LIFE SPAN AND CONSISTENCY OF SOLUTIONS FOR WHITHAM–BOUSSINESQ TYPE EQUATIONS

Tilahun Deneke Weldegiorgis

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy
in Applied Mathematics (Specialization in Applied Differential Equations)

Office of Graduate Studies
Adama Science and Technology University

May, 2023
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Declaration

I hereby declare that this Dissertation entitled " Life Span and Consistency of Solutions for Whitham–Boussinesq Type Equations " is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials used for this dissertation have been duly acknowledged through appropriate citations.

Tilahun Deneke Weldegiorgis	____________________	____________________
Name of Student	Signature	Date

Recommendation

We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled "Life Span and Consistency of Solutions for Whitham–Boussinesq Type Equations " prepared under our guidance by Tilahun Deneke Weldegiorgis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics. Therefore, we recommend the submission of the dissertation to the department for further review and defense.

Dr. Achenef Tesfahun	____________________	____________________
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Approval page

We hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled " Life Span and Consistency of Solutions for Whitham–Boussinesq Type Equations" by Tilahun Deneke Weldegiorgis.

Dr. Achenef Tesfahun	________________	________________
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We, the undersigned, members of the Board of Examiners of the dissertation open defense by Tilahun Deneke Weldegiorgis have read and evaluated the dissertation entitled " Life Span and Consistency of Solutions for Whitham–Boussinesq Type Equations " and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics (Specialization in Applied Differential Equations).

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Dedication

This work is dedicated for the Jesus Christ.

Acknowledgments

I have especially grateful to my main supervisor Dr. Achenef Tesfahun for his guidance, advice , and direction during course work as well as research work. I sincerely appreciate the time and energy he put into my supervision through every detail of my work and giving me constructive comments. I would also like to thank my co-supervisor Dr. Tamirat Temesgen for his guidance, and support during my study.

I wish to express my special thanks to my family for their patient during my study. It is also my pleasure to thanks my colleagues for their support, comment, and suggestions during my work. Finally I thanks Adama Science and Technology university, in particular applied mathematics department.

Contents

Notations

α Multi-index in $\mathbb{N}_0^d$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$

$|\alpha|$ The length of multi-index α, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$

x^α The multi-index power of $x \in \mathbb{R}^d$, $x^\alpha = x_1^{\alpha_1}.x_2^{\alpha_2} \ldots x_d^{\alpha_d}$

∂^α $\partial^\alpha := \partial_1^{\alpha_1} \ldots \partial_1^{\alpha_d}$, where $\partial_j = \frac{\partial}{\partial x_j}$ $j = 1, \ldots, d$.

$\mathscr{S}$ The space of Schwartz function

$\mathscr{S}'$ The space of continuous linear functional on $\mathscr{S}$

H^k The Sobolev space of order k, where $k \in \mathbb{R}$

$a \lesssim b$ For $a, b > 0$ it stands for $a \leq cb$, where $c > 0$ that may change from line to line.

$a \ll b$ Represent $a \leq cb$, where $0 < c < 1$

$a \sim b$ Stands for $a \lesssim b$ and $b \lesssim a$

$\langle x \rangle$ Stands for $(1 + |x|^2)^{\frac{1}{2}}$

$\|f\|_{L_T^q L_x^r}$ Represent the mixed space-time norm $\left(\int_0^T \|f(.,t)\|_{L_x^r}^q dt\right)^{\frac{1}{q}}$ when $1 \leq q, r < \infty$, with the usual modification when $q, r = \infty$

List of publications

Part of this dissertation has been published in the form of the following papers in peer reviewed indexed journals in web of science and scopus.

- T. Deneke, T. T. Dufera and A. Tesfahun (2022): Comparison between Boussinesq and Whitham–Boussinesq-type systems. Mathematical Methods in the Applied Sciences. Volume 45 issue 16.
  https://doi.org/10.1002/mma.8304

- T. Deneke, T. T. Dufera and A. Tesfahun (2023): Dispersive Estimates for Linearized Water Wave Type Equations in $\mathbb{R}^d$. Annales Henri Poincaré.
  https://doi.org/10.1007/s00023-023-01322-0

Abstract

In this dissertation we consider a Whitham-Boussinesq type system modeling surface water waves of an inviscid incompressible fluid layer. The system describes the evolution with time of surface waves of a liquid layer in $d+1$ *dimensional physical space. We are interested in well-posedness at a low level regularity. We diagonalized the system and reduce it to integral form. Then we derived solution propagators. We established a lower bound for derivatives of all order for the Fourier symbol of the pseudo-differential operator that appears in linearized water wave type equations for both purely gravity waves and capillary-gravity waves. We then derived dispersive and Strichartz estimate for linearized water wave type equations in* $\mathbb{R}^d$*. We also derived bilinear estimates which are important to bound nonlinear terms. Using these estimates together with contraction mapping principle we prove low regularity well-posedness for Whitham-Boussinesq type system in* $\mathbb{R}^d$*,* $d \geq 3$*. This dissertation also contains a study of the long time existence of solutions of Whitham–Boussinesq type system in one dimensional space. Using fixed point argument we prove that the system is well-posed on the time scale of order* $\mathscr{O}(1/\sqrt{\varepsilon})$*, where* $\varepsilon > 0$ *is the shallowness parameter measuring the ratio of amplitude of the wave to mean depth of fluid. Thus if the amplitude of the wave is very small compared to the mean depth of the fluid that is* $0 < \varepsilon \leq 1$ *then the solution exists for arbitrarily large time. We also show that the solution to the Whitham–Boussinesq type system approximates the solution of a Boussinesq system on the time scale of order* $\mathscr{O}(1/\sqrt{\varepsilon})$ *for one dimensional space.*

Keywords: Surface waves; Whitham–Boussinesq system; Dispersive estimates; Well-posedness; Long time existence; Comparison estimate.

Chapter 1

Introduction

1.1 Background of the study

Fluid dynamics is one of the most important of all areas of physics. As we know life would not exist without fluids and without the behaviors that fluids exhibit. Therefore we have to know some basic properties of fluids to categorize the fluids flow according to their behaviors. There are many behaviors of fluids flow, in this dissertation we will discuss only those which have particular relevance for us. If we consider the flow, not a fluid, a flow will be incompressible if its density is constant. Rotational flows are those containing many swirls or vertices, that is the fluids elements are rotating. Conversely, fluids not exhibiting such effects might be considered to be irrotational.

The waves on a surface of an incompressible fluid are one of the most common phenomena we observe in our life. The theory of surface waves attracted the attention of many mathematicians. Among them the most famous mathematicians are Boussinesq, Stokes and Airy. In spite of the fact that the mathematical aspect of wave propogation is one of the classical subjects of mathematical physics, the theory of surface waves for many decades weakly connected with the theory of sound and the theory of electromagnetic waves (Zakharov, 1998). One of the reasons for this was a dispersion. The waves on the surface of an incompressible fluid are strongly dispersive. Their phase velocity depends essentially on a wave number. If the group velocity c is a non constant function one can says that the wave (or rather the equation) is dispersive (Linares and Ponce, 2014). In the physical context this means that when time evolves, the different waves disperse in the medium, with the result that a single hump breaks into wave trains. The basic equations describing waves on a surface of an incompressible fluid flow in their classical formulation are catheterized as dispersive partial differential equations (Zakharov, 1998).

The theory of surface waves became a beginning of the modern theory of waves in nonlinear dispersive media. The fast development of plasma and solid state physics showed that a strong dispersion is a common thing for waves in real media, sound and light waves which mostly are strongly dispersive (Zakharov, 1968). In the twenty century the surface waves became a subject of intense study and (Zakharov, 1967) have found that the waves on a surface of ideal fluid in mathematical sense are Hamiltonian systems and their theory can be treated in a general theory of dispersive waves. This fact made possible to establish numerous parallels between the theory of surface waves and theories of waves of different type in plasmas and nonlinear optics.

The study of nonlinear evolution equations arising from mathematical models for fluid flow is of great interest both for mathematicians and engineers. For example, nonlinear and shallow water wave characteristics become important for coastal engineers to understand the influence of shallow water waves on nonlinear forms (Mohapatra et al., 2016, 2019). The character of coastal wave dynamics can be very complex; waves are strongly affected by the bottom topology through shoaling and the resulting variations in local linear wave speed, with the subsequent effects of refraction, diffraction, and reflection (Craig et al., 2005). Many researchers formulate mathematical models based on Boussinesq regime, small amplitude long wave length, to study propagation of shallow water wave (Dinvay et al., 2019; Saut and Xu, 2012). In(Mohapatra et al., 2019, 2022) different Boussinesq type models are proposed and studied analytically as well as numerically.

The theory of surface waves have a wide range of applications both mathematically and physically. Most of equations of mathematical physics arise from the water wave equations, in particular from surface waves model's. Engineers also use the model's of surface wave to describe the characteristics of solitary waves in coastal area. So in next section we introduce how these models are developed.

1.2 The equations of fluid motion

Motion of fluid is governed by the Euler system consisting of the Laplace's equation in this domain

$$\Delta\phi = 0 \quad \text{for} \quad x \in \mathbb{R}^d, \quad -h < z < \eta(x,t), \tag{1.2.1}$$

the Neumann boundary condition at the flat bottom

$$\partial_z \phi = 0 \quad \text{at} \quad z = -h, \tag{1.2.2}$$

the kinematic condition at the free surface

$$\partial_t \eta = \partial_z \phi - \sum_{i=1}^{d} \partial_{x_i} \phi \partial_{x_i} \eta \quad \text{for}\, x \in \mathbb{R}^d, \quad z = \eta(x,t), \tag{1.2.3}$$

and the Bernoulli equation

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + g\eta = 0 \quad \text{at}\, z = \eta(x,t). \tag{1.2.4}$$

The unknowns are surface elevation $\eta(x,t)$ and velocity potential $\phi(x,z,t)$. The fluid domain is the set $\{(x,z) \in \mathbb{R}^{d+1} : -h < z < \eta(x,t)\}$ extending to infinity in the positive and negative x-direction. System (1.2.1)-(1.2.4) poses a conserved quantity having the meaning of the total energy

$$\mathscr{H} = \int_{\mathbb{R}^d} \int_0^{\eta} gz dz dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{-h}^{\eta} |\nabla \phi|^2 dz dx, \tag{1.2.5}$$

that is the sum of the potential energy due to gravity and the kinetic energy (Dinvay et al., 2019).

If we introduce the trace of the potential at the free surface as $\Phi(x,t) = \phi(x,\eta(x,t),t)$ with associate Dirichlet–Neumann operator $G(\eta)$ given by the formula (Alazard et al., 2011; Lannes, 2013)

$$G(\eta)\Phi = \sqrt{1 + |\nabla \eta|^2}\partial_n \phi, \tag{1.2.6}$$

where $\partial_n \phi$ is the projection of the surface fluid velocity on the outer normal, then the system (1.2.1)-(1.2.4) is reduced to

$$\partial_t \eta = G(\eta)\Phi, \tag{1.2.7}$$

$$\partial_t \Phi = -g\eta - \frac{1}{2}|\nabla \Phi|^2 + \frac{(\nabla \eta . \nabla \Phi + G(\eta)\Phi)^2}{2(1 + |\nabla \eta|^2)}, \tag{1.2.8}$$

with $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$. The pair (η, Φ) solve system (1.2.7)-(1.2.8) describing the surface waves completely. This formulation is known to be equivalent to the Euler system (1.2.1)-(1.2.4) (Dinvay et al., 2019). However, it does not simplify the water wave problem since in general there is no explicit expression for the operator $G(\eta)$. As shown by (Zakharov, 1998), the system (1.2.7)-(1.2.8) has Hamiltonian structure

$$\partial_t \eta = \frac{\delta \mathscr{H}}{\delta \Phi}, \quad \partial_t \Phi = -\frac{\delta \mathscr{H}}{\delta \eta}. \tag{1.2.9}$$

Together with the trace $\Phi(x,t) = \phi(x.\eta(x,t),t)$ of the potential at the free surface and the Dirichlet–Neumann operator $G(\eta)$ the total energy (1.2.5) takes the form (Dinvay et al., 2019)

$$\mathscr{H} = \frac{1}{2}\int_{\mathbb{R}^d} \left(g\eta^2 + \Phi G(\eta)\Phi\right) dx. \tag{1.2.10}$$

Thus, one can simplify the water wave problem further approximating the Dirichlet–Neumann operator by different explicit expression.

The simplest possible approximation of (1.2.7)-(1.2.8) is the linearization

$$\partial_t \eta = G_0 \Phi, \tag{1.2.11}$$

$$\partial_t \Phi = -g\eta, \tag{1.2.12}$$

where $G_0 = G(0)$ is the Dirichlet–Neumann operator corresponding to the undisturbed surface $\eta = 0$. For any given potential trace Φ the elliptic problem (1.2.1) with fluid domain $\{(x,z) \in \mathbb{R}^{d+1} : -h < z < \eta(x,t)\}$, (1.2.2) and $\phi = \Phi$ at $z = 0$ can be easily solved in the frequency space. Indeed, defining $\widehat{\phi}(\xi,z)$ to be Fourier transform of ϕ with respect to the horizontal variables one obtains

$$\begin{cases} \partial_z^2 \widehat{\phi}(\xi,z) = |\xi|^2 \widehat{\phi}(\xi,z) \quad \text{for } -h < z < 0, \\ \partial_z \widehat{\phi}(\xi,-h) = 0, \\ \widehat{\phi}(\xi,0) = \widehat{\Phi}(\xi), \end{cases}$$

that is a second order ordinary differential equation in z with boundary conditions at $z = -h$ and $z = 0$. It has a unique solution given by

$$\widehat{\phi}(\xi,z) = \frac{\cosh((h+z)|\xi|)}{\cosh(h|\xi|)} \widehat{\Phi}(\xi),$$

where ξ lies in the frequency space and $z \in [-h,0]$. Thus, from Dirichlet–Neumann operator (1.2.6) we have $G_0\Phi = \partial_z \phi$ at $z = 0$ and so

$$\mathscr{F}(G_0\Phi)(\xi) = \partial_z \widehat{\phi}(\xi,0) = |\xi| \tanh(h|\xi|) \widehat{\Phi}(\xi).$$

That is, G_0 is a Fourier multiplier operator given by the formula

$$G_0 = |D|\tanh(h|D|), \tag{1.2.13}$$

where $D = -i\partial_x$ if $d = 1$ and $D = -i\nabla$ if $d > 1$. Then we can solve system (1.2.11)-(1.2.12) analytically.

Nonlinear long wave approximation of the water wave problem (1.2.7)-(1.2.8) can be done through expansion of the Dirichlet–Neumann operator (1.2.6) developed by (Craig et al., 1997, 1992). It was shown in (Zakharov, 1998) that this operator depends analytically on the unknown η and can therefore be expanded in the power series

$$G(\eta)\Phi = \sum_{j=0}^{\infty} G_j(\eta)\Phi, \tag{1.2.14}$$

where each operator $G_j(\eta)$ is homogeneous of degree j in η. The first term G_0 for zero order was computed in (1.2.13). The next term has the form

$$G_1(\eta)\Phi = D(\eta D\Phi) - G_0(\eta G_0\Phi) \tag{1.2.15}$$

and the rest terms in the power series can be computed using a recursion formula Craig and Groves (1994); Craig and Sulem (1993). Introducing the new variable

$$\mathbf{v} = K_h \nabla\Phi,$$

where Φ is the surface trace of the potential ϕ and

$$K_h = \frac{\tanh(h|D|)}{h|D|}, \tag{1.2.16}$$

the Hamiltonian structure (1.2.9) transforms to (Dinvay et al., 2019, 2020)

$$\partial_t(\eta, \mathbf{v})^T = \mathbf{J}\nabla(\eta, \mathbf{v})$$

when we used change of variables $\Phi \mapsto \mathbf{v}$, with skew adjoint matrix

$$\mathbf{J} = \begin{pmatrix} 0 & -K_h\partial_{x_1} & \cdots & -K_h\partial_{x_{d-1}} & -K_h\partial_{x_d} \\ -K_h\partial_{x_1} & 0 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & \\ -K_h\partial_{x_d} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Applying the Hamiltonian long wave approximation, and keeping untouched the part of the Hamiltonian responsible for the linear wave, one simplifies the energy functional to the form (Dinvay et al., 2019, 2020; Craig and Groves, 1994; Craig et al., 1992)

$$\mathscr{H}(\eta,\mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^d} \left(g\eta^2 + h|K_h^{-1}\mathbf{v}|^2 + \eta|\mathbf{v}|^2\right) dx. \tag{1.2.17}$$

Such Hamiltonian structure generates evolution described by the model

$$\begin{cases} \partial_t\eta + h\nabla.\mathbf{v} = -K_h\nabla.(\eta\mathbf{v}), \\ \partial_t\mathbf{v} + gK_h\nabla\eta = -K_h\nabla(|\mathbf{v}|^2/2), \end{cases} \tag{1.2.18}$$

for two dimensional case (Dinvay, 2020). There are at least two conserved quantities for this system. The first one is energy (1.2.17). The second one has the meaning of momentum and has a form (Dinvay, 2020)

$$J(\eta,\mathbf{v}) = \int_{\mathbb{R}^d} \eta K_h^{-1}\mathbf{v}dx. \tag{1.2.19}$$

The latter conserves under the restriction that $\mathbf{v}$ is a curl free vector field that is, $\nabla \times \mathbf{v} = 0$.

In this section we have seen how Whitham–Boussinesq type equations (1.2.18) was developed. In the derivation of this model (Dinvay et al., 2019) used Boussinesq regime, that is, small amplitude long wave length. This model is an alternative to other water wave type equations. The model has many interesting future of water wave problems and in good agreements with experiments (Carter, 2018). Thus, we are interested in improving the well-posedness results of this model.

1.3 Statement of the problem

Our principal aim here is to study the well-posedness of the Cauchy problem for the Whitham-Boussinesq system (1.2.18). Concerning the local well-posedness question, some results are obtained for the Whitham–Boussinesq system (1.2.18). Dinvay (2019) proved that (1.2.18) is locally well-posedness in Sobolev space $H^s(\mathbb{R})$ for $s > 1/2$ in one dimension. More recently, (Dinvay et al., 2020) improved local well-posedness of the Cauchy problem (1.2.18) in one dimension and extend it to two dimensions. They also proved global well-posedness of this Cauchy problem for sufficiently small initial data in the $L^2(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R})$ norm. To our knowledge there is no well-posedness result for this system in $\mathbb{R}^d$ for $d \geq 3$. Also the long time existence of solution and it's consistence with other Boussinesq system were not studied. Therefore, we are interested to study well-posedness of Whitham–Boussinesq type system posed on $\mathbb{R}^{d+1}$ written as

$$\begin{cases} \partial_t \eta + \nabla.\mathbf{v} = -K\nabla.(\eta \mathbf{v}), \\ \partial_t \mathbf{v} + L_\kappa^2 \nabla \eta = -K\nabla(|\mathbf{v}|^2/2), \end{cases} \tag{1.3.1}$$

where $\eta : \mathbb{R}^{d+1} \mapsto \mathbb{R}, \mathbf{v} : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ is curl free vector field that is $\nabla \times \mathbf{v} = 0$ and

$$L_\kappa := L_\kappa(D) = \sqrt{(1+\kappa|D|^2)K(D)}, \tag{1.3.2}$$

with

$$K(D) = \frac{\tanh|D|}{|D|} \quad (D = -i\nabla = -i\partial_x),$$

and κ measure surface tension effect.

In Particular, we study the following:

1. Dispersive and Strichartz estimates for linearized water wave type equations in $\mathbb{R}^d$.

2. Long time existence of solution for Whitham–Boussinesq type system in $\mathbb{R}$.

3. Comparison between Boussinesq and Whitham–Boussinesq type systems in $\mathbb{R}$.

1.4 General and specific objectives

1.4.1 General objective of the study

The general objective is to study the well-posedness of Whitham–Boussinesq type system (1.3.1) and it's consistency with Boussinesq system.

1.4.2 Specific objectives

The specific objectives of this study are to

1. derive frequency localized despersive estimates for linearized water wave type equations in $\mathbb{R}^d$.

2. derive frequency localized Strichartz estimates for linearized water wave type equations in $\mathbb{R}^d$.

3. prove local well-posedness of Whitham–Boussinesq type system in $\mathbb{R}^d$ for $d \geq 3$.

4. determine the long time existance of solution of Whitham–Boussinesq type system in $\mathbb{R}$.

5. formulate the difference estimate between Whitham–Boussinesq type and Boussinesq systems in $\mathbb{R}$.

1.5 Significance of the study

The results obtained in this study are important for both engineers and mathematicians. This study can contribute useful tools for coastal engineers to study nonlinear water wave characteristics which are important to understand the effect of shallow water wave on coastal area of lake, sea and ocean. Air strips and train tracks can be built on thick ice covers. Even these activities met varied success, sometimes results in loss of life and equipment. So there were a need to improve understanding of engineering of ice covers. These were done by including mathematical treatments based on water wave theory. The results in this dissertation can also be used by mathematicians to study the water wave problems which are modeled based on Boussinesq regime.

1.6 Scope of the study

This dissertation is limited to the study of dispersive and Strichartz estimates for linearized water wave type equations in $\mathbb{R}^d$, well-possedness of Whitham–Bossinesq type equations (1.3.1) and comparison estimate between Boussinesq and Whitham–Boussinesq system.

1.7 Organization of the dissertation

The remaining of this dissertation is organized as follow. In Chapter 2 we review some related literature and Chapter 3 is concerned with method we used to achieve the stated objectives. In Chapter 4 we state some mathematical tools which we use later in the dissertation.

In Chapter 5 using change of variables we derived solution propagators. Then, we establish a lower bound estimates for derivatives of all order of $m_\kappa(r) = \sqrt{r(1+\kappa r^2)\tanh r}$. Here $m_\kappa(r)$ is the Fourier symbol of pseudo-differential operator $m_\kappa(D)$ that appears in linearized water wave type equations. Using these estimates we derive dispersive and Strichartz estimates. We also derive bilinear estimates. Finally we proved that the Whitham–Boussnesq type equations (1.3.1) is locally well-posed in $\mathbb{R}^d$ for $d \geq 3$.

In Chapter 6, first we derive some useful estimates which helps us to prove long time existence of solutions. Again using change of variable we write Whitham–Boussinesq type system in integral form. Then using contraction mapping principle we proved that the Whitham–Boussinesq type system is well-posed on the time scale of order $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$, where $\varepsilon > 0$ is shallowness parameter. We also derived a comparison estimate between Whitham–Boussinesq type and Boussinesq systems.

Chapter 2

Review of Related Literature

2.1 The water wave models

The motion of a perfect, incompressible and irrotational fluid under the influence of gravity is described by the free surface Euler (or water-waves) equations. The complexity of Euler equations led physicists and mathematicians to derive simpler sets of equations most likely to describe the dynamics of the water waves equations in some specific physical regimes. In fact, many of the most famous equations of mathematical physics were historically obtained as formal asymptotic limits of the water waves equations. For example, (Korteweg and De Vries, 1895) and (Kadomtsev and Petviashvili, 1970) equations are derived in these regime. Each of these asymptotic limits corresponds to a very specific physical regime whose range of validity is determined in terms of the characteristics of the flow such as amplitude, wavelength, bottom topography, depth, and so on.

It is generally understood that the Korteweg-de Vries equation, appear to describe the approximate behavior of unidirectional long waves in nonlinear dispersive media (Zakharov, 1968). The derivation of such model equations in specific physical situations is presented in (Benjamin et al., 1972). Considerable stimulus to the study of the Korteweg-De Vries equation was provided by the research group at the Plasma Physics Laboratory of Princeton University. They built on earlier work of (Whitham, 1967) and obtain very interesting results.

From Korteweg-De Vries, which is known to be a good model for waves of long wavelength in shallow water, (Whitham, 2011) proposed the mode

$$v_t + \sqrt{K} v_x + v v_x = 0,$$

where the operator K is a Fourier multiplier with symbol $K(\xi) = \frac{\tanh(\xi)}{\xi}$, that incorporates the full unidirectional dispersion relation for the Euler equations in an effort to better capture mid and high frequency phenomena found in the Euler equations, such as wave steepening or peaking. Whitham equation, has received considerable attention, having been investigated analytically in (Ehrnström and Kalisch, 2009, 2013; Ehrnström et al., 2019; Hur and Johnson, 2015; Hur and Pandey, 2019) as well as numerically (Sanford et al., 2014; Remonato and Kalisch, 2017).

Whitham's idea has been extended to the study of systems of evolution equations which allow for bidirectional wave propagation. In particular, in (Aceves-Sánchez et al., 2013; Ehrnström et al., 2019; Hur and Pandey, 2019; Hur and Tao, 2018; Carter, 2018; Dinvay et al., 2019) the bidirectional Whitham models that capture more of the important qualitative properties of the full Euler equations, such as existence of smooth solitary waves and peaked traveling waves, and both high frequency and modulational instabilities of small periodic wave trains are developed.

In (Klein et al., 2018), the following bidirectional Whitham–Boussinesq system was considered

$$\begin{cases} \partial_t \eta + K_\varepsilon \partial_x v + \varepsilon \partial_x(\eta v) = 0, \\ \partial_t v + \partial_x \eta + \varepsilon v \partial_x v = 0, \end{cases} \tag{2.1.1}$$

where

$$K_\varepsilon := K_\varepsilon(D) = \frac{\tanh(\sqrt{\varepsilon} D)}{\sqrt{\varepsilon} D}.$$

Taking the limit $\sqrt{\varepsilon}|\xi| \mapsto 0$ in K_ε , (2.1.1) reduces formally to

$$\begin{cases} \partial_t \eta + \partial_x v + \frac{\varepsilon}{3}\partial_x^3 v + \varepsilon \partial_x(\eta v) = 0, \\ \partial_t v + \partial_x \eta + \varepsilon v \partial_x v = 0. \end{cases} \tag{2.1.2}$$

This system is locally well-posed under a non physical condition on the initial data (Klein et al., 2018). System (2.1.2) is also known in the inverse scattering community as the Kaup-Kupperschmidt system. The Boussinesq system (2.1.1) can be seen as a well-posed regularization of the Kaup-Kupperschmidt system (Klein et al., 2018).

Dinvay et al. (2019) derived the bidirectional Whitham–Boussinesq type model

$$\begin{cases} \partial_t \eta + \partial_x v = -K\partial_x(\eta v), \\ \partial_t v + K\partial_x \eta = -K\partial_x(v^2/2), \end{cases} \tag{2.1.3}$$

where

$$K := K(D) = \frac{\tanh(D)}{D}.$$

One can observe that the nonlinear part of system (2.1.3) contains only the bounded operator $K\partial_x$. So among all bidirectional Whitham systems this is numerically the most stable one (Dinvay et al., 2019). There are also other bidirectional Whitham system considered in (Dinvay et al., 2019) which was written as

$$\begin{cases} \eta_t = -Kv_x - (\eta v)_x, \\ v_t = -\eta_x - vv_x. \end{cases} \tag{2.1.4}$$

It was shown in (Moldabayev et al., 2015) how this system arises as a Hamiltonian system from the Zakharov-Craig-Sulem formulation of the water wave problem using an exponential long wave scaling. Also system (2.1.4) has been studied numerically in the presence of an uneven bottom in (Vargas-Magana and Panayotaros, 2016).

2.2 Well-posedness theory for Whitham–Boussinesq type equations

It was known that the model (2.1.3) can be obtained in the long wave framework from the Zakharov–Craig–Sulem formulation of the water wave problem (Lannes, 2013), also known to be Hamiltonian. Thus, system (2.1.3) provide an example of a nonlinear Hamiltonian system that is locally well posed. The model (2.1.3) has Hamiltonian structure, which is directly comparable with the one of the full water wave system (Dinvay et al., 2020). Numerical simulations done in (Dinvay et al., 2019) indicate how the Hamiltonian functional differ from the corresponding energy levels of the full water problem. It was known that the system (2.1.3) outperforms other bidirectional Whitham models both in the sense of numerical stability and accuracy of approximation of Euler equations (Dinvay et al., 2019). This is due to the reason that nonlinear part of this system has bounded operator.

System (2.1.3) appeared in literature as an alternative to other linearly fully dispersive models able to describe two wave propagation (Dinvay et al., 2019). Those models capture many interesting features of the full water wave problem and are in a good agreement with experiments (Carter, 2018). System (2.1.3) was derived and studied numerically in (Dinvay et al., 2019). Then, (Dinvay, 2019) proved the local well-posedness based on an energy method and a compactness argument. Pei and Wang (2019) proved the local well-posedness of the systems considered in (Dinvay et al., 2019) with additional nonphysical condition

$\eta \geq C > 0$. This system is probably ill-posed for large data if one removes the assumption $\eta > 0$. An heuristic argument is given in (Klein et al., 2018). Without the assumption $\eta > 0$, (Kalisch and Pilod, 2019) proved local well posedness for a surface tension regularization of the system from (Pei and Wang, 2019). However, the maximal time of existence for their regularization is bounded by the capillary parameter. Dinvay et al. (2020) able to remove regularization or special nonphysical conditions to prove the well posedness for (2.1.3).

In some sense (2.1.3) can be regarded itself as a regularization of the system introduced by (Hur and Pandey, 2019). The latter was also investigated numerically in (Dinvay et al., 2019) and compared with other models of Whitham–Boussinesq type. Indeed, if one formally admits $\tanh D \sim D$ for small frequencies, and substituting D instead of $\tanh D$ to the nonlinear part of system (2.1.3), one arrives to the system considered in (Hur and Pandey, 2019). This approximation inline with the long wave framework. If one in addition formally discards the term $\eta \partial_x u$ in the system given in (Hur and Pandey, 2019), then a new alternative system turns out to be locally well-posed and features wave breaking (Hur and Tao, 2018). However, the latter does not belong to the class of Whitham–Boussinesq models since nonlinear non dispersive terms have been neglected.

If we consider the system regarded in (Duchêne et al., 2016), they modified the bilinear Green–Naghdi model improving frequency dispersion. Due to its fully dispersive nature, their model is closely related to system (2.1.3). Even their system is Hamiltonian as well. Moreover, they have justified the Green–Naghdi modification proving the well-posedness, consistency, and convergence to the full water wave problem in the Boussinesq regime (Duchêne et al., 2016). Also, the energy levels of their approximate model is comparable with full water energy. Existence of solitary waves for their system is also proved in (Duchêne et al., 2018). Therefore, the well posedness of the modified Green–Naghdi model is satisfactory, in the sense that it needs neither surface tension nor any nonphysical initial condition. All this together makes it a promising system. And indeed, as noticed in (Duchêne et al., 2016), their modification gives more reliable results than other models of the Green–Naghdi type.

System (2.1.3) has a couple of advantages when compared with the modified Green-Naghdi model (Duchêne et al., 2016). First, it is derived, though not rigorously, from the Zakharov–Craig–Sulem formulation, and as a result one knows the relation between variables (η, v) and those describing the full potential fluid flow (Dinvay et al., 2019). As to the modification discussed, it is presented in variables where the first one has the meaning of the surface elevation and so coincides with η. Its dual variable is called the layer averaged horizontal velocity (Duchêne et al., 2016). In the Boussinesq regime it definitely coincides with the same object associated with the full Euler equations. The second issue is that it does not seem obvious how the modified Green–Naghdi system can be generalized to a three dimensional model, whereas system (2.1.3) can be generalized easily(Dinvay et al., 2020).

2.3 Conclusion of literature review

As we have seen from the literature starting from (Korteweg and De Vries, 1895) equations to (Dinvay et al., 2019) there are different types of models which can approximate the dynamics of water problem. Most of them are modeled based on Boussinesq regime, small amplitude long wave length. Some of the models are locally well-posed with imposed conditions on initial data. For some of the models researchers are able to improve the well-posedness result by removing the imposed conditions and make the well-posedness result more strong. The other thing we observe is that some models are derived not rigorously and satisfies most of the interesting future of water wave problem. Among the Whitham–Boussinesq system we have seen only system (2.1.3) has bounded operator $K\partial_x$ on its nonlinear part that make it more interesting than the other systems. From above review process we also observe that the existing well-posedness results for some of the Whitham–Bousinesq systems are not satisfactory. That is for some of them we have only local well-posedness, also the local well-posedness results are not generalized to $\mathbb{R}^d$ and there are still imposed conditions. If these conditions are removed the models may be ill posed. Therefore, in this study we are interested to improve the well-posedness result of Whitham–Boussinesq type equations (2.1.3).

Chapter 3

Materials and Methods

In this study, we analysis the existence and uniqueness of solutions of Whitham–Boussinesq type system and its consistence with other Boussinesq system in the Sobolev space.To do this, we choose the appropriate contraction spaces via their norms. Then we analyze the solution of Whitham–Boussinesq type equations in these spaces. To make these analysis we follow some procedure that help us to achieve our objectives. The mathematical tools we used in this study is the Banach fixed point theorem also known as contraction mapping principle. The Banach fixed point theorem is one of the most important tool used to solve nonlinear equations. It can also be used to prove existence and uniqueness of solutions of initial value problem for differential equations.

Here we state results from real analysis which are important to our study.

Definition 3.0.1. (Royden and Fitzpatrick, 1988) Let $(X,\|.\|_X)$ be a normed space. A map $T : X \mapsto X$ is called contraction if there exists a constant $\tilde{\theta} \in [0,1)$ such that

$$\|T(x)-T(y)\|_X \leq \tilde{\theta}\|x-y\|_X,$$

for all $x,y \in X$.

Theorem 3.0.1. *(Royden and Fitzpatrick, 1988) Let $(X,\|.\|_X)$ be a Banach space. If $T : X \mapsto X$ is a contraction, then the fixed point equation $T(x) = x$ has a unique solution.*

In this study, first we diagonalize the system (1.3.1) and convert it to differential equation of the form

$$\begin{cases} \dfrac{du}{dt} = f(t,u(t)) \\ u(0) = u_0 \in \mathbb{R}^d, \end{cases} \tag{3.0.1}$$

where $f : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d$ with the properties that it is bounded and Lipschitz, i.e.,

$$\|f(t,x(t))\| \leq C\|x(t)\|, \tag{3.0.2}$$

$$\|f(t,x(t))-f(t,y(t))\| \leq C\|x(t)-y(t)\|, \tag{3.0.3}$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$, where $\|x\|$ is length of $x \in \mathbb{R}^d$.

We derived solutions propagators for diagonalized water wave models. Then we established a lower bound for derivatives of all orders for the Fourier symbol of the pseudo-differential operator that appears in linearized water wave type equations for both purely gravity waves and capillary gravity waves. Using these estimates we derived dispersive and Strichartz estimate for linearized water wave type equations in $\mathbb{R}^d$. We also derived bilinear estimates which are important to bound nonlinear terms.

Now we want to answer the following questions:

i. Does a solution to the initial value problem (3.0.1) exists?

ii. If it exists, is the solution unique?

To do this first we transform (3.0.1) into an integral equation of the form

$$u(t) = u_0 + \int_0^t f(s, u(s))ds.$$

Now define the mapping

$$\Gamma(u)(t) := u_0 + \int_0^t f(s, u(s))ds. \tag{3.0.4}$$

Then we reduce the questions of existence and uniqueness to finding a fixed point of the mapping

$$\Gamma : X \mapsto X$$

on some appropriate Banach space X. As a contraction space we choose X to be the space of continuous functions on a compact interval $[0, T]$ with values in $\mathbb{R}^d$, i.e.,

$$X := C\left([0, T], \mathbb{R}^d \times \mathbb{R}^d\right)$$

defined via the appropriate norms like sup norm or some Strichartz norms.

Next we show that there exists a time $T > 0$ which will be determine later, such that

- Γ maps X into X.
- Γ is a contraction in X.

To answer Γ maps X into X we used (3.0.2) to obtain

$$\begin{aligned}
\|\Gamma(u(t))\| &\leq \|u_0\| + \int_0^t \|f(s,u(s))\| ds \\
&\leq \|u_0\| + C\int_0^T \|u(s)\| ds \\
&\leq \|u_0\| + CT\|u\|_X.
\end{aligned}$$

Therefore,

$$\|\Gamma(u)\|_X \leq \|u_0\| + CT\|u\|_X,$$

and hence

$$u \in X \quad \text{implies} \quad \Gamma(u) \in X.$$

That is $\Gamma(u)$ is well defined. To show that Γ is contraction, we used the Lipschitz condition (3.0.3) on f to obtain

$$\begin{aligned}
\|\Gamma(u(t)) - \Gamma(v(t))\| &\leq \left\| \int_0^t f(s,u(s)) - f(s,v(s)) ds \right\| \\
&\leq C\int_0^T \|u(s) - v(s)\| ds \\
&\leq CT\|u - v\|_X.
\end{aligned}$$

Therefore,

$$\|\Gamma(u) - \Gamma(v)\|_X \leq 2CT\|u - v\|_X.$$

Now if we choose $2CT = \frac{1}{2}$, we obtain

$$\|\Gamma(u) - \Gamma(v)\|_X \leq \frac{1}{2}\|u - v\|_X$$

and thus Γ becomes a contraction in X. Thus, we can conclude by Banach fixed point theorem, there is a unique solution u of the integral equation (3.0.4) in X , i.e.,

$$u(t) = \Gamma(u(t)) \quad \forall t \in [0,T].$$

Chapter 4

Mathematical Preliminaries

In this chapter we briefly define the function spaces we used throughout the dissertation, and state a few important results. The results are all relatively well known and for some of them we omit the proofs. Some of the references we used for this chapter are (Sogge, 1995; Folland, 2009; Linares and Ponce, 2014; Mitrea, 2013; Bhattacharyya, 2012).

4.1 Distributions

We denote by $L^1_{loc}(\mathbb{R}^d)$ the space of locally integrable functions $f : \mathbb{R}^d \mapsto \mathbb{R}$. These are the Lebesgue measurable functions which are integrable over every bounded domain. For a positive integer k, the space of functions that are k times continuously differentiable is denoted by

$$C^k = \{f \in C(\mathbb{R}^d) : \partial_x^\alpha f \in C(\mathbb{R}^d)\,\text{for each}\,\alpha\,\text{with}\,|\alpha| \leq k\}.$$

Let $\Omega \subset \mathbb{R}^d$ be open subset of $\mathbb{R}^d$, then the support of ϕ, denoted by $supp\,\phi$ is the closure of the set $\{x : x \in \Omega,\, \phi(x) \neq 0\}$ in $\mathbb{R}^d$, i.e.

$$supp\,\phi = \overline{\{x : x \in \Omega,\, \phi(x) \neq 0\}} \quad \text{in}\,\mathbb{R}^d.$$

By $C_0^\infty(\Omega)$ we denote the space of continuous function $\phi : \Omega \mapsto \mathbb{R}^d$, having continuous partial derivatives of all orders and whose support is a compact subset of Ω.

Definition 4.1.1. (Bhattacharyya, 2012) Let $\Omega \subset \mathbb{R}^d$ be open subset of $\mathbb{R}^d$. Then the space $C_0^\infty(\Omega)$ is called the test space(or, equivalently, the space of test functions) and denoted by $\mathscr{D}(\Omega)$ using the notation of Schwartz; i.e $\mathscr{D}(\Omega) = C_0^\infty(\Omega) = \{\phi : \phi \in C^\infty,\, supp\,\phi \subset \Omega\}$ if it is equipped with the following notion of convergence: A sequence $\{\phi_n\}_{n=1}^\infty$ in $\mathscr{D}(\Omega)$ is said to converge to $\phi \in \mathscr{D}(\Omega)$ if and only if

(i) there exists a compact subset $\mathtt{K} \subset \Omega$ such that $supp\,(\phi_n - \phi) \subset \mathtt{K}\ \forall n \in \mathbb{N}$.

(ii) $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly in Ω as $n \to \infty$ for all fixed α with $|\alpha| \in \mathbb{N}_0$.

Definition 4.1.2. (Bhattacharyya, 2012) Let Ω be open subset of $\mathbb{R}^d$ and $\mathscr{D}(\Omega)$ is the space of test functions. The space of distribution denoted by $\mathscr{D}'(\Omega)$ is the collection of all complex-valued functionals f over $\mathscr{D}(\Omega)$ where the value of f acting over ϕ denoted by $f(\phi) = \langle f, \phi \rangle$ satisfies the following conditions

(i) for any $a_1, a_2 \in \mathbb{C}$, $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$, $\langle f, a_1\phi_1 + a_2\phi_2 \rangle = a_1\langle f, \phi_2 \rangle + a_2\langle f, \phi_2 \rangle$.

(ii) $\langle f, \phi_n \rangle \to \langle f, \phi \rangle$, whenever $\phi_n \to \phi$ in $\mathscr{D}(\Omega)$ as $n \to \infty$.

Convergence in $\mathscr{D}'(\Omega)$**.** Let $\{f_n\}$ be an infinite sequence in $\mathscr{D}'(\Omega)$ and let $f \in \mathscr{D}'(\Omega)$. Then we say that $f_n \to f$ in $\mathscr{D}'(\Omega)$ if

$$\langle f_n, \phi \rangle \to \langle f, \phi \rangle \text{ in } \mathbb{C} \text{ as } n \to \infty \text{ for all } \phi \in \mathscr{D}(\Omega).$$

Definition 4.1.3. (Mitrea, 2013) For $\alpha \in \mathbb{N}_0^d$ and $f \in \mathscr{D}'(\Omega)$, the distributional derivative (or the derivative in the sense of distribution) of order α of distribution f is the mapping $\partial^\alpha f : \mathscr{D} \mapsto \mathbb{C}$ defined by

$$\langle \partial^\alpha f, \phi \rangle = (-1)^{|\alpha|}\langle f, \partial^\alpha \phi \rangle \quad \forall \phi \in C_0^\infty(\Omega).$$

Definition 4.1.4. (Mitrea, 2013) If $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$, we say that $\partial^\alpha f$ belongs to $L^1_{loc}(\Omega)$ in a weak sense provided there exists some function $g \in L^1_{loc}(\Omega)$ with property that

$$\int_\Omega g\phi dx = (-1)^{|\alpha|}\int_\Omega f\partial^\alpha \phi dx \qquad \text{for every } \phi \in C_0^\infty(\Omega).$$

Whenever this happens, we shall write $\partial^\alpha f = g$ and call g the weak derivative of order α of f.

4.2 The Schwartz space and tempered distributions

In this section we will introduce an important class of distributions called a tempered distributions. The theory of distributions in the form widely used today was created by Laurent Schwartz at the end of 1940's. Then since now, this theory has played a central role in analysis of partial differential equations. This space is the most suitable for the theory of Fourier transform. This is for the reason that Fourier transform of functions in Schwartz space are themselves in Schwartz space.

Definition 4.2.1. (Mitrea, 2013) The Schwartz class of rapidly decreasing functions is defined as

$$\mathscr{S}(\mathbb{R}^d) := \{f \in C^\infty : \sup_{x\in\mathbb{R}^d} |x^\beta \partial^\alpha f(x)| < \infty, \text{ for all } \alpha, \beta \in \mathbb{N}_0^d\}.$$

If $f \in \mathscr{S}(\mathbb{R}^d)$, then for each $\alpha \in \mathbb{N}_0^d$ and $N \in \mathbb{N}_0$ there exists $C \in (0,\infty)$ such that

$$|\partial^\alpha f(x)| \leq \frac{C}{(1+|x|)^{2N}} \quad \text{for each } x \in \mathbb{R}^d. \tag{4.2.1}$$

Using this to deduce that $\mathscr{S}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for every $P \in [1,\infty]$. In particular, we have $\mathscr{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. The space $\mathscr{S}(\mathbb{R}^d)$ has several properties not shared by $C_0^\infty(\mathbb{R}^d)$. One of these turn out to the fact that the Fourier transform is an isomorphism from $\mathscr{S}(\mathbb{R}^d)$ on to $\mathscr{S}(\mathbb{R}^d)$ while the Fourier transform of $f \in C_0^\infty(\mathbb{R}^d)$ is never in $C_0^\infty(\mathbb{R}^d)$ except in the trivial case $f = 0$.

The algebraic dual of $\mathscr{S}(\mathbb{R}^d)$ is the vector space

$$\{f : \mathscr{S}(\mathbb{R}^d) \to \mathbb{C} : f \text{ is a linear and continous}\}.$$

Functionals f belonging to this space are called tempered distributions. The space of tempered distribution is denoted by $\mathscr{S}'(\mathbb{R}^d)$.

Definition 4.2.2. (Bhattacharyya, 2012) A sequence $\{f_n\}$ of distribution belonging to $\mathscr{S}'(\mathbb{R}^d)$ converges to $f \in \mathscr{S}'(\mathbb{R}^d)$ if, for every $\phi \in \mathscr{S}(\mathbb{R}^d)$, $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ in which case we write $f_n \to f$ as $n \to \infty$.

4.3 Fourier transform

Definition 4.3.1. The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$, denoted by, $\widehat{f}(\xi)$ is defined as:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix.\xi} f(x)dx \quad \text{for } \xi \in \mathbb{R}^d,$$

where $x.\xi = x_1\xi_1 + \cdots + x_d\xi_d$.

Since $\mathscr{S}(\mathbb{R}^d)$ is contained in $L^1(\mathbb{R}^d)$, it is possible to define the Fourier transform on the Schwartz class. We now turn to the Fourier inversion formula, that is, the procedure for recovering f from $\widehat{f}$. The inverse Fourier transform on $\mathscr{S}(\mathbb{R}^d)$ is defined as

$$\mathscr{F}^{-1}[f](x) = \int_{\mathbb{R}^d} e^{i(x.\xi)} \widehat{f}(\xi) d\xi.$$

We also denote $f^{\vee}(x) := \mathscr{F}^{-1}[f](x)$.

Theorem 4.3.1 ($L^2(\mathbb{R}^d)$ inner product under Fourier transform)**.** *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$ Then*

$$\langle \widehat{f}, g \rangle_{L^2(\mathbb{R}^d)} = \langle f, g^{\vee} \rangle_{L^2(\mathbb{R}^d)}. \tag{4.3.1}$$

Proof. We use Fubini's Theorem to change the order of integration

$$\begin{aligned}
\langle \widehat{f}, g \rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} e^{-ix.y} f(y) dy \right] \overline{g(x)} dx \\
&= \int_{\mathbb{R}^d} \overline{\left[\int_{\mathbb{R}^d} e^{ix.y} g(x) dx \right]} f(y) dy \\
&= \int_{\mathbb{R}^d} f(y) \overline{g^{\vee}(y)} dy \\
&= \langle f, g^{\vee} \rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

Corollary 4.3.2 (Plancherel identity)**.** *Let $f, g \in \mathscr{S}(\mathbb{R}^d)$. Then*

$$\langle \widehat{f}, \widehat{g} \rangle_{L^2(\mathbb{R}^d)} = \langle f, g \rangle_{L^2(\mathbb{R}^d)}. \tag{4.3.2}$$

In particular, we have

$$\|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \|f\|_{L^2(\mathbb{R}^d)}. \tag{4.3.3}$$

Proof. To show (4.3.2) we replace g by $\widehat{g}$ in Theorem 4.3.1 and then apply the Fourier inversion formula. The identity (4.3.3) follow from (4.3.2) by setting $f = g$. □

Definition 4.3.2. (Folland, 1999) If f and g are function on $\mathbb{R}^d$, their convolution is the function defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy, \tag{4.3.4}$$

provide the integral exists.

By a simple change of variable, one easily gets $(f * g)(x) = (g * f)(x)$. Various conditions can be imposed on f and g to ensure the integral (4.3.4) will be absolutely convergent for all $x \in \mathbb{R}^d$. For example if $f \in L^1(\mathbb{R}^d)$ and g is a bounded function.

Theorem 4.3.3 ((Folland, 1999), Theorem 6.18)**.** *Let $k(x,y)$ be measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ such that for some $c > 0$,*

$$\int_{\mathbb{R}^d} |k(x,y)|dx \leq c \quad for\, a.e\ y \in \mathbb{R}^d, \text{ and}$$
$$\int_{\mathbb{R}^d} |k(x,y)|dy \leq c \quad for\, a.e\ x \in \mathbb{R}^d.$$

Then for $1 \leq p \leq \infty$ and $f \in L^P(\mathbb{R}^d)$, the function

$$Tf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

and $\|Tf\|_{L^p(\mathbb{R}^d)} \leq c\|f\|_{L^p(\mathbb{R}^d)}$, where c is as in the hypothesis.

Remark 4.3.1. *It may not be clear at the outset why the integral*

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)dy$$

makes sense for $f \in L^p(\mathbb{R}^d)$. The proof will show that $\int_{\mathbb{R}^d} |k(x,y)f(y)|dy < \infty$ for almost every x, implies that $Tf(x)$ is defined almost everywhere.

Proof. Observe that the Theorem is trivial for $p = \infty$, an so assume $1 \leq p < \infty$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} |k(x,y)||f(y)|dy &= \int_{\mathbb{R}^d} |k(x,y)|^{\frac{1}{q}}| \left(|k(x,y)|^{\frac{1}{q}}|f(y)|\right) dy \\
&\leq \left(\int_{\mathbb{R}^d} |k(x,y)|dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |k(x,y)||f(y)|^p dy\right)^{\frac{1}{p}} \\
&\leq c^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} |k(x,y)||f(y)|^p dy\right)^{\frac{1}{p}}
\end{aligned}$$

where we used Holder's inequality to get the first inequality.

Integrating with x we get:

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|k(x,y)||f(y)|^p dy\right)^p dx &\le c^{\frac{p}{q}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|k(x,y)||f(y)|^p dy\right)dx\\ &\le c^{\frac{p}{q}}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|k(x,y)|dx\right)|f(y)|^p dy\\ &\le c^{\frac{p}{q}+1}\|f\|_{L^p}^p\end{aligned}$$

here we used Tonelli to get the second inequality.

$$\int_{\mathbb{R}^d}|Tf(x)|^p dx \le c^{\frac{p}{q}+1}\|f\|_{L^p}^p.$$

□

Lemma 4.3.4. *(Folland, 1999)[Young's inequality] If $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ for $1 \le p \le \infty$, then $f * g \in L^p(\mathbb{R}^d)$ and*

$$\|f*g\|_{L^p(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}.$$

Proof. This is a special case of Theorem 4.3.3, with $K(x,y) = f(x-y)$. □

Corollary 4.3.5 (Generalization of Young's inequality)**.** *Suppose $1 \le p,q,r \le \infty$ and $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}+1$. If $f \in L^q(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$, then $f*g \in L^r(\mathbb{R}^d)$ and*

$$\|f*g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^q(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}.$$

4.4 Sobolev spaces

One of the most satisfactory ways of measuring smoothness of functions and distribution is in terms of $L^2(\mathbb{R}^d)$ norms (Folland, 1999). There are two reasons for this: $L^2(\mathbb{R}^d)$ has the advantage of being a Hilbert space, and a Fourier transform, which converts differentiation into multiplication by coordinate functions, is an isometry on $L^2(\mathbb{R}^d)$.

Definition 4.4.1. (Folland, 1999) Let $k \in \mathbb{N}_0$. The Sobolev space $H^k(\mathbb{R}^d)$ is the space of all functions $f \in L^2(\mathbb{R}^d)$ whose distributional derivatives $\partial^\alpha f$ are $L^2(\mathbb{R}^d)$ functions for all $|\alpha| \le k$.

The notation $W^{k,2}(\mathbb{R}^d)$ is also frequently used instead of $H^k(\mathbb{R}^d)$. One then regards $H^k(\mathbb{R}^d)$ as a member of the more general family $W^{k,p}(\mathbb{R}^d)$ of Sobolev space in which $L^2(\mathbb{R}^d)$ is replaced by $L^p(\mathbb{R}^d)$.

Lemma 4.4.1. *Let $m_k(\xi) = \sum_{|\alpha|\leq k} |\xi^\alpha|^2$. Then $f \in H^k(\mathbb{R}^d)$ if and only if*

$$\int_{\mathbb{R}^d} m_k(\xi)|\widehat{f}(\xi)|^2 d\xi < \infty. \tag{4.4.1}$$

Moreover,

$$\|f\|_{H^k(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} m_k(\xi)|\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}. \tag{4.4.2}$$

Proof. If $f \in L^2(\mathbb{R}^d)$, then the distributional derivative satisfy $\widehat{(\partial^\alpha f)}(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$. If $f \in H^k(\mathbb{R}^d)$, we therefore obtain the inequality (4.4.2) which implies (4.4.1). On the other hand, if (4.4.1) holds, we obtain that all the derivatives of order less than or equal to k are in $L^2(\mathbb{R}^d)$. We therefore obtain that $f \in H^k(\mathbb{R}^d)$ and that (4.4.2) holds. □

This result suggests a generalization of the Sobolev spaces. We introduce the notation

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$

A simple modification in the proof of Proposition 8.3 in (Folland, 1999) shows that there exist $C_1, C_2 > 0$ such that

$$C_1 \langle \xi \rangle^{2k} \leq m_k(\xi) \leq C_2 \langle \xi \rangle^{2k} \qquad \forall \xi \in \mathbb{R}^d.$$

We can therefore replace $m_k(\xi)$ by $\langle \xi \rangle^{2k}$ in the definition of the norm of $H^k(\mathbb{R}^d)$ above. This in particular allows us to extend the definition of Sobolev spaces for any $k \in \mathbb{R}$.

Definition 4.4.2. (Linares and Ponce, 2014) Let $k \in \mathbb{R}$, we define $H^k(\mathbb{R}^d)$, the Sobolev spaces of order k on $\mathbb{R}^d$ by

$$H^k(\mathbb{R}^d) = \{ f \in \mathscr{S}'(\mathbb{R}^d) : \langle \xi \rangle^k \widehat{f}(\xi) \in L^2(\mathbb{R}^d) \},$$

with norm

$$\|f\|_{H^k(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2k} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Lemma 4.4.2. *Let $\xi, \eta \in \mathbb{R}^d$ and $k \geq 1$, then*

$$\langle \xi + \eta \rangle^k \leq 2^{k-1} \left(\langle \xi \rangle^k + \langle \eta \rangle^k \right).$$

Proof. Clearly,

$$\begin{aligned} \langle \xi + \eta \rangle^2 &= 1 + |\xi + \eta|^2 \leq 1 + |\xi|^2 + |\eta|^2 + 2|\xi||\eta| \\ &\leq \langle \xi \rangle^2 + \langle \eta \rangle^2 + 2\langle \xi \rangle \langle \eta \rangle \leq (\langle \xi \rangle + \langle \eta \rangle)^2, \end{aligned}$$

which implies

$$\langle \xi + \eta \rangle \leq \langle \xi \rangle + \langle \eta \rangle. \tag{4.4.3}$$

The function $f(r) = r^k$ is convex for $k \geq 1$. Hence $f(\frac{r_1}{2} + \frac{r_2}{2}) \leq 1/2(f(r_1) + f(r_2))$. Thus

$$(r_1 + r_2)^k \leq 2^{k-1}(r_1^k + r_2^k). \tag{4.4.4}$$

Combing (4.4.3) and (4.4.4) we obtain

$$\langle \xi + \eta \rangle^k \leq (\langle \xi \rangle + \langle \eta \rangle)^k \leq 2^{k-1}(\langle \xi \rangle^k + \langle \eta \rangle^k).$$

□

Theorem 4.4.3. *(Linares and Ponce, 2014), If $k > d/2$, then $H^k(\mathbb{R}^d)$ is an algebra with respect to the product functions. That is, if $f, g \in H^k(\mathbb{R}^d)$, then $fg \in H^k(\mathbb{R}^d)$ with*

$$\|fg\|_{H^k(\mathbb{R}^d)} \leq c_{k,d} \|f\|_{H^k(\mathbb{R}^d)} \|g\|_{H^k(\mathbb{R}^d)}. \tag{4.4.5}$$

Proof. By the Fourier transform and convolution theorem we have for all $f, g \in H^k(\mathbb{R}^d)$, we have

$$\widehat{fg}(\xi) = (\widehat{f} * \widehat{g})(\xi).$$

Then,

$$\|fg\|_{H^k(\mathbb{R}^d)} = \|\langle \xi \rangle^k |\widehat{f} * \widehat{g}|\|_{L^2_\xi} = \left(\int_{\mathbb{R}^d} \langle \xi \rangle^{2k} |\widehat{f} * \widehat{g}|^2 d\xi \right)^{\frac{1}{2}}.$$

Now observe that

$$\langle\xi\rangle^k|(\widehat{f}*\widehat{g})(\xi)| \leq \int_{\mathbb{R}^d}\langle\xi\rangle^k|\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|d\eta.$$

Hence, using Lemma 4.4.2

$$\langle\xi\rangle^k|(\widehat{f}*\widehat{g})(\xi)| \leq 2^{k-1}\left(\int_{\mathbb{R}^d}\langle\eta\rangle^k|\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|d\eta + \int_{\mathbb{R}^d}\langle\xi-\eta\rangle^k|\widehat{f}(\eta)||\widehat{g}(\xi-\eta)|d\eta\right).$$

Applying Young's inequality Lemma 4.3.4 we have

$$\|\langle\xi\rangle^k(\widehat{f}*\widehat{g})(\xi)\|_{L^2(\mathbb{R}^d)} \leq 2^{k-1}\left(\|\langle\xi\rangle^k\widehat{f}\|_{L^2(\mathbb{R})^d}\|\widehat{g}\|_{L^1(\mathbb{R}^d)} + \|\widehat{f}\|_{L^1(\mathbb{R})^d}\|\langle\xi\rangle^k\widehat{g}\|_{L^2(\mathbb{R})^d}\right).$$

Next we use Cauchy-Schwarz to estimate

$$\begin{aligned}
\|\widehat{f}\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|\widehat{f}(\xi)|d\xi &= \int_{\mathbb{R}^d}\langle\xi\rangle^{-k}\langle\xi\rangle^k|\widehat{f}(\xi)|d\xi \\
&\leq \left(\int_{\mathbb{R}^d}\langle\xi\rangle^{-2k}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\langle\xi\rangle^{2k}|\widehat{f}(\xi)|^2\right)^{\frac{1}{2}} \\
&\leq c_k\|f\|_{H^k(\mathbb{R}^d)},
\end{aligned}$$

where

$$c_k = \left(\int_{\mathbb{R}^d}\langle\xi\rangle^{-2k}d\xi\right)^{\frac{1}{2}} < \infty \quad for\, k > \frac{d}{2}.$$

By similar arguments

$$\|\widehat{g}\|_{L^1(\mathbb{R}^d)} \leq c_k\|g\|_{H^k(\mathbb{R}^d)}.$$

This complete the proof of the Theorem. □

4.5 Littlewood-Paley theory

In this section we present the theory of Littlewood-Paley projections and discuss a few of their immediate applications. The philosophy behind the Littlewood-Paley theory is to decompose functions by localizing at different frequencies in the Fourier domain. Localization is a fundamental notion in analysis. Given a function, localization means restricting it into a smaller region in a physical space, or frequency space. Littlewood-Paley decomposition of function is based on frequency space localization.

There is a certain amount of flexibility in setting up the Littlewood-Paley decomposition on $\mathbb{R}^d$. One standard way is as follows. Let $\chi(\xi)$ be a real bump function with

$$\chi(\xi) = \begin{cases} 1, & |\xi| \le 1; \\ 0, & |\xi| \ge 2. \end{cases}$$

Let $\rho(\xi)$ be a function

$$\rho(\xi) = \chi(\xi) - \chi(2\xi).$$

Then χ and ρ are supported on $\{|\xi| \le 2\}$ and $\{\frac{1}{2} \le |\xi| \le 2\}$, respectively. We can re-scale χ and ρ as

$$\chi_j(\xi) = \chi(\xi/2^j) \qquad \text{for} \quad j \in \mathbb{Z},$$

$$\rho_j(\xi) = \rho(\xi/2^j) \qquad \text{for} \quad j \in \mathbb{Z}.$$

Then, by construction we have

$$\sum_{j \in \mathbb{Z}} \rho_j(\xi) = 1 \qquad \text{for} \quad \xi \neq 0. \tag{4.5.1}$$

The Littlewood-Paley projection onto a frequencies $|\xi| \sim j$, denoted by P_j, is defined in the Fourier domain of Schwartz function via

$$\widehat{P_j f}(\xi) = \rho_j(\xi)\widehat{f}(\xi).$$

The Littlewood-Paley projection onto low frequencies, denoted by $P_{\le j}$, is defined in the Fourier domain for Schwartz function f via

$$\widehat{P_{\le j} f}(\xi) = \chi_j(\xi)\widehat{f}(\xi).$$

The Littlewood-Paley projection onto high frequencies is

$$P_{>j} = I - P_{\le j}.$$

Using the Littlewood-Paley projections, we can decompose any $L^2(\mathbb{R}^d)$ function into the sum of frequency localized functions.

4.5.1 Characterization of $H^k(\mathbb{R}^d)$ in Littlewood-Paley decomposition

We want to characterize the Sobolev space $H^k(\mathbb{R}^d)$ in terms of the Littlewood-Paley decomposition. Observe that if $2^{j-1} \le |\xi| \le 2^{j+1}$, then

$$(1+|\xi|^2)^{\frac{k}{2}} \sim (1+2^{2j})^{\frac{k}{2}}$$

in the sense that there is a constant $C > 0$ such that

$$C^{-1}(1+2^{2j})^{\frac{k}{2}} \le (1+|\xi|^2)^{\frac{k}{2}} \le C(1+2^{2j})^{\frac{k}{2}}.$$

Then it follows that

$$\|P_j f\|_{H^k(\mathbb{R}^d)} \sim \|(1+2^{2j})^{\frac{k}{2}} P_j f\|_{L^2(\mathbb{R}^d)}. \tag{4.5.2}$$

Lemma 4.5.1. *The operator P_j are self adjoint operator and $P_{j_1}P_{j_2} = 0$ whenever $|j_1 - j_2| \ge 2$.*

Proof. For any $f, h \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle P_j f, h\rangle_{L^2(\mathbb{R}^d)} &= \langle \widehat{P_j f}, \widehat{h}\rangle_{L^2(\mathbb{R}^d)} = \langle \rho(2^{-j}.)\widehat{f}, \widehat{h}\rangle_{L^2(\mathbb{R}^d)} = \langle \widehat{f}, \rho(2^{-j}.)\widehat{h}\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \widehat{f}, \widehat{P_j h}\rangle_{L^2(\mathbb{R}^d)} = \langle f, P_j h\rangle_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Hence P_j is self adjoint. Observe that if

$$\operatorname{supp}(\rho_{j_1}) \cap \operatorname{supp}(\rho_{j_2}) \subset \{2^{j_1-1} \le |\xi| \le 2^{j_1+1}\} \cap \{2^{j_2-1} \le |\xi| \le 2^{j_2+1}\} = \emptyset,$$

then $j_1 - 1 \ge j_2 + 1$ and $j_1 + 1 \le j_2 - 1$. That is $|j_1 - j_2| \ge 2$. Since $\rho(\xi/2^{j_1})\rho(\xi/2^{j_2}) = 0$ whenever $|j_1 - j_2| \ge 2$, we have

$$\widehat{P_{j_1}P_{j_2}f}(\xi) = \rho(\xi/2^{j_1})\rho(\xi/2^{j_2})\widehat{f}(\xi) = 0.$$

Thus, we get $P_{j_1}P_{j_2} = 0$ whenever $|j_1 - j_2| \ge 2$. $\square$

Lemma 4.5.2. *Let $k \in \mathbb{R}$. Then for any $f \in H^k(\mathbb{R}^d)$*

$$\|f\|^2_{H^k(\mathbb{R}^d)} \sim \sum_j (1+2^{2j})^k \|P_j f\|^2_{L^2(\mathbb{R}^d)}.$$

Proof. In view of (4.5.2), it is suffices to show

$$\|f\|^2_{H^k(\mathbb{R}^d)} \sim \sum_j \|P_j f\|^2_{H^k(\mathbb{R}^d)}.$$

Using Lemma 4.5.1 and Cauchy- Schwarz inequality

$$\begin{aligned}
\|f\|^2_{H^k(\mathbb{R}^d)} &= \langle f, f\rangle_{H^k(\mathbb{R}^d)} = \langle \sum_j P_j f, \sum_{j'} P_{j'} f\rangle_{H^k(\mathbb{R}^d)} \\
&\leq \sum_{l=-1}^{1} \sum_j \langle P_j f, P_{j+l} f\rangle_{H^k(\mathbb{R}^d)} \\
&\lesssim \sum_{l=-1}^{1} \sum_j \|P_j f\|_{H^k} \|P_{j+l} f\|_{H^k(\mathbb{R}^d)} \\
&\lesssim \sum_{l=-1}^{1} \left(\sum_j \|P_j f\|^2_{H^k(\mathbb{R}^d)}\right)^{\frac{1}{2}} \left(\sum_j \|P_{j+l} f\|^2_{H^k(\mathbb{R}^d)}\right)^{\frac{1}{2}} \\
&\lesssim 3 \sum_j \|P_j f\|^2_{H^k(\mathbb{R}^d)}
\end{aligned}$$

On the other hand, we have by (4.5.1)

$$\begin{aligned}
\sum_j \|P_j f\|^2_{H^k(\mathbb{R}^d)} &= \sum_j \int_{\mathbb{R}^d} (1+|\xi\|^2)^k |\rho_j(\xi)\widehat{f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^d} \left(\sum_j \rho_j^2(\xi)\right) (1+|\xi\|^2)^k |\widehat{f}(\xi)|^2 d\xi \\
&\leq \int_{\mathbb{R}^d} (1+|\xi\|^2)^s |\widehat{f}(\xi)|^2 d\xi \\
&\leq \|f\|^2_{H^k(\mathbb{R}^d)},
\end{aligned}$$

this complete the proof. □

The Littlewood-Paley theory can be used to give alternative description of Sobolev space and introduce, more refine, spaces of functions. In view of Lemma (4.5.2) we can give a Littlewood-Paley description of Sobolev norm as

$$\|f\|_{H^k(\mathbb{R}^d)} = \left(\sum_{j\in\mathbb{Z}} \langle 2^j\rangle^{2k} \|P_j f\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{1}{2}},$$

where

$$\langle 2^j \rangle = \sqrt{1+2^{2j}}.$$

The Littlewood-Paley calculus is essential for nonlinear estimates. Let f and g be two functions on $\mathbb{R}^d$, then we can write

$$P_j(fg) = P_j \left(\sum_{j',j'' \in \mathbb{Z}} P_{j'} f . P_{j''} g \right). \tag{4.5.3}$$

Since $P_{j'}$ has Fourier support $D' = \{2^{j'-1} \leq |\xi| \leq 2^{j'+1}\}$ and $P_{j''}$ has Fourier support $D'' = \{2^{j''-1} \leq |\xi| \leq 2^{j''+1}\}$. It follow that $P_{j'} P_{j''}$ has Fourier support $D' + D''$. We get a nonzero contribution in the sum (4.5.3) if $D' + D''$ intersects $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$.

Chapter 5

Dispersive Estimates for Linearized Water Wave Type Equations in $\mathbb{R}^d$

5.1 Introduction

In this chapter, we drive $L^1_x(\mathbb{R}^d) - L^\infty_x(\mathbb{R}^d)$ time decay estimate for the linear propagators

$$\mathrm{S}_{m_\kappa}(D) := \exp\left(\mp itm_\kappa(D)\right),$$

where

$$m_\kappa(D) = \sqrt{|D|\,(1+\kappa|D|^2)\tanh|D|}$$

with $\kappa \in \{0,1\}$ and $D = -i\nabla$. The pseudo-differential operator $m_\kappa(D)$ appears in linearized water wave type equations. The case $\kappa = 0$ and $\kappa = 1$ correspond respectively to purely gravity waves and capillary-gravity waves.

We fix a smooth cutoff function χ such that

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \le \chi \le 1, \quad \chi|_{[-1,1]} = 1 \quad \textit{and} \quad \mathrm{supp}(\chi) \subset [-2,2].$$

We set

$$\rho(s) = \chi(s) - \chi(2s).$$

Thus, $supp\,\rho = \{s \in \mathbb{R} : 1/2 \le |s| \le 2\}$. For $\lambda \in 2^{\mathbb{Z}}$ we set $\rho_\lambda(s) := \rho(s/\lambda)$ and define the frequency projection P_λ by

$$\widehat{P_\lambda f}(\xi) = \rho_\lambda(|\xi|)\widehat{f}(\xi).$$

Sometimes, we write $f_\lambda = P_\lambda f$.

5.2 Solution propagators

Consider the Whitham equation with or without surface tension, see for example (Dinvay et al., 2017; Remonato and Kalisch, 2017)

$$u_t + L_\kappa u_x + u u_x = 0, \qquad (\kappa \geq 0) \tag{5.2.1}$$

where $u : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, and the non local operator $L_\kappa(D)$ is related to the dispersion relation of the linearized water waves system and it is as defined in (1.3.2). The linear part of (5.2.1) can be written as

$$iu_t - \widetilde{m}_\kappa(D)u = 0, \tag{5.2.2}$$

where

$$\widetilde{m}_\kappa(D) = DL_\kappa(D) = \frac{D}{|D|} m_\kappa(D).$$

In terms of the Fourier symbol we have $\widetilde{m}_\kappa(\xi) = sgn(\xi) m_\kappa(\xi)$. So the solution propagator for (5.2.2) is given by $\mathtt{S}_{\widetilde{m}_\kappa}(t) = \exp(-it\widetilde{m}_\kappa(D))$. In fact, both $\mathtt{S}_{\widetilde{m}_\kappa}(t)$ and $\mathtt{S}_{m_\kappa}(t)$ satisfy the same $L^1_x(\mathbb{R}) - L^\infty_x(\mathbb{R})$ time-decay estimate.

As an other example, consider the fully dispersion Boussinesq system see (Klein et al., 2018; Lannes, 2013)

$$\begin{cases} \eta_t + L_\kappa^2 \nabla.\mathbf{v} + \nabla.(\eta \mathbf{v}) = 0, \\ \mathbf{v}_t + \nabla \eta + \nabla |\mathbf{v}|^2 = 0, \end{cases} \tag{5.2.3}$$

where

$$\eta : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}, \quad \mathbf{v} : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d.$$

This system describes the evolution with time of surface waves of a liquid layer, where η and $\mathbf{v}$ denote the surface elevation and the fluid velocity, respectively. One can derive an equivalent system of (5.2.3), by diagonalizing its linear part. Indeed, define

$$w_\pm = \frac{\eta \mp iL_\kappa \mathfrak{R}.\mathbf{v}}{2L_\kappa}$$

where $(\eta, \mathbf{v})$ are solution of (5.2.3), and $\mathfrak{R} = |D|^{-1}\nabla$ is Riesz transform. The linear part of system (5.2.3) is transformed to

$$i\partial_t w_\pm \mp m_\kappa(D) w_\pm = 0 \tag{5.2.4}$$

whose corresponding solution propagators are $\mathtt{S}_{m_\kappa}(\pm t) = \exp(\mp itm_\kappa(D))$.

As a third example, consider the Whitham–Boussinesq type system see (Carter, 2018; Dinvay, 2019; Dinvay et al., 2019)

$$\begin{cases} \partial_t \eta + \nabla.\mathbf{v} = -K\nabla.(\eta\mathbf{v}), \\ \partial_t \mathbf{v} + L_\kappa^2 \nabla\eta = -K\nabla(|v|^2/2). \end{cases} \tag{5.2.5}$$

Again, by defining the new variables

$$u_\pm = \frac{L_\kappa \eta \mp i\mathfrak{R}.\mathbf{v}}{2L_\kappa}, \tag{5.2.6}$$

we see that the linear part of (5.2.5) transforms to

$$i\partial_t u_\pm \mp m_\kappa(D)u_\pm = 0 \tag{5.2.7}$$

whose solution propagators are again $\mathtt{S}_{m_\kappa}(\pm t) = \exp(\mp itm_\kappa(D))$.

So the linear propagators $\mathtt{S}_{m_\kappa}(\pm t)$ appear in all of the equations (5.2.1), (5.2.3) and (5.2.5). Since the symbol m_κ is non homogeneous, we will derive a time decay estimate from

$$\mathtt{S}_{m_\kappa}(\pm t) : L_x^1(\mathbb{R}^d) \mapsto L_x^\infty(\mathbb{R}^d)$$

for frequency localised functions.

5.3 Linear estimates

In this section, we drive some useful estimates on the derivatives of all order for function

$$m_\kappa(r) = \sqrt{r(1+\kappa r^2)\tanh(r)} \qquad \kappa \in \{0,1\},$$

where $m_\kappa(r)$ is the Fourier symbol of the pseudo-differential operator that appear in linearized water wave type equations. Estimates for the first and second order derivatives of the function $m_\kappa(r)$ is derived recently in (Pilod et al., 2021).

Since

$$\frac{\tanh r}{r} \sim \langle r \rangle^{-1}$$

we have,

$$m_\kappa(r) \sim r\langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{1}{2}} \quad \text{for } r > 0.$$

Lemma 5.3.1. *Let $\kappa \in \{0,1\}$ and $r > 0$. Then*

$$m'_\kappa(r) \sim \langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{1}{2}}, \tag{5.3.1}$$

$$|m''_\kappa(r)| \sim r\langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{5}{2}}. \tag{5.3.2}$$

Moreover,

$$|m_\kappa^{(k)}(r)| \lesssim r^{1-k}\langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{1}{2}} \qquad (k \geq 3). \tag{5.3.3}$$

Proof. The estimates (5.3.1) and (5.3.2) are proved in [(Pilod et al., 2021), Lemma 3.2]. So we only prove (5.3.3).

Let

$$T(r) = \tanh(r), \quad S(r) = sech(r).$$

Then

$$T' = S^2, \qquad S' = -TS, \qquad T'' = -2S^2T,$$

in general, we have

$$T^j(r) = S^2 P_{j-1}(S,T) \qquad (j \geq 1)$$

for some polynomial P_{j-1} of degree $j-1$.

Clearly,

$$T(r) \sim r\langle r\rangle^{-1} \qquad \text{and} \qquad S(r) \sim e^{-r}.$$

So $P_{j-1}(S,T) \lesssim 1$, and hence

$$|T^{(j)}(r)| \lesssim e^{-2r} \qquad (j \geq 1). \tag{5.3.4}$$

Write

$$m_\kappa(r) = f_\kappa(r)T_0(r),$$

where $f_\kappa(r) = \sqrt{r}\langle\sqrt{\kappa}r\rangle$ and $T_0(r) = \sqrt{T(r)}$. Then one can show that

$$|f_\kappa^{(j)}(r)| \lesssim r^{\frac{1}{2}-j}\langle\sqrt{\kappa}r\rangle \qquad (j \geq 1). \tag{5.3.5}$$

To verify (5.3.5) we find the general form of n^{th} derivative of $f_\kappa(r)$ as follow:

$$\begin{aligned}
f'_\kappa(r) &= \frac{1}{2}r^{-\frac{1}{2}}\langle\sqrt{\kappa}r\rangle + \kappa r^{\frac{3}{2}}\langle\sqrt{\kappa}r\rangle^{-1} \\
f''_\kappa(r) &= -\frac{1}{4}r^{-\frac{3}{2}}\langle\sqrt{\kappa}r\rangle + 2\kappa r^{\frac{1}{2}}\langle\sqrt{\kappa}r\rangle^{-1} - \kappa^2 r^{\frac{5}{2}}\langle\sqrt{\kappa}r\rangle^{-3} \\
f'''_\kappa(r) &= \frac{3}{8}r^{-\frac{5}{2}}\langle\sqrt{\kappa}r\rangle + \frac{3}{4}\kappa r^{-\frac{1}{2}}\langle\sqrt{\kappa}r\rangle^{-1} - \frac{9}{2}\kappa^2 r^{\frac{3}{2}}\langle\sqrt{\kappa}r\rangle^{-3} + 3\kappa^3 r^{\frac{7}{2}}\langle\sqrt{\kappa}r\rangle^{-5},
\end{aligned}$$

continuing this way, we have

$$f_\kappa^{(j)} = r^{\frac{1}{2}-j}\left(c_1\langle\sqrt{\kappa}r\rangle + c_2\kappa r^2\langle\sqrt{\kappa}r\rangle^{-1} + c_3(\kappa r^2)^2\langle\sqrt{\kappa}r\rangle^{-3} + \cdots + c_{j+1}(\kappa r^2)^j\langle\sqrt{\kappa}r\rangle^{1-2j}\right), \tag{5.3.6}$$

where $c_1, c_2, c_3 \ldots, c_{j+1}$ are nonzero constants for $j \geq 1$. Hence (5.3.5) follow from (5.3.6). Combining (5.3.4) with $T(r) \sim r\langle r\rangle^{-1}$, we obtain

$$T_0(r) \sim r^{\frac{1}{2}}\langle r\rangle^{-\frac{1}{2}}, \qquad |T_0^{(j)}(r)| \lesssim r^{\frac{1}{2}-j}\langle r\rangle^{j-\frac{1}{2}}e^{-2r} \qquad (j \geq 1). \tag{5.3.7}$$

Finally, we use (5.3.5) and (5.3.7) to obtain for all $k \geq 3$

$$\begin{aligned}
|m_\kappa^{(k)}(r)| &= \left| f_\kappa^{(k)}(r)T_0(r) + \sum_{j=1}^{k}\binom{k}{j} f_\kappa^{(k-j)}(r)T_0^{(j)}(r)\right| \\
&\lesssim r^{1-k}\langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{1}{2}} + \sum_{j=1}^{k}\binom{k}{j} r^{\frac{1}{2}-(k-j)}\langle\sqrt{\kappa}r\rangle r^{\frac{1}{2}-j}\langle r\rangle^{j-\frac{1}{2}}e^{-2r} \\
&\lesssim r^{1-k}\langle\sqrt{\kappa}r\rangle\langle r\rangle^{-\frac{1}{2}},
\end{aligned}$$

here we also used the fact that exponential decays faster than any polynomial. □

Corollary 5.3.2. *For $\lambda, r > 0$ define $m_{\kappa,\lambda}(r) = m_\kappa(\lambda r)$. Then*

$$\max_{r\sim 1}\left|\partial_r^k\left(\frac{1}{m'_{\kappa,\lambda}(r)}\right)\right| \lesssim \lambda^{-1}\langle\sqrt{\kappa}\lambda\rangle^{-1}\langle\lambda\rangle^{\frac{1}{2}} \qquad (k \geq 0). \tag{5.3.8}$$

Proof. Observe that

$$m_{\kappa,\lambda}^{(k)}(r) = \lambda^k m_\kappa^{(k)}(\lambda r) \qquad (k \geq 1).$$

By (5.3.1), we have, for $r \sim 1$

$$|m'_{\kappa,\lambda}(r)| \sim \lambda\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}$$

and by (5.3.2)-(5.3.3)

$$|m^{(k)}_{\kappa,\lambda}(r)| \lesssim \lambda\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}} \qquad (k \geq 2).$$

Finally, one can combine these two estimates with the differentiation formula

$$\partial_r^k\left(\frac{1}{f}\right) = \sum_{p=1}^{k} \sum_{k_1,\dots,k_p\in\mathbb{N}k_1+\cdots+k_p=k} C_{p,k_1,\dots,k_p}\frac{\partial_r^{k_1} f \dots \partial_r^{k_p} f}{f^{p+1}}$$

to obtain the desired estimate (5.3.8). □

Van der Corput's Lemma will be useful in the proof of Theorem 5.4.1.

Lemma 5.3.3. *[(Stein and Murphy, 1993), Van der Corput's Lemma] Assume $g \in C^1(a,b)$, $\psi \in C^2(a,b)$ and $|\psi''(r)| \gtrsim A$ for $r \in (a,b)$. Then*

$$\left|\int_a^b e^{it\psi(r)} g(r)dr\right| \leq C(At)^{-\frac{1}{2}}\left[|g(b)| + \int_a^b |g'(r)|dr\right], \tag{5.3.9}$$

for some constant $C > 0$ that is independent of a, b and t.

Lemma 5.3.3 holds even if $\psi'(r) = 0$ for some $r \in (a,b)$. However, if $|\psi'(r)| > 0$ for all $r \in (a,b)$ one can use integration by parts to obtain the following Lemma.

Lemma 5.3.4. *Suppose $g \in C_0^\infty(a,b)$ and $\psi \in C^\infty(a,b)$ with $|\psi'(r)| > 0$ for all $r \in (a,b)$. If*

$$\max_{a\leq r\leq b} |\partial_r^j g(r)| \leq A \quad \max_{a\leq r\leq b}\left|\partial_r^j\left(\frac{1}{\psi'(r)}\right)\right| \leq B \tag{5.3.10}$$

for all $0 \leq j \leq N \in \mathbb{N}_0$, then

$$\left|\int_a^b e^{it\psi(r)} g(r)dr\right| \lesssim AB^N|t|^{-N}. \tag{5.3.11}$$

Proof. Let

$$I(t) = \int_a^b e^{it\psi(r)} g(r)dr$$

For $m, N \in \mathbb{N}_0$, define

$$S_{m,N} = \{k_1,\dots,k_N \in \mathbb{N}_0 : k_1 < \cdots < k_N \leq N \,\&\, k_1 + \cdots + k_N = m\}.$$

Integration by part yields

$$
\begin{aligned}
I(t) = & I_1(t) = -it^{-1}\int_a^b \partial_r\left(e^{it\psi(r)}\right)\frac{1}{\psi'(r)}g(r)dr \\
& = (-it)^{-1}\int_a^b e^{it\psi(r)}\left[\frac{1}{\psi'(r)}g'(r)+\partial_r\left(\frac{1}{\psi'(r)}\right)g(r)\right]dr.
\end{aligned}
$$

And

$$
\begin{aligned}
I(t) = I_2(t) = & -(-it)^{-2}\int_a^b \partial_r\left(e^{it\Psi(r)}\right)\frac{1}{\psi'(r)}\left[\frac{1}{\psi'(r)}g'(r)+\partial_r\left(\frac{1}{\psi'(r)}\right)g(r)\right]dr \\
= & (-it)^{-2}\int_a^b e^{it\psi(r)}\left(\frac{1}{\psi'(r)}\right)\left[\frac{1}{\psi'(r)}g''(r)+2\partial_r\left(\frac{1}{\psi'(r)}\right)g'(r)+\partial_r^2\left(\frac{1}{\psi'(r)}\right)g(r)\right]dr \\
& +(-it)^{-2}\int_a^b e^{it\psi(r)}\partial_r\left(\frac{1}{\psi'(r)}\right)\left[\frac{1}{\psi'(r)}g'(r)+\partial_r\left(\frac{1}{\psi'(r)}\right)g(r)\right]dr.
\end{aligned}
$$

Repeating the integration by parts N-times we get

$$
I(t) = I_N(t) = (-it)^{-N}\sum_{m=0}^{N}\sum_{(k_1,\dots,k_N)\in S_{m,N}} C_{k,m,N}\int_a^b E_{k,m,N}(r)dr,
$$

where $C_{k,m,N}$ are constants and

$$
E_{k,m,N}(r) = \prod_{j=1}^{N}\partial_r^{k_j}\left(\frac{1}{\psi'(r)}\right)g^{(N-m)}(r).
$$

Applying (5.3.10) we obtain

$$
|E_{k,m,N}| \le \left(\prod_{j=1}^{N} B\right)A = B^N A
$$

and hence

$$
|I(t)| = |I_N(t)| \lesssim AB^N|t|^{-N}
$$

as desired. $\square$

5.4 Dispersive and Strichartz estimates

The following is the key dispersive estimate that is important to the proof of Theorem (5.5.1).

Theorem 5.4.1. *(Localised dispersive estimate). Let $\kappa \in \{0,1\}, d \geq 1$ and $\lambda \in 2^{\mathbb{Z}}$. Then*

$$\|\mathrm{S}_{m_\kappa}(\pm t) f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \lesssim c_{\kappa,d}(\lambda)|t|^{-\frac{d}{2}}\|f\|_{L_x^1(\mathbb{R}^d)} \tag{5.4.1}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$, where

$$c_{\kappa,d}(\lambda) = \lambda^{\frac{d}{2}-1}\langle\sqrt{\kappa}\lambda\rangle^{-\frac{d}{2}}\langle\lambda\rangle^{\frac{d}{4}+1}. \tag{5.4.2}$$

Remark 5.4.1. *In view of* (5.4.2) *the loss of derivative (this corresponds to the exponent of λ) is $3d/4$ in the case $\kappa = 0$, whereas the loss is $d/4$ when $\kappa = 1$*

Proof. Without loss of generality we may assume $\pm = -$ and $t > 0$. Now we can write

$$(S_{m_\kappa}(-t)f_\lambda)(x) = \mathscr{F}_x^{-1}\left(e^{itm_\kappa(\xi)}\rho_\lambda(|\xi|)\widehat{f}\right)(x) = (I_\lambda(.,t) * f)(x),$$

where

$$I_\lambda(x,t) = \lambda^d \int_{\mathbb{R}^d} e^{i\lambda x.\xi + itm_\kappa(\lambda\xi)}\rho(|\xi|)d\xi.$$

By Young's inequality Lemma 4.3.4

$$\|\mathrm{S}_{m_\kappa}(-t)f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \leq \|I_\lambda(.,t)\|_{L_x^\infty(\mathbb{R}^d)}\|f\|_{L_x^1(\mathbb{R}^d)},$$

and therefore (5.4.1) reduces to proving

$$\|I_\lambda(x,t)\|_{L_x^\infty(\mathbb{R}^d)} \leq c_{\kappa,d}(\lambda)t^{-\frac{d}{2}}. \tag{5.4.3}$$

Observe that

$$t \lesssim \lambda^{-\frac{1}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-1} \quad \text{implies} \quad c_{\kappa,d}(\lambda)t^{-d/2} \gtrsim \lambda^d$$

in which case (5.4.3) follows from simple estimate

$$\|I_\lambda(x,t)\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^d.$$

We may therefore assume from now on

$$t \gg \lambda^{-\frac{1}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-1}. \tag{5.4.4}$$

Proof of (5.4.3) *when* $d=1$. In the case $d=1$ we have

$$I_\lambda(x,t)=\lambda\int_{\mathbb{R}}e^{it\phi_\lambda(\xi)}\rho(|\xi|)d\xi,$$

where

$$\phi_\lambda(\xi):=\lambda\xi x/t+m_\kappa(\lambda\xi).$$

We want to prove

$$\|I_\lambda(x,t)\|_{L^\infty_x(\mathbb{R})}\lesssim\lambda^{-\frac{1}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-\frac{1}{2}}\langle\lambda\rangle^{\frac{5}{4}}t^{-\frac{1}{2}} \tag{5.4.5}$$

under condition (5.4.4).

Since $supp\rho=\{\xi\in\mathbb{R}:1/2\le|\xi|\le 2\}$, we can write

$$I_\lambda(x,t)=\underbrace{\lambda\int_{1/2}^{2}e^{it\phi_\lambda(\xi)}\rho(\xi)d\xi}_{:=I^+_\lambda(x,t)}+\underbrace{\lambda\int_{1/2}^{2}e^{it\phi_\lambda(-\xi)}\rho(\xi)d\xi}_{:=I^-_\lambda(x,t)}.$$

We estimate only $I^+_\lambda(x,t)$ as the estimate for $I^-_\lambda(x,t)$ can be derived in exactly the same way. Since

$$\phi'_\lambda(\xi)=\lambda\left[x/t+m'_\kappa(\lambda\xi)\right],\quad \phi''_\lambda(\xi)=\lambda^2m''_\kappa(\lambda\xi) \tag{5.4.6}$$

it follows from Lemma 5.3.1 that

$$|\phi''_\lambda(\xi)|\sim\lambda^3\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{5}{2}}\quad\text{for all}\,\xi\in[1/2,2]. \tag{5.4.7}$$

Now we prove (5.4.5) by dividing the region of integration into sets of non stationary contribution:$\{\xi:\phi'_\lambda(\xi)\neq 0\}$ and stationary contribution $\{\xi:\phi'_\lambda(\xi)=0\}$.

Non stationary contribution

Since m'_κ is positive, the non stationary contribution occurs if either

$$x\ge 0\quad\text{or}\quad 0<-x\ll\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t\quad\text{or}\quad -x\gg\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t.$$

In this case we have

$$|\phi'_\lambda(\xi)|\gtrsim\lambda\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}\quad\text{for all}\,\xi\in[1/2,2],$$

where Lemma 5.3.1 is also used. Combing this estimate with (5.4.7) we get

$$\max_{1/2\le\xi\le 2}\left|\partial_\xi\left(\frac{1}{\phi_\lambda'(\xi)}\right)\right| \lesssim \lambda\langle\sqrt{\kappa}\lambda\rangle^{-1}\langle\lambda\rangle^{-\frac{3}{2}}.$$

Now this estimate can be combined with Lemma 5.3.4 for $N=1$ to estimate $I_\lambda^+(x,t)$ as

$$\begin{aligned}|I_\lambda^+(x,t)| &\lesssim \lambda\lambda\langle\sqrt{\kappa}\lambda\rangle^{-1}\langle\lambda\rangle^{-\frac{3}{2}}t^{-1}\\ &\lesssim \lambda^{-\frac{1}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-\frac{1}{2}}\langle\lambda\rangle^{\frac{5}{4}}t^{-\frac{1}{2}}\end{aligned}$$

where to get the second line we used (5.4.4).

Stationary contribution

This occurs if

$$0<-x\sim\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t.$$

In this case we use Lemma 5.3.3 and (5.4.7) to obtain

$$\begin{aligned}|I_\lambda^+(x,t)| &\lesssim \lambda\left(\lambda^3\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{5}{2}}t\right)^{-\frac{1}{2}}\left[|\rho(2)|+\int_{1/2}^2|\rho'(r)|dr\right]\\ &\lesssim \lambda^{-\frac{1}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-\frac{1}{2}}\langle\lambda\rangle^{\frac{5}{4}}t^{-\frac{1}{2}}.\end{aligned}$$

□

Proof of (5.4.3) *when* $d\ge 2$. To prove (5.4.3) first observe that $I_\lambda(x,t)$ is radially symmetric with respect to x, as it is the inverse Fourier transform of radial function $e^{itm_\kappa(\lambda\xi)}\rho(\xi)$, and so we can write see (Loukas Grafakos, 2014), appendix B.5

$$I_\lambda(x,t)=\lambda^d(2\pi)^{\frac{d}{2}}\int_{1/2}^2 e^{itm_\kappa(\lambda r)}(\lambda r|x|)^{-\frac{d-2}{2}}J_{\frac{d-2}{2}}(\lambda r|x|)r^{d-1}\rho(r)dr, \tag{5.4.8}$$

where $J_\sigma(r)$ is Bessel function:

$$J_\sigma(r)=\frac{(r/2)^\sigma}{\Gamma(\sigma+1/2)\sqrt{\pi}}\int_{-1}^1 e^{irs}(1-s^2)^{\sigma-1/2}ds \quad \text{for}\,\sigma>-1/2.$$

The Bessel function $J_\sigma(r)$ satisfies the following properties for $\sigma>-1/2$ and $r>0$ see

(Loukas Grafakos, 2014) appendix B and (Stein and Murphy, 1993)):

$$J_\sigma(r) \le Cr^\sigma, \tag{5.4.9}$$

$$J_\sigma(r) \le Cr^{-\frac{1}{2}}, \tag{5.4.10}$$

$$\partial_r \left[r^{-\sigma} J_\sigma(r)\right] = -r^{-\sigma} J_{\sigma+1}(r). \tag{5.4.11}$$

Moreover, it was known that (John, 1981)

$$r^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(s) = e^{is} h(s) + e^{-is} \bar{h}(s) \tag{5.4.12}$$

for some function h satisfying the decay estimate

$$\left|\partial_r^k h(r)\right| \le C_k \langle r \rangle^{-\frac{d-1}{2}-k} \quad (k \ge 0). \tag{5.4.13}$$

We use the short hand

$$m_{\kappa,\lambda}(r) = m_\kappa(\lambda r), \quad \tilde{\rho}(r) = r^{d-1}\rho(r), \quad \tilde{J}_a(r) = r^{-a} J_a(r).$$

Hence,

$$I_\lambda(x,t) = \lambda^d (2\pi)^{\frac{d}{2}} \int_{1/2}^{2} e^{itm_{\kappa,\lambda}(r)} \tilde{J}_{\frac{d-2}{2}}(\lambda r|x|) \tilde{\rho}(r) dr. \tag{5.4.14}$$

We prove (5.4.3) by treating the cases $|x| \lesssim \lambda^{-1}$ and $|x| \gg \lambda^{-1}$ separately.

Case 1 $|x| \lesssim \lambda^{-1}$:

By (5.4.9) and (5.4.11) we have for all $r \in (1/2, 2)$ the estimate

$$|\partial_r^k [\tilde{J}_{\frac{d-2}{2}}(\lambda r|x|)\tilde{\rho}(r)]| \lesssim 1 \quad (k \ge 0). \tag{5.4.15}$$

From Corollary 5.3.2 we have

$$\max_{1/2 \le r \le 2} \left|\partial_r^k \left(\frac{1}{m'_{\kappa,\lambda}(r)}\right)\right| \lesssim \lambda^{-1} \langle \sqrt{\kappa}\lambda \rangle^{-1} \langle \lambda \rangle^{\frac{1}{2}} \quad (k \ge 0). \tag{5.4.16}$$

We denote

$$\lceil d/2 \rceil = \begin{cases} d/2, & \text{if } d \text{ is even,} \\ (d+1)/2, & \text{if } d \text{ is odd.} \end{cases}$$

Applying Lemma 5.3.4 with (5.4.15)- (5.4.16) and $N = \lceil d/2 \rceil$ to (5.4.14) we obtain

$$\begin{aligned} |I_\lambda(x,t)| &\lesssim \lambda^d \left(\lambda^{-1} \langle \sqrt{\kappa}\lambda \rangle^{-1} \langle \lambda \rangle^{\frac{1}{2}} \right)^{\lceil d/2 \rceil} t^{-\lceil \frac{d}{2} \rceil} \\ &\lesssim c_{\kappa,d}(\lambda) t^{-\frac{d}{2}}, \end{aligned}$$

where to get the second line we used (5.4.4).

Case 2: $|x| \gg \lambda^{-1}$ Using (5.4.12) and (5.4.14) we write

$$I_\lambda(x,t) = \lambda^d (2\pi)^{\frac{d}{2}} \left[\int_{1/2}^{2} e^{it\phi_\lambda^+(r)} h(\lambda r|x|)\tilde{\rho}(r) dr + \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)} \bar{h}(\lambda r|x|)\tilde{\rho}(r) dr \right],$$

where

$$\phi_\lambda^\pm(r) = \lambda r|x|/t \pm m_{\kappa,\lambda}(r).$$

Set $H_\lambda(|x|,r) = h(\lambda r|x|)\tilde{\rho}(r)$. In view of (5.4.13) we have

$$\max_{1/2 \le r \le 2} \left| \partial_r^k H_\lambda(|x|,r) \right| \lesssim (\lambda|x|)^{-\frac{d-1}{2}}, \quad (k \ge 0) \tag{5.4.17}$$

where we also use the fact $\lambda|x| \gg 1$.

Now we write

$$I_\lambda(x,t) = I_\lambda^+(x,t) + I_\lambda^-(x,t),$$

where

$$\begin{aligned} I_\lambda^+(x,t) &= \lambda^d (2\pi)^{\frac{d}{2}} \int_{1/2}^{2} e^{it\phi_\lambda^+(r)} H_\lambda(|x|,r) dr, \\ I_\lambda^-(x,t) &= \lambda^d (2\pi)^{\frac{d}{2}} \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)} \bar{H}_\lambda(|x|,r) dr. \end{aligned}$$

Observe that

$$\partial_r \phi_\lambda^\pm(r) = \lambda \left[|x|/t \pm m_\kappa'(\lambda r) \right], \quad \partial_r^2 \phi_\lambda^\pm(r) = \pm \lambda^2 m_\kappa''(\lambda r),$$

and hence by Lemma 5.3.1,

$$|\partial_r\phi_\lambda^+(r)| \gtrsim \lambda\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}, \quad |\partial_r^2\phi_\lambda^\pm(r)| \sim \lambda^3\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{5}{2}}, \tag{5.4.18}$$

for all $r \in (1/2, 2)$, where we used the fact that m'_κ is positive.

Estimate for $I_\lambda^+(x,t)$: Following as in the proof of Corollary 5.3.2, we have

$$\max_{1/2\leq r\leq 2}\left|\partial_r^k\left([\partial_r\phi_\lambda^+(r)]^{-1}\right)\right| \lesssim_k \lambda^{-1}\langle\sqrt{\kappa}\lambda\rangle^{-1}\langle\lambda\rangle^{\frac{1}{2}} \quad (k \geq 0) \tag{5.4.19}$$

Applying Lemma 5.3.4 with (5.4.17), (5.4.19) and $N = \lceil d/2\rceil$ to $I_\lambda^+(x,t)$ we obtain

$$\begin{aligned} |I_\lambda^+(x,t)| &\lesssim \lambda^d(\lambda|x|)^{-\frac{d-1}{2}}(\lambda^{-1}\langle\sqrt{\kappa}\lambda\rangle^{-1}\langle\lambda\rangle^{\frac{1}{2}})^{\lceil d/2\rceil}t^{-\lceil d/2\rceil} \\ &\lesssim c_{\kappa,d}(\lambda)t^{-\frac{d}{2}}, \end{aligned} \tag{5.4.20}$$

where to get the second line we used the fact that $\lambda|x| \gg 1$ and the condition in (5.4.4).

Estimate for $I_\lambda^-(x,t)$: We treat the non stationary and stationary cases separately. In the non stationary case, where

$$|x| \gg \langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t \quad \text{or} \quad |x| \ll \langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t,$$

we have

$$|\partial_r\phi_\lambda^-(r)| \gtrsim \lambda\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}},$$

and hence $I_\lambda^-(x,t)$ can be estimated in exactly the same way as $I_\lambda^+(x,t)$ above and satisfy the same bound in (5.4.20).

So it remain to treat the stationary case:

$$|x| \sim \langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t.$$

In this case we use Lemma 5.3.3, (5.4.18) and (5.4.17) to obtain

$$\begin{aligned} |I_\lambda^-(x,t)| &\lesssim \lambda^d(\lambda^3\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{5}{2}}t)^{-\frac{1}{2}}\left[|\bar{H}_\lambda(|x|,2)| + \int_{1/2}^2|\partial_r\bar{H}_\lambda(|x|,2)|dr\right] \\ &\lesssim \lambda^{d-\frac{3}{2}}\langle\sqrt{\kappa}\lambda\rangle^{-\frac{1}{2}}\langle\lambda\rangle^{\frac{5}{4}}t^{-\frac{1}{2}}(\lambda|x|)^{-\frac{d-1}{2}} \\ &\leq c_{\kappa,d}(\lambda)t^{-\frac{d}{2}}, \end{aligned}$$

where we also used the fact that $\bar{H}_\lambda(|x|,2)=0$ and $|x|\sim\langle\sqrt{\kappa}\lambda\rangle\langle\lambda\rangle^{-\frac{1}{2}}t$. $\square$

$\square$

Once Theorem 5.4.1 is proved, the corresponding Strichartz estimates are deduced from a classical TT^* argument.

Theorem 5.4.2. *(Localised Strichartz estimate). Let $\kappa\in\{0,1\}, d\geq 1$ and $\lambda\in 2^{\mathbb{Z}}$. Assume that the pair (p,r) satisfies the following conditions:*

$$2<q\leq\infty,\quad 2\leq r\leq\infty,\quad \frac{2}{q}+\frac{d}{r}=\frac{d}{2}.$$

Then

$$\|\mathrm{S}_{m_\kappa}(\pm t)f_\lambda\|_{L_t^qL_x^r(\mathbb{R}^{d+1})}\lesssim[c_{\kappa,d}(\lambda)]^{\frac{1}{2}-\frac{1}{r}}\|f_\lambda\|_{L_x^2(\mathbb{R}^d)},\tag{5.4.21}$$

$$\left\|\int_0^t\mathrm{S}_{m_\kappa}(\pm(t-s))F_\lambda(s)ds\right\|_{L_t^qL_x^r(\mathbb{R}^{d+1})}\lesssim[c_{\kappa,d}(\lambda)]^{\frac{1}{2}-\frac{1}{r}}\|F_\lambda\|_{L_t^1L_x^2(\mathbb{R}^{d+1})},\tag{5.4.22}$$

for all $f\in\mathscr{S}(\mathbb{R}^d)$ and $F\in\mathscr{S}(\mathbb{R}^{d+1})$ where, $c_{\kappa,d}(\lambda)$ is as in (5.4.2).

We shall use the Hardy-Littlewood-Sobolev inequality which asserts that (Tao, 2006)

$$\||.|^{-\tilde{\gamma}}*f\|_{L^a(\mathbb{R})}\lesssim\|f\|_{L^b(\mathbb{R})},\tag{5.4.23}$$

whenever $1<b<a<\infty$ and $0<\tilde{\gamma}<1$ obeying the scaling condition

$$\frac{1}{b}=\frac{1}{a}+1-\tilde{\gamma}.$$

Proof of Theorem 5.4.2. We only prove (5.4.21) since (5.4.22) follows from (5.4.21). First note that (5.4.21) holds true for the pair $(q,r)=(\infty,2)$ as this is just the energy inequality. So we may assume $2<q<\infty$ and $r>2$.

Let q' and r' be conjugate of q and r, respectively, i.e., is $q'=\frac{q}{q-1}$ and $r'=\frac{r}{r-1}$. By standard TT^* argument (5.4.21) is equivalent to

$$\|TT^*F\|_{L_t^qL_x^r(\mathbb{R}^{d+1})}\lesssim\left(c_{\kappa,d}(\lambda)\right)^{1-\frac{2}{r}}\|F\|_{L_t^{q'}L_x^{r'}(\mathbb{R}^{d+1})},\tag{5.4.24}$$

where

$$TT^*F(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{ix.\xi \pm i(t-s)m_\kappa(\xi)} \rho_\lambda^2(\xi) \widehat{F}(\xi,s) ds d\xi \qquad (5.4.25)$$
$$= \int_{\mathbb{R}} k_{\lambda,t-s} * F(.,s) ds,$$

with

$$k_{\lambda,t}(x) = \int_{\mathbb{R}^d} e^{ix.\xi \pm itm_\kappa(\xi)} \rho_\lambda^2(\xi) d\xi.$$

Since

$$k_{\lambda,t} * g(x) = e^{itm_\kappa(D)} P_\lambda g_\lambda,$$

it follows from (5.4.1)

$$\|k_{\lambda,t} * g\|_{L_x^\infty(\mathbb{R}^d)} \lesssim c_{\kappa,d}(\lambda) |t|^{-\frac{d}{2}} \|g\|_{L_x^1(\mathbb{R}^d)}. \qquad (5.4.26)$$

On the other hand by Corollary 4.3.2 ,i.e., Plancherel identity

$$\|k_{\lambda,t} * g\|_{L_x^2(\mathbb{R}^d)} \lesssim \|g\|_{L_x^2(\mathbb{R}^d)}. \qquad (5.4.27)$$

So interpolation between (5.4.26) and (5.4.27) yields

$$\|k_{\lambda,t} * g\|_{L_x^r(\mathbb{R}^d)} \lesssim [c_{\kappa,d}(\lambda)]^{1-\frac{2}{r}} |t|^{-d(\frac{1}{2}-\frac{1}{r})} \|g\|_{L_x^{r'}(\mathbb{R}^d)}, \qquad (5.4.28)$$

for all $r \in [2,\infty]$.

Applying Minkowski inequality to (5.4.25), and then (5.4.28) and (5.4.23) with

$$(a,b) = (q,q'), \quad \tilde{\gamma} = \frac{d}{2} - \frac{d}{r} = \frac{2}{q},$$

we get

$$
\begin{aligned}
\|TT^*F\|_{L_t^qL_x^r(\mathbb{R}^{d+1})} &\lesssim \left\| \int_{\mathbb{R}} \|k_{\lambda,t} * F(.,s)\|_{L_x^r(\mathbb{R}^d)} \right\|_{L_t^q(\mathbb{R})} \\
&\lesssim [c_{\kappa,d}(\lambda)]^{1-\frac{2}{r}} \left\| \int_{\mathbb{R}} |t-s|^{-d(\frac{1}{2}-\frac{1}{r})} \|F(.,s)\|_{L_x^{r'}(\mathbb{R}^d)} ds \right\|_{L_t^q(\mathbb{R})} \\
&\lesssim [c_{\kappa,d}(\lambda)]^{1-\frac{2}{r}} \left\| \|F\|_{L_x^{r'}(\mathbb{R}^d)} \right\|_{L_t^{q'}(\mathbb{R})} \\
&\lesssim [c_{\kappa,d}(\lambda)]^{1-\frac{2}{r}} \|F\|_{L_t^{q'}L_x^{r'}(\mathbb{R}^{d+1})},
\end{aligned}
$$

which is the desired estimate (5.4.24). □

5.5 Well-posedness of Whitham–Boussinesq type system in $\mathbb{R}^d$ for $d \geq 3$

As an application of Theorem 5.4.2 we prove low regularity well-posedness for the system (5.2.5) with $\kappa = 0$ (i.e., for purely gravity waves) in $\mathbb{R}^d$, $d \geq 3$. To this end, we complement the system (5.2.5) with initial data

$$
\eta(0) = \eta_0 \in H^s(\mathbb{R}^d) \qquad \mathbf{v}(0) = \mathbf{v}_0 \in \left[H^{s+1/2}(\mathbb{R}^d)\right]^d. \tag{5.5.1}
$$

Theorem 5.5.1. *Let $\kappa = 0$, $d \geq 3$, $0 < \nu \ll 1$ and $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$. Suppose that $\boldsymbol{v}_0$ is curl free, i.e., $\nabla \times \boldsymbol{v}_0 = 0$, and*

$$
\left(\|\eta_0\|_{H^s(\mathbb{R}^d)} + \|\boldsymbol{v}_0\|_{[H^{s+1/2}(\mathbb{R}^d)]^d} \right) \sim \mathtt{D}_0.
$$

Then there exists a solution

$$
(\eta, \boldsymbol{v}) \in C\left([0,T]; H^s(\mathbb{R}^d) \times [H^{s+1/2}(\mathbb{R}^d)]^d\right)
$$

of the Cauchy problem (5.2.5), (5.5.1), *with existence time $T \sim \mathtt{D}_0^{-\frac{2}{1+2\nu}}$.*
Moreover, the solution is unique in some subspace of the above solution space and depends continuously on the initial data.

Remark 5.5.1. *The followings are known results:*

1. *Theorem 5.4.1 and Theorem 5.4.2 are proved in (Dinvay et al., 2020) when $\kappa = 0$,*

$d \in \{1,2\}$ and $\lambda \gtrsim 1$.

2. *Theorem 5.5.1 is also proved in (Dinvay et al., 2020) when $d = 2$. In the case $d = 1$ local well-posedness for $s > -1/10$ and global well-posedness for small initial data in $L^2(\mathbb{R})$ is also established in (Dinvay et al., 2020). Long time existence of solution in the case $d = 2$ is also obtained in (Tesfahun, 2022).*

We consider the system (5.2.5) with $\kappa = 0$ and a curl free vector field $\mathbf{v}$, i.e., $\nabla \times \mathbf{v} = 0$. Observe that

$$L_0(D) = \sqrt{K(D)} \qquad m_0(D) = |D|\sqrt{K(D)}.$$

The transformation (5.2.6) (with $\kappa = 0$) yields

$$\eta = u_+ + u_- \qquad \mathbf{v} = -i\sqrt{K}\mathfrak{R}(u_+ - u_-),$$

where we have used the fact that $\mathbf{v}$ is curl free, in which case

$$\nabla(\nabla.\mathbf{v}) = \Delta\mathbf{v} = -|D|^2\mathbf{v}.$$

Then, the equation for u_+ becomes

$$\begin{aligned}
2\partial_t u_+ &= \partial_t\eta - iK^{-\frac{1}{2}}|D|^{-1}\nabla.\partial_t\mathbf{v} \\
&= -\nabla.\mathbf{v} - K\nabla.(\eta\mathbf{v}) + iK^{-\frac{1}{2}}|D|^{-1}\nabla.\left[K\nabla\eta + K\nabla(|\mathbf{v}|^2/2)\right] \\
&= -\nabla.\mathbf{v} - K\nabla.(\eta\mathbf{v}) - iK^{\frac{1}{2}}|D|\eta - iK^{\frac{1}{2}}|D|(|\mathbf{v}|^2/2) \\
&= -i\sqrt{K}|D|\left[\eta - iK^{-\frac{1}{2}}|D|^{-1}\nabla.\mathbf{v}\right] + i\sqrt{K}|D|\left[\sqrt{K}|D|^{-1}i\nabla.(\eta\mathbf{v}) - |\mathbf{v}|^2/2\right].
\end{aligned}$$

Thus,

$$\partial_t u_+ = -i\sqrt{K}|D|u_+ + \frac{i\sqrt{K}|D|}{2}\left[\sqrt{K}|D|^{-1}i\nabla.(\eta\mathbf{v}) - |\mathbf{v}|^2/2\right].$$

Similarly,

$$\begin{aligned}
2\partial_t u_- &= \partial_t\eta + iK^{-\frac{1}{2}}|D|^{-1}\nabla.\partial_t\mathbf{v} \\
&= -\nabla.\mathbf{v} - K_\nabla.(\eta\mathbf{v}) - iK^{-\frac{1}{2}}|D|^{-1}\nabla.\left[K\nabla\eta + K\nabla(|\mathbf{v}|^2/2)\right] \\
&= -\nabla.\mathbf{v} - K\nabla.(\eta\mathbf{v}) + iK^{\frac{1}{2}}|D|\eta + iK^{\frac{1}{2}}|D|(|\mathbf{v}|^2/2) \\
&= i\sqrt{K}|D|\left[\eta + iK^{-\frac{1}{2}}|D|^{-1}\nabla.\mathbf{v}\right] + i\sqrt{K}|D|\left[\sqrt{K}|D|^{-1}i\nabla.(\eta\mathbf{v}) + |\mathbf{v}|^2/2\right].
\end{aligned}$$

Thus,

$$\partial_t u_- = i\sqrt{K}|D|u_- + \frac{i\sqrt{K}|D|}{2}\left[\sqrt{K}|D|^{-1}i\nabla.(\eta\mathbf{v}) + |\mathbf{v}|^2/2\right].$$

Consequently, the Cauchy problem (5.2.5), (5.5.1) transforms to

$$\begin{cases} i\partial_t u_+ - m_0(D)u_+ = -\mathscr{B}^+(u_+,u_-), \\ i\partial_t u_- + m_0(D)u_- = -\mathscr{B}^-(u_+,u_-), \\ u_\pm(0) = f_\pm, \end{cases} \tag{5.5.2}$$

where

$$\begin{aligned} \mathscr{B}^\pm(u_+,u_-) = 2^{-1}|D|K\Re.\left[(u_+ + u_-)\Re\sqrt{K}(u_+ - u_-)\right] \\ \mp 4^{-1}|D|\sqrt{K}\left[\Re\sqrt{K}(u_+ - u_-)\right]^2 \end{aligned} \tag{5.5.3}$$

and

$$f_\pm = \frac{\sqrt{K}\eta_0 \mp i\Re.\mathbf{v}_0}{2\sqrt{K}} \in H^s(\mathbb{R}^d). \tag{5.5.4}$$

Thus, Theorem 5.5.1 reduces to the following:

Theorem 5.5.2. *Let $d \geq 3$, $0 < \nu \ll 1$ and $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$. If the initial data has size*

$$\sum_\pm \|f_\pm\|_{H^s} \leq \mathtt{D}_0,$$

then there exists a solution

$$(u_+,u_-) \in C\left([0,T];H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)\right)$$

of the Cauchy problem (5.5.2)-(5.5.4) *with existence time $T \sim \mathtt{D}_0^{-\frac{2}{1+2\nu}}$. Moreover, the solution is unique in some subspace of $C\left([0,T];H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)\right)$ and the solution depends continuously on the initial data.*

5.5.1 Reduction of Theorem 5.5.2 to bilinear estimates

The Duhamel's representation of (5.5.2) is given by

$$u_\pm(t) = \mathtt{S}_{m_0}(\pm t)f_\pm + \frac{i}{2}\mathscr{B}_1^\pm(\mathtt{u},\mathtt{v})(t) \mp \frac{i}{4}\mathscr{B}_2^\pm(\mathtt{u},\mathtt{v})(t), \tag{5.5.5}$$

where $\mathtt{S}_{m_0}(t) = e^{itm_0(D)}$ and

$$\mathscr{B}_1^{\pm}(\mathtt{u},\mathtt{v})(t) := \int_0^t \mathtt{S}_{m_0}(\pm(t-t'))DK\mathfrak{R}.\left(\mathtt{u}\mathfrak{R}\sqrt{K}\mathtt{v}\right)(t')dt' \tag{5.5.6}$$

$$\mathscr{B}_2^{\pm}(\mathtt{u},\mathtt{v})(t) := \int_0^t \mathtt{S}_{m_0}(\pm(t-t'))D\sqrt{K}\left(\mathfrak{R}\sqrt{K}\mathtt{u}.\mathfrak{R}\sqrt{K}\mathtt{v}\right)(t')dt' \tag{5.5.7}$$

with $\mathtt{u} = u_+ + u_-$ and $\mathtt{v} = u_+ - u_-$. Setting $\kappa = 0$ in Theorem 5.4.2 we obtain the following.

Corollary 5.5.3. *Let $\lambda \in 2^{\mathbb{Z}}$ and $d \geq 3$. Assume the pair (q,r) satisfies*

$$2 < q \leq \infty, \quad 2 \leq r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

Then

$$\|\mathtt{S}_{m_0}(t)f_\lambda\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \langle\lambda\rangle^{\frac{3d}{4}(\frac{1}{2}-\frac{1}{r})}\|f_\lambda\|_{L_x^2(\mathbb{R}^d)}, \tag{5.5.8}$$

$$\left\|\int_0^t \mathtt{S}_{m_0}(\pm t)(t-s)F_\lambda(s)ds\right\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \langle\lambda\rangle^{\frac{3d}{4}(\frac{1}{2}-\frac{1}{r})}\|F_\lambda\|_{L_t^1 L_x^2(\mathbb{R}^{d+1})}, \tag{5.5.9}$$

for all $f \in \mathscr{S}(\mathbb{R}^d)$ and $F \in \mathscr{S}(\mathbb{R}^{d+1})$.

Now we define the contraction space X_T^s, via the norm

$$\|u\|_{X_T^s} = \left[\sum_\lambda \langle\lambda\rangle^{2s}\|u\|_{X_\lambda}^2\right]^{\frac{1}{2}},$$

where

$$\|u\|_{X_\lambda} = \left[\|P_\lambda u\|_{L_T^\infty L_x^2}^2 + \langle\lambda\rangle^{-\frac{3}{2}(1-2\nu)}\|P_\lambda u\|_{L_T^q L_x^r}^2\right]^{1/2}$$

with

$$\frac{1}{q} = \frac{1}{2} - \nu \quad \textit{and} \quad \frac{1}{r} = \frac{d-2+4\nu}{2d}.$$

Observe that

$$\begin{aligned} &\|P_\lambda u\|_{L_T^\infty L_x^2} \leq \|u\|_{X_\lambda}, \\ &\|P_\lambda u\|_{L_T^q L_x^r} \leq \langle\lambda\rangle^{\frac{3}{4}(1-2\nu)}\|u\|_{X_\lambda}. \end{aligned} \tag{5.5.10}$$

We estimate the linear part of (5.5.5) using (5.5.8) as follows:

$$\begin{aligned} \|\mathtt{S}_{m_0}(\pm)f_\pm\|_{X^s_T} &= \left[\sum_\lambda \langle\lambda\rangle^{2s}\|\mathtt{S}_{m_0}(\pm)f_\pm\|^2_{X_\lambda}\right]^{1/2} \\ &\lesssim \left[\sum_\lambda \langle\lambda\rangle^{2s}\|P_\lambda f_\pm\|^2_{L^2_x}\right]^{1/2} \\ &\sim \|f_\pm\|_{H^s}. \end{aligned} \tag{5.5.11}$$

So Theorem 5.5.2 reduces to proving bilinear estimates on the terms $\mathscr{B}_1^\pm(\mathtt{u},\mathtt{v})$ and $\mathscr{B}_2^\pm(\mathtt{u},\mathtt{v})$ that are defined in (5.5.6) and (5.5.7), respectively.

Here we state the lemma which we used to prove local well-posedness in $\mathbb{R}^d$. We give the proof of this lemma in the next section.

Lemma 5.5.4. *Let $d \geq 3$, $T > 0$, $0 < \nu \ll 1$ and $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$. Then*

$$\|\mathscr{B}_1^\pm(\mathtt{u},\mathtt{v})\|_{X^s_T} \lesssim T^{\frac{1}{2}+\nu}\|\mathtt{u}\|_{X^s_T}\|\mathtt{v}\|_{X^s_T}, \tag{5.5.12}$$

$$\|\mathscr{B}_2^\pm(\mathtt{u},\mathtt{v})\|_{X^s_T} \lesssim T^{\frac{1}{2}+\nu}\|\mathtt{u}\|_{X^s_T}\|\mathtt{v}\|_{X^s_T}, \tag{5.5.13}$$

for all $\mathtt{u},\mathtt{v} \in X^s_T$, where $\mathscr{B}_1^\pm$ and $\mathscr{B}_2^\pm$ are as in (5.5.6) *and* (5.5.7)*, respectively.*

5.5.2 Local well-posedness

Given that Lemma 5.5.4 holds, we solve the integral equation (5.5.5) by contraction mapping techniques as follows. Define the mapping

$$(u_+,u_-) \mapsto (\Phi^+(u_+,u_-),\Phi^-(u_+,u_-)),$$

where $\Phi^\pm(u_+,u_-)$ is given by the right hand side of (5.5.5). Now given the initial data with norm

$$\sum_\pm \|f_\pm\|_{H^s} \leq \mathtt{D}_0,$$

we look for solution in the set

$$E_T = \left\{u_\pm \in X^s_T : \sum_\pm \|u_\pm\|_{X^s_T} \leq 2C\mathtt{D}_0\right\}.$$

Then by (5.5.11) and Lemma 5.5.4 we have

$$\begin{aligned}\|\Phi^{\pm}(u_+,u_-)\|_{X_T^s} &\leq C\sum_{\pm}\|f_{\pm}\|_{H^s}+CT^{\frac{1}{2}+\nu}\left(\sum_{\pm}\|u_{\pm}\|_{X_T^s}\right)^2\\ &\leq C\mathtt{D}_0+CT^{\frac{1}{2}+\nu}(2C\mathtt{D}_0)^2\\ &\leq 2C\mathtt{D}_0,\end{aligned}$$

provided that

$$T\leq(8C^2\mathtt{D}_0)^{-\frac{2}{1+2\nu}}. \tag{5.5.14}$$

Similarly, for two pair of solutions (u_+,u_-) and (v_+,v_-) in E_T with the same initial data, one can drive the difference estimate

$$\begin{aligned}&\|\sum_{\pm}\Phi^{\pm}(u_+,u_-)-\sum_{\pm}\Phi^{\pm}(v_+,v_-)\|_{X_T^s}\\ &\quad\leq CT^{\frac{1}{2}+\nu}\left(\sum_{\pm}\|u_{\pm}\|_{X_T^s}+\|v_{\pm}\|_{X_T^s}\right)\left(\sum_{\pm}\|u_{\pm}-v_{\pm}\|_{X_T^s}\right)\\ &\quad\leq 4C^2T^{\frac{1}{2}+\nu}\mathtt{D}_0\left(\sum_{\pm}\|u_{\pm}-v_{\pm}\|_{X_T^s}\right)\\ &\quad\leq\frac{1}{2}\left(\sum_{\pm}\|u_{\pm}-v_{\pm}\|_{X_T^s}\right),\end{aligned}$$

where in the last inequality we used (5.5.14).

Thus, (Φ^+,Φ^-) is contraction on E_T and therefore it has a unique fixed point $(u_+,u_-)\in E_T$ solving the integral equation (5.5.5)-(5.5.7) on $\mathbb{R}^d\times[0,T]$, where $T\sim\mathtt{D}_0^{-\frac{2}{1+1\nu}}$. Continuous dependence of solution on the initial data can be shown in the similar way by difference estimates. This conclude the proof of Theorem (5.5.2).

5.6 Bileaner estimates in $\mathbb{R}^d$ for $d\geq 3$

First we prove some bilinear estimates in Lemma 5.6.1 below that will be crucial in the proof of Lemma 5.5.4. To do so, we need the following Bernstein inequality, which is valid for $1\leq a\leq b\leq\infty$ see for instance (Tao, 2006) appendix A.

$$\|f_{\lambda}\|_{L_x^b}\leq\lambda^{\frac{d}{a}-\frac{d}{b}}\|f_{\lambda}\|_{L_x^a}. \tag{5.6.1}$$

Moreover, we have for all $s_1, s_2 \in \mathbb{R}$ and $p \geq 1$,

$$\| |D|^{s_1} K^{s_2} \mathfrak{R} f_\lambda \|_{L_x^p} \leq \lambda^{s_1} \langle \lambda \rangle^{-s_2} \| f_\lambda \|_{L_x^p}, \tag{5.6.2}$$

where we used $K(\xi) \sim \langle \xi \rangle^{-1}$.

Lemma 5.6.1. *Let $d \geq 3, T > 0$, $0 < \nu \ll 1$ and $\lambda_j \in 2^{\mathbb{Z}}$ $(j = 0, 1, 2)$. Then*

$$\| |D| K P_{\lambda_0} \mathfrak{R}. \left(\mathrm{u}_{\lambda_1} \mathfrak{R} \sqrt{K} \mathrm{v}_{\lambda_2} \right) \|_{L_T^1 L_x^2} \lesssim \frac{T^{\frac{1}{2}+\nu} \min(\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{d}{2} - \frac{1}{4} + \frac{\nu}{2}}}{\langle \lambda_2 \rangle^{\frac{1}{2}}} \| \mathrm{u} \|_{X_{\lambda_1}} \| \mathrm{v} \|_{X_{\lambda_2}}, \tag{5.6.3}$$

$$\| |D| \sqrt{K} P_{\lambda_0} \left(\mathfrak{R} \sqrt{K} \mathrm{u}_{\lambda_1} \mathfrak{R} \sqrt{K} \mathrm{v}_{\lambda_2} \right) \|_{L_T^1 L_x^2} \lesssim \frac{T^{\frac{1}{2}+\nu} \langle \lambda_0 \rangle^{\frac{1}{2}} \min(\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{d}{2} - \frac{1}{4} + \frac{\nu}{2}}}{\langle \lambda_1 \rangle^{\frac{1}{2}} \langle \lambda_2 \rangle^{\frac{1}{2}}} \| \mathrm{u} \|_{X_{\lambda_1}} \| \mathrm{v} \|_{X_{\lambda_2}}. \tag{5.6.4}$$

for all $\mathrm{u} \in X_{\lambda_1}$, $\mathrm{v} \in X_{\lambda_2}$

Proof. To prove (5.6.3), by symmetry we may assume $\lambda_1 \leq \lambda_2$. Let

$$\frac{1}{q} = \frac{1}{2} - \nu \qquad r = \frac{2d}{d - 2 + 4\nu} \qquad d \geq 2.$$

Then by Holder, (5.6.2), (5.6.1) and (5.5.10) we obtain

$$\begin{aligned}
&\| |D| K P_{\lambda_0} \mathfrak{R}. (\mathrm{u}_{\lambda_1} \mathfrak{R} \sqrt{K} \mathrm{v}_{\lambda_2}) \|_{L_T^1 L_x^2} \\
&\quad \lesssim T^{\frac{1}{2}} \lambda_0 \langle \lambda_0 \rangle^{-1} \| \mathfrak{R}. \left(\mathrm{u}_{\lambda_1} \mathfrak{R} \sqrt{K} \mathrm{v}_{\lambda_2} \right) \|_{L_{T,x}^2} \\
&\quad \lesssim T^{\frac{1}{2}} \lambda_0 \langle \lambda_0 \rangle^{-1} \langle \lambda_2 \rangle^{-\frac{1}{2}} \| \mathrm{u}_{\lambda_1} \|_{L_T^2 L_x^\infty} \| \mathrm{v}_{\lambda_2} \|_{L_T^\infty L_x^2} \\
&\quad \lesssim T^{\frac{1}{2}+\nu} \langle \lambda_2 \rangle^{-\frac{1}{2}} \lambda_1^{\frac{d}{r}} \| \mathrm{u}_{\lambda_1} \|_{L_T^q L_x^r} \| \mathrm{v}_{\lambda_2} \|_{L_T^\infty L_x^2} \\
&\quad \lesssim T^{\frac{1}{2}+\nu} \langle \lambda_2 \rangle^{-\frac{1}{2}} \lambda_1^{\frac{d}{2} - 1 + 2\nu} \langle \lambda_1 \rangle^{\frac{3}{4}(1 - 2\nu)} \| \mathrm{u} \|_{X_{\lambda_1}} \| \mathrm{v} \|_{X_{\lambda_2}} \\
&\quad \lesssim T^{\frac{1}{2}+\nu} \langle \lambda_2 \rangle^{-\frac{1}{2}} \langle \lambda_1 \rangle^{\frac{d}{2} - \frac{1}{4} + \frac{\nu}{2}} \| \mathrm{u} \|_{X_{\lambda_1}} \| \mathrm{v} \|_{X_{\lambda_2}},
\end{aligned}$$

which proves (5.6.3).

By similarly arguments,

$$
\begin{aligned}
&\||D|\sqrt{K}P_{\lambda_0}\left(\Re\sqrt{K}\mathrm{u}_{\lambda_1}\Re\sqrt{K}\mathrm{v}_{\lambda_2}\right)\|_{L^1_TL^2_x}\\
&\lesssim T^{\frac12}\lambda_0\langle\lambda_0\rangle^{-\frac12}\|\left(\Re\sqrt{K}\mathrm{u}_{\lambda_1}\Re\sqrt{K}\mathrm{v}_{\lambda_2}\right)\|_{L^2_{T,x}}\\
&\lesssim T^{\frac12}\lambda_0\langle\lambda_0\rangle^{-\frac12}\langle\lambda_1\rangle^{-\frac12}\langle\lambda_2\rangle^{-\frac12}\|\mathrm{u}_{\lambda_1}\|_{L^2_TL^\infty_x}\|\mathrm{v}_{\lambda_2}\|_{L^\infty_TL^2_x}\\
&\lesssim T^{\frac12+\nu}\langle\lambda_0\rangle^{\frac12}\langle\lambda_1\rangle^{-\frac12}\langle\lambda_2\rangle^{-\frac12}\lambda_1^{\frac{d}{r}}\|\mathrm{u}_{\lambda_1}\|_{L^q_TL^r_x}\|\mathrm{v}_{\lambda_2}\|_{L^\infty_TL^2_x}\\
&\lesssim T^{\frac12+\nu}\langle\lambda_0\rangle^{\frac12}\langle\lambda_1\rangle^{-\frac12}\langle\lambda_2\rangle^{-\frac12}\langle\lambda_1\rangle^{\frac{d}{2}-\frac14+\frac{\nu}{2}}\|\mathrm{u}\|_{X_{\lambda_1}}\|\mathrm{v}\|_{X_{\lambda_2}}.
\end{aligned}
$$

□

Now we are ready to prove Lemma 5.5.4, estimates (5.5.12) and (5.5.13). To this end, we decompose $\mathrm{u}=\sum_\lambda \mathrm{u}_\lambda$ and $\mathrm{v}=\sum_\lambda \mathrm{v}_\lambda$. Note that by denoting

$$
a_\lambda := \|\mathrm{u}\|_{X_\lambda}, \qquad b_\lambda := \|\mathrm{v}\|_{X_\lambda},
$$

we can write

$$
\|\mathrm{u}\|_{X^s_T} = \|(\langle\lambda\rangle^s a_\lambda)\|_{l^2_\lambda}, \qquad \|\mathrm{v}\|_{X^s_T} = \|(\langle\lambda\rangle^s b_\lambda)\|_{l^2_\lambda}. \tag{5.6.5}
$$

We shall make a frequent use of the following dyadic summation estimate, for $\mu,\lambda\in 2^{\mathbb{Z}}$ and $c_1,c_2>0$:

$$
\sum_{\mu\sim\lambda} a_\mu \sim a_\lambda, \qquad \sum_{c_1\lesssim\lambda\lesssim c_2}\lambda^p \lesssim \begin{cases} c_2^p, & \text{if } p>0,\\ c_1^p, & \text{if } p<0.\end{cases}
$$

Applying (5.5.9) to $\mathscr{B}_1^\pm(\mathrm{u},\mathrm{v})$ in (5.5.6) we get

$$
\|\mathscr{B}_1^\pm(\mathrm{u},\mathrm{v})\|^2_{X^s_T} \lesssim \sum_{\lambda_0}\langle\lambda_0\rangle^{2s}\left\||D|K\Re.P_{\lambda_0}\left(\mathrm{u}\Re\sqrt{K}\mathrm{v}\right)\right\|^2_{L^1_TL^2_x}.
$$

By the dyadic decomposition, we obtain

$$
\left\||D|K\Re.P_{\lambda_0}\left(\mathrm{u}\Re\sqrt{K}\mathrm{v}\right)\right\|^2_{L^1_TL^2_x} \lesssim \sum_{\lambda_1,\lambda_2}\left\||D|K\Re.P_{\lambda_0}\left(\mathrm{u}_{\lambda_1}\Re\sqrt{K}\mathrm{v}_{\lambda_2}\right)\right\|^2_{L^1_TL^2_x}. \tag{5.6.6}
$$

Now let $\lambda_{min},\lambda_{med}$ and λ_{max} denote the minimum, median and maximum of $\lambda=\{\lambda_0,\lambda_1,\lambda_2\}$, respectively. By checking the support properties in the Fourier space of the bilinear term on the right hand side of (5.6.6) one can see that this term vanishes unless $\lambda=(\lambda_0,\lambda_1,\lambda_2)\in\Lambda$,

where

$$\Lambda = \{\lambda : \lambda_{med} \sim \lambda_{max}\}.$$

Thus, we only get a nonzero contribution in (5.6.6) if $\lambda \in \cup_{j=0}^{2} \Lambda_j$,where

$$\begin{cases} \Lambda_0 = \{\lambda : \lambda_0 \lesssim \lambda_1 \sim \lambda_2\}, \\ \Lambda_1 = \{\lambda : \lambda_2 \ll \lambda_1 \sim \lambda_0\}, \\ \Lambda_2 = \{\lambda : \lambda_1 \ll \lambda_2 \sim \lambda_0\}. \end{cases} \tag{5.6.7}$$

By using these facts, and applying (5.6.3) to the right hand side of (5.6.6), we get

$$\|\mathscr{B}_1^{\pm}(\mathrm{u},\mathrm{v})\|_{X_T^s}^2 \lesssim T^{1+2\nu} \sum_{j=0}^{2} \mathscr{I}_j.$$

where

$$\mathscr{I}_j = \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left[\sum_{\lambda_1, \lambda_2 : \lambda \in \Lambda_j} \frac{\min(\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{d}{2} - \frac{1}{4} + \frac{\nu}{2}}}{\langle \lambda_2 \rangle^{\frac{1}{2}}} a_{\lambda_1} b_{\lambda_2} \right]^2. \tag{5.6.8}$$

So (5.5.12) is reduced to proving

$$\mathscr{I}_j \lesssim \|\mathrm{u}\|_{X_T^s}^2 \|\mathrm{v}\|_{X_T^s}^2 \qquad (j = 0, 1, 2), \tag{5.6.9}$$

for $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$. These are shown to hold as follows:

$$\begin{aligned} \mathscr{I}_0 &\lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{2s} \left(\sum_{\lambda_1 \sim \lambda_2 \gtrsim \lambda_0} a_{\lambda_1} \langle \lambda_2 \rangle^{\frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}} b_{\lambda_2} \right)^2 \\ &\lesssim \sum_{\lambda_0} \langle \lambda_0 \rangle^{-2\left(s - \frac{d}{2} + \frac{3}{4} - \frac{\nu}{2}\right)} \left(\sum_{\lambda_1 \sim \lambda_2 \gtrsim \lambda_0} \langle \lambda_1 \rangle^s a_{\lambda_1} \langle \lambda_2 \rangle^s b_{\lambda_2} \right)^2 \\ &\lesssim \|\mathrm{u}\|_{X_T^s}^2 \|\mathrm{v}\|_{X_T^s}^2 \end{aligned}$$

for all $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$, where to obtain the last line we used Cauchy Schwarz inequality in $\lambda_1 \sim \lambda_2$ and (5.6.5).

Similarly,

$$
\begin{aligned}
\mathscr{J}_1 &\lesssim \sum_{\lambda_0} \langle\lambda_0\rangle^{2s} \left(\sum_{\lambda_2 \ll \lambda_1 \sim \lambda_0} a_{\lambda_1} \langle\lambda_2\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}} b_{\lambda_2} \right)^2 \\
&\lesssim \sum_{\lambda_0} \langle\lambda_0\rangle^{2s} a_{\lambda_0}^2 \left(\sum_{\lambda_2} \langle\lambda_2\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}} b_{\lambda_2} \right)^2 \\
&\lesssim \|\mathrm{u}\|_{X_T^s}^2 \|\mathrm{v}\|_{X_T^s}^2
\end{aligned}
$$

for all $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$, where to get the last two inequalities we used $\sum_{\lambda_1 \sim \lambda_0} a_{\lambda_1} \sim a_{\lambda_0}$ and Cauchy Schwarz

$$
\begin{aligned}
\sum_{\lambda_2} \langle\lambda_2\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}} b_{\lambda_2} &= \sum_{\lambda_2} \langle\lambda_2\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}-s} \langle\lambda_2\rangle^s b_{\lambda_2} \\
&\lesssim \|(\langle\lambda_2\rangle^s b_{\lambda_2})\|_{l^2_{\lambda_2}} \\
&\lesssim \|\mathrm{v}\|_{X_T^s}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathscr{J}_2 &\lesssim \sum_{\lambda_0} \langle\lambda_0\rangle^{2s} \left(\sum_{\lambda_1 \ll \lambda_2 \sim \lambda_0} \langle\lambda_1\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}} a_{\lambda_1} b_{\lambda_2} \right)^2 \\
&\lesssim \sum_{\lambda_0} \langle\lambda_0\rangle^{2s} b_{\lambda_0}^2 \left(\sum_{\lambda_2} \langle\lambda_1\rangle^{\frac{d}{2}-\frac{3}{4}+\frac{\nu}{2}} a_{\lambda_1} \right)^2 \\
&\lesssim \|\mathrm{u}\|_{X_T^s}^2 \|\mathrm{v}\|_{X_T^s}^2,
\end{aligned}
$$

for all $s > \frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}$.

Arguing as in the preceding case we apply (5.5.9) to $\mathscr{B}_2^{\pm}(\mathrm{u},\mathrm{v})$ in (5.5.7) and then use (5.6.4) to obtain

$$
\|\mathscr{B}_2^{\pm}(\mathrm{u},\mathrm{v})\|_{X_T^s}^2 \lesssim T^{1+2\nu} \sum_{j=0}^{2} \tilde{\mathscr{J}}_j,
$$

where

$$
\tilde{\mathscr{J}}_j = \sum_{\lambda_0} \langle\lambda_0\rangle^{2s} \left[\sum_{\lambda_1,\lambda_2 : \lambda \in \Lambda_j} C(\lambda_0,\lambda_1,\lambda_2) a_{\lambda_1} b_{\lambda_2} \right]^2 \tag{5.6.10}
$$

with

$$
C(\lambda_0,\lambda_1,\lambda_2) = \frac{\langle\lambda_0\rangle^{\frac{1}{2}} \min(\langle\lambda_1\rangle,\langle\lambda_2\rangle)^{\frac{d}{2}-\frac{1}{4}+\frac{\nu}{2}}}{\langle\lambda_1\rangle^{\frac{1}{2}} \langle\lambda_2\rangle^{\frac{1}{2}}}.
$$

So (5.5.13) is reduced to proving

$$\tilde{\mathscr{I}}_j \lesssim \|\mathtt{u}\|_{X_T^s}^2 \|\mathtt{v}\|_{X_T^s}^2 \qquad (j = 0, 1, 2). \tag{5.6.11}$$

These can be proved following the argument of the preceding case by using the fact that

$$C(\lambda_0, \lambda_1, \lambda_2) \lesssim \min(\langle \lambda_1 \rangle, \langle \lambda_2 \rangle)^{\frac{d}{2} - \frac{3}{4} + \frac{\nu}{2}} \tag{5.6.12}$$

for all $\lambda_0, \lambda_1, \lambda_2 \in \Lambda$.

Chapter 6

Comparison between Boussinesq and Whitham–Boussinesq Type Systems

6.1 Introduction

In this chapter we consider the Whitham–Boussinesq type system

$$\begin{cases} \eta_t + v_x + \varepsilon K_\varepsilon(\eta v)_x = 0, \\ v_t + K_\varepsilon \eta_x + \varepsilon K_\varepsilon(v^2/2)_x = 0, \end{cases} \tag{6.1.1}$$

with given initial data

$$\eta(x,0) = f(x) \in H^s(\mathbb{R}), \quad v(x,0) = g(x) \in H^{s+\frac{1}{2}}(\mathbb{R}). \tag{6.1.2}$$

Here K_ε is a non local operator related to the dispersion relation of the linearized water waves system, and is defined as

$$K_\varepsilon := K_\varepsilon(D) = \frac{\tanh(\sqrt{\varepsilon}D)}{\sqrt{\varepsilon}D}, \tag{6.1.3}$$

where $D = -i\partial_x$. The shallowness parameter $\varepsilon > 0$ is proportional to the ratio of the amplitude of the wave to the mean depth of the fluid. The system (6.1.1) approximates the two dimensional water waves problem for an inviscid incompressible potential flow. The variables η and v denotes the surface elevation and fluid velocity, respectively. For some discussion on its precise physical meaning we refer the reader to the work of (Dinvay et al., 2019) where the system (6.1.1) appear for the first time.

There has been several studies by (Saut et al., 2017; Saut and Xu, 2012, 2020a; Linares et al., 2012; Saut and Xu, 2020b) regarding the long time existence of solution for Boussinesq type systems with initial data of size $\mathscr{O}(1)$. In (Saut et al., 2017; Saut and Xu, 2012, 2020a) the authors showed that the time of existence of solution for Boussinesq type system is of scale $\mathscr{O}(1/\varepsilon)$. For example, in (Saut et al., 2017) using quasilinearization the authors proved that the one dimensional dispersive Boussinesq type system

$$\begin{cases} \eta_t + v_x + \varepsilon(\eta v)_x = 0, \\ v_t + \eta_x + \varepsilon v v_x - \varepsilon \eta_{xxx} = 0, \end{cases} \tag{6.1.4}$$

is locally well-posed with existence time of scale $\mathscr{O}(1/\varepsilon)$.

In this chapter, we are interested in the problem of the long time existence of solution to (6.1.1)-(6.1.2) assuming that the shallowness parameter ε is sufficiently small. We also drive a comparison estimate between solutions of (6.1.1) and (6.1.4). In particular, we prove that (6.1.1)-(6.1.2) is locally well-posed with existence time of order $\mathscr{O}(1/\sqrt{\varepsilon})$ provided $(\eta_0, v_0) \in H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$ for $s > 1/2$. This result was conjectured by (Dinvay, 2019) but the proof was not given. Our comparison result is related to the work of (Alazman et al., 2006) comparing solutions of Boussinesq system to solutions of the Korteweg- De Vries equation and Benjamin-Bona-Mahony equations.

6.2 Preliminary estimates

Note that $K_\varepsilon(\xi) = \frac{\tanh(\sqrt{\varepsilon}\xi)}{\sqrt{\varepsilon}\xi}$ is even and real analytic, $K_\varepsilon(0) = 1$, and decreases to zero monotonically away from origin. In fact,

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \mathscr{O}(x^7) \quad \text{for} \quad | x |< 1.$$

We need the following technical result which compares the symbol of $K_\varepsilon \partial_x$ and $\partial_x + \frac{\varepsilon}{3}\partial_x^3$ that will be used later, where K_ε is as in (6.1.3).

Lemma 6.2.1. *Assume that $\sqrt{\varepsilon} \mid \xi \mid< 1$. Then,*

$$\mid \left(K_\varepsilon(\xi)\xi - (\xi - \frac{\varepsilon}{3}\xi^3) \right) \mid \lesssim \varepsilon^2 \mid \xi \mid^5 .$$

Proof. The proof follow from the expansion

$$\xi K_\varepsilon(\xi) = \xi \left(1 - \frac{1}{3}\varepsilon\xi^2 + \frac{2}{15}\varepsilon^2\xi^4 - \mathscr{O}((\sqrt{\varepsilon}\xi)^6) \right) \quad \text{for} \quad | \sqrt{\varepsilon}\xi |< 1.$$

□

We need a bound on $\pi_\varepsilon(D)\eta$ in $L^2(\mathbb{R})$, where (η, v) is the solution of the system (6.1.1) and $\pi_\varepsilon(D)$ is the Fourier multiplier operator of symbol $\pi_\varepsilon(\xi)$ defined via

$$\pi_\varepsilon(\xi) = -i\left(K_\varepsilon(\xi)\xi - (\xi - \frac{\varepsilon}{3}\xi^3)\right). \tag{6.2.1}$$

Lemma 6.2.2. *Let* $(\eta, v) \in H^{s+8}(\mathbb{R}) \times H^{s+8}(\mathbb{R})$ *be the solution of Cauchy problem* (6.1.1)-(6.1.2) *for* $s \geq 0$*. Then, for all* $j \in \mathbb{N}$*,* $j \geq 0$*, there exists* $C > 0$ *that depends on the norm of the solutions such that*

$$\|\partial_x^j \pi_\varepsilon(D)\eta(t)\|_{L^2} \leq C\varepsilon^{\frac{3}{2}},$$

for all $0 \leq t \leq \varepsilon^{-\frac{1}{2}}$.

Proof. We apply the operator $\partial_x^j \pi_\varepsilon(D)$ to the first equation of system (6.1.1), multiply this equation by $\partial_x^j \pi_\varepsilon(D)\eta$ integrating in the space variable over $\mathbb{R}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \partial_x^j \pi_\varepsilon(D)\eta_t \partial_x^j \pi_\varepsilon(D)\eta \, dx \\ &\qquad = -\int_{\mathbb{R}} \left(\partial_x^{j+1}\pi_\varepsilon(D)v + \varepsilon\partial_x^{j+1}\pi_\varepsilon(D)K_\varepsilon(\eta v)\right)\partial_x^j \pi_\varepsilon(D)\eta \, dx.\end{aligned}$$

Then using the Cauchy-Schwarz inequality we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_x^j \pi_\varepsilon(D)\eta\|_{L^2}^2 \\ &\leq \left(\|\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2} + \varepsilon\|\partial_x^{j+1}\pi_\varepsilon(D)K_\varepsilon(\eta v)\|_{L^2}\right)\|\partial_x^j \pi_\varepsilon(D)\eta\|_{L^2}.\end{aligned} \tag{6.2.2}$$

Now, we fix cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that the $supp\,(\chi) \subset [-1,1]$ and $\chi = 1$ on the interval $[-1/2, 1/2]$ and we define the Fourier multiplier $P_{<1/\sqrt{\varepsilon}}$ of symbol $\chi(\sqrt{\varepsilon}.)$ and $P_{\geq 1/\sqrt{\varepsilon}} = 1 - P_{<1/\sqrt{\varepsilon}}$. Then, using Minkowski's inequality we get

$$\|\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2} \leq \|P_{<1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2} + \|P_{\geq 1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2}. \tag{6.2.3}$$

We have to estimate each norm on the right hand side of (6.2.3).

To bound the first term in (6.2.3), we use Corollary 4.3.2 and Lemma 6.2.1 to get

$$\|P_{<1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2} \lesssim \varepsilon^2\|v\|_{H^{j+6}}. \tag{6.2.4}$$

Since $K_\varepsilon(\xi) \leq 1$, we have

$$| \pi_\varepsilon(\xi) | \leq 2 | \xi | + | \frac{\varepsilon}{3}\xi^3 | \lesssim | \xi | (1+ | \xi |^2) \lesssim (1+ | \xi |^2)^{\frac{3}{2}}. \tag{6.2.5}$$

Then using Plancherel's identity and inequality (6.2.5) we get

$$\begin{aligned}\|P_{\geq 1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2}^2 &\lesssim \int_{|\xi|\geq 1/\sqrt{\varepsilon}} | \xi |^{2(j+1)}\, \pi_\varepsilon^2(\xi) \,|\, \widehat{v}(\xi) |^2\, d\xi \\ &\lesssim \int_{|\xi|\geq 1/\sqrt{\varepsilon}} | \xi |^{2(j+1)}\, (1+ | \xi |^2)^3 \,|\, \widehat{v}(\xi) |^2\, d\xi.\end{aligned}$$

On the support of the integral $1 \leq \varepsilon^4 | \xi |^8$. Thus, we have

$$\|P_{\geq 1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)v\|_{L^2} \lesssim \varepsilon^2\|v\|_{H^{j+8}}. \tag{6.2.6}$$

Similarly, using Plancherel's identity , Lemma 6.2.1, boundedness of K_ε over $\mathbb{R}$ and Theorem 4.4.3 for $s > 1/2$, we have the estimate

$$\begin{aligned}\|\partial_x^{j+1}\pi_\varepsilon(D)K_\varepsilon(\eta v)\|_{L^2} &\leq \|\partial_x^{j+1}\pi_\varepsilon(D)(\eta v)\|_{L^2} \\ &\leq \|P_{<1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)(\eta v)\|_{L^2} + \|P_{\geq 1/\sqrt{\varepsilon}}\partial_x^{j+1}\pi_\varepsilon(D)(\eta v)\|_{L^2} \\ &\lesssim \varepsilon^2\|\eta v\|_{H^{j+6}} + \varepsilon^2\|\eta v\|_{H^{j+8}} \\ &\lesssim \varepsilon^2\|\eta\|_{H^{j+6}}\|v\|_{H^{j+6}} + \varepsilon^2\|\eta\|_{H^{j+8}}\|v\|_{H^{j+8}}.\end{aligned} \tag{6.2.7}$$

Combining estimates (6.2.4), (6.2.6) and (6.2.7) the inequality (6.2.2) becomes

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^j\pi_\varepsilon(D)\eta(t)\|_{L^2}^2 \leq C\varepsilon^2\|\partial_x^j\pi_\varepsilon(D)\eta\|_{L^2}, \tag{6.2.8}$$

where C depends on the norm of solutions. Therefore, integrating (6.2.8) between 0 and t we obtain

$$\begin{aligned}\|\partial_x^j\pi_\varepsilon(D)\eta(t)\|_{L^2} &\leq \|\partial_x^j\pi_\varepsilon(D)\eta(0)\|_{L^2} + C\varepsilon^2 t \\ &\leq \|\partial_x^j\pi_\varepsilon(D)\eta(0)\|_{L^2} + C\varepsilon^{\frac{3}{2}},\end{aligned} \tag{6.2.9}$$

for all $0 \le t \le \varepsilon^{-\frac{1}{2}}$. Finally, using similar arguments as above, we have

$$\begin{aligned}\|\partial_x^j \pi_\varepsilon(D)\eta(0)\|_{L^2} &\le \|P_{<1/\sqrt{\varepsilon}}\partial_x^j \pi_\varepsilon(D)\eta(0)\|_{L^2} + \|P_{\ge 1/\sqrt{\varepsilon}}\partial_x^j \pi_\varepsilon(D)\eta(0)\|_{L^2} \\ &\lesssim \varepsilon^2\|\eta(0)\|_{H^{j+5}} + \varepsilon^2\|\eta(0)\|_{H^{j+7}} \le C\varepsilon^2. \end{aligned} \tag{6.2.10}$$

Thus, combining inequalities (6.2.9) and (6.2.10) we complete the proof of the Lemma. □

We state the following Gronwall's type Lemma which is used to prove the comparison estimate.

Lemma 6.2.3 ((Albert and Bona, 1991), Lemma 2). *Let $\mu > 0$, $\vartheta > 0$ and $\sigma > 1$ be given. Define*

$$\mathtt{T} = \vartheta^{-\frac{1}{\sigma}}\mu^{\frac{1-\sigma}{\sigma}}\int_0^\infty (1+x^\sigma)^{-1}dx.$$

Then there exists a constant $C > 0$, which is independent of μ and ϑ, such that for any $\mathtt{T}_1 \in [0,\mathtt{T}]$, if $A(t)$ is non-negative, continuous function defined on $[0,\mathtt{T}_1]$ satisfying $A(0) = 0$ and

$$A^2(t) \le \int_0^t \left[\mu A(\tau) + \vartheta A^{\sigma+1}(\tau)\right] d\tau$$

for all $t \in [0,\mathtt{T}_1]$, then

$$A(t) \le C\mu t$$

for all $t \in [0,\mathtt{T}_1]$.

Using these estimates we prove long time existence of solutions for Whitam–Boussinesq type system and it's consistence with Boussinesq system.

6.3 Well-posedness of Whitham–Boussinesq type system in $\mathbb{R}$

In this section we discuss the well-posedness theory for the initial value problem (6.1.1)-(6.1.2).

Theorem 6.3.1 (Well-posedness). *Let $s > 1/2$, $0 < \varepsilon \le 1$ and the initial data has size*

$$\left(\|\eta_0\|_{H^s(\mathbb{R})} + \|v_0\|_{H^{s+1/2}(\mathbb{R})}\right) \sim \mathtt{D}_0.$$

Then there exists a solution

$$(\eta, v) \in C\left([0,T_\varepsilon]; H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})\right)$$

of the Cauchy problem (6.1.1)-(6.1.2) *with existence time T_ε given by*

$$T_\varepsilon \sim \mathtt{D}_0^{-1}\varepsilon^{-\frac{1}{2}}.$$

Moreover, the solution is unique in some subspace of the above solution space and depends continuously on the initial data.

6.3.1 Diagonalization of (6.1.1) and reformulation of existence theorem

Here we diagonalize system (6.1.1), and then reduce Theorem 6.3.1 to Theorem 6.3.2 below which correspond to the diagonalized system. We define the new variables

$$u^\pm = \frac{\eta \pm K_\varepsilon^{-\frac{1}{2}} v}{2}$$

then,

$$\eta = u^+ + u^-, \qquad v = K_\varepsilon^{\frac{1}{2}}(u^+ - u^-). \tag{6.3.1}$$

Then, the equation for u^+ becomes

$$\begin{aligned}
2\partial_t u^+ &= \partial_t \eta + K_\varepsilon^{-\frac{1}{2}} \partial_t v \\
&= -iDv - i\varepsilon DK_\varepsilon(\eta v) + K_\varepsilon^{-\frac{1}{2}}[-iDK_\varepsilon \eta - i\varepsilon DK_\varepsilon(v^2/2)] \\
&= -iDv - i\varepsilon DK_\varepsilon(\eta v) - iDK_\varepsilon^{\frac{1}{2}}\eta - i\varepsilon DK_\varepsilon^{\frac{1}{2}}(v^2/2) \\
&= -iDK_\varepsilon^{\frac{1}{2}}[\eta + K_\varepsilon^{-\frac{1}{2}}v] - i\varepsilon DK_\varepsilon^{\frac{1}{2}}[K_\varepsilon^{\frac{1}{2}}(\eta v) + v^2/2].
\end{aligned}$$

Therefore, we have

$$\partial_t u^+ = -iDK_\varepsilon^{\frac{1}{2}} u^+ - \frac{i\varepsilon DK_\varepsilon^{\frac{1}{2}}}{2}[K_\varepsilon^{\frac{1}{2}}(\eta v) + v^2/2]. \tag{6.3.2}$$

Similarly,

$$\begin{aligned}
2\partial_t u^- &= \partial_t \eta - K_\varepsilon^{-\frac{1}{2}} \partial_t v \\
&= -iDv - i\varepsilon DK_\varepsilon(\eta v) - K_\varepsilon^{-\frac{1}{2}}[-iDK_\varepsilon \eta - i\varepsilon DK_\varepsilon(v^2/2)] \\
&= -iDv - i\varepsilon DK_\varepsilon(\eta v) + iDK_\varepsilon^{\frac{1}{2}} \eta + i\varepsilon DK_\varepsilon^{\frac{1}{2}}(v^2/2) \\
&= iDK_\varepsilon^{\frac{1}{2}}[\eta - K_\varepsilon^{-\frac{1}{2}} v] - i\varepsilon DK_\varepsilon^{\frac{1}{2}}[K_\varepsilon^{\frac{1}{2}}(\eta v) - v^2/2].
\end{aligned}$$

Hence,

$$\partial_t u^- = iDK_\varepsilon^{\frac{1}{2}} u^- - \frac{i\varepsilon DK_\varepsilon^{\frac{1}{2}}}{2}[K_\varepsilon^{\frac{1}{2}}(\eta v) - v^2/2]. \tag{6.3.3}$$

From (6.3.2) and (6.3.3) system (6.1.1) transforms to

$$\begin{cases} \partial_t u^+ + iDK_\varepsilon^{\frac{1}{2}} u^+ = -i\varepsilon B_\varepsilon^+(u^+, u^-), \\ \partial_t u^- - iDK_\varepsilon^{\frac{1}{2}} u^- = -i\varepsilon B_\varepsilon^-(u^+, u^-), \end{cases} \tag{6.3.4}$$

where

$$\begin{aligned}
B_\varepsilon^\pm(u^+, u^-) = \frac{1}{2} & DK_\varepsilon[(u^+ + u^-)\sqrt{K_\varepsilon}(u^+ - u^-)] \\
& \pm \frac{1}{4} D\sqrt{K_\varepsilon}[\sqrt{K_\varepsilon}(u^+ - u^-)]^2.
\end{aligned}$$

The initial data (6.1.2) transforms to

$$f^\pm = \frac{\eta_0 \pm K_\varepsilon^{-\frac{1}{2}} v_0}{2}. \tag{6.3.5}$$

By Duhamel's principle the Cauchy problem (6.3.4)-(6.3.5) can be written as integral equation

$$u^\pm(t) = e^{\mp itm_\varepsilon(D)} f^\pm - i\varepsilon \int_0^t e^{\mp i(t-s)m_\varepsilon(D)} B_\varepsilon^\pm(u^+, u^-)(s) ds, \tag{6.3.6}$$

where

$$m_\varepsilon(D) := D\sqrt{K_\varepsilon(D)}.$$

Now we can state Theorem 6.3.1 in terms of the new variables as follow.

Theorem 6.3.2. *Let $s > 1/2$, $0 < \varepsilon \leq 1$ and the initial data has size*

$$\|f^+\|_{H^s} + \|f^-\|_{H^s} \sim \mathtt{D}_0.$$

Then there exists a solution

$$(u^+, u^-) \in C\left([0, T_\varepsilon]; H^s(\mathbb{R}) \times H^s(\mathbb{R})\right)$$

of the Cauchy problem (6.3.4)-(6.3.5) *with existence time T_ε as in Theorem 6.3.1. Moreover, the solution is unique in some subspace of the above solution space and solution depends continuously on the initial data.*

6.3.2 Bilinear estimates in $\mathbb{R}$

Here, we proved the key bilinear estimates which are important in the proof of Theorem 6.3.2 in next subsection.

Lemma 6.3.3. *Let $0 < \varepsilon \leq 1$ and $s > 1/2$. Then , we have*

$$\|DK_\varepsilon\left(\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} \lesssim \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\|_{H^s}\|\mathtt{v}\|_{H^s}, \tag{6.3.7}$$

$$\|D\sqrt{K_\varepsilon}\left(\sqrt{K_\varepsilon}\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} \lesssim \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\|_{H^s}\|\mathtt{v}\|_{H^s}, \tag{6.3.8}$$

for all $\mathtt{u}, \mathtt{v} \in H^s(\mathbb{R})$.

Proof. Since $|\tanh(\sqrt{\varepsilon}\xi)| \leq 1$, we have

$$|\xi K_\varepsilon(\xi)| = |\xi| \frac{\tanh(\sqrt{\varepsilon}|\xi|)}{\sqrt{\varepsilon}|\xi|} = \frac{\tanh(\sqrt{\varepsilon}|\xi|)}{\sqrt{\varepsilon}} \leq \varepsilon^{-\frac{1}{2}}. \tag{6.3.9}$$

Then using Corollary 4.3.2, inequality (6.3.9), algebra property of $H^s(\mathbb{R})$ Theorem 4.4.3 and boundedness of K_ε over $\mathbb{R}$ we have

$$\begin{aligned}
\|DK_\varepsilon\left(\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} &= \|\langle\xi\rangle^s \mid \xi K_\varepsilon(\xi)\mathscr{F}(\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v})(\xi) \mid\|_{L^2_\xi} \\
&\leq \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\|_{H^s} \\
&\lesssim \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\|_{H^s}\|\sqrt{K_\varepsilon}\mathtt{v}\|_{H^s} \\
&\lesssim \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\|_{H^s}\|\mathtt{v}\|_{H^s},
\end{aligned}$$

for all $s > 1/2$.

Using similar arguments as above with triangular inequality via Fourier transform, we have estimate

$$
\begin{aligned}
&\|D\sqrt{K_\varepsilon}\left(\sqrt{K_\varepsilon}\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} \\
&\quad \le \|\sqrt{DK_\varepsilon}\left(\sqrt{DK_\varepsilon}\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} + \|\sqrt{DK_\varepsilon}\left(\sqrt{K_\varepsilon}\mathtt{u}\sqrt{DK_\varepsilon}\mathtt{v}\right)\|_{H^s} \\
&\quad \le \varepsilon^{-\frac{1}{4}}\|\left(\sqrt{DK_\varepsilon}\mathtt{u}\sqrt{K_\varepsilon}\mathtt{v}\right)\|_{H^s} + \varepsilon^{-\frac{1}{4}}\|\left(\sqrt{K_\varepsilon}\mathtt{u}\sqrt{DK_\varepsilon}\mathtt{v}\right)\|_{H^s} \\
&\quad \lesssim \varepsilon^{-\frac{1}{4}}\|\sqrt{DK_\varepsilon}\mathtt{u}\|_{H^s}\|\sqrt{K_\varepsilon}\mathtt{v}\|_{H^s} + \varepsilon^{-\frac{1}{4}}\|\sqrt{K_\varepsilon}\mathtt{u}\|_{H^s}\|\sqrt{DK_\varepsilon}\mathtt{v}\|_{H^s} \\
&\quad \lesssim \varepsilon^{-\frac{1}{2}}\|\mathtt{u}\|_{H^s}\|\mathtt{v}\|_{H^s},
\end{aligned}
$$

for all $s > 1/2$, which completes the proof of the Lemma. □

6.3.3 Long time existence

We define contraction space X_T, via the norm

$$
\|u\|_{X_T} = \sup_{0\le t\le T}\|u(t)\|_{H^s(\mathbb{R})},
$$

where T is to be determine later. We estimate the linear part of (6.3.6) as follows;

$$
\begin{aligned}
\|e^{\mp itm_\varepsilon(D)}f^\pm\|_{X_T} &= \sup_{0\le t\le T}\|e^{\mp itm_\varepsilon(D)}f^\pm\|_{H^s} \\
&\le \sup_{0\le t\le T}\|f^\pm\|_{H^s} \qquad (6.3.10)\\
&\le \|f^\pm\|_{X_T}.
\end{aligned}
$$

We solve the integral equation (6.3.6) by contraction mapping techniques as follows. Define the mapping

$$
(u^+, u^-) \mapsto (\Psi_\varepsilon^+(u^+, u^-), \Psi_\varepsilon^-(u^+, u^-)),
$$

where $\Psi_\varepsilon^\pm(u^+, u^-)$ is given by the right hand side of (6.3.6). Now let

$$
\sum_\pm \|f^\pm\|_{X_T} \le \mathtt{D}_0,
$$

we find solution in the set

$$E_{\mathtt{D}_0}=\{u^\pm\in X_T:\sum_\pm\|u^\pm\|_{X_T}\le 2\mathtt{D}_0\}.$$

Then by (6.3.10) and Lemma 6.3.3 we get

$$\begin{aligned}\sum_\pm\|\Psi_\varepsilon^\pm(u^+,u^-)\|_{X_T}&\le\sum_\pm\left(\|f^\pm\|_{X_T}+\varepsilon\|\int_0^t B_\varepsilon^\pm(u^+,u^-)(s)ds\|_{X_T}\right)\\&\le\sum_\pm\|f^\pm\|_{X_T}+CT\varepsilon^{\frac12}\left(\sum_\pm\|u^\pm\|_{X_T}\right)^2\\&\le\mathtt{D}_0+CT\varepsilon^{\frac12}(2\mathtt{D}_0)^2\\&\le 2\mathtt{D}_0,\end{aligned}$$

where

$$T\le\frac{1}{8C\mathtt{D}_0\varepsilon^{\frac12}}.\tag{6.3.11}$$

Let (u^+,u^-) and (v^+,v^-) are two solution pairs of the integral equation (6.3.6) in $E_{\mathtt{D}_0}$ with the same initial data, then we can drive the difference estimate

$$\begin{aligned}&\sum_\pm\|\Psi_\varepsilon^\pm(u^+,u^-)-\Psi_\varepsilon^\pm(v^+,v^-)\|_{X_T}\\&\quad\le CT\varepsilon^{\frac12}\left(\sum_\pm\|u^\pm\|_{X_T}+\|v^\pm\|_{X_T}\right)\left(\sum_\pm\|u^\pm-v^\pm\|_{X_T}\right)\\&\quad\le 4CT\varepsilon^{\frac12}\mathtt{D}_0\sum_\pm\|u^\pm-v^\pm\|_{X_T}\\&\quad\le\frac12\sum_\pm\|u^\pm-v^\pm\|_{X_T},\end{aligned}$$

where in the last inequality we use (6.3.11).

Therefore, $(\Psi_\varepsilon^+,\Psi_\varepsilon^-)$ is contraction on $E_{\mathtt{D}_0}$ and has a unique fixed point $(u^+,u^-)\in E_{\mathtt{D}_0}$ solving the integral equation (6.3.6) on $\mathbb{R}\times[0,T_1]$, where $T_1\sim\mathtt{D}_0^{-1}\varepsilon^{-\frac12}$. Continuous dependence on the initial data can be shown in a similar way, by the difference estimates.

Finally, since

$$(u^+,u^-)\in C([0,T_\varepsilon];H^s(\mathbb{R})\times H^s(\mathbb{R}))$$

is the unique solution to (6.3.4)- (6.3.5), we conclude that

$$(\eta, v) = (u^+ + u^-, K_\varepsilon^{\frac{1}{2}}(u^+ - u^-)) \in C\left([0, T_\varepsilon]; H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})\right)$$

is the unique solution to (6.1.1)-(6.1.2). This recovers the result of Theorem 6.3.1.

6.4 Comparison result

This section is devoted to prove comparison between Boussinesq and Whitham–Boussinesq type equations with the same initial data.

Theorem 6.4.1 (Comparison estimate)**.** *Let $(f, g) \in H^{m+8}(\mathbb{R}) \times H^{m+8}(\mathbb{R})$ for integer $m \geq 0$. Suppose that the pairs (η, v) and $(\tilde{\eta}, \tilde{v})$ are solutions in*

$$C\left([0, \varepsilon^{-\frac{1}{2}}\mathtt{D}_0]; H^{m+8}(\mathbb{R})\right) \times C\left([0, \varepsilon^{-\frac{1}{2}}\mathtt{D}_0]; H^{m+8}(\mathbb{R})\right)$$

of systems (6.1.1) *and* (6.1.4)*, respectively with the same initial data* (6.1.2)*. Then there exists a constant $C > 0$ depending on the norm of solution, such that*

$$\|\partial_x^m(\eta - \tilde{\eta})\|_{L^2} + \|\partial_x^m(v - \tilde{v})\|_{L^2} \leq C\varepsilon t,$$

for $0 < \varepsilon \leq 1$ and $0 \leq t \leq \varepsilon^{-\frac{1}{2}}\mathtt{D}_0$, where $\mathtt{D}_0$ is as in Theorem 6.3.1.

Remark 6.4.1. *We state Theorem 6.4.1 in the space $H^{m+8}(\mathbb{R}) \times H^{s+8}(\mathbb{R})$ since we need η, v and their derivatives with respect to x up to order eight are bounded in L^2 by constants which depends on initial data norm. Hence we will use this fact to proof Theorem 6.4.1.*

Let

$$\omega = \eta - \tilde{\eta} \qquad \textit{and} \qquad \theta = v - \tilde{v},$$

where (η, v) and $(\tilde{\eta}, \tilde{v})$ are solutions of the system (6.1.1) and (6.1.4), respectively with the same initial data. Then one can verify that the pair (ω, θ) satisfies the initial value problem

$$\begin{cases} \omega_t + \theta_x + \varepsilon(\omega\theta)_x + \varepsilon(\tilde{\eta}\theta)_x + \varepsilon(\tilde{v}\omega)_x - \varepsilon\omega_{xxt} - \varepsilon\theta_{xxx} = H_1 + \varepsilon G_1, \\ \theta_t + \omega_x + \varepsilon\theta\theta_x + \varepsilon(\tilde{v}\theta)_x - \varepsilon\theta_{xxt} - \varepsilon\omega_{xxx} = H_{21} + H_{22} + \varepsilon G_2, \\ \omega(x,0) = \theta(x,0) = 0, \end{cases} \tag{6.4.1}$$

where

$$
\begin{aligned}
H_1 &= \varepsilon\left((\eta v)_x - K_\varepsilon(\eta v)_x\right)\\
G_1 &= \varepsilon\left(K_\varepsilon(\eta v)_{xxx} - (\tilde{\eta}\tilde{v})_{xxx}\right)\\
H_{21} &= \varepsilon\left(vv_x - K_\varepsilon(vv_x) - \frac{1}{3}\partial_x^3\eta - \partial_x^3\tilde{\eta}\right)\\
H_{22} &= \pi_\varepsilon(D)\eta - \varepsilon\partial_x^2\pi_\varepsilon(D)\eta\\
G_2 &= \varepsilon\left(K_\varepsilon(vv_x)_{xx} - (\tilde{v}\tilde{v}_x)_{xx} + \frac{1}{3}\partial_x^5\eta + \partial_x^5\tilde{\eta}\right)
\end{aligned}
$$

and $\pi_\varepsilon(D)$ is a Fourier multiplier of symbol $\pi_\varepsilon(\xi)$ defined in (6.2.1).

To verify that the pair (ω, θ) satisfies the initial value problem (6.4.1) we calculate the following derivatives

$$
\begin{aligned}
\omega_t &= -\theta_x - \varepsilon K_\varepsilon(\eta v)_x + \varepsilon(\tilde{\eta}\tilde{v})_x,\\
\varepsilon(\omega\theta)_x &= \varepsilon(\eta_x v - \eta_x\tilde{v} - \tilde{\eta}_x v + \tilde{\eta}_x\tilde{v} + \eta v_x - \eta\tilde{v}_x - \tilde{\eta}v_x + \tilde{\eta}\tilde{v}_x),\\
\varepsilon(\tilde{\eta}\theta)_x &= \varepsilon(\tilde{\eta}_x v - \tilde{\eta}_x\tilde{v} + \tilde{\eta}v_x - \tilde{\eta}\tilde{v}_x),\\
\varepsilon(\tilde{v}\omega)_x &= \varepsilon(\eta\tilde{v}_x - \tilde{\eta}\tilde{v}_x + \eta_x\tilde{v} - \tilde{\eta}_x\tilde{v}),\\
-\varepsilon\omega_{xxt} &= \varepsilon\theta_{xxx} + \varepsilon^2 K_\varepsilon(\eta v)_{xxx} - \varepsilon^2(\tilde{\eta}\tilde{v})_{xxx}.
\end{aligned}
$$

Adding the above equations we get

$$
\begin{aligned}
&\omega_t + \theta_x + \varepsilon(\omega\theta)_x + \varepsilon(\tilde{\eta}\theta)_x + \varepsilon(\tilde{v}\omega)_x - \varepsilon\omega_{xxt} - \varepsilon\theta_{xxx}\\
&\qquad = \varepsilon(\eta v)_x - \varepsilon K_\varepsilon(\eta v)_x + \varepsilon^2 K_\varepsilon(\eta v)_{xxx} - \varepsilon^2(\tilde{\eta}\tilde{v})_{xxx}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\theta_t &= -\omega_x - K_\varepsilon\eta_x + \eta_x + \frac{\varepsilon}{3}\partial_x^3\eta - \varepsilon K_\varepsilon(vv_x) + \varepsilon\tilde{v}\tilde{v}_x - \varepsilon(\frac{1}{3}\partial_x^3\eta + \partial_x^3\tilde{\eta}),\\
&= -\omega_x + \pi_\varepsilon(D)\eta - \varepsilon K_\varepsilon(vv_x) + \varepsilon\tilde{v}\tilde{v}_x - \varepsilon(\frac{1}{3}\partial_x^3\eta + \partial_x^3\tilde{\eta}),\\
\varepsilon\theta\theta_x &= \varepsilon\left(vv_x - v\tilde{v}_x - \tilde{v}v_x + \tilde{v}\tilde{v}_x\right),\\
\varepsilon(\tilde{v}\theta)_x &= \varepsilon\left(v\tilde{v}_x - \tilde{v}\tilde{v}_x + \tilde{v}v_x - \tilde{v}\tilde{v}_x\right),\\
-\varepsilon\theta_{xxt} &= \varepsilon\omega_{xxx} - \varepsilon\partial_x^2\pi_\varepsilon(D)\eta + \varepsilon^2 K_\varepsilon(vv_x)_{xx} - \varepsilon^2(\tilde{v}\tilde{v}_x)_{xx} + \varepsilon^2(\frac{1}{3}\partial_x^5\eta + \partial_x^5\tilde{\eta}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\theta_t + \omega_x + \varepsilon\theta\theta_x + \varepsilon(\tilde{v}\theta)_x - \varepsilon\theta_{xxt} - \varepsilon\omega_{xxx} \\
&\quad = \pi_\varepsilon(D)\eta + \varepsilon v v_x - \varepsilon K_\varepsilon(v v_x) - \varepsilon(\frac{1}{3}\partial_x^3\eta + \partial_x^3\tilde{\eta}) \\
&\qquad - \varepsilon\partial_x^2\pi_\varepsilon(D)\eta + \varepsilon^2 K_\varepsilon(v v_x)_{xx} - \varepsilon^2(\tilde{v}\tilde{v}_x)_{xx} + \varepsilon^2(\frac{1}{3}\partial_x^5\eta + \partial_x^5\tilde{\eta}).
\end{aligned}
$$

We accomplish the proof of Theorem 6.4.1 in two steps. In Subsection 6.4.1 we proved Theorem 6.4.1 in the case of $m = 0$. Then using induction with the case $m = 0$ at hand we proved for $m \geq 1$ in Subsection 6.4.2.

6.4.1 Comparison result in case $m = 0$

Multiplying the first equation of (6.4.1) by ω and the second by θ, integrating over space variable we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} (\omega\omega_t + \theta\theta_t - \varepsilon\omega\omega_{xxt} - \varepsilon\theta\theta_{xxt} + \omega\theta_x + \theta\omega_x - \varepsilon\omega\theta_{xxx} - \varepsilon\theta\omega_{xxx})\, dx \\
&\quad = -\varepsilon\int_{-\infty}^{\infty} \big(\omega(\omega\theta)_x + \omega(\tilde{\eta}\theta)_x + \omega(\tilde{v}\omega)_x + \theta^2\theta_x + \theta(\tilde{v}\theta)_x\big)\, dx \\
&\qquad + \int_{-\infty}^{\infty} (\varepsilon\omega G_1 + \omega H_1 + \varepsilon\theta G_2 + \theta(H_{21} + H_{22}))\, dx.
\end{aligned} \tag{6.4.2}
$$

First, we simplify the left hand side of equation (6.4.2). Therefore, using the assumption θ, ω and their derivatives vanishes at infinity we obtain

$$
\int_{-\infty}^{\infty} (\omega_x\theta + \omega\theta_x) dx = \int_{-\infty}^{\infty} \partial_x(\omega\theta)\, dx = 0.
$$

By similar arguments as above, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} (\omega\theta_{xxx} + \theta\omega_{xxx})\, dx &= -\int_{-\infty}^{\infty} (\theta_{xx}\omega_x + \theta_x\omega_{xx}) dx \\
&= -\int_{-\infty}^{\infty} \partial_x(\theta_x\omega_x)\, dx = 0.
\end{aligned}
$$

And also

$$
\int_{-\infty}^{\infty} \theta^2\theta_x dx = \frac{1}{3}\int_{-\infty}^{\infty} \partial_x(\theta^3) dx = 0.
$$

Using integration by parts we obtain

$$\int_{-\infty}^{\infty} (\omega\omega_t + \theta\theta_t - \varepsilon\omega\omega_{xxt} - \varepsilon\theta\theta_{xxt})\, dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(\omega^2 + \theta^2 + \varepsilon\omega_x^2 + \varepsilon\theta_x^2\right) dx.$$

Thus, the integral equation (6.4.2) can be reduced to the form

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(\omega^2 + \theta^2 + \varepsilon\omega_x^2 + \varepsilon\theta_x^2\right) dx \\ &\quad = -\varepsilon\int_{-\infty}^{\infty} (\omega(\omega\theta)_x + \omega(\tilde{\eta}\theta)_x + \omega(\tilde{v}\omega)_x + \theta(\tilde{v}\theta)_x)\, dx \\ &\qquad + \int_{-\infty}^{\infty} (\varepsilon\omega G_1 + \omega H_1 + \varepsilon\theta G_2 + \theta(H_{21} + H_{22}))\, dx. \end{aligned} \tag{6.4.3}$$

Since $\omega(x,0) = \theta(x,0) = 0$, integrating the integral equation (6.4.3) over the interval $[0,t]$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{-\infty}^{\infty} \left(\omega^2 + \theta^2 + \varepsilon\omega_x^2 + \varepsilon\theta_x^2\right) dx \\ &\quad = -\varepsilon\int_0^t\int_{-\infty}^{\infty} (\omega(\omega\theta)_x + \omega(\tilde{\eta}\theta)_x + \omega(\tilde{v}\omega)_x + \theta(\tilde{v}\theta)_x)\, dxd\tau \\ &\qquad + \int_0^t\int_{-\infty}^{\infty} (\varepsilon\omega G_1 + \omega H_1 + \varepsilon\theta G_2 + \theta(H_{21} + H_{22}))\, dxd\tau. \end{aligned} \tag{6.4.4}$$

Let $A_0(t)$ denotes the positive square root of the integral equation on the left hand side of (6.4.4), i.e.,

$$A_0^2(t) = \int_{-\infty}^{\infty} \left(\omega^2 + \theta^2 + \varepsilon\omega_x^2 + \varepsilon\theta_x^2\right) dx.$$

Then, we have

$$\begin{aligned} \|\omega\|_{L^2} &\lesssim A_0(t), \\ \|\omega_x\|_{L^2} &\lesssim \varepsilon^{-\frac{1}{2}} A_0(t). \end{aligned} \tag{6.4.5}$$

Using the inequality

$$\|f\|_{L^\infty}^2 \lesssim \|f\|_{L^2}\|f_x\|_{L^2}, \tag{6.4.6}$$

it follows that

$$\|\omega\|_{L^\infty} \lesssim \varepsilon^{-\frac{1}{4}} A_0(t).$$

In fact, the same estimates also hold for θ.

Since $h = \eta, v, \tilde{\eta}, \tilde{v} \in H^8(\mathbb{R})$ for all $t \in [0, \varepsilon^{-\frac{1}{2}} \mathrm{D}_0]$ and $0 \le i \le 8$, we have

$$\|\partial_x^i h\|_{L^2} \lesssim 1. \tag{6.4.7}$$

Using these observations we can bound the terms on the right hand side of the integral equation (6.4.4) as follow:

$$\begin{aligned}
-\varepsilon \int_0^t \int_{-\infty}^{\infty} \omega(\omega\theta)_x dx d\tau &= \varepsilon \int_0^t \int_{-\infty}^{\infty} \omega_x \omega \theta dx d\tau, \\
&\le \varepsilon \int_0^t \|\omega_x\|_{L^2} \|\omega\|_{L^2} \|\theta\|_{L^\infty} d\tau \\
&\lesssim \varepsilon \int_0^t \|\omega_x\|_{L^2} \|\omega\|_{L^2} \|\theta\|_{L^2}^{\frac{1}{2}} \|\theta_x\|_{L^2}^{\frac{1}{2}} d\tau \\
&\lesssim \varepsilon^{\frac{1}{4}} \int_0^t A_0^3(\tau) d\tau,
\end{aligned} \tag{6.4.8}$$

where we used Holder's inequality, estimate (6.4.6) and (6.4.5) to obtain the first, second and third inequalities, respectively. Using similar arguments as above, we have the following estimates

$$\begin{aligned}
\varepsilon \int_0^t \int_{-\infty}^{\infty} \omega_x \tilde{\eta} \theta dx d\tau &\lesssim \varepsilon^{\frac{1}{2}} \int_0^t A_0^2(\tau) d\tau, \\
\varepsilon \int_0^t \int_{-\infty}^{\infty} \omega_x \omega \tilde{v} dx d\tau &\lesssim \varepsilon^{\frac{1}{2}} \int_0^t A_0^2(\tau) d\tau, \\
\varepsilon \int_0^t \int_{-\infty}^{\infty} \theta_x \theta \tilde{v} dx d\tau &\lesssim \varepsilon^{\frac{1}{2}} \int_0^t A_0^2(\tau) d\tau,
\end{aligned} \tag{6.4.9}$$

here we also used estimate (6.4.7). Using Holder's, Minkowski's inequalities and boundedness of K_ε over $\mathbb{R}$, we have

$$\begin{aligned}
\varepsilon \int_0^t \int_{-\infty}^{\infty} \omega G_1 dx d\tau &\le \varepsilon^2 \int_0^t \|\omega\|_{L^2} \|K_\varepsilon(\eta v)_{xxx} - (\tilde{\eta}\tilde{v})_{xxx}\|_{L^2} d\tau \\
&\lesssim \varepsilon^2 \int_0^t \|\omega\|_{L^2} (\|(\eta v)_{xxx}\|_{L^2} + \|(\tilde{\eta}\tilde{v})_{xxx}\|_{L^2}) d\tau \\
&\lesssim \varepsilon^2 \int_0^t A_0(\tau) d\tau,
\end{aligned} \tag{6.4.10}$$

where we used estimates (6.4.6), (6.4.7) and (6.4.5) to get the last inequality. By similar arguments, we have the following estimates

$$
\begin{aligned}
\int_0^t \int_{-\infty}^{\infty} \omega H_1 dx d\tau &\lesssim \varepsilon \int_0^t A_0(\tau) d\tau, \\
\varepsilon \int_0^t \int_{-\infty}^{\infty} \theta G_2 dx d\tau &\lesssim \varepsilon^2 \int_0^t A_0(\tau) d\tau, \\
\int_0^t \int_{-\infty}^{\infty} \theta H_{21} dx d\tau &\lesssim \varepsilon \int_0^t A_0(\tau) d\tau.
\end{aligned}
\tag{6.4.11}
$$

Using Minkowski's inequality, estimate (6.4.5) and Lemma 6.2.2, it follows that

$$
\int_0^t \int_{-\infty}^{\infty} \theta H_{22} dx d\tau \lesssim \varepsilon^{\frac{3}{2}} \int_0^t A_0(\tau) d\tau, \tag{6.4.12}
$$

for all $0 \leq t \leq \varepsilon^{-\frac{1}{2}}$.

Combining the estimates (6.4.8)-(6.4.12) we get the differential inequality

$$
A_0^2(t) \leq C \int_0^t \Big(\varepsilon^2 A_0(\tau)) + \varepsilon^{\frac{3}{2}} A_0(\tau) + \varepsilon A_0(\tau) + \varepsilon^{\frac{1}{2}} A_0^2(\tau) + \varepsilon^{\frac{1}{4}} A_0^3(\tau) \Big) d\tau. \tag{6.4.13}
$$

Using Young's inequality we can write

$$
\varepsilon^{\frac{1}{2}} A_0^2 = (\varepsilon A_0)^{\frac{1}{2}} A_0^{\frac{3}{2}} \leq \frac{\varepsilon A_0}{2} + \frac{A_0^3}{2}. \tag{6.4.14}
$$

From (6.4.14) and the fact that $0 < \varepsilon \leq 1$ the differential inequality (6.4.13) can written as

$$
A_0^2(t) \leq C \int_0^t \left(\varepsilon A_0(\tau) + A_0^3(\tau) \right) d\tau. \tag{6.4.15}
$$

Now applying Lemma 6.2.3 on the differential inequality (6.4.15) with $\mu = C\varepsilon$, $\vartheta = C$ and $\sigma = 2$, we have

$$
A_0(t) \leq C\varepsilon t,
$$

for all $0 \leq t \leq \varepsilon^{-\frac{1}{2}} \mathtt{D}_0$. Thus, we conclude that

$$
\|\omega\|_{L^2} + \|\theta\|_{L^2} \leq C\varepsilon t,
$$

for $t \in [0, \varepsilon^{-\frac{1}{2}} \mathtt{D}_0]$. This complete the proof of Theorem 6.4.1 in the case $m = 0$.

6.4.2 Comparison result in case $m \geq 1$

In this subsection, we compare the solutions (η, v) and $(\tilde{\eta}, \tilde{v})$ of system (6.1.1) and (6.1.4), respectively in the Sobolev spaces $H^m(\mathbb{R})$ for $m \geq 1$. We can proceed the proof of the theorem by induction on the derivatives. We assume that for $0 \leq i \leq m-1$ and $0 \leq t \leq \varepsilon^{-\frac{1}{2}}\mathtt{D}_0$ there exists a constant $C > 0$ that depends on the norm of solutions, such that

$$\|\partial_x^i \omega\|_{L^2} + \|\partial_x^i \theta\|_{L^2} \leq C\varepsilon t.$$

In particular,

$$\|\partial_x^i \omega\|_{L^2} + \|\partial_x^i \theta\|_{L^2} \lesssim C\varepsilon^{\frac{1}{2}}, \tag{6.4.16}$$

for all $t \in [0, \varepsilon^{-\frac{1}{2}}\mathtt{D}_0]$. We will use these estimate in the induction step. Then we have to show that the result holds for m.

Multiply the first equation in (6.4.1) by $\partial_x^{2m}\omega$ and the second by $\partial_x^{2m}\theta$, add the results, integrating over space variable we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\partial_x^{2m}\omega\omega_t + \partial_x^{2m}\theta\theta_t - \varepsilon\partial_x^{2m}\omega\omega_{xxt} - \varepsilon\partial_x^{2m}\theta\theta_{xxt}\right) dx \\
&\quad = \int_{-\infty}^{\infty} \left(\varepsilon\partial_x^{2m}\omega\theta_{xxx} + \varepsilon\partial_x^{2m}\theta\omega_{xxx} - \partial_x^{2m}\omega\theta_x - \partial_x^{2m}\theta\omega_x\right) dx \\
&\quad - \varepsilon\int_{-\infty}^{\infty} \left(\partial_x^{2m}\omega((\omega\theta)_x + (\tilde{\eta}\theta)_x + (\tilde{v}\omega)_x) + \partial_x^{2m}\theta(\theta\theta_x + (\tilde{v}\theta)_x)\right) dx \\
&\quad + \int_{-\infty}^{\infty} \left(\partial_x^{2m}\omega(\varepsilon G_1 + H_1) + \partial_x^{2m}\theta(\varepsilon G_2 + H_{21} + H_{22})\right) dx.
\end{aligned} \tag{6.4.17}$$

Then, applying integration by parts we can write the left hand side of integral equation (6.4.17) as

$$(-1)^m \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left((\partial_x^m\omega)^2 + (\partial_x^m\theta)^2 + \varepsilon(\partial_x^{m+1}\omega)^2 + \varepsilon(\partial_x^{m+1}\theta)^2\right) dx.$$

Again using integration by parts the right hand side of integral equation (6.4.17) can be simplified as

$$\begin{aligned}
\int_{-\infty}^{\infty} (\partial_x^{2m}\omega\theta_x + \partial_x^{2m}\theta\omega_x) dx &= (-1)^m \int_{-\infty}^{\infty} (\partial_x^m\omega\partial_x^{m+1}\theta + \partial_x^m\theta\partial_x^{m+1}\omega) dx \\
&= (-1)^m \int_{-\infty}^{\infty} \partial_x(\partial_x^m\omega\partial_x^m\theta) dx = 0,
\end{aligned}$$

here we used the fact that ω, θ and their derivatives vanishes at $\pm\infty$. By similar arguments, we have

$$\begin{aligned}\int_{-\infty}^{\infty}(\partial_x^{2m}\omega\theta_{xxx}+\partial_x^{2m}\theta\omega_{xxx})dx &= (-1)^{m-1}\int_{-\infty}^{\infty}(\partial_x^{m+1}\omega\partial_x^{m+2}\theta+\partial_x^{m+1}\theta\partial_x^{m+2}\omega)dx\\ &= (-1)^{m-1}\int_{-\infty}^{\infty}\partial_x(\partial_x^{m+1}\omega\partial_x^{m+1}\theta)dx = 0.\end{aligned}$$

Since $\omega(x,0)=\theta(x,0)=0$, the integral equation (6.4.17) can be written as

$$\begin{aligned}&\frac{1}{2}A_m^2(t) = \\ &\quad -\varepsilon\int_0^t\int_{-\infty}^{\infty}\left(\partial_x^m\omega\left(\partial_x^{m+1}(\omega\theta+\tilde{\eta}\theta+\tilde{v}\omega)\right)+\partial_x^m\theta\left(\partial_x^{m+1}(\theta^2/2+\tilde{v}\theta)\right)\right)dxd\tau\\ &\qquad +\int_0^t\int_{-\infty}^{\infty}(\partial_x^m\omega\partial_x^m(\varepsilon G_1+H_1)+\partial_x^m\theta\partial_x^m(\varepsilon G_2+H_{21}+H_{22}))\,dxd\tau,\end{aligned} \tag{6.4.18}$$

where

$$A_m^2(t)=\int_{-\infty}^{\infty}\left((\partial_x^m\omega)^2+(\partial_x^m\theta)^2+\varepsilon(\partial_x^{m+1}\omega)^2+\varepsilon(\partial_x^{m+1}\theta)^2\right)dx$$

for $m\in\mathbb{N}$ and $m\geq 1$. Then, we have

$$\begin{aligned}\|\partial_x^m\omega\|_{L^2}+\|\partial_x^m\theta\|_{L^2}&\lesssim A_m(t),\\ \|\partial_x^{m+1}\omega\|_{L^2}+\|\partial_x^{m+1}\theta\|_{L^2}&\lesssim \varepsilon^{-\frac{1}{2}}A_m(t).\end{aligned} \tag{6.4.19}$$

Recall that, for all $t\in[0,\varepsilon^{-\frac{1}{2}}\mathtt{D}_0]$, $h=\eta, v, \tilde{\eta}, \tilde{v}\in H^{m+8}(\mathbb{R})$ and $0\leq i\leq m+8$, we have

$$\|\partial_x^i h\|_{L^2}\lesssim 1. \tag{6.4.20}$$

Next we will bound the terms on the right hand side of integral equation (6.4.18). Using Leibniz's rule to expand the differentiated term in the integral, one obtain expression which are easily estimated using the assumed bound in the induction hypothesis. Thus we have the following estimates:

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}(\omega\tilde{v})dxd\tau &= \int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\left(\sum_{i=0}^{m+1}\binom{m+1}{i}\partial_x^{m+1-i}\omega\partial_x^i\tilde{v}\right)dxd\tau\\ &= \int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}\omega\tilde{v}+(m+1)(\partial_x^m\omega)^2\partial_x\tilde{v}\\ &\quad+\partial_x^m\omega\left(\sum_{i=2}^{m+1}\binom{m+1}{i}\partial_x^{m+1-i}\omega\partial_x^i\tilde{v}\right)dxd\tau.\end{aligned}$$

However,

$$\begin{aligned}\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}\omega\tilde{v}dxd\tau &= \frac{1}{2}\int_0^t\int_{-\infty}^{\infty}\partial_x(\partial_x^m\omega)^2\tilde{v}dxd\tau\\ &= -\frac{1}{2}\int_0^t\int_{-\infty}^{\infty}(\partial_x^m\omega)^2\partial_x\tilde{v}dxd\tau.\end{aligned}$$

Then, we have

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}(\omega\tilde{v})dxd\tau\\ &=\int_0^t\int_{-\infty}^{\infty}(m+\frac{1}{2})(\partial_x^m\omega)^2\partial_x\tilde{v}+\sum_{i=2}^{m+1}\binom{m+1}{i}\partial_x^m\omega\partial_x^{m+1-i}\omega\partial_x^i\tilde{v}dxd\tau.\end{aligned}\tag{6.4.21}$$

Using Holder's inequality, estimate (6.4.6), (6.4.20) and (6.4.19) we have

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}\sum_{i=2}^{m+1}\binom{m+1}{i}\partial_x^m\omega\partial_x^{m+1-i}\omega\partial_x^i\tilde{v}dxd\tau\\ &\quad\lesssim\int_0^t\sum_{i=2}^{m+1}\binom{m+1}{i}\|\partial_x^m\omega\|_{L^2}\|\partial_x^{m+1-i}\omega\|_{L^2}\|\partial_x^i\tilde{v}\|_{L^2}^{\frac{1}{2}}\|\partial_x^{i+1}\tilde{v}\|_{L^2}^{\frac{1}{2}}d\tau\\ &\quad\lesssim\sum_{i=2}^{m+1}\binom{m+1}{i}\int_0^t A_m(\tau)d\tau,\end{aligned}\tag{6.4.22}$$

where we also used induction hypothesis (6.4.16) to get the last inequality. Hence, from (6.4.21) and (6.4.22) we obtain the estimate

$$\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}(\omega\tilde{v})dxd\tau\lesssim\varepsilon\int_0^t\left(A_m^2(\tau)+A_m(\tau)\right)d\tau.\tag{6.4.23}$$

Using similar arguments, we have the estimates

$$\begin{aligned}
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^{m+1}(\theta\tilde{v})dxd\tau &\lesssim \varepsilon\int_0^t\left(A_m^2(\tau)+A_m(\tau)\right)d\tau,\\
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^{m+1}(\theta^2)dxd\tau &\lesssim \varepsilon\int_0^t\left(A_m^2(\tau)+A_m(\tau)\right)d\tau,\\
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}(\omega\theta)dxd\tau &\lesssim \varepsilon\int_0^t\left(1+\varepsilon^{-\frac{1}{2}})A_m^2(\tau)+A_m(\tau)\right)d\tau,\\
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^{m+1}(\theta\tilde{\eta})dxd\tau &\lesssim \varepsilon\int_0^t\left((1+\varepsilon^{-\frac{1}{2}})A_m^2(\tau)+A_m(\tau)\right)d\tau.
\end{aligned} \tag{6.4.24}$$

On the other hand from Holder's, Minkowski's inequalities, estimate (6.4.6), (6.4.20) and (6.4.19) we have

$$\begin{aligned}
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^m G_1dxd\tau &= \varepsilon^2\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^m\left(K_\varepsilon(\eta v)_{xxx}-(\tilde{\eta}\tilde{v})_{xxx}\right)dxd\tau\\
&\lesssim \varepsilon^2\int_0^t\|\partial_x^m\omega\|_{L^2}\left(\|\partial_x^{m+3}(\eta v)\|_{L^2}+\|\partial_x^{m+3}(\tilde{\eta}\tilde{v})\|_{L^2}\right)d\tau\\
&\lesssim \varepsilon^2\sum_{i=0}^{m+3}\binom{m+3}{i}\int_0^t A_m(\tau)d\tau.
\end{aligned} \tag{6.4.25}$$

By similar arguments we have the following estimates

$$\begin{aligned}
\int_0^t\int_{-\infty}^{\infty}\partial_x^m\omega\partial_x^m H_1dxd\tau &\lesssim \varepsilon\sum_{i=0}^{m+1}\binom{m+1}{i}\int_0^t A_m(\tau)d\tau\\
\int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^m H_{21}dxd\tau &\lesssim \varepsilon\sum_{i=0}^{m+3}\binom{m+3}{i}\int_0^t A_m(\tau)d\tau\\
\varepsilon\int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^m G_2dxd\tau &\lesssim \varepsilon^2\sum_{i=0}^{m+5}\binom{m+5}{i}\int_0^t A_m(\tau)d\tau.
\end{aligned} \tag{6.4.26}$$

Finally, applying Cauchy-Schwarz, Minkowski's inequality, Lemma 6.2.2 and estimate (6.4.19) it follows that

$$\begin{aligned}
\int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^m H_{22}dxd\tau &= \int_0^t\int_{-\infty}^{\infty}\partial_x^m\theta\partial_x^m(\pi_\varepsilon(D)\eta-\varepsilon\pi_\varepsilon(D)\eta_{xx})dxd\tau\\
&\le \int_0^t\|\partial_x^m\theta\|_{L^2}(\|\partial_x^m\pi_\varepsilon(D)\eta\|_{L^2}+\|\varepsilon\partial_x^m\pi_\varepsilon(D)\eta_{xx}\|_{L^2})d\tau\\
&\lesssim \varepsilon^{\frac{3}{2}}\int_0^t A_m(\tau)d\tau.
\end{aligned} \tag{6.4.27}$$

Hence, gathering the estimates (6.4.23)-(6.4.27) the integral equation (6.4.18) can be bounded as

$$A_m^2(t) \leq C \int_0^t \Big(\varepsilon A_m(\tau) + \varepsilon^{\frac{3}{2}} A_m(\tau) + \varepsilon^2 A_m(\tau) + \varepsilon^{\frac{1}{2}} A_m^2(\tau) + \varepsilon A_m^2(\tau) \Big) \, d\tau. \tag{6.4.28}$$

Thus, using Young's inequality (6.4.28) can be reduced to the form

$$A_m^2(t) \leq C \int_0^t \big(\varepsilon A_m(\tau) + A_m^3(\tau) \big) \, d\tau. \tag{6.4.29}$$

Therefore, we conclude the proof of Theorem 6.4.1 using Lemma 6.2.3 as for the initial step with $\mu = C\varepsilon$, $\vartheta = C$ and $\sigma = 2$.

Chapter 7

Conclusions, Recommendations and Future Works

7.1 Conclusion

In this dissertation we consider a Whitham–Boussinesq type system modeling surface water waves of an inviscid incompressible potential flow. We derive a lower bound estimate for derivative of all order for Fourier symbol of pseudo-differential operator that appear in linearized water wave type equations. We also derive dispersive and Strichartz estimates for linearized water waves type equations in $\mathbb{R}^d$. Then for pure gravity case we proved that the system is locally well posed in $\mathbb{R}^d$ for $d \geq 3$. This generalizes a recent result by (Dinvay et al., 2020), where they proved low regularity well-posedness in one and two dimensions. Using contraction mapping principle we proved that the one dimensional Whitham–Boussinesq type system is well-posed on the time scale of order $\mathscr{O}(\varepsilon^{-\frac{1}{2}})$, where $\varepsilon > 0$ is shallowness parameter. This result was conjectured by (Dinvay, 2019), but the proof was not given. In particular, if the amplitude of the wave is very small compared to the mean depth of the fluid that is $0 < \varepsilon \leq 1$, then the solutions of Whitham–Boussinesq type equations exist for arbitrarily large time. Thus, the solution has a life span of order $\mathscr{O}(\varepsilon^{-\frac{1}{2}})$. We also derived the difference estimate between Boussinesq and Whitham–Boussinesq type systems on the time scale of order $\mathscr{O}(\varepsilon^{-\frac{1}{2}})$. Hence, on this time interval we can approximate Boussinesq system with Whitham–Boussinesq type system.

7.2 Recommendations

The results of this study can be used to study wave propagation on an ice sheet over a body of water. Coastal engineers can also use results of this dissertation to study characteristics of solitary waves. For example, using the characteristics of solitary waves coastal engineers can study about sedimentation of sands and rocks around the coastal area.

7.3 Future works

In the present work we only consider the pure gravity case to prove the well-posedness results. In future, it would be interesting to consider gravity-capillary case and extending the result of this study to a global well-posedness. We have also an interest to extend the long time existence of solutions of Whitham–Boussinesq type equations to three dimensional space. Using the results of this study one can also prove well-posedness of other water wave models.

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