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Title: **Solving Mixed boundary value problem using Boundary Integral equation**

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APPROVAL SHEET

This is to certify that the thesis prepared by Fikadu Lechisa entitled Three dimensional heat transfer problem using boundary integral equation. Submitted in partial fulfillment of the requirement for the Degree of Master of Science in Applied Mathematics with the regulation of the University accepted standards with respect.

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Abstract

In this thesis we consider to solve mixed boundary value problem (BVP) in three dimension.To solve We change that boundary value problem to boundary integral equation in consider of Laplace operator. Using fundamental solution in Green's representation formula we reduce the mixed boundary value problem to boundary integral equation.We treat the boundary integral equation for classical solution and weak solution with in of Sobolev spaces. Finally, cite how to estimate boundary integral equation numerically.

Key words : boundary value problem, boundary integral equation, Green's identities, three-dimensional heat equation, Sobolev spaces.

Contents

Chapter 1

INTRODUCTION

1.1 Background of the study

The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing a boundary value problem (BVP) for a partial differential equation in a domain to an integral equation on the domain. This approach diminishes the problem dimensionality by one which is very important for construction of various numerical algorithms requiring small computer resources. The main ingredient necessary for reduction of a BVP to a boundary integral equation (BIE) is a fundamental solution to the original partial differential equation, available in an analytical form and /or cheaply calculated.

After the fundamental solution is used in the corresponding Green formulae, as a first step, one can reduce the BVP to a BIE. The next essential step is to establish the equivalence of the original BVP and the corresponding BIE, and to show the invertibility of the BIE operator which is a crucial moment for further numerical analysis.

Boundary integral method is also a classical tool for the analysis of boundary value problem for partial differential equations. The term boundary element method denotes any method for the approximate numerical solution of the boundary integral equations. Only the boundary of the domain needs to be discretized in one, two or three dimensional heat transfer problem. In some application, the physical relevant data are given not by the solution in the boundary of the domain but rather by the boundary value of the solution or its derivatives.

The boundary element methods recently become a popular and powerful numerical method next to finite element method. In the finite element method the domain is divided in to sub domain commonly referred to as finite elements [1] . The fundamental building block in the finite element method is the sub domain element and straight forward. The theory behind finite element method is relatively abstract. The biggest disadvantage of finite element method is mesh generation, which becomes more difficult and time consuming for complicated problem. Again in finite difference method the domain of solution of the given partial differential equation is first sub divided by a net with a finite number of mesh point. The derivative at each boundary is then replaced by a finite difference approximation. One can consider as a solution of the partial differential equation with an interpolation polynomial and presented by using Taylor series [11]
.

The advantage of boundary integral method is that the equations apply only on the boundary of the solution domain. An immediate consequence of this is that the order of the system of algebraic equations generated by the method is considerably smaller than those generated by finite element method or finite difference method.

Another advantage is that on the boundary of the domain, the solution obtained by this method is more accurate than from the either finite element method or finite difference methods [9, 10], so it is preferable to use the boundary element method when one needs a solution only on the boundary of the domain when using the methods, one approximates the solution of the boundary integral equation [1].

The solution obtained inside the domain satisfies the differential equation exactly, approximate only occur due to the fact that boundary condition are only satisfies approximately. Since functions are defined globally, there is no need to subdivide the domain in to the element.

Although the boundary integral formulation for different PDE_s will necessarily change due to different Green function . Thus for the most part, it suffices to discuss the simplest boundary integral equations, for Laplace equation. Therefore, the objective of this thesis is to transform boundary value problem for steady state heat transfer problem in to boundary integral equation and the solution of boundary integral equation for the boundary value problem by using Greens identity and fundamental solution.

This thesis contains five sections. section one discusses about introduction and background of the study, statement of the problem, objectives of the study, significance of the study and scope of the study. Section two deals with review of related literature. In section three as preliminaries that have the definition of related terms were discussed . In section four Boundary integral equation for the mixed boundary value problem, Green's Identity and representation formula, fundamental solution of BVP for heat equation in $\mathbb{R}^3$, Boundary integral expression for classical and weak solution are discussed and on the final section conclusion of the thesis were drawn.

1.2 Statement of the problem

If we consider a homogeneous isotropic heat conducting body Ω is a simple connected and bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $S_1 \bigcup S_2 \bigcup S_3 \bigcup S_4$ and when S_1, S_2, S_3 and S_4 are disjoint parts of S. It is supposed that convection in to ambient medium occurs on boundary S_1 and S_2, temperature is kept constant at the prescribed value T_3 on S_3, and S_4 is insulated. The state equation for the system is described by the mixed boundary value problem as :

$$\begin{cases} \triangle u = 0, \quad in \quad \Omega \\ \frac{\partial u}{\partial n} = -\frac{h}{k}(u - u_\infty), \quad on \quad S_1 \bigcup S_2 \\ u = 0, \quad on \quad S_3 \\ \frac{\partial u}{\partial n} = 0, \quad on \quad S_4 \end{cases} \tag{1.1}$$

where $u = T - T_3$, $u_\infty = T_\infty - T_3$ when T is the temperature in the domain, T_∞ is the ambient temperature, n is the out ward unit normal vector, h is the convection coefficient, k is conduction coefficient

Since the classical solution $u \in C^1(\overline{\Omega}) \bigcap C^2(\Omega)$ to the problem does not exist, where $\overline{\Omega}$ is closure of the domain Ω, we can concerned with the variational solution $H^1(\Omega)$. For the steady state heat transfer problem with initial and boundary conditions the boundary integral equation formulation is on the basis of Greens formula,

$$\int_\Omega (v \nabla^2 u - u \nabla^2 v) dx = \int_S (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) ds \tag{1.2}$$

With the domain Ω and Lipschitz boundary S the fundamental solution holds where ds is a boundary element. Besides, we apply fundamental solution and Greens identity in three dimensional heat transfer problem of boundary value problem to obtain the solution by means of boundary integral equations, such that Ω be a bounded open three dimensional region of $\mathbb{R}^3$, we assume that the boundary $S = \partial\Omega$ is simply connected, closed, infinitely smooth surface. $\bar{S}_1 \bigcup \bar{S}_2$ where S_1 and S_2 are non empty and $S_1 \bigcap S_2 = \emptyset$ with infinitely smooth boundary surface $\partial S_1 = \partial S_2 \in C^\infty$ we consider the elliptic differential equation with the unknown function and the given function in Ω. When the constant in that equation is equal to 1 the operator becomes the Laplace operator Δ.

Instead of the above, the study will find out whether the problem stated are the main contributing factor for the effectiveness of mixed boundary value problem using boundary integral equation in the study.

1. How to transform boundary value problem for steady state heat transfer problem in to boundary integral equation?

2. How we can apply fundamental solution and Greens formula to find the solution for the heat transfer problem?

1.3 Objectives of the study

1.3.1 General objectives

The general objective of the study is to solve mixed boundary value problem (BVP) using boundary integral equation.

1.3.2 Specific objectives

The specific objectives are:- To

1. transform boundary value problem for steady state heat transfer problem in to boundary integral equation.

2. prove the unique existence of solution of mixed boundary value problem for steady state heat transfer problem.

3. apply fundamental solution and Greens formula to find the solution of BVP.

4. prove the equivalence of solution of the boundary value problem with boundary integral equation.

1.4 Significance of the study

The significance (purpose) of the study is:-

For many practical boundary value problems it is not possible to obtain a solution by means of analytical technique. Instead, solving them requires the use of boundary integral equation which in many cases allows such problem to be solved quickly.

To provided advice and how to deal with variance three-dimensional heat transfer problem using boundary integral equation, further more this study serves as reference for individual who have interested to conduct further research in three-dimensional heat transfer problem using boundary integral equation.

1.5 Scope of the study

There are different types of dimensional heat transfer problem were solved by using boundary integral equation. However, for the manageability of the paper, this study was delimited in to the solution of mixed value problem using boundary integral equation.

1.6 Preliminaries

Partial differential equation

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

using the notation we can write out symbolically a typical PDE, as follows. Fix an integer $k \geq 1$ and let U denote an open subset of $\mathbb{R}^n$.

Definition: An expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0 \quad (x \in U) \tag{1.3}$$

is the k^{th} order partial differential equation, where

$F : R^{n^k} \times R^{n^{k-1}} \times ... \times R^n \times R \times U \to R$ is given, and $u : U \to R$ is the unknown.

Lipschitz boundary

A Lipschitz domain (or domain with lipschitz boundary) is a domain in euclidean space whose boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a lipschitz continuous function.

Test function

Let Ω be an open subset of $\mathbb{R}^n$, for $\phi : \Omega \to C$ the support of ϕ, is written as supp ϕ is defined as the closure in Ω of the set of the $x \in \Omega$ for which $\phi(x) \neq 0$. The class of test functions $\mathscr{D}(\Omega)$ consists of all functions $\phi(x)$ defined in Ω, vanishing outside a bounded subset of Ω that stays away from the boundary of Ω, and such that all partial derivatives of all orders of ϕ are continuous.

$$\mathscr{D}(\Omega) = \phi \in C^{\infty}(\Omega) \quad supp \quad \phi \quad is \quad compact \quad in \quad \Omega$$

The set $\mathscr{D}(\Omega)$ is also denoted by $C_0^{\infty}(\Omega)$.

It mean that a Test function is an infinitely differentiable function (smooth function) on $\mathbb{R}^2$ vanishing out side of some boundary set.

Distribution

A distribution is a linear mapping $\phi \mapsto \langle u, \phi \rangle$ from $\mathscr{D}(\Omega)$ to $\mathbb{R}$, which is continues in the following sense: If $\phi_n \to \phi$ in $D(\Omega)$, then $\phi \to \langle u, \phi_n \rangle \to \phi \to \langle u, \phi \rangle$.

The set of all distribution is called $\mathscr{D}'(\Omega)$. In other words, $u : \mathscr{D}(\Omega) \to \mathbb{R}$ where the value of u acting in φ denoted by:

$$u : \varphi \mapsto u(\varphi) = u\varphi = \langle u, \varphi \rangle$$

satisfies linearity and continuity criteria . In other word A distribution f on, $\mathbb{R}^2$ we mean a continuous linear functional on $\mathscr{D}(\mathbb{R}^2\ )$. The space of all distributions is denoted by $\mathscr{D}(\mathbb{R}^2)$ or simply by $\mathscr{D}$' . A distribution $f \in \mathscr{D}'(\mathbb{R}^2\ )$ is a linear functional on the space of basic Test function, that is, if $\varphi, \psi \in \mathscr{D}$ and $\lambda, \mu \in \mathbb{C}$ then $\langle f, \lambda\varphi + \mu\psi \rangle = \lambda \langle f, \varphi \rangle + \mu \langle f, \psi \rangle$.

One of the most important examples of distribution is the Dirac Delta. It is denoted by δ and defined by $\langle \delta, \varphi \rangle = \varphi(0)$ for any test function φ.

Weak derivative

Suppose $u, v \in L^1_{loc}(u)$, and α is a multi index we say that v is the α^{th} weak partial derivative of u ,written as

$D^{\alpha}u = v$ provided

$$\int_{u} UD^{\alpha}\phi dx = (-1)^{|\alpha|}\int_{u} v\phi dx \tag{1.4}$$

For all test functions $\phi \in C_c^{\infty}(u)$

Green's identities

Consider two function u(x) and v(x) defined on a domain $\Omega \subseteq \mathbb{R}^n$. Calculus identities for integrations by parts give the following formulae, which is known as Greens first identity

$$\int_{\Omega} v\triangle u dx = -\int_{\Omega} \nabla v.\nabla u dx + \int_{\partial\Omega} v\partial_N u dS_x \tag{1.5}$$

The notation is that the differential of surface area in the integral over the boundary is dS_x. To ensure that the manipulations in this formula are valid we ask that $u,v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$, that is, all derivatives of u and v up to second order are continuous in the interior of Ω and at least their first derivatives have continuous limits on the boundary $\partial\Omega$. Integrating again by parts (or alternatively using $x \in \Omega : \lambda\nabla^2 T^*(\xi,x) + cu_1\frac{\partial T^*(\xi,x)}{\partial x_1} = -\delta(\xi,x)$ in a symmetric way with the roles of u and v reversed) we obtain Greens second identity

$$\int_{\Omega} v\triangle u dx - \int_{\Omega} \triangle v u dx = \int_{\partial\Omega} (v\partial_N u - \partial_N v u) dS_x \tag{1.6}$$

for $u,v \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$. The integral over the boundary $\partial\Omega$ is the analog of the Wronskian in ODEs.

Fundamental solution

To describe the reaction of an infinite homogeneous domain do to a single load fundamental solutions are essential for the solution of Boundary integral equation.Suppose we want to solve the Cauchy problem.

$$\begin{cases} u_t = Lu, x \in \mathbb{R}^n, t > 0 \\ u(x,0) = g(x), x \in \mathbb{R}^n \end{cases} \tag{1.7}$$

where L is a differential operator in $\mathbb{R}^n$ with constant coefficients. Suppose K(x,t) is a distribution in $\mathbb{R}^n$ for each value of $t \geq 0$, K is C^1 in t and satisfies

$$\begin{cases} k_t - LK = 0 \\ K(x,0) = \delta(x) \end{cases} \tag{1.8}$$

We call K a fundamental solution for the initial value problem. The solution of eqn. (1.7) is then given by convolution in the space variables:

$$u(x,t) = \int_{R^n} k(x-y,t)g(y)dy \tag{1.9}$$

For operators of the form $\partial_t - L$, the fundamental solution of the initial value problem, K(x, t) as defined in eqn.(1.8), coincides with the free space fundamental solution, which Satisfies:-

$$(\partial_t - L)K(x,t) = \delta(x,t) \tag{1.10}$$

provided we extend K(x, t) by zero to $t < 0$. For the heat equation, consider

$$K(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x|^2}{4t}}, t > 0 \\ 0, t \leq 0 \end{cases} \tag{1.11}$$

Notice that $\tilde{K}$ is smooth for (x, t) $\neq (0,0)$.

$\tilde{K}$ defined as in eqn. (1.11) is the fundamental solution of the "$freespace$" heat equation. For the heat transfer problem, the boundary integral equation is formulated on the bases of Green formula with fundamental solution (1.12) and

$$\frac{\partial \theta}{\partial n} = 0 \tag{1.12}$$

The fundamental solution for the Laplace operator is given by Let $E \in \mathscr{D}'(\mathbb{R}^3)$ such that

$$\nabla^2 E = -\delta(x-\xi); x, \xi \in \mathbb{R}^3 \tag{1.13}$$

Sobolev space

The Sobolev space $H^1(\Omega)$ is the set off all $U \in L_2(\Omega)$ such that all the first partial derivatives $\frac{\partial u}{\partial x_i}$ belongs to $L_2(\Omega)$.

$$H^1(\Omega) = \{U \in L_2(\Omega) : \nabla u \in L_2(\Omega)\}$$

Fix $1 \leq p \leq \infty$ and let K be a nonnegative integer. We define a function that have weak derivative of various orders lying in various L^p spaces.Thus the Sobolev space $W^{(k,p)}(u)$ consists of all locally summable functions $u : U \to R$ such that for each multi index α with $|\alpha| \leq k, D^\alpha u$ exists in the weak sense and belongs to $L^\alpha(u)$.

Boundary value problem

There are three type of boundary value problem those should define as follow. Assume $\Omega \subset \mathbb{R}^3$ be a connected domain

Dirichlet Boundary Value Problem

The Dirichlet Boundary value problem or first boundary value problem is to find a function $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ which satisfies

$$\begin{cases} \Delta u = 0, \quad in \quad \Omega \\ \\ u = \phi, \quad on \quad \partial\Omega \end{cases}$$

where ϕ is given and continuous on $\partial\Omega$.

The Neumann Boundary Value problem

The Neumann boundary value problem or second boundary value problem is to find a function $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ which satisfies

$$\begin{cases} \Delta u = 0, \quad in \quad \Omega \\ \frac{\partial u}{\partial n} = \phi, \quad on \quad \partial\Omega \end{cases}$$

where ϕ is given and continuous on $\partial\Omega$.

Mixed boundary value problem

The mixed boundary value problem or the third boundary value problem is to find a function $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ which satisfies both **The Dirichlet** and **The Neumann** boundary conditions. It mean

$$\begin{cases} \Delta u = 0, \quad in \quad \Omega \\ \frac{\partial u}{\partial n} + hu = \phi, \quad on \quad \partial\Omega \end{cases}$$

where ϕ and h are given and continuous on $\partial\Omega$ $\quad n(n_x, n_y, n_z)$ is the vector normal to the boundary of the isolator .

Boundary Integral Equation

The efficient numerical solution of partial differential equations (PDE) via boundary integral formulation plays an important role in diverse applications. Consider the Laplace equation:

$$-\Delta u(x) = 0 \tag{1.14}$$

In some domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary Γ, then Greens representation theorem allows us to write the solution u as:

$$u(x) = \int_S G(x,y)\frac{\partial}{\partial n}u(y)dS(y) - \int_S \frac{\partial}{\partial n(y)}G(x,y)u(y)dS(y), for x \in \Omega \tag{1.15}$$

Where, n is the unit outward pointing normal at Γ and G(x,y) is a fundamental solution defined as

$$G(x,y) = -\frac{1}{4\pi r} = \frac{1}{4\pi|x-y|} \tag{1.16}$$

Hence, in principle if either u or $\frac{\partial u}{\partial n}$ is known on S we can recover the unknown quantity by restricting equation (1.16) to the boundary and solving to unknown boundary.

Chapter 2

Review of Related Literature

Boundary integral equation is used to solve n-dimensional heat transfer problem where n = 1,2,or 3 using boundary integral expression and checking the valid for classical solution and unique solution of boundary integral equation is identical to the given space. Even it can give idea if the classical solution does not exist.

YOUNG W. CHUN Discuss the mathematical properties of variational solution and solution of BIE of 2D- heat transfer problem and also check that a unique solution of BIE is identical to the variational solution in Soboleve space. If the classical solution doesn't exist the researcher is utilized Green's formula. This shows that it is possible to get a numerical approximation of the variational solution by BEM. The procedure can also applied to the 3D heat transfer problem using the same procedure. I also want to solve such a problem [1].

For three dimensional heat transfer this researcher (Y. Guetal) described steady state heat conduction subject to Dirichelet and Neumann boundary condition to mathematical formulation and fundamental solution then after the researcher use singular boundary method which doesn't requires of integration to found fundamental solution then matrix form of discretization can be expressed [2]. Agree with the problems arisen in connection with various physical and engineering problems and the researcher state that the difficulty of finding analytical solution instead he use to approximate numerical solution and use reproducing kernel particle method (RKPM) is used to obtain the numerical solution of a three dimensional steady-state conduction problem [5].

Also study a mixed (Dirichlet-Neumann) boundary value problem (BVP) for the ”stationary heat transfer” partial differential equation with variable coefficient by using the method of reducing to some systems of nonstandard segregated direct parametrix-based boundary-domain integral equations (BDIEs). And that BDIE systems contain integral operators defined on the domain under consideration as well as potential-type and pseudo-differential operators defined on open sub-manifolds of the boundary. It is shown that the BDIE systems are equivalent to the original mixed BVP, and the operators of the BDIE systems are invertible in appropriate Sobolev spaces.As that boundary integral expression for weak solution [7].

Also solve the steady state 3D problem with the method of BEM as follow by formulating then transform to the 2^{nd} Green’s formula and integrating by part and obtain that of fundamental solution as follow [3]. By considering solving of three - dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial u}{\partial t} \tag{2.1}$$

for the unknown function $u(x,y,z,t)$ (the temperature) for $(x,y,z) \in V$ and $t \geq 0$, subject to the initial and boundary conditions

$$u(x,y,z,0) = f(x,y,z), \quad for \quad (x,y,z) \in V, \tag{2.2}$$

$$u(x,y,z,t) = g(x,y,z,t) \quad for \quad (x,y,z) \in S_1 \quad and \quad t \geq 0, \tag{2.3}$$

$$u(x,y,z,t) = h(x,y,z)q(t) \quad for \quad (x,y,z) \in S_2 \quad and \quad t \geq 0, \tag{2.4}$$

$$\frac{\partial}{\partial n}[u(x,y,z,t)] = k(x,y,z,t) \quad for \quad (x,y,z) \in S_3 \quad and \quad t \geq 0, \tag{2.5}$$

and the non-local (integral) condition

$$\int\int_V\int u(x,y,z,t)dxdydz = m(t) \quad for \quad t \geq 0, \tag{2.6}$$

Where V is a three- dimensional region bounded by a simple closed surface S, S_1, S_2, S_3, are non intersecting surface such that $S_1 \bigcup S_2 \bigcup S_3 = Sf, g, h, k, and, m$ are suitably prescribed functions and q is an unknown function to be determined. Notice that $\frac{\partial u}{\partial n} = \mathbf{n}.\nabla u$ where $\mathbf{n}$ is the unit normal vector on S pointing away from V. Notice that eqn. (2.6) specifies the total energy present in the region V at any time $t \geq 0$.

This researcher is used a boundary integral method (BIM) for the numerical solution of the three- dimensional problem defined by (2.1) - (2.6) in the Laplace transform (LT) space. For solving three dimensional heat transfer problems the proposed BIM should proved to the finite-difference methods.

Again the researcher is formulated in LT space by defining the LT operator on the function $r(x,y,z,t)$ $(t \geq 0)$ by

$$\ell r(x,y,z,t); t \to p = \int_0^\infty r(x,y,z,t) \quad exp(-pt)dt \tag{2.7}$$

Where p is the LT parameter. For the purpose here, he shall take p to be real and positive.And obtain

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}-pT=0 \tag{2.8}$$

$$T(x,y,z;p)=G(x,y,z;p)-U_{part}(x,y,z;p) \quad for \quad (x,y,z)\in S_1 \tag{2.9}$$

$$T(x,y,z;p)=h(x,y,z)\Phi(p)-U_{part}(x,y,z;p) \quad for \quad (x,y,z)\in S_2 \tag{2.10}$$

$$\frac{\partial}{\partial n}[T(x,y,z;p)]=K(x,y,z;p)-\frac{\partial}{\partial n}[U_{part}(x,y,z;p)] \quad for \quad (x,y,z)\in S_3 \tag{2.11}$$

$$\int\int_S \frac{\partial}{\partial n}[T(x,y,z,p)]ds=pM(p)-p\int\int_V\int U_{part}(x,y,z;p) \quad dxdydz \tag{2.12}$$

After formulation in LT space he can use BIM to solve the equation as follow If $\Phi(p)$ is known, the BIM for solving the above equations is standard . AS $\Phi(p)$ is unknown, the usual boundary integral procedure has slightly modified to take in to consideration the extra equation (2.13) is given by

$$\lambda(\xi,\eta,\psi)T(\xi,\eta,\psi;p)=\int\int_S\{T(x,y,z;p)\frac{\partial}{\partial n}[\Gamma(x,y,z;\xi,\eta,\psi;p)]-\Gamma(x,y,z;\xi,\eta,\psi;p)\frac{\partial}{\partial n}[T(x,y,z,p)]\}ds \tag{2.13}$$

where $\lambda(\xi,\eta,\psi)$ = 1 if $(\xi,\eta,\psi)\in V$ and $0<\lambda(\xi,\eta,\psi)<1$ if $(\xi,\eta,\psi)\in S$ and

$$\Gamma(x,y,z;\xi,\eta,\psi;p)=-\frac{1}{4\pi r}exp(-r\sqrt{p}), \tag{2.14}$$

Where $r=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\psi)^2}$ at the last the surface boundary S is discretized into N surface elements $S^{(1)},S^{(2)},....,S^{(N-1)}andS^{(N)}$ and find the solution.

The three dimensional heat transfer problem (steady state) is considered and solved by means of the boundary element method the results obtained by the method is the numerical once and compared with analytical solution. To find that result the first thing done by the researcher is formulation of the problem as such [4]

The following Fourier-Kirchhoff equation (steady state, 3D problem) is considered

$$x \in \Omega : \lambda \nabla^2 T(x) - cu.\nabla T(x) + Q(x) = 0 \tag{2.15}$$

where λ is the thermal conductivity and c is the volumetric specific heat, respectively $u = [u_1, u_2, u_3]$ is the velocity, Q(x) is the source function, T denotes the temperature and $x = [x_1, x_2, x_3]$ are the spatial co-ordinates.

For $u = [u_1, 0, 0]$ the equation (2.16) takes a form

$$x \in \Omega : \lambda \nabla^2 T(x) - cu_1 \frac{\partial T(x)}{\partial x_1} + Q(x) = 0 \tag{2.16}$$

The boundary conditions in the form

$$\begin{aligned} x \in \Gamma_1 &: T(x) = T_b \\ x \in \Gamma_2 &: q(x) = -\lambda \frac{\partial T(x)}{\partial n} = q_b \end{aligned} \tag{2.17}$$

are also known, at the same time $\frac{\partial T}{\partial n}$ denotes the normal derivative, $n = [\cos\alpha_1, \cos\alpha_2, \cos\alpha_3]$ is the normal outward vector T_b, q_b are the known boundary temperature and boundary heat flux, respectively.

After formulation of the problem for equation (2.15) the researcher is find the solution of the formulated problem (2.16) and the above equation by the means of boundary element method. For that he obtained the boundary integral equation.

$$B(\xi)T(\xi) + \int_\Gamma T^*(\xi,x)q(x)d\Gamma = \int_\Gamma [q^*(\xi,x) - cu_1 T^*(\xi,x)\cos\alpha_1]T(x)d\Gamma + \int_\Omega Q(x)T^*(\xi,x)d\Omega \tag{2.18}$$

For $\xi \in \Gamma$ and where $T^*(\xi,x)$ is the fundamental solution and $q^*(\xi,x)$ is obtain from that of fundamental solution.

At the end to solve equation (2.18) he divided the boundary in to N boundary elements and the interior is divided in to L internals cells and find the result.

Chapter 3

Mixed Boundary Value Problem (BVP)

The most commonly occurring form of problem that is associated with Laplaces equation is a boundary value problem, normally posed on a domain $\Omega \subseteq \mathbb{R}^n$. That is, Ω is an open set of $\mathbb{R}^n$ whose boundary is smooth enough so that integrations by parts may be performed, thus at the very least rectifiable.

3.1 Green's Identities and representation formula

Green's first identity

Before Green's first identity let as look at the following theorem [8]

Theorem

(Divergence Theorem) Let D be a bounded solid region with a piecewise C^1 boundary surface ∂D. Let n be the unit outward normal vector on ∂D. Let f be any C^1 vector field on $\bar{D} = D \bigcup \partial D$. Then

$$\int\int\int_D \vec{\nabla}.f dV = \int\int_{\partial D} f.nds$$

where dV is the volume element in D and dS is the surface element on ∂D

By integrating the identity

$$\vec{\nabla}.(v\vec{\nabla}u) = \vec{\nabla}v.\vec{\nabla}.u + v\triangle u$$

over D and applying the divergence theorem. one gets

$$\int\int_{\partial D} v\frac{\partial u}{\partial n}dS = \int\int\int_D \vec{\nabla}v.\vec{\nabla}udV + \int\int\int_D v\triangle udV$$

where $\frac{\partial u}{\partial n}$ = n . $\vec{\nabla}$ u is the directional derivative in the outward normal direction. This is Green's first identity.

Constraint on Neumann problems

Let $v = 1$ Green's first identity becomes

$$\int\int_{\partial D} \frac{\partial u}{\partial n}dS = \int\int\int_D \triangle udV \tag{3.1}$$

Consider the Neumann problem in any domain D

$$\begin{aligned} \triangle u &= f(x) \quad in \quad D \\ \frac{\partial u}{\partial n} &= h(x) \quad on \quad \partial D \end{aligned} \tag{3.2}$$

Applying equation (1.16) on equation (1.15) one gets

$$\int\int_{\partial D} hdS = \int\int\int_D fdV$$

There for h and f cannot be freely selected, else the problem may have no solution.

Mean value property

In three dimensions, the average value of any harmonic function over any sphere equals it's value at the center. Applying equation (1.15) to a sphere with radius a in spherical coordinates, one get's

$$\int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)a^2\sin\theta \quad d\theta d\phi = 0 = a^2\int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)\sin\theta \quad d\theta d\phi$$

This gives

$$\frac{d}{dr}[\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)r^2\sin\theta \quad d\theta d\phi] = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u_r(r,\theta,\phi)\sin\theta \quad d\theta d\phi = 0$$

Thus the mean value inside the rectangular bracket is independent of r and

$$\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)r^2\sin\theta \quad d\theta d\phi = u(0)$$

Maximum principle

If D is a connected solid region , a non-constant harmonic function in D cannot take its maximum value inside D, but only on ∂D.

Green's second identity

Switch u and v in Green's first identity, then subtract it from the original form of the identity. The result is

$$\int\int\int_D (u\triangle v - v\triangle u)dV = \int\int_{\partial D}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS$$

This is Green's second identity. It is valid for any pair of function u and v. Specially for boundary conditions can be imposed on the functions to make the right hand side of these identity zero. So that

$$\int\int\int_D u\triangle v = \int\int\int_D v\triangle u$$

Definition :

A boundary condition is called symmetric for the operator $\triangle$ on D if $\int\int_{\partial D}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS = 0$

for all pairs of functions u,v that satisfy the boundary condition. Dirichlet, Neumann, and Robin BCs are symmetric.[8]

Representation formula

Let $K(X,X_0) = -\frac{1}{(4\pi|X-X_0|)}$. Any harmonic function u in an open solid region D can be express as an integral over the boundary ∂D as

$$u(X_0) = \int\int_{\partial D} [u(X)\frac{\partial}{\partial n}K(X,X_0) - K(X,X_0)\frac{\partial u}{\partial n}]dS$$

where $X_0 \in D$ and X is on ∂D

<u>Proof</u>

First note that $\triangle K(X,X_0) = 0$ except at $X = X_0$. Therefore both u and K are harmonic in the domain $D^{'} = D - B_{\varepsilon}(X_0)$ where $B_{\varepsilon}(X_0)$ a ball in D centered at X_0 and with radius $\varepsilon > 0$. Applying Green's second identity to u and K in D'. We get

$$\int\int_{\partial(D-B_{\varepsilon})} [u(X)\frac{\partial}{\partial n}K(X,X_0) - K(X,X_0)\frac{\partial u}{\partial n}]dS = 0$$

With X_0 put at the origin of a spherical coordinate system the unit normal for the directional derivative at the surface of B_{ε} is pointing towards the origin. Flipping the unit normal out ward introduces a minus sign in front of the flux integral over ∂B_{ε}. The equation can be rewritten as:-

$$\int\int_{\partial D} [u(X)\frac{\partial}{\partial n}K(X,X_0) - K(X,X_0)\frac{\partial u}{\partial n}]dS = \frac{1}{4\pi}\int\int_{\partial B_{\varepsilon}} [u(x)\frac{\partial}{\partial r}(\frac{-1}{r}) - (\frac{-1}{r})\frac{\partial u}{\partial r}]dS$$
$$= \frac{1}{4\pi}\int\int_{r=\varepsilon} [u + \varepsilon\frac{\partial u}{\partial r}]\sin\theta \quad d\theta d\phi \to u(0) as \varepsilon \to 0^{+}$$

The formula is thus obtained. This formula is not useful for finding a solution. For a Dirichlet problem, u is uniquely determined by its value on ∂D. There is no freedom in choosing $\frac{\partial u}{\partial n}$. However, this formula is a step towards Green's function, the use of which eliminates the $\frac{\partial u}{\partial n}$

term.

Generally those Green's identities should be summarized as follow :

Consider two function u(x) and v(x) defined on a domain $\Omega \subseteq R^n$. Calculus identities for integrations by parts give the following formulae, which is known as Greens first identity

$$\int_\Omega v \triangle u dx = -\int_\Omega \nabla v . \nabla u dx + \int_{\partial\Omega} v \partial_N u dS_x \tag{3.3}$$

The notation is that the differential of surface area in the integral over the boundary is dS_x. To ensure that the manipulations in this formula are valid we ask that $u, v \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$, that is, all derivatives of u and v up to second order are continuous in the interior of Ω and at least their first derivatives have continuous limits on the boundary $\partial\Omega$. Integrating again by parts (or alternatively using $x \in \Omega : \lambda \nabla^2 T^*(\xi, x) + cu_1 \frac{\partial T^*(\xi,x)}{\partial x_1} = -\delta(\xi, x)$ in a symmetric way with the roles of u and v reversed) we obtain Greens second identity

$$\int_\Omega v \triangle u dx - \int_\Omega \triangle v u dx = \int_{\partial\Omega} (v \partial_N u - \partial_N v u) dS_x \tag{3.4}$$

for $u, v \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$. The integral over the boundary $\partial\Omega$ is the analog of the Wronskian in ODEs.[8]

Fundamental solution

To describe the reaction of an infinite homogeneous domain do to a single load fundamental solutions are essential for the solution of Boundary integral equation.We found that starting with the Laplacian in $\mathbb{R}^3$ for a radially symmetric function K,

$$K'' + \frac{2}{r} K' = 0$$

and letting $K = \frac{1}{r}w(r)$, we obtained the equation $w = c_1 r + c_2$, which implied

$$K = c_1 + \frac{c_2}{r}$$

we now find the constant c_2 that ensures that for $v \in C_0^{\infty}(\mathbb{R}^3)$, we have

$$\int_{R^3} K(|x|) \triangle v(x) dx = v(0)$$

suppose $v(x) \equiv 0$ for $|x| \geq R$ and let $\Omega = B_R(0)$; for small $\varepsilon > 0$ let

$$\Omega_\varepsilon = \Omega - B_\varepsilon(0)$$

$K(|x|)$ is harmonic $\triangle K(|x|) = 0$ in Ω_ε. Consider Green's identity ($\partial\Omega_\varepsilon = \partial\Omega \bigcup \partial B_\varepsilon(0)$):

$$\int_{\Omega_\varepsilon} K(|x|) \triangle v dx = \underbrace{\int_{\partial\Omega} (K(|x|) \frac{\partial v}{\partial n} - v \frac{\partial K(|x|)}{\partial n}) ds}_{=0, since v \equiv 0 for x \geq R} + \int_{\partial B_\varepsilon(0)} (K(|x|) \frac{\partial v}{\partial n} - v \frac{\partial K(|x|)}{\partial n}) ds.$$

$$\lim_{\varepsilon \to 0} [\int_{\Omega_\varepsilon} K(|x|) \triangle v dx] = \int_{\Omega} K(|x|) \triangle v dx. (\quad since \quad K(r) = c_1 + \frac{c_2}{r} \quad is \quad integrable \quad at \quad x = 0)$$

on $\partial B_\varepsilon(0), K(|x|) = K(\varepsilon)$. Thus,

$$|\int_{\partial B_\varepsilon(0)} (K(|x|) \frac{\partial v}{\partial n} dS| = |K(\varepsilon)| \int_{\partial B_\varepsilon(0)} \frac{|}{\partial v} \partial n | dS \leq |c_1 + \frac{c_2}{\varepsilon}| 4\pi\varepsilon^2 max |\nabla v| \to 0, \quad as \quad \varepsilon \to 0.$$

$$\begin{aligned} \int_{\partial B_\varepsilon(0)} v(x) \frac{\partial K(|x|)}{\partial n} dS &= \int_{\partial B_\varepsilon(0)} \frac{c_2}{\varepsilon^2} v(x) dS \\ &= \int_{\partial B_\varepsilon(0)} \frac{c_2}{\varepsilon^2} v(0) dS + \int_{\partial B_\varepsilon(0)} \frac{c_2}{\varepsilon^2} [v(x) - v(0)] dS \\ &= \frac{c_2}{\varepsilon^2} v(0) 4\pi\varepsilon^2 + \underbrace{4\pi c_2 \quad max_{x \in \partial B_\varepsilon(0)} |v(x) - v(0)|}_{\to 0, (v \quad is \quad continuous)} \\ &= 4\pi c_2 v(0) \to -v(0). \end{aligned}$$

Thus, taking $4\pi c_2 = -1, i.e. c_2 = -\frac{1}{4\pi}$, we obtain

$$\int_\Omega K(|x|) \triangle v dx = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} K(|x|) \triangle v dx = v(0),$$

that is $K(r) = -\frac{1}{4\pi r}$ is the fundamental solution of $\triangle$

3.2 Boundary integral expression for the classical solution

Let u and v be $C^1(\overline{\Omega})\bigcap C^2(\Omega)$ function, then Greens formula (1.2) holds. As follow if $u \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$ and $V \in C^1(\bar{\Omega})$ then Green's first formula holds as:-

$$\int_{\Omega} vŁudx = \int_{\Omega} p\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \int_{s} pv\frac{\partial u}{\partial n}ds + \int_{\Omega} quvdx \tag{3.5}$$

This obtain by taking any arbitrary region Ω' with a piecewise smooth boundary s' strictly lying in the region Ω . Since $u \in C^2(\bar{\Omega}')$ and so

$$\int_{\Omega'} vŁudx = \int_{\Omega'} v[-div(p \quad grad \quad u) + qu]dx$$
$$-\int_{\Omega'} div(pv \quad grad \quad u)dx + \int_{\Omega'} p\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx + \int_{\Omega'} quvdx$$

then using the Gauss-artrogradski formula we obtain

$$\int_{\Omega'} vŁudx = \int_{\Omega'} p\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}dx - \int_{s'} pv\frac{\partial u}{\partial n}ds' + \int_{\Omega'} quvdx$$

Allowing Ω' to tend to Ω in this equation and using the fact that u and v $\in C'(\bar{\Omega})$ we conclude that the limit of the right hand side exists, so that there is a limit of the left-hand side and eqn. (3.5) is true. For this the integral on the left on (3.5) must be understood to be non singular.

If u and v $\in C^2(\Omega)\bigcap C^1(\bar{\Omega})$, then Green's second formula holds by interchange the place of u and v on eqn. (3.5) we obtain

$$\int_{\Omega} uŁvdx = \int_{\Omega} p\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx - \int_{s} pu\frac{\partial v}{\partial n}ds + \int_{\Omega} qvudx \tag{3.6}$$

and subtracting (3.6) from (3.5) as a result we obtain that Green's second formula

$$\int_{\Omega}(vŁu - uŁv)dx = \int_{s} p(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})ds \tag{3.7}$$

Specifically for p = 1 and q = 0 the eqn.(3.7) is equivalent to that of equation (1.2) . If the classical solution $\theta \in C^1(\overline{\Omega})\bigcap C^2(\Omega)$ exist, we can substitute u by θ on Greens formula. However, the singularity of E in equation is preventing one from substituting v by E in the equation. One way of overcoming the difficulty is to replace Ω by $\Omega - B_\rho(\xi)$ where $B_\rho(\xi)$ is a sphere with the small radius ρ centered at a singular point ξ. One can conclude from equation that

$$\int_{\Omega - B_\rho}(E\Delta\theta - \theta\Delta E)dx = \int_v (E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n})ds + \int_{\partial B_\rho}(E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n})ds \quad for \quad \theta \in C^1(\overline{\Omega})\bigcap C^2(\Omega) \tag{3.8}$$

Where $\Omega - B_\rho$ is the boundary of B_ρ in eqn. (3.8) since $\Delta\theta = 0$ and $\Delta E = -\delta(x-\xi) = 0$ on $\Omega - B_\rho$ we have

$$\int_{\Omega - B_\rho}(E\Delta\theta)dx = 0 \tag{3.9}$$

$$\int_{\Omega - B_\rho}(\theta\Delta E)dx = 0 \tag{3.10}$$

The first term in the integral over ∂B_ρ in equation (3.8) becomes:-

$$\int_{\partial B_\rho} E\frac{\partial\theta}{\partial n}ds = E(\rho)\int_{\partial B_\rho}\frac{\partial\theta}{\partial n}ds \leq E(\rho)\omega_2\rho \quad supp_{\partial B} \quad |\frac{\partial\theta}{\partial n}| \to 0, as\rho \to 0 \tag{3.11}$$

Since

$$|C\rho^{-1}\int_{\partial B_\rho}(\theta(x) - \theta(\xi))ds| \leq C\rho^{-1}\int_{\partial B_\rho}|(\theta(x)-\theta(\xi))|ds \leq C\omega_2 \quad max_{x\in\partial B_\rho}|\theta(x)-\theta(\xi)| \to 0, as\rho \to 0 \tag{3.12}$$

The second term in integration over ∂B_ρ in equation (3.8) becomes

$$\int_{\partial B_\rho}\theta\frac{\partial E}{\partial n}ds = C\rho^{-1}\int_{\partial B_\rho}\theta(x)ds \to C\rho^{-1}\int_{\partial B_\rho}\theta(\xi)ds = C\omega_2\theta(\xi), \quad as \quad \rho \to 0 = \theta(\xi) \tag{3.13}$$

where $\omega_2 = 4\pi$ is the boundary area of the unit sphere in R^3 and $\omega_2\rho$ is the boundary of the sphere with radius ρ. if one choose $C = \frac{1}{\omega_2}$ and substituting equations (3.9), (3.10), (3.11),and (3.13) in the equation (3.8) then

$$\theta(\xi) = \int_v (E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n})ds \quad for \quad all \quad \xi \in \Omega, \theta \in C^1(\overline{\Omega})\bigcap C^2(\Omega) \tag{3.14}$$

holds as ρ goes to zero. If ξ is on v, equation (3.14) has a singularity. Then we can divided the boundary v by v_ε and $v-v_\varepsilon$ where v_ε hemisphere with small radius ε centered at a singular ξ. Then equation (3.14) becomes

$$\theta(\xi)=\int_{v_\varepsilon}(E\frac{\partial\theta}{\partial n}-\theta\frac{\partial E}{\partial n})ds-\int_{v-v_\varepsilon}(E\frac{\partial\theta}{\partial n}-\theta\frac{\partial E}{\partial n})ds \tag{3.15}$$

The first term of the boundary integration over v_ε in equation (3.15) becomes

$$\int_{v_\varepsilon}E\frac{\partial\theta}{\partial n}ds=E(\varepsilon)\int_{\partial B_\rho}\frac{\partial\theta}{\partial n}ds\leq E(\varepsilon)\omega_2\varepsilon \quad supp_{B_\varepsilon} \quad |\frac{\partial\theta}{\partial n}|\to 0, as\varepsilon\to 0 \tag{3.16}$$

While the second term becomes

$$-\int_{v_\varepsilon}\theta\frac{\partial E}{\partial n}ds=C\varepsilon^{-1}\int_{v_\varepsilon}\theta(x)ds \tag{3.17}$$

Since

$$|C\varepsilon^{-1}\int_{v_\varepsilon}(\theta(x)-\theta(\xi))ds|\leq C\varepsilon^{-1}\int_{v_\varepsilon}|(\theta(x)-\theta(\xi))|ds\leq\frac{1}{2}C\omega_2 \quad max_{x\in\Gamma_\varepsilon} \quad |\theta(x)-\theta(\xi)|\to 0, \quad as \quad \varepsilon\to 0 \tag{3.18}$$

Assume $C=\frac{1}{\omega_2}$ then equation (3.17) becomes

$$C\varepsilon^{-1}\int_{v_\varepsilon}\theta(x)ds\to C\varepsilon^{-1}\int_{v_\varepsilon}\theta(\xi)ds=\frac{1}{2}C\omega_2\theta(\xi), as\varepsilon\to 0=\frac{1}{2}\theta(\xi) \tag{3.19}$$

by substituting equation (3.16) and equation (3.19) in to equation (3.15) and let ε go to zero then we obtain

$$\frac{1}{2}\theta(\xi)=\int_v(E\frac{\partial\theta}{\partial n}-\theta\frac{\partial E}{\partial n})ds \quad for \quad all \quad \xi\in v,\theta\in C^1(\overline{\Omega})\bigcap C^2(\Omega) \tag{3.20}$$

When we use a boundary element method for the problem with $\xi\in v,\theta(\xi)$ is obtained numerically from equation (3.20), while it is obtained from equation (3.14) when $\xi\in\Omega$. From the above result, one can see that the classical solution, if it exist, can be approximate numerically from boundary integral equation (3.14) and (3.20) by dividing the boundary into the small segment. However, since the classical solution does not exist in the mixed boundary value problem when x and ξ are at the same point then it has a singular andξ is singularity of fundamental solution, then we cant use equation (3.14) and (3.20) directly.It is also possible to construct the same things for boundary integral expression for weak solution.

3.3 Boundary Integral Expression for the Weak Solution

The variational form of the state equation of Eqs (1.1) is given as

$$\int_{\Omega} \nabla\theta\nabla v dx + \int_{S_1 \bigcup S_2} \frac{h}{k}(\theta - \theta_\infty) v ds = 0 \quad for \quad all \quad v \in K \tag{3.21}$$

where K is the admissible set given by $K = v : v \in H^1(\Omega), v = 0 \quad on \quad S_3$. It is known that the weak solution of Eq. (3.21) is unique in $H^1(\Omega)$ by the Lax-Milgram theorems.To represent the boundary integral equation for the variational weak solution $\theta \in H^1(\Omega)$, one needs Greens formula in Sobolev space theorem and that Greens formula in Sobolev space holds for the domain Ω with Lipschitz boundary S if u, v $\in H^1(\Omega, \nabla^2)$, where $H^1(\Omega, \nabla^2) = u \backslash u \in H^1(\Omega)$ such that $\nabla^2 u \in L^2(\Omega)$.

The variational solution θ is in $H^1(\Omega, \nabla^2)$, but the fundamental solution is not.In fact, it is in $C^\infty(R^3 - \xi)$. Then by making similar procedure as that of Boundary Integral Expression for the classical solution done for Boundary Integral Expression for the Weak Solution.And obtain

$$\theta(\xi) = \int_S (E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n}) ds \quad for \quad \xi \in \Omega, \theta \in H^1(\Omega, \nabla^2) \tag{3.22}$$

as ρ goes to zero. Likewise, if ξ is on S, Eq (3.22) becomes

$$\frac{1}{2}\theta(\xi) = \int_S (E\frac{\partial\theta}{\partial n} - \theta\frac{\partial E}{\partial n}) ds \quad for \quad all \quad \xi \in S, \theta \in H^{\frac{1}{2}}(S) \tag{3.23}$$

If one inserts the boundary conditions of Eqs (1) into Eqs (3.22) and (3.23),respectively, one can get

$$\theta(\xi) = -\int_{S_1 \bigcup S_2} E\frac{h}{k}(\theta - \theta_\infty)ds + \int_{S_3} E\frac{\partial \theta}{\partial n}ds - \int_{S_1 \bigcup S_2 \bigcup S_4} \theta \frac{\partial E}{\partial n}ds \tag{3.24}$$

$$\frac{1}{2}\theta(\xi) = -\int_{S_1 \bigcup S_2} E\frac{h}{k}(\theta - \theta_\infty)ds + \int_{S_3} E\frac{\partial \theta}{\partial n}ds - \int_{S_1 \bigcup S_2 \bigcup S_4} \theta \frac{\partial E}{\partial n}ds \tag{3.25}$$

From the above results it is evident that if one finds the solution of Eqs (3.24) and (3.25) in $H^1(\Omega)$, it is identical to the variational solution since it is unique in $H^1(\Omega)$. For the solution of the problem, Eqs (3.24) and (3.25) can be approximated numerically by dividing the boundary into small regions.

Chapter 4

Discussion and Conclusions

In this thesis it is shown that for the problem governed by the Laplace operator, a unique solution of the boundary integral equation, the fundamental solution on Green's identities exists and is identical to the variational solution in $H^1(\Omega)$, which can be served as the basic mathematical theory of the boundary element method for the problem. Therefore, it is possible to get a numerical approximation of the variational solution by the boundary element method even when the classical solution does not exist. It is expected that a similar procedure can be applied to other elliptic boundary value problems.

4.1 Recommendation

To solve mixed boundary value problem one can use other different method and get the same solutions. In this case I used the boundary integral to approximate numerical solution of boundary integral equations with in Sobolov space. Additionally it is also possible to solve in other space than that of sobolov. And also such type of solving BVP is done for other type of BVP not only for heat equation.

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