Quantum Analysis of Twin Light Beams by Parametric Oscillator

by

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Abstract

A detailed quantum analysis of twin light beams with the same and different frequencies produced by the process of parametric oscillation has been presented. We have described the process of the parametric oscillation with first-order Hamiltonian regardless of weather the twin light beams have the same or different frequencies. Applying this Hamiltonian, we have obtained the pertinent master equation which then have been used to obtain the steady state solutions of the equation of evolution of the expectation values of the cavity operators. Applying resulting steady state solutions, we have studied the photon statistics, the quadrature squeezing, and entanglement for the radiation. Obviously, this time the mean photon number of the signal-signal modes has been found to be exactly the same as that of the signal-idler modes and they also have exhibited entanglement property to each other with the sum of the variances of cavity EPR-like operators exactly equal to the quadrature variance of the cavity mode. However, the fundamental modes(signal- signal or signal-idler) and the reside mode were not squeezed as well as entangled. In addition, we have found that the maximum global quadrature squeezing of the signal-signal or the signal-idler modes is 50% below the vacuum-state level and the maximum local quadrature squeezing of the signal-signal modes is 74.9% below the same level.

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Introduction

Light has played a special role in our attempt to understand nature both classically and quantum mechanically. In classical description light consists of waves with well defined amplitude and phase, but this is not the case when we treat light quantum mechanically. Quantum optics is a field of quantum physics that deals with the interaction of photons with matter and the quantum properties of the light generated by various quantum optical system. The properties of quantized light is largely determine by the well known quantum states of light: the number state, the coherent state, the chaotic state and the squeezed state. Squeezing is one of the interesting non-classical feature of light that has been attracting attention and studied by many authors [1-28]. In squeezed light the noise in one quadrature is below the vacuum or coherent level at the expense of enhancing fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Squeezed light has potential applications in low noise communications and precision measurements [10,11]. Squeezed light can be generated by quantum optical processes such as parametric oscillation [1-8], second harmonic generation [1,7,8] and three-level lasing at certain conditions [13-17].

Entanglement is another interesting non-classical feature of light that has been drawing attention and studied by many authors[22,23]. It is believed that the key ingredient of quantum information is entanglement which has been recognized as the essential resource for quantum teleportation, quantum decoding, quantum computation and and quantum cryptography. Researchers in quantum entanglement were initiated since a 1935 paper[22], by Albert Einstein, Boris Podolsky and Nathan Rosen describing the EPR Paradox and several papers by Erwin Schrodinger [28] shortly. Recently much attention has been paid to the generation and detection of continuous variable entanglement as it might be easier to manipulate than discrete counter parts, quantum bits, in order to perform quantum information processing. On the other hand, the efficiency of the quantum information processing highly depends on the degree of entanglement. Hence, it is desirable to generate strongly entangled continuous variable state.

A parametric oscillator has been considered as an important source of squeezed and entangled light [1-8]. It is one of the most interesting and well characterized optical devices in quantum optics. In this device a pump photon of frequency ω interacts with a nonlinear crystal inside a cavity and is down-converted into two highly correlated photons of frequencies ω_1 and ω_2. If these photons have the same frequency the device is called a degenerate parametric oscillator and the two photons are labeled as signal-signal, otherwise it is called a nondegenerate parametric oscillator and the photons are labeled as signal-idler. The quantum dynamical properties of the twin light beams produced by the degenerate and the non-degenerate parametric oscillators have been studied by several authors [3,5].

We may envision parametric down conversion as a process in which a three-level atom absorbs a pump photon and makes a transition from the bottom to the top level. Then it emits a photon and decays to the intermediate level. Finally, the atom emits another photon and decays to the bottom level. We certainly expect several three-level atoms to be involved in this process. The photons emitted from the top constitute one light mode and the other emitted from the intermediate level form another type. Due to the fact that these two photons shared the same intermediate level and they are said to have coherence which is responsible for the squeezing and entanglement exposition of the twin light beams.Due to the inherent coherence of the two-photon, the parametric oscillator can be taken as a conventional source of a squeezed light.

The squeezing and statistical properties of the degenerate parametric oscillators were studied intensively. The maximum achievable squeezing for the degenerate parametric oscillator coupled to a vacuum reservoir is found to be 50$\%$ below coherent or vacuum state level by many authors following different approaches [1,3] such as G. J. Milburn and D. F. Walls [9]. B. Teklu [6] using stochastic differential equations. However, K. Fesseha [3] showed, using quantum Langevin equations, that if the ordinary vacuum reservoir is replaced by squeezed vacuum reservoir the squeezing could be enhanced exponentially with the squeezed input.

The squeezing, statistical and entanglement properties of a nondegenerate parametric oscillator (NOPO) coupled to ordinary or two uncorrelated squeezed vacuum reservoirs were analyzed by many authors following different procedures [1,3]. The studies showed that the maximum achievable squeezing for the non-degenerate parametric oscillator coupled is to be the same as that of the degenerate parametric oscillator[1,3]. However, the mean photon number is found to be twice that of the degenerate [1,3]. Furthermore, using different criteria [23,24], some authors studied the entanglement of the radiation produced by NOPO. Su-Yong Lee *et al* [29] considered parametric converter as a source of entangled radiation and examined two recently derived criteria for entanglement. They showed that for different initial states, the two criteria give different conditions that ensure that the sates are entangled.

All the above studies assumed the pump mode to be strong and treated classically. Although this consideration appears to be mathematically friendly, it denies the very existence of the pump mode in the cavity, which emerges out of the nonlinear crystal without down converted. Some authors [20,25,26] designated it as residue mode and the down converted pair the signal-signal or signal-idler modes as the fundamental. Contrary to the well established claim that a degenerate parametric oscillator has only showed one mode squeezing, different authors [19,20] showed following different approaches that the two modes in the cavity (the fundamental and residue) exhibit squeezing and entanglement. S. Tesfa [19] analyzed squeezing properties of a two-mode radiation by a process of driven degenerate parametric down conversion, when the cavity is coupled to independent squeezed vacuum reservoirs employing the linearization procedure. He found that the superimposed radiation exhibits a higher degree of two-mode squeezing. He also claimed that the two-mode cavity and output radiations exhibit considerable squeezing even when the oscillator is coupled to a vacuum reservoir. In his other work S. Tesfa [20] also studied the entanglement of the radiation generated in a degenerate down-conversion process when the cavity is coupled to a two-mode vacuum reservoir. He reported that there is a quadrature entanglement between the related fundamental and residual modes.

It is worth noting that the previous works on the quantum analysis of the twin light beams generated by degenerate parametric oscillator were presented in the conventional Hamiltonian by $\hat{a}^2$ and $\hat{a}^{\dagger 2}$. For a given pump mode, we anticipate the mean photon number of the twin beams with the same or different frequencies to be the same. However, studies showed that the mean photon number of the twin light beams with the same frequency obtained using, the conventional Hamiltonian, is found to be half that of the twin light beams with different frequencies. We maintain the stand point that the number operator $\hat{a}^\dagger \hat{a}$ holds only for a light mode represented in the pertinent Hamiltonian by first-order annihilation and creation operators, the unexpected result (different mean photon number) obtained for the twin light beams with the same frequency must then be due to representation of these twin light beams by second-order annihilation. Realizing this fact K. Fesseha [1] describe the process of down conversation by the same first-order Hamiltonian by $\hat{a}_1\hat{a}_2$ and $\hat{a}_1^\dagger\hat{a}_2^\dagger$ and calculated the photon statistical and quadrature squeezing of the signal-signal and signal-idler modes. This time he obtained that the mean photon number and the maximum global quadrature squeezing of the signal-signal modes is exactly the same as that of the signal-idler modes.

Provoked by Fesseha's idea we also keep up the perspective that the down conversion process for the same and different frequencies shall exhibit entanglement and must have the same entanglement quantification. The main task of this thesis is devoted to the quantum analysis of the twin light beams with the same or different frequencies generated by a parametric down converter coupled to a vacuum reservoir via a single-port mirror which is represented by a first-order Hamiltonian.

In chapter two we wish to determine, using the pertinent master equation obtained applying a first-order Hamiltonian, the steady state solutions of the expectation values of the cavity mode operators. In chapter three we calculate, applying the steady state solutions of the expectation values of the cavity mode variables, the mean photon number, the variance of the photon number of the fundamental modes, the mean photon number of the residual mode and the photon number distribution applying the Q-function. In chapter four we seek to determine, using the steady state solutions of the expectation values of the cavity mode variables, the global and local quadrature squeezing for the fundamental modes. In chapter five we analyze, applying the Duan *etal* criteria [23], the entanglement of the fundamental modes, the fundamental-residual modes and the out-put modes.

2

Operator Dynamics

Here we first wish to get the master equation for a cavity coupled to a vacuum reservoir. Then we construct the Hamiltonian for the system under consideration, using this Hamiltonian we find the master equation for the system. Finally, we obtain the operator dynamics.

2.1 Conventional Hamiltonian

Now we seek to calculate, employing the conventional Hamiltonian, the mean photon number of twin light beams with the same frequency produced by a degenerate subharmonic generator. The process of subharmonic generation leading to the creation of the twin light beams is usually described by the Hamiltonian

$$\hat{H} = i\gamma(\hat{b}^{\dagger} - \hat{b}) + \frac{i\varepsilon}{2}(\hat{b}^{\dagger}\hat{a}^{2} - \hat{b}\hat{a}^{\dagger 2}), \tag{2.1}$$

in which $\hat{a}$ is the annihilation operator for the twin light beams, $\hat{b}$ is the annihilation operator for pump mode, ε is the coupling constant, and γ is propertional to the amplitude of the coherent light deriving the pump mode. We may refer to a Hamiltonian of the form described by Eq.(2.1) as second-order Hamiltonian. We consider here the case in which the twin light beams and the pump mode are in a cavity coupled to a vacuum reservoir via a single port-mirror. The equations of evolution of $\langle\hat{a}^{\dagger}\hat{a}\rangle$ and $\langle\hat{a}^{2}\rangle$ can be written as

$$\frac{d}{dt}\langle\hat{a}^{\dagger}\hat{a}\rangle = -i\langle[\hat{a}^{\dagger}\hat{a}.\hat{H}]\rangle + \frac{1}{2}\kappa Tr[(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a})\hat{a}^{\dagger}\hat{a}] \tag{2.2}$$

and

$$\frac{d}{dt}\langle\hat{a}^{2}\rangle = -i\langle[\hat{a}^{2},\hat{H}]\rangle + \frac{1}{2}\kappa Tr[(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a})\hat{a}^{2}], \tag{2.3}$$

in which κ is the cavity damping constant. Now taking into account Eq.(2.1) and the fact that

$$\frac{1}{2}\kappa Tr[(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{\dagger}\hat{a})\hat{a}^{\dagger}\hat{a}] = -\kappa\langle\hat{a}^{\dagger}\hat{a}\rangle \tag{2.4}$$

and

$$\frac{1}{2}\kappa Tr[(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a})\hat{a}^2] = -\kappa\langle\hat{a}^2\rangle, \tag{2.5}$$

we easily get

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - \varepsilon\langle\hat{b}^\dagger\hat{a}^2\rangle - \varepsilon\langle\hat{b}\hat{a}^{\dagger 2}\rangle \tag{2.6}$$

and

$$\frac{d}{dt}\langle\hat{a}^2\rangle = -\kappa\langle\hat{a}^2\rangle - 2\varepsilon\langle\hat{b}\hat{a}^\dagger\hat{a}\rangle - \varepsilon\langle\hat{b}\rangle. \tag{2.7}$$

The steady-state solutions of these equations are

$$\langle\hat{a}^\dagger\hat{a}\rangle = -\frac{\varepsilon}{\kappa}\langle\hat{b}^\dagger\hat{a}^2\rangle - \frac{\varepsilon}{\kappa}\langle\hat{b}\hat{a}^{\dagger 2}\rangle \tag{2.8}$$

and

$$\langle\hat{a}^2\rangle = -\frac{2\varepsilon}{\kappa}\langle\hat{b}\hat{a}^\dagger\hat{a}\rangle - \frac{\varepsilon}{\kappa}\langle\hat{b}\rangle. \tag{2.9}$$

Now substitution of Eq.(2.60) and Eq.(2.66) referred below, into Eqs. (2.8) and (2.9) yields

$$\langle\hat{a}^\dagger\hat{a}\rangle = -\frac{\lambda}{\kappa}\langle\hat{a}^2\rangle - \frac{\lambda}{\kappa}\langle\hat{a}^{\dagger 2}\rangle \tag{2.10}$$

$$\langle\hat{a}^2\rangle = -\frac{2\varepsilon}{\kappa}\langle\hat{a}^\dagger\hat{a}\rangle - \frac{\varepsilon}{\kappa}. \tag{2.11}$$

Finally, on account of Eqs. (2.10) and (2.11), we find the mean photon number of the twin light beams to be

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}. \tag{2.12}$$

2.2 Master Equation

2.2.1 Master Equation for a Cavity Coupled to a Vacuum Reservoir

Now we seek to find the master equation for a cavity mode coupled to a two-mode vacuum reservoir. A system coupled to a reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{2.13}$$

where $\hat{H}_{SR}$ describes the interaction between the system and the reservoir and $\hat{H}_S$ describes some other interaction in a cavity. Suppose $\hat{\chi}(t)$ is the density operator for the system and the reservoir. Then the equation of evolution of the density operator can be written as

$$\frac{d\hat{\chi}(t)}{dt} = -i[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)]. \tag{2.14}$$

We are interested in the quantum dynamics of the system in a cavity alone. And the density operator for the system known as the reduced density operator is given by

$$\hat{\rho}(t) = Tr_R \hat{\chi}(t), \tag{2.15}$$

in which Tr_R indicates trace over the reservoir variables only. Hence taking into account Eq.(2.14), we see that the reduced density operator evolves in time according to

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - iTr_R[\hat{H}_{SR}, \chi(t)]. \tag{2.16}$$

In order to obtain a mathematically manageable equation for evolution of the reduced density operator, we need to adopt some approximation scheme. We thus seek to obtain this equation of evolution applying the approximately valid relation

$$\frac{d\hat{\chi}(t)}{dt} = \frac{\hat{\chi}(t) - \hat{\chi}(t-h)}{h} = -i[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)], \tag{2.17}$$

where h is a small parameter. Now in view of Eqs. (2.14) and (2.17), we see that

$$\hat{\chi}(t) = -ih[\hat{H}_S + \hat{H}_{SR}, \hat{\chi}(t)] + \hat{\chi}(t-h), \tag{2.18}$$

so that on substituting this into Eq.(2.16), there follows

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - iTr_R[\hat{H}_{SR}, \hat{\chi}(t-h)] - hTr_R[\hat{H}_{SR}, [\hat{H}_S, \hat{\chi}(t)]] - hTr_R[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\chi}(t)]] \tag{2.19}$$

We next replace $\hat{\chi}(t)$ by some approximately valid expression. Then, in the first place, we would arrange the reservoir in such a way that its density operator $\hat{R}$ remains constant in time. Moreover, decoupling the system and reservoir density operators, we have

$$\hat{\chi}(t) = \hat{\rho}(t)\hat{R}. \tag{2.20}$$

On introducing this in to Eq.(2.19) and using trace properties, we obtain

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}_S, \hat{\rho}(t)] - i[\langle \hat{H}_{SR} \rangle_R, \hat{\rho}(t-h)] - h[\langle \hat{H}_{SR} \rangle_R, [\hat{H}_S, \hat{\rho}(t)]] - hTr_R[\hat{H}_{SR}, [\hat{H}_{SR}, \hat{\rho}(t)R]], \tag{2.21}$$

where the subscript R indicates that the expectation is calculated using the reservoir density operator $\hat{R}$.

Furthermore, the Hamiltonian describing the interaction of cavity coupled to a reservoir is expressible as

$$\hat{H}_{SR} = i\lambda_1'^2(\hat{a}_1^\dagger \hat{a}_{1in} - \hat{a}_1 \hat{a}_{1in}^\dagger) + \lambda_2'^2(\hat{a}_2^\dagger \hat{a}_{2in} - \hat{a}_2 \hat{a}_{2in}^\dagger). \tag{2.22}$$

In fact, the expectation of Eq.(2.22) found to be

$$\langle \hat{H}_{SR}\rangle = iTr_R\hat{R}[\lambda_1'^2(\hat{a}_1^\dagger\hat{a}_{1in} - \hat{a}_{1in}^\dagger\hat{a}_1) + \lambda_2'^2(\hat{a}_2^\dagger\hat{a}_{2in} - \hat{a}_{2in}^\dagger\hat{a}_2)]. \tag{2.23}$$

Now applying the density operator for a vacuum reservoir, $\hat{R} = |0_1, 0_2\rangle\langle 0_2, 0_1|$, we find

$$\begin{aligned}\langle \hat{H}_{SR}\rangle = iTr_R[&\lambda_1'^2(\hat{a}_1^\dagger\langle 0_2, 0_1|\hat{a}_{1in}|0_1, 0_2\rangle - \langle 0_2, 0_1|\hat{a}_{1in}^\dagger|0_1, 0_2\rangle\hat{a}_1)\\ &+ \lambda_2'^2(\hat{a}_2^\dagger\langle 0_2, 0_1|\hat{a}_{2in}|0_1, 0_2\rangle - \langle 0_2, 0_1|\hat{a}_{2in}^\dagger|0_1, 0_2\rangle\hat{a}_2)].\end{aligned} \tag{2.24}$$

Thus, with the aid of the eigenvalue equation for the number state

$$\hat{a}_1|n_1, n_2\rangle = \sqrt{n_1}|n_1 - 1, n_2\rangle, \tag{2.25}$$

and

$$\hat{a}_2|n_1, n_2\rangle = \sqrt{n_2}|n_1, n_2 - 1\rangle, \tag{2.26}$$

one can find

$$\langle \hat{H}_{SR}\rangle = 0. \tag{2.27}$$

Moreover, in view of Eq.(2.22) and boson commutation relation, we can readily obtain

$$\langle \hat{H}_{SR}^2\rangle = \lambda_1'^2\hat{a}_1^\dagger\hat{a}_1\langle\hat{a}_{1in}^\dagger\hat{a}_{1in}\rangle + \lambda_1'^2\hat{a}_1^\dagger\hat{a}_1 + \lambda_1'^2\langle\hat{a}_{1in}^\dagger\hat{a}_{1in}\rangle\hat{a}_1\hat{a}_1^\dagger + \lambda_2'^2\hat{a}_2^\dagger\hat{a}_2\langle\hat{a}_2^\dagger\hat{a}_2\rangle + \lambda_2'^2\hat{a}_2^\dagger\hat{a}_2 + \lambda_2'^2\langle\hat{a}_2^\dagger\hat{a}_2\rangle\hat{a}_2\hat{a}_2^\dagger], \tag{2.28}$$

in which

$$[\hat{a}_{1in}, \hat{a}_{1in}^\dagger] = [\hat{a}_{2in}, \hat{a}_{2in}^\dagger] = 1. \tag{2.29}$$

Now considering the mean photon number of vacuum reservoir to be zero, one can easily get

$$\langle \hat{H}_{SR}^2\rangle = \lambda_1'^2\hat{a}_1^\dagger\hat{a}_1 + \lambda_2'^2\hat{a}_2^\dagger\hat{a}_2. \tag{2.30}$$

In addition to this, using Eq.(2.22), it can not be hard to prove

$$\begin{aligned}Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = -Tr_R[&\lambda_1'^2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}\hat{R}\hat{a}_{1in} - \lambda_1'^2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1\hat{a}_{1in}\hat{R}\hat{a}_{1in}^\dagger + \lambda_1'\lambda_2'\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho}\hat{a}_{1in}\hat{R}\hat{a}_{2in}\\ &- \lambda_1'\lambda_2'\hat{a}_1^\dagger\hat{\rho}\hat{a}_2\hat{a}_{1in}\hat{R}\hat{a}_{2in}^\dagger - \lambda_1'^2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{1in} + \lambda_1'^2\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{1in}^\dagger\hat{a}_1\hat{\rho}\hat{a}_1\\ &- \lambda_1'\lambda_2'\hat{a}_1^\dagger\hat{R}\hat{a}_{2in}\hat{a}_1\hat{\rho}\hat{a}_2^\dagger + \lambda_1'\lambda_2'\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{2in}\hat{a}_1\hat{\rho}\hat{a}_2^\dagger - \lambda_1'\lambda_2'\hat{a}_2^\dagger\hat{\rho}\hat{a}_1^\dagger\hat{a}_{2in}\hat{R}\hat{a}_{1in}\\ &- \lambda_1'\lambda_2'\hat{a}_2^\dagger\hat{\rho}\hat{a}_1\hat{a}_{2in}\hat{R}\hat{a}_{1in}^\dagger + \lambda_2'^2\hat{a}_2^\dagger\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}\hat{R}\hat{a}_{2in} - \lambda_2'^2\hat{a}_2^\dagger\hat{\rho}\hat{a}_2\hat{a}_{2in}\widehat{R}\hat{a}_{2in}^\dagger - \lambda_1'\lambda_2'\hat{a}_2\hat{\rho}\hat{a}_1^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{1in}\\ &+ \lambda_1'\lambda_2'\hat{a}_2\hat{\rho}\hat{a}_1\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{1in}^\dagger - \lambda_2'^2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in} + \lambda_2'^2\hat{a}_2\hat{\rho}\hat{a}_2\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in}^\dagger].\end{aligned} \tag{2.31}$$

Then it follows

$$\begin{aligned}Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\widehat{H}_{SR}) = Tr_R[&-\lambda_1'^2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1\hat{a}_{1in}\hat{R}\hat{a}_{1in}^\dagger - \lambda_1'^2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_{1in}^\dagger\hat{R}\hat{a}_{1in}\\ &- \lambda_2'^2\hat{a}_2^\dagger\hat{\rho}\hat{a}_2\hat{a}_{2in}\hat{R}\hat{a}_{2in}^\dagger - \lambda_2'^2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_{2in}^\dagger\hat{R}\hat{a}_{2in}].\end{aligned} \tag{2.32}$$

Using Eq.(2.29), we see that

$$\hat{a}_{1in}\hat{a}^{\dagger}_{1in} = \hat{a}^{\dagger}_{1in}\hat{a}_{1in} + 1 \tag{2.33}$$

and taking the mean photon number of vacuum reservoir to be zero, one can easily get

$$Tr_R(\hat{H}_{SR}\hat{\rho}\hat{R}\hat{H}_{SR}) = \lambda_1'^2\hat{a}_1\hat{\rho}\hat{a}_1^{\dagger} + \lambda_2'^2\hat{a}_2\hat{\rho}\hat{a}_2^{\dagger}. \tag{2.34}$$

Therefore, on account of Eq.(2.27), Eq.(2.30), and Eq.(2.34), Eq.(2.21) becomes

$$\frac{d\hat{\rho}}{dt} = [\hat{H}_S, \hat{\rho}] + \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^{\dagger} - \hat{a}_1^{\dagger}\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^{\dagger}\hat{a}_1) + \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^{\dagger} - \hat{a}_2^{\dagger}\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^{\dagger}\hat{a}_2). \tag{2.35}$$

This represents the master equation for a cavity coupled to a vacuum reservoir, and $\kappa_1 = 2h\lambda_1'^2$, is the cavity damping constant for light mode $\hat{a}_1$ as well as $\kappa_2 = 2h\lambda_2'^2$, is the cavity damping constant for light mode $\hat{a}_2$.

2.2.2 Master Equation for the System Under Consideration

The process of parametric oscillation leading to the creation of twin light modes with the same or different frequencies can be described by the Hamiltonian

$$\hat{H} = i\gamma(\hat{b}^{\dagger} - \hat{b}) + i\varepsilon(\hat{b}^{\dagger}\hat{a}_1\hat{a}_2 - \hat{b}\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}) \tag{2.36}$$

where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators for the light modes emitted from the top and intermediate levels of the fundamental mode, $\hat{b}$ is the annihilation operator for the pump mode and part of the pump mode that emerges from non-linear crystal with out being down-converted (residue mode), ε is the coupling constant, and γ is proportional to the amplitude of the coherent light deriving the pump mode. We assume that the operators $\hat{a}_1$ and $\hat{a}_2$ commute and satisfy the commutation relation

$$[\hat{a}_1, \hat{a}_1^{\dagger}] = [\hat{a}_2, \hat{a}_2^{\dagger}] = 1. \tag{2.37}$$

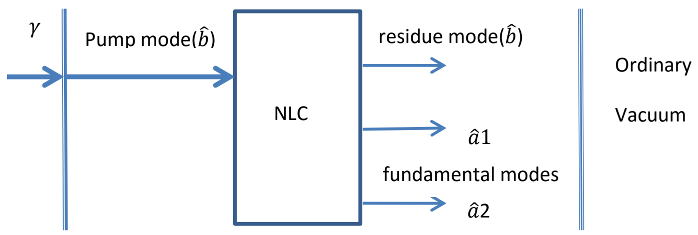

Fig. 2.1: Plot of the external pumping radiation of frequency (ω) is down converted to the fundamental(signal-signal or signal-idler) mode of frequencies (ω_s,ω_i) by the nonlinear crystal.

We may refer to a Hamiltonian of the form described by Eq.(2.36) as first order Hamiltonian. We next seek to calculate, the master equation, operator dynamics and Q function for the twin light modes by applying the pertinent Hamiltonian described by Eq.(2.36).
Substituting the system Hamiltonian given in Eq.(2.36) in to Eq.(2.35), the master equation for the system under consideration turns out to be

$$\frac{d\hat{\rho}}{dt} = \gamma(\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger + \hat{\rho}\hat{b} - \hat{b}\hat{\rho}) + \varepsilon(\hat{b}^\dagger\hat{a}_1\hat{a}_2\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}_1\hat{a}_2 + \hat{\rho}\hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho})$$
$$+ \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1) + \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2). \tag{2.38}$$

2.3 Operator Dynamics

At issue here is to find, the equation of evolution of the expectation values of the cavity mode operators, and their steady state solutions. Using the relation

$$\frac{d\langle\hat{A}\rangle}{dt} = Tr(\frac{d\hat{\rho}}{dt}\hat{A}), \tag{2.39}$$

Using Eq.(2.38), one can get

$$\frac{d\langle\hat{a}_1\rangle}{dt} = Tr[(-i[\hat{H},\hat{\rho}]\hat{a}_1 + \frac{\kappa_1}{2}(2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_1\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1^2) + \frac{\kappa_2}{2}(2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2\hat{a}_1)]. \tag{2.40}$$

On account of the properties of the trace operator, and Eq.(2.37) along with the fact that the operators $\hat{a}_1$, and $\hat{a}_2$ commute, we find

$$\frac{d\langle\hat{a}_1\rangle}{dt} = -i\langle[\hat{a}_1,\hat{H}]\rangle - \frac{\kappa_1}{2}\langle\hat{a}_1\rangle. \tag{2.41}$$

In similar way, one can get

$$\frac{d\langle\hat{a}_2\rangle}{dt} = -i\langle[\hat{a}_2,\hat{H}]\rangle - \frac{\kappa_2}{2}\langle\hat{a}_2\rangle, \tag{2.42}$$

$$\frac{d\langle\hat{a}_1^\dagger\hat{a}_1\rangle}{dt} = -i\langle[\hat{a}_1^\dagger\hat{a}_1,\hat{H}]\rangle - \kappa_1\langle\hat{a}_1^\dagger\hat{a}_1\rangle, \tag{2.43}$$

$$\frac{d\langle\hat{a}_2^\dagger\hat{a}_2\rangle}{dt} = -i\langle[\hat{a}_2^\dagger\hat{a}_2,\hat{H}]\rangle - \kappa_2\langle\hat{a}_2^\dagger\hat{a}_2\rangle, \tag{2.44}$$

.

$$\frac{d\langle\hat{a}_1\hat{a}_2\rangle}{dt} = -i\langle[\hat{a}_1\hat{a}_2,\hat{H}]\rangle - \frac{1}{2}(\kappa_1+\kappa_2)\langle\hat{a}_1\hat{a}_2\rangle, \tag{2.45}$$

$$\frac{d\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle}{dt} = -i\langle[\hat{a}_1^\dagger\hat{a}_2^\dagger,\hat{H}]\rangle - \frac{1}{2}(\kappa_1+\kappa_2)\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle. \tag{2.46}$$

Substituting the Hamiltonian Eq.(2.36) into Eq.(2.41) , we find

$$\frac{d\langle \hat{a}_1\rangle}{dt} = -i\langle[\hat{a}_1, i\gamma(\hat{b}^\dagger - \hat{b}) + i\varepsilon(\hat{b}^\dagger \hat{a}_1\hat{a}_2 - \hat{b}\hat{a}_1^\dagger\hat{a}_2^\dagger)]\rangle - \frac{\kappa_1}{2}\langle\hat{a}_1\rangle.$$

It then follows

$$\frac{d\langle \hat{a}_1\rangle}{dt} = -\frac{\kappa_1}{2}\langle\hat{a}_1\rangle - \varepsilon\langle\hat{b}\hat{a}_2^\dagger\rangle, \tag{2.47}$$

Following a similar procedure, we get

$$\frac{d\langle \hat{a}_2\rangle}{dt} = -\frac{\kappa_2}{2}\langle\hat{a}_2\rangle - \varepsilon\langle\hat{b}\hat{a}_1^\dagger\rangle, \tag{2.48}$$

$$\frac{d\langle \hat{a}_1^\dagger\hat{a}_1\rangle}{dt} = -\varepsilon\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \varepsilon\langle b\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \kappa_1\langle\hat{a}_1^\dagger\hat{a}_1\rangle, \tag{2.49}$$

$$\frac{d\langle \hat{a}_2^\dagger\hat{a}_2\rangle}{dt} = -\varepsilon\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \varepsilon\langle b\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle - \kappa_2\langle\hat{a}_2^\dagger\hat{a}_2\rangle, \tag{2.50}$$

$$\frac{d\langle \hat{a}_1\hat{a}_2\rangle}{dt} = -\varepsilon\langle\hat{b}\widehat{a}_1^\dagger\hat{a}_1\rangle - \varepsilon\langle\hat{b}\hat{a}_2^\dagger\hat{a}_2\rangle - \varepsilon\hat{b} - (\kappa_1+\kappa_2)\langle\hat{a}_1\hat{a}_2\rangle. \tag{2.51}$$

On taking $\kappa_1 = \kappa_2 = \kappa$, the steady-state solutions of the above equations become

$$\langle\hat{a}_1\rangle = -\frac{2\varepsilon}{\kappa}\langle\hat{b}\hat{a}_2^\dagger\rangle, \tag{2.52}$$

$$\langle\hat{a}_2\rangle = -\frac{2\varepsilon}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\rangle, \tag{2.53}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = -\frac{\varepsilon}{\kappa}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \frac{\varepsilon}{\kappa}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{2.54}$$

$$\langle\hat{a}_2^\dagger\hat{a}_2\rangle = -\frac{\varepsilon}{\kappa}\langle\hat{b}^\dagger\hat{a}_1\hat{a}_2\rangle - \frac{\varepsilon}{\kappa}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{2.55}$$

$$\langle\hat{a}_1\hat{a}_2\rangle = -\frac{\varepsilon}{\kappa}\langle\hat{b}\hat{a}_1^\dagger\hat{a}_1\rangle - \frac{\varepsilon}{\kappa}\langle\hat{b}\hat{a}_2^\dagger\hat{a}_2\rangle - \frac{\varepsilon}{\kappa}\langle\hat{b}\rangle. \tag{2.56}$$

On the other hand, the evolution of the expectation value of the pump mode $\hat{b}$ is given by

$$\frac{d\langle\hat{b}\rangle}{dt} = -iTr([\hat{H},\hat{\rho}]\hat{b}) + \frac{\kappa}{2}Tr\{(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b})\hat{b}\}, \tag{2.57}$$

in the absance of parametric oscillation ($\varepsilon = 0$), the Eq.(2.36) becomes

$$\hat{H} = i\gamma(\hat{b}^\dagger - \hat{b}),$$

Substituting into Eq.(2.57), one can get

$$\frac{d\langle\hat{b}\rangle}{dt} = -\frac{\kappa}{2}\langle\hat{b}\rangle + \gamma. \tag{2.58}$$

Upon dropping the noise operator, we can write the quantum langevin equation as

$$\frac{d\hat{b}}{dt} = -\frac{1}{2}\kappa\hat{b} + \gamma, \tag{2.59}$$

where κ is the cavity damping constant. The steady-state solution of this equation is

$$\hat{b} = \frac{2\gamma}{\kappa}. \tag{2.60}$$

Now substituting the value of $\hat{b}$ in Eqs.(2.52)-(2.56),we get

$$\langle\hat{a}_1\rangle = -\frac{2\varepsilon}{\kappa}\langle\frac{2\gamma}{\kappa}\hat{a}_2^\dagger\rangle = -\frac{4\varepsilon\gamma}{\kappa^2}\langle\hat{a}_2^\dagger\rangle = -\frac{2\lambda}{\kappa}\langle\hat{a}_2^\dagger\rangle, \tag{2.61}$$

$$\langle\hat{a}_2\rangle = -\frac{2\varepsilon}{\kappa}\langle\frac{2\gamma}{\kappa}\hat{a}_1^\dagger\rangle = -\frac{4\varepsilon\gamma}{\kappa^2}\langle\hat{a}_1^\dagger\rangle = -\frac{2\lambda}{\kappa}\langle\hat{a}_1^\dagger\rangle, \tag{2.62}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = -\frac{\lambda}{\kappa}\langle\hat{a}_1\hat{a}_2\rangle - \frac{\lambda}{\kappa}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{2.63}$$

$$\langle\hat{a}_2^\dagger\hat{a}_2\rangle = -\frac{\lambda}{\kappa}\langle\hat{a}_1\hat{a}_2\rangle - \frac{\lambda}{\kappa}\langle\hat{a}_1^\dagger\hat{a}_2^\dagger\rangle, \tag{2.64}$$

$$\langle\hat{a}_1\hat{a}_2\rangle = -\frac{\lambda}{\kappa}\langle\hat{a}_1^\dagger\hat{a}_1\rangle - \frac{\lambda}{\kappa}\langle\hat{a}_2^\dagger\hat{a}_2\rangle - \frac{\lambda}{\kappa}, \tag{2.65}$$

in which λ is defined by

$$\lambda = \frac{2\gamma\varepsilon}{\kappa}. \tag{2.66}$$

Applying Eq. (2.61) and Eq.(2.62), we easily find

$$\langle\hat{a}_1\rangle = \langle\hat{a}_2\rangle = 0. \tag{2.67}$$

Moreover, using Eqs. (2.63),(2.64),and (2.65), we get

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}, \tag{2.68}$$

$$\langle\hat{a}_1^\dagger\hat{a}_1\rangle = \langle\hat{a}_2^\dagger\hat{a}_2\rangle, \tag{2.69}$$

$$\langle\hat{a}_1\hat{a}_2\rangle = -\frac{\lambda\kappa}{\kappa^2 - 4\lambda^2}. \tag{2.70}$$

It can also be readily verified that

$$\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = \langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0. \tag{2.71}$$

We take

$$\hat{a} = \hat{a}_1 + \hat{a}_2. \tag{2.72}$$

to be the annihilation operator for superposition of light modes $\hat{a}_1$ and $\hat{a}_2$, produced by the parametric oscillator. One can easily check that

$$[\hat{a}, \hat{a}^\dagger] = 2. \tag{2.73}$$

We actualize that the superposition of the two light modes, with the same or different frequencies, constitutes a two-mode light. We wish to call superposed light modes with the same frequency the signal-signal modes and the superposed light modes with different frequencies the signal-idler modes. It also proves to be convenient to the parametric oscillator which produces the signal-signal modes as the degenerate parametric oscillator and the one which produces the signal-idler modes as the non-degenerate parametric oscillator. Finally, we would like to mention that the result described by Eqs. (2.66)-(2.70) are valid for the signal-signal or signal-idler modes.

3

Photon Statistics

In this chapter we wish to calculate the mean photon number, the variance of the photon number and the photon number distribution for the signal-signal modes as well as the signal-idler modes.

3.1 The Q Function

At issue here is to determine the Q function for the two-mode subharmonic light (the signal-signal or the signal-idler modes).The Q function is expressible [1] as

$$Q(\alpha_1, \alpha_2, t) = \frac{1}{\pi^4} \int d^2z d^2\eta \phi_a(z, \eta, t) exp(z^* \alpha_1 - z\alpha_1^* + \eta^* \alpha_2 - \eta\alpha_2^*), \tag{3.1}$$

in which

$$\phi_a(z, \eta, t) = Tr(\rho(0) e^{-z^* \hat{a}_1(t)} e^{z\hat{a}_1^\dagger(t)} e^{-\eta^* \hat{a}_2(t)} e^{\eta \hat{a}_2^\dagger(t))}. \tag{3.2}$$

Employing the antinormally-ordered characteristic function $\phi_a(z, \eta, t)$ is defined in the Heisenberg picture is identity

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]}, \tag{3.3}$$

and taking into account the fact that $\hat{a}_1$ and $\hat{a}_2$ are Gaussian variables with zero mean, Eq.(3.2) one can be put in the form

$$\phi_a(z, \eta, t) = exp[-\frac{1}{2}(zz^* + \eta\eta^*)] exp[\frac{1}{2}\langle(z\hat{a}_1^\dagger - z^* \hat{a}_1 + \eta\hat{a}_2^\dagger - \eta^* \hat{a}_2)^2\rangle]. \tag{3.4}$$

It becomes

$$\begin{aligned}
\phi_a(z, \eta, t) = exp[-\frac{1}{2}(zz^* + \eta\eta^*)] exp[-\frac{1}{2} zz^* \left(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle\right) + \frac{1}{2} z^{*2} \langle \hat{a}_1^2 \rangle + \frac{1}{2} z^2 \langle \hat{a}_1^{\dagger 2} \rangle \\
- \frac{1}{2}\eta\eta^* \left(\langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \langle \hat{a}_2 \hat{a}_2^\dagger \rangle\right) + \frac{1}{2}\eta^{*2} \langle \hat{a}_2^2 \rangle + \frac{1}{2}\eta^2 \langle \hat{a}_2^\dagger \rangle + z^* \eta^* \langle \hat{a}_1 \hat{a}_2 \rangle + z\eta \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \\
- z\eta^* \langle \hat{a}_1^\dagger \hat{a}_2 \rangle - z^* \eta \langle \hat{a}_1 \hat{a}_2^\dagger \rangle].
\end{aligned} \tag{3.5}$$

Now on account of Eqs.(2.67), (2.68), (2.69), (2.70) and (2.71), Eq.(3.5)

$$\phi_a(z,\eta,t) = exp[-a(zz^* + \eta\eta^*) - b(z^*\eta^* + z\eta)], \tag{3.6}$$

where

$$a = \frac{\kappa^2 - 2\lambda^2}{\kappa^2 - 4\lambda^2} \tag{3.7}$$

and

$$b = \frac{\kappa\lambda}{\kappa^2 - 4\lambda^2}. \tag{3.8}$$

Finally, upon substituting Eq.(3.6) into Eq.(3.1), the Q function for the two-mode subharmonic light is

$$Q(\alpha_1,\alpha_2,t) = \frac{1}{\pi^4}\int d^2zd^2\eta exp[-a(zz^* + \eta\eta^*) - b(z^*\eta^* + z\eta) + z^*\alpha_1 - z\alpha_1^* + \eta^*\alpha_2 - \eta\alpha_2^*].$$

Carrying out integration over η with the help of

$$\int \frac{d^2z}{\pi} exp(-azz^* + bz + cz^* + Az^2 + Bz^{*2}) = [\frac{1}{a^2 - 4AB}]^{\frac{1}{2}} exp[\frac{abc + Ac^2 + Bb^2}{a^2 - 4AB}]. \tag{3.9}$$

We get

$$\begin{aligned} Q(\alpha_1,\alpha_2,t) &= \frac{1}{\pi^3}\int d^2z exp[-azz^* + z^*\alpha_1 - z\alpha_1^*]\int \frac{d^2\eta}{\pi} exp[-a\eta\eta^* - bz^*\eta^* - bz\eta + \eta^*\alpha_2 - \eta\alpha_2^*] \\ &= \frac{1}{a\pi^2}\int \frac{d^2z}{\pi} exp[-azz^* + \frac{b^2zz^*}{a} + z^* + \frac{bz^*\alpha_2^*}{a} - z\alpha_1^* - \frac{bz\alpha_2}{a} - \frac{\alpha_2\alpha_2^*}{a}] \end{aligned} \tag{3.10}$$

and on performing the integration over z, there follows

$$\begin{aligned} Q(\alpha_1,\alpha_2,t) &= \frac{1}{\pi^2(a^2 - b^2)} exp[-\frac{a}{a^2 - b^2}(\alpha_1\alpha_1^* + \alpha_2\alpha_2^*) - \frac{b}{a^2 - b^2}(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)] \\ &= \frac{1}{\pi^2}[u^2 - v^2] exp[-u(\alpha_1\alpha_1^* + \alpha_2\alpha_2^*) - v(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)], \end{aligned} \tag{3.11}$$

where

$$u = \frac{a}{a^2 - b^2}, \tag{3.12}$$

$$v = \frac{b}{a^2 - b^2}. \tag{3.13}$$

3.2 The Mean and Variance of the Photon Number

We define the photon number of the two-mode subharmonic light by $\overline{n} = \langle\hat{a}^\dagger\hat{a}\rangle$. Then using Eq.(2.72) and taking into account Eq.(2.71), we easily find

$$\overline{n} = \langle\hat{a}_1^\dagger\hat{a}_1\rangle + \langle\hat{a}_2^\dagger\hat{a}_2\rangle, \tag{3.14}$$

so considering Eqs.(2.68) and (2.69) there follows

$$\overline{n} = \frac{4\lambda^2}{\kappa^2 - 4\lambda^2}. \tag{3.15}$$

This represents the mean photon number of the signal-signal or signal-idler modes. We observe that the mean photon number for the conventional Hamiltonian is half of the result given by Eq.(3.15). This unexpected result must be due to representation of the twin light beams with the same frequency by second- order annihilation and creation operators in the conventional Hamiltonian.

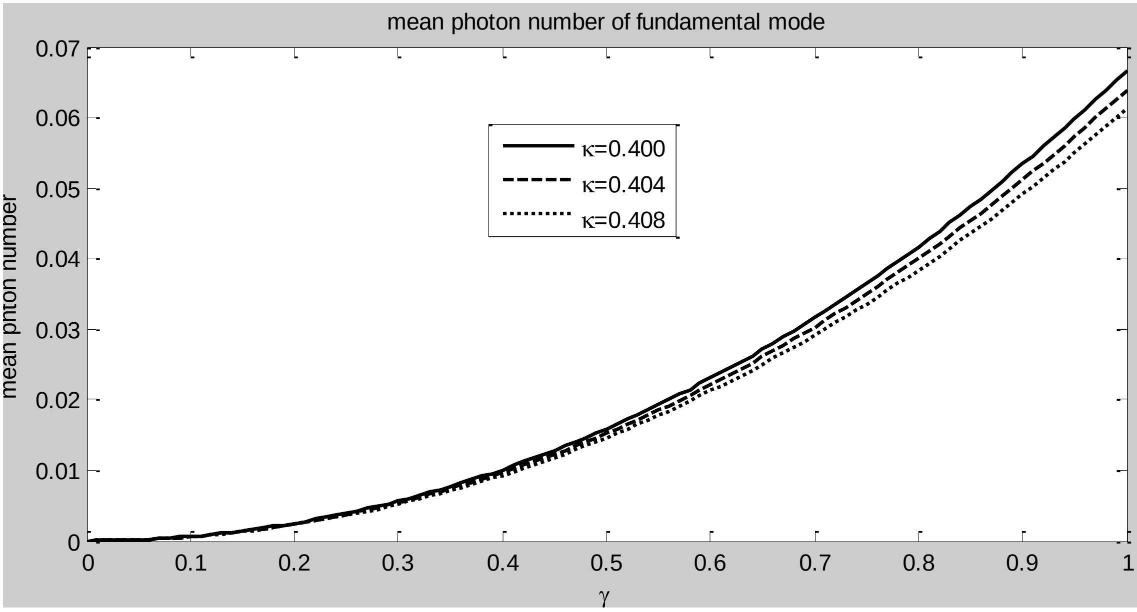

Fig. 3.1: Plots of the the mean photon number [Eq.(3.15)] versus γ for $\varepsilon = 0.01$, and different values of κ ,for $\kappa = 0.400$(solid curve), $\kappa = 0.404$ (dashed curve) and $\kappa = 0.408$ (doted curve).

It can readily be observed that the mean photon number of the cavity mode decrease with the cavity damping constant increase. Since more photons escape from the cavity for large damping constant. Now appling Eq.(2.73), the photon-number variance of the two-mode subharmonic light is defined by

$$(\triangle n)^2 = \langle(\hat{a}^\dagger \hat{a})^2\rangle - \overline{n}^2, \tag{3.16}$$

can be put in form

$$(\triangle n)^2 = \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + 2\overline{n} - \overline{n}^2. \tag{3.17}$$

Now applying the fact that $\hat{a}$ is a Gaussian variable with zero mean, we get

$$(\triangle n)^2 = 2\overline{n} + \overline{n}^2 + \langle \hat{a}^{\dagger 2} \rangle \langle \hat{a}^2 \rangle \tag{3.18}$$

and on taking into account Eq.(2.73) along with Eq.(2.72), we arrive at

$$(\triangle n)^2 = 2\overline{n} + \overline{n}^2 + 4\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \langle \hat{a}_1 \hat{a}_2 \rangle. \tag{3.19}$$

Hence in view of Eq.(3.15) along with Eq.(2.70), the photon-number variance of the two-mode parametric light takes the form

$$(\triangle n)^2 = \frac{8\lambda^2}{\kappa^2 - 4\lambda^2} + \frac{16\lambda^4}{(\kappa^2 - 4\lambda^2)^2} + \frac{4\kappa^2\lambda^2}{(\kappa^2 - 4\lambda^2)^2}. \tag{3.20}$$

In addition, we note that the equation of evolution of the mean photon number for the pump mode can be written as

$$\frac{d}{dt}\langle \hat{b}^\dagger \hat{b} \rangle = -i\langle [\hat{b}^\dagger \hat{b}, \hat{H}] \rangle + \frac{1}{2}\kappa Tr[(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger \hat{b}\hat{\rho} - \rho\hat{b}^\dagger \hat{b})\hat{b}^\dagger \hat{b}]. \tag{3.21}$$

Then using Eq.(2.36) and the fact that

$$\frac{1}{2}\kappa Tr[(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger \hat{b}\hat{\rho} - \rho\hat{b}^\dagger \hat{b})\hat{b}^\dagger \hat{b}] = -\kappa\langle \hat{b}^\dagger \hat{b} \rangle, \tag{3.22}$$

we readily get

$$\frac{d}{dt}\langle \hat{b}^\dagger \hat{b} \rangle = -\kappa\langle \hat{b}^\dagger \hat{b} \rangle + \gamma\langle \hat{b} \rangle + \gamma\langle \hat{b}^\dagger \rangle + \varepsilon\langle \hat{b}^\dagger \hat{a}_1 \hat{a}_2 \rangle + \varepsilon\langle \hat{b}\hat{a}_1^\dagger \hat{a}_2^\dagger \rangle. \tag{3.23}$$

The steady-state solution of this equation is

$$\langle \hat{b}^\dagger \hat{b} \rangle = \frac{\gamma}{\kappa}\langle (\hat{b} + \hat{b}^\dagger) \rangle + \frac{\varepsilon}{\kappa}(\langle \hat{b}^\dagger \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{b}\hat{a}_1^\dagger \hat{a}_2^\dagger \rangle), \tag{3.24}$$

so that in view of Eq.(2.60) and Eq.(2.69), the mean photon number of the pump mode takes the form

$$\langle \hat{b}^\dagger \hat{b} \rangle = \frac{4\gamma^2}{\kappa^2} - \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}. \tag{3.25}$$

The first term of Eq.(3.25) represents the mean photon number of the pump mode in the absence of the parametric oscillator and the second one represents the mean photon number of light mode a_1 or light mode a_2 or the mean photon number of the down converted pump mode. The difference of the two terms is the mean photon number of the pump mode that emerges from the nonlinear crystal without down-converted(residue mode). This is exactly what we would expect the mean photon number of the pump mode to be.

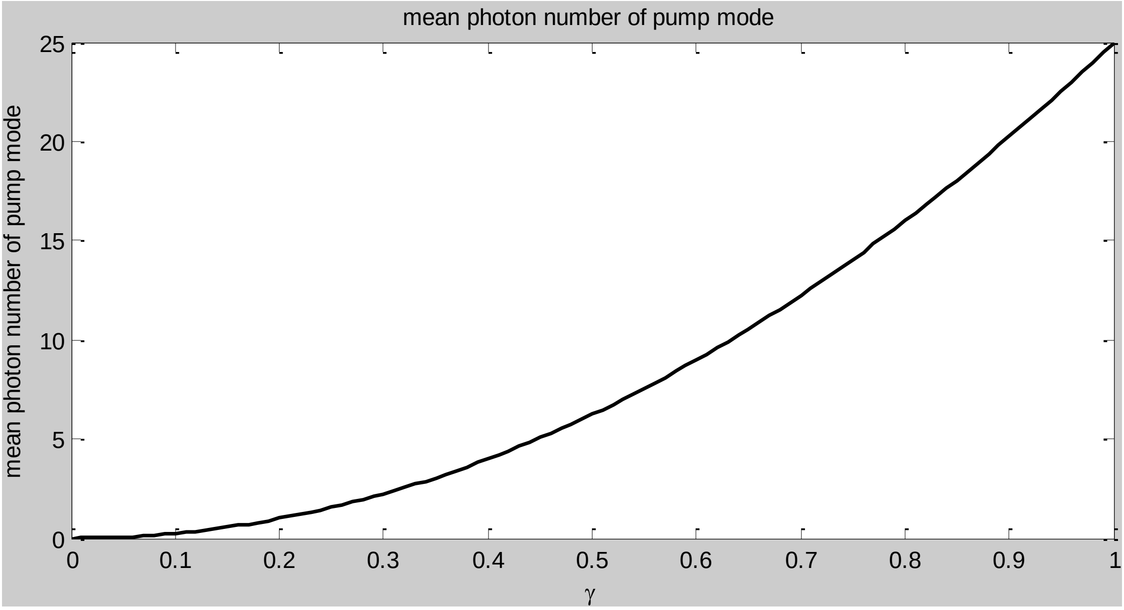

Fig. 3.2: Plots of the the mean photon number of residue pump mode [Eq.(3.24)] versus γ for $\varepsilon = 0.01$, and $\kappa = 0.4$.

Fig.(3.2) shows the mean photon number of residue pump mode increase with the amplitude of pump mode.

Setting Eq.(3.25) to zero, one can find $\gamma = \frac{\kappa}{4\varepsilon}\sqrt{(\kappa^2 - 2\varepsilon^2)}$, at this point the incident photons were fully down-converted to fundamental (signal-signal or signal-idler) mode.

3.3 The Photon Number Distribution

We finally proceed to calculate the photon number distribution for the signal-signal or signal-idler modes. The probability for observing $x + y$ signal photons or the probability for observing x signal photons and y idler photons is expressible in terms of the Q function in [3.11], as cited in [1]

$$P(x,y) = \frac{\pi^2}{x!y!}\frac{\partial^{2x}}{\partial\alpha_1^x\partial\alpha_1^{x*}}\frac{\partial^{2y}}{\partial\alpha_2^y\alpha_2^{y*}}[Q(\alpha_1,\alpha_2)e^{\alpha_1\alpha_1^*+\alpha_2\alpha_2^*}]|_{\alpha_1=\alpha_1^*=\alpha_2=\alpha_2^*=0}. \tag{3.26}$$

$$P(x,y) = \frac{\pi^2}{x!y!}\frac{\partial^{2x}}{\partial\alpha_1^x\partial\alpha_1^{x*}}\frac{\partial^{2y}}{\partial\alpha_2^y\alpha_2^{y*}}[\frac{1}{\pi^2}[u^2 - v^2]exp[-u(\alpha_1\alpha_1^* + \alpha_2\alpha_2^*) - v(\alpha_1\alpha_2 + \alpha_1^*\alpha_2^*)]$$
$$\times e^{\alpha_1\alpha_1^*+\alpha_2\alpha_2^*}]|_{\alpha_1=\alpha_1^*=\alpha_2=\alpha_2^*=0}$$

$$P(x,y) = \frac{[u^2 - v^2]}{x!y!}\frac{\partial^{2x}}{\partial\alpha_1^x\partial\alpha_1^{x*}}\frac{\partial^{2y}}{\partial\alpha_2^y\alpha_2^{y*}} \times [e^{(1-u)(\alpha_1\alpha_1^*}e^{(1-u)(\alpha_2\alpha_2^*)}$$
$$\times e^{-v(\alpha_1\alpha_2)}e^{-v(\alpha_1^*\alpha_2^*)}]|_{\alpha_1=\alpha_1^*=\alpha_2=\alpha_2^*=0}. \tag{3.27}$$

Moreover, using the identity

$$\frac{\partial^m}{\partial q^m} q^n = \frac{n!}{(n-m)!} q^{n-m}. \tag{3.28}$$

$$e^x = \sum_n \frac{x^n}{n!}, \tag{3.29}$$

one finds

$$P(x,y) = \frac{[u^2 - v^2]}{x!y!} \sum_{ijk} \frac{(-1)^{k+l}(1-u)^{i+j} v^{k+l}(i+l)!(i+k)!(j+k)!(j+l)!}{(i+l-x)!(i+k-x)!(j+k-y)!(j+l-y)!}$$

$$\times \alpha_1^{i+k-x} \alpha_1^{*(i+l-x)} \alpha_2^{j+k-y} \alpha_2^{*(j+l-y)}|_{\alpha_1=\alpha_1^*=\alpha_2=\alpha_2^*=0}$$

Now by applying the condition $\alpha_1 = \alpha_1^* = \alpha_2 = \alpha_2^* = 0$,we get

$$P(x,y) = \frac{[u^2 - v^2]}{x!y!} \sum_{ijk} \frac{(-1)^{k+l}(1-u)^{i+j} v^{k+l}(i+l)!(i+k)!(j+k)!(j+l)!}{(i+l-x)!(i+k-x)!(j+k-y)!(j+l-y)!}$$

$$\times i + l, x\delta i + k, x\delta j + k, y\delta j + l, y. \tag{3.30}$$

We note that $k = l = x - i = y - j'i = j$.Therefore, for $x = y$

$$P(y,y) = [u^2 - v^2] \sum_{i=0}^{y} \frac{y!^2 (1-u)^{2i} v^{2(y-i)}}{i!^2 [(y-i)!]^2} \tag{3.31}$$

where u, and v are referred in Eq.(3.12) and Eq.(3.13) respectively. We observe that Eq.(3.31) represents the probability to observe $2y$ signal photons or y signal photons and y idler photons.

4

Quadrature Squeezing

In this chapter we need to calculate the global and local quadrature squeezing for the two-mode subharmonic light.

4.1 Global Quadrature Squeezing

4.1.1 Quadrature Variances

Here we wish to determine the global quadrature squeezing (in the entire frequency interval) for the two-mode subharmonic light. We remember that the variance of the plus and minus quadrature for a two-mode light is given by

$$(\triangle a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle - \langle \hat{a}_{\pm} \rangle^2, \tag{4.1}$$

where

$$\hat{a}_{+} = \hat{a}^{\dagger} + \hat{a}, \tag{4.2}$$

$$\hat{a}_{-} = i(\hat{a}^{\dagger} - \hat{a}), \tag{4.3}$$

with $\hat{a}$ beng the annihilation operator for the two-mode light. Now on account of Eq.(2.67) along with Eqs.(2.71), (4.2), and, (4.3) we see that

$$\langle \hat{a}_{\pm} \rangle = 0. \tag{4.4}$$

Hence in view of this, equation (4.1) is expressible as

$$(\triangle a_{\pm})^2 = \langle \hat{a}_{\pm}^2 \rangle. \tag{4.5}$$

We note that

$$(\triangle a_{\pm})^2 = \langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a} \hat{a}^{\dagger} \rangle \pm \langle \hat{a}^2 + \widehat{a}^{\dagger 2} \rangle, \tag{4.6}$$

and applying Eq.(2.73) the quadrature variance can be put in the form

$$(\triangle a_{\pm})^2 = 2 + 2\langle \hat{a}^{\dagger}\hat{a}\rangle \pm \langle \hat{a}^2 + \hat{a}^{\dagger 2}\rangle. \tag{4.7}$$

With the aid of Eq.(2.72) we can rewrite equation Eq.(4.7) as

$$(\triangle a_{\pm})^2 = 2 + 2\langle(\hat{a}_1^{\dagger} + \hat{a}_2^{\dagger})(\hat{a}_1 + \hat{a}_2)\rangle \pm \langle \hat{a}_1^2 + \hat{a}_1^{\dagger 2} + \hat{a}_2^2 + \hat{a}_2^{\dagger 2} + 2\hat{a}_1\hat{a}_2 + 2\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\rangle \tag{4.8}$$

and taking Eq.(2.71), we get

$$(\triangle a_{\pm})^2 = 2 + 2\langle(\hat{a}_1^{\dagger}\hat{a}_1 + \hat{a}_2^{\dagger}\hat{a}_2)\rangle \pm 2\langle(\hat{a}_1\hat{a}_2 + \hat{a}_1^{\dagger}\hat{a}_2^{\dagger})\rangle. \tag{4.9}$$

Therefore considering Eqs.(2.67), (2.68), (2.69), and (2.70) the quadrature variance takes the form

$$(\triangle a_{+})^2 = 2 - \frac{4\lambda}{\kappa + 2\lambda}, \tag{4.10}$$

$$(\triangle a_{-})^2 = 2 + \frac{4\lambda}{\kappa - 2\lambda}. \tag{4.11}$$

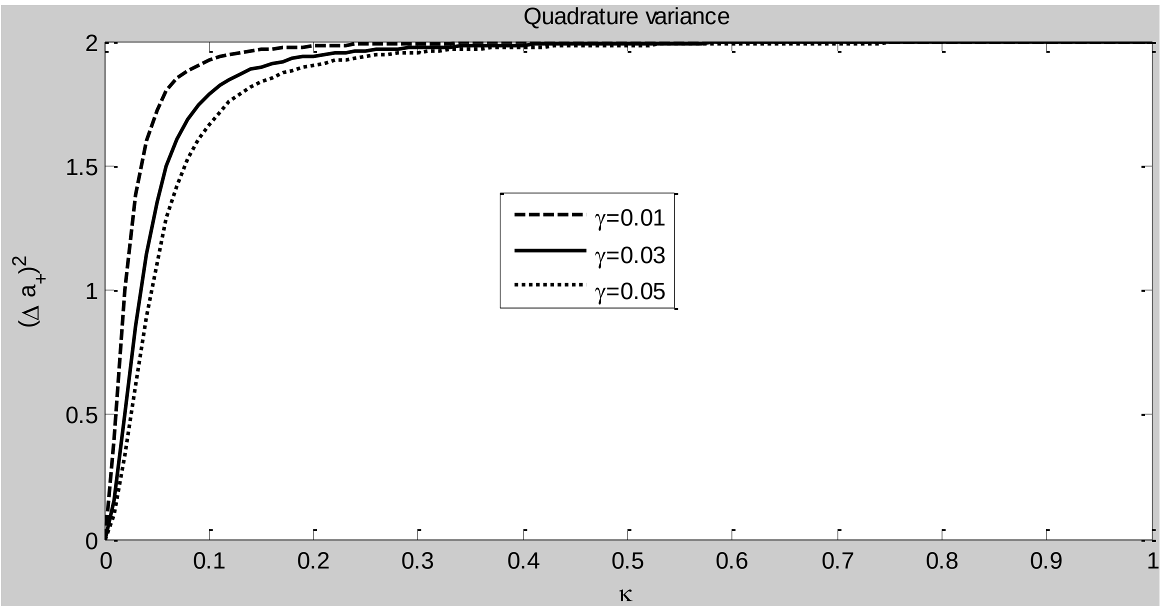

Fig. 4.1: Plots of the quadrature variance [Eq.(4.10)] versus κ, for, $\varepsilon = 0.01$, and different values of γ, for $\gamma = 0.01$ (dashed curve), $\gamma = 0.03$ (solid curve), and $\gamma = 0.05$ (doted curve).

We immediately observe from Eq.(4.10) that the quadrature variance of the signal-signal or signal-idler modes less than the quadrature variance of the two-mode vacuum reservoir. The plots in Fig.(4.1) indicate that the quadrature variance decrease with the amplitude of the pump mode and it approaches the values of the quadrature variance of a two-mode vacuum state as κ approaches to 1. At threshold, one readily gets $(\triangle a_+)^2 = 1$ and $(\triangle a_-)^2 = \infty$. Upon setting $\lambda = 0$, in Eq. (4.10) and Eq.(4.11), we find

$$(\triangle a_\pm)_v^2 = 2. \tag{4.12}$$

We thus see that for $\lambda = 0$, the cavity light is in a two-mode vacuum state in which the uncertainties in the two quadrature are equal and satisfy the minimum uncertainty relation.

4.1.2 Global Quadrature Squeezing of the Cavity Mode

We need to determine the quadrature squeezing of the two-mode cavity subharmonic light relative to the quadrature variance of the two-mode cavity vacuum states. We therefore define the quadrature squeezing of the two-mode cavity subharmonic light by

$$S = \frac{2 - (\triangle a_+)^2}{2}. \tag{4.13}$$

Now combination of Eq.(4.10) and Eq.(4.13) yields

$$S = \frac{2\lambda}{\kappa + 2\lambda}. \tag{4.14}$$

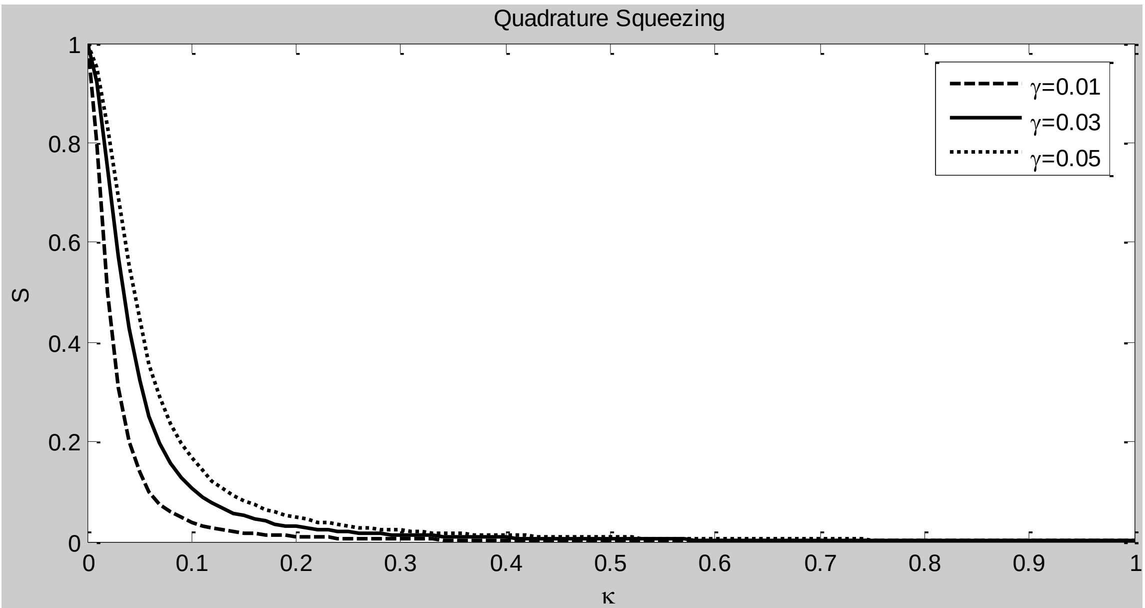

Fig. 4.2: Plots of the quadrature squeezing [Eq.(4.14)] versus κ for $\varepsilon = 0.01$, and different values of γ, for $\gamma = 0.01$ (dashed curve), $\gamma = 0.03$ (solid curve) and $\gamma = 0.05$ (doted curve).

Eq.(4.14) shows that the two-mode cavity(signal-signal or signal-idler modes) are squeezed and the squeezing occurs in the plus quadrature. We also note that at steady state and at the threshold there is a 50% quadrature squeezing below the vacuum state level. The plots in Fig.(4.2) clearly indicate that global quadrature squeezing increase with the amplitude of pump mode and the squeezing approaches to zero as κ approaches one.

4.1.3 Global Quadrature Squeezing of the Out-Put Mode

We next attempt to get the quadrature squeezing of two-mode output subharmonic light. At this end, we need that the quadrature variance of the two-mode output subharmonic light must be the sum of the quadrature variance of the transmitted two-mode cavity subharmonic light and the quadrature variance of the two-mode reflected input modes. On can then write

$$(\triangle a_{\pm out})^2 = \kappa(\triangle a_{\pm})^2 + (1-\kappa)(\triangle a_{\pm in})^2, \tag{4.15}$$

where, $\hat{a}_{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}(t)$, in [1] so on the account of Eq.(4.10) and Eq.(4.11) and the fact that

$$(\triangle a_{\pm in})^2 = 2, \tag{4.16}$$

we easily find

$$(\triangle a_{+}^{out})^2 = 2 - \frac{4\kappa\lambda}{\kappa + 2\lambda}, \tag{4.17}$$

and

$$(\triangle a_{-}^{out})^2 = 2 + \frac{4\kappa\lambda}{\kappa - 2\lambda}. \tag{4.18}$$

Moreover, the quadrature variance of the two-mode output vacuum state can be written as

$$(\triangle a_{\pm out})_v^2 = \kappa(\triangle a_{\pm})_v^2 + (1-\kappa)(\triangle a_{\pm in})^2. \tag{4.19}$$

Hence in view of Eq.(4.12) and Eq.(4.16), there follows

$$(\triangle a_{\pm out})_v^2 = 2. \tag{4.20}$$

Now we define the quadrature squeezing of the two-mode output subharmonic light by

$$S_{out} = \frac{2 - (\triangle a_{+out})^2}{2}. \tag{4.21}$$

Then with the aid of Eq.(4.17), we obtain

$$S_{out} = \frac{2\kappa\lambda}{\kappa + 2\lambda}. \tag{4.22}$$

Finally, at the threshold ($\kappa = 2\lambda$), it becomes

$$S_{out} = \frac{\kappa}{2}. \tag{4.23}$$

From Eq.(4.22), $\kappa = 0$ means no light is transmitted, the port mirror is 100% reflective and $\kappa = 1$, means no light is transmitted from the cavity, there is no port mirror.Thus the value of κ should be $0 \leq \kappa \leq 1$, implies, the quadrature squeezing of the two-mode output subharmonic light is always less than that of the two-mode cavity subharmonic light.

4.1.4 Global Quadrature Squeezing of the Fundamental-Residue Mode

Some authors [19, 20] have studied the quantum dynamics of parametric oscillators using different methods. They referred the signal mode as fundamental mode and the pump mode that emerges from the non-linear crystal without being down converted as the residue mode. They have reported that the superposition of the fundamental mode and the residue mode are in squeezed state. We next seek to see whether, in our case, the superposition of the fundamental and the residue mode is squeezed or not. The superposition of the fundamental and residue pump mode can be defined using the operator

$$\hat{c} = \hat{a}_1 + \hat{b}$$

where, $\hat{c}_+ = \hat{c} + \hat{c}^\dagger$, $\hat{c}_- = \widehat{c}^\dagger - \hat{c}$, and $[\hat{c}, \hat{c}^\dagger] = 2$.
Their quadrature variance becomes

$$(\triangle c_\pm)^2 = \langle \widehat{c}_\pm^2 \rangle - \langle \widehat{c}_\pm \rangle^2$$

but

$$\langle \hat{c}_\pm \rangle^2 = 0,$$

which implies

$$\begin{aligned}
(\triangle c_\pm)^2 &= \langle \hat{c}_\pm^2 \rangle \\
&= \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}\hat{c}^\dagger \rangle \pm \langle (\hat{c}^2 + \hat{c}^{\dagger 2}) \rangle \\
&= 2 + 2\langle \hat{c}^\dagger \hat{c} \rangle \\
&= 2 + 2\langle (\hat{a}_1^\dagger \hat{a}_1 + \hat{b}^\dagger \hat{b}) \rangle \\
&= 2 + 2[\frac{2\lambda^2}{\kappa^2 - 4\lambda^2} + \frac{4\gamma^2}{\kappa^2} - \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}] \\
&= 2 + \frac{8\gamma^2}{\kappa^2}.
\end{aligned} \tag{4.24}$$

From Eq.(4.24) we can see that the quadrature variance of the superposition of the fundamental-residue mode is always greater than 2 for all values of γ, and κ. This shows that the fundamental-residue mode is not squeezed. This may be due to the fact that the fundamental and residue photons do not have coherence.

4.2 Local Quadrature Squeezing

In this section we seek to calculate the local quadrature squeezing (in a given frequency interval) for the signal-signal modes. To this end, we define the spectrum of quadrature fluctuations for the cavity signal-signal modes with central frequency ω_0 by

$$S_{\pm}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle\hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle_{ss}e^{i(\omega-\omega_0)}d\tau. \tag{4.25}$$

Upon integrating both sides of Eq.(4.25) over ω, we easily get

$$\int_{-\infty}^{\infty}S_{\pm}(\omega)d\omega = (\triangle a_{\pm})^2, \tag{4.26}$$

with $(\triangle a_{\pm})^2$ being the steady-state global quadrature variance. On the basis of this relation, we observe that $S_{\pm}(\omega)d\omega$ is the quadrature variance of the cavity signal-signal modes in the interval between ω and $\omega + d\omega$. The quadrature variance in the interval between $\omega' = -\gamma$ and $\omega' = \gamma$ can then be expressed as

$$(\triangle a_{\pm})^2_{\pm\gamma} = \int_{-\gamma}^{\gamma}S_{\pm}(\omega')d\omega', \tag{4.27}$$

in which $\omega' = \omega - \omega_0$. We now proceed to obtain the spectrum of quadrature fluctuations for the cavity signal-signal modes produced by the degenerate subharmonic generator. Using the notation

$$\langle\hat{a}_{\pm}(t), \hat{a}_{\pm}(t+\tau)\rangle_{ss} = \langle\hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle_{ss} - \langle\hat{a}_{\pm}(t)\rangle\langle\hat{a}_{\pm}(t+\tau)\rangle_{ss}, \tag{4.28}$$

in view of Eq.(4.4), one can write Eq.(4.25) as

$$S_{\pm}(\omega) = \frac{Re}{\pi}\int_{0}^{\infty}\langle\hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle_{ss}e^{i(\omega-\omega')\tau}d\tau. \tag{4.29}$$

Moreover, upon setting $\kappa_1 = \kappa_2 = \kappa$ and applying Eq.(2.60) in Eqs.(2.47) and (2.48), we get

$$\frac{d}{dt}\langle\hat{a}_1\rangle = -\frac{1}{2}\kappa\langle\hat{a}_1\rangle - \lambda\langle\hat{a}_2^{\dagger}\rangle \tag{4.30}$$

and

$$\frac{d}{dt}\langle\hat{a}_2\rangle = -\frac{1}{2}\kappa\langle\hat{a}_2\rangle - \lambda\langle\hat{a}_1^{\dagger}\rangle. \tag{4.31}$$

Now addition of Eqs.(4.30) and (4.31) results in

$$\frac{d}{dt}\langle\hat{a}\rangle = -\frac{1}{2}\kappa\langle\hat{a}\rangle - \lambda\langle\hat{a}^{\dagger}\rangle, \tag{4.32}$$

where $\hat{a}$ is defined by Eq. (2.72). Hence using Eq.(4.32) and its complex conjugate, one readily finds

$$\frac{d}{dt}\langle\hat{a}_{\pm}(t)\rangle = -\eta_{\pm}\langle\hat{a}_{\pm}(t)\rangle, \tag{4.33}$$

in which $\langle\hat{a}_{+}(t)\rangle$ and $\langle\hat{a}_{-}(t)\rangle$ are given by Eq.(4.2) and Eq.(4.3) and

$$\eta_{\pm} = \frac{\kappa}{2} \pm \lambda. \tag{4.34}$$

We note that the solution of Eq. (4.33) can be written as

$$\frac{d\langle\hat{a}_{\pm}(t')\rangle}{\langle\hat{a}_{\pm}(t')\rangle} = -\eta_{\pm}dt', \tag{4.35}$$

$$\int_{t}^{t+\tau}\frac{d\langle\hat{a}_{\pm}(t')\rangle}{\langle\hat{a}_{\pm}(t')\rangle} = -\eta_{\pm}\int_{t}^{t+\tau}dt',$$

$$\ln[\langle\hat{a}_{\pm}(t')\rangle]|_{t}^{t+\tau} = -\eta_{\pm}t'|_{t}^{t+\tau},$$

one can readily gets

$$\langle\hat{a}_{\pm}(t+\tau)\rangle = \langle\hat{a}_{\pm}(t)\rangle e^{-\eta_{\pm}\tau}, \tag{4.36}$$

and applying the quantum regression theorem, we have

$$\langle\hat{a}_{\pm}(t)\hat{a}_{\pm}(t+\tau)\rangle = \langle\hat{a}_{\pm}^{2}(t)\rangle e^{-\eta_{\pm}\tau}. \tag{4.37}$$

Therefore, on account of Eq. (4.37) one can put Eq. (4.29) in the form

$$S_{\pm}(\omega) = \frac{1}{\pi}(\triangle a_{\pm})^{2}Re\int_{0}^{\infty}e^{-(\eta_{\pm}-i(\omega-\omega_{0}))\tau}d\tau, \tag{4.38}$$

so that on carrying out the integration, let

$$u = (\eta_{\pm} - i(\omega-\omega_{0}))\tau.$$

$$du = (\eta_{\pm} - i(\omega-\omega_{0}))d\tau,$$

$$d\tau = \frac{du}{\eta_{\pm} - i(\omega-\omega_{0})},$$

then substituting in to Eq. (4.38), it results

$$S_{\pm}(\omega) = \frac{1}{\pi}(\triangle a_{\pm})^{2}Re\int_{0}^{\infty}e^{-u}\frac{du}{\eta_{\pm} - i(\omega-\omega_{0})}$$

$$S_{\pm}(\omega) = \frac{1}{\pi}(\triangle a_{\pm})^{2}Re[\frac{\eta_{\pm} + i(\omega+\omega_{0})}{\eta_{\pm}^{2} + (\omega-\omega_{0})^{2}}]. \tag{4.39}$$

The spectrum of the plus quadrature fluctuations for the cavity signal-signal modes is found to be

$$S_+(\omega) = (\triangle a_+)^2 \left(\frac{\frac{(\frac{\kappa}{2}+\lambda)}{\pi}}{(\omega-\omega_0)^2 + [\frac{\kappa}{2}+\lambda]^2}\right) \tag{4.40}$$

and on taking in to account Eq. (4.10), we get

$$S_+(\omega) = \left(\frac{\frac{(\frac{\kappa}{2}+\lambda)}{\pi}}{(\omega-\omega_0)^2 + [\frac{\kappa}{2}+\lambda]^2}\right)\left(\frac{2\kappa}{\kappa+2\lambda}\right). \tag{4.41}$$

Furthermore, taking

$$a = \frac{\kappa}{2} + \lambda.$$

and

$$\omega' = \omega - \omega_0.$$

then Eq. (4.41) became

$$S_+(\omega') = \frac{2a\kappa}{\pi(\kappa+2\lambda)(\omega'^2 + a^2)}, \tag{4.42}$$

as we seen from Eq. (4.27)

$$(\triangle a_+)^2_{\pm\gamma} = \int_{-\gamma}^{\gamma} \frac{2a\kappa d\omega'}{\pi(\kappa+2\lambda)(\omega'^2+a^2)} \tag{4.43}$$

$$= \frac{2a\kappa}{\pi(\kappa+2\lambda)} \int_{-\gamma}^{\gamma} \frac{d\omega'}{\omega'^2+a^2},$$

applying the fact that

$$\int_{-\gamma}^{\gamma} \frac{d\omega'}{\omega'^2+a^2} = \frac{2}{a}\tan^{-1}\left(\frac{\gamma}{a}\right), \tag{4.44}$$

we arrive at

$$(\triangle a_+)^2_{\pm\gamma} = \frac{2}{\pi}\tan^{-1}\left(\frac{\gamma}{\frac{\kappa}{2}+\lambda}\right)\left(\frac{2\kappa}{\kappa+2\lambda}\right). \tag{4.45}$$

On the other hand, on setting $\lambda = 0$ in Eq. (4.45), we find

$$(\triangle a_+)^2_{v\pm\gamma} = \frac{4}{\pi}\tan^{-1}\left(\frac{2\gamma}{\kappa}\right). \tag{4.46}$$

This represents the quadrature variance of the two-mode cavity vacuum state in the same frequency interval. We define the quadrature squeezing of the cavity signal-signal modes in the interval between $\omega' = -\gamma$ and $\omega' = \gamma$ by

$$S_{\pm\gamma} = \frac{(\triangle a_+)^2_{v\pm\gamma} - (\triangle a_+)^2_{\pm\gamma}}{(\triangle a_+)^2_{v\pm\gamma}} \tag{4.47}$$

Therefore on account of Eq. (4.45) and Eq. (4.46), we see that

$$S_{\pm\gamma} = \frac{\frac{4}{\pi}\tan^{-1}(\frac{2\gamma}{\kappa}) - \frac{2}{\pi}\tan^{-1}(\frac{\gamma}{\frac{\kappa}{2}+\lambda})(\frac{2\kappa}{\kappa+2\lambda})}{\frac{4}{\pi}\tan^{-1}(\frac{2\gamma}{\kappa})}$$

$$= \frac{2\tan^{-1}(\frac{2\gamma}{\kappa}) - \tan^{-1}(\frac{\gamma}{\frac{\kappa}{2}+\lambda})(\frac{2\kappa}{\kappa+2\lambda})}{2\tan^{-1}(\frac{2\gamma}{\kappa})}$$

$$S_{\pm\gamma} = 1 - \frac{\tan^{-1}(\frac{\gamma}{\frac{\kappa}{2}+\lambda})}{2\tan^{-1}(\frac{2\gamma}{\kappa})}(\frac{2\kappa}{\kappa+2\lambda}) \tag{4.48}$$

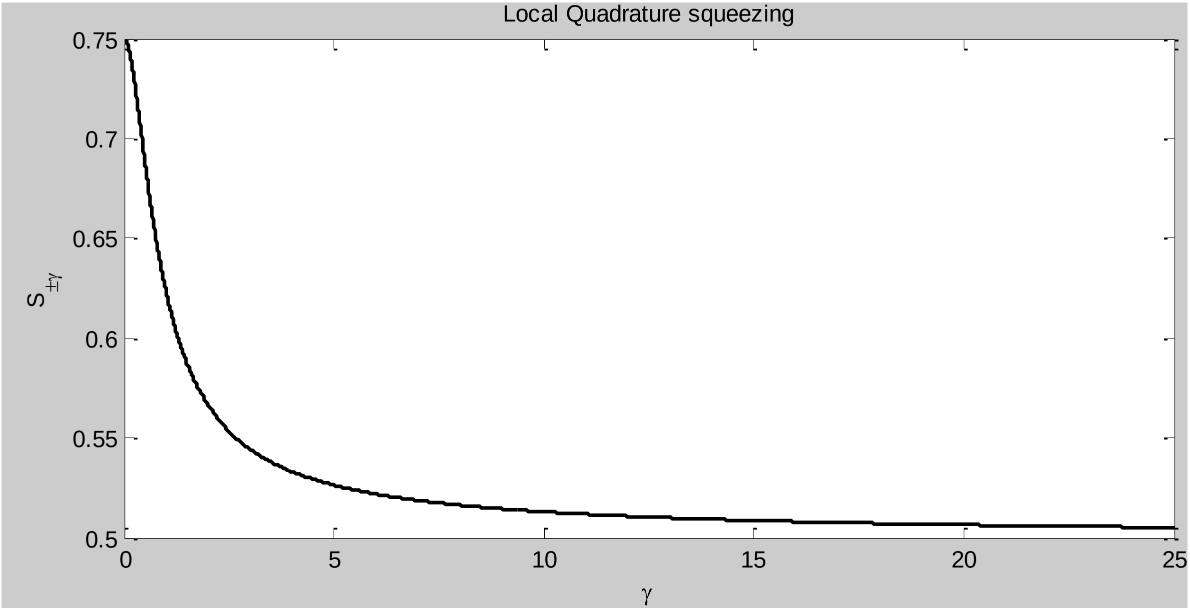

Fig. 4.3: Plots of the local quadrature squeezing [(Eq. 4.48)] versus γ for $\lambda = 0.4$ $\kappa = 0.8$.

The plot in Fig. (4.3) shows that the maximum local quadrature squeezing is 74.9% below the vacuum-state level. This occurs in the frequency interval $\gamma_{\pm} = 0.05$. In addition, we note that the local quadrature squeezing approaches the global quadrature squeezing as γ increases.

5

Entanglement Quantification

At issue here is to study the entanglement of the fundamental mode (signal-signal or signal-idler), the fundamental-residue and the entanglement output modes. It is a well-established fact that a quantum system is said to be entangled if it is not separable. That is, if density operator for the combined state can not be expressed as a product of the density operators of the individual constituents.

$$\hat{\rho} \neq \sum_{j} p_j \hat{\rho}_j^{(1)} \otimes \hat{\rho}_j^{2}, \tag{5.1}$$

where $p_j \geq 0$ and $\sum_j p_j = 1$ to ensure the normalization of the combined density operator. On the other hand, entangled continuous variable state can be expressed as a co-eigenstate of a pair of EPR-type operators such as $\hat{x}_1 + \hat{x}_2$ and $\hat{p}_1 - \hat{p}_2$ [30]. The total variance of these two operators reduce to zero for maximally entangled continuous variable states. Nonetheless according to the criterion set by Duan et al [23] quantum states of the system are claimed to be entangled.

5.1 Entanglement of the Cavity Mode

In this section we need to find the entanglement of the cavity modes. A state of the system is entangled, if the sum of the quantum fluctuations of the two Einstein-Podosky-Rosen (EPR)-like operators $\hat{u}$ and $\hat{v}$ of the two modes satisfies the inequality

$$\triangle u^2 + \triangle v^2 < 2 \tag{5.2}$$

where,

$$\hat{u} = \hat{x}_1 + \hat{x}_2, \tag{5.3}$$

$$\hat{v} = \hat{p}_1 - \hat{p}_2, \tag{5.4}$$

with $\hat{x}_j = \frac{1}{\sqrt{2}}(\hat{a}_j + \hat{a}_j^\dagger)$, and $\hat{p}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j)$, (with j=1,2) are the quadratures for the two-modes of the cavity. But

$$\begin{aligned}(\triangle u^2) &= \langle \hat{u}^2 \rangle - \langle \hat{u} \rangle^2 \\ &= \langle (\hat{x}_1 + \hat{x}_2)^2 \rangle - \langle \hat{x}_1 + \hat{x}_2 \rangle^2 \\ &= \langle \hat{x}_1^2 \rangle + \langle \hat{x}_1 \hat{x}_2 \rangle + \langle \hat{x}_2 \hat{x}_1 \rangle + \langle \hat{x}_2^2 \rangle \\ &- (\langle \hat{x}_1 \rangle^2 + \langle \hat{x}_1 \rangle \langle \hat{x}_2 \rangle + \langle \hat{x}_2 \rangle \langle \hat{x}_1 \rangle + \langle \hat{x}_2 \rangle^2),\end{aligned} \tag{5.5}$$

substituting, $\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_1^\dagger)$,$\hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{a}_2 + \hat{a}_2^\dagger)$ and using Eqs.(2.71), (2.67), one can get

$$(\triangle u^2) = \frac{1}{2}(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle). \tag{5.6}$$

Following the same step and substituting $\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1)$, and $\hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_2)$, one can readily find

$$\begin{aligned}(\triangle v^2) &= \langle \hat{v}^2 \rangle - \langle \hat{v} \rangle^2 \\ &= \langle (\hat{p}_1 - \hat{p}_2)^2 \rangle - \langle \hat{p}_1 - \hat{p}_2 \rangle^2 \\ &= \langle \hat{p}_1^2 \rangle - \langle \hat{p}_1 \hat{p}_2 \rangle - \langle \hat{p}_2 \hat{p}_1 \rangle + \langle \hat{p}_2^2 \rangle \\ &- (\langle \hat{p}_1 \rangle^2 - \langle \hat{p}_1 \rangle \langle \hat{p}_2 \rangle - \langle \hat{p}_2 \rangle \langle \hat{p}_1 \rangle + \langle \hat{p}_2 \rangle^2),\end{aligned} \tag{5.7}$$

$$(\triangle v^2) = \frac{1}{2}(\langle \hat{a}_1^\dagger \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_1^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle + \langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle). \tag{5.8}$$

On the basis of the boson commutation relation that are mentioned in Eq.(2.37), we can find that

$$\triangle u^2 + \triangle v^2 = 2 + 2\langle (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger) \rangle. \tag{5.9}$$

Taking Eq.(2.69) and Eq.(4.9) under consideration, one can simplifies

$$\triangle u^2 + \triangle v^2 = 2 + 4\langle (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_1 \hat{a}_2) \rangle = (\triangle a_+)^2, \tag{5.10}$$

which implies in the system under consideration, the entanglement and quadrature variance exactly equal. Now the various correlations involved in equation (5.9) would be determined by using the steady-state equations, (2.67),(2.68),(2.69) and (2.70), we can get

$$\begin{aligned}\triangle u^2 + \triangle v^2 &= 2 + 2[2(\frac{2\lambda^2}{\kappa^2 - 4\lambda^2}) + 2(-\frac{\kappa\lambda}{\kappa^2 - 4\lambda^2})] \\ &= 2 + 2[\frac{4\lambda^2}{\kappa^2 - 4\lambda^2} - \frac{2\kappa\lambda}{\kappa^2 - 4\lambda^2}] \\ &= 2 + 2[\frac{4\lambda^2 - 2\kappa\lambda}{\kappa^2 - 4\lambda^2}] \\ &= 2 + 2[\frac{-2\lambda}{\kappa + 2\lambda}] \\ &= \frac{2\kappa}{\kappa + 2\lambda}. \end{aligned} \tag{5.11}$$

At threshold ($\kappa = 2\lambda$) the entanglement of the two-mode light of the cavity results 1. Since the cavity modes are squeezed in plus quadrature. i.e. $(\triangle a_+)^2 < 2$. One can easily deduce based on Eq.(5.11) that the cavity modes in the system that we have considered are entangled, implying that

$$\triangle u^2 + \triangle v^2 < 2.$$

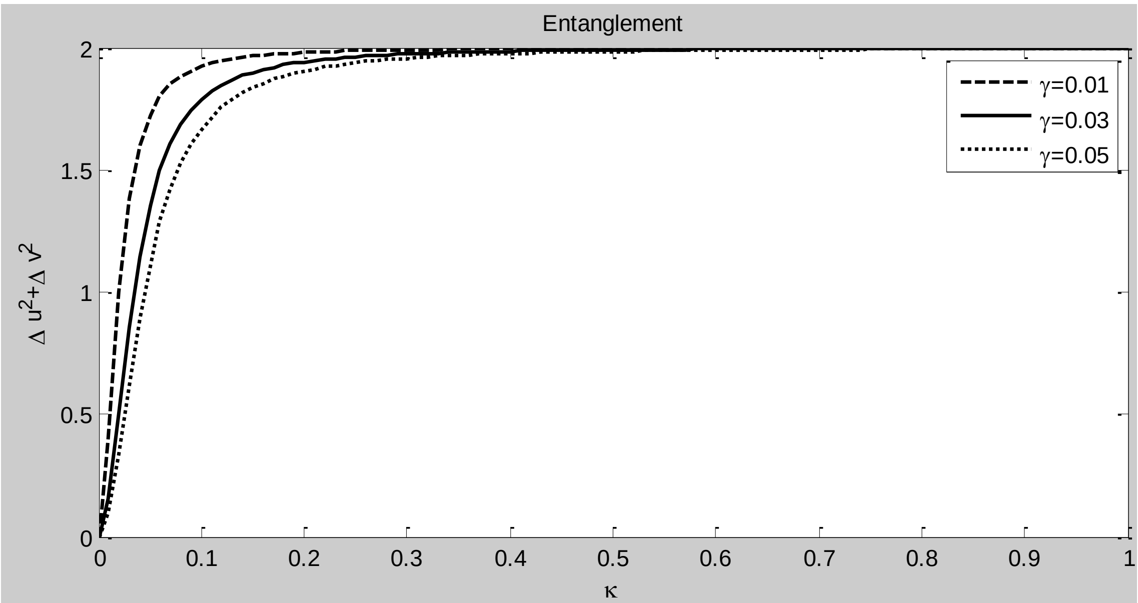

Fig. 5.1: Plots of the sum of the variance cavity modes for EPR-like operators [Eq.(5.11)] at steady state versus κ, for $\varepsilon = 0.01$,and different value of γ, for $\gamma = 0.01$ (dashed curve), $\gamma = 0.03$ (solid curve), $\gamma = 0.05$ (dotted curve).

The plots in Fig(5.1) indicate that the degree of entanglement of the cavity modes for the system under consideration increase with the amplitude of the pump mode γ, and decrease with the cavity damping constant, which means that the lesser the cavity damping the more probable for the radiation to stay in the cavity which in turn strengths the correlation that leads to entangled. The entanglement criterion given by Eq.(5.2) is also satisfied.

5.2 Entanglement of the Fundamental-Residue Mode

Here we wish to get the entanglement of fundamental and residual pump modes $\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_1^\dagger)$,$\hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger)$, $\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1)$, and $\hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{b}^\dagger - \hat{b})$.
On the basis of the boson commutation relation in Eq.(2.37), we can get

$$\begin{aligned}\triangle u^2 + \triangle v^2 = 2[&\langle \hat{a}_1^\dagger \hat{a}_1\rangle + \langle \hat{b}^\dagger \hat{b}\rangle + \langle \hat{a}_1\rangle\langle \hat{b}\rangle + \langle \hat{a}_1^\dagger\rangle\langle \hat{b}^\dagger\rangle + \langle \hat{a}_1 \hat{b}\rangle \\ &+ \langle \hat{a}_1^\dagger \hat{b}^\dagger\rangle + \langle \hat{a}_1^\dagger\rangle\langle \hat{a}_1\rangle + \langle \hat{b}^\dagger\rangle\langle \hat{b}\rangle] + 2.\end{aligned} \tag{5.12}$$

Now the various correlations involved in Eq. (5.12) would be determined by Eq. (2.67), and Eq. (3.25), one can reduce Eq.(5.12) into

$$\triangle u^2 + \triangle v^2 = 2[\langle \hat{a}_1^\dagger \hat{a}_1\rangle + \langle \hat{b}^\dagger \hat{b}\rangle] + 2,$$

in which $\langle \widehat{a}_1 \widehat{b}\rangle = 0$, and applying Eq.(2.68) we can get

$$\begin{aligned}\triangle u^2 + \triangle v^2 &= 2[\frac{2\lambda^2}{\kappa^2 - 4\lambda^2} + \frac{4\mu^2}{\kappa^2} - \frac{2\lambda^2}{\kappa^2 - 4\lambda^2}] + 2 \\ &= \frac{8\gamma^2}{\kappa^2} + 2,\end{aligned} \tag{5.13}$$

which implies

$$\frac{8\gamma^2}{\kappa^2} + 2 > 2.$$

It shows that the sum of variance of a cavity mode for EPR-like operators is always greater than 2 for all values of γ and κ. From this we conclude that the fundamental and residual pump modes are not entangled.

5.3 Entanglement of Out-Put Mode

The entanglement of the output modes using the EPR-like operators is defined as

$$\triangle u_{out}^2 + \triangle v_{out}^2 = \langle u_{out}^2\rangle - \langle u_{out}\rangle^2 + \langle v_{out}^2\rangle - \langle v_{out}\rangle^2, \tag{5.14}$$

where $u_{out} = \hat{x}_{1out} + \hat{x}_{2out}$, and $v_{out} = \hat{p}_{1out} - \hat{p}_{2out}$, with $\hat{x}_{jout} = \frac{1}{\sqrt{2}}(\hat{a}_{jout} + \hat{a}_{jout}^\dagger)$ and $\hat{p}_{jout} = \frac{i}{\sqrt{2}}(\hat{a}_{jout}^\dagger - \hat{a}_{jout})$ where j=1,2. Therefore,

$$\begin{aligned}(\triangle u_{out}^2) = &\langle \hat{x}_{1out}^2\rangle + \langle \hat{x}_{1out}\hat{x}_{2out}\rangle + \langle \hat{x}_{2out}\hat{x}_{1out}\rangle + \langle \hat{x}_{2out}^2\rangle \\ &- (\langle \hat{x}_{1out}\rangle^2 + \langle \hat{x}_{1out}\rangle\langle \hat{x}_{2out}\rangle + \langle \hat{x}_{2out}\rangle\langle \hat{x}_{1out}\rangle + \langle \hat{x}_{2out}\rangle^2),\end{aligned} \tag{5.15}$$

$$(\triangle v_{out}^2) = \langle \hat{p}_{1out}^2 \rangle - \langle \hat{p}_{1out}\hat{p}_{2out} \rangle - \langle \hat{p}_{2out}\hat{p}_{1out} \rangle + \langle \hat{p}_{2out}^2 \rangle$$
$$- (\langle \hat{p}_{1out} \rangle^2 - \langle \hat{p}_{1out} \rangle \langle \hat{p}_{2out} \rangle - \langle \hat{p}_{2out} \rangle \langle \hat{p}_{1out} \rangle + \langle \hat{p}_{2out} \rangle^2). \quad (5.16)$$

Employing the output relation as cited on [1]

$$\hat{a}_{out}(t) = \sqrt{\kappa}\hat{a}(t) - \hat{a}_{in}, \quad (5.17)$$

together with the properties

$$\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle = N,$$

$$\langle \hat{a}_{out}(t)^\dagger \hat{a}_{out}(t) \rangle = \kappa \langle \hat{a}^\dagger \hat{a} \rangle + (1 - \kappa)N,$$

and

$$\langle \hat{a}^\dagger \hat{a}_{in} \rangle + \langle \hat{a}_{in}^\dagger \hat{a} \rangle.$$

Considering the mean photon number of vacuum reservoir to be zero, one can get

$$\langle \hat{a}_{out}(t)^\dagger \hat{a}_{out}(t) \rangle = \kappa \langle \hat{a}^\dagger \hat{a} \rangle,$$

$$\langle \hat{a}_{out}(t) \hat{a}_{out}^\dagger(t) \rangle = \kappa \langle \hat{a}^\dagger \hat{a} \rangle + 1.$$

$$\langle \langle \hat{a}_{out}^\dagger(t) \rangle = \langle \hat{a}_{out}^2(t) = 0.$$

From this point of view, we have

$$\langle u \rangle^2 = \langle v \rangle^2 = 0.$$

Therefore, the sum of variance of EPR-like operators reduced to

$$\triangle u_{out}^2 + \triangle v_{out}^2 = \langle u_{out}^2 \rangle + \langle v_{out}^2 \rangle, \quad (5.18)$$

$$\triangle u_{out}^2 + \triangle v_{out}^2 = 2\kappa \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + 2\kappa \langle \hat{a}_2^\dagger \hat{a}_2 \rangle + \kappa(\langle \hat{a}_1 \hat{a}_2 \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle) + \kappa(\langle \hat{a}_2 \hat{a}_1 \rangle + \langle \hat{a}_2^\dagger \hat{a}_1^\dagger \rangle) + 2. \quad (5.19)$$

Using Eq.(2.68), Eq.(2.69), and Eq.(2.70), Eq.(5.19) becomes

$$\triangle u_{out}^2 + \triangle v_{out}^2 = 2 - \frac{4\kappa\lambda}{\kappa + 2\lambda}. \quad (5.20)$$

Eq.(5.20) shows that the entanglement of output mode. Fig.(5.2) indicates that the entanglement of output mode is always less than that of cavity mode and both of the approaches to zero as cavity damping approaches to 1.

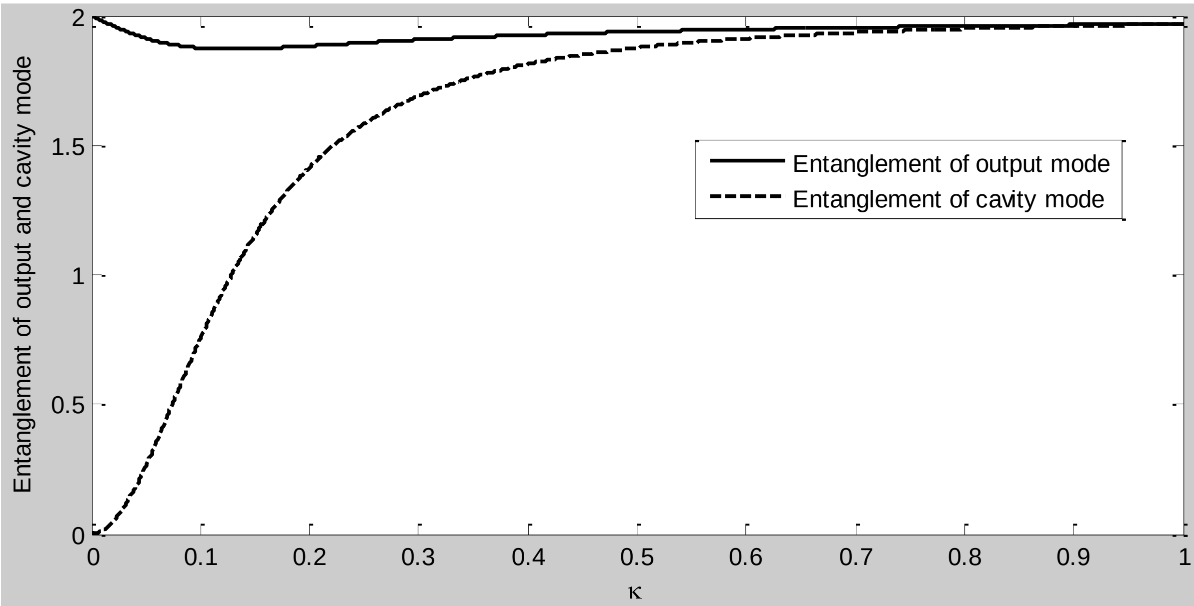

Fig. 5.2: Plots of the cavity and output mode for EPR-like operators [Eq.(5.11) and Eq.(5.20)] at steady state versus κ, for $\varepsilon = 0.055$,and $\gamma = 0.075$.

6

Conclusion

The mean photon number of the twin light beams with the same frequency, obtained applying the conventional Hamiltonian by many authors, is found to be half of the twin light beams with different frequencies.this unexpected result must be due to representation of the twin light beams with the same frequency by second-order annihilation and creation operators in the conventional Hamiltonian. We then realize that the twin light beams with the same or different frequencies must be represented in the Hamiltonian by 1^{st}-order annihilation and creation operators. We thus describe the process of parametric oscillation leading to the creation of twin light beams with the same or different frequencies by the first-order Hamiltonian .

Evidently, the mean photon number calculated using this Hamiltonian turns out to be the same for both the signal-signal and signal-idler modes. In addition, the mean photon number of the pump mode is found to be the mean photon number of the pump mode in the absence of the parametric oscillation minus the mean photon number of either the signal-signal or the signal-idler modes. This is exactly what we would expect the mean photon number of the pump mode to be. On the other hand, we have established that the maximum global quadrature squeezing of the signal-signal or the signal-idler modes is 50% below the vacuum-state level and the maximum local quadrature squeezing of the signal-signal modes is 74.9% below the same level. Moreover, the strong correlation between signal-idler or signal-signal modes leads not only quadrature squeezing but also to entanglement, but fundamental-residue pump modes are not squeezed as well as entangled. This may be due to the fact that the fundamental and residue photons do not have coherence.

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DECLARATION

I hereby declare that this Msc thesis is my original work and has not been presented for a degree in any other universities, and that all sources of material used for the dissertation have been duly acknowledged.

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