COMPACT FINITE DIFFERENCE METHOD AND FINITE ELEMENT METHOD FOR SECOND ORDER WAVE EQUATION

BY

Hundasa Chala Nagari

A Thesis Submitted to Department of Applied Mathematics

School of Applied Natural Science

Office of Graduate Studies

Adama Science and Technology University

July, 2020

Adama, Ethiopia

COMPACT FINITE DIFFERENCE METHOD AND FINITE ELEMENT METHOD FOR SECOND ORDER WAVE EQUATION

By

Hundasa Chala Nagari

Advisor: Dr. Tekle Gemechu

A Thesis Submitted to Department of Applied Mathematics

School of Applied Natural Science

Office of Graduate Studies

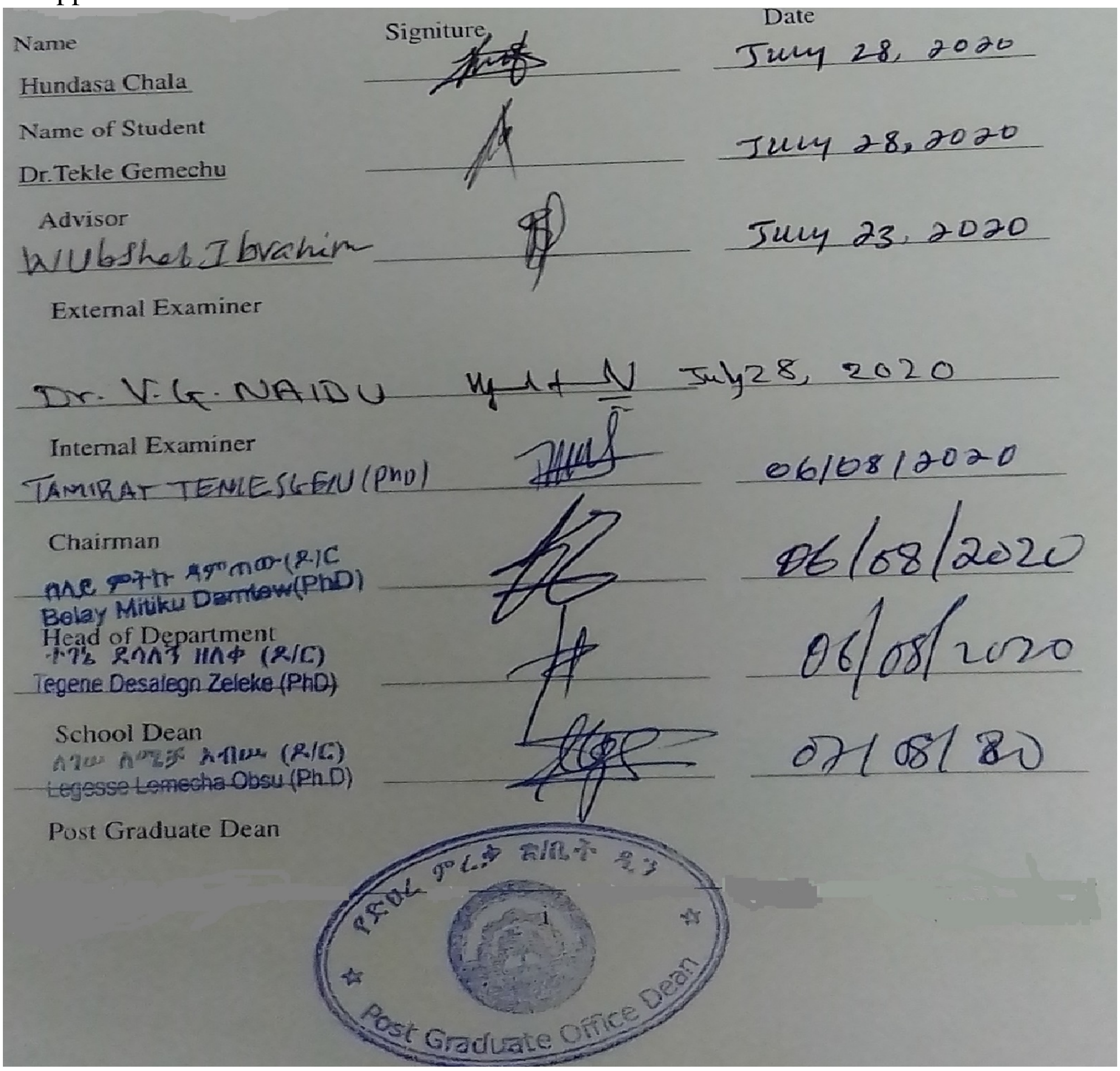
Adama Science and Technology University

July, 2020

Adama, Ethiopia

Approval Page

This is to certify that the thesis prepared by Hundasa Chala, entitled: Compact Finite Difference Method and Finite Element Method for Second Order Wave Equation and submitted in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics(Specialization in Numerical Analysis). As a member of the board of examiners of the MSc. Thesis open defense examination, we certify that we have read, evaluated the thesis prepared by Hundasa Chala and examined the candidate. We recommended that the thesis be accepted as fulfilling the thesis requirement for the degree of Master's of Science in Applied Mathematics.



Name	Signiture	Date
Hundasa Chala Name of Student		July 28, 2020
Dr.Tekle Gemechu Advisor		July 28, 2020
Wubshet Ibrahim External Examiner		July 23, 2020
Dr. V. G. NAIDU Internal Examiner		July 28, 2020
TAMIRAT TEMESGEN (PhD) Chairman		06/08/2020
በላይ ምትኩ ዳምጠው (ዶ/ር) Belay Mitiku Damtew(PhD) Head of Department		06/08/2020
ተገኜ ደሳለኝ ዘለቀ (ዶ/ር) Tegene Desalegn Zeleke (PhD) School Dean		06/08/2020
ለገሠ ለሜቻ ኦብሱ (ዶ/ር) Legesse Lemecha Obsu (Ph.D) Post Graduate Dean		07/08/20

Acknowledgment

First of all, I would like to express my deepest gratitude to God for giving me patience; I am also grateful to my advisor Dr. Tekle Gemechu for his helpful discussion, comments and providing the necessary materials in the preparation of this paper.

Secondly I would like to extend my thanks to my families and all who encouraged me to complete my project and their heart full help while writing the paper. Lastly I would like thanks to Department of mathematics, Adama Science and Technology University for giving the necessary materials throughout the preparation of this paper.

Table of Contents

List of Tables

List of Figures

List of Acronyms

CFDM	Compact Finite difference method
FEM	Finite element method
PDEs	Partial differential equations
BVP	Boundary Value Problem
1D	One Dimension
2D	Two Dimension
FDM	Finite Difference Method
BEM	Boundary Element Method
CPU	Central Processing Unit
FVM	Finite Volume Method

Abstract

In this thesis, the hyperbolic (PDE) or wave equation with Dirichlet boundary condition is solved numerically using the compact finite difference method (CFDM) and the finite element method (FEM) to approximate the analytical solution. To transform partial differential equation into weak formulation we used Galerkin finite element method. The discretizing procedure transforms the initial value problem of the PDE into a linear system of algebraic equations that can be solved by matrix inversion method. For finite element method, the convergence, stability and consistency depend on the trial solutions. Sixth order compact finite difference method and its stability, consistency and convergence were analyzed. It is based on meshes sizes. error analysis of compact finite difference method is based on Taylor's theorem. The test example was investigated and the sixth order compact finite difference method has small absolute and small maximum error related with Finite element method. Thus, sixth order compact finite difference method is more accurate than finite element method.

Chapter 1

Introduction

1.1 Background of the Study

Partial differential equations(PDEs) are equations that seeks unknown functions of several variables and their partial derivatives. Partial differential equations can be classified based on the properties of the characteristics of the equation. There are three basic types of equations: hyperbolic, parabolic and elliptic. Hyperbolic and parabolic equations are initial value problems, whereas an elliptic equation is a boundary value problem. The general form of second order linear partial differential equations is given as follows:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial t} + fu = 0 \tag{1.1}$$

Where u is dependent variable, x and t are independent variables a, b , c, d, e, f are constant coefficient of PDE. The summarized form of the classification of partial differential equations, characteristics directions, condition and its examples are given in table (1.1)

Table 1.1: Characteristics of Hyperbolic, Parabolic and Elliptic PDES

Type	Characteristic directions	Condition	Example
Hyperbolic	Real	$b^2 - 4ac > 0$	Wave equation
Parabolic	Imaginary	$b^2 - 4ac = 0$	Diffusion equation
Elliptic	non-existent	$b^2 - 4ac < 0$	Poisson equation

But in this study we considered only hyperbolic type of Partial differential equation which is wave equation. Wave equation is a hyperbolic second order linear partial differential equation which identifies the nature of waves appears in various physical phenomena given as vibration motion. Partial differential equation of hyperbolic type appearing in different fields like: sound wave, elastic, vibrations, fluid dynamics and so on[1] . In physics, propagation of sound, light and water waves are modeled by hyperbolic partial differential equations. The efficient and accurate numerical method for the wave equations are fundamental importance for the simulation of time dependent acoustic, electromagnetic or elastic wave phenomena [2].

The wave equation can be defined in n-dimensions as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \text{ with } c > 0 \tag{1.2}$$

where u is a function from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$ and Δ is the Laplacian operator over the spatial coordinates defined as

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{1.3}$$

A special case is the one-dimensional wave equation which can be more simply written as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ with } c > 0 \tag{1.4}$$

The function u may be the displacement of a string or membrane of a drum, the height of a water wave or air pressure for sound waves, dependent on the field the equation is used. The constant c in the above equations defines the speed at which the wave "moves". To see this we can look at the one dimensional wave equation for which, if we have no other conditions, $u(x,t) = g(x+ct)$ and $u(x,t) = f(x-ct)$ are solutions if the function $f(\eta)$ and $g(\mu)$ are twice differentiable. These two solutions have a constant shape which move in time with speed c to the left and right respectively. A general solution to the one dimensional wave equation was actually derived by d'Alembert and is given by $u(x,t) = f(x-ct) + g(x+ct)$, being the sum of a left and right moving wave.

The one-dimensional wave equations over a bounded domain D where the spatial boundary of the domain ∂D is fixed at zero and we have an initial value in time. These initial conditions are specified as follows

$$u(x,t) = 0 \text{ for all } x \in \partial D \text{ and } t \in \mathbb{R} \tag{1.5}$$

$$u(x,0) = u_0(x) \text{ for all } x \in D \text{ and} \tag{1.6}$$

$$\frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ for all } x \in D \tag{1.7}$$

The compact finite difference method is a newly developed numerical technique with excellent capabilities and resolution leading to it being in demand for application in different kinds of engineering problems. An efficient sixth order compact finite difference discretization was introduced for vibration analysis of a nonlocal Euler Bernoulli beam embedded in an elastic medium. Sixth order accuracy schemes were developed for discretization of both the governing equation and boundary conditions. Sixth order accuracy schemes were calculated from simply supported, damped and free boundary conditions. The numerical results included comparison with exact solutions and with previously published works were presented for the fundamental frequencies. And also demonstrated the effects of nonlocal parameter slenderness ratios and boundary conditions on the dynamics characteristics of the beam [8].

The two conservative and fourth order compact finite difference schemes were proposed and analyzed for solving the regularized long wave equation. The first compact finite difference scheme was two level and nonlinear implicit. The second scheme was three level and linearized implicit. Conservations of the discrete mass and energy, and unique solvability of the numerical solutions were proved. Convergence and unconditional stability were also derived with any restrictions on the grid ratios by using discrete energy method. The optimal errors norm $||.||$ and $||.||_{L^\infty}$ were the fourth order and second order accuracy for the spatial and temporal step sizes, respectively. Numerical examples were presented to simulate the collision of different solitary waves and support the theoretical analysis[10].

The finite element method (FEM) is the general method for the numerical solution of partial differential equations covering all three main types of equations, namely elliptic, parabolic, and hyperbolic equations. It can be implemented to any type of PDE. FEM is flexible and accurate method but requires a good knowledge in coding. The finite element method (FEM) is good enough to give a vision of numerical solution [13], and it was introduced by engineers in the late 1950's and early 1960's for the numerical solution of partial differential equations in structural engineering (elasticity equations, plate equations).

In the present work, we considered sixth order compact finite difference method and finite element method for solving 1D wave equation. The advantage of our work is that among the two method which one is more suitable to solve 1D wave equation. There was no comparison between sixth order compact finite difference method and finite element method to solve 1D wave equation. So, the result of this study was verified the comparison of sixth order compact finite difference method and finite element method.

1.2 Statement of the Problem

There are many methods to solve wave equation, such as finite element method, finite difference method and shooting method. The finite element method constitutes a general tool for the numerical solution of partial differential equations in engineering and applied science [12]. The proposed study intended to focus on comparison of finite element method and sixth order compact finite difference method for partial differential equation. A number of researchers have addressed solution of wave equation by using finite element method and other methods. But in each of the reviews, there is no solution of wave equation by using sixth order compact finite difference method and detailed comparison of these methods on their efficiency, accuracy and convergence with FEM. This study is intended to consider the following:

1. comparison of sixth order CFDM with FEM for wave equation.
2. The convergence and stability of sixth order CFDM for wave equation.
3. Developing matlab code for FEM and CFDM.
4. Analysis of finite element method for wave equation

1.3 Objective of the Study

1.3.1 General Objective

The general objective of this thesis is to compare finite element method and sixth order compact finite difference method solution for wave equation.

1.3.2 Specific objectives

The specific objectives of this study are to:

1. Determine the consistency, stability and convergence of sixth order CFDM.
2. Compare the results of FEM, sixth order CFDM with exact solution.
3. To test the validation of the methods with examples and write matlab code for the problem.

1.4 Significance of the Study

The main purposes of the study are:

- ✓ To contribute the knowledge of FEM and sixth order CFDM for wave equation for scientific community.
- ✓ For the growth of research in the area.

1.5 Limitation

There are problems or shortages under this study of research work. Some of them are:

- ✓ This study only comparsion of sixth order compact finite difference method and finite element method.
- ✓ This study only concerning the one dimension wave equation, anybody will study two or more dimension wave equation.

Chapter 2

Review of Related Literature

Hyperbolic partial differential equations (PDEs) play an important role in science, technology and arise very frequently in physical applications as models of waves, such as acoustic, elastic, seismic, shock, electromagnetic, and gravitational waves[1]. In fact, most of hyperbolic PDEs that arise in mathematical models of physical phenomena are very difficult to solve analytically, so numerical methods become necessary to approximate the solution of such hyperbolic PDEs. For many hyperbolic partial differential problems, FDM and FEM are the techniques of choice [14].

The second order PDEs with non-homogeneous derivative source terms are of common occurrence in mathematical physics and several other area of science and engineering such as flow and heat transfer through porous media. The fourth order CFDM was proposed for solving general second order steady PDE in 2D on geometries having nonuniform curvilinear grids. The authors focused on nonorthogonal curvilinear grids with the presence of mixed derivative term and non-homogeneous derivative source terms in the governing equation(i.e., Naiver-Stokes equations) on geometries having nonuniform and nonorthogonal curvilinear grids with Dirichlet as well as Neumann boundary conditions. Those authors compared the data produced by the proposed scheme for all the test cases were provided and with existing analytical solutions [11].

The development of numerical techniques for the solution of the hyperbolic nonlocal boundary value problems has been an important research topic in many branches of science and engineering . There are many papers that deal with the numerical solution of wave equations. For instance exponential B-spline collocation method was presented to estimate the solution of second order 1D nonlinear wave equation. Exponential B-splines were used for spatial variable and derivatives. That method produced two systems of first order ordinary differential equations and solved those systems using strong stability preserving methods. Numerical experiments were given to illustrate the accuracy and efficiency of the proposed method [3].

The numerical scheme introduced to solve the second-order wave equation with given initial boundary condition and an integral condition in place of the classical boundary condition. The shifted Legendre Tau method was applied to solve that problem. The stability and convergence of the Legendre Tau approximations made that approach very attractive and contributed to the good agreement between approximated and exact values for the numerical tests. The obtained results showed that the approach could solve the problem effectively and it needs less CPU time. That method described technique produced very accuracy results even when employing a small number of collocation points[4].

The FDM had been used to discrete the time derivative whereas quintic spline was applied as an interpolation function in the space dimension. The accuracy of the method by expanded the equation based on Taylor series and minimize the error. The proposed method had eighth order accuracy in space and fourth order accuracy in time variables. According to those authors the solution gotten by that method was best approximated among the previous works and also it was efficient to use. Numerical examples were given to show the applicability and efficiency of the method[5].

The CFDM is designed to obtain quick and accurate solutions to PDE problems [9]. Numerical results were presented to indicate the accuracy of the method based on the sixth-order discretization for predicting the vibrational response of embedded nanobeams subject to various boundary conditions and also they have introduced the comparison of the sixth order FDM and the exact solution [8]. But for 1D wave equation the authors consider fourth order CFDM rather than sixth order CFDM[6].

The FEM was based on Galerkin method to solve linear second order 1D inhomogeneous wave equation numerically with accuracy of the developed scheme had been analyzed by comparing the numerical solution with exact solution given by the authors in [2]. The numerical methods for solving the wave equation has been developed by many researchers. Accordingly, Karadoc and Bhowmik[17] have implemented a numerical technique supported a Petrov-Galerkin method using quadratic weight functions and cubic B-spline finite elements have been proffered to get numerical solution. They experimented with their algorithm along with single solitary wave during which the precise solution was thought and extended it to look at the interaction of two solitary waves and Maxwellian initial condition where the precise solutions were unknown during the interaction. Variational formulation and semi-discrete Galerkin schemes of the equation were generated. Stability analysis was done and also the linearized numerical scheme was obtained unconditionally stable. The author of [16] has discussed on BEM coupling with the FEM fictitious domain approach for the solution of the exterior Poisson problem and of wave scattering by rotating rigid bodies. Pandit and Chattopadhyay [15] have used a robust higher order compact scheme for solving general second order PDE with derivative source terms on nonuniform curvilinear meshes.

Recently, fourth order CFDM for solving homogeneous and non-homogeneous 1D wave equations was presented [6]. They have presented and analyzed a weak Galerkin FEM for solving the symmetric hyperbolic systems[7]. Many researcher solved one dimensional wave equation by using several numerical methods. However, they did not introduce comparison sixth order CFDM and FEM, also linear system obtained from both sixth order CFDM and FEM in previous works. In this study sixth order CFDM has small absolute error and maximum error relative to finite element method. Thus, the result of this study indicated sixth order compact finite difference method gives more accurate result than finite element method.

Chapter 3

Research Methodology

This section consists of methods and materials used to undertake the study. These are study design and Procedures of the study.

3.1 Study Design

The design of this study is solving by discretizing the partial differential equation (wave equation) using variational formulation and sparse system for CFDM. We also use graphical representation and matlab as a tool.

3.2 Procedures of the study

To attain the objective of the study, the following procedures were undertaken:

★ Defining the problem/formulating the method.

★ Discretizing the domain/interval.

★ Replacing the given equation by the finite element method and sixth order compact finite difference approximation.

★ Transform partial differential equation into weak formulation we use Galerkin's method.

★ Writing a code for the problem solving by using matlab language.

★ Validating the methods using numerical examples.

★ Discussing about the methods from the graphs.

Chapter 4

FEM and 6th order CFDM for 1D Wave equation

4.1 Finite Element Method for 1D Wave equation

The FEM is a numerical technique for solving problems which are described by PDES or can be formulated as functional minimization. A domain of interest is represented as an assembly of finite elements. Approximating functions in finite elements are determined in terms of nodal values of a physical field which is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown nodal values. Values inside finite elements can be recovered using nodal values. Several approaches can be used to transform the physical formulation of the problem to its finite element discrete analogue. If the mathematical formulation of the problem is known as a differential equation then the most popular method of its finite element formulation is the Galerkin method. If the physical problem can be formulated as minimization of a functional then variational formulation of the finite element equations is usually used[21].

Consider the second order one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < b \tag{4.1}$$

with boundary conditions

$$u(0,t) = u(b,t) = 0 \qquad 0 < t \leq T \tag{4.2}$$

and initial conditions

$$u(x,0) = \phi(x) \qquad 0 \leq x \leq b$$

and

$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad 0 \leq x \leq b$$

First we will derive a different but equivalent formulation for the 1D wave equation defined in equation (4.1). Let $W(x)$ be a differentiable functions such that $W(0) = W(b) = 0$ and therefore zero on the boundaries of the domain. By multiplying the wave equation with this function and integrating it over the $[0, b]$ interval we get following equation[21].

$$0 = \int_0^b W(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2})dx \tag{4.3}$$

This equation can be rewritten, using integration by parts, to

$$0 = \int_0^b W\frac{\partial^2 u}{\partial t^2}dx - [c^2 W\frac{\partial u}{\partial x}]_0^b + \int_0^b c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx$$

$$0 = \int_0^b W\frac{\partial^2 u}{\partial t^2}dx + \int_0^b c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx = a(u(x,t), W(x)) \tag{4.4}$$

$$\text{with } a(u(x,t), W(x)) : - \int_0^b W\frac{\partial^2 u}{\partial t^2}dx + \int_0^b c^2\frac{\partial W}{\partial x}\frac{\partial u}{\partial x}dx$$

For the integrals above to exist we require certain properties of $u,\ W$ and their derivatives. For this we will use the L_2-norm on a domain D. For a function f this is defined by

$$||f||_2 := \ \Big(\int_D |f(x)|^2 dx\Big)^{\frac{1}{2}}. \tag{4.5}$$

Using this norm a function f is called square-integrable if the L_2-norm of f is

finite or more specific.

$$||f||_2 < \infty \tag{4.6}$$

The set of all square-integrable functions is denoted by L_2. An important property of square-integrable functions is that the integral of the product of two square-integrable functions over the domain D is also finite[21]. This property is important for the existence of the integrals in equation (4.4).

We want the functions u, W and their first derivatives with respect to x to be square-integrable. The set of functions which are square-integrable and whose first derivative is square-integrable is called a **first order Sobolev space** and can be denoted by H^1 . The set of functions which are also zero at the boundary of the domain is denoted by H_0^1.

We now want to find a function u which is in H_0^1 for each moment in time t such that for every function W in H_0^1 the equations of (4.4) holds. This gives rise to the weak formulation of our one-dimensional wave equation, which can now be stated as follows.

Definition 4.1.1 (Weak formulation)**.** *[21] Find $u(x,t)$ such that for all $t \in [0,T]$ it holds that $u(.,t) \in H_0^1$ and for all $W \in H_0^1$*

$$a(u(x,t), W(x)) = 0 \tag{4.7}$$

and the initial boundary conditions equation(4.2) holds.

This is called the weak formulation as the function u only has to hold in eq.(4.7) with respect to certain functions W called test functions.
Basic steps of finite element method [21].

1. *Discretize the domain*. The first step is to divide the solution region into smaller regions that we call **elements**. The elements contain inside a certain number of points we call **nodes**.

2. *Select interpolation functions*. Interpolation functions are used to interpolate the field variables over the element. Often, polynomials are selected as interpolation functions. The degree of the polynomial depends on the number of nodes assigned to the element[21].

3. *Find the element properties*. The element equation for the finite element should be established which relates the nodal values of the unknown function to other parameters. For this task different approaches can be used; the most convenient are: the variational approach and the Galerkin method.

4. *Assemble the element equations*. To find the global equation system for the whole solution region we must assemble all the element equations. In other words we must combine local element equations for all elements used for discretization. Element connectivities are used for the assembly process. Before solution, boundary conditions (which are not accounted in element equations) should be imposed.

5. *Solve the global equation system.* The finite element global equation system is typically sparse, symmetric and positive definite. Inversion methods can be used for solution. The nodal values of the sought function are produced as a result of the solution.

4.1.1 Discretize the domain

Let us consider the global domain $[0, b] \times [0, T]$ as shown in figure (4.1) in which we have to approximate the solution of equation (4.1). We divide the global domain into finite number of rectangular elements. Let there be k nodes and $k-1$ linear elements in spatial direction. In the figure (4.1), i represents i^{th} node and (i) represents i^{th} the element. Each element has two nodes. The element (i) has left node i and right node $i+1$. The length of element i is given by $\Delta x_i = x_{i+1} - x_i$. Then we can apply central difference approximation for time and step size $\Delta t^n = t^{n+1} - t^n$,

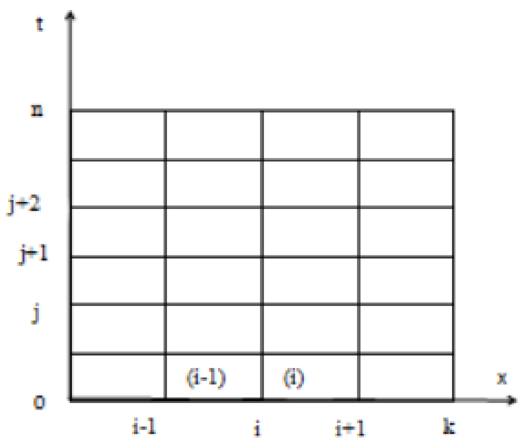


Figure 4.1: Discretization of $x - t$ plane

4.1.2 Interpolating Functions

Let us approximate the solution of equation (4.1) by $u(x,t)$[21], where $u(x,t)$ is defined as

$$u(x,t) = u^{(1)}(x,t) + \ldots + u^{(i)}(x,t) + \ldots + u^{(k-1)}(x,t) \tag{4.8}$$

where each $u^{(i)}(x,t)$ $(i = 1, 2, \ldots, k-1)$ represent the local interpolating polynomials over the element (i). Write $u^{(i)}(x,t)$ for i^{th} element as

$$u^{(i)}(x,t) = u_i(t)N_i^{(i)} + u_{i+1}(t)N_{i+1}^{(i)} \tag{4.9}$$

where $u_i(t)$ $i = 1, 2, \ldots, k$ represents the nodal values and $N_i^{(i)}$ and $N_{i+1}^{(i)}$ represent the shape functions for the element (i) at nodes i and $i+1$ respectively and $N_i's$ are Lagrangian polynomials of degree one. Basis functions are piecewise linear functions and the corresponding elements are line segments in 1D illustrated in figure 4.2.

We define

$$N_i^{(i)} = \frac{x - x_{i+1}}{x_i - x_{i+1}} = -\frac{x - x_{i+1}}{\Delta x_i} \tag{4.10}$$

and

$$N_{i+1}^{(i)} = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{\Delta x_i} \tag{4.11}$$

In algebraic form, these basis functions are defined as

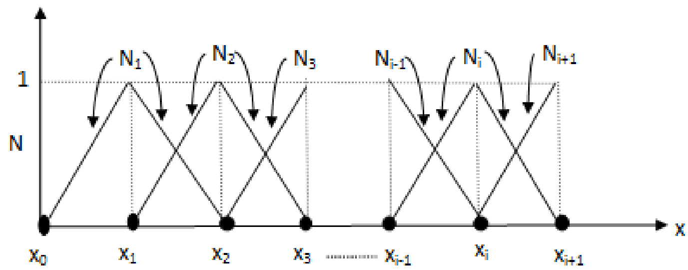


Figure 4.2: Finite element basis functions.

$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & , x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & , x_i \le x \le x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

Substituting the values from equations (4.10) and (4.11) into equation (4.9), we have

$$u^{(i)}(x,t) = u_i(-\frac{x - x_{i+1}}{\Delta x_i}) + u_{i+1}(\frac{x - x_i}{\Delta x_i}) \tag{4.13}$$

4.1.3 Element Equations

In this section we apply Galerkin method to approximate the solution of 1D wave equation given by eq.(4.1) i.e.,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Steps of Galerkin method for 1D wave equation.

1. Write 1D wave equation as residual form. The difference between the exact value and the approximate value is called the residual, we will denote it by R.

$$R(x,t) = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \tag{4.14}$$

2. Multiplying residual $R(x,t)$ by weighting functions W_k $(k = 1, 2, 3, \dots)$

3. Integrating that integral over $[0, b]$ and set this integral equal to zero in addition $W(x)$ take as the general weighting function.

$$l(u(x,t)) = \int_0^b W(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2})dx = 0$$

$$l(u(x,t)) = \int_0^b W \frac{\partial^2 u}{\partial t^2} dx - \int_0^b c^2 W \frac{\partial^2 u}{\partial x^2} dx = 0 \tag{4.15}$$

4. Applying boundary conditions. Now solving the second integral in (4.15) by parts That is

$$\int_0^b W \frac{\partial^2 u}{\partial t^2} dx = [c^2 W \frac{\partial u}{\partial x}]_0^b - \int_0^b c^2 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx \tag{4.16}$$

The factor $\frac{\partial u}{\partial x}$ in first term on R.H.S., of equation (4.16) cancels out at all interior points when we assemble the element equations. It exists only at first and last node when there are boundary conditions on derivatives. Therefore, we will drop this term so that equation (4.15) will take the form

$$l(u(x,t)) = \int_0^b W \frac{\partial^2 u}{\partial t^2} dx + \int_0^b c^2 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx \tag{4.17}$$

5. Expressed as sum of weighted residual integral of each element. Now the weighted residual integral $l(u(x,t))$ for the entire domain is expressed as sum of weighted residual integrals of each element (i) $(i = 1, 2, \dots, k-1)$.

$$l(u(x,t)) = l^{(1)}u(x,t) + \ldots + l^{(i)}u(x,t) + \ldots + l^{(k-1)}u(x,t) = 0 \tag{4.18}$$

where

$$l^{(i)}(u(x,t)) = \int_{x_i}^{x_{i+1}} W\frac{\partial^2 u^{(i)}}{\partial t^2}dx + \int_{x_i}^{x_{i+1}} c^2\frac{\partial W}{\partial x}\frac{\partial u^{(i)}}{\partial x}dx \tag{4.19}$$

To evaluate $l^{(i)}(u(x,t))$ given by equation (4.19), we require $u^{(i)}(x,t)$ and its partial derivatives with respect to t and differentiating (4.9) two times partially with respect to t and substituting values of $N_i's$ from equations (4.10) and (4.11) we get

$$\frac{\partial^2 u^{(i)}}{\partial t^2} = u_i''(-\frac{x-x_{i+1}}{\Delta x_i}) + u_{i+1}''(\frac{x-x_i}{\Delta x_i}) \tag{4.20}$$

Now differentiating equation (4.9) w.r.t, x

$$\frac{\partial u^{(i)}}{\partial x} = u_i(-\frac{1}{\Delta x_i}) + u_{i+1}(\frac{1}{\Delta x_i})$$

$$\frac{\partial u^{(i)}}{\partial x} = \frac{1}{\Delta x_i}(u_{i+1} - u_i) \tag{4.21}$$

Substituting the values from equations (4.20) and (4.21) into equation (4.19)

$$l^{(i)}(u(x,t)) = \int_{x_i}^{x_{i+1}} W[u_i''(-\frac{x-x_{i+1}}{\Delta x_i}) + u_{i+1}''(\frac{x-x_i}{\Delta x_i})]dx + \int_{x_i}^{x_{i+1}} c^2\frac{\partial W}{\partial x}[\frac{u_{i+1}-u_i}{\Delta x_i}]dx \tag{4.22}$$

Writing equation (4.22) symbolically as

$$l^{(i)}(u(x,t)) = A + B \tag{4.23}$$

Where A and B represents the integrals in equation (4.22). In Galerkin method, the weighted functions W_k $(k = 1, 2, \ldots,)$ are considered to be test functions. Here $N_i^{(i)}(x)$ and $N_{i+1}^{(i)}(x)$ are the shape function. Therefore, let

$W(x) = N_i^{(i)}(x) = -\frac{x-x_{i+1}}{\Delta x_i}$ and $\frac{\partial W}{\partial x} = -\frac{1}{\Delta x_i}$

Substituting the value of $W(x)$ and $\frac{\partial W}{\partial x}$ in equation (4.22) and solving the integrals for A and B

$$A = \int_{x_i}^{x_{i+1}} [-\frac{x-x_{i+1}}{\Delta x_i}][u_i''(-\frac{x-x_{i+1}}{\Delta x_i}) + u_{i+1}''(\frac{x-x_i}{\Delta x_i})]dx$$

$$A = \frac{\Delta x_i}{6}[2u_i'' + u_{i+1}''] \tag{4.24}$$

Next solving for B

$$B = \int_{x_i}^{x_{i+1}} c^2(-\frac{1}{\Delta x_i})[\frac{u_{i+1} - u_i}{\Delta x_i}]dx$$

$$B = -\frac{c^2}{\Delta x_i}[u_{i+1} - u_i] \tag{4.25}$$

Now put the values of equations (4.24)and (4.25) in equation (4.23) we have

$$l^{(i)}(u(x,t)) = \frac{\Delta x_i}{6}[2u_i'' + u_{i+1}''] - \frac{c^2}{\Delta x_i}[u_{i+1} - u_i] = 0 \tag{4.26}$$

Further we let, $W(x) = N_{i+1}^{(i)}(x)$, then

$$W(x) = \frac{x - x_i}{\Delta x_i} \text{ and } \frac{\partial W}{\partial x} = \frac{1}{\Delta x_i}$$

Next we use these values of $W(x)$ and $\frac{\partial W}{\partial x}$ equation (4.22), and solving for A and B we get.

$$A = \int_{x_i}^{x_{i+1}} [\frac{x - x_i}{\Delta x_i}][u_i''(-\frac{x - x_{i+1}}{\Delta x_i}) + u_{i+1}''(\frac{x - x_i}{\Delta x_i})]dx$$

$$A = \frac{\Delta x_i}{6}[u_i'' + 2u_{i+1}''] \tag{4.27}$$

Now

$$B = \int_{x_i}^{x_{i+1}} c^2(\frac{1}{\Delta x_i})[\frac{u_{i+1} - u_i}{\Delta x_i}]dx$$

$$B = \frac{c^2}{\Delta x_i}[u_{i+1} - u_i] \tag{4.28}$$

Now substituting values from equations (4.27) and (4.28) into equation (4.23), we have:

$$l^{(i)}(u(x,t)) = \frac{\Delta x_i}{6}[u_i'' + 2u_{i+1}''] + \frac{c^2}{\Delta x_i}[u_{i+1} - u_i] = 0 \tag{4.29}$$

equations (4.26) and (4.29) are the element equations for i^{th} element.

4.1.4 Assembly of Element Equations

Since i^{th} node is common between $(i-1)$ and (i) element therefore in order to get the nodal equation for i^{th} node we assemble the elements equation for node i in $(i-1)$ and (i) element. The physical domain for element $(i-1)$ and (i) is shown in figure (4.3).

As we can see from the figure (4.3) that the node $i+1$ for the element (i) corresponds to node i for the element $(i-1)$ and node i in element (i) corresponds to node $(i-1)$ in the

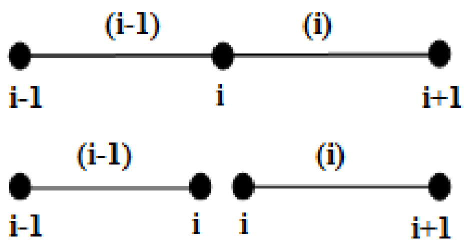


Figure 4.3: Linear Elements

element $(i-1)$. Therefore, to write the element equations for node i of element $(i-1)$ we will replace i by $i-1$ in the element equation of element (i) for the node $i+1$. Thus we have:

$$\frac{\Delta x_{i-1}}{6}[u''_{i-1} + 2u''_i] + \frac{c^2}{\Delta x_{i-1}}[u_i - u_{i-1}] = 0 \tag{4.30}$$

In general $\Delta x_i = x_{i+1} - x_i$ and $\Delta x_{i-1} = x_i - x_{i-1}$. Now let $\Delta x_i = \Delta x_+$ and $\Delta x_{i-1} = \Delta x_-$. So equations (4.26) and (4.30) for i^{th} node of (i) and $(i-1)$ element will be

$$\frac{\Delta x_+}{6}[2u''_i + u''_{i+1}] - \frac{c^2}{\Delta x_+}[u_{i+1} - u_i] = 0 \tag{4.31}$$

$$\frac{\Delta x_-}{6}[u''_{i-1} + 2u''_i] + \frac{c^2}{\Delta x_-}[u_i - u_{i-1}] = 0 \tag{4.32}$$

Multiplying equation (4.31) by $\frac{6}{\Delta x_+}$ and equation (4.32) by $\frac{6}{\Delta x_-}$ and then adding we get

$$[u''_{i-1} + 4u''_i + u''_{i+1}] + \frac{c^2}{(\Delta x_-)^2}[u_i - u_{i-1}] - \frac{c^2}{(\Delta x_+)^2}[u_{i+1} - u_i] = 0 \tag{4.33}$$

In eq.(4.33) u is differentiation with respect to t.

4.1.5 Approximation of Time Derivative

The central finite difference approximation for j^{th} time level is given as[21]:

$$u_{tt}(i,j) = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta t)^2} \tag{4.34}$$

Substitute equation (4.34) into equation (4.33). We get

$$\begin{aligned} 0 = {} & \frac{u_{i-1,j-1} - 2u_{i-1,j} + u_{i-1,j+1}}{(\Delta t)^2} + 4\frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta t)^2} + \frac{u_{i+1,j-1} - 2u_{i+1,j} + u_{i+1,j+1}}{(\Delta t)^2} \\ & + \frac{c^2}{(\Delta x_-)^2}[u_{i,j} - u_{i-1,j}] - \frac{c^2}{(\Delta x_+)^2}[u_{i+1,j} - u_{i,j}] \end{aligned} \tag{4.35}$$

By simplifying we get

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = (-u_{i-1,j-1} - 4u_{i,j-1} - u_{i+1,j-1}) + (2u_{i-1,j} + 8u_{i,j} + 2u_{i+1,j}) - (\frac{6c^2(\Delta t)^2}{(\Delta x_-)^2}[u_{i,j} - u_{i-1,j}]) + \frac{6c^2(\Delta t)^2}{(\Delta x_+)^2}[u_{i+1,j} - u_{i,j}] \tag{4.36}$$

Equation (4.36) represents the FEM for equation (4.1) when we have non-uniform grid. Now for uniform grids we have

$$\Delta x_+ = \Delta x_- = \Delta x$$

Therefore from equation(4.36) we have

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = (-u_{i-1,j-1} - 4u_{i,j-1} - u_{i+1,j-1}) + (2u_{i-1,j} + 82u_{i,j} + 2u_{i+1,j}) + (\frac{6c^2(\Delta t)^2}{(\Delta x)^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]) \tag{4.37}$$

Let $d = c^2\frac{(\Delta t)^2}{(\Delta x)^2}$. Therefore we have:

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = -(u_{i-1,j-1} + 4u_{i,j-1} + u_{i+1,j-1}) + (2+6d)u_{i-1,j} + (8-12d)u_{i,j} + (2+6d)u_{i+1,j} \tag{4.38}$$

Equation (4.38) represents the FEM for linear second order 1D wave equation with uniform mesh points and to be solved system of linear equations by using inversion method.

4.1.6 Convergence and Error Estimate for Finite Element Method

In this section we are going to bound the discretization error $e_h = u - u_h$ of Galerkin's method, consistency estimates on the discretization error in the solution space V in terms of "distance" between the solution space and the approximation space: in a Sobolev[18].

$$||u - u_h|| \leq C||v - v_h||_V \ v_h \in V$$

with $C > 0$ a constant real number.

Theorem 4.1.1 (Interpolation inequality in $H_0^1(D)$ and $L^2(D)$[18]). *let $v \in H^2(D), \exists C_1$ such that*

$$|v - v_h|_{H^1(D)} \leq C_1 h_\tau |v|_{H_2(D)} \tag{4.39}$$

and $\exists C_0 > 0$ such that

$$|v - v_h|_{L^2(D)} \leq C_0 h_\tau |v|_{H_2(D)} \tag{4.40}$$

with $h_\tau = \max_{K \in \tau_h}(h_K)$

Proof. The proof is based on the Mean-Value Theorem and a decomposition of the error per element. The global interpolation error is then recovered by summing over the cells. This makes sense since the polynomial estimate is defined point-wise: this is then a local property. In the same spirit the stability of the interpolation operator is also a local property, defined element-wise[18]. □

The discretization error being bounded in $O(h_\tau)$ the method is first order in $H_0^1(D)$.

Remark 4.1.1 (Convergence order $H^1(D)$). *On the other hand we know that $\exists C_1 > 0$ such that, $\forall v \in H^2(D)$:*

$$\|v - \pi_1 v\|_{L^2(\Omega)} \leq C_1 h_\tau^2 |v|_{H^2(D)}$$

Using the definition of the norm

$$\|v - \pi_1 v\|_{H^1(D)}^2 = \|v - \pi_1 v\|_{L^2(D)}^2 + \|v - \pi_1 v\|_{H^1(D)}^2$$

we get:

$$\|v - \pi_1 v\|_{H^1(\Omega)}^2 \leq C_1^2 (h_\tau^4 |v|_{H^2(D)} + h_\tau^2 |v|_{H^2(D)})$$

$$\|v - \pi_1 v\|_{H^1(\Omega)} \leq C_1 h_\tau (1 + h_\tau^2)^{\frac{1}{2}} |v|_{H^2(D)}$$

Thus, we verify that the approximation is also first order in $H^1(D)$.

4.2 Compact Finite Difference Method for 1D wave equation

Consider the one dimensional wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < b \tag{4.41}$$

with boundary conditions

$$u(0,t) = u(b,t) = 0 \qquad 0 < t \leq T \tag{4.42}$$

and initial conditions

$$u(x,0) = \phi(x) \ \ , \frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad 0 \leq x \leq b \tag{4.43}$$

where c is speed of wave, $\phi(x)$ is initial deflection (displacement) $\psi(x)$, initial velocity and $u(x,t)$ is the displacement with position x and time t. In order for this equation to be solvable the initial conditions Eq.(4.43) as well as the boundary conditions Eq.(4.42) should be provided[6]. To describe the scheme, we divide interval $[0,b]$ and $[0,T]$ in k and n equal subintervals of mesh length Δx and Δt respectively. Let $0 = x_0 < x_1 < \ldots < x_{k-1} = b$ and $0 = t_0 < t_1 < \ldots < t_{n-1} = T$ be the mesh points with $x_i = x_0 + i\Delta x$ and $t_j = t_0 + j\Delta t$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$. Assume that $u(x,t)$ has continuous higher order partial derivatives on the region $[0,b] \times [0,T]$. For the sake of simplicity we use $u(x_i,t_j) = u(i,j)$, $\frac{\partial^s u}{\partial x^s}(x_i,t_j) = \frac{\partial^s u}{\partial x^s}(i,j)$ and $\frac{\partial^s u}{\partial t^s}(x_i,t_j) = \frac{\partial^s u}{\partial t^s}(i,j)$ ($s \geq 1$ said to be s^{th} order derivatives) by using Taylor series expression, we have:

$$\begin{aligned} u(i+1,j) = u(i,j) + \Delta x\frac{\partial u}{\partial x}(i,j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}(i,j) + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}(i,j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}(i,j) + \\ \frac{(\Delta x)^5}{5!}\frac{\partial^5 u}{\partial x^5}(i,j) + \frac{(\Delta x)^6}{6!}\frac{\partial^6 u}{\partial x^6}(i,j) + \frac{(\Delta x)^7}{7!}\frac{\partial^7 u}{\partial x^7}(i,j) + \frac{(\Delta x)^8}{8!}\frac{\partial^8 u}{\partial x^8}(i,j) \\ + \frac{(\Delta x)^9}{9!}\frac{\partial^9 u}{\partial x^9}(i,j) + \ldots \end{aligned} \tag{4.44}$$

$$\begin{aligned} u(i-1,j) = u(i,j) - \Delta x\frac{\partial u}{\partial x}(i,j) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u}{\partial x^2}(i,j) - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u}{\partial x^3}(i,j) + \frac{(\Delta x)^4}{4!}\frac{\partial^4 u}{\partial x^4}(i,j) - \\ \frac{(\Delta x)^5}{5!}\frac{\partial^5 u}{\partial x^5}(i,j) + \frac{(\Delta x)^6}{6!}\frac{\partial^6 u}{\partial x^6}(i,j) - \frac{(\Delta x)^7}{7!}\frac{\partial^7 u}{\partial x^7}(i,j) + \frac{(\Delta x)^8}{8!}\frac{\partial^8 u}{\partial x^8}(i,j) \\ - \frac{(\Delta x)^9}{9!}\frac{\partial^9 u}{\partial x^9}(i,j) + \ldots \end{aligned} \tag{4.45}$$

$$u(i,j+1) = u(i,j) + \Delta t \frac{\partial u}{\partial t}(i,j) + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}(i,j) + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}(i,j) + \frac{(\Delta t)^4}{4!}\frac{\partial^4 u}{\partial t^4}(i,j) + \frac{(\Delta t)^5}{5!}\frac{\partial^5 u}{\partial t^5}(i,j) + \frac{(\Delta t)^6}{6!}\frac{\partial^6 u}{\partial t^6}(i,j) + \frac{(\Delta t)^7}{7!}\frac{\partial^7 u}{\partial t^7}(i,j) + \frac{(\Delta t)^8}{8!}\frac{\partial^8 u}{\partial t^8}(i,j) + \frac{(\Delta t)^9}{9!}\frac{\partial^9 u}{\partial t^9}(i,j) + \dots \quad (4.46)$$

$$u(i,j-1) = u(i,j) - \Delta t \frac{\partial u}{\partial t}(i,j) + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u}{\partial t^2}(i,j) - \frac{(\Delta t)^3}{3!}\frac{\partial^3 u}{\partial t^3}(i,j) + \frac{(\Delta t)^4}{4!}\frac{\partial^4 u}{\partial t^4}(i,j) - \frac{(\Delta t)^5}{5!}\frac{\partial^5 u}{\partial t^5}(i,j) + \frac{(\Delta t)^6}{6!}\frac{\partial^6 u}{\partial t^6}(i,j) - \frac{(\Delta t)^7}{7!}\frac{\partial^7 u}{\partial t^7}(i,j) + \frac{(\Delta t)^8}{8!}\frac{\partial^8 u}{\partial t^8}(i,j) - \frac{(\Delta t)^9}{9!}\frac{\partial^9 u}{\partial t^9}(i,j) + \dots \quad (4.47)$$

Subtracting Eq. (4.45) from Eq. (4.44) and Eq. (4.47) from Eq. (4.46), we obtain the second order finite $\delta_c^1 u(i,j)$ difference for the first derivative of $u(i,j)$:

$$\delta_{cx}^1 u(i,j) = \frac{u(i+1,j) - u(i-1,j)}{2\Delta x} + \tau_1 \text{ and } \delta_{ct}^1 u(i,j) = \frac{u(i,j+1) - u(i,j-1)}{2\Delta t} + \tau_2 \quad (4.48)$$

where $\tau_1 = -\frac{(\Delta x)^2}{6}\frac{\partial^3 u}{\partial x^3}(i,j)$ and $\tau_2 = -\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}(i,j)$.

Similarly, adding Eq. (4.44) with Eq. (4.45) and Eq. (4.46) with Eq. (4.47), we obtain the second order finite difference $\delta_c^2 u(i,j)$ for the second derivative of $u(i,j)$:

$$\delta_{cx}^2 u(i,j) = \frac{u(i+1,j) - 2u(i,j) + u(i-1,j)}{(\Delta x)^2} + \tau_3 \quad (4.49)$$

$$\delta_{ct}^2 u(i,j) = \frac{u(i,j+1) - 2u(i,j) + u(i,j-1)}{(\Delta t)^2} + \tau_4 \quad (4.50)$$

where $\tau_3 = -\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(i,j)$ and $\tau_4 = -\frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}(i,j)$.

Substituting Eqs. (4.44) - (4.47) into Eqs. (4.49) and (4.50) yields:

$$\delta_{cx}^2 u(i,j) = \frac{\partial^2 u}{\partial x^2}(i,j) + \frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(i,j) + \frac{(\Delta x)^4}{360}\frac{\partial^6 u}{\partial x^6}(i,j) + \tau_5 \quad (4.51)$$

where $\tau_5 = \frac{(\Delta x)^6}{20160}\frac{\partial^8 u}{\partial x^8}(i,j)$.

$$\delta_{ct}^2 u(i,j) = \frac{\partial^2 u}{\partial t^2}(i,j) + \frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}(i,j) + \frac{(\Delta t)^4}{360}\frac{\partial^6 u}{\partial t^6}(i,j) + \tau_6 \quad (4.52)$$

where $\tau_6 = \frac{(\Delta t)^6}{20160}\frac{\partial^8 u}{\partial t^8}(i,j)$.

Using from Eq.(4.41) and successive differentiation, we have:

$$\frac{\partial^4 u}{\partial x^4}(i,j) = \frac{1}{c^2}\frac{\partial^2}{\partial x^2}(\frac{\partial^2 u}{\partial t^2}(i,j)) = \frac{1}{c^2}\delta_{cx}^2(\delta_{ct}^2 u(i,j)) \tag{4.53}$$

$$\frac{\partial^6 u}{\partial x^6}(i,j) = \frac{1}{c^2}\frac{\partial^2}{\partial x^2}(\frac{\partial^2}{\partial x^2}(\frac{\partial^2 u}{\partial t^2}(i,j))) = \frac{1}{c^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) \tag{4.54}$$

$$\frac{\partial^4 u}{\partial t^4}(i,j) = c^2\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u}{\partial x^2}(i,j)) = c^2\delta_{ct}^2(\delta_{cx}^2 u(i,j)) \tag{4.55}$$

$$\frac{\partial^6 u}{\partial t^6}(i,j) = c^2\frac{\partial^2}{\partial t^2}(\frac{\partial^2}{\partial t^2}(\frac{\partial^2 u}{\partial x^2}(i,j))) = c^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) \tag{4.56}$$

Substituting Eqs. (4.53), (4.54), (4.55) and (4.56) into Eqs.(4.51) and (4.52) respectively, gives:

$$\delta_{cx}^2 u(i,j) = \frac{\partial^2 u}{\partial x^2}(i,j) + \frac{(\Delta x)^2}{12}\frac{1}{c^2}\delta_{cx}^2(\delta_{ct}^2 u(i,j)) + \frac{(\Delta x)^4}{360}\frac{1}{c^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) + \tau_5$$

$$\delta_{ct}^2 u(i,j) = \frac{\partial^2 u}{\partial t^2}(i,j) + \frac{(\Delta t)^2}{12}c^2\delta_{ct}^2(\delta_{cx}^2 u(i,j)) + \frac{(\Delta t)^4}{360}c^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) + \tau_6$$

$$\frac{\partial^2 u}{\partial x^2}(i,j) = \delta_{cx}^2 u(i,j) - \frac{(\Delta x)^2}{12}\frac{1}{c^2}\delta_{cx}^2(\delta_{ct}^2 u(i,j)) - \frac{(\Delta x)^4}{360}\frac{1}{c^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) - \tau_5 \tag{4.57}$$

$$\frac{\partial^2 u}{\partial t^2}(i,j) = \delta_{ct}^2 u(i,j) - \frac{(\Delta t)^2}{12}c^2\delta_{ct}^2(\delta_{cx}^2 u(i,j)) - \frac{(\Delta t)^4}{360}c^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) - \tau_6 \tag{4.58}$$

Again, substituting Eqs.(4.57) and (4.58) into Eq.(4.41), we obtain:

$$\begin{aligned}\delta_{ct}^2 u(i,j) - \frac{(\Delta t)^2}{12}c^2\delta_{ct}^2(\delta_{cx}^2 u(i,j)) - \frac{(\Delta t)^4}{360}c^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) - \tau_6 = \\ c^2[\delta_{cx}^2 u(i,j) - \frac{(\Delta x)^2}{12}\frac{1}{c^2}\delta_{cx}^2(\delta_{ct}^2 u(i,j)) - \frac{(\Delta x)^4}{360}\frac{1}{c^2}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) - \tau_5]\end{aligned}$$

$$\begin{aligned}\delta_{ct}^2 u(i,j) - \frac{(\Delta t)^2}{12}c^2\delta_{ct}^2(\delta_{cx}^2 u(i,j)) - \frac{(\Delta t)^4}{360}c^2\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) - \\ c^2\delta_{cx}^2 u(i,j) + \frac{(\Delta x)^2}{12}\delta_{cx}^2(\delta_{ct}^2 u(i,j)) + \frac{(\Delta x)^4}{360}\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) = \tau_6 - c^2\tau_5\end{aligned} \tag{4.59}$$

Also,

$$\begin{aligned}\delta_{ct}^2(\delta_{cx}^2 u(i,j)) = \frac{1}{(\Delta x)^2(\Delta t)^2}[u(i+1,j+1) - 2u(i+1,j) + u(i+1,j-1) - 2u(i,j+1) \\ + 4u(i,j) - 2u(i,j-1) + u(i-1,j+1) - 2u(i-1,j) + u(i-1,j-1)]\end{aligned} \tag{4.60}$$

$$\delta_{cx}^2(\delta_{ct}^2 u(i,j)) = \frac{1}{(\Delta x)^2(\Delta t)^2}[u(i+1,j+1) - 2u(i,j+1) + u(i-1,j+1) - 2u(i+1,j) + 4u(i,j) - 2u(i-1,j) + u(i+1,j-1) - 2u(i,j-1) + u(i-1,j-1)] \quad (4.61)$$

$$\delta_{cx}^2(\delta_{cx}^2(\delta_{ct}^2 u(i,j))) = \frac{1}{(\Delta x)^4(\Delta t)^2}[u(i+2,j+1) - 4u(i+1,j+1) + 6u(i,j+1) - 4u(i-1,j+1) + u(i-2,j+1) - 2u(i+2,j) + 8u(i+1,j) - 12u(i,j) + 8u(i-1,j) - 2u(i-2,j) + u(i+2,j-1) - 4u(i+1,j-1) + 6u(i,j-1) - 4u(i-1,j-1) + u(i-2,j-1)] \quad (4.62)$$

$$\delta_{ct}^2(\delta_{ct}^2(\delta_{cx}^2 u(i,j))) = \frac{1}{(\Delta x)^2(\Delta t)^4}[u(i+1,j+2) - 4u(i+1,j+1) + 6u(i+1,j) - 4u(i+1,j-1) + u(i+1,j-2) - 2u(i,j+2) + 8u(i,j+1) - 12u(i,j) + 8u(i,j-1) - 2u(i,j-2) + u(i-1,j+2) - 4u(i-1,j+1) + 6u(i-1,j) - 4u(i-1,j-1) + u(i-1,j-2)] \quad (4.63)$$

Substituting Eqs.(4.49), (4.50),(4.60),(4.61), (4.62) and (4.63) into Eq.(4.59) and after specification, we obtain:

$$[306(\Delta x)^2 + 52(\Delta t)^2 c^2](u(i,j+1) + u(i,j-1)) + [-612(\Delta x)^2 + 612(\Delta t)^2 c^2]u(i,j) + [26(\Delta x)^2 - 26(\Delta t)^2 c^2](u(i+1,j+1) + u(i+1,j-1) + u(i-1,j-1) + u(i-1,j+1)) + [-52(\Delta x)^2 - 306(\Delta t)^2 c^2](u(i+1,j) + u(i-1,j)) + [-(\Delta t)^2 c^2](u(i+1,j+2) + u(i-1,j+2) + u(i+1,j-2) + u(i-1,j-2)) + [2(\Delta t)^2 c^2](u(i,j+2) + u(i,j-2)) + [(\Delta x)^2](u(i+2,j+1) + u(i-2,j+1) + u(i+2,j-1) + u(i-2,j-1)) + [-2(\Delta x)^2](u(i+2,j) + u(i-2,j)) - \tau_7 = 0 \quad (4.64)$$

where, $\tau_7 = 360(\Delta x)^2(\Delta t)^2[\tau_6 - c^2\tau_5 - \tau_4 + c^2\tau_3]$ is a local truncation error with order eight.

$$A(u(i+1,j+2) + u(i-1,j+2) + u(i+1,j-2) + u(i-1,j-2)) + B(u(i,j+2) + u(i,j-2)) = C(u(i,j+1) + u(i,j-1)) + Du(i,j) + E(u(i+1,j+1) + u(i+1,j-1) + u(i-1,j-1) + u(i-1,j+1)) + F(u(i+1,j) + u(i-1,j)) + G(u(i+2,j+1) + u(i-2,j+1) + u(i+2,j-1) + u(i-2,j-1)) + H(u(i+2,j) + u(i-2,j)) \quad (4.65)$$

for $i = 2, 3, \ldots, k-2$ and $j = 2, 3, \ldots, n-2$.

Where, $A = -(\Delta t)^2 c^2$, $B = 2(\Delta t)^2 c^2$, $C = -306(\Delta x)^2 - 52(\Delta t)^2 c^2$, $D = 612(\Delta x)^2 - 612(\Delta t)^2 c^2$, $E = -26(\Delta x)^2 + 26(\Delta t)^2 c^2$, $F = 52(\Delta x)^2 + 306(\Delta t)^2 c^2$, $G = -(\Delta x)^2$ and $H = 2(\Delta x)^2$.

In order to satisfy the initial boundary conditions and the reduce algorithm complexity we can use the following talyor series estimates.

$$u(i+1,j+2) + u(i+1,j-2) = 4u(i+1,j+1) - 6u(i+1,j) + 4u(i+1,j-1) \quad (4.66)$$

$$u(i,j+2) + u(i,j-2) = 4u(i,j+1) - 6u(i,j) + 4u(i,j-1) \quad (4.67)$$

$$u(i-1,j+2) + u(i-1,j-2) = 4u(i-1,j+1) - 6u(i-1,j) + 4u(i-1,j-1) \quad (4.68)$$

$$u(i+2,j+1) + u(i-2,j+1) = 4u(i+1,j+1) - 6u(i,j+1) + 4u(i-1,j+1) \quad (4.69)$$

$$u(i+2,j) + u(i-2,j) = 4u(i+1,j) - 6u(i,j) + 4u(i-1,j) \quad (4.70)$$

$$u(i+2,j-1) + u(i-2,j-1) = 4u(i+1,j-1) - 6u(i,j-1) + 4u(i-1,j-1) \quad (4.71)$$

Substituting equations (4.66), (4.67), (4.68), (4.69), (4.70) and (4.71) into equation (4.65) and by rearranging we get the following.

$$\alpha u(i-1,j+1) + \beta u(i,j+1) + \alpha u(i+1,j+1) + \gamma u(i-1,j) + \eta u(i,j) + \gamma u(i+1,j) + \\ \alpha u(i-1,j-1) + \beta u(i,j-1) + \alpha u(i+1,j-1) = 0 \quad (4.72)$$

where $\alpha = -30(\Delta t)^2 c^2 + 30(\Delta x)^2$, $\beta = 60(\Delta t)^2 c^2 + 300(\Delta x)^2$, $\gamma = -300(\Delta t)^2 c^2 - 60(\Delta x)^2$ and $\eta = 600(\Delta t)^2 c^2 - 600(\Delta x)^2$.

Thus, equation (4.72) is CFDM with 9 points for 1D wave equation. The system of equations obtained from this equation can be solved by matrix inversion method.

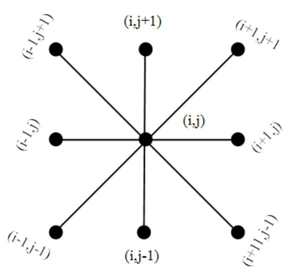


Figure 4.4: Nine points Molecule for equation (4.72)

4.3 Stability, Consistency and Convergence Analysis

Stability: The error caused by a small perturbation in the numerical method remains bounded. A method is said to be stable if it produces a bounded solution which imitates the exact solution otherwise it is said to be unstable. When the difference between the computed numerical solution and the exact solution of the discrete equation remains bounded as time grows[6]. A numerical method is said to be stable if numerical errors, e.g. due to round off, are not amplified, and the approximate solution remains bounded.

Consistence: The numerical scheme is called consistent with the differential equation if the truncation error goes to zero when the numerical parameters are made arbitrarily small[6].

Convergence: means that the solution to the numerical scheme approaches the true solution of the PDE when the mesh is refined. The numerical scheme formulation must have the same set of initial conditions and boundary conditions[6].

Von neumann stability analysis is based on decomposing the numerical solution into fourier harmonics on the spatial grid. Although this method does not capture the influence of the boundary conditions, it is quite easy to formulate and usually accurate enough to provide practical stability criteria. We can decompose the solution into fourier modes on the mesh as cited in [6], assume that the solution of Eq. (4.65) at the grid point $(i\Delta x, j\Delta t)$ is:

$$u(i,j) = \sum_{m} u_m(t_j)e^{i\Delta xq\theta} \tag{4.73}$$

$$u(i,j) = \lambda^j e^{i\Delta xq\theta} \tag{4.74}$$

where $q = \sqrt{-1}$, θ is a real number and $|\lambda^j|$ is the amplitude of j^{th} component.

By linearity it is enough to examine the behavior of a single fourier component so we replace $u(i,j)$ by $\lambda^j e^{i\Delta xq\theta}$ and rearrange to get

$$\lambda = \frac{\lambda^{j+1}}{\lambda^j} \leq 1$$

This is the Von Neumann condition for stability. λ is the amplification factor. Substituting Eq.(4.74) into Eq.(4.65) gives:

$$A(\lambda^{j+2}e^{(i+1)q\theta}+\lambda^{j+2}e^{(i-1)q\theta}+\lambda^{j-2}e^{(i+1)q\theta}+\lambda^{j-2}e^{(i-1)q\theta})+B(\lambda^{j+2}e^{(i)q\theta}+\lambda^{j-2}e^{(i)q\theta})$$
$$=C(\lambda^{j+1}e^{(i)q\theta}+\lambda^{j-1}e^{(i)q\theta})+D\lambda^{j}e^{(i)q\theta}+E(\lambda^{j+1}e^{(i+1)q\theta}+\lambda^{j-1}e^{(i+1)q\theta}+\lambda^{j-1}e^{(i-1)q\theta}+\lambda^{j+1}e^{(i-1)q\theta})+$$
$$F(\lambda^{j}e^{(i+1)q\theta}+\lambda^{j}e^{(i-1)q\theta})+G(\lambda^{j+1}e^{(i+2)q\theta}+\lambda^{j+1}e^{(i-2)q\theta}+\lambda^{j-1}e^{(i+2)q\theta}+\lambda^{j-1}e^{(i-2)q\theta})+$$
$$H(\lambda^{j}e^{(i+2)q\theta}+\lambda^{j}e^{(i-2)q\theta})$$

$$\Rightarrow 0=\lambda^{j+2}[A(e^{(i+1)q\theta}+e^{(i-1)q\theta})+Be^{(i)q\theta}]-\lambda^{j+1}[Ce^{(i)q\theta}+E(e^{(i+1)q\theta}+e^{(i-1)q\theta})+G(e^{(i+2)q\theta}+$$
$$e^{(i-2)q\theta})]-\lambda^{j}[De^{(i)q\theta}+F(e^{(i+1)q\theta}+e^{(i-1)q\theta})+H(e^{(i+2)q\theta}+e^{(i-2)q\theta})]-\lambda^{j-1}[Ce^{(i)q\theta}+$$
$$E(e^{(i+1)q\theta}+e^{(i-1)q\theta})+G(e^{(i+2)q\theta}+e^{(i-2)q\theta})]+\lambda^{j-2}[A(e^{(i+1)q\theta}+e^{(i-1)q\theta})+Be^{(i)q\theta}]$$

$$\Rightarrow 0=\lambda^{4}[A(e^{3q\theta}+e^{q\theta})+Be^{2q\theta}]-\lambda^{3}[Ce^{2q\theta}+E(e^{3q\theta}+e^{q\theta})+G(e^{4q\theta}+1)]-$$
$$\lambda^{2}[De^{2q\theta}+F(e^{3q\theta}+e^{q\theta})+H(e^{4q\theta}+1)]-\lambda[Ce^{2q\theta}+E(e^{3q\theta}+e^{q\theta})+G(e^{4q\theta}+1)]+$$
$$[A(e^{3q\theta}+e^{q\theta})+Be^{2q\theta}]$$

$$a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0 \tag{4.75}$$

Where, $a_1=-[A(e^{3q\theta}+e^{q\theta})+Be^{2q\theta}]$, $a_2=[Ce^{2q\theta}+E(e^{3q\theta}+e^{q\theta})+G(e^{4q\theta}+1)]$, $a_3=[De^{2q\theta}+F(e^{3q\theta}+e^{q\theta})+H(e^{4q\theta}+1)]$, $a_4=[Ce^{2q\theta}+E(e^{3q\theta}+e^{q\theta})+G(e^{4q\theta}+1)]$ and $a_5=-[A(e^{3q\theta}+e^{q\theta})+Be^{2q\theta}]$.

Routh Hurwitz stability criteria provides a simple algorithm to decide whether or not the zeros of polynomial are in the left of the complex plane (such a polynomial at times “Hurwitz”). A Hurwitz polynomial is a key requirement for a linear continuous time invariant to be stable(all bounded inputs produce bounded outputs). By using Routh Hurwitz criteria and using the transformation [6], $\lambda=\frac{1+z}{1-z}$ substitute into Eq.(4.75), we have

$$(a_1-a_2+a_3-a_4+a_5)\lambda^4+(3a_1+2a_2-2a_4-3a_5)\lambda^3+(6a_1-2a_3+6a_5)\lambda^2+$$
$$(3a_1+2a_2-2a_4-3a_5)\lambda+(a_1+a_2+a_3+a_4+a_5)=0 \tag{4.76}$$

If $|\lambda|\leq 1$, then the CFDM of Eq.(4.65) is stable. It is sufficient to show that:

$$a_2+a_4<0 \quad \text{and} \quad a_1+a_3+a_5>0$$

From Eq. (4.75),

$$\begin{aligned}
a_2 + a_4 &= [Ce^{2q\theta} + E(e^{3q\theta} + e^{q\theta}) + G(e^{4q\theta} + 1)] + [Ce^{2q\theta} + E(e^{3q\theta} + e^{q\theta}) + G(e^{4q\theta} + 1)] \\
&= 2[-306(\Delta x)^2 - 52(\Delta t)^2 c^2]e^{2q\theta} + 2[26(\Delta x)^2 - 26(\Delta t)^2 c^2](e^{3q\theta} + e^{q\theta}) \\
&\quad + 2[-(\Delta x)^2(e^{4q\theta} + 1)] \\
&= -2[(\Delta x)^2(1 + 26e^{q\theta} + 306e^{2q\theta} + 26e^{3q\theta} + e^{4q\theta}) + 26(\Delta t)^2 c^2(-e^{q\theta} + 2e^{2q\theta} - e^{3q\theta})]
\end{aligned}$$

$$\begin{aligned}
a_2 + a_3 + a_5 &= -[A(e^{3q\theta} + e^{q\theta}) + Be^{2q\theta}] + [De^{2q\theta} + F(e^{3q\theta} + e^{q\theta}) + H(e^{4q\theta} + 1)] - \\
&\quad [A(e^{3q\theta} + e^{q\theta}) + Be^{2q\theta}] \\
&= (F - 2A)e^{q\theta} + e^{2q\theta}(D - 2B) + (F - 2A)e^{3q\theta} + He^{4q\theta} + H \\
&= 2[(\Delta x)^2(1 + 26e^{q\theta} + 306e^{2q\theta} + 26e^{3q\theta} + e^{4q\theta}) + (\Delta t)^2 c^2(154e^{q\theta} - 307e^{2q\theta} + \\
&\quad 154e^{3q\theta})]
\end{aligned}$$

Clearly, for sufficiently small Δt , $a_2 + a_4 < 0$ and $a_1 + a_3 + a_5 > 0$. Thus, the CFDM given in Eq.(4.65) is absolutely stable for wave equation. Now, expand Eq. (4.64) in Taylor series and replace the derivatives involving x and t for the relation,

$$\frac{\partial^2 u}{\partial t^2}(i,j) = c^2 \frac{\partial^2 u}{\partial x^2}(i,j) \tag{4.77}$$

and then we drive the local truncation error. The principal part of the local truncation error of the proposed method using Eqs. (4.50), (4.52) and (4.64) for the wave equation is:

$$\begin{aligned}
T(i,j) &= 360(\Delta x)^2(\Delta t)^2[\tau_6 - c^2\tau_5 - \tau_4 + c^2\tau_3] \\
&= 360(\Delta x)^2(\Delta t)^2[(\frac{(\Delta t)^6}{20160}\frac{\partial^8 u}{\partial t^8}(i,j)) \quad - c^2(\frac{(\Delta x)^6}{20160}\frac{\partial^8 u}{\partial x^8}(i,j)) - \\
&\quad (\Delta x)^2(-\frac{(\Delta t)^2}{12}\frac{\partial^4 u}{\partial t^4}(i,j)) - c^2(-\frac{(\Delta x)^2}{12}\frac{\partial^4 u}{\partial x^4}(i,j))] \\
&= 30(-\Delta x)^4(\Delta t)^2 c^2(\frac{\partial^4 u}{\partial x^4}(i,j) + (\Delta x)^2(\Delta t)^4\frac{\partial^4 u}{\partial x^4}(i,j)) + O((\Delta x)^2(\Delta t)^8 + (\Delta x)^8(\Delta t)^2)
\end{aligned}$$

Thus, $T \to 0$ as Δx and $\Delta t \to 0$.

So that, the scheme is consistent with the order of $O((\Delta x)^2(\Delta t)^4 + (\Delta x)^4(\Delta t)^2)$.

The **Lax equivalence theorem** on the convergence of the solution of the discrete problem to that of the given properly posed initial-value problem states that if the numerical scheme is consistent, then stability is necessary and sufficient for convergence[23].

Chapter 5

Result and Discussion

Matlab version of the finite element method and Compact finite difference method of 1D wave equation is presented and demonstrated in this section.

5.1 Test problem

Consider the initial and boundary problem[6]

$$\begin{aligned}
&u_{tt} = u_{xx}, \\
&u(0,t) = u(\pi,t) = 0, && 0 \le t \le \tfrac{1}{25} \\
&u(x,0) = \sin(\pi x) && \text{for } \ 0 \le x \le 1, \\
&u_t(x,0) = 0 && \text{for } \ 0 \le x \le 1.
\end{aligned}$$

The exact analytical solution of the above wave equation is: $u(x,t) = \sin(\pi x)\cos(\pi t)$.
Solve this Problem by using:

1. Finite element method.

2. Sixth order compact finite difference method.

Finite element method:
Let $\Delta x = \frac{1}{4}$, $\Delta t = \frac{1}{100}$ and for simplicity $u(x_i,t_j) = u_{ij}$ for $i,j = 0,1,2,3,4$. Discretize spatial as $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$ and discretize time $t_0 = 0$, $t_1 = \frac{1}{100}$, $t_2 = \frac{1}{50}$, $t3 = \frac{3}{100}$ and $t_4 = \frac{1}{25}$. We can see in the figure (5.1). The initial condition is given as $u_{i0} = \sin(\pi x_i) = \phi_i$ and $\frac{\partial u_{i0}}{\partial t} = 0 = \psi_i$, for $i = 0,1,2,3,4$. The boundary condition of the given problem $u_{0j} = 0$ and $u_{4j} = 0$ for $j = 0,1,2,3,4$. $u_{00} = 0$, $u_{10} = \frac{\sqrt{2}}{2}$, $u_{20} = 1$, $u_{30} = \frac{\sqrt{2}}{2}$ and $u_{40} = 0$
From difference equation, it can be approximated for $\frac{\partial u_{i0}}{\partial t}$

$$\psi_i = \frac{\partial u_{i0}}{\partial t} = \frac{u_{i1} - u_{i-1}}{2\Delta t} = 0$$

$$\Rightarrow u_{i,-1} = u_{i,1} \tag{5.1}$$

Using eqn.(4.38) with $d = \frac{1}{400}$, $c = 1$

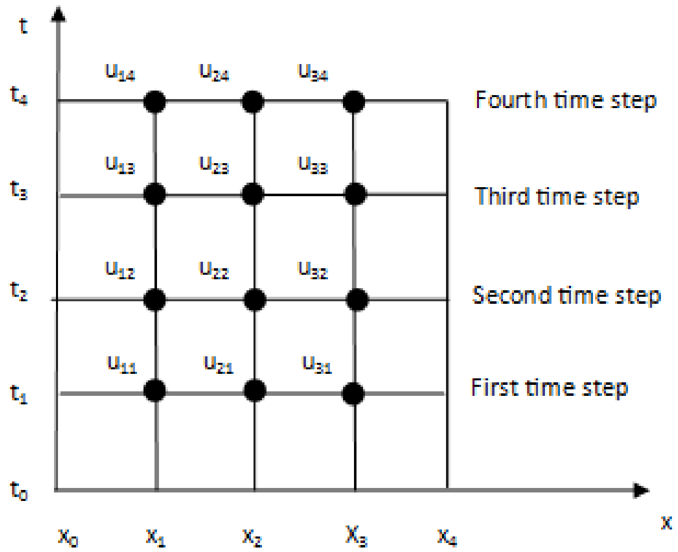


Figure 5.1: Discretization of the domain

$$u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1} = -u_{i-1,j-1} - 4u_{i,j-1} - u_{i+1,j-1} + \frac{403}{200}u_{i-1,j} +$$

$$\frac{797}{100}u_{i,j} + \frac{403}{200}u_{i+1,j} \tag{5.2}$$

For first time step substitute $j = 0$ and equation(5.1) into equation (5.2). We get

$$u_{i-1,1} + 4u_{i,1} + u_{i+1,1} = \frac{403}{400}u_{i-1,0} + \frac{797}{200}u_{i,0} + \frac{403}{400}u_{i+1,0}$$

This equation is used to find first time step, so, for $i = 1$

$$u_{0,1} + 4u_{1,1} + u_{2,1} = \frac{403}{400}u_{0,0} + \frac{797}{200}u_{1,0} + \frac{403}{400})u_{2,0}$$

$$\Rightarrow 4u_{1,1} + u_{2,1} = \frac{2387}{624} \tag{5.3}$$

For i=2, then we have

$$u_{1,1} + 4u_{2,1} + u_{3,1} = \frac{403}{400}u_{1,0} + \frac{797}{200}u_{2,0} + \frac{403}{400}u_{3,0}$$

$$\Rightarrow u_{1,1} + 4u_{2,1} + u_{3,1} = \frac{5729}{1059} \tag{5.4}$$

For i=3, then we get

$$u_{2,1} + 4u_{3,1} + u_{4,1} = \frac{403}{400}u_{2,0} + \frac{797}{200}u_{3,0} + \frac{403}{400}u_{4,0}$$

$$\Rightarrow u_{2,1} + 4u_{3,1} = \frac{2387}{624} \tag{5.5}$$

From equation (5.3), (5.4) and (5.5) we have the following

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} = \begin{pmatrix} \frac{2387}{624} \\ \frac{5729}{1059} \\ \frac{2387}{624} \end{pmatrix} \tag{5.6}$$

For second time step substitute $j = 1$ into equation (5.2). We get

$$u_{i-1,2} + 4u_{i,2} + u_{i+1,2} = -u_{i-1,0} - 4u_{i,0} - u_{i+1,0} + \frac{403}{200}u_{i-1,1} + \frac{797}{100}u_{i,1} + \frac{403}{200}u_{i+1,1}$$

For i=1, then we get

$$u_{0,2} + 4u_{1,2} + u_{2,2} = -u_{0,0} - 4u_{1,0} - u_{2,0} + \frac{403}{200}u_{0,1} + \frac{797}{100}u_{1,1} + \frac{403}{200}u_{2,1}$$

$$-\frac{797}{100}u_{1,1} - \frac{403}{200}u_{2,1} + 4u_{1,2} + u_{2,2} = -4u_{1,0} - u_{2,0}$$

$$\Rightarrow -\frac{797}{100}u_{1,1} - \frac{403}{200}u_{2,1} + 4u_{1,2} + u_{2,2} = \frac{-4552}{1189} \tag{5.7}$$

For i=2, then we get

$$u_{1,2} + 4u_{2,2} + u_{3,2} = -u_{1,0} - 4u_{2,0} - u_{3,0} + \frac{403}{200}u_{1,1} + \frac{797}{100}u_{2,1} + \frac{403}{200}u_{3,1}$$

$$-\frac{403}{200}u_{1,1} - \frac{797}{100}u_{2,1} - \frac{403}{200}u_{3,1} + u_{1,2} + 4u_{2,2} + u_{3,2} = -u_{1,0} - 4u_{2,0} - u_{3,0}$$

$$\Rightarrow -\frac{403}{200}u_{1,1} - \frac{797}{100}u_{2,1} - \frac{403}{200}u_{3,1} + u_{1,2} + 4u_{2,2} + u_{3,2} = \frac{-2209}{408} \tag{5.8}$$

For i=3, then we get

$$u_{2,2} + 4u_{3,2} + u_{4,2} = -u_{2,0} - 4u_{3,0} - u_{4,0} + \frac{403}{200}u_{2,1} + \frac{797}{100}u_{3,1} + \frac{403}{200}u_{4,1}$$

$$-\frac{403}{200}u_{2,1}-\frac{797}{100}u_{3,1}-\frac{403}{200}u_{4,1}+u_{2,2}+4u_{3,2}+u_{4,2}=-u_{2,0}-4u_{3,0}-u_{4,0}$$

$$\Rightarrow -\frac{403}{200}u_{2,1}-\frac{797}{100}u_{3,1}-\frac{403}{200}u_{4,1}+u_{2,2}+4u_{3,2}+u_{4,2}=\frac{-4552}{1189} \tag{5.9}$$

From equation (5.7), (5.8) and (5.9) we have the following

$$\begin{pmatrix} \frac{-797}{100} & \frac{-403}{200} & 0 \\ \frac{-403}{200} & \frac{-797}{100} & \frac{-403}{200} \\ 0 & \frac{-403}{200} & \frac{-797}{100} \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix}+\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{pmatrix}=\begin{pmatrix} \frac{-4552}{1189} \\ \frac{-2209}{408} \\ \frac{-4552}{1189} \end{pmatrix} \tag{5.10}$$

For third time step substitute $j=2$ into equation (5.2). We get

$$u_{i-1,3}+4u_{i,3}+u_{i+1,3}=-u_{i-1,1}-4u_{i,1}-u_{i+1,1}+\frac{403}{200}u_{i-1,2}+\frac{797}{100}u_{i,2}+\frac{403}{200}u_{i+1,2}$$

For i=1, then we get

$$u_{0,3}+4u_{1,3}+u_{2,3}=-u_{0,1}-4u_{1,1}-u_{2,1}+\frac{403}{200}u_{0,2}+\frac{797}{100}u_{1,2}+\frac{403}{200}u_{2,2}$$

$$4u_{1,1}+u_{2,1}-\frac{797}{100}u_{1,2}-\frac{403}{200}u_{2,2}+4u_{1,3}+u_{2,3}=0$$

$$\Rightarrow 4u_{1,1}+u_{2,1}-\frac{797}{100}u_{1,2}-\frac{403}{200}u_{2,2}+4u_{1,3}+u_{2,3}=0 \tag{5.11}$$

For i=2, then we get

$$u_{1,3}+4u_{2,3}+u_{3,3}=-u_{1,1}-4u_{2,1}-u_{3,1}+\frac{403}{200}u_{1,2}+\frac{797}{100}u_{2,2}+\frac{403}{200}u_{3,2}$$

$$u_{1,1}+4u_{2,1}+u_{3,1}-\frac{403}{200}u_{1,2}-\frac{797}{100}u_{2,2}-\frac{403}{200}u_{3,2}+u_{1,3}+4u_{2,3}+u_{3,3}=0$$

$$\Rightarrow u_{1,1}+4u_{2,1}+u_{3,1}-\frac{403}{200}u_{1,2}-\frac{797}{100}u_{2,2}-\frac{403}{200}u_{3,2}+u_{1,3}+4u_{2,3}+u_{3,3}=0 \tag{5.12}$$

For i=3, then we get

$$u_{2,3}+4u_{3,3}+u_{4,3}=-u_{2,1}-4u_{3,1}-u_{4,1}+\frac{403}{200}u_{2,2}+\frac{797}{100}u_{3,2}+\frac{403}{200}u_{4,2}$$

$$u_{2,1}+4u_{3,1}+u_{4,1}-\frac{403}{200}u_{2,2}-\frac{797}{100}u_{3,2}-\frac{403}{200}u_{4,2}+u_{2,3}+4u_{3,3}+u_{4,3}=$$

$$\Rightarrow u_{2,1}+4u_{3,1}-\frac{403}{200}u_{2,2}-\frac{797}{100}u_{3,2}+u_{2,3}+4u_{3,3}=0 \tag{5.13}$$

From equation (5.11), (5.12) and (5.13) we have the following

$$\begin{pmatrix}4&1&0\\1&4&1\\0&1&4\end{pmatrix}\begin{pmatrix}u_{1,1}\\u_{2,1}\\u_{3,1}\end{pmatrix}+\begin{pmatrix}\frac{-797}{100}&\frac{-403}{200}&0\\\frac{-403}{200}&\frac{-797}{100}&\frac{-403}{200}\\0&\frac{-403}{200}&\frac{-797}{100}\end{pmatrix}\begin{pmatrix}u_{1,2}\\u_{2,2}\\u_{3,2}\end{pmatrix}+\begin{pmatrix}4&1&0\\1&4&1\\0&1&4\end{pmatrix}\begin{pmatrix}u_{1,3}\\u_{2,3}\\u_{3,3}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \tag{5.14}$$

For fourth time step substitute $j=3$ into equation (5.2). We get

$$u_{i-1,4}+4u_{i,4}+u_{i+1,4}=-u_{i-1,2}-4u_{i,2}-u_{i+1,2}+\frac{403}{200}u_{i-1,3}+\frac{797}{100}u_{i,3}+\frac{403}{200}u_{i+1,3}$$

For i=1, then we get

$$u_{0,4}+4u_{1,4}+u_{2,4}=-u_{0,2}-4u_{1,2}-u_{2,2}+\frac{403}{200}u_{0,3}+\frac{797}{100}u_{1,3}+\frac{403}{200}u_{2,3}$$

$$4u_{1,2}+u_{2,2}-\frac{797}{100}u_{1,3}-\frac{403}{200}u_{2,3}+4u_{1,4}+u_{2,4}=0$$

$$\Rightarrow 4u_{1,2}+u_{2,2}-\frac{797}{100}u_{1,3}-\frac{403}{200}u_{2,3}+4u_{1,4}+u_{2,4}=0 \tag{5.15}$$

For i=2, then we get

$$u_{1,4}+4u_{2,4}+u_{3,4}=-u_{1,2}-4u_{2,2}-u_{3,2}+\frac{403}{200}u_{1,3}+\frac{797}{100}u_{2,3}+\frac{403}{200}u_{3,3}$$

$$u_{1,2}+4u_{2,2}+u_{3,2}-\frac{403}{200}u_{1,3}-\frac{797}{100}u_{2,3}-\frac{403}{200}u_{3,3}+u_{1,4}+4u_{2,4}+u_{3,4}=0$$

$$\Rightarrow u_{1,2}+4u_{2,2}+u_{3,2}-\frac{403}{200}u_{1,3}-\frac{797}{100}u_{2,3}-\frac{403}{200}u_{3,3}+u_{1,4}+4u_{2,4}+u_{3,4}=0 \tag{5.16}$$

For i=3, then we get

$$u_{2,4}+4u_{3,4}+u_{4,4}=-u_{2,2}-4u_{3,2}-u_{4,2}+\frac{403}{200}u_{2,3}+\frac{797}{100}u_{3,3}+\frac{403}{200}u_{4,3}$$

$$u_{2,2}+4u_{3,2}+u_{4,2}-\frac{403}{200}u_{2,3}-\frac{797}{100}u_{3,3}-\frac{403}{200}u_{4,3}+u_{2,4}+4u_{3,4}+u_{4,4}=0$$

$$\Rightarrow u_{2,2}+4u_{3,2}-\frac{403}{200}u_{2,3}-\frac{797}{100}u_{3,3}+u_{2,4}+4u_{3,4}=0 \tag{5.17}$$

From equation (5.15), (5.16) and (5.17) we have the following

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{pmatrix} + \begin{pmatrix} \frac{-797}{100} & \frac{-403}{200} & 0 \\ \frac{-403}{200} & \frac{-797}{100} & \frac{-403}{200} \\ 0 & \frac{-403}{200} & \frac{-797}{100} \end{pmatrix}\begin{pmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{pmatrix} + \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} u_{1,4} \\ u_{2,4} \\ u_{3,4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.18}$$

Thus, combine the equation (5.6), (5.10), (5.14) and (5.18). We get

$$\begin{bmatrix}
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-797}{100} & \frac{-403}{200} & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-403}{200} & \frac{-797}{100} & \frac{-403}{200} & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-403}{200} & \frac{-797}{100} & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & \frac{-797}{100} & \frac{-403}{200} & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & \frac{-403}{200} & \frac{-797}{100} & \frac{-403}{200} & 1 & 4 & 1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 & \frac{-403}{200} & \frac{-797}{100} & 0 & 1 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 1 & 0 & \frac{-797}{100} & \frac{-403}{200} & 0 & 4 & 1 & 0 \\
0 & 0 & 0 & 1 & 4 & 1 & \frac{-403}{200} & \frac{-797}{100} & \frac{-403}{200} & 1 & 4 & 1 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & \frac{-403}{200} & \frac{-797}{100} & 0 & 1 & 4
\end{bmatrix}
\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \\ u_{14} \\ u_{24} \\ u_{34} \end{bmatrix}
=
\begin{bmatrix} \frac{2387}{624} \\ \frac{5729}{1059} \\ \frac{2387}{624} \\ \frac{-4552}{1189} \\ \frac{-2209}{408} \\ \frac{-4552}{1189} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve this system of equation by using matlab software and shown in table (5.1).Graphical visualized in figure (5.2)

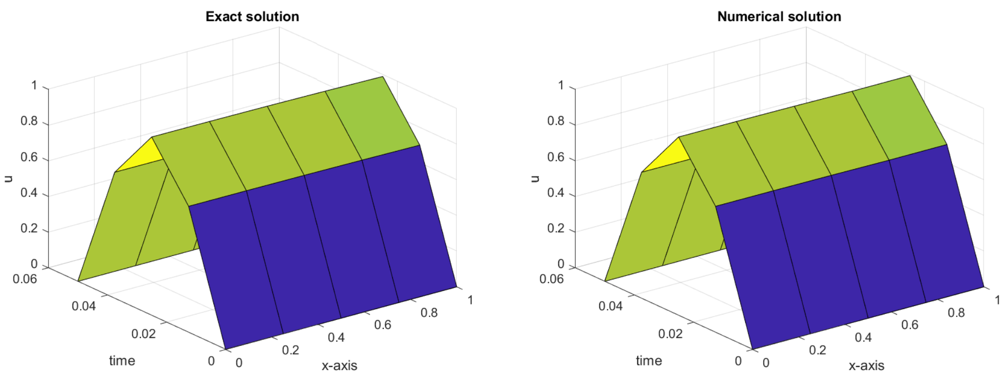


Figure 5.2: Exact solution and Numerical solution of the 1D wave equation

Table 5.1: Exact, Numerical solution, Absolute Error of wave equation by FEM

Grid point	Exact Solution	Numerical Solution	Absolute Error
u11	0.706561627484314	0.706532994858762	2.86326255513059e-05
u21	0.999229036240723	0.999188543593342	4.04926473805967e-05
u31	0.706561627484314	0.706532994858762	2.86326255513059e-05
u12	0.704927006965107	0.70481256708059	0.000114439884516759
u22	0.996917333733128	0.996755491296368	0.000161842436760096
u32	0.704927006965107	0.70481256708059	0.000114439884516759
u13	0.702205440095289	0.701948289956318	0.000257150138971762
u23	0.993068456954926	0.992704791740826	0.000363665214100584
u33	0.702205440095289	0.701948289956318	0.000257150138971762
u14	0.69840112333371	0.69794481195799	0.000456311375720664
u24	0.987688340595138	0.987043018858928	0.000645321736210236
u34	0.69840112333371	0.69794481195799	0.000456311375720664
Maximum error			0.000645321736210236

Sixth order compact finite difference method

$$\alpha u(i-1,j+1)+\beta u(i,j+1)+\alpha u(i+1,j+1)+\gamma u(i-1,j)+\eta u(i,j)+\gamma u(i+1,j)+$$
$$\alpha u(i-1,j-1)+\beta u(i,j-1)+\alpha u(i+1,j-1)=0$$

For first time step substitute $j=0$ and equation(5.1) into equation (4.72). We get

$$\alpha u(i-1,1)+\beta u(i,1)+\alpha u(i+1,1)+\gamma u(i-1,0)+\eta u(i,0)+\gamma u(i+1,0)+$$
$$\alpha u(i-1,-1)+\beta u(i,-1)+\alpha u(i+1,-1)=0$$

This equation is used to find first time step, so, for $i=1$

$$2(\alpha u(0,1)+\beta u(1,1)+\alpha u(2,1))+\gamma u(0,0)+\eta u(1,0)+\gamma u(2,0)=0$$

$$\beta u(1,1)+\alpha u(2,1)=-\frac{1}{2}(\frac{\sqrt{2}}{2}\eta+\gamma) \tag{5.19}$$

For i=2, then we have

$$\alpha u_{1,1}+\eta u_{2,1}+\alpha u_{3,1}=-\frac{1}{2}(\gamma u_{1,0}+\eta u_{2,0}+\gamma u_{3,0})$$

$$\Rightarrow \alpha u_{1,1} + \beta u_{2,1} + \alpha u_{3,1} = -\frac{1}{2}(\sqrt{2}\gamma + \eta) \tag{5.20}$$

For i=3, then we get

$$\alpha u_{2,1} + \beta u_{3,1} + \alpha u_{4,1} = -\frac{1}{2}(\frac{\sqrt{2}}{2}\eta + \gamma) \tag{5.21}$$

From equation (5.3), (5.4) and (5.5) we have the following

$$\begin{pmatrix} \beta & \alpha & 0 \\ \alpha & \beta & \alpha \\ 0 & \alpha & \beta \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(\frac{\sqrt{2}}{2}\eta + \gamma) \\ -\frac{1}{2}(\sqrt{2}\gamma + \eta) \\ -\frac{1}{2}(\frac{\sqrt{2}}{2}\eta + \gamma) \end{pmatrix} \tag{5.22}$$

For second time step substitute $j = 1$ into equation (4.72). We get

$$\alpha u(i-1,2) + \beta u(i,2) + \alpha u(i+1,2) + \gamma u(i-1,1) + \eta u(i,1) + \gamma u(i+1,1) +$$
$$\alpha u(i-1,0) + \beta u(i,0) + \alpha u(i+1,0) = 0$$

For i=1, then we get

$$\alpha u(0,2) + \beta u(1,2) + \alpha u(2,2) + \gamma u(0,1) + \eta u(1,1) + \gamma u(2,1) +$$
$$\alpha u(0,0) + \beta u(1,0) + \alpha u(2,0) = 0$$

$$\Rightarrow \eta u(1,1) + \gamma u(2,1) + \beta u(1,2) + \alpha u(2,2) + = -\frac{\sqrt{2}}{2}\beta - \alpha \tag{5.23}$$

For i=2, then we get

$$\alpha u(1,2) + \beta u(2,2) + \alpha u(3,2) + \gamma u(1,1) + \eta u(2,1) + \gamma u(3,1) +$$
$$\alpha u(1,0) + \beta u(2,0) + \alpha u(3,0) = 0$$

$$\gamma u(1,1) + \eta u(2,1) + \gamma u(3,1) + \alpha u(1,2) + \beta u(2,2) + \alpha u(3,2) +$$
$$= -\alpha u(1,0) - \beta u(2,0) - \alpha u(3,0)$$

$$\Rightarrow \gamma u(1,1) + \eta u(2,1) + \gamma u(3,1) + \alpha u(1,2) + \beta u(2,2) + \alpha u(3,2) = -\sqrt{2}\alpha - \beta \tag{5.24}$$

For i=3, then we get

$$\alpha u(2,2) + \beta u(3,2) + \alpha u(4,2) + \gamma u(2,1) + \eta u(3,1) + \gamma u(4,1) +$$
$$\alpha u(2,0) + \beta u(3,0) + \alpha u(4,0) = 0$$

$$\gamma u(2,1) + \eta u(3,1) + \alpha u(2,2) + \beta u(3,2) = -\alpha u(2,0) - \beta u(3,0)$$

$$\Rightarrow \gamma u(2,1) + \eta u(3,1) + \alpha u(2,2) + \beta u(3,2) = -\frac{\sqrt{2}}{2}\beta - \alpha \tag{5.25}$$

From equation (5.23), (5.24) and (5.25) we have the following

$$\begin{pmatrix} \eta & \gamma & 0 \\ \gamma & \eta & \gamma \\ 0 & \gamma & \eta \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} + \begin{pmatrix} \beta & \alpha & 0 \\ \alpha & \beta & \alpha \\ 0 & \alpha & \beta \end{pmatrix} \begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2}\beta - \alpha \\ -\sqrt{2}\alpha - \beta \\ -\frac{\sqrt{2}}{2}\beta - \alpha \end{pmatrix} \tag{5.26}$$

For third time step substitute $j = 2$ into equation (5.2). We get

$$\alpha u(i-1,3) + \beta u(i,3) + \alpha u(i+1,3) + \gamma u(i-1,2) + \eta u(i,2) + \gamma u(i+1,2) +$$
$$\alpha u(i-1,1) + \beta u(i,1) + \alpha u(i+1,1) = 0$$

For i=1, then we get

$$\alpha u(0,3) + \beta u(1,3) + \alpha u(2,3) + \gamma u(0,2) + \eta u(1,2) + \gamma u(2,2) +$$
$$\alpha u(0,1) + \beta u(1,1) + \alpha u(2,1) = 0$$

$$\Rightarrow \beta u(1,1) + \alpha u(2,1) + \eta u(1,2) + \gamma u(2,2) + \beta u(1,3) + \alpha u(2,3) = 0 \tag{5.27}$$

For i=2, then we get

$$\alpha u(1,3) + \beta u(2,3) + \alpha u(3,3) + \gamma u(1,2) + \eta u(2,2) + \gamma u(3,2) +$$
$$\alpha u(1,1) + \beta u(2,1) + \alpha u(3,1) = 0$$

$$\alpha u(1,1)+\beta u(2,1)+\alpha u(3,1)+\gamma u(1,2)+\eta u(2,2)+\gamma u(3,2)+\alpha u(1,3)+\beta u(2,3)+\alpha u(3,3) = 0 \tag{5.28}$$

For i=3, then we get

$$\alpha u(2,1) + \beta u(3,1) + \gamma u(2,2) + \eta u(3,2) + \alpha u(2,3) + \beta u(3,3) = 0 \tag{5.29}$$

From equation (5.27), (5.28) and (5.29) we have the following

$$\begin{pmatrix} \beta & \alpha & 0 \\ \alpha & \beta & \alpha \\ 0 & \alpha & \beta \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} + \begin{pmatrix} \eta & \gamma & 0 \\ \gamma & \eta & \gamma \\ 0 & \gamma & \eta \end{pmatrix} \begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \end{pmatrix} + \begin{pmatrix} \beta & \alpha & 0 \\ \alpha & \beta & \alpha \\ 0 & \alpha & \beta \end{pmatrix} \begin{pmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.30}$$

For fourth time step substitute $j = 3$ into equation (5.2). We get

$$\alpha u(i-1,4)+\beta u(i,4)+\alpha u(i+1,4)+\gamma u(i-1,3)+\eta u(i,3)+\gamma u(i+1,3)+ \alpha u(i-1,2)+\beta u(i,2)+\alpha u(i+1,2)=0$$

For i=1, then we get

$$\beta u(1,2)+\alpha u(2,2)+\eta u(1,3)+\gamma u(2,3)+\beta u(1,4)+\alpha u(2,4)=0 \tag{5.31}$$

For i=2, then we get

$$\alpha u(1,2)+\beta u(2,2)+\alpha u(3,2)+\gamma u(1,3)+\eta u(2,3)+\gamma u(3,3)+\alpha u(1,4)+\beta u(2,4)+\alpha u(3,4)=0 \tag{5.32}$$

For i=3, then we get

$$\alpha u(2,2)+\beta u(3,2)+\gamma u(2,3)+\eta u(3,3)+\alpha u(2,4)+\beta u(3,4)=0 \tag{5.33}$$

From equation (5.31), (5.32) and (5.33) we have the following

$$\begin{pmatrix}\beta & \alpha & 0\\ \alpha & \beta & \alpha\\ 0 & \alpha & \beta\end{pmatrix}\begin{pmatrix}u_{1,2}\\ u_{2,2}\\ u_{3,2}\end{pmatrix}+\begin{pmatrix}\eta & \gamma & 0\\ \gamma & \eta & \gamma\\ 0 & \gamma & \eta\end{pmatrix}\begin{pmatrix}u_{1,3}\\ u_{2,3}\\ u_{3,3}\end{pmatrix}+\begin{pmatrix}\beta & \alpha & 0\\ \alpha & \beta & \alpha\\ 0 & \alpha & \beta\end{pmatrix}\begin{pmatrix}u_{1,4}\\ u_{2,4}\\ u_{3,4}\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{5.34}$$

Thus, combine the equation (5.22), (5.26), (5.30) and (5.34). We get

$$\begin{bmatrix}
\beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\alpha & \beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\eta & \gamma & 0 & \beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\gamma & \eta & \gamma & \alpha & \beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \gamma & \eta & 0 & \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0\\
\beta & \alpha & 0 & \eta & \gamma & 0 & \beta & \alpha & 0 & 0 & 0 & 0\\
\alpha & \beta & \alpha & \gamma & \eta & \gamma & \alpha & \beta & \alpha & 0 & 0 & 0\\
0 & \alpha & \beta & 0 & \gamma & \eta & 0 & \alpha & \beta & 0 & 0 & 0\\
0 & 0 & 0 & \beta & \alpha & 0 & \eta & \gamma & 0 & \beta & \alpha & 0\\
0 & 0 & 0 & \alpha & \beta & \alpha & \gamma & \eta & \gamma & \alpha & \beta & \alpha\\
0 & 0 & 0 & 0 & \alpha & \beta & 0 & \gamma & \eta & 0 & \alpha & \beta
\end{bmatrix}
\begin{bmatrix}u_{11}\\ u_{21}\\ u_{31}\\ u_{12}\\ u_{22}\\ u_{32}\\ u_{13}\\ u_{23}\\ u_{33}\\ u_{14}\\ u_{24}\\ u_{34}\end{bmatrix}
=
\begin{bmatrix}-\frac{1}{2}(\frac{\sqrt{2}}{2}\eta+\gamma)\\ -\frac{1}{2}(\sqrt{2}\gamma+\eta)\\ -\frac{1}{2}(\frac{\sqrt{2}}{2}\eta+\gamma)\\ -\frac{\sqrt{2}}{2}\beta-\alpha\\ -\sqrt{2}\alpha-\beta\\ -\frac{\sqrt{2}}{2}\beta-\alpha\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

we solve this using matlab and results are shown in table (5.2) and Graphically illustrated by figure (5.3).

Table 5.2: Exact Solution, numerical solution by CFDM and Absolute Error

Grip point	Exact Solution	Numerical Solution of CFDM	Absolute Error
u11	0.706561627484314	0.706547923590673	1.37038936404155e-05
u21	0.999229036240723	0.999209656008479	1.9380232243571e-05
u31	0.706561627484314	0.706547923590673	1.37038936404155e-05
u12	0.704927006965107	0.704872234182536	5.47727825710442e-05
u22	0.996917333733128	0.996839873321167	7.74604119608835e-05
u32	0.704927006965107	0.704872234182536	5.47727825710442e-05
u13	0.702205440095289	0.702082361704247	0.000123078391041975
u23	0.993068456954926	0.99289439782508	0.000174059129846138
u33	0.702205440095289	0.702082361704247	0.000123078391041975
u14	0.69840112333371	0.698182716073708	0.000218407260002551
u24	0.987688340595138	0.987379466085922	0.000308874509215684
u34	0.69840112333371	0.698182716073708	0.000218407260002551
Maximum error			0.000308874509215684

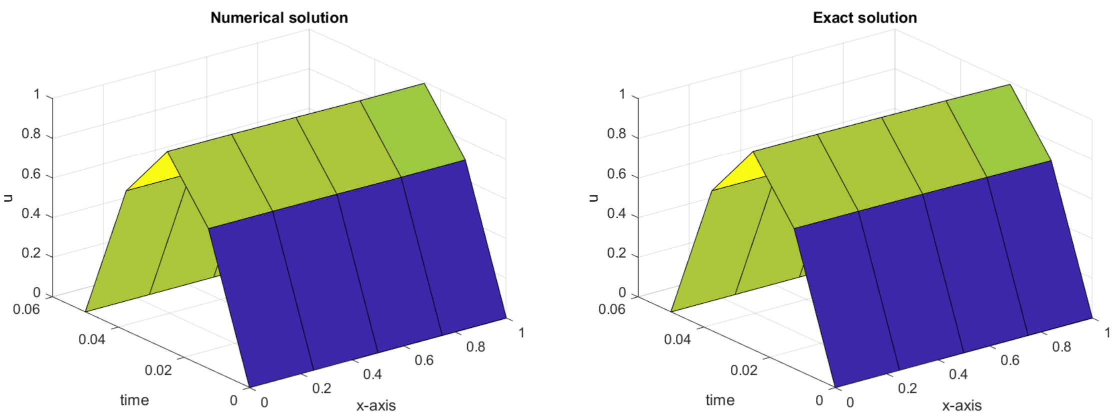


Figure 5.3: Exact Solution and Numerical Solution by CFDM for 1D wave equation

5.2 Discussion

In this study we applied sixth-order CFDM and FEM for 1D wave equation. The compact techniques in FDM are usually used to obtain the numerical solution of differential equations. Therefore, the CFDM can obtain by repeated differentiation and estimated the higher derivative. In FEM we use Galerkin's method to transform wave equation into weak formulation. We developed the linear system of equation used for the matlab simulation with exact and approximate solution. From the system of equation to understand the behavior of numerical and exact solution at grid points. The method reduces the underlying problem to linear system of algebraic equations, which can be solved successively to obtain a numerical solution at varied time-levels. From the above tables (5.1) and (5.2) have shown exact, numerical solution and absolute error when the solution is convergent. Sixth order CFDM more accurate than FEM as given in above example. Numerical experiments which were shown in the above tables and graphs are good agreement with the exact ones. The tables and graphs shown that CFDM gives good numerical results than FEM.

Chapter 6

Conclusion and Recommendation

6.1 Conclusion

This thesis presents the basic finite element method and the compact finite Difference method to solve 1D wave PDE. The main idea of finite element method is to choose the appropriate basis functions and then expressing the unknown as linear combination of the basis functions. To transform partial differential equation into weak formulation we use Galerkin's method. To investigate the stability, consistency and convergence of finite element method, one may observe the results which were obtained in the test example in table (5.1). For compact finite difference method, we observe the results which were obtained in tested example, using table (5.2) which shows the absolute errors and maximum error. This shows that as mesh sizes becomes closer and closer to zero, it produces a bounded solution which approaches the exact solution and the absolute and maximum error are smaller, which shows that the method is stable. The linear system of equations that is obtained from both FEM and CFDM is sparse matrix which contains more zero elements. In both methods matlab is used to produce the results, based on matrix inversion.

Generally, sixth order compact finite difference method has small absolute and maximum error with relative to Finite element method. Hence sixth order compact finite difference method is more accurate than the result from finite element method.

6.2 Recommendation

The results obtained in this thesis on the comparison of FEM and sixth order CFDM for solving wave equation on stability, consistency and convergence in 1D. This thesis can be extended further: One may make analysis of the three methods with FVM or other methods.

Research Fund Acknowledgment

This research project is funded by Adama Science and Technology University under the grant number ASTU/SM-R/015/19, Adama, Ethiopia.

References

[1] Jain, M. K., Iyengar, S. R. K., & and Jian, R. K. (2007). Numerical Methods for Scientific and Engineering Computation, new Age International (P) Ltd. *Publishers New Delhi*.

[2] Zafar, Z. U., Rafiq, M., Pervaiz, A., & Ahmed, M. O. (2016). Finite Element Model for Linear Second Order One Dimensional Inhomogeneous Wave Equation. *Pakistan Journal of Engineering and Applied Sciences*.

[3] Rajni A., Suruchi, S., & Swarn, S.(2015) Exponential B-Spline Collocation Method For The Numerical Solution Of One-Space Dimensional Nonlinear Wave Equation With Strong Stability Preserving Time Integration. *International Journal of Advance Research in Science and Engineering, 4(11)*, 102-113.

[4] Saadatmandi, A., & Dehghan, M. (2007). Numerical solution of the one-dimensional wave equation with an integral condition. *Numerical Methods for Partial Differential Equations: An International Journal, 23(2)*, 282-292.

[5] Rashidinia, J., & Mohsenyzade, M. (2015). Numerical solution of one-dimensional heat and wave equation by non-polynomial quintic spline. *Int. J. Math. Model. Comput, 5(4)*, 291-305.

[6] Duressa, G. F., Bullo, T. A. & Kiltu G. G(2016). Fourth Order Compact Finite Difference Method for Solving One Dimensional Wave. *International Journal of Engineering & Applied Sciences, 8(4)*, 30-39.

[7] Zhang, T., & Zhang, S. (2020). The weak Galerkin finite element method for the symmetric hyperbolic systems. *Journal of Computational and Applied Mathematics, 365*, 112375.

[8] Mohamed, S. A., Shanab, R. A., & Seddek, L. F. (2016). Vibration analysis of Euler-

Bernoulli nanobeams embedded in an elastic medium by a sixth-order compact finite difference method. *Applied Mathematical Modelling, 40(3)*, 2396-2406.

[9] Akbari, R., Mokhtari, R., & Jahandideh, M. T. (2019). A combined compact difference scheme for option pricing in the exponential jump-diffusion models. *Advances in Difference Equations, 2019(1)*, 1-13.

[10] Wang, B., Sun, T., & Liang, D. (2019). The conservative and fourth-order compact finite difference schemes for regularized long wave equation. *Journal of Computational and Applied Mathematics, 356*, 98-117.

[11] Pandit, S. K., & Chattopadhyay, A. (2017). A robust higher order compact scheme for solving general second order partial differential equation with derivative source terms on nonuniform curvilinear meshes. *Computers & Mathematics with Applications, 74(6)*, 1414-1434.

[12] Ganzha, V. G. E.,& Vorozhtsov, E. V. (2017). *Numerical solutions for partial differential equations: problem solving using Mathematica.* CRC Press.

[13] Pepper, D. W., & Heinrich, J. C. (2017). The finite element method: basic concepts and applications with MATLAB, MAPLE, and COMSOL. CRC press.

[14] Möller, T., & Friederich, W. (2016, April). 1D and 2D simulations of seismic wave propagation in fractured media. In *EGU General Assembly Conference Abstracts* (Vol. 18).

[15] Kapuria, S., & Kumar, A. (2020). A wave packet enriched finite element for electroelastic wave propagation problems. *International Journal of Mechanical Sciences, 170*, 105081.

[16] Falletta, S. (2018). BEM coupling with the FEM fictitious domain approach for the solution of the exterior Poisson problem and of wave scattering by rotating rigid bodies. *IMA Journal of Numerical Analysis, 38(2)*, 779-809.

[17] Karakoc, S. B. G., & Bhowmik, S. K. (2019). Numerical approximation of the generalized regularized long wave equation using Petrov-Galerkin finite element method. *arXiv preprint arXiv:1904.03354.*

[18] Gorynina, O., Lozinski, A., & Picasso, M. (2019). An easily computable error estimator in space and time for the wave equation. *ESAIM: Mathematical Modelling and Numerical Analysis, 53(3)*, 729-747.

[19] Karunanithi, S., Gajalakshmi, N., Malarvishi, M., & Saileshwari, M. (2018). A Study on comparison of Jacobi, Gauss-Seidel and SOR methods for the solution in system of linear equations. *Int. J. of Math. Trends and Technology,(IJMTT), 56(4).*

[20] Everink, J. M. (2018). *Numerically solving the wave equation using the finite element method* (Bachelor's thesis).

[21] Nikishkov, G. P. (2004). Introduction to the finite element method. *University of Aizu*, 1-70.

[22] Christiansen, S. H. (2009). Foundations of finite element methods for wave equations of Maxwell type. *In Applied wave mathematics* (pp. 335-393). Springer, Berlin, Heidelberg.

[23] Harwood, R. C., & Main, M. (2017). Eigenvalue Dependence of Numerical Oscillations in Parabolic Partial Differential Equations. *arXiv preprint arXiv:1701.04798.*