic approximations. He has studied for the two systems, the cavity radiation as well as the out put radiation is in squeezed state for values of η between zero and one. He found that almost perfect squeezing can be achieved for sufficiently large values of the linear gain coefficient. Furthermore, He found that the probability of finding an even number of photons is greater than the probability of finding an odd number of photons for the light produced by a degenerate three-level laser. This is because the photons are always generated in pairs and existence of some finite probability to find an odd number of photons is due to damping of the cavity mode. On the other hand, the distribution function for a coherently prepared non-degenerate three-level laser has the same form as the distribution function for the signal mode from a non-degenerate parametric oscillator. He has also seen that the photon number distribution in both cases decreases with the photon number.

Abraham[13] has studied the squeezing and entanglement of light produced by a non degenerate three-level laser whose cavity contains two degenerate parametric amplifier coupled to the squeezed vacuum reservoir. He has found that the degree of squeezed vacuum reservoir increases the degree of squeezing and entanglement.

Eyob[14] has studied the squeezing and statistical properties of the cavity and output modes of a degenerate three-level laser whose cavity contains a parametric amplifier and coupled to a squeezed vacuum. He has studied that the effect of the squeezed vacuum reservoir and the parametric amplifier is to increase the mean photon number and the degree of squeezing of the cavity and output modes. Also, he has studied that squeezed vacuum reservoir increases the degree of squeezing significantly over and above the squeezing attainable from a degenerate three-level laser with a parametric amplifier. He have also found that the mean photon number of the cavity mode is greater than that of the output mode.

Dawit[16] studied a degenerate three-level laser in which top and bottom levels are coupled by strong coherent light and with half probability for the atoms to be in the top or bottom levels coupled to a squeeze vacuum reservoir applying the solution of stochastic differential equations. He found that the squeezing increases with the linear gain coefficient.

Alebachew and Fesseha [17] have studied the squeezing properties of the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier by applying the solution of the stochastic differential equations, with the top and bottom levels of injected atoms coupled by the pump mode emerging from the parametric amplifier. In this study they showed that the optical system generates light in a squeezed state with a maximum interacavity squeezing of 93% below the coherent state level.

Tewodros and Fesseha [18] have studied the squeezing properties of the cavity mode produced by a degenerate three-level laser whose cavity contains a parametric amplifier and with the cavity mode driven by strong coherent light and the three-level atoms injected into the cavity are initially prepared in a coherent superposition of the top and bottom levels, with these levels coupled by the pump mode emerging from the parametric amplifier by applying the solution of c-number Langevin equations. Their study showed that the system generates squeezed light with maximum squeezing of 94% below the coherent state level for certain conditions.

Misrak [19] has studied the squeezing properties of cavity mode produced by degenerate three-level laser with parametric amplifier by applying the solution of stochastic differential equations. This study showed that the quantum optical system generates squeezed light and the degree of squeezing increases with the linear gain coefficient with maximum interacavity squeezing of 96.5% below the coherent state level.

Eyob[23] has analyzed the squeezing and entanglement properties of the two-mode light produced by non degenerate three-level cascade laser with a subthreshold non degenerate parametric oscillator coupled to a vacuum reservoir inside and outside the cavity by applying the pertinent master equation. This study showed that the light generated by this system is in a two-mode squeezed state and the state of the system is strongly entangled at steady state. Moreover, the presence of the parametric oscillator leads to an increase in the degree of squeezing and entanglement. He also found that the intermode correlation decreases as the injected atomic coherence decreases in the system.

3

METHODOLOGY

We first get the master equation for a cavity coupled to a vacuum reservoir. Then we construct the Hamiltonian for the system under consideration, using this Hamiltonian we find the master equation for our system. Finally, we obtain the operator dynamics.

3.1 Master Equation

3.1.1 Master Equation for a Cavity Coupled to a Vacuum Reservoir

Now we seek to find the master equation for the same and different frequencies of a cavity modes coupled to vacuum reservoir. A system coupled to a reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{3.1}$$

where $\hat{H}_{SR}$ describes the interaction between the system and the reservoir and $\hat{H}_S$ describes some other interaction of the system. Suppose $\hat{\chi}(t)$ the density operator for a system and the reservoir. Then the equation of evolution of the density operator can be written as

$$\frac{d\hat{\chi}(t)}{dt} = -i[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)]. \tag{3.2}$$

We are interested in the quantum dynamics of the system alone. And the density operator for the system also known as the reduced density operator is given by

$$\hat{\rho}(t) = T_{rR}\hat{\chi}(t), \tag{3.3}$$

in which T_{rR} indicates trace over the reservoir variables only. Hence taking in to account Eq.(3.2), we see that the reduced density operator evolves in time according to

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - iT_{rR}[\hat{H}_{SR}, \hat{\chi}(t)]. \tag{3.4}$$

In order to obtain a mathematically manageable equation of evolution of the reduced density operator, we need to adopt some approximation scheme. Thus seek to obtain this equation of evolution applying the approximately valid relation

$$\frac{d\hat{\chi}(t)}{dt} = \frac{\hat{\chi}(t) - \hat{\chi}(t-h)}{h} = -i[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)], \tag{3.5}$$

where h is small parameter. Now in view of Eq.(3.2) and Eq. (3.5), we see that

$$\hat{\chi}(t) = -ih[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)] + \hat{\chi}(t-h), \tag{3.6}$$

So that on substituting this in to Eq.(3.4), there follows

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - iT_{rR}[\hat{H}_{SR}, \hat{\chi}(t-h)] - hT_{rR}[\hat{H}_{SR}, [\hat{H}_S, \hat{\chi}(t)]] - hT_{rR}[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\chi}(t-h)]]. \tag{3.7}$$

We next replace $\hat{\chi}(t)$ by some approximately valid expression. Then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. Moreover, decoupling the system and reservoir density operators, we have

$$\hat{\chi}(t) = \hat{\rho}(t)\hat{R}, \tag{3.8}$$

So that on introducing this in to Eq.(3.7) and using trace properties, we obtain

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - i[\langle\hat{H}_{SR}\rangle_R, \hat{\rho}(t-h)] - h[\langle\hat{H}_{SR}\rangle_R, [\hat{H}_S, \hat{\rho}(t)]] - hT_{rR}[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\rho}(t)\hat{R}]], \tag{3.9}$$

where the subscript R indicates that the expectation value is to be calculated using the reservoir density operator R. Furthermore, the interaction of a cavity modes with a squeezed vacuum reservoir can be described by the Hamiltonian

$$\hat{H}_{SR} = i\lambda_1(\hat{a}_1^\dagger \hat{a}_{1in} - \hat{a}_1\hat{a}_{1in}^\dagger) + i\lambda_2(\hat{a}_2^\dagger \hat{a}_{2in} - \hat{a}_{2in}\hat{a}_2^\dagger), \tag{3.10}$$

where λ_1 and λ_2 are coupling constant, $\hat{a}_{1in}$ and $\hat{a}_{2in}$ are annihilation operator for the same and different frequencies of squeezed vacuum modes and $\hat{a}_1$ and $\hat{a}_2$ are annihilation operator for the same and different frequencies of light modes. One can establish that for a squeezed vacuum

$$\langle\hat{a}_{in}\rangle = 0. \tag{3.11}$$

The expectation of Eq.(3.10) is found to be

$$\langle\hat{H}_{SR}\rangle_R = i\lambda_1(\hat{a}_1^\dagger\langle\hat{a}_{1in}\rangle_R - \hat{a}_1\langle\hat{a}_{1in}^\dagger\rangle_R) + i\lambda_2(\hat{a}_2^\dagger\langle\hat{a}_{2in}\rangle_R - \langle\hat{a}_{2in}\rangle_R\hat{a}_2^\dagger). \tag{3.12}$$

So that on account of Eq.(3.11), we have

$$[\langle \hat{H}_{SR}\rangle_R, \hat{\rho}(t-h)] = [\langle \hat{H}_{SR}\rangle_R, [\hat{H}_S, \hat{\rho}(t)]] = 0. \tag{3.13}$$

Now on view of this result, expression (3.9) reduces to

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - hT_{rR}(\hat{H}_{SR}^2\hat{R})\hat{\rho}(t) - h\hat{\rho}(t)T_{rR}(\hat{R}\hat{H}_{SR}^2) + 2hT_{rR}(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}). \tag{3.14}$$

Furthermore, using the Hamiltonian described by (3.10) and ignoring the non community of the cavity and in put mode operators, we get

$$\begin{aligned} hT_{rR}(\hat{H}_{SR}^2\hat{R})\hat{\rho} &= h\lambda_1^2(\langle \hat{a}_{1in}^\dagger \hat{a}_{1in}\rangle_R \hat{a}_1^\dagger \hat{a}_1\hat{\rho} + \langle \hat{a}_{1in}\hat{a}_{1in}^\dagger\rangle_R \hat{a}_1\hat{a}_1^\dagger\hat{\rho}) \\ &+ h\lambda_2^2(\langle \hat{a}_{2in}^\dagger \hat{a}_{2in}\rangle_R \hat{a}_2^\dagger \hat{a}_2\hat{\rho} + \langle \hat{a}_{2in}\hat{a}_{2in}^\dagger\rangle_R \hat{a}_2\hat{a}_2^\dagger\hat{\rho}), \end{aligned} \tag{3.15}$$

in which

$$[\hat{a}_{1in}, \hat{a}_{1in}^\dagger] = [\hat{a}_{2in}, \hat{a}_{2in}^\dagger] = 1. \tag{3.16}$$

For squeezed vacuum,

$$\langle \hat{a}_{1in}^\dagger \hat{a}_{1in}\rangle_R = \langle \hat{a}_{2in}^\dagger \hat{a}_{2in}\rangle_R = 0. \tag{3.17}$$

So that by using Eq.(3.16) and (3.17), Eq.(3.15) will be

$$hT_{rR}(\hat{H}_{SR}^2\hat{R})\hat{\rho} = \frac{\kappa_1}{2}\hat{a}_1\hat{a}_1^\dagger\hat{\rho} + \frac{\kappa_2}{2}\hat{a}_2\hat{a}_2^\dagger\hat{\rho}, \tag{3.18}$$

where $\kappa_1 = 2h\lambda_1^2$, is the cavity damping constant for light mode $\hat{a}_1$ as well as $\kappa_2 = 2h\lambda_2^2$, is the cavity damping constant for light mode $\hat{a}_2$. Following the same procedure, one can easily show that

$$\hat{\rho}hT_{rR}(\hat{R}\hat{H}_{SR}^2) = \frac{\kappa_1}{2}\hat{\rho}\hat{a}_1\hat{a}_1^\dagger + \frac{\kappa_2}{2}\hat{\rho}\hat{a}_2\hat{a}_2^\dagger. \tag{3.19}$$

In addition, employing (3.10) and the cyclic property of the trace operation, we readily find

$$\begin{aligned} hT_{rR}(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}) &= h\lambda_1^2(\langle \hat{a}_{1in}^\dagger \hat{a}_{1in}\rangle_R \hat{a}_1^\dagger\hat{\rho}\hat{a}_1 + \langle \hat{a}_{1in}\hat{a}_{1in}^\dagger\rangle_R \hat{a}_1\hat{\rho}\hat{a}_1^\dagger) \\ &+ h\lambda_2^2(\langle \hat{a}_{2in}^\dagger \hat{a}_{2in}\rangle_R \hat{a}_2^\dagger\hat{\rho}\hat{a}_2 + \langle \hat{a}_{2in}\hat{a}_{2in}^\dagger\rangle_R \hat{a}_2\hat{\rho}\hat{a}_2^\dagger). \end{aligned} \tag{3.20}$$

So that application of (3.16) and (3.17) leads to

$$hT_{rR}(\hat{H}_{SR}\hat{\rho}(t)\hat{R}\hat{H}_{SR}) = \frac{\kappa_1}{2}\hat{a}_1\hat{\rho}\hat{a}_1^\dagger + \frac{\kappa_2}{2}\hat{a}_2\hat{\rho}\hat{a}_2^\dagger. \tag{3.21}$$

Therefore, on account of (3.18), (3.19), and (3.21), expression (3.14) takes the form

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] + \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1)$$
$$+ \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2), \quad (3.22)$$

this represents the master equation for a cavity coupled to a vacuum reservoir.

3.1.2 Master Equation for the System Under Consideration

We consider here a three-level atom in a cascade configuration and initially prepared in a superposition of the top and bottom levels. We assume that such an atom emits two photons of the same and different frequencies of ω_1 and ω_2 as it makes a transition from the top level to the intermediate and then the bottom level, respectively. In this optical system the three-level atoms we denote the top, middle and bottom levels by eigen states $\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$ and they are injected at a constant rate of r_a in to a cavity coupled to a vacuum reservoir via a single-port mirror as shown in fig 3.1. The atoms are removed from the cavity after time τ. The process of a three-level cascade laser leading to the creation of the same and different light modes of frequencies can be described by the Hamiltonian

$$\hat{H} = ig[\mid a\rangle\langle b \mid \hat{a}_1 + \mid b\rangle\langle c \mid \hat{a}_2 - \hat{a}_1^\dagger \mid b\rangle\langle a \mid - \hat{a}_2^\dagger \mid c\rangle\langle b \mid], \quad (3.23)$$

where g is the coupling constant between cavity modes and the atom chosen to be the same for all transitions for convenience equal for both transitions and $\hat{a_1}$ and $\hat{a_2}$ are the annihilation operators for the same and different frequencies of cavity modes.

Initially a single atom is prepared in a coherent superposition of the top and bottom levels:

$$\mid \psi_A(0)\rangle = c_a(0) \mid a\rangle + c_c(0) \mid c\rangle, \quad (3.24)$$

in which $c_a(0)$ and $c_c(0)$ are the probability amplitudes for the atom to be initially in states $\mid a\rangle$ and $\mid c\rangle$, respectively and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)} \mid a\rangle\langle a \mid + \rho_{ac}^{(0)} \mid a\rangle\langle c \mid + \rho_{ca}^{(0)} \mid c\rangle\langle a \mid + \rho_{cc}^{(0)} \mid c\rangle\langle c \mid, \quad (3.25)$$

in which $\rho_{aa}^{(0)} = |c_a|^2$ and $\rho_{cc}^{(0)} = |c_c|^2$ are probabilities of the atom to be in the states $\mid a\rangle$ and $\mid c\rangle$, respectively and $\rho_{ac}^{(0)} = c_c^* c_a$ and $\rho_{ca}^{(0)} = c_a^* c_c$ are the coherence between levels $\mid a\rangle$ and $\mid c\rangle$

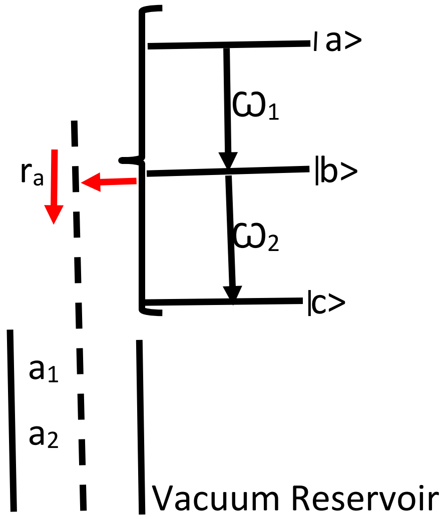

Fig. 3.1: Schematic representation of three-level cascade laser with the same and different frequencies coupled to vacuum reservoir

and indicative of non-classical features. The density operator for all atoms in a cavity plus the cavity modes at time t can then be written as

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t, \tag{3.26}$$

where $r_a\Delta t$ represents the number of atoms injected into the cavity at time t_j. Now converting the summation into integration ,we get

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt', \tag{3.27}$$

and on differentiating with respect to t, there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t, t) - \hat{\rho}_{AR}(t, t-\tau)) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt', \tag{3.28}$$

We observe that $\hat{\rho}_{AR}(t, t)$ is the density operator for the cavity mode plus an atom injected at time t. This operator can thus be expressed as

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(0)\hat{\rho}(t), \tag{3.29}$$

in which $\hat{\rho}(t)$ is the density operator for the cavity modes alone. We also observe that $\hat{\rho}_{AR}(t, t-\tau)$ is the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. We can also express this operator as

$$\hat{\rho}_{AR}(t, t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{3.30}$$

Now with the aid of (3.29) and (3.30),we can rewrite (3.30) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(0) - \hat{\rho}_A(t, t-\tau))\hat{\rho}(t) + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt', \tag{3.31}$$

In the absence of damping of the cavity mode by an ordinary vacuum reservoir, the density operator $\hat{\rho}_{AR}(t, t')$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t') = -i[\hat{H}, \hat{\rho}_{AR}(t, t')], \tag{3.32}$$

so that using this relation along with (3.27),one can put Eq. (3.31) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(0) - \hat{\rho}_A(t, t-\tau))\hat{\rho}(t) - i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{3.33}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity mode by an ordinary vacuum reservoir, we have

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -iTr_A[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1) \\ &+ \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2),\end{aligned} \tag{3.34}$$

where we have used the fact that

$$Tr_A(t) = Tr\hat{\rho}_A(t-\tau) = 1. \tag{3.35}$$

Employing (3.23) the master equation for the cavity modes can put in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= g[\hat{\rho}_{ab}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{\rho}_{bc} + \hat{a}_1\hat{\rho}_{ba} - \hat{\rho}_{ba}\hat{a}_1 + \hat{a}_2\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{a}_2] \\ &+ \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1) + \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2),\end{aligned} \tag{3.36}$$

in which the matrix element

$$\hat{\rho}_{\alpha\beta} = \langle\alpha \mid \hat{\rho}_{\alpha\beta} \mid \beta\rangle \tag{3.37}$$

with $\alpha, \beta = a, b, c$

$$\begin{aligned}\frac{d\hat{\rho}_{\alpha\beta}}{dt} &= r_a(\langle\alpha \mid \hat{\rho}_A(0) \mid \beta\rangle - \langle\alpha \mid \hat{\rho}_A(t, t-\tau) \mid \beta\rangle)\hat{\rho}(t) \\ &- i(\langle\alpha \mid \hat{H}\hat{\rho}_{AR} \mid \beta\rangle - \langle\alpha \mid \hat{\rho}_{AR}\hat{H} \mid \beta\rangle) - \gamma\hat{\rho}_{\alpha\beta},\end{aligned} \tag{3.38}$$

where the last term is included to account for the decay of the atoms due to spontaneous photon emission. And γ, considered to be the same for all of a three levels, is the atomic decay rate. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle \alpha \mid \hat{\rho}_A(t, t-\tau) \mid \beta \rangle = 0, \tag{3.39}$$

and hence Eq.(3.38) reduces to

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a(\langle \alpha \mid \hat{\rho}_A(0) \mid \beta \rangle)\hat{\rho}(t) - i(\langle \alpha \mid \hat{H}\hat{\rho}_{AR} \mid \beta \rangle - \langle \alpha \mid \hat{\rho}_{AR}\hat{H} \mid \beta \rangle) - \gamma\hat{\rho}_{\alpha\beta}. \tag{3.40}$$

Applying this equation and taking into account (3.23) and (3.25) one readily obtains

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{a}_1\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a}_1 + \hat{\rho}_{ac}\hat{a}_2^\dagger) - \gamma\hat{\rho}_{ab}, \tag{3.41}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{a}_2\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{a}_2 + \hat{a}_1^\dagger\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \tag{3.42}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\rho_{aa}^{(o)}\hat{\rho} - g(\hat{a}_1\hat{\rho}_{ba} + \hat{\rho}_{ab}\hat{a}_1^\dagger) - \gamma\hat{\rho}_{aa}, \tag{3.43}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = -g(\hat{a}_1^\dagger\hat{\rho}_{ab} - \hat{a}_2\hat{\rho}_{cb} - \hat{\rho}_{ba}\hat{a}_1 - \hat{\rho}_{bc}\hat{a}_2^\dagger) - \gamma\hat{\rho}_{bb}, \tag{3.44}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a\rho_{cc}^{(o)}\hat{\rho} - g(\hat{a}_2^\dagger\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{a}_2) - \gamma\hat{\rho}_{cc}, \tag{3.45}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a\rho_{ac}^{(o)}\hat{\rho} + g(\hat{a}_1\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{a}_2) - \gamma\hat{\rho}_{ac}. \tag{3.46}$$

We carry out our analysis by applying the linear and adiabatic approximation schemes. The linear approximation is achieved by dropping g terms in equations above. In addition we assume that $\kappa << g, \gamma$ (the good-cavity limit). Under this assumption, the cavity mode variables change slowly compered with the atomic variables. Since the atomic variable reach steady state in the relatively short period of γ^{-1}, one can take the time derivatives of such variables to be zero, while keeping the zero-order atomic and cavity mode variables at time t. This procedure may be referred to as the adiabatic approximation scheme. So by applying those schemes, we get from Eqs. (3.43), (3.44), (3.45), and (3.46) that

$$\hat{\rho}_{aa} = \frac{r_a\rho_{aa}^{(o)}}{\gamma}\hat{\rho}, \tag{3.47}$$

$$\hat{\rho}_{bb} = 0, \tag{3.48}$$

$$\hat{\rho}_{cc} = \frac{r_a \rho_{cc}^{(o)}}{\gamma}\hat{\rho}, \tag{3.49}$$

$$\hat{\rho}_{ac} = \frac{r_a \rho_{ac}^{(o)}}{\gamma}\hat{\rho}. \tag{3.50}$$

Substituting the above results into Eqs. (3.41) and (3.42), we have

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{gr_a}{\gamma}(\rho_{ac}^{(0)}\hat{\rho}\hat{a}_2^\dagger - \rho_{aa}\hat{\rho}\hat{a}_1) - \gamma\hat{\rho}_{ab}, \tag{3.51}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{gr_a}{\gamma}(\hat{a}_2\rho_{cc}^{(0)}\hat{\rho} - \hat{a}_1^\dagger\rho_{ac}^{(0)}\hat{\rho}) - \gamma\hat{\rho}_{bc}. \tag{3.52}$$

So that employing once more the adiabatic approximation scheme, we get

$$\rho_{ab} = \frac{gr_a}{\gamma^2}(\rho_{ac}^{(0)}\hat{\rho}\hat{a}_2^\dagger - \rho_{aa}^{(0)}\hat{\rho}\hat{a}_1), \tag{3.53}$$

$$\rho_{bc} = \frac{gr_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{a}_2\hat{\rho} - \rho_{ac}^{(0)}\hat{a}_1^\dagger\hat{\rho}). \tag{3.54}$$

The complex conjugate of Eqs.(3.53) and (3.54) is obtain as

$$\rho_{bc} = \frac{gr_a}{\gamma^2}(\rho_{ca}^{(0)}\hat{a}_2\hat{\rho} - \rho_{aa}^{(0)}\hat{a}_1^\dagger\hat{\rho}), \tag{3.55}$$

$$\rho_{cb} = \frac{gr_a}{\gamma^2}(\rho_{cc}^{(0)}\hat{\rho}\hat{a}_2^\dagger - \rho_{ca}^{(0)}\hat{\rho}\hat{a}_1). \tag{3.56}$$

Finally on account of Eqs.(3.53), (3.54), (3.55) and (3.56), the equation of evolution of density operator for the cavity modes given by Eq.(3.36) takes the form

$$\begin{aligned}
\frac{d\hat{\rho}}{dt} &= \frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_1^\dagger - \hat{a}_1\hat{a}_1^\dagger\hat{\rho}) \\
&+ \frac{1}{2}A\rho_{cc}^{(0)}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_2\hat{\rho}) \\
&- \frac{1}{2}A\rho_{ac}^{(0)}(2\hat{a}_1^\dagger\hat{\rho}\hat{a}_2^\dagger - \hat{\rho}\hat{a}_2^\dagger\hat{a}_1^\dagger - \hat{a}_2^\dagger\hat{a}_1^\dagger\hat{\rho}) \\
&- \frac{1}{2}A\rho_{ca}^{(0)}(2\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2\hat{\rho}) \\
&+ \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1) \\
&+ \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2),
\end{aligned} \tag{3.57}$$

in which k_1 and k_2 are the damping constant for light modes of $\hat{a}_1$ and $\hat{a}_2$, respectively and $A = \frac{2g^2 r_a}{\gamma^2}$ is the linear gain coefficient with r_a is the rate at which atoms are injected into the cavity as well as γ is atomic decay constant. It is worth mentioning that the quantum properties of light with the same and different frequencies generated by three-level cascade laser are determined by the master equation(3.57). It is not hard to see that with $\rho_{ac}^{(0)} = \rho_{cc}^{(0)} = 0$ and $\rho_{aa}^{(0)} = 1$, these equation goes over to the master equation of a two-level laser operating below threshold. The term involving $\rho_{aa}^{(0)}$ corresponds to the usual gain and that proportional to $\rho_{cc}^{(0)}$, to absorption. We will se that the atomic coherence $\rho_{ac}^{(0)}$ as well as $\rho_{ca}^{(0)}$ responsible for squeezing.

3.2 Operator Dynamics

At issue here is to find operator dynamics, the equation of the expectation values of the cavity operators, and their steady state solutions by using the relation time evolution of the expectation value of an operator $\hat{A}$ in Schr$\ddot{o}$dinger picture which can be expressed as[9]

$$\frac{d\langle\hat{A}\rangle}{dt} = T_r(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{3.58}$$

where $\frac{d\hat{\rho}}{dt}$ is expressed in above Eq. (3.57) one can get

$$\begin{aligned}
\frac{d\langle\hat{a}_1\rangle}{dt} &= T_r\{\frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{\rho}\hat{a}_1)\\
&+ \frac{1}{2}A\rho_{cc}^{(0)}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_1 - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\hat{\rho}\hat{a}_1)\\
&- \frac{1}{2}A\rho_{ac}^{(0)}(2\hat{a}_1^\dagger\hat{\rho}\hat{a}_2^\dagger\hat{a}_1 - \hat{\rho}\hat{a}_2^\dagger\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_1^\dagger\hat{\rho}\hat{a}_1)\\
&- \frac{1}{2}A\rho_{ca}^{(0)}(2\hat{a}_2\hat{\rho}\hat{a}_1\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_2\hat{a}_1 - \hat{a}_1\hat{a}_2\hat{\rho}\hat{a}_1)\\
&+ \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_1\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1\hat{a}_1)\\
&+ \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2\hat{a}_1)\}.
\end{aligned} \tag{3.59}$$

Applying the cyclic property of the trace operation and the commutation relations

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1,$$

$$[\hat{a}_1, \hat{a}_2] = [\hat{a}_1^\dagger, \hat{a}_2^\dagger] = [\hat{a}_1, \hat{a}_2^\dagger] = 0,$$

and with the aid of the identity

$$[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C},$$

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B},$$

We readily get

$$\frac{d\langle\hat{a}_1\rangle}{dt} = \frac{1}{2}(A\rho_{aa}^{(0)} - \kappa_1)\langle\hat{a}_1\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_2^\dagger\rangle. \tag{3.60}$$

Following a similar procedure, we get

$$\frac{d\langle\hat{a}_2\rangle}{dt} = \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\rangle - \frac{1}{2}(A\rho_{cc}^{(0)} + \kappa_2)\langle\hat{a}_2\rangle, \tag{3.61}$$

$$\frac{d\langle\hat{a}_1^\dagger\hat{a}_1\rangle}{dt} = (A\rho_{aa}^{(0)} - \kappa_1)\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}_1\hat{a}_2\rangle + A\rho_{aa}^{(0)}, \tag{3.62}$$

$$\frac{d\langle\hat{a}_2^\dagger\hat{a}_2\rangle}{dt} = -(A\rho_{cc}^{(0)} + \kappa_2)\langle\hat{a}_2^\dagger\hat{a}_2\rangle + \frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}_1\hat{a}_2\rangle, \tag{3.63}$$

$$\frac{d\langle\hat{a}_1\hat{a}_2\rangle}{dt} = \frac{1}{2}[A\rho_{aa}^{(0)} - \kappa_1 - (A\rho_{cc}^{(0)} + \kappa_2)]\langle\hat{a}_1\hat{a}_2\rangle + \frac{1}{2}A\rho_{ac}^{(0)}(\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\rangle - \langle\hat{a}_2^\dagger\hat{a}_2\rangle + 1). \tag{3.64}$$

On taking $\kappa_1 = \kappa_2 = \kappa$, the steady-state solutions of the above equations becomes

$$\langle\hat{a}_1\rangle = \frac{A\rho_{ac}^{(0)}\langle\hat{a}_2^\dagger\rangle}{\mu_1}, \tag{3.65}$$

$$\langle\hat{a}_2\rangle = \frac{A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\rangle}{\mu_2}, \tag{3.66}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \frac{\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}_1\hat{a}_2\rangle - A\rho_{aa}^{(0)}}{\mu_1}, \tag{3.67}$$

$$\langle\hat{a}_2^\dagger\hat{a}_2\rangle = \frac{\frac{1}{2}A\rho_{ac}^{(0)}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \frac{1}{2}A\rho_{ca}^{(0)}\langle\hat{a}_1\hat{a}_2\rangle}{\mu_2}, \tag{3.68}$$

$$\langle\hat{a}_1\hat{a}_2\rangle = \frac{-A\rho_{ac}^{(0)}(\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\rangle - \langle\hat{a}_2^\dagger\hat{a}_2\rangle + 1)}{\mu_1 - \mu_2}. \tag{3.69}$$

where

$$\mu_1 = A\rho_{aa}^{(0)} - \kappa, \tag{3.70}$$

$$\mu_2 = A\rho_{cc}^{(0)} + \kappa. \tag{3.71}$$

Applying Eq.(3.65) and (3.66), we easily find

$$\langle\hat{a}_1\rangle = \langle\hat{a}_2\rangle = 0. \tag{3.72}$$

Moreover, using Eqs.(3.67), (3.68), and (3.69), we get

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \frac{-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_2 + A\rho_{aa}^{(0)}(\mu_1 - \mu_2)\mu_2}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2)}, \tag{3.73}$$

$$\langle \hat{a}_2^\dagger \hat{a}_2 \rangle = \frac{-A^3 \rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_1}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1 \mu_2](\mu_1 - \mu_2)}, \tag{3.74}$$

$$\langle \hat{a}_1 \hat{a}_2 \rangle = \frac{(-A\rho_{aa}^{(0)} + \mu_1) A \rho_{ac}^{(0)} \mu_2}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1 \mu_2](\mu_1 - \mu_2)}. \tag{3.75}$$

It can also be readily verified that

$$\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0. \tag{3.76}$$

We take

$$\hat{a} = \hat{a}_1 + \hat{a}_2, \tag{3.77}$$

to be the annihilation operator for superposition of light modes $\hat{a}_1$ and $\hat{a}_2$, produced by the three-level cascade laser. One can easily check that

$$[\hat{a}, \hat{a}^+] = 2. \tag{3.78}$$

We wish to study various quantities of interest on the superposed light modes with the same and different frequencies.

4

RESULT AND DISCUSSION

4.1 Quadrature Fluctuations

We proceed to determine the quadrature variance for the light modes with the same and different frequencies that produced by a three-level cascade laser inside and outside the cavity that coupled to vacuum reservoir.

4.1.1 Quadrature Variance of Cavity Modes

Cavity modes with the same and different frequencies are said to be in squeezed state if either $\Delta a_+ < 1$ and $\Delta a_- > 1$ or $\Delta a_+ > 1$ and $\Delta a_- < 1$ such that $\Delta a_+ \Delta a_- \geq 1$. We remember that the variance of the plus and minus quadrature operators for the same and different frequencies of cavity modes is given by

$$(\Delta a_\pm)^2 = \langle \hat{a}_\pm^2 \rangle - \langle \hat{a}_\pm \rangle^2, \tag{4.1}$$

where quadrature operators that describes the properties of squeezed light are

$$\hat{a}_+ = \hat{a}^\dagger + \hat{a} \tag{4.2}$$

and

$$\hat{a}_- = i(\hat{a}^\dagger - \hat{a}). \tag{4.3}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of light obtained from the superposition of the same and different frequencies of cavity modes. Now on account of Eq.(3.72) along with Eqs.(3.76), (4.2) and (4.3) we see that

$$\langle \hat{a}_\pm \rangle = 0. \tag{4.4}$$

Hence in view of this, equation (4.1) is expressible as

$$(\Delta a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle. \tag{4.5}$$

We note that

$$(\Delta a_{\pm})^2 = \langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{a}\hat{a}^{\dagger} \rangle \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle, \tag{4.6}$$

and applying Eq.(3.78) the quadrature variance can be put in the form

$$(\Delta a_{\pm})^2 = 2 + 2\langle \hat{a}^{\dagger}\hat{a} \rangle \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2} \rangle, \tag{4.7}$$

With the aid of Eq.(3.77) we can rewrite Eq.(4.7) as

$$(\Delta a_{\pm})^2 = 2 + 2\langle (\hat{a}_1^{\dagger} + \hat{a}_2^{\dagger})(\hat{a}_1 + \hat{a}_2) \rangle \pm \langle \hat{a}_1^2 + \hat{a}_2^{\dagger 2} + \hat{a}_2^2 + \hat{a}_2^{\dagger 2} + 2\hat{a}_1\hat{a}_2 + 2\hat{a}_1^{\dagger}\hat{a}_2^{\dagger} \rangle, \tag{4.8}$$

and taking Eq.(3.76), we get

$$(\Delta a_{\pm})^2 = 2[1 + \langle (\hat{a}_1^{\dagger}\hat{a}_1 + \hat{a}_2^{\dagger}\hat{a}_2) \rangle \pm \langle (\hat{a}_1\hat{a}_2 + \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) \rangle]. \tag{4.9}$$

Therefore by considering Eqs.(3.73), (3.74) and (3.75) the quadrature variance takes the form

$$\begin{aligned}(\Delta a_{\pm})^2 &= \frac{2\{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2) - A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_2 + A\rho_{aa}^{(0)}(\mu_1 - \mu_2)\mu_2\}}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2)} \\ &+ \frac{2\{-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_1 \pm (-A\rho_{aa}^{(0)} + \mu_1)A\mu_2(\rho_{ac}^{(0)} + \rho_{ca}^{(0)})\}}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2)}.\end{aligned} \tag{4.10}$$

It proves to be more convenient to introduce a new parameter defined by

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{4.11}$$

so that in view of the fact that

$$\rho_{aa}^{(0)} + \rho_{cc}^{(0)} = 1 \tag{4.12}$$

and

$$\mid \rho_{ac}^{(0)} \mid^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}, \tag{4.13}$$

one easily finds

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2} \tag{4.14}$$

and

$$\mid \rho_{ac}^{(0)} \mid = \frac{1}{2}(1-\eta^2)^{\frac{1}{2}}. \tag{4.15}$$

Upon setting

$$\rho_{ac}^{(0)} = \mid \rho_{ac}^{(0)} \mid e^{i\theta} \tag{4.16}$$

and taking into account (3.70), and (3.71), the quadrature variance takes the form

$$(\Delta a_{\pm})^2 = 2\{\frac{-\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2 \rho_{cc}^{(0)2} - 3\kappa^2 A \rho_{cc}^{(0)} + \kappa^2 A \rho_{aa}^{(0)} - 2\kappa^3}{2\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2 \rho_{cc}^{(0)2} - \kappa A^2 \rho_{aa}^{(0)2} - 3\kappa^2 A \rho_{cc}^{(0)} + 3\kappa^2 A \rho_{aa}^{(0)} - 2\kappa^3}$$
$$\pm \frac{-2\kappa A \mid \rho_{ac}^{(0)} \mid (\rho_{cc}^{(0)} + \kappa)\cos\theta}{2\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2 \rho_{cc}^{(0)2} - \kappa A^2 \rho_{aa}^{(0)2} - 3\kappa^2 A \rho_{cc}^{(0)} + 3\kappa^2 A \rho_{aa}^{(0)} - 2\kappa^3}\}. \tag{4.17}$$

Applying the relations (4.11) to (4.16), the quadrature variance can be put in the form

$$(\Delta a_{+})^2 = \frac{(A^2 + 4\kappa A)\eta + (A^2 + 2\kappa A + 4\kappa^2) + [A^2(\eta^2 - \eta^4)^{\frac{1}{2}} + (2\kappa A + A^2)(1 - \eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2} \tag{4.18}$$

$$(\Delta a_{-})^2 = \frac{(A^2 + 4\kappa A)\eta + (A^2 + 2\kappa A + 4\kappa^2) - [A^2(\eta^2 - \eta^4)^{\frac{1}{2}} + (2\kappa A + A^2)(1 - \eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2}. \tag{4.19}$$

(4.20)

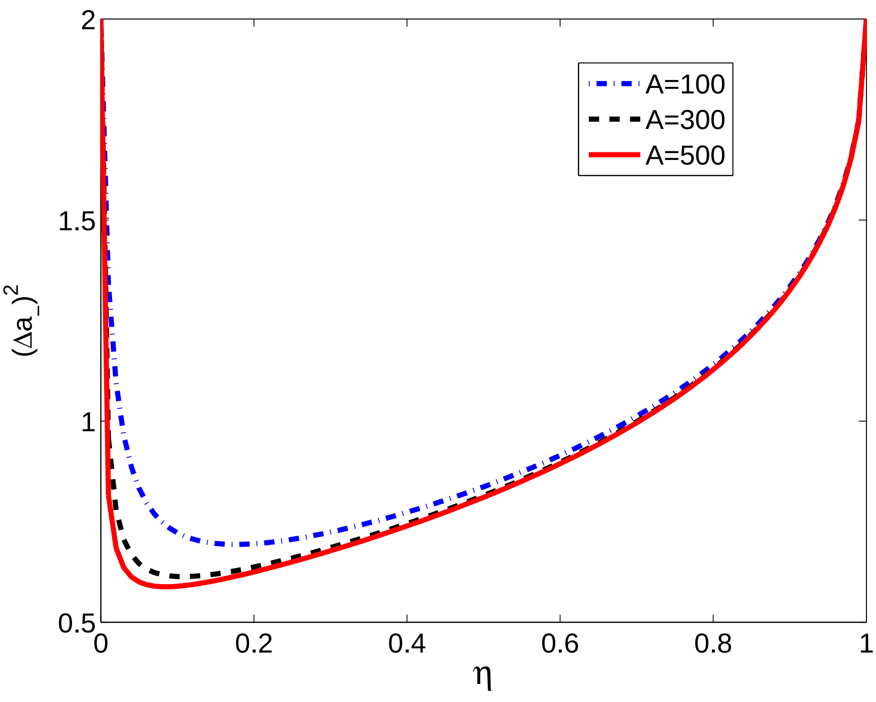

Fig. 4.1: plots of minus quadrature variance of cavity mode($(\Delta a_{-})^2$) [Eq.(4.19)] versus η at steady state for $\kappa = 0.8$, $\theta = 0$ and different values of linear gain coefficient, for A=100(dash-dot curve), A=300(dash curve) and A=500.(solid curve)

The plots in Fig.4.1 indicates that as the linear gain coefficient increases, the values of η at which the minimum value of the quadrature variance occurs approaches zero. We thus realize that better squeezing can be achieved by initially preparing the atoms in such a way slightly more atoms are in the lower level than in the upper level and by increasing the linear gain coefficient his result is agrees with the report in[9].

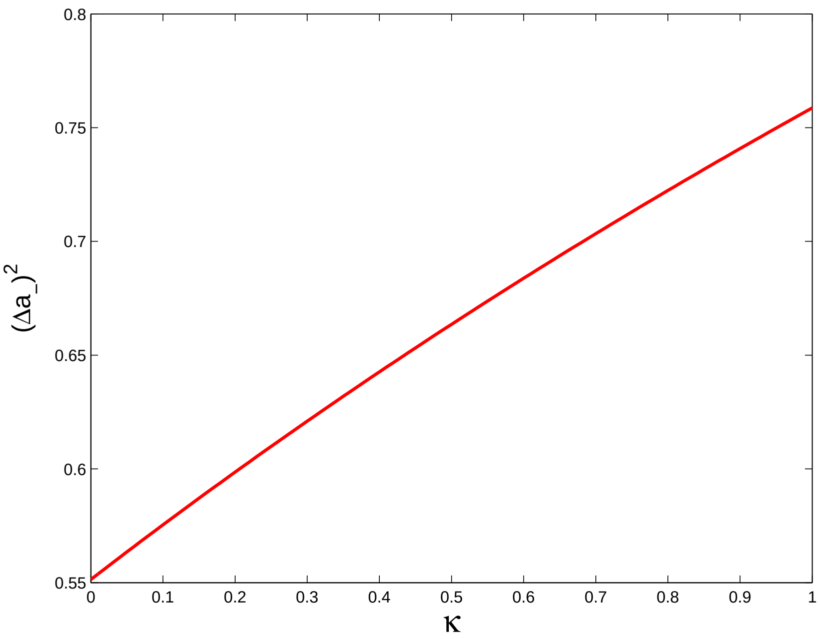

Fig. 4.2: plots of minus quadrature variance of cavity mode($(\Delta a_-)^2$) [Eq.(4.19)] versus κ at steady state for η=0.1, $\theta = 0$ and A=100.

Fig.4.2 shows that as the cavity damping constant κ increases, then the values of minus quadrature variance of cavity modes increases. However, squeezing of light inside the cavity is decreases.

4.1.2 Quadrature Variance of Output Modes

The variance of the output mode quadrature operators

$$\hat{a}_{+}^{out} = \hat{a}_{out}^{\dagger} + \hat{a}_{out}, \tag{4.21}$$

$$\hat{a}_{-}^{out} = i(\hat{a}_{out}^{\dagger} - \hat{a}_{out}), \tag{4.22}$$

in which

$$\hat{a}_{out} = \hat{a}_{1,out} + \hat{a}_{2,out}, \tag{4.23}$$

where, $\hat{a}_{out}$ is the annihilation operator for superposition of output mode of $\hat{a}_{1,out}$ and $\hat{a}_{2,out}$, produced by the three-level cascade laser and transmitted from cavity to outside. One can easily check that

$$[\hat{a}_{out}, \hat{a}_{out}^{\dagger}] = 2, \tag{4.24}$$

with

$$\hat{a}_{1,out} = \sqrt{\kappa}\hat{a}_1, \hat{a}_{2,out} = \sqrt{\kappa}\hat{a}_2, \tag{4.25}$$

can be expressed as

$$(\Delta a_{\pm,out})^2 = \langle \hat{a}_{\pm,out}^2 \rangle - \langle \hat{a}_{\pm,out} \rangle^2. \tag{4.26}$$

Now applying Eq.(4.20)-(4.24) and Eq.(4.4) one can get

$$(\Delta a_{\pm,out})^2 = \langle \hat{a}_{\pm,out}^2 \rangle, \tag{4.27}$$

then, the steady quadrature variance of the same and different frequencies of light outside the cavity can be put in the form

$$(\Delta a_{\pm,out})^2 = 2 + 2\kappa[\langle(\hat{a}_1^{\dagger}\hat{a}_1 + \hat{a}_2^{\dagger}\hat{a}_2)\rangle \pm (\langle \hat{a}_1\hat{a}_2\rangle + \langle \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle)]. \tag{4.28}$$

Up on substituting Eqs.(3.73), (3.74) and (3.75) into Eq.(4.27)

$$\begin{aligned}(\Delta a_{\pm,out})^2 = 2 &+ \frac{2\kappa[-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_2 + A\rho_{aa}^{(0)}(\mu_1 - \mu_2)\mu_2]}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2)} \\ &+ \frac{2\kappa[-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_1 \pm (-A\rho_{aa}^{(0)} + \mu_1)A\mu_2(\rho_{ac}^{(0)} + \rho_{ca}^{(0)})]}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1 - \mu_2)},\end{aligned} \tag{4.29}$$

and taking Eqs.(3.70) and (3.71), also Eqs.(4.11)-(4.16) into account, then the quadrature variance of output light mode takes the form

$$\begin{aligned}(\Delta a_{+,out})^2 = 2 &+ \frac{(\kappa A^2 - 2\kappa A^2\eta^2 + \kappa A^2\eta + 2\kappa^2 A - 2\kappa^2 A\eta)}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2} \\ &+ \frac{[\kappa A^2(\eta^2 - \eta^4)^{\frac{1}{2}} + (2\kappa^2 A + \kappa A^2)(1 - \eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2},\end{aligned} \tag{4.30}$$

and

$$(\Delta a_{-,out})^2 = 2 + \frac{(\kappa A^2 - 2\kappa A^2\eta^2 + \kappa A^2\eta + 2\kappa^2 A - 2\kappa^2 A\eta)}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2} - \frac{[\kappa A^2(\eta^2 - \eta^4)^{\frac{1}{2}} + (2\kappa^2 A + \kappa A^2)(1-\eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2}. \tag{4.31}$$

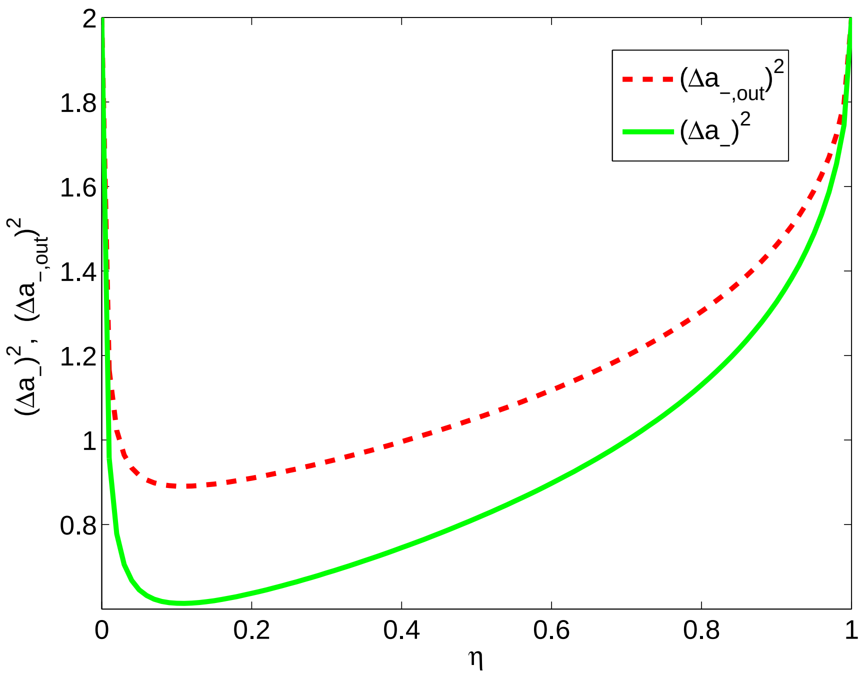

Fig. 4.3: plots of minus quadrature variance of output mode($(\Delta a_{-,out})^2$) [Eq.(4.30)] and cavity mode($(\Delta a_{-})^2$) versus η that represented by dash-curve and solid curve,respectively and for $\kappa = 0.8, \theta = 0$ and A=300.

It can readily be observed from the plots in Fig.(4.3) that the degree of squeezing of the same and different frequencies of light outside the cavity is less than that of the interacavity. Photons with the same and different frequencies in the cavity are generated pairwise and highly correlated and this correlation is the source of squeezing in the system. However, when photons leave the cavity through the port mirror, correlated pairs of photons may not leave at the same time, and hence, there is some finite probability observing an odd number of photons inside and outside the cavity. It is obvious that the probability of observing an odd number of photons outside the cavity is greater than the interacavity one. As a result of this, the correlation between the pairs could be destroyed to some extent depending on the property of the port mirror(or the value of κ). This leads to the decreases in the degree of squeezing of the output mode. This result is agrees with the report in[23].

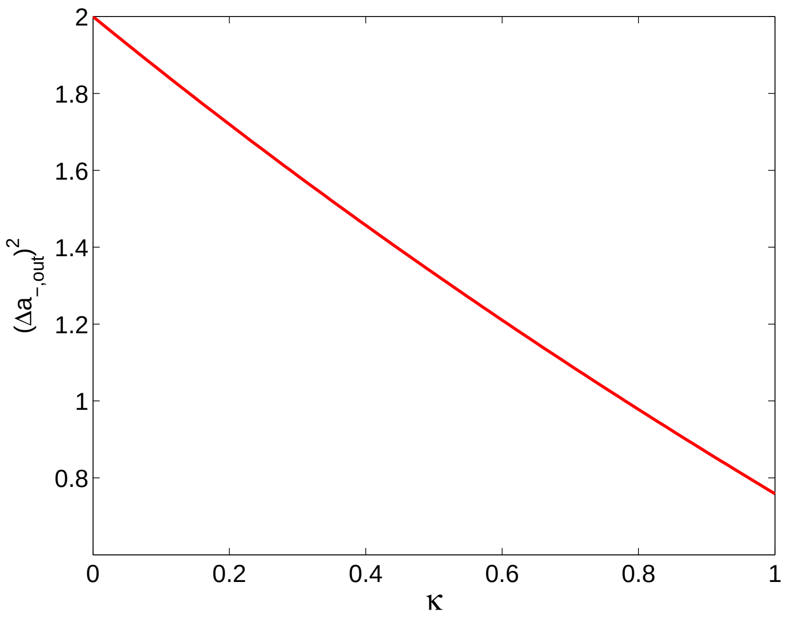

Fig. 4.4: plots of minus quadrature variance of output mode($(\Delta a_{-,out})^2$) [Eq.(4.30)] versus κ for η=0.1,$\theta = 0$ and A=100.

Fig.4.4 shows that as the cavity damping constant κ increases, then the values of minus quadrature variance of output mode decreases. However, the squeezed light available at outside the cavity is increasing.

4.2 Photon Statistics

The statistical properties of the same and different frequencies of cavity modes described in terms of the mean and variance of the photon number. Here we wish to calculate the mean and variance of photon number for the same and different frequencies of cavity modes generated by a three-level cascade laser. Photon statistics of light modes based on the relation between the variance and the mean of the photon number. Hence the photon statistics of a light modes for which $(\Delta n) = \bar{n}$ is referred to as Poissonian and the photon statistics of a light mode for $(\Delta n) > \bar{n}$ which is called super-Poissonian. Otherwise the photon statistics is said to be sub-Poissonian.

4.2.1 The Mean Photon Number

We define the mean photon number of the same and different frequencies of light mode by

$$\bar{n} = \langle \hat{a}^{\dagger} \hat{a} \rangle. \tag{4.32}$$

Then using Eq.(3.77) and taking into account (3.76), we easily find

$$\bar{n} = \langle \hat{a}_1^{\dagger} \hat{a}_1 \rangle + \langle \hat{a}_2^{\dagger} \hat{a}_2 \rangle. \tag{4.33}$$

so by considering the Eqs.(3.73) and (3.74) there follows

$$\bar{n} = \frac{-A^3 \rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_2 + A(\mu_1 - \mu_2)\mu_2}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1 \mu_2](\mu_1 - \mu_2)} + \frac{-A^3 \rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_1}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1 \mu_2](\mu_1 - \mu_2)}. \tag{4.34}$$

in terms of η Eq.(4.33) can be put in the form

$$\bar{n} = \frac{\frac{-3\kappa A^2}{4}\eta^2 - (\kappa^2 A - \frac{\kappa A^2}{2})\eta + \frac{\kappa A^2}{4} + \kappa^2 A}{\kappa A^2 \eta^2 + 3\kappa^2 A \eta + 2\kappa^3} + \frac{\frac{-\kappa A^2}{4}\eta^2 + \frac{\kappa A^2}{4}}{\kappa A^2 \eta^2 + 3\kappa^2 A \eta + 2\kappa^3}. \tag{4.35}$$

this equation is simplified as follow

$$\bar{n} = \frac{-A^2\eta^2 - (\kappa A - \frac{A^2}{2})\eta + \frac{A^2}{2} + \kappa A}{A^2\eta^2 + 3\kappa A \eta + 2\kappa^2}. \tag{4.36}$$

This represents the mean photon number of the same and different frequencies of cavity modes.

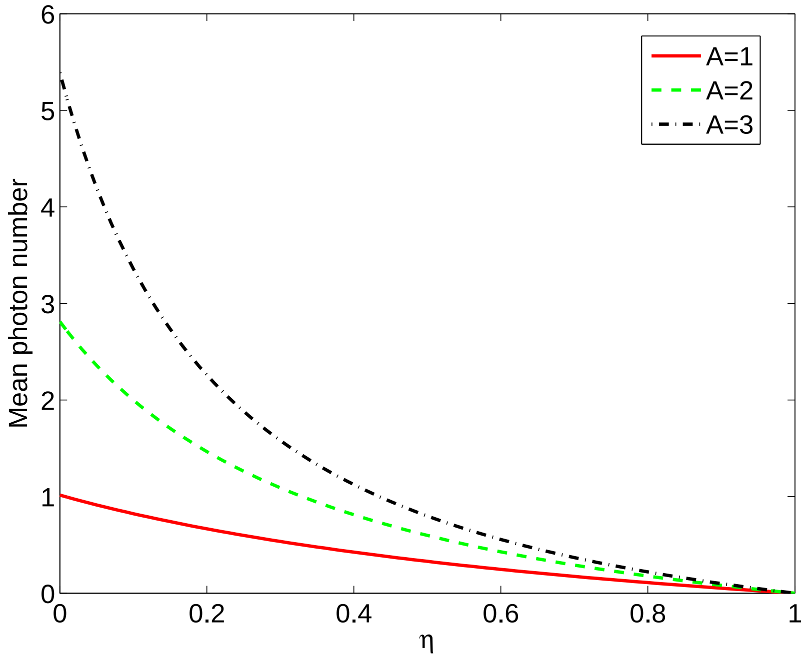

Fig. 4.5: plots of mean photon number($\bar{n}$) [Eq.(4.35)] versus η for $\kappa = 0.8$, and different values of linear gain coefficient, for A=1(solid curve), A=2(dash curve) and A=3(dash-dot curve)

It can be observed from the plots in Fig.(4.5) that the mean photon number decreases with linear gain coefficient.

4.2.2 The Variance of the Photon Number

The variance of photon number of cavity modes with the same and different frequencies is defined as

$$(\Delta n)^2 = \langle(\hat{a}^\dagger \hat{a})^2\rangle - \bar{n}^2. \tag{4.37}$$

This can be put in the form

$$(\Delta n)^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + 2\bar{n} - \bar{n}^2. \tag{4.38}$$

Now applying the fact that $\hat{a}$ is a Gaussian variable with zero mean, we get

$$(\Delta n)^2 = 2\bar{n} + \bar{n}^2 + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle. \tag{4.39}$$

and on taking into account Eq.(3.78) along with Eq.(3.77), we arrive at

$$(\Delta n)^2 = 2\bar{n} + \bar{n}^2 + 4\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \langle \hat{a}_1 \hat{a}_2 \rangle. \tag{4.40}$$

Hence in the view of Eq.(4.35) and Eq.(3.75), the photon number variance of cavity modes with the same and different frequencies takes the form

$$(\Delta n)^2 = \frac{(\frac{-A^4}{4} - \kappa A^3)\eta^4 + (\frac{-3A^4}{2} - 2\kappa A^3)\eta^3 + (A^4 - \kappa^2 A^2 - \frac{3\kappa A^3}{2})\eta^2}{(A^2\eta^2 + 3\kappa A\eta + 2\kappa^2)^2} + \frac{(\frac{5A^4}{4} + 2\kappa A^3 - 2\kappa^3 A + 3\kappa^2 A^2)\eta + (A^4 + 2\kappa^2 A^2 + 2\kappa^3 A + \frac{5}{2}\kappa A^3)}{(A^2\eta^2 + 3\kappa A\eta + 2\kappa^2)^2}. \tag{4.41}$$

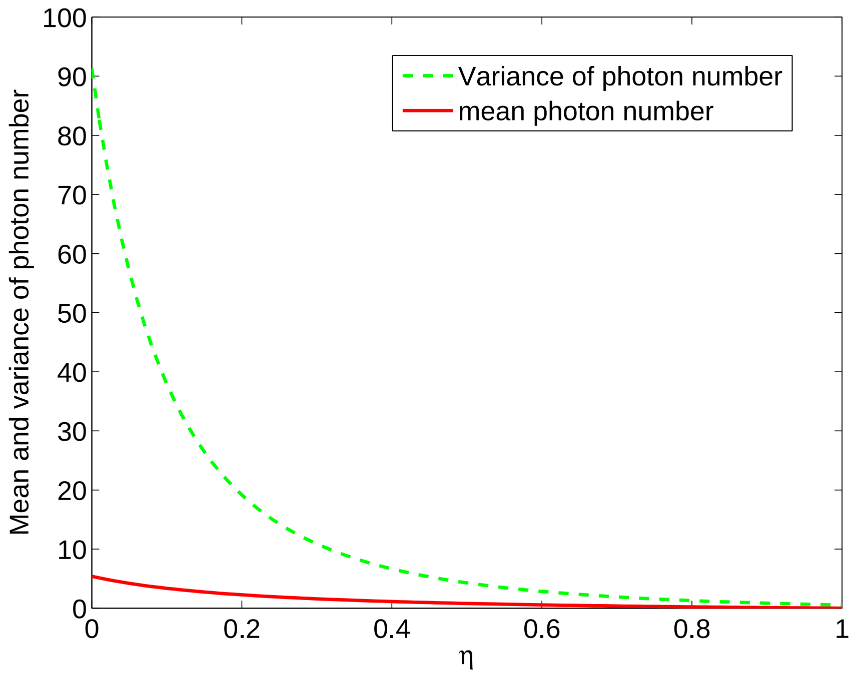

Fig. 4.6: plots of mean and variance of photon number($\bar{n}$ and $(\Delta n)^2$) [Eqs.(4.35) and (4.40)] versus η for $\kappa = 0.8$ and A=3

.

We see from Fig.4.6 that the variance of the photon number is greater than the mean of the photon number. This indicates that the photon number statistics is super-poissonian. This result agrees with a report in[22]. Furthermore, we observe that the mean and variance of the photon number decreases as η increases and as linear gain coefficient decreases.

4.3 Quantum Entanglement

In this section, we wish to investigate the entanglement properties of modes with the same and different frequencies inside and outside the cavity at a steady state. It is known that a state of a system $\hat{\rho}$ of the same and different frequencies of cavity modes $\hat{a_1}$ and $\hat{a_2}$ is said to be entangled or not separable if it is not possible to express in the form

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j^{(\hat{a}_1)} \hat{\rho}_j^{(\hat{a}_2)}, \tag{4.42}$$

where $\hat{\rho}_j^{(\hat{a}_1)}$ and $\hat{\rho}_j^{(\hat{a}_2)}$ are assumed to be the normalized density operators of modes $\hat{a_1}$ and $\hat{a_2}$ respectively, with $p_j \geq 0$ and $\sum_j p_j = 1$. A maximally entangled continuous variable state can be expressed as a co-eigen state of a pair of Einstein-Podolsky-Rosen (EPR)-type operators[21] such as $\hat{x}_1 - \hat{x}_2$ and $\hat{p}_1 + \hat{p}_2$. Thus the total variance of this operators reduced to zero for the maximally entangled continuous variable states. In addition, according to the criterion set by Duan et al.[25] quantum states of the system claimed to be entangled.

4.3.1 Entanglement of the Cavity Modes

A quantum state of a system is said to be entangled if the sum of the quantum fluctuations of the two Einstein-Podolsky-Rosen(EPR)-like operators $\hat{\upsilon}$ and $\hat{\nu}$ satisfies the inequality

$$(\Delta\upsilon)^2 + (\Delta\nu)^2 < 2, \tag{4.43}$$

where

$$\hat{\upsilon} = \hat{x}_1 - \hat{x}_2, \tag{4.44}$$

$$\hat{\nu} = \hat{p}_1 + \hat{p}_2. \tag{4.45}$$

with $\hat{x}_j = \frac{1}{\sqrt{2}}(\hat{a}_j + \hat{a}_j^\dagger)$, and $\hat{p}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j)$, (with j= 1,2) are the quadrature for the same and different frequencies of cavity modes. But

$$\begin{aligned}(\Delta\upsilon)^2 &= \langle\hat{\upsilon}^2\rangle - \langle\hat{\upsilon}\rangle^2 \\ &= \langle(\hat{x}_1 - \hat{x}_2)^2\rangle - \langle\hat{x}_1 - \hat{x}_2\rangle^2 \\ &= \langle\hat{x}_1^2\rangle + \langle\hat{x}_2^2\rangle - (\langle\hat{x}_1\hat{x}_2\rangle + \langle\hat{x}_2\hat{x}_1\rangle) \\ &- [\langle\hat{x}_1\rangle^2 + \langle\hat{x}_2\rangle^2 - (\langle\hat{x}_1\rangle\langle\hat{x}_2\rangle + \langle\hat{x}_2\rangle\langle\hat{x}_1\rangle)].\end{aligned} \tag{4.46}$$

substituting, $\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_1^\dagger)$, and $\hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_2^\dagger)$. and using Eqs. (3.72) and (3.76), one can get

$$(\Delta\upsilon)^2 = \frac{1}{2}[\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_1\hat{a}_1^\dagger\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle + \langle\hat{a}_2\hat{a}_2^\dagger\rangle - (\langle\hat{a}_1\hat{a}_2\rangle + \langle\hat{a}_2\hat{a}_1\rangle + \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \langle\hat{a}_2^\dagger\hat{a}_1^\dagger\rangle)]. \tag{4.47}$$

Following the same step and substituting $\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1)$, and $\hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_2)$. one can readily find

$$\begin{aligned}(\Delta\nu)^2 &= \langle\hat{\nu}^2\rangle - \langle\hat{\nu}\rangle^2\\ &= \langle(\hat{p}_1+\hat{p}_2)^2\rangle - \langle\hat{p}_1+\hat{p}_2\rangle^2\\ &= \langle\hat{p}_1^2\rangle + \langle\hat{p}_2^2\rangle + \langle\hat{p}_1\hat{p}_2\rangle + \langle\hat{p}_2\hat{p}_1\rangle\\ &- (\langle\hat{p}_1\rangle^2 + \langle\hat{p}_2\rangle^2 + \langle\hat{p}_1\rangle\langle\hat{p}_2\rangle + \langle\hat{p}_2\rangle\langle\hat{p}_1\rangle),\end{aligned} \tag{4.48}$$

$$(\Delta\nu)^2 = \frac{1}{2}[\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_1\hat{a}_1^\dagger\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle + \langle\hat{a}_2\hat{a}_2^\dagger\rangle - (\langle\hat{a}_1\hat{a}_2\rangle + \langle\hat{a}_2\hat{a}_1\rangle + \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle + \langle\hat{a}_2^\dagger\hat{a}_1^\dagger\rangle)]. \tag{4.49}$$

On the basis of the boson commutation relation that are mentioned as

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1. \tag{4.50}$$

we can find that

$$(\Delta\upsilon)^2 + (\Delta\nu)^2 = 2 + 2[\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle - (\langle\hat{a}_1\hat{a}_2\rangle + \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle). \tag{4.51}$$

By taking Eq.(4.9) under consideration, one can obtain

$$(\Delta\upsilon)^2 + (\Delta\nu)^2 = 2 + 2[\langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle - (\langle\hat{a}_1\hat{a}_2\rangle + \langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle)]. \tag{4.52}$$

which implies in the system under consideration, the entanglement and quadrature variance exactly equal. Now the various correlations involved in Eq.(4.51) would be determined by using the steady-state equations, (3.72), (3.73), (3.74), and (3.75), we can get

$$\begin{aligned}(\Delta\upsilon)^2 + (\Delta\nu)^2 = 2 + 2[&\frac{-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_2 + A(\mu_1-\mu_2)\mu_2}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1-\mu_2)}\\ &- \frac{-A^3\rho_{aa}^{(0)} \mid \rho_{ac}^{(0)} \mid^2 + A^2 \mid \rho_{ac}^{(0)} \mid^2 \mu_1}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1-\mu_2)}\\ &+ \frac{(-A\rho_{aa}^{(0)} + \mu_1)A\mu_2(\rho_{ac}^{(0)} + \rho_{ca}^{(0)})}{[A^2 \mid \rho_{ac}^{(0)} \mid^2 - \mu_1\mu_2](\mu_1-\mu_2)}].\end{aligned} \tag{4.53}$$

by inserting Eqs.(3.70), (3.71) and (4.16) in above expression, one can get

$$\begin{aligned}(\Delta\upsilon)^2 + (\Delta\nu)^2 = 2\{&\frac{-\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2\rho_{cc}^{(0)2} - 3\kappa^2 A\rho_{cc}^{(0)} + \kappa^2 A\rho_{aa}^{(0)} - 2\kappa^3}{2\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2\rho_{cc}^{(0)2} - \kappa A^2\rho_{aa}^{(0)2} - 3\kappa^2 A\rho_{cc}^{(0)} + 3\kappa^2 A\rho_{aa}^{(0)} - 2\kappa^3}\\ &- \frac{-2\kappa A \mid \rho_{ac}^{(0)} \mid (\rho_{cc}^{(0)} + \kappa)\cos\theta}{2\kappa A^2 \mid \rho_{ac}^{(0)} \mid^2 - \kappa A^2\rho_{cc}^{(0)2} - \kappa A^2\rho_{aa}^{(0)2} - 3\kappa^2 A\rho_{cc}^{(0)} + 3\kappa^2 A\rho_{aa}^{(0)} - 2\kappa^3}\}.\end{aligned} \tag{4.54}$$

By applying Eqs.(4.11)-(4.15), one can obtain

$$\begin{aligned}(\Delta\upsilon)^2+(\Delta\nu)^2 &= (\Delta a_-)^2\\ &= \frac{(A^2+4kA)\eta+(A^2+2\kappa A+4\kappa^2)}{A^2\eta^2+3\kappa A\eta+2\kappa^2}\\ &- \frac{[A^2(\eta^2-\eta^4)^{\frac{1}{2}}+(2\kappa A+A^2)(1-\eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2+3\kappa A\eta+2\kappa^2}. \end{aligned} \tag{4.55}$$

which implies in the system under consideration, the total variance of the EPR-types operators equal with the minus quadrature variance of the cavity modes.

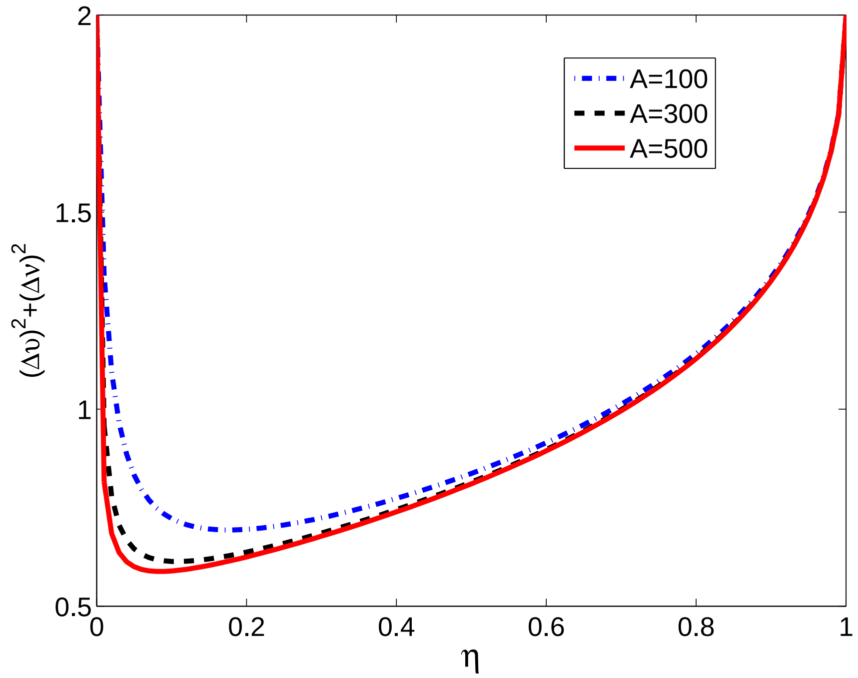

Fig. 4.7: plots of the the total variance of EPR-likes operators of cavity modes($(\Delta\upsilon)^2+(\Delta\nu)^2$) at steady state versus η for $\kappa=0.8$, $\theta=0$ and different values of linear gain coefficient, for A=100(dash-dot curve), A=300(dash curve) and A=500.(solid curve)

We easily see from Fig.4.7 that $(\Delta\upsilon)^2+(\Delta\nu)^2$ less than 2 for all values of η except $\eta=1$, hence the entanglement criteria [Eq.(4.42)] is satisfied. This indicates that the state of the system is entangled at a steady state provided that there is injected atomic coherence. Moreover, the degree of entanglement increases with the linear gain coefficient. On the other hand, comparison of Fig.4.7 and Fig.4.1 shows that there is strong entanglement of the cavity modes when there is a substantial degree of squeezing. This strong values entanglement is produced by small values of η, that is, when slightly more atoms are in the lower level at the initial time. It is also easy to see that the squeezing and entanglement are directly related as given in Eq.(4.54).

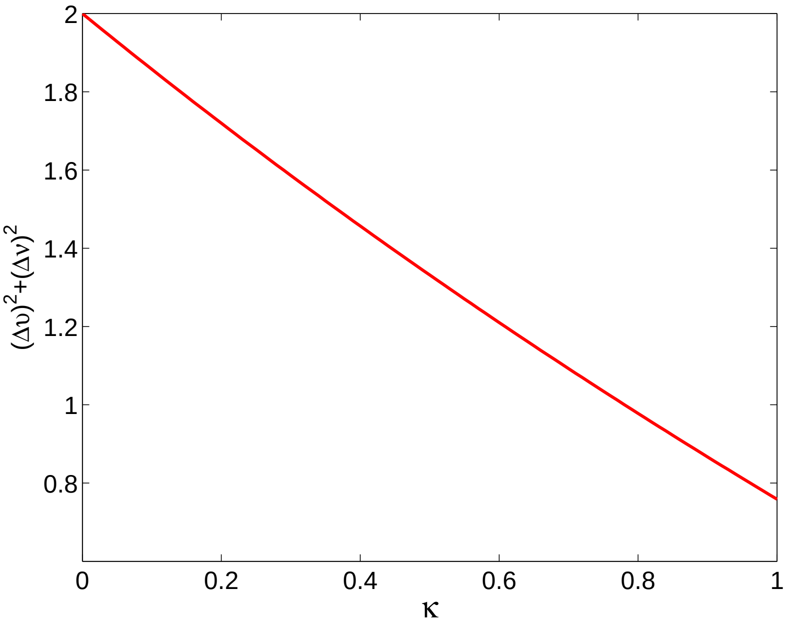

Fig. 4.8: plots of the variance of the sum of the EPR-likes operators of cavity modes($(\Delta\upsilon)^2 + (\Delta\nu)^2$) at steady state versus κ for η=0.1, $\theta = 0$ and A=100

Fig.4.8 shows that as cavity damping constant κ increases, then the values of entanglement of cavity modes increases, as a result of this degree of entanglement of light inside the cavity is decreasing.

4.3.2 Entanglement of the Output Modes

The EPR-like operators for the output modes have the form

$$\hat{\upsilon}^{out} = \hat{x}_1^{out} - \hat{x}_2^{out}, \tag{4.56}$$

$$\hat{\nu}^{out} = \hat{p}_1^{out} + \hat{p}_2^{out}. \tag{4.57}$$

in which

$$\hat{x}_1^{out} = \frac{\sqrt{\kappa}}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_1),$$

$$\hat{x}_2^{out} = \frac{\sqrt{\kappa}}{\sqrt{2}}(\hat{a}_2^\dagger + \hat{a}_2),$$

$$\hat{p}_1^{out} = \frac{i\sqrt{\kappa}}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1)$$

and

$$\hat{p}_2^{out} = \frac{i\sqrt{\kappa}}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_2).$$

are quadrature operators for the output modes. Hence, the entanglement criterion for the output modes can be written as

$$(\Delta\upsilon)^2_{out} + (\Delta\nu)^2_{out} < 2. \tag{4.58}$$

By considering the mean photon number of a vacuum reservoir to be zero, one can get

$$\langle \hat{a}^\dagger_{out}(t)\hat{a}_{out}(t)\rangle = \kappa\langle \hat{a}^\dagger\hat{a}\rangle,$$

$$\langle \hat{a}_{out}(t)\hat{a}^\dagger_{out}(t)\rangle = \kappa\langle \hat{a}^\dagger\hat{a}\rangle + 1,$$

$$\langle \hat{a}^\dagger_{out}(t)\rangle = \langle \hat{a}^{\dagger 2}\rangle = 0.$$

The steady-state variance of the operators $\hat{\upsilon}_{out}$ and $\hat{\nu}_{out}$ can then be expressed as

$$(\Delta\upsilon)^2_{out} = (\Delta\nu)^2_{out} = 1 + \kappa\langle \hat{a}_1^\dagger\hat{a}_1\rangle + \kappa\langle \hat{a}_2^\dagger\hat{a}_2\rangle - \kappa(\langle \hat{a}_1\hat{a}_2\rangle + \langle \hat{a}_1^\dagger\hat{a}_2^\dagger\rangle). \tag{4.59}$$

So that on account of Eqs.(4.28) and (4.30), the total variance of the operators $\hat{\upsilon}_{out}$ and $\hat{\nu}_{out}$ becomes

$$\begin{aligned}(\Delta\upsilon)^2_{out} + (\Delta\nu)^2_{out} &= (\Delta a_{-,out})^2 \\ &= 2 + \frac{(\kappa A^2 - 2\kappa A^2\eta^2 + \kappa A^2\eta + 2\kappa^2 A - 2\kappa^2 A\eta)}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2} \\ &- \frac{[\kappa A^2(\eta^2-\eta^4)^{\frac{1}{2}} + (2\kappa^2 A + \kappa A^2)(1-\eta^2)^{\frac{1}{2}}]\cos\theta}{A^2\eta^2 + 3\kappa A\eta + 2\kappa^2}.\end{aligned} \tag{4.60}$$

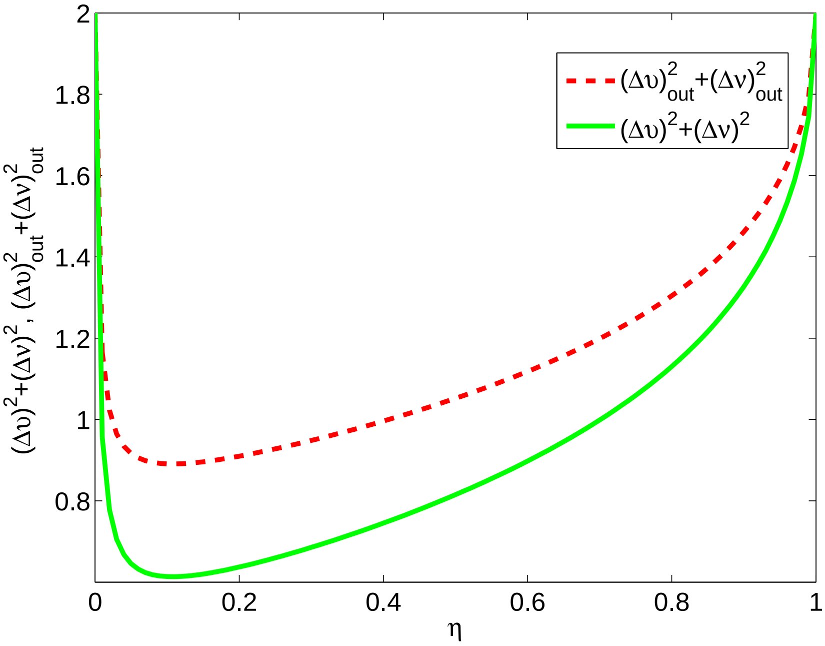

Fig. 4.9: plots of the variance of the sum of EPR-likes operators for both of the light modes in the cavity$[(\Delta\upsilon)^2 + (\Delta\nu)^2]$ and out side the cavity$[(\Delta\upsilon)^2_{out} + (\Delta\nu)^2_{out}]$ at steady state versus η for $\kappa = 0.8$, $\theta = 0$ and for A=300.

Fig.4.9 shows that the entanglement criterion for output modes, Eq.(4.57), is satisfied except η=1 as before. In addition, as can be seen from the same figure, the degree of entanglement of the output modes is less than that of the cavity modes.This is because the degree of squeezing of the output modes is less than that of the cavity modes.

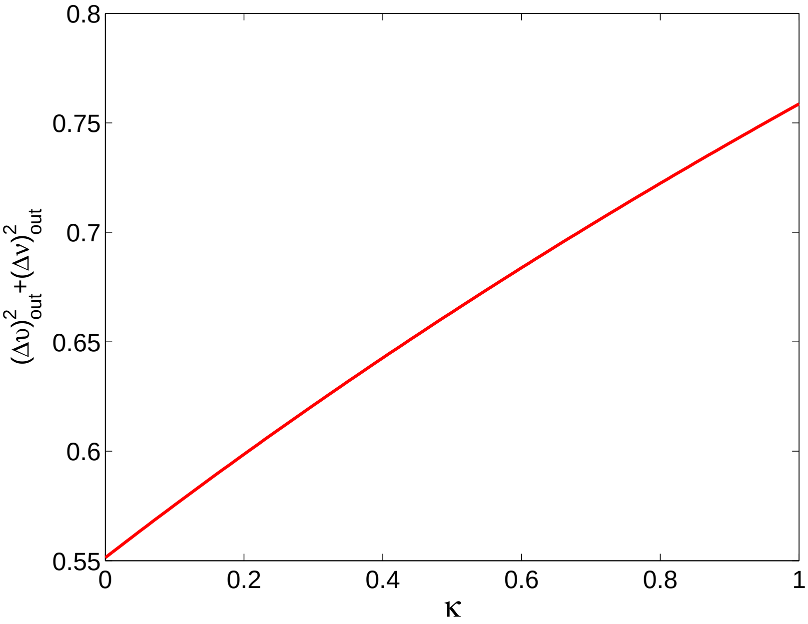

Fig. 4.10: plots of the total variances of EPR-likes operators of light modes outside the cavity$((\Delta \upsilon)^2_{out} + (\Delta \nu)^2_{out})$ at steady state versus κ for η=0.1, $\theta = 0$ and A=100

Fig.4.10 shows that as cavity damping constant κ increases, then the sum of variance of EPR-likes operators of output modes decreases and as a result of this the degree of entanglement of light outside the cavity is increases.

5

CONCLUSION

In this thesis, we have studied the photon statistical, squeezing and entanglement properties of light inside and outside the cavity produced by the three-level cascade laser with the same and different frequencies coupled to vacuum reservoir. We first calculated the master equation of a system under consideration. Using this pertinent master equation, we obtained the steady state solutions of the expectation values of cavity operators. Applying this steady state solutions, we have calculated the mean and variance of photon number, the squeezing and entanglement of cavity and output modes generated by the system. Our results show that the light generated by the system is in squeezing state and the state of the system is entangled, also the squeezing inside and outside the cavity is related with the entanglement. We have seen that the mean and variance of photon number, the degree of squeezing and entanglement of cavity and output modes increases with increase in the values of A, small value of η(if more atoms are initially prepared at bottom level). We have also observed that squeezing and entanglement of the cavity light increases with the value of κ. However, the squeezing and entanglement of output modes decreases. In addition, we also seen that the mean photon number is less than variance, and this indicates that the photon statistics is super-poissonian. Furthermore, it is found that the state of the system is strongly entangled at steady state. We have shown that the degree of entanglement in the cavity and output modes light is directly related to their squeezing. Whenever there is squeezing in light modes, there exists entanglement in the system.

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