ANALYSIS OF QUANTUM CORRELATION FOR TWO-MODE LIGHT PRODUCED BY NON-DEGENERATE THREE LEVEL LASER

BY

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Approval Page

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ABSTRACT

We studied the various quantum correlations for the input and output two-mode light generated by non-degenerate three level laser coupled to vacuum reservoir. First we have constructed the Hamiltonian for our system. Applying this Hamiltonian, we have obtained the pertinent master equation which have been used to obtain the steady state solutions of the equation of evolution of the expectation values of the cavity mode operators. Applying the resulting steady state solutions, we have studied the quadrature variance, entanglement, discord and steering of the radiation. The obtained results showed that the quantum correlations of the radiation generated by non degenerate three level laser increase with linear gain coefficient (A)*, for small values of population inversion* (η)*. On the other hand applying the input-out put relation we have also shown that the quantum correlation of the cavity mode is greater than the out put mode.*

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INTRODUCTION

1.1 Background of the Study

Quantum optics deals mainly with the quantum properties of the light generated by various quantum optical systems such as lasers and with the effect of light on the dynamics of atoms. It is also a field of quantum physics that deals specifically with the interaction of photon with matter. The quantum properties of light are largely determined by the quantum state of the light mode. Squeezing is one of the non-classical feature of light that has been attracting attention and studied by many authors. It is generated as a result of quantum optical process such as subharmonic generation and second harmonic generation. It has also been studied that three-level cascade lasers can be considered as a source of squeezed light under certain conditions. In such lasers, the squeezing is due to the atomic coherence which can be introduced either by initially preparing the three-level atoms in a coherent superposition of the top and bottom level or coupling these levels by external coherent light. When a three-level atom makes a transition from the top to bottom level through the intermediate level, two highly correlated photons are generated. If these photons are identical, the laser is referred to as a degenerate three-level laser, otherwise it is a nondegenerate three-level laser.

The interest in the study of quantum correlation of confined systems such as quantum discord, quantum entanglement and quantum steering has increased with the recent progress in quantum information. Correlations describe the dependence of certain properties for different parts of a composite object. If these properties of different parts are independent of each other, then we say that there do not exist correlations between (or among) them. If they are correlated, then how to characterize the correlation, both qualitatively and quantitatively, becomes an essential task.

Until recently the quantum aspects of correlation were attributed to inseparable quantum states [1] (i.e., all nonclassical correlations in a composite quantum state was regarded as entanglement). However, the discovery that mixed separable (unentangled) states can also have nonclassical correlation [2, 3] and that the use of such states can improve performance in some computational tasks (compared to classical computing) [4, 5] has opened a new perspective on the study and comprehension of such correlations. The distinction between quantum and classical aspects of correlation in a composite quantum state is an important issue in quantum information theory (QIT). It is largely accepted that quantum mutual information is a measure of the total correlation contained in a bipartite quantum state [6, 7]. Quantum Information Theory is the study of how such tasks can be accomplished using quantum mechanical systems.

In quantum information theory it is common to distinguish between purely classical information, measured in bits, and quantum information, which is measured in qubits. These differ in the channel resources required to communicate them. Qubits may not be sent by a classical channel alone, but must be sent either via a quantum channel which preserves coherence or by teleportation through an entangled channel with two classical bits of communication. Any bipartite quantum state may be used as a communication channel with some degree of success, and so it is of interest to determine how to separate the correlations it contains into a classical and an entangled part.

Entanglement is a property of correlations between two or more quantum systems. These correlations defy classical description and are associated with intrinsically quantum phenomena. For this reason, entanglement has played an important role in the development and testing of quantum theory, it is also a central element in quantum information [8]. Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles cannot be described independently instead, a quantum state may be given for the system as a whole. Classical information and quantum information based on entangled quantum systems can be used for quantum communication purposes such as teleporting quantum states [9]. The efficiency of quantum information processing highly depends on the degree of entanglement. Hence, it is desirable to generate strongly entangled continuous variable state. The idea of a nonlocal effect due to entanglement was pointed out by Einstein, Podolsky, and Rosen; hence, the name EPR state [10].

To quantify the quantumness of the correlation contained in a bipartite quantum state Olliver and Zurek [3] proposed a measure for quantum correlation known as quantum discord and based on a distinction between quantum information theory (QIT) and classical information theory (CIT). A related quantity concerning classical correlation was proposed by Henderson and Vedral. A recent result that almost all quantum states have a nonvanishing quantum discord shows up the relevance of studying such correlation. Quantum discord measures quantum correlation by comparing the quantum mutual information with the maximal amount of mutual information accessible to a quantum measurement [11].

In 1935, Einstein-Podolsky-Rosen (EPR) published their famous paper proposing a now well-known paradox (the EPR paradox) that cast doubt on the completeness of quantum mechanics [10]. To investigate the EPR paradox, Schrödinger introduced the concept of steer [12], now known as the EPR steering [13]. As an asymmetric concept, EPR steering describes the ability of a system to nonlocally affect the states of another system through local measurements. EPR steering exists between the concepts of entanglement and Bell nonlocality [14]. Not all entangled states are steerable, and not all steerable states exhibit nonlocality [13, 15], but states that exhibit steering allow for the verification of their entanglement in a semi-device-independent way: there is no need to trust the devices used by the steering party [13, 15, 16]. From a quantum information perspective [13], steering corresponds to the task of verifiable entanglement distribution by an untrusted party.

Postulates for general measures of quantum correlations have been proposed in [17]. There are three necessary conditions every measure of quantum correlations Q should satisfy. These conditions are; if Q is nonnegative, if Q is invariant under local unitary operations and if Q is zero on classically correlated states.

1.2 Statement of the Problem

In previous time, several authors have studied the quantum correlation property generated by a non-degenerate three-level cascade laser alone using different methods. In this study, we analyze the quantum correlation property generated by a non-degenerate three-level cascade laser coupled to vacuum reservoir together by using steady state solution.

1.3 Objectives of the Study

1.3.1 General Objectives of the Study

The general objective of this study is to analyze quantum correlation for two-mode light generated by non-degenerate three level laser coupled to vacuum reservoir.

1.3.2 Specific Objectives of the Study

The specific objectives of this studies are:

- ▶ constructing Hamiltonian for the system under consideration
- ▶ determining the master equation for system under consideration and to obtain the steady state of the expectation values of the cavity mode
- ▶ obtaining quadrature variance of cavity mode and output mode for non degenerate three level laser
- ▶ analyzing both quantum entanglement using Duan *et al* criterion and quantum discord for cavity mode and output mode for the system under consideration
- ▶ determining quantum steering for non degenerate three level laser

1.4 Significance of the Study

Quantum correlation has many applications in teleportation [18] and quantum computation [19]. So, the results of this research can be applied for technological advancement in the area of quantum information. In addition it can be used as input for other researchers or it will serve as a platform for other scientific researchers.

2

LITERATURE REVIEW

2.1 Overview of the Study

Quantum optics deals mainly with quantum properties of the light generated by various quantum optical system, such as; Laser and maser. A three-level cascade laser is a quantum optical system in which three level atoms in a cascade configuration. In this system the top, intermediate and bottom levels of a three level cascade atom can be denoted by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively. In quantum optics, the annihilation and creation operator describing a single-mode radiation can be decomposed into two component operators, referred to as quadrature operators. For a single-mode radiation in any state, the product of the fluctuations in the two quadratures satisfies the uncertainty relation $\Delta a_+ \Delta a_- \geq 1$. In particular, for some class of light states called minimum uncertainty states, the uncertainty product is equal to the vacuum level, which represent the standard quantum limit to reduction of noise in a signal. One such a minimum uncertainty state is a coherent state, the closest quantum counter-part to a classical radiation [20]. In this state, the product of the fluctuations is minimum and randomly distributed between the two quadratures.

Quantum information is concerned with using the special features of quantum physics for the processing and transmission of information. This recent branch of quantum physics, involving elements of quantum optics, the physical system used for encoding the information is governed by the laws of quantum mechanics. This gives rise to properties, which are classically not possible. Among those quantum steering, quantum entanglement and quantum discord are mentioned. Like squeezing those correlation have non-classical features.

2.2 Squeezing

In a squeezed state the quantum noise in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the conjugate quadrature, with the product of the variance in the two quadratures satisfying the uncertainty relation. Because of the quantum noise reduction achievable below the vacuum level in one of the quadratures, squeezed light has potential applications in the detection of week signals (such as gravitational wave) and in low-noise communications [21].

Dawit H. [22] studied a degenerate three-level laser in which top and bottom levels are coupled by strong coherent light and with half probability for the atoms to be in the top or bottom levels coupled to a squeeze vacuum reservoir applying the solution of stochastic differential equations. He found that the squeezing increases with the linear gain coefficient (A).

2.3 Entanglement

It is now believed that the key ingredient of quantum information is entanglement which has been recognized as the essential resource for quantum teleportation [23]. Traditionally, quantum entangled states are considered to be non-classical correlations between individual qubits. In the field of quantum information the fundamental element is the qubit, or the quantum bit [24]. It is a state in a two-dimensional Hilbert space and it is a generalization of the classical bit. Whereas a classical bit can take one of the binary values 0 or 1, the qubit can analogously be in one of the two orthogonal basis states $|0\rangle$ or $|1\rangle$. Additionally the qubit can be in any superposition of the two basis states

$$|Q\rangle = c_0|0\rangle + c_1|1\rangle,$$

where c_0 and c_1 are arbitrary complex numbers that satisfy the normalization condition $|c_0|^2 + |c_1|^2 = 1$.

However, it has been proved that continuous-variable (CV) entanglement has many advantages in some cases [25]. A pair of particles is taken to be entangled in quantum theory, if its states cannot be expressed as a product of the states of its individual constituents. The preparation and manipulation of these entangled states that have nonclassical and nonlocal properties lead to a better understanding of the basic quantum principles. In other words, it is a well-known fact that a quantum system is said to be entangled, if it is not separable. That is, if the density operator for the combined state cannot be described as a combination of the product density operators of the constituents

$$\hat{\rho} \neq \sum_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}, \tag{2.1}$$

in which $p_k > 0$ and $\sum_k p_k = 1$ to verify the normalization of the combined density states. The criterion to detect whether a given state is entangled or not are in the form of inequality and in most instances provide sufficient condition of inseparability. Some of the criteria includes Duan *et al* criterion [26], logarithmic negativity criterion [27, 28], Hillery-Zubairy criterion [29] and positive partial transpose (PPT) criterion [30]. A maximally entangled CV state can be expressed as a coeigenstate of a pair of EPR-type operators [10] such as $\hat{x}_a + \hat{x}_b$ and $\hat{p}_a - \hat{p}_b$. The total variance of these two operators reduces to zero for maximally entangled CV states.

Eyob A. [31] has studied CV entanglement in a non-degenerate three-level laser with a parametric amplifier. In his model the injected atomic coherence introduced by initially preparing the atoms in a coherent superposition of the top and bottom levels. This combined system exhibits a two-mode squeezed light and produces light in an entangled state.

2.4 Discord

Quantum discord measures a very general form of non-classical correlation, which can be present in quantum systems even in the absence of entanglement. Quantum discord was originally suggested as a measure of quantumness of correlations [32], and has since been studied in variety of systems and settings [33, 34]. Discord can be regarded as a measure of a violation of classicality of a joint state of two quantum subsystems, while the discord is a fundamental notion allowing for the description of the quantumness of the correlations present in the state of a quantum system, its evaluation requires an optimization procedure over the set of all measurements on a given subsystem, and thus attacking the general case is a formidable task [32]. For separable state we have [35]

$$\hat{\rho} = \sum_k p_k \hat{\rho}_k^{(1)} \otimes \hat{\rho}_k^{(2)}. \tag{2.2}$$

Like entangled state separable state also have quantum behavior that studied by Discord. A state is said to be classically correlated if it can be fully determined without disturbing it with the aid of local operations and classical communication (LOCC).

Sentayehu T. [36] has studied continuous variable quantum discord in a correlated emission laser. This study showed that the quantum discord increases with the prepared atomic coherent superposition and disappears when all the atoms are initially prepared to be in the lower energy level.

2.5 Steering

Steering is the quantum mechanical phenomenon that allows one party, to change (i.e., to steer) the state of a distant party, by exploiting their shared entanglement. steering, unlike entanglement, is an asymmetric property: a quantum state may be steerable from the first party (state) to another party (state), but not vice versa. On the operational side, it has been recently realized that steering provides security in one-sided device-independent quantum key distribution (QKD) [37], where the measurement apparatus of one party only is untrusted. Quantum steering, the ability of one party to perform a measurement on their side of an entangled system with different outcomes leading to different sets of states for another part of the entangled system arbitrarily far away, is a purely quantum phenomena with no classical analogue. However, it is closely related (and indeed equivalent for pure states) to entanglement and Bell nonlocality [38].

Ullah *et al* [39] has studded hierarchy of temporal quantum correlations using a correlated spontaneous emission laser. They showed that the steering is a witness of entanglement but the reverse is not always true. Likewise, entanglement in the system always shows quantum discord but the latter may not necessarily indicates the former. Therefore, all the four kinds of temporal quantum correlations satisfy a strict hierarchy relation in the quantum system of CEL (correlated emission laser) such that quantum discord $\supseteq$ quantum entanglement $\supseteq$ quantum steering $\supseteq$ Bell non-locality, where X $\supseteq$ Y means X is the superset of Y, stating such that entangled states are a strict subset of the states that may exhibit discord, and a strict superset of the steerable states.

3

METHODOLOGY

In this chapter we seek to get the master equation and operator dynamics at steady state solution for the cavity mode produced by non degenerate three-level laser coupled to vacuum reservoir employing the constructed Hamiltonian for the system .

3.1 Master Equation

Here we wish to obtain the master equation for a cavity mode of non degenerate three-level laser in which a non degenerate three-level atoms in a cascade configuration (Fig.3.1) are injected at a constant rate r_a into a cavity coupled to vacuum reservoir via a single port-mirror and removed after a large enough decay time τ. We denote the top, middle and bottom levels by eigen states $|a\rangle, |b\rangle$ and $|c\rangle$. We assume the cavity mode to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between $|a\rangle$ and $|c\rangle$ is to be electric-dipole forbidden. The interaction of a three-level atom with a cavity mode can be described in the interaction picture by the Hamiltonian [21]

$$\hat{H} = ig\Big[|a\rangle\langle b|\hat{a} - \hat{a}^{\dagger}|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^{\dagger}|c\rangle\langle b|\Big], \tag{3.1}$$

where g is the coupling constant between the cavity and a three level atom and $\hat{a}$ and $\hat{b}$ is the annihilation operator for the cavity mode. Considering the atom to be initially in the state

$$|\psi_A(0)\rangle = c_a(0)|a\rangle + c_c(0)|c\rangle \tag{3.2}$$

and hence the initial density operator for a single atom can be written as

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{3.3}$$

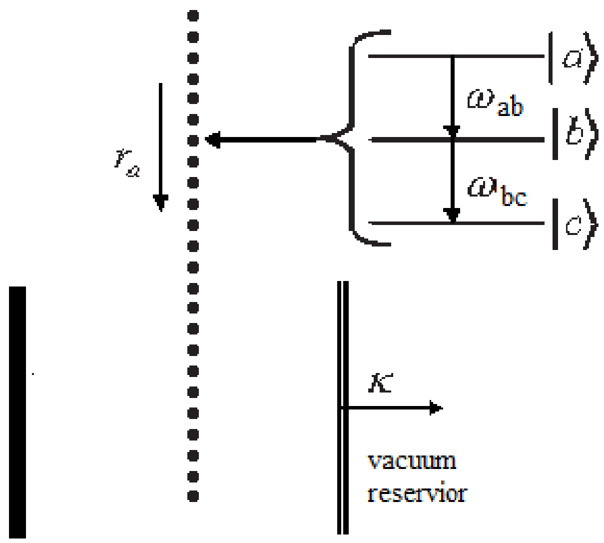

Fig. 3.1: Nondegenerate three-level atom coupled to vacuum reservoir.

where,

$$\rho_{aa}^{(0)} = |c_a|^2 \quad and \quad \rho_{cc}^{(0)} = |c_c|^2 \tag{3.4}$$

are the probability for the atom to be in the upper and the lower levels at the initial time respectively,

$$\rho_{ac}^{(0)} = c_a c_c^* \quad and \quad \rho_{ca}^{(0)} = c_c c_a^*. \tag{3.5}$$

represent the atomic coherence at the initial time. Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity mode at time t, with the atom injected at time t_j such that $(t - \tau) \leq t_j \leq t$. The density operator for all atoms in the cavity plus the cavity mode at time t can then be written as

$$\hat{\rho}_{AR}(t) = r_a \sum_j \rho_{AR}(t, t_j) \Delta t_j, \tag{3.6}$$

where $r_a \Delta t_j$ represents the number of atoms injected into the cavity in a time Δt_j. Now converting the summation into integration in the limit $\Delta t_j \rightarrow 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t') dt', \tag{3.7}$$

and differentiating with respect to t, and taking into account the Leibnitz rule

$$\frac{d}{dx}\int_{u(x)}^{\upsilon(x)} f(x,x')dx' = \Big(f(x,\upsilon)\frac{d\upsilon(x)}{dx} - f(x,u)\frac{du(x)}{dx}\Big) + \int_{u(x)}^{\upsilon(x)} \frac{\partial}{\partial x} f(x,x')dx',$$

there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t,t) - \hat{\rho}_{AR}(t,t-\tau)) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{3.8}$$

We observe that $\hat{\rho}_{AR}(t,t)$is the density operator for the cavity mode plus an atom injected at time t. This operator can be expressed as

$$\hat{\rho}_{AR}(t,t) = \hat{\rho}_A(t)\hat{\rho}(t), \tag{3.9}$$

with $\hat{\rho}(t)$ being the density operator for the cavity mode alone.

We also note that $\hat{\rho}_{AR}(t,t-\tau)$ represents the density operator for an atom plus the cavity mode at time t, with the atom being removed from the cavity at this time. This operator can also be put in the form

$$\hat{\rho}_{AR}(t,t-\tau) = \hat{\rho}_A(t-\tau)\hat{\rho}(t). \tag{3.10}$$

Now in view of of Eq. (3.9) and (3.10),one can rewrite Eq. (3.8) as

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau))\hat{\rho}(t) + r_a\int_{t-\tau}^{t}\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t')dt'. \tag{3.11}$$

In the absence of the damping of the cavity mode by vacuum reservoir, the density operator $\hat{\rho}_{AR}(t,t)$ evolves in time according to

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t,t') = i[\hat{H},\hat{\rho}_{AR}(t,t')]. \tag{3.12}$$

so that using this relation along with Eq. (3.7),one can put Eq. (3.11) in the form

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_A(t) - \hat{\rho}_A(t-\tau))\hat{\rho}(t) - i[\hat{H},\hat{\rho}_{AR}(t)]. \tag{3.13}$$

Furthermore, tracing over the atomic variables and taking into account the damping of the cavity mode by the vacuum reservoir, we have

$$\frac{d\hat{\rho}}{dt} = -iTr_A[\hat{H},\hat{\rho}_{AR}(t)] + \frac{1}{2}\kappa\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \frac{1}{2}\kappa\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\Big),$$

where κ is the cavity damping constant and we have used the fact

$$Tr\hat{\rho}_A(t) = Tr\hat{\rho}_A(t-\tau) = 1. \tag{3.14}$$

The master equation associated with the interactions described by the Hamiltonian can be put in the form

$$\frac{d\hat{\rho}}{dt} = g\Big(\hat{\rho}_{ab}\hat{a}^\dagger - \hat{a}^\dagger\hat{\rho}_{ab} + \hat{\rho}_{bc}\hat{b}^\dagger - \hat{b}^\dagger\hat{\rho}_{bc} + \hat{a}\hat{\rho}_{ba} - \hat{\rho}_{ab}\hat{a} + \hat{b}\hat{\rho}_{cb} - \hat{\rho}_{cb}\hat{b}\Big) + \frac{1}{2}\kappa\Big(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\Big) + \frac{1}{2}\kappa\Big(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\Big), \tag{3.15}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ is defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}|\beta\rangle, \tag{3.16}$$

with $\alpha, \beta = a, b, c$.

On the other hand, we see from Eq. (3.13) that

$$\frac{d\hat{\rho}_{\alpha\beta}}{dt} = r_a(\langle\alpha|\hat{\rho}_A(0)|\beta\rangle - \langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle)\hat{\rho}(t) - i(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle) - \gamma\hat{\rho}_{\alpha\beta}, \tag{3.17}$$

where the last term is included to account for the decay of the atoms due to spontaneous emission. Here γ, considered to be the same for all the three levels, is the atomic decay constant. We assume that the atoms are removed from the cavity after they have decayed to a level other than the middle or bottom level. We then see that

$$\langle\alpha|\hat{\rho}_A(t-\tau)|\beta\rangle = 0 \tag{3.18}$$

and hence Eq. (3.17) reduced to

$$\frac{\hat{\rho}_{\alpha\beta}}{dt} = r_a\langle\alpha|\hat{\rho}_A(0)|\beta\rangle\hat{\rho}(t) - i(\langle\alpha|\hat{H}\hat{\rho}_{AR}|\beta\rangle - \langle\alpha|\hat{\rho}_{AR}\hat{H}|\beta\rangle) - \gamma\hat{\rho}_{\alpha\beta}. \tag{3.19}$$

Applying this equation and taking into account Eq. (3.1) and (3.3) one readily obtain

$$\frac{d\hat{\rho}_{ab}}{dt} = g(\hat{\rho}_{ac}\hat{b}^\dagger + \hat{a}\hat{\rho}_{bb} - \hat{\rho}_{aa}\hat{a}) - \gamma\hat{\rho}_{ab}, \tag{3.20}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = g(\hat{b}\hat{\rho}_{cc} - \hat{\rho}_{bb}\hat{b} - \hat{a}^\dagger\hat{\rho}_{ac}) - \gamma\hat{\rho}_{bc}, \tag{3.21}$$

$$\frac{d\hat{\rho}_{aa}}{dt} = r_a\hat{\rho}_{aa}^{(0)}\hat{\rho} + g(\hat{\rho}_{ab}\hat{a}^\dagger + \hat{a}\hat{\rho}_{ba}) - \gamma\hat{\rho}_{aa}, \tag{3.22}$$

$$\frac{d\hat{\rho}_{bb}}{dt} = g(\hat{\rho}_{bc}\hat{b}^\dagger + \hat{b}\hat{\rho}_{cb} - \hat{a}^\dagger\hat{\rho}_{ab} - \hat{\rho}_{ba}\hat{a}) - \gamma\hat{\rho}_{bb}, \tag{3.23}$$

$$\frac{d\hat{\rho}_{ac}}{dt} = r_a \hat{\rho}_{ac}^{(0)} \hat{\rho} + g(\hat{a}\hat{\rho}_{bc} - \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{ac}, \tag{3.24}$$

$$\frac{d\hat{\rho}_{cc}}{dt} = r_a \hat{\rho}_{cc}^{(0)} \hat{\rho} - g(\hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}. \tag{3.25}$$

We confine ourselves to linear analysis and this can be achieved by dropping the g terms in Eq. (3.22), (3.23), (3.24) and (3.25) and applying the first-order. Thus upon dropping the g-term and applying the first-order approximation scheme, we get from Eq. (3.22), (3.23), (3.24) and (3.25) that

$$\hat{\rho}_{aa} = \frac{r_a \hat{\rho}_{aa}^{(0)}}{\gamma}\hat{\rho}, \tag{3.26}$$

$$\hat{\rho}_{bb} = 0, \tag{3.27}$$

$$\hat{\rho}_{ac} = \frac{r_a \hat{\rho}_{ac}^{(0)}}{\gamma}\hat{\rho}, \tag{3.28}$$

$$\hat{\rho}_{cc} = \frac{r_a \hat{\rho}_{cc}^{(0)}}{\gamma}\hat{\rho}. \tag{3.29}$$

Now combination of Eq. (3.20), (3.26), (3.27) and (3.28) as well as Eq. (3.21), (3.27), (3.28) and (3.29) leads to

$$\frac{d\hat{\rho}_{ab}}{dt} = \frac{g r_a}{\gamma}\left(\hat{\rho}_{ac}^{(0)}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}_{aa}^{(0)}\hat{\rho}\hat{a}\right) - \gamma\hat{\rho}_{ab}, \tag{3.30}$$

$$\frac{d\hat{\rho}_{bc}}{dt} = \frac{g r_a}{\gamma}\left(\hat{\rho}_{cc}^{(0)}\hat{b}\hat{\rho} - \hat{\rho}_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right) - \gamma\hat{\rho}_{bc}. \tag{3.31}$$

Using once more the adiabatic approximation scheme, we see that

$$\hat{\rho}_{ab} = \frac{g r_a}{\gamma^2}\left(\hat{\rho}_{ac}^{(0)}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}_{aa}^{(0)}\hat{\rho}\hat{a}\right), \tag{3.32}$$

$$\hat{\rho}_{bc} = \frac{g r_a}{\gamma^2}\left(\hat{\rho}_{cc}^{(0)}\hat{b}\hat{\rho} - \hat{\rho}_{ac}^{(0)}\hat{a}^{\dagger}\hat{\rho}\right). \tag{3.33}$$

Finally on account of Eq. (3.32) and (3.33), the master equation for the cavity mode given by

Eq. (3.15) takes the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \frac{1}{2}A\hat{\rho}_{aa}^{(0)}(2\hat{a}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger} - \hat{a}\hat{a}^{\dagger}\hat{\rho}) \\ &+ \frac{1}{2}(A\hat{\rho}_{cc}^{(0)} + \kappa)(2\hat{b}\hat{\rho}\hat{b}^{\dagger} - \hat{\rho}\hat{b}^{\dagger}\hat{b} - \hat{b}^{\dagger}\hat{b}\hat{\rho}) \\ &+ \frac{1}{2}A\hat{\rho}_{ac}^{(0)}(\hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger} + \hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho} - 2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}) \\ &+ \frac{1}{2}A\hat{\rho}_{ca}^{(0)}(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}) \\ &+ \frac{1}{2}\kappa(2\hat{a}\hat{\rho}\hat{a}^{\dagger} - \hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}), \end{aligned} \tag{3.34}$$

where

$$A = 2r_a g^2/\gamma^2, \tag{3.35}$$

is the linear gain coefficient.

3.2 Operator Dynamics

Here we seek to find the equation of evolution of the expectation values of the cavity mode operators, and their steady state. Using the relation

$$\frac{d\langle A\rangle}{dt} = Tr(\frac{d\hat{\rho}}{dt}A), \tag{3.36}$$

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{a}(t)). \tag{3.37}$$

In view of Eq. (3.34) and (3.37), we find

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}\rangle &= \frac{1}{2}A\hat{\rho}_{aa}^{(0)}\ Tr\Big(2\hat{a}^{\dagger}\hat{\rho}\hat{a}\hat{a} - \hat{\rho}\hat{a}\hat{a}^{\dagger}\hat{a} - \hat{a}\hat{a}^{\dagger}\hat{\rho}\hat{a}\Big) \\ &+ \frac{1}{2}(A\hat{\rho}_{cc}^{(0)} + \kappa)\ Tr\Big(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a}\Big) \\ &+ \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\ Tr\Big(\hat{\rho}\hat{b}^{\dagger}\hat{a}^{\dagger}\hat{a} + \hat{b}^{\dagger}\hat{a}^{\dagger}\hat{\rho}\hat{a} - 2\hat{a}^{\dagger}\hat{\rho}\hat{b}^{\dagger}\hat{a}\Big) \\ &+ \frac{1}{2}A\hat{\rho}_{ca}^{(0)}\ Tr\Big(\hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{a}\hat{b}\hat{\rho}\hat{a} - 2\hat{b}\hat{\rho}\hat{a}\hat{a}\Big) \\ &+ \frac{1}{2}\kappa\ Tr\Big(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a}\Big), \end{aligned}$$

Using the cyclic property of trace operator, we get

$$\frac{d}{dt}\langle\hat{a}\rangle = \frac{1}{2}(A\hat{\rho}_{aa}^{(0)} - \kappa)\langle\hat{a}\rangle - \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\langle\hat{b}^{\dagger}\rangle. \tag{3.38}$$

In the same fashion, one can show that

$$\frac{d}{dt}\langle\hat{b}\rangle = \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\langle\hat{a}^{\dagger}\rangle - \frac{1}{2}(A\hat{\rho}_{cc}^{(0)} + \kappa)\langle\hat{b}\rangle, \tag{3.39}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger\hat{a}\rangle = (A\hat{\rho}_{aa}^{(0)} - \kappa)\langle\hat{a}^\dagger\hat{a}\rangle - \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle - \frac{1}{2}A\hat{\rho}_{ca}^{(0)}\langle\hat{a}\hat{b}\rangle + A\hat{\rho}_{aa}^{(0)}, \tag{3.40}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -(A\hat{\rho}_{cc}^{(0)} + \kappa)\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \frac{1}{2}A\hat{\rho}_{ca}^{(0)}\langle\hat{a}\hat{b}\rangle, \tag{3.41}$$

$$\frac{d}{dt}\langle\hat{a}\hat{b}\rangle = \frac{1}{2}\Big(A\hat{\rho}_{aa}^{(0)} - \kappa - (A\hat{\rho}_{cc}^{(0)} + \kappa)\Big)\langle\hat{a}\hat{b}\rangle + \frac{1}{2}A\hat{\rho}_{ac}^{(0)}\Big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{b}^\dagger\hat{b}\rangle + 1\Big). \tag{3.42}$$

On taking the steady-state solutions of the above equations it becomes

$$\langle\hat{a}\rangle = \frac{A\hat{\rho}_{ac}^{(0)}}{A\hat{\rho}_{aa}^{(0)} - \kappa}\langle\hat{b}^\dagger\rangle, \tag{3.43}$$

$$\langle\hat{b}\rangle = \frac{A\hat{\rho}_{ac}^{(0)}}{A\hat{\rho}_{cc}^{(0)} + \kappa}\langle\hat{a}^\dagger\rangle, \tag{3.44}$$

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{\frac{1}{2}A\hat{\rho}_{ac}^{(0)}\Big(\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle\Big)}{A\hat{\rho}_{aa}^{(0)} - \kappa} - \frac{A\hat{\rho}_{aa}^{(0)}}{A\hat{\rho}_{aa}^{(0)} - \kappa}, \tag{3.45}$$

$$\langle\hat{b}^\dagger\hat{b}\rangle = \frac{\frac{1}{2}A\hat{\rho}_{ac}^{(0)}\Big(\langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{a}\hat{b}\rangle\Big)}{A\hat{\rho}_{cc}^{(0)} + \kappa}, \tag{3.46}$$

$$\langle\hat{a}\hat{b}\rangle = \frac{-A\hat{\rho}_{ac}^{(0)}\Big(\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{b}^\dagger\hat{b}\rangle + 1\Big)}{(A\hat{\rho}_{aa}^{(0)} - \kappa) - (A\hat{\rho}_{cc}^{(0)} + \kappa)}, \tag{3.47}$$

Now it proves to be more convenient to introduce a new parameter defined by

$$\hat{\rho}_{aa}^{(0)} = \frac{1-\eta}{2}, \tag{3.48}$$

with $-1 \leq \eta \leq 1$. In view of the fact that $\hat{\rho}_{aa}^{(0)} + \hat{\rho}_{cc}^{(0)} = 1$

one easily finds,

$$\hat{\rho}_{cc}^{(0} = \frac{1+\eta}{2} \tag{3.49}$$

and

$$\hat{\rho}_{ac}^{(0)} = \frac{1}{2}\sqrt{1-\eta^2}. \tag{3.50}$$

Applying Eqs. (3.43) and (3.44) , we easily find

$$\langle\hat{a}\rangle = \langle\hat{b}\rangle = 0. \tag{3.51}$$

Moreover, using Eqs. (3.45), (3.46) (3.47)(3.48) (3.49) and (3.50) we get

$$\langle\hat{a}^\dagger\hat{a}\rangle = \frac{A^2\big(\eta^2-1\big)}{4\kappa\big(2\kappa + A\eta\big)} + \frac{A\big(1-\eta\big)\big(A\big(1+\eta\big) + 2\kappa\big)}{4\kappa\big(\kappa + A\eta\big)}, \tag{3.52}$$

$$\langle \hat{b}^\dagger \hat{b} \rangle = \frac{A^2\big(1-\eta^2\big)}{4\big(2\kappa + A\eta\big)\big(\kappa + A\eta\big)}, \tag{3.53}$$

$$\langle \hat{a}\hat{b} \rangle = \frac{A\sqrt{1-\eta^2}\ \big(A(1+\eta) + 2\kappa\big)}{4\big(2\kappa + A\eta\big)\big(\kappa + A\eta\big)}. \tag{3.54}$$

It can also be readily verified that

$$\langle \hat{a}^2 \rangle = \langle \hat{b}^2 \rangle = \langle \hat{a}^\dagger \hat{b} \rangle = 0. \tag{3.55}$$

We observe that Eqs. (3.52), (3.53) and (3.54) respectively represent the steady state mean photon number of the cavity modes a and b.

4

RESULT AND DISCUSSION

4.1 Quadrature Variance

We proceed to determine the quadrature variance that are produced by a non degenerate three-level cascade laser inside and outside the cavity which coupled to vacuum reservoir.

4.1.1 Quadrature Variance of Cavity Modes

The squeezing properties for the two-mode lights are described by two quadrature operators defined as,

$$\hat{c}_+ = \hat{c} + \hat{c}^\dagger, \tag{4.1}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{4.2}$$

with

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b}), \tag{4.3}$$

where $\hat{a}_+$ and $\hat{a}_-$ are Hermitian operators representing the physical quantities called plus and minus quadratures, respectively while $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of light obtained from non degenerate three-level laser. These quadrature operators satisfy the commutation relation

$$\left[\hat{c}_+, \hat{c}_-\right] = 2i.$$

The quadrature variance can be expressed in terms of the quadrature operators as

$$(\Delta\hat{c}_\pm)^2 = \langle\hat{c}_\pm^2\rangle - \langle\hat{c}_\pm\rangle^2. \tag{4.4}$$

Now with the aid of Eq (3.51) along with Eqs (4.1), (4.2) and (4.3), we easily see that

$$\langle\hat{c}_\pm\rangle^2 = 0 \tag{4.5}$$

and hence Eq (4.4) reduces to

$$(\Delta\hat{c}_{\pm})^2 = \langle \hat{c}_{\pm}^2 \rangle. \tag{4.6}$$

We note that

$$(\Delta\hat{c}_{\pm})^2 = \langle \hat{c}^{\dagger}\hat{c} \rangle + \langle \hat{c}\hat{c}^{\dagger} \rangle \pm \Big(\langle \hat{c}^{\dagger 2} \rangle + \langle \hat{c}^2 \rangle \Big). \tag{4.7}$$

Making use of Eqs (3.55), (4.1), (4.2) and (4.3), we find

$$(\Delta\hat{c}_{\pm})^2 = \langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{a}\hat{a}^{\dagger} \rangle + \langle \hat{b}^{\dagger}\hat{b}^{\dagger} \rangle + \langle \hat{b}\hat{b}^{\dagger} \rangle - (\langle \hat{a}^{\dagger}\hat{b}^{\dagger} \rangle + \langle \hat{b}^{\dagger}\hat{a}^{\dagger} \rangle + \langle \hat{a}\hat{b} \rangle + \langle \hat{b}\hat{a} \rangle). \tag{4.8}$$

On the basis of the boson commutation relation properties we have

$$[\hat{a}, \hat{a}^{\dagger}] = [\hat{b}, \hat{b}^{\dagger}] = 1 \;\; and \;\; [\hat{a}, \hat{b}] = 0, \tag{4.9}$$

then applying Eq (4.9), the intracavity quadrature variance of two-mode light becomes

$$(\Delta\hat{c}_{\pm})^2 = 1 + \langle \hat{a}^{\dagger}\hat{a} \rangle + \langle \hat{b}^{\dagger}\hat{b} \rangle \pm 2\langle \hat{a}\hat{b} \rangle. \tag{4.10}$$

Thus, with the help of Eqs. (3.52), (3.53) and (3.54), the above equation takes the form

$$\begin{aligned}(\Delta\hat{c}_{\pm})^2 = 1 &+ \frac{A^2\big(\eta^2 - 1\big)}{4\kappa\big(2\kappa + A\eta\big)} + \frac{A\big(1-\eta\big)\big(A\big(1+\eta\big) + 2\kappa\big)}{4\kappa\big(\kappa + A\eta\big)} \\ &+ \frac{A^2\big(1-\eta^2\big)}{4\big(2\kappa + A\eta\big)\big(\kappa + A\eta\big)} \pm \frac{A\sqrt{1-\eta^2}\,\big(A(1+\eta) + 2\kappa\big)}{2\big(2\kappa + A\eta\big)\big(\kappa + A\eta\big)}. \end{aligned} \tag{4.11}$$

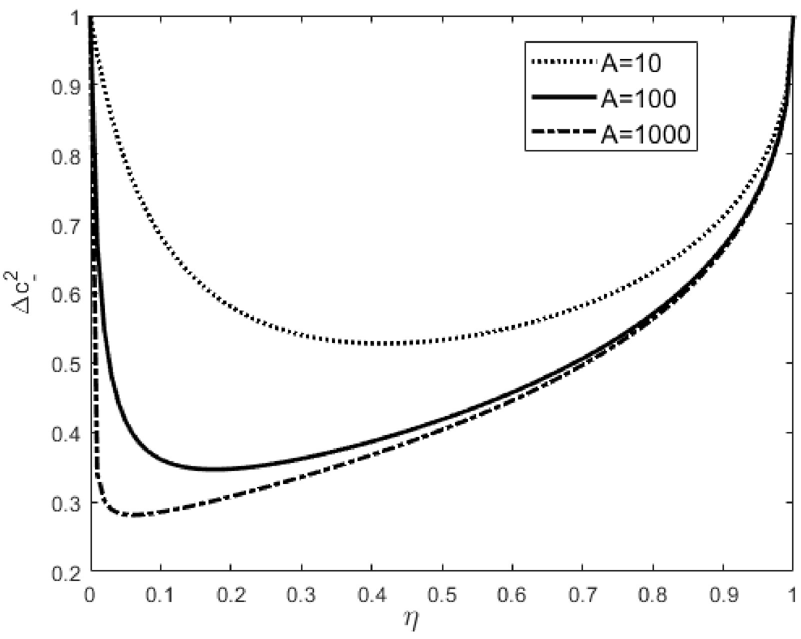

Fig. 4.1: Plots of the quadrature variance, Eq. (4.11), versus η for κ = 0.8 and for different values of the linear gain coefficient, A

We clearly see from Fig. (4.1) that the squeezing occurs at minus quadrature for values of η between 0 and 1. This corresponds to the case when the atoms are initially prepared in such a way that there are more atoms in the bottom level than the upper level. Moreover, it is possible to realize that the degree of squeezing increases with the linear gain coefficient (A).

4.1.2 Quadrature Variance of Output Modes

The variance of the output mode quadrature operators

$$\hat{c}_+^{out} = \hat{c}_{out} + \hat{c}_{out}^\dagger, \tag{4.12}$$

$$\hat{c}_-^{out} = i(\hat{c}_{out}^\dagger - \hat{c}_{out}), \tag{4.13}$$

in which

$$\hat{c}_{out} = \frac{1}{\sqrt{2}}(\hat{a}_{out} + \hat{b}_{out}), \tag{4.14}$$

with

$$\hat{a}_{out} = \sqrt{\kappa}\hat{a} - \hat{a}_{in} \quad and \quad \hat{b}_{out} = \sqrt{\kappa}\hat{b} - \hat{b}_{in}, \tag{4.15}$$

we note that for a cavity mode coupled to a vacuum reservoir, the output and intracavity variables are related by

$$\hat{a}_{out} = \sqrt{\kappa}\hat{a}, \quad and \quad \hat{b}_{out} = \sqrt{\kappa}\hat{b}, \tag{4.16}$$

can be expressed as

$$(\Delta\hat{c}_{\pm,out})^2 = \langle\hat{c}^2_{\pm,out}\rangle - \langle\hat{c}_{\pm,out}\rangle^2. \tag{4.17}$$

Now applying Eqs. (4.12), (4.13), (4.14), (4.16) and (4.5), the steady quadrature variance of the two-mode light outside the cavity can be put in the form,

$$(\Delta\hat{c}_{\pm,out})^2 = 1 + \kappa\langle\hat{a}^\dagger\hat{a}\rangle + \kappa\langle\hat{b}^\dagger\hat{b}\rangle \pm 2\kappa\langle\hat{a}\hat{b}\rangle. \tag{4.18}$$

Upon substituting Eqs. (3.52), (3.53) and (3.54) into Eq.(4.18), we get

$$\begin{aligned}(\Delta\hat{c}_{\pm,out})^2 = 1 &+ \frac{A^2(\eta^2-1)}{4(2\kappa+A\eta)} + \frac{A(1-\eta)(A(1+\eta)+2\kappa)}{4(\kappa+A\eta)} \\ &+ \frac{\kappa A^2(1-\eta^2)}{4(2\kappa+A\eta)(\kappa+A\eta)} \pm \frac{\kappa A\sqrt{1-\eta^2}\,(A(1+\eta)+2\kappa)}{2(2\kappa+A\eta)(\kappa+A\eta)}.\end{aligned} \tag{4.19}$$

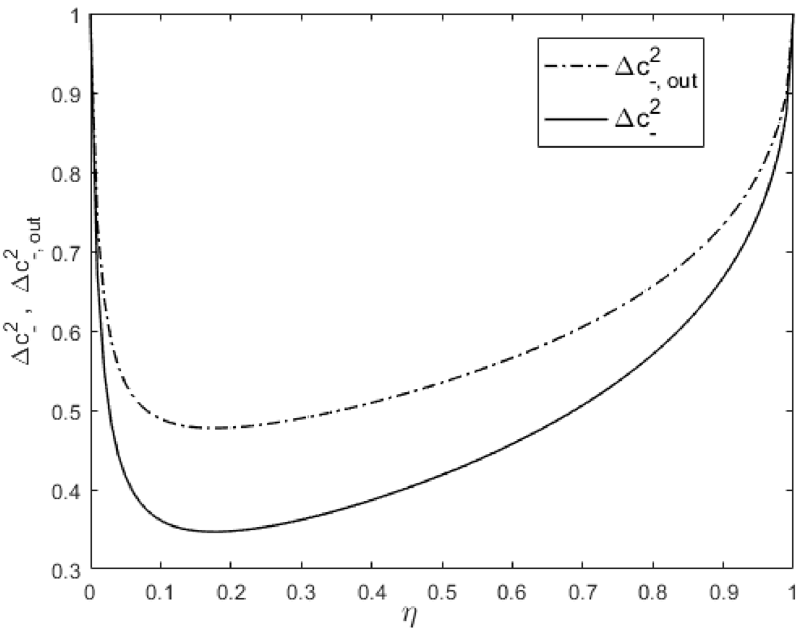

Fig. 4.2: Plots of the quadrature variance for cavity mode (Eq. (4.11)) and output mode (Eq. (4.19)), versus η for κ = 0.8 and A= 100

It can readily be observed from the plots in Fig. (4.2) that the degree of squeezing for outside the cavity is less than that of the cavity inside. It is known that the photons in the two cavity modes are generated pairwise and highly correlated and this correlation is the source of squeezing in this system. However, when photons leave the cavity through the port mirror, correlated pairs photons may not leave at the same time, and hence, there is some finite probability of observing an odd number of photons inside and outside the cavity. It is obvious that the probability of observing an odd number of photons outside the cavity is greater than the inside the cavity one.

4.2 Entanglement

In this section, we wish to investigate the entanglement properties of two modes light inside and outside the cavity at a steady state using Duan *et al* criteria.

4.2.1 Entanglement of the Cavity Modes

In this section we need to find the entanglement of the cavity modes, according to the criteria given by Duan *et al* [26], cavity photons of a system are entangled, if the sum of the variance of a pair of EPR-like operators

$$\hat{u} = \hat{x}_a - \hat{x}_b, \tag{4.20}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b, \tag{4.21}$$

with

$$\hat{x}_a = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}),$$
$$\hat{x}_b = \frac{1}{\sqrt{2}}(\hat{b}^\dagger + \hat{b}),$$
$$\hat{p}_a = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$$

and

$$\hat{p}_b = \frac{i}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}),$$

are quadrature operators for modes a and b , satisfy

$$(\Delta u)^2 + (\Delta v)^2 < 2 \tag{4.22}$$

and recalling that the cavity mode operators $\hat{a}$ and $\hat{b}$ are Gaussian variables with zero mean. The variance of the operators $\hat{u}$ and $\hat{v}$ can be put at steady state in the form

$$\begin{aligned}(\Delta u)^2 &= \langle \hat{u}^2 \rangle - \langle \hat{u} \rangle^2 \\ &= \langle (\hat{x}_a - \hat{x}_b)^2 \rangle - \langle \hat{x}_a - \hat{x}_b \rangle^2 \\ &= \langle \hat{x}_a^2 \rangle + \langle \hat{x}_b^2 \rangle - (\langle \hat{x}_a \hat{x}_b \rangle + \langle \hat{x}_b \hat{x}_a \rangle) \\ &- \Big[\langle \hat{x}_a \rangle^2 + \langle \hat{x}_b \rangle^2 - \big(\langle \hat{x}_a \rangle \langle \hat{x}_b \rangle + \langle \hat{x}_b \rangle \langle \hat{x}_a \rangle \big) \Big]. \end{aligned} \tag{4.23}$$

Substituting for $\hat{x}_a$, $\hat{x}_b$ and using equation Eqs. (3.51) and (3.55), one can get

$$(\Delta u)^2 = \frac{1}{2} \Big[\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle + \langle \hat{b} \hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{b} \rangle - \big(\langle \hat{a} \hat{b} \rangle + \langle \hat{b} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle + \langle \hat{b}^\dagger \hat{a}^\dagger \rangle \big) \Big]. \tag{4.24}$$

Following a similar step for $\hat{v}$, one can get

$$(\Delta v)^2 = \frac{1}{2}\Big[\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}\hat{a}^\dagger\rangle + \langle\hat{b}\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - \big(\langle\hat{a}\hat{b}\rangle + \langle\hat{b}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle + \langle\hat{b}^\dagger\hat{a}^\dagger\rangle\big)\Big]. \tag{4.25}$$

Using Eq (4.9)we can find that

$$(\Delta u)^2 + (\Delta v)^2 = 2 + 2\Big[\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle - \big(\langle\hat{a}\hat{b}\rangle + \langle\hat{a}^\dagger\hat{b}^\dagger\rangle\big)\Big]. \tag{4.26}$$

Now making use of Eqs (3.52), (3.53) and (3.54), we see that

$$\begin{aligned}(\Delta u)^2 + (\Delta v)^2 &= 2(\Delta c_-^2)\\ &= 2 + \frac{A^2\big(\eta^2-1\big)}{2\kappa\big(2\kappa+A\eta\big)} + \frac{A\big(1-\eta\big)\big(A\big(1+\eta\big)+2\kappa\big)}{2\kappa\big(\kappa+A\eta\big)}\\ &+ \frac{A^2\big(1-\eta^2\big)}{2\big(2\kappa+A\eta\big)\big(\kappa+A\eta\big)} - \frac{A\sqrt{1-\eta^2}\,\big(A(1+\eta)+2\kappa\big)}{\big(2\kappa+A\eta\big)\big(\kappa+A\eta\big)}.\end{aligned} \tag{4.27}$$

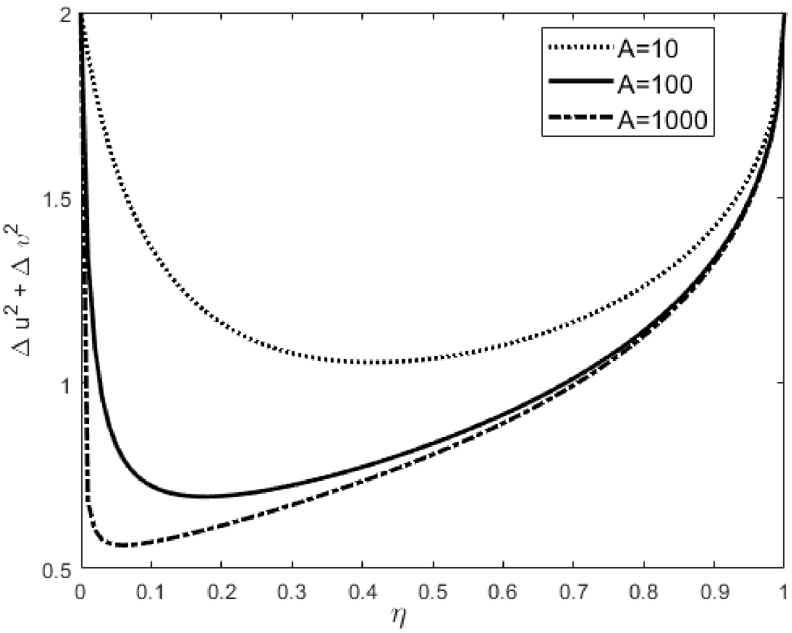

Fig. 4.3: Plots of the variance of the sum of EPR-likes operators for the light modes in the cavity Eq. (4.27) ($(\Delta u)^2 + (\Delta v)^2$) at steady state versus vs η for κ =0.8 and A=10, 100 and 1000

We easily see from Fig. (4.3) that $(\Delta u)^2 + (\Delta v)^2$ is less than 2 for all values of η except for η = 0 and 1, hence the entanglement criterion (Eq. (4.22)) is satisfied. One can immediately notice that, this particular entanglement measure is directly related the two-mode squeezing. This direct relationship shows that whenever there is a two-mode squeezing in the system there will be entanglement in the system as well. It is noted that the entanglement disappears when the squeezing vanishes at η=0 and η=1. This is due to the fact that the entanglement is directly related to the squeezing as given by Eq. (4.27). Moreover, the degree of entanglement increases with the linear gain coefficient.

4.2.2 Entanglement of the Output Modes

The EPR-like operators for the output modes have the form

$$\hat{u}^{out} = \hat{x}_a^{out} - \hat{x}_b^{out}, \tag{4.28}$$

$$\hat{v}^{out} = \hat{p}_a^{out} + \hat{p}_b^{out}, \tag{4.29}$$

in which

$$\hat{x}_a^{out} = \frac{\sqrt{\kappa}}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}),$$
$$\hat{x}_b^{out} = \frac{\sqrt{\kappa}}{\sqrt{2}}(\hat{b}^\dagger + \hat{b}),$$
$$\hat{p}_a^{out} = \frac{i\sqrt{\kappa}}{\sqrt{2}}(\hat{a}^\dagger - \hat{a})$$

and

$$\hat{p}_b^{out} = \frac{i\sqrt{\kappa}}{\sqrt{2}}(\hat{b}^\dagger - \hat{b}),$$

are quadrature operators for the output modes. Hence, the entanglement criterion for the output modes can be written as

$$\Delta u_{out}^2 + \Delta v_{out}^2 < 2. \tag{4.30}$$

The steady-state variance of the operators $\hat{u}_{out}$ and $\hat{v}_{out}$ can then be expressed as

$$\Delta u_{out}^2 = \Delta v_{out}^2 = 2 + 2\kappa\langle\hat{a}^\dagger\hat{a}\rangle + 2\kappa\langle\hat{b}^\dagger\hat{b}\rangle - 4\kappa\langle\hat{a}\hat{b}\rangle, \tag{4.31}$$

so that on account of Eqs. (3.52), (3.53) and (3.54), the total variance of the operators $\hat{u}_{out}$ and $\hat{v}_{out}$ becomes

$$\begin{aligned}(\Delta u)_{out}^2 + (\Delta v)_{out}^2 = 2 &+ \frac{A^2(\eta^2-1)}{2(2\kappa + A\eta)} + \frac{A(1-\eta)\big(A(1+\eta)+2\kappa\big)}{2(\kappa + A\eta)} \\ &+ \frac{\kappa A^2(1-\eta^2)}{2(2\kappa+A\eta)(\kappa+A\eta)} - \frac{\kappa A\sqrt{1-\eta^2}\,\big(A(1+\eta)+2\kappa\big)}{(2\kappa+A\eta)(\kappa+A\eta)}.\end{aligned} \tag{4.32}$$

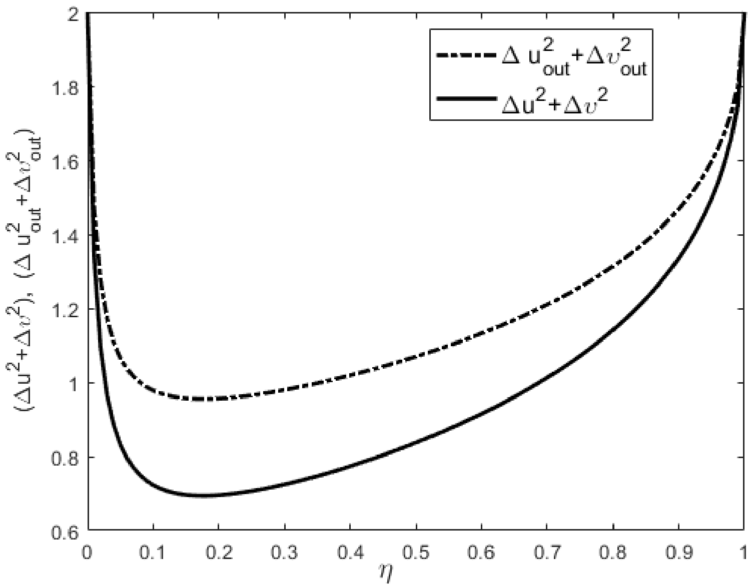

Fig. 4.4: Plots of the variance of the sum of EPR-likes operators for both of the light modes in the cavity Eq (4.27) $((\Delta u)^2 + (\Delta v)^2)$ and out side the cavity Eq. (4.32) $((\Delta u)^2_{out} + (\Delta v)^2_{out})$ at steady state versus vs η for κ =0.8 and A=100

Plots in Fig. (4.4) show that the entanglement criterion for the output modes, Eq. (4.30), is satisfied (except η= 0 and 1 as before). In addition, as can be seen from the same figure, the degree of entanglement of the output modes is less than that of the cavity modes. This is because the degree of squeezing of the output modes is less than that of the cavity modes.

4.3 Discord

Quantum discord measures the non-classical correlations which goes beyond quantum entanglement in a quantum state of a composite system. The Gaussian quantum discord can be measured in the domain of generalized Gaussian measurements on a bipartite quantum system of two-mode covariance matrix

$$\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^T & \beta \end{pmatrix}, \tag{4.33}$$

where α=diag$(\sigma_{11}, \sigma_{22})$, β=diag$(\sigma_{33}, \sigma_{44})$, γ=diag$(\sigma_{13}, \sigma_{24})$ with

$$\sigma_{ij} = \frac{1}{2}(\langle \hat{x}_i \hat{x}_j + \hat{x}_j \hat{x}_i \rangle) - \langle \hat{x}_i \hat{x}_j \rangle \tag{4.34}$$

and the corresponding quadrature operators are $\hat{x}_1 = \hat{a} + \hat{a}^\dagger$, $\hat{x}_2 = i(\hat{a}^\dagger - \hat{a})$, $\hat{x}_3 = \hat{b} + \hat{b}^\dagger$ and $\hat{x}_4 = i(\hat{b}^\dagger - \hat{b})$.

The covariance matrix corresponds to a physical state provided that $\lambda_\pm \geq 1$, where the symplectic eigenvalues are given by

$$\lambda_\pm^2 = \frac{\Delta \pm \sqrt{\Delta^2 - 4D}}{2}, \tag{4.35}$$

with Δ = A+B+2C in which A = detα, B = detβ, C = detγ and D = detσ.

The expanded covariance matrix is

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}. \tag{4.36}$$

4.3.1 Discord of the Cavity Mode

Here we want to obtain the quantum discord for non degenerate three level cascade laser coupled to vacuum reservoir of the cavity mode. By using Eq (4.34) and quadrature operators, we can find the submatrices

$$\alpha = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad where \;\; m = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle, \tag{4.37}$$

$$\beta = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}, \quad where \;\; n = \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{b} \hat{b}^\dagger \rangle, \tag{4.38}$$

$$\gamma = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \quad where \;\; c = \langle \hat{a}\hat{b} \rangle + \langle \hat{a}^\dagger \hat{b}^\dagger \rangle. \tag{4.39}$$

Furthermore, making use of Eqs (4.33), (4.37), (4.38) and (4.39)

$$A = det\alpha = 4\langle \hat{a}^\dagger \hat{a} \rangle^2 + 4\langle \hat{a}^\dagger \hat{a} \rangle + 1, \tag{4.40}$$

$$B = det\beta = 4\langle \hat{b}^\dagger \hat{b} \rangle^2 + 4\langle \hat{b}^\dagger \hat{b} \rangle + 1, \tag{4.41}$$

$$C = det\gamma = -4\langle \hat{a}\hat{b} \rangle^2, \tag{4.42}$$

$$D = det\sigma = 16\left[\frac{1}{4} + \frac{1}{2}\Big(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle\Big) + \langle \hat{a}^\dagger \hat{a} \rangle \langle \hat{b}^\dagger \hat{b} \rangle - \Big(\langle \hat{a}\hat{b} \rangle\Big)^2\right]^2. \tag{4.43}$$

For the Gaussian variable the two-mode quantum discord is expressed as

$$D(\rho_{AB}) = f(\sqrt{A}) - f(\lambda_+) - f(\lambda_-) + f(\frac{\sqrt{A} + 2\sqrt{AB} + 2C}{1 + 2\sqrt{A}}), \tag{4.44}$$

with

$$f(x) = \Big(x + \frac{1}{2}\Big) \log \Big(x + \frac{1}{2}\Big) - \Big(x - \frac{1}{2}\Big) \log \Big(x - \frac{1}{2}\Big). \tag{4.45}$$

When $D_{B/A} = 0$ it represent classical behavior and $D_{B/A} > 0$ indicate quantumness or move generally non-classical correlation. Obviously, upon replacing A by B one can go from the A discord to the B discord. In order to analyze the nature of the quantum discord in depth, D_A is plotted against η using Eqs (3.52), (3.53), (3.54), (4.40), (4.41), (4.42), (4.43) and (4.45) along with Eq. (4.44)

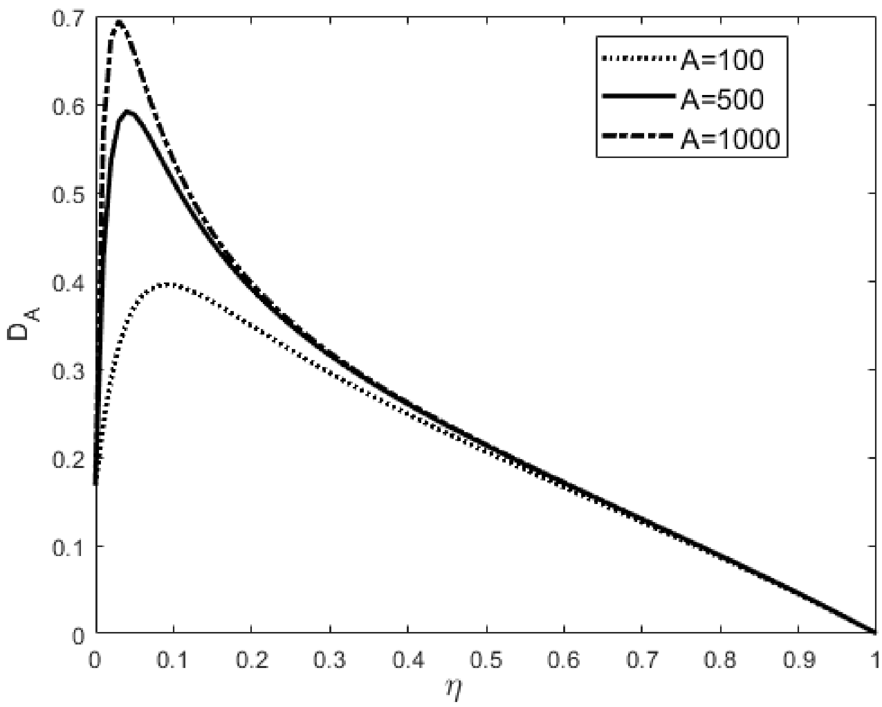

Fig. 4.5: Plots of the forward quantum discord (D_A) of the cavity radiation vs η for κ =0.8 and different values of A

From Fig. (4.5) we observe that the correlated radiation of cavity mode exhibits quantum features as expected and it is clearly shown in figure that the quantum discord increases with the linear gain coefficient (Eq. (3.35)). This entails that the quantum discord increases with the rate at which the atoms are injected into the cavity and the strength of the coupling of the atoms with the cavity radiation, but it decreases with the rate at which the atoms spontaneously decay to the energy levels that do not involve in the lasing process. Quite remarkably, the quantum discord is found to be zero for η=1, irrespective of the values of the linear gain coefficient. This result mainly asserts that if the atoms are initially prepared in the lower energy level (η=1), there is no way that they emit a radiation capable of forging the nonclassical correlations. Since there is no radiation in the cavity in this case, obviously nonclassical correlation is not expected. It is also vividly indicated that the quantum discord is almost independent of the linear gain coefficient except for when the atoms are initially prepared in a relatively stronger coherent superposition (η small) except η=0.

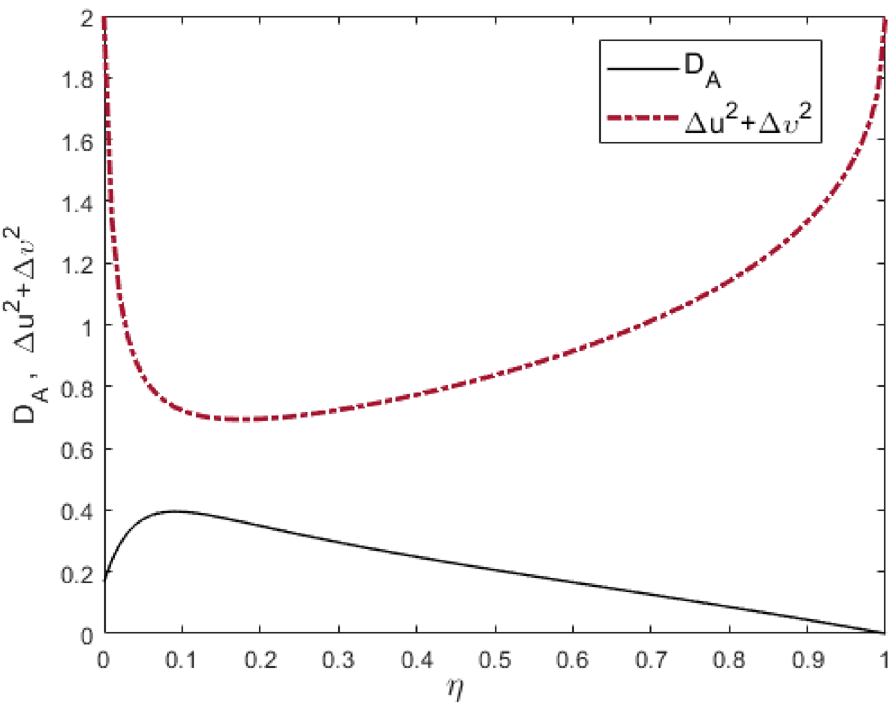

Fig. 4.6: Plots of the forward discord (D_A) and the variance of the EPR-operators ($\Delta u^2 + \Delta \upsilon^2$) of the cavity radiation vs η for κ =0.8 and A=100

In Fig (4.6) according to Duan *et al* criteria there is no entanglement at η=0, but the quantum discord is found at that point. This reveals that, quantum discord measures a very general form of non-classical correlation, which can be present in quantum systems even in the absence of entanglement.

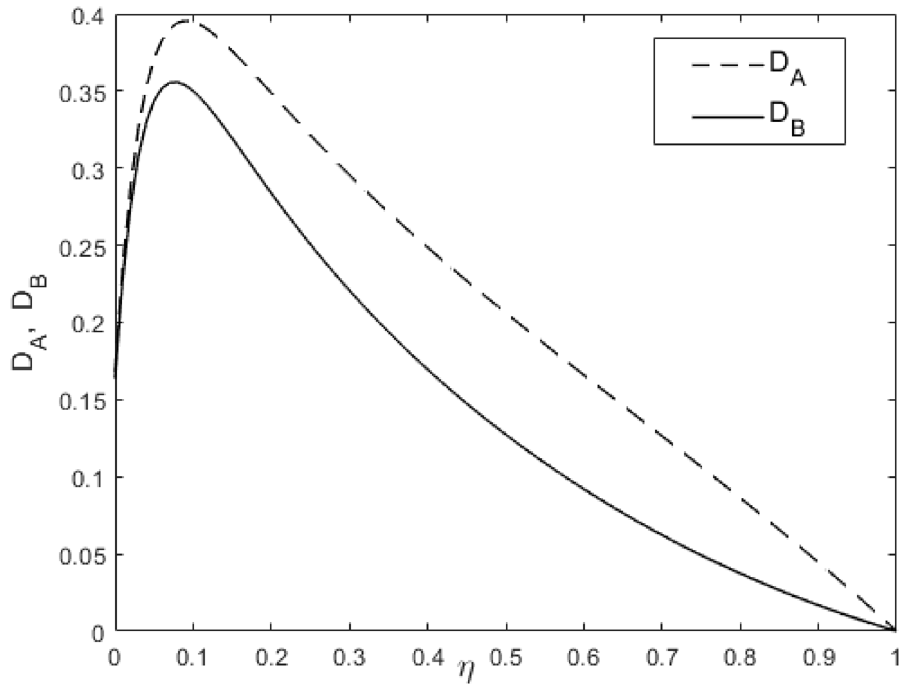

Fig. 4.7: Plots of the forward and backward quantum discord (D_A and D_B) of the cavity radiation vs η for κ =0.8 and A=100

Fig (4.7) show that, the forward quantum discord is greater than the backward quantum discord. This is due to that the mean photon number of mode a is greater than that of mode b.

4.3.2 Discord of the Output Mode

In here we wish to obtain the quantum discord for non degenerate three level cascade laser coupled to vacuum reservoir of the output mode.

$$m = 1 + \kappa\langle\hat{a}^{\dagger}\hat{a}\rangle, \tag{4.46}$$

$$n = 1 + \kappa\langle\hat{b}^{\dagger}\hat{b}\rangle, \tag{4.47}$$

$$c = \kappa\Big(\langle\hat{a}\hat{b}\rangle + \langle\hat{a}^{\dagger}\hat{b}^{\dagger}\rangle\Big). \tag{4.48}$$

With the help of Eqs (4.33), (4.46), (4.47) and (4.48)

$$A = det\alpha = 4\kappa^2\langle\hat{a}^{\dagger}\hat{a}\rangle^2 + 4\kappa\langle\hat{a}^{\dagger}\hat{a}\rangle + 1, \tag{4.49}$$

$$B = det\beta = 4\kappa^2\langle\hat{b}^\dagger\hat{b}\rangle^2 + 4\kappa\langle\hat{b}^\dagger\hat{b}\rangle + 1, \tag{4.50}$$

$$C = det\gamma = -4\kappa^2\langle\hat{a}\hat{b}\rangle^2, \tag{4.51}$$

$$D = det\sigma = 16\left[\frac{1}{4} + \frac{1}{2}\kappa\Big(\langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{b}^\dagger\hat{b}\rangle\Big) + \kappa^2\Big(\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle\Big) - \Big(\kappa\langle\hat{a}\hat{b}\rangle\Big)^2\right]^2. \tag{4.52}$$

To analyze the nature of quantum discord for the output mode, D_A is plotted against η using Eqs (3.52), (3.53), (3.54), (4.49), (4.50), (4.51), (4.52) and (4.45) along with Eq. (4.44)

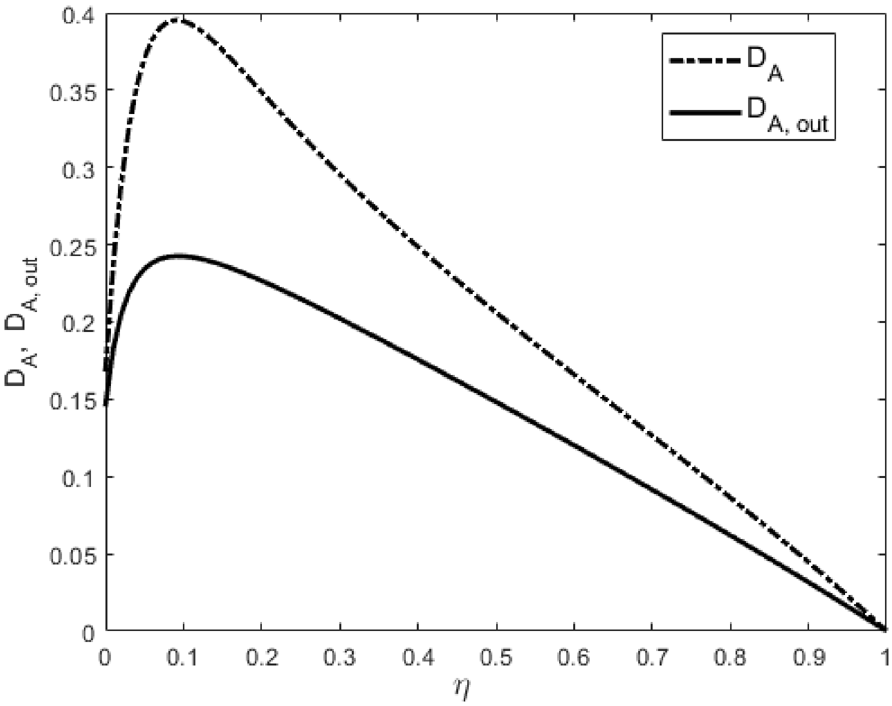

Fig. 4.8: Plots of quantum discord for inside and outside (D_A and $D_{A,out}$) of the cavity radiation vs η for κ =0.8 and A= 100

From Fig.(4.8) we clearly observe, quantum discord of the cavity mode is greater than discord the output mode this is due to the probability of observing an odd number of photons outside the cavity is greater than inside the cavity.

4.4 Steering

Conjuring quantum entanglement as a paradox by EPR [10], Schrodinger [12] introduced another kind of quantum correlation, the so-called, quantum steering, to support quantum information theory. According to quantum steering, through a local operation, one party of a bipartite entangled quantum system steers other party at a non-local distance. Distinct from quantum entanglement, steering exhibits an intrinsic asymmetry elucidating a novel behavior of non-local correlation [40]. As a result, new techniques of the one-sided device-independent quantum key distribution [37] and quantum teleportation [41] are devised. Using continuous variables (CV) quantum systems, quantum steering has been studied both theoretically and experimentally [42]. To this aim, Gaussian quantum steering have been widely investigated [43], the condition becomes necessary and sufficient for steering $(A \rightarrow B)$ for two mode gaussian system, this is called steerability from $(A \rightarrow B)$

$$G^{A \rightarrow B}(\sigma) = max\Big\{0, \frac{1}{2}\ln\Big(\frac{det\alpha}{det\sigma}\Big)\Big\}. \tag{4.53}$$

Every Covariant matrix σ (Eq (4.33)) that corresponds to a physical quantum state has to satisfy the bona fide condition,

$$\Big[(m^2-1)(n^2-1)+2c^2-mnc^2+c^2(c^2-mn)\Big] \geq 0, \tag{4.54}$$

where m, n $\geq$1.

With the help of Eqs (4.37), (4.38) and (4.39) along with Eqs (3.52), (3.53) and (3.54) the bona-fide condition is satisfied. Using Eqs. (3.52), (3.53), (3.54), (4.40), (4.43) along with Eq. (4.53) and also using Eqs. (3.52), (3.53), (3.54), (4.49), (4.52) along with Eq. (4.53), we can analyze the steering of the cavity mode and the out put mode, respectively.

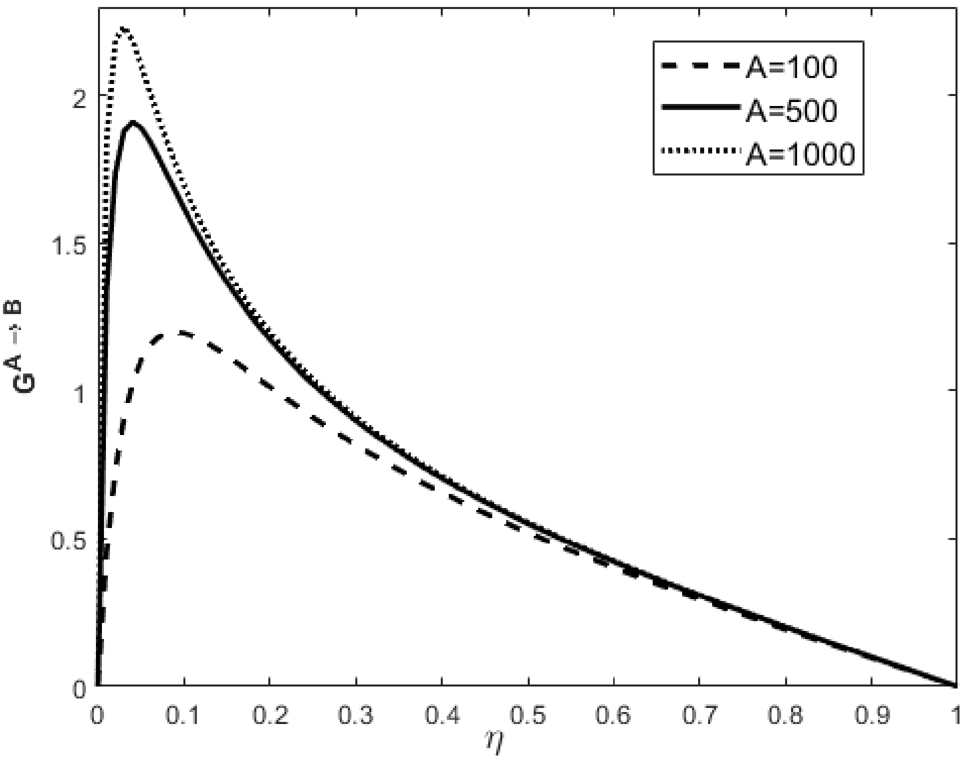

Fig. 4.9: Plots of quantum steering ($G^{A\to B}$) for the cavity mode vs η for κ =0.8 and A= 100, 500 and 1000

The plots in Fig. (4.9) shows that the steering ($G^{A\to B}$) at η=0 and η=1 is vanish as our entanglement is also vanish at the same point. This exhibit steerability reveals the presence of entanglement in cavity mode, because all steerable state are entangled state.

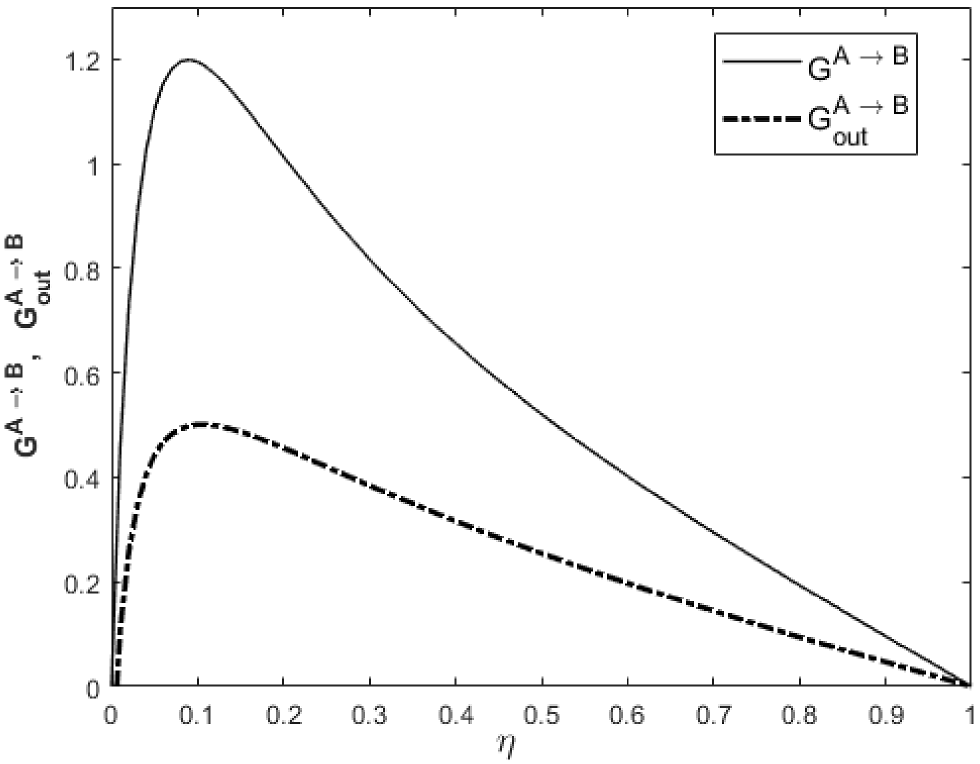

Fig. 4.10: Plots of quantum steering inside ($G^{A\to B}$) and outside ($G_{out}^{A\to B}$)of the cavity radiation vs η for κ =0.8 and A= 100

From Fig. (4.10) we clearly observe, quantum steering of the cavity mode ($G^{A\to B}$) is greater than the output mode ($G_{out}^{A\to B}$) this is due to the probability of observing an odd number of photons outside the cavity is greater than the inside the cavity one.

5

CONCLUSION

In this thesis quantum correlations such as squeezing, entanglement, discord and steering of a non degenerate three-level cascade laser in which the atoms was initially prepared in a coherent superposition of the top and bottom levels and the cavity is coupled to vacuum reservoir were analyzed for both cavity mode and output mode. We have obtained, using the master equation, steady state solutions of the expectation values of cavity operators. Applying this steady state solutions, we have calculated quadrature variance, entanglement, discord and steering of cavity and output modes generated by the system. Our results showed that the light generated by the system is in squeezing state. It was found that degree of squeezing increases with the linear gain coefficient (A) for small values of population inversion (η).

On the other hand, using Duan *et al* criteria we calculated the entanglement for cavity mode and the output mode. We clearly seen that entanglement is directly related to squeezing. Moreover, entanglement increases with linear gain coefficient (A) for slightly more atoms are initially prepared at bottom level (small value of η). In addition, we have also seen that the quantum discord increases linear gain coefficient (A) but disappears when all the atoms are initially prepared to be in the lower energy level. On the other hand we have seen the quantum discord can capture the quantumness of the radiation, when the entanglement criterion failed at η=0. We also analyzed the forward quantum discord (D_A) and backward quantum discord (D_B). Our result showed that the forward quantum discord (D_A) is greater to the backward quantum discord (D_B).

Finally we have evaluated the steering properties for cavity mode and output mode. For the cavity mode our steering increases with linear gain coefficient (A) for small values of population inversion (η), except at η=0 and η=1 which is the point both steering and entanglement

vanished. This revealed that the steering property exhibit in our system is a witness of entanglement.
We have noticed that for all quantum correlation properties the cavity mode is greater than that of the output mode for some values of population inversion (η).

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