BIFURCATION ANALYSIS OF A DIFFUSIVE PREDATOR-PREY SYSTEM

By
Woliyu Salo Temam

A Thesis Submitted to Applied Mathematics Program

School of Applied Natural science

Presented in partial fulfillment of the requirement for

Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

Adama , Ethiopia
June 2017

BIFURCATION ANALYSIS OF A DIFFUSIVE PREDATOR-PREY SYSTEM

By
Woliyu Salo Temam

Advisor:
Legesse Lemecha (Ph.D)

A Thesis Submitted to Applied Mathematics Program

School of Applied Natural science

Presented in partial fulfillment of the requirement for

Degree of Master's of Science in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

Approval Page

Name of Student	Signiture	Date
Advisor	Signiture	Date
Examiner	Signiture	Date
External Examiner	Signiture	Date
Chair person	Signiture	Date
HoD	Signiture	Date
School Dean	Signiture	Date
SGS Dean	Signiture	Date

Abstract

For ecological systems, one of the main characters is the relationship between different species and their living environment. Thus modeling predator-prey interactions is one of the important issues in mathematical ecology. For the classical predator-prey models, the crucial components are the growth rate of prey species in the absence of predator, the mortality rate of predator species in the absence of prey and the functional response the predator to the prey. This thesis involves a diffusive predator-prey system, in which the predator species Holling type II functional response and linear mortality rate. The stability of positive constant equilibrium, Hopf bifurcations, and diffusion-driven Turing instability are investigated. The explicit condition for the occurrence of the diffusion-driven Turing instability is derived. Finally, numerical simulations are carried out to verify and extend the theoretical results. As we observe the numerical result match the analytical result as expected. When we change the values of parameters the stability of equilibrium changed stable one become unstable and vice versa.

Keywords:
Diffusive predatorprey system, Holling type-II functional response, Hopf bifurcation.

Acknowledgements

First of all, I would like to thank God for his permission to do our daily activity including this work.

Second, I would like to thank my advisor Legesse Lemecha (Ph.D) for his most support and advice. He kindly read my thesis and offered invaluable detailed advices on organization and the theme of the thesis. This thesis is made possible through the help and his support. and also I would like to thank to Adama science and technology university administrations to their support by any means financially and non financially
lastly, I would like to thank all my families and my friends to their useful advice and comment.

Woliyu Salo
Adama Science and Technology University
June 2017

Contents

Chapter 1

Introduction

Ecosystems are characterized by the interaction between different species and natural environment. Predator-prey models have been in the focus of ecological science since the great work of Lotka and Volterra. Modelling predator-prey interaction has been one of the main thing in mathematical ecology. Given two species of animals, interdependence might arise because one species (the "prey") serves as a food source for the other species (the "predator"). Models of this type are called predator-prey models.

Logistic growth and death is a common choice made to model prey birth and death in the absence of predators. A linear death rate for predators is common. If predator density is not so large that they interfere with each other while searching for prey, then one often assumes that the death rate due to predators is linear in predator density. Also, it is common to assume that predator reproduction rate is proportional to the predator kill rate.

Predation is one of the most important types of interaction which has effects on population dynamics of all species. As a result, predator-prey models have long been and will continue to be one of the dominant themes in both ecosystems and mathematical ecology. There are many factors which affect population dynamics in predator-prey models. One crucial component of predator-prey relationships is the functional response. The other crucial component in the predator-prey models is the formulation of the mortality terms for the predator. Usually, the predator mortality is described by either the linear mortal-

ity version, or the quadratic mortality version.

1.1 Mathematical preliminaries

1.1.1 Equilibrium points

An equilibrium point of a dynamical system represents a stationary condition for the dynamics. We say that x_0 is an equilibrium point for a dynamical system $x'(t) = f(x)$ if $f(x_0) = 0$. Equilibrium points are one of the most important features of a dynamical system since they define the states corresponding to constant operating conditions. A dynamical system can have zero, one or more equilibrium points.

In dynamical systems, only the solutions of linear systems may be found explicitly. The problem is that in general real life problems may only be modeled by nonlinear systems. The main idea is to approximate a nonlinear system by a linear one (around the equilibrium point).Of course, we do hope that the behavior of the solutions of the linear system will be the same as the nonlinear one. But this is not always true.

1.1.2 Linear stability analysis

Theorem 1.1.1. *(Stability Theorem)*
Suppose that $f(x)$ is twice continuously differentiable, that the autonomous system $\frac{dx}{dt} = f(x)$ has an equilibrium point at $x = p$ and that $J(p)$ is the Jacobean of f evaluated at p. Then:
(i) The system $\frac{dx}{dt} = f(x)$ is asymptotically stable at $x = p$ if all eigen values of $J(p)$ have negative real part.
(ii) The system $\frac{dx}{dt} = f(x)$ is unstable at $x = p$ if $f(p)$ has at least one eigenvalue with a positive real part.

The proof and detail about this theorem is available on (Walter, 2010).

1.1.3 Bifurcation analysis

In dynamical systems , a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative" or topological change in its behavior (Celic, 2012). Generally, at a bifurcation, the local stability properties of equilibria, periodic orbits or other invariant sets changes. It has two types;

Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds;

Global bifurcations,which often occur when larger invariant sets of the system collide with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria.

When parameter λ crosses some point $\lambda = \lambda_0$, the phase portrait of the system for $\lambda > \lambda_0$ is topologically different from the phase portrait of the system for $\lambda < \lambda_0$. The point $\lambda < \lambda_0$ is called a bifurcation point at which the system undergoes a bifurcation (Celic, 2012).

The local bifurcations of fixed points (equilibrium points) are classified into static bifurcations or dynamic bifurcations. Saddle-node bifurcation, pitchfork bifurcation and transcritical bifurcation are examples of static bifurcations as only branches of fixed points or static solutions meet. Hopf bifurcation is an example of dynamic bifurcation.

Saddle-node bifurcation:is a collision and disappearance of two equilibria in dynamical systems. In autonomous systems, this occurs when the critical equilibrium has one zero eigenvalue. This phenomenon is also called fold or limit point bifurcation. A saddle-node bifurcation of a fixed point of the system $x' = f(x, \lambda)$ where $x \epsilon R^n$ and $\lambda \epsilon R$ occurs at (x_0, λ_0) if

(i) there is an equilibrium at $x = x_0$ for $\lambda = \lambda_0$ that is $f(x_0, \lambda_0) = 0$ or

(ii) the Jacobian matrix $D_x f(x_0, \lambda_0)$ has a zero eigenvalue (Celic,2012).

Transcritical bifurcation: two families of fixed points collide and exchange their stability properties. The family that was stable before the bifurcation is unstable after it. The other fixed point goes from being unstable to being stable.

The pitchfork bifurcation:In pitchfork bifurcation one family of fixed points transfers its stability properties to two families after or before the bifurcation point. If this occurs after the bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial fixed points occur for values of the parameter lower than the bifurcation value.

Hopf Bifurcation consider a two dimensional system of the form (Celic, 2012).

$$\frac{dx}{dt} = f(x, y, \lambda)$$

$$\frac{dy}{dt} = g(x, y, \lambda)$$

where λ is a parameter. The system has a fixed point (x^*, y^*), which may depend on λ Let the eigenvalues of the linearized system about this fixed point be given by

$\mu(\lambda) = \alpha(\lambda) + i\beta(\lambda)$ and $\mu(\lambda) = \alpha(\lambda) - i\beta(\lambda)$

Hopf bifurcation of the fixed point of the two dimensional system occurs at some critical value of the parameter, $\lambda = \lambda_0$, if the following conditions are satisfied:

i. $f(x^*, y^*, \lambda_0) = 0$ and $g(x^*, y^*, \lambda_0) = 0$

ii. The Jacobian matrix $\begin{pmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{pmatrix}$ has a pair of purely imaginary eigenvalues $\pm \ \omega i$ at (x^*, y^*, λ_0)

iii $\frac{d\alpha(\lambda)}{d\lambda} \neq 0$ at $\lambda = \lambda_0$

1.2 Lotka-Volterra model

The Italian mathematician Vito Volterra in 1926 first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. Similarly American chemist and biologist Alfred James Lotka in 1920 develop the same model (Murray, 2001).

$$\frac{dN}{dt} = N(a - bp) \tag{1.1}$$

$$\frac{dP}{dt} = P(cN - d) \tag{1.2}$$

Where $N(t)$ is the prey population and $P(t)$ that of the predator at a time t and the positive constants a, b, c, d are interpreted as:

a represents the natural growth rate of the prey in the absence of predators,

b represents the effect of predation on the prey,

c represents the efficiency and propagation rate of the predator in the presence of prey.

d represents the natural death rate of the predator in the absence of prey (Murray, 2001).

The assumptions in the model are

1. The prey in the absence of any predation grows unboundedly in a Malthusian way;this is the term aN in (1.1)

2. The effect of the predation is to reduce the preys per capita growth rate by a term proportional to the prey and predator populations; this is the term $-bNP$.

3. In the absence of any prey for sustenance the predators death rate results in exponential decay, that is, the term $-dP$.

4. The preys contribution to the predators growth rate is cNP ; that is It is proportional to the available prey as well as to the size of the predator population.

1.3 Functional response

Predator-prey models have always been considered to be classical, and the source of all the research work in population models during the past century is the Lotka-Volterra model. In the predator-prey models, The functional response is one of the crucial factors. The relationship between predation rate (i.e., number of prey eaten per predator per unit time) and prey density is termed the functional response. It is specific for each predator-prey system which affect population dynamics.Theory and laboratory testing suggest that the functional response of an animal may take one of three forms called the Type I,Type II and Type III functional responses (Hurkova,2013).

Type I is the simplest functional response in which prey consumption rises linearly with prey density to a threshold level. This is called a filter feeder's functional response. However, increasing prey density beyond a threshold does not result in an increased per capita predation rate. This is due to digestion and handling components of predation.

The Type I functional response is characterized by a linear relationship between capture rate C and prey density N

$$C = \alpha N$$

where α is a proportionality constant set by the rate at which predators encounter prey. This was the functional response assumed by Lotka and Volterra in their classic theoretical work on predatorprey interactions.

The Type II functional response is a cyrtoid functional response in which the attack rate increases at a decreasing rate with prey density until it becomes constant at satiation. Cyrtoid behavioral responses are typical of predators that specialize on one or a few prey. Type II functional response is given by the formula

$$c = \frac{\alpha N}{1 + \alpha t N}$$

In the Type III response, prey consumption remains low until a threshold density is

reached. The predation rate then increases exponentially until levels out. The shape of this curve is described as sigmoidal. The predator ignores the prey when densities are very low in order to make hunting energetically profitable (Hurkova,2013). Type III functional response can be expressed as:

$$C = \frac{\alpha N^n}{1 + \alpha t N^n}$$

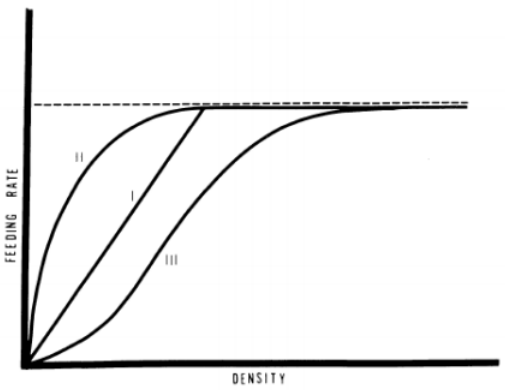

Figure 1.1: Three types of functional response (Hurkova,2013)

1.4 Holling type I-III

Holling justified the functional forms of types I, II and III using a straightforward argument based on the division of an individual predators time into two periods: searching for and handling of prey (Souza, 2011).

Holling reserved the term type I for the case in which such a plot shows a linear relationship between numbers of prey consumed and prey density, and termed type II the case in which the gradient of the curve decreases monotonically with increasing prey density, eventually saturating at a constant value of prey consumption. Type III behaviour occurs when the gradient of the curve first increases and then decreases with increasing prey density. This sigmoidal behaviour has been attributed to the existence of learning behaviour in the predator population. Real in fact lists six distinct behavioural shifts

which might move the functional response from type II to type III, for example predators learning more specialised techniques for hunting or prey handling, or learning to focus their search in particular places within the environment.

(a) Holling Type I functional response which is the standard mass action or linear response

$$f(N) = aN$$

Here $a > 0$ is the attack rate of the predator. This type of response is found in passive predators like spiders. The number of flies caught in the net is proportional to fly density. Prey mortality due to predation is constant.

(b) Holling type II functional response, describes the average feeding rate of a predator when the predator spends some time searching for prey and some time processing the captured prey (that is, handling time),and takes the form:

$$f(N) = \frac{bN}{1 + cN}$$

Where N represents the density of the prey population, and the positive constants b (units: 1/times) and c (units: 1/prey) describe the effects of capture rate and handling time, respectively, on the feeding rate. Note that the Holling type II function is prey-dependent and is not affected by the abundance of the predator population.

(c) Holling Type III functional response is represented by the equation below

$$f(N) = \frac{dN^n}{F + N^n}$$

where $x > 1$ is the encounter rate between predator and prey before the predator reaches maximum efficiency. Holling Type III functional response occurs in predators which increase their search activity with increasing prey density (Edwin,2010).

1.5 Statement of the problem

Our study intended to focus on the dynamics of a diffusive predator-prey system. In which prey and predator populations move randomly and predator have Holling type II

functional response.

Hence the present study aims to answer the following thesis problems.

i. under what condition does diffusion-driven instability will occur subject to Neumann boundary conditions ?

ii. what are the conditions for the occurrence of hopf bifurcation ?

iii. what is the biological implication of these conditions ?

1.6 Objective

The general objective of the present study is to investigate a bifurcation points of a diffusive predator-prey system with Holling type II functional response and the linear mortality.

1.6.1 Specific objective

The specific objectives of the present study are to :

i. investigate the stability condition for equilibrium points.

ii. investigate the occurrence of Hopf bifurcations and its properties.

iii. investigate the diffusion-driven Turing instability under stated boundary condition.

iii. study the biological implication of model solution.

1.7 Significance of the study

It is hoped that if the effect of parameters such as intrinsic growth rate, carrying capacity, mortality rate, biomass conversion etc, on the long term stable coexistence of two species is known, this will enable the park authorities to manage the species, specially taking well established measures to avoid extinction of any species. It also serve as reference

for other researchers to advance knowledge on this area. Further it serve as baseline for policy makers to protect extinction of species.

1.8 Research methodology

In this thesis, we have formulated a mathematical model which describe diffusive prey predator with Holling type II functional response using non-linear differential equation.We non-dimensionalize in order to reduce number of parameters involved in the model. We computed an equilibrium points and linearize the model at positive equilibrium point. After linearizing we were tried to find the characteristic equation. Using characteristic equation we have found the conditions for stability of positive equilibrium point, conditions for occurrence of Hopf bifurcation and Turing instability conditions. After computing the theoretical part we sketch the graph using matlab and observe numerical simulation to check stability of positive equilibrium.

Chapter 2

Literature Review

Mathematical population models have been used to study the dynamics of predator-prey systems since Lotka and Volterra proposed the simple model of predator-prey interactions now called the Lotka-Volterra model. Since then, many mathematical models have been developed and studied. In this chapter we will review some recent literature.
Some of the aspects that need to be critically considered in a realistic and plausible mathematical model include; carrying capacity which is the maximum number of prey that the ecosystem can sustain in absence of predator, competition among prey and predators which can be intraspecific or interspecific and functional responses of predators.

Predator-prey models have always been considered to be classical, and the source of all the research work in population models during the past century is the Lotka-Volterra model (Murray, 2001). In the predator-prey models, the functional response is one of the crucial factor which affect population dynamics. Typically, the Lotka-Volterra interaction term can be classified into many different types, for instance, Beddington-DeAngelis type (Beddington,1975) , Holling-Tanner type (Tanner, 1975) , Holling type I-III (Souza, 2011) and Ivlev type (Kooji,1996).

As a mathematical consequence of the herd behavior, Authors considered competition models and predator-prey systems in which interaction terms use the square root of the prey population rather than simply the prey population. The use of the square root properly accounts for the assumption that the interactions occur along the boundary

of the population. Then, (Braza,2012) has shown that the solution behavior near the origin is more subtle and interesting than standard models and makes sense ecologically. The dynamics of this kind of predator-prey model is governed by the following ordinary differential equation

$$\begin{cases} \frac{d}{dt}X(t) = r(1 - \frac{X(t)}{k})X(t) - \alpha\sqrt{X(t)}Y(t) \\ \frac{d}{dt}Y(t) = -sY(t) + c\alpha\sqrt{X(t)}Y(t) \end{cases} \tag{2.1}$$

where $X(t)$ and $Y(t)$ denote the prey and predator densities, respectively; the parameter r is the net reproduction of the prey species; k is the carrying capacity for the prey species; c is the conversion coefficient; and s and α are predators mortality and hunting rates,respectively. System (2.1) is based on the fact that while predators are moving freely in the environment actively searching for their prey, the latter are gathering together in herds for defensive purposes, namely, the interactions between predators and prey affect mainly on the prey individuals occupying the outermost positions in the herd. The dynamics of system (2.1) such as the global stability are investigated in (Brentnall). The predators and prey are usually abundant in space with different densities at different positions, that is, they are diffusive. (Venturino, 2013) extended (2.1) to a spatially distributed model, and some specific spatio-temporal features are discovered. Different Authors use different dimension (Song and Zou, 2014) and (Xu,2015) use one dimension and also (Venturino, 2013) and (Yuan et al.2013) use two dimension. For our work we also choose one dimension.

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = r(1 - \frac{X(x,t)}{k})X(x,t) - \alpha\sqrt{X(x,t)}Y(x,t) + d_1X_{xx}(x,t) \\ \frac{\partial Y(x,t)}{\partial t} = -sY(x,t) + c\alpha\sqrt{X(x,t)}Y(x,t) + d_2Y_{xx}(x,t) \end{cases} \tag{2.2}$$

where d_1 and d_2 represent the diffusive rates of the prey and predator, respectively. Considering the importance of the death rate of the predator species on the dynamics (Yuan et al.2013) introduced the quadratic mortality to the diffusive predatorprey system with herd behavior. The model introduced by (Yuan et al.2013) with one-dimension spatial

variable x is given by

$$\begin{cases} \frac{\partial X(x,t)}{\partial t} = r(1 - \frac{X(x,t)}{k})X(x,t) - \alpha\sqrt{X(x,t)}Y(x,t) + d_1 X_{xx}(x,t) \\ \frac{\partial Y(x,t)}{\partial t} = -sY^2(x,t) + c\alpha\sqrt{X(x,t)}Y(x,t) + d_2 Y_{xx}(x,t) \end{cases} \tag{2.3}$$

Xu (2016) studied Bifurcation analysis of a diffusive predator-prey system with a herd behavior and quadratic mortality subject to Neumann boundary condition was studied . Recently many authors study bifurcation analysis of diffusive prey predator model with different functional responses.

(Sambath,2015) studied bifurcation analysis of a diffusive predator-prey model with Beddington-DeAngeles type and, (Baek, 2014) studied Bifurcation analysis of a predator-prey system with self- and cross-diffusion and constant harvesting rate is considered. And also in (Zhou, 2015) bifurcation analysis of a diffusive predator-prey model with ratio-dependent Holling type III functional response and, in (Wang et al. 2016) Global bifurcation analysis and pattern formation in homogeneous diffusive predator-prey systems is also studied. Hopf bifurcation and Turing instability in spatial homogeneous and inhomogeneous predator-prey models is considered in (Zhang, 2011). Stability and Hopf bifurcation in a diffusive predatory-prey system with delay effect is also studied in (Zuo, 2011). We can mention many study in this area . In the present study we investigate the stability conditions and occurrence of Hopf bifurcation for both diffusive and non diffusive predator-prey system with Holling type II functional response and linear mortality rate under Neumann boundary condition.

Chapter 3

Model Formulation and Analysis

In the present study we formulate the model (2.3) to the following form that means we change functional to Holling type II and mortality rate of predator to linear mortality :

$$\begin{cases} \frac{\partial X}{\partial t} = rX(1-\frac{X}{K}) - \frac{\alpha XY}{\beta+X} + d_1 X_{xx} \\ \frac{\partial Y}{\partial t} = -sY + \frac{bXY}{\beta+X} + d_2 Y_{xx} \end{cases} . \tag{3.1}$$

The prey is denoted by $X(t)$ and the predator by $Y(t)$. The parameter r is the growth rate of the prey, K is its carrying capacity, and s is the death rate of the predator in the absence of prey. The parameter α is the search efficiency of Y for X , c is biomass conversion or consumption rate and $b = c\alpha$,α and b represent the strength of the relative effect of the interaction on the two species. The positive parameter β measures the saturation effect: the consumption of prey by a unit number of predator.

3.1 Assumptions of the Model

The model (3.1) that we consider has the following assumptions:

(i) there is logistic growth of the prey in absence of the predator.

(ii) the prey and predator move randomly in the bounded domain Ω ,that means they are diffusive.

(iii) the mortality rate of predator population is described by linear.

- (iv) Holling type II functional response.

3.2 Boundedness of system

Definition 1. Let f be a real valued function defined on a domain D.The function f is said to be bounded on D if and only if there is a positive number M such that $|f(x,y)| \leq M$ for all $(x,y) \in D$.

Theorem 3.2.1 (comparison principle of parabolic equation)**.** *Let v and w satisfies*

$$v_t - d\triangle v = f_1 \quad and \quad w_t - d\triangle w = f_2$$

Then if $v \geq w$ on $\partial\Omega$ and $f_1 \geq f_2$ in Ω then $v \geq w$ in all Ω.

proof and detail about this theorem is available on (Sandro,2008).

Theorem 3.2.2. *All solutions of the system (3.1) that starts in R^2_+ are uniformly bounded.*

Proof. let $X(x,t), Y(x,t)$ is any solution of equation (3.1) then X satisfies

$$\frac{\partial X}{\partial t} = rX(1 - \frac{X}{K}) - \frac{\alpha XY}{\beta + X} + d_1 X_{xx}$$

This implies

$$\frac{\partial X}{\partial t} - d_1 X_{xx} \leq rX(1 - \frac{X}{K})$$

Let $z(t)$ be a solution of ODE system then

$$z'(t) = rz(1 - \frac{z}{K})$$

$$\frac{dz}{dt} = \frac{rz(K-z)}{K}$$

$$\frac{K}{rz(K-z)}dz = dt$$

By using decomposition we have

$$\left(\frac{1}{z} - \frac{1}{z-K}\right)dz = dt$$

Integrating both sides we have

$$\frac{z}{z-K} = rt + c$$

$$\frac{z}{z-K} = e^{rt+c}$$

On rearranging it gives:

$$z(t) = \frac{Ke^{rt+c}}{e^{rt+c} - 1}$$

By Lihoptal‘s rule

$$\lim_{t\to\infty} z(t) = K.$$

From comparison principle of parabolic equation, it follows that $X(x,t) \leq z(t)$. Hence

$$\limsup_{t\to\infty} \max_{x\epsilon\overline{\Omega}} X(x,t) \leq K.$$

As a result, for any small enough $\varepsilon > 0$, there exists $T > 0$ such that $X(x,t) \leq K + \varepsilon$ for all $x\epsilon\overline{\Omega}$ and $t \geq T$. Multiplying the first equation by c and adding it to the second equation in (3.1), we have

$$w_t - dw_{xx} = crX(1 - \frac{X}{K}) - sY$$

Where $w = cX + Y$. Since

$$crX(1 - \frac{X}{K}) - sY = c(r+s)X - \frac{crX^2}{K} - rw \leq c(r+s)X \leq c(r+s)(k+\varepsilon)$$

In $[T,\infty) \times \Omega$, the comparison argument shows that

$$\limsup_{t\to\infty} Y(x,t) \leq \limsup_{t\to\infty} w(t,x) \leq c(r+s)(k+\varepsilon)$$

on $\overline{\Omega}$ which implies Y is bounded. □

In order to minimize number of parameters involved in (3.1) some scaling needs to take place.

$u = \frac{1}{\beta}X$, $v = \frac{\alpha}{b\beta}Y$, $\bar{t} = rt$,

With these changes and dropping the bares for simplification of notations, system (3.1) becomes

$$\begin{cases} u_t = du_{xx} + u(1 - \frac{u}{k}) - \frac{muv}{1+u} \\ v_t = v_{xx} - \theta v + \frac{muv}{1+u} \end{cases} \tag{3.2}$$

where $\theta = \frac{s}{r}, m = \frac{b}{r}c, d = \frac{d_1}{d_2}$. For system of equation (3.2), d is considered as the relative diffusion rate of two species, θ and m are often called as the scaled death rate and biomass conversion rate respectively. In general, to ensure that Turing pattern is determined by reaction-diffusion mechanism, we choose the following nonzero initial condition.

$u(x,0) > 0$, $v(x,0) > 0$ where $x \epsilon \Omega = (0, l\pi)$ and Neumann (zero-flux) boundary condition:

$$u_x(0,t) = u_x(l\pi,t) = v_x(0,t) = v_x(l\pi,t) = 0. \tag{3.3}$$

where $l\pi$ denotes the size of the system in the directions of u and v.

3.3 Linear analysis for positive equilibrium

Equilibrium points

An equilibrium solution is that state in a dynamical system where the net rate of change in population density is zero. The homogeneous steady state solutions are obtained from the equation

$$f(u,v) = 0$$

$$g(u,v) = 0 \quad ,$$

where

$$f(u,v) = u(1-\frac{u}{k}) - \frac{muv}{1+u} \quad and \tag{3.4}$$

$$g(u,v) = -\theta v + \frac{muv}{1+u}. \tag{3.5}$$

Since the solutions are constant in space and time we have

$\frac{\partial u}{\partial t} = 0, u_{xx} = 0$ and $\frac{\partial v}{\partial t} = 0, v_{xx} = 0$

From equation (3.5) we have

$$-\theta v + \frac{muv}{1+u} = 0$$

$$v(-\theta + \frac{mu}{1+u}) = 0$$

$$v = 0 \ \ and \ / \ or \ \ \ -\theta + \frac{mu}{1+u} = 0.$$

From these equations we get

$$v = 0 \ \ and \ \ u = \frac{\theta}{m-\theta}.$$

Then substituting into (3.4) We can verify that system (3.2) always has two boundary equilibrium's $E_0 = (0,0)$ and $E_1 = (k,0)$, and a positive constant equilibrium $E_*(\delta, v_\delta)$ with $\delta = \frac{\theta}{m-\theta}$, $v_\delta = \frac{(k-\delta)(1+\delta)}{km}$.

Theorem 3.3.1. *The prey and predator species coexist if the conditions $m > \theta$ and $\delta < k$ are satisfied.*

Proof. for coexistence of both species both δ and $v_\delta = \frac{(k-\delta)(1+\delta)}{km}$ must be positive. Thus for

$$\delta > 0$$

we have

$$\frac{\theta}{m-\theta} > 0$$

Since m and θ are positive we get

$$m > \theta$$

And also

$$\frac{(k-\delta)(1+\delta)}{km} > 0$$
$$(k-\delta)(1+\delta) > 0$$
$$k - \delta > 0 \quad since \quad \delta > 0$$
$$k > \delta$$

Therefore $\delta < k$ □

Setting $U = (u, v)^T$ and taking δ as parameter system (3.2) can be rewritten as the following abstract differential equation

$$\frac{\partial U}{\partial t} = D\Delta U + L(\delta)U + F(U) \tag{3.6}$$

where

$$L(\delta) = \begin{pmatrix} A(\delta) & B(\delta) \\ C(\delta) & D(\delta) \end{pmatrix},\ D\Delta = \text{diag}\ \left(d\frac{\partial^2}{\partial x^2} \quad , \frac{\partial^2}{\partial x^2}\right)$$

$F(u) = \begin{pmatrix} u(1-\frac{u}{k}) - \frac{muv}{1+u} - A(\theta)u - B(\theta)v \\ -\theta v + \frac{muv}{1+u} - C(\theta)u - D(\theta)v \end{pmatrix}$ and

$$A(\delta) = \frac{\partial f}{\partial u} = \frac{k-2u}{k} - \frac{mv}{(1+u)^2}$$
$$B(\delta) = \frac{\partial f}{\partial v} = -\frac{mu}{1+u}$$
$$C(\delta) = \frac{\partial g}{\partial u} = \frac{mv}{(1+u)^2}$$
$$D(\delta) = \frac{\partial g}{\partial v} = -\theta + \frac{mu}{1+u}\ .$$

For positive constant equilibrium (δ, v_δ) we have the following

$$A(\delta) = \frac{\delta(k-1-2\delta)}{k(1+\delta)}, B(\delta) = -\theta, C(\delta) = \frac{k-\delta}{k(1+\delta)}, D(\delta) = 0 \tag{3.7}$$

The linearized system of equation (3.6) at $E_*(u_*, v_*) = (\delta, v_\delta)$ is

$$\frac{\partial U}{\partial t} = D\Delta U + L(\theta)U \tag{3.8}$$

Setting

$$U = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} cos(\frac{n}{l}x)e^{\lambda t}$$

which satisfy the boundary conditions at $x = (0, l\pi)$ and then substituting it into (3.8) we can obtain characteristic equation as follow :

$$(L_n(\delta) - \lambda I) \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

where

$$L_n(\delta) = \begin{pmatrix} A(\delta) - d(\frac{n}{L})^2 & B(\delta) \\ C(\delta) & D(\delta) - (\frac{n}{L})^2 \end{pmatrix}.$$

Solving the equation

$$|L_n(\delta) - \lambda I| = 0$$

we have characteristic polynomial

$$\lambda^2 - T_n(\delta)\lambda + J_n(\delta) = 0 \tag{3.9}$$

where

$$T_n(\delta) = A(\delta) + D(\delta) - (1 + d)(\frac{n}{l})^2 \tag{3.10}$$

$$J_n(\delta) = d(\frac{n}{l})^4 - (A(\delta) + dD(\delta))(\frac{n}{l})^2 + A(\delta)D(\delta) - B(\delta)C(\delta) \tag{3.11}$$

The roots of equation (3.9) can be obtained by the following form:

$$\lambda = \frac{1}{2}\left(T_n(\delta) \pm \sqrt{(T_n(\delta))^2 - 4J_n(\delta)}\right) \tag{3.12}$$

Stability is understood to mean that the interacting species maintain relatively steady population sizes within a given parametric region, for increasing time.

From equation (3.9) we can give the following definition.

The Turing condition is the one in which the positive steady state is stable for the non-diffusive model, but it is unstable for the reaction-diffusion model. And the condition for the positive steady state to be stable for the corresponding non-diffusive model is given by :

$$A(\delta) + D(\delta) < 0 \; and \qquad A(\delta)D(\delta) - B(\delta)C(\delta) > 0 \tag{3.13}$$

Theorem 3.3.2. *The conditions that the given system with diffusion have eigenvalues with $Re\lambda < 0$ are:*

$$A(\delta) + D(\delta) - d(\frac{n}{l})^2 < 0 \; and$$

$$(A(\delta) - d(\frac{n}{l}^2))(D(\delta) - \frac{n}{l}^2)) - B(\delta)C(\delta) > 0$$

Proof. To linearize the given non-linear system at equilibrium point. Let

$$\begin{cases} u = u_1 + u_* \\ v = v_1 + v_* \end{cases}$$

be small non homogeneous perturbation.

$$\frac{\partial u_1}{\partial t} = \frac{\partial u}{\partial t} = f(u, v) + du_{xx}$$

$$= f(u_1 + u_*, v_1 + v_*) + du_{1xx}$$

similarly

$$\frac{\partial v_1}{\partial t} = g(u_1 + u_*, v_1 + v_*) + v_{1xx} \; .$$

Thus after applying Taylor series we get:

$$\begin{cases} \frac{\partial u_1}{\partial t} = u_1 \left(\frac{\partial f}{\partial u}\right)_{(u_*, v_*)} + v_1 \left(\frac{\partial f}{\partial v}\right)_{(u_*, v_*)} + du_{1xx} \\ \frac{\partial v_1}{\partial t} = u_1 \left(\frac{\partial f}{\partial u}\right)_{(u_*, v_*)} + v_1 \left(\frac{\partial g}{\partial v}\right)_{(u_*, v_*)} + v_{1xx} \; . \end{cases} \tag{3.14}$$

Together with boundary condition setting

$$u_1(x, t) = A_1 e^{\lambda t} cos((\frac{n}{l})x)$$

$$v_1(x, t) = A_2 e^{\lambda t} cos((\frac{n}{l})x)$$

And substituting into equation (3.14)we have the following,

$$A_1(\lambda - A(\delta) + d(\frac{n}{l})^2) - A_2 B(\delta) = 0$$

$$-C(\delta)A_1 + (\lambda - D(\delta) + (\frac{n}{l})^2)A_2 = 0.$$

Where

$$A(\delta) = \left(\frac{\partial f}{\partial u}\right)_{(u_*,v_*)}, B(\delta) = \left(\frac{\partial f}{\partial v}\right)_{(u_*,v_*)}$$

$$C(\delta) = \left(\frac{\partial g}{\partial u}\right)_{(u_*,v_*)} D(\delta) = \left(\frac{\partial g}{\partial v}\right)_{(u_*,v_*)}$$

clearly, $A_1 = 0$, $A_2 = 0$ is a solution, which is trivial and is we are interested in non-trivial solutions. Existence of a non-trivial solution implies

$$\begin{vmatrix} \lambda - A(\delta) + d(\frac{n}{l})^2 & -B(\delta) \\ C(\delta) & \lambda - D(\delta) + (\frac{n}{l})^2 \end{vmatrix} = 0$$

$$This\ implies\ \lambda^2 - (A(\delta) + D(\delta) - (\frac{n}{l})^2 - d\frac{n^2}{l})\lambda + (A(\delta) - d(\frac{n}{l})^2)(D(\delta) - (\frac{n}{l})^2) - B(\delta)C(\delta) = 0$$

The conditions that the given system with diffusion have eigenvalues with $Re\lambda < 0$ are

$$A(\delta) + D(\delta) - d(\frac{n}{l})^2 < 0\ and$$

$$(A(\delta) - d(\frac{n}{l})^2)(D(\delta) - (\frac{n}{l})^2) - B(\delta)C(\delta) > 0$$

In the absence of diffusion, the condition for stability is

$$A(\delta) + D(\delta) < 0\ and$$

$$A(\delta)D(\delta) - B(\delta)C(\delta) > 0$$

Which is the proof of above two theorems.

□

Theorem 3.3.3. *For $\theta > 1$ an equilibrium point $E_0(0, 0)$ that means the two species will die out with increased time corresponding to non-diffusive model is stable.*

Proof. From the condition for stability given by

$$A(\delta) + D(\delta) < 0 \quad and \qquad A(\delta)D(\delta) - B(\delta)C(\delta) > 0$$

For equilibrium point $E_0(0, 0)$ we have

$$A(\delta) = 1 \quad B(\delta) = 0 \quad C(\delta) = 0 \quad and \quad D(\delta) = -\theta$$

substituting this values into the stability condition we get $\theta > 1$ □

Theorem 3.3.4. *For $\delta > k$ an equilibrium point $E_1(k, 0)$ that means the prey species persists while the predator species eventually disappear due to predation. corresponding to non-diffusive model is stable.*

Proof. From the condition of stability we have

$$A(\delta) + D(\delta) < 0 \quad and \qquad A(\delta)D(\delta) - B(\delta)C(\delta) > 0$$

For equilibrium point $(k, 0)$ we get

$$A(\delta) = -1 \quad B(\delta) = -\frac{mk}{1+k} \quad C(\delta) = 0 \quad and \quad D(\delta) = -\theta + \frac{mk}{1+k}$$

From condition $A(\delta) + D(\delta) < 0$ we have

$$-1 + (-\theta) + \frac{mk}{1+k} < 0$$

$$-\frac{-1-k-\theta-\theta k+mk}{1+k} < 0$$

$$\theta(-1-k) < -mk + 1 + k$$

$$\theta > \frac{k(m-1)-1}{1+k} \tag{3.15}$$

And also from the second condition $A(\delta)D(\delta) - B(\delta)C(\delta) > 0$ we get

$$-1(\frac{-\theta-\theta k+mk}{1+k}) > 0$$

$$\theta > \frac{mk}{1+k} \tag{3.16}$$

From equation (3.15) and equation (3.16) we get

$$\theta > \frac{mk}{1+k} \tag{3.17}$$

And solving for θ in $\delta = \frac{\theta}{m-\theta}$ we get

$$\theta = \frac{\delta m}{1+\delta}$$

Substituting into equation (3.17) we get $\delta > k$ □

Theorem 3.3.5. *For $\frac{k-1}{2} < \delta < k$ an equilibrium point $E_*(\delta, v_\delta)$ that means the non zero coexistence equilibrium corresponding to non-diffusive model is stable.*

Proof. From stability conditions we considered above with

$$A(\delta) = \frac{\delta(k-1-2\delta)}{k(1+\delta)} \qquad B(\delta) = -\theta \qquad C(\theta) = \frac{k-\delta}{k(1+\delta)} \qquad D(\delta) = 0.$$

Then

$$A(\delta) + D(\delta) < 0$$

$$\frac{\delta(k-1-2\delta)}{k(1+\delta)}+0<0$$
$$\delta(k-1-2\delta)<0$$
$$\delta>\frac{k-1}{2} \tag{3.18}$$

And from condition

$$A(\delta)D(\delta)-B(\delta)C(\delta)>0 \quad we \quad have$$
$$0-(-\theta)(\frac{k-\delta}{k(1+\delta)})$$
$$\theta(k-\delta)>0$$
$$\delta<k \tag{3.19}$$

combining equation (3.18) and equation (3.19) we get

$$\frac{k-1}{2}<\delta<k$$

□

3.4 Hopf bifurcation

Theorem 3.4.1. *For non-diffusive system with positive equilibrium $\delta=\frac{k-1}{2}$ is bifurcation point in which Hopf bifurcation occur.*

Proof. We know that the necessary conditions for Hopf bifurcation to occur are $T_n(\delta)=0$ and $J_n(\theta)>0$ from equation (3.10),

$$T_n(\delta)=0$$
$$\frac{\delta(k-1-2\delta)}{k(1+\delta)}=0$$
$$\delta(k-1-2\delta)=0$$
$$(k-1-2\delta)=0$$
$$\delta=\frac{k-1}{2}$$

Therefore $\delta=\frac{k-1}{2}$ is a point in which Hopf bifurcation occur. □

Similarly for diffusive system of equation we have the following results.

$$T_n(\delta)=0$$

$$A(\delta) - (1+d)(\frac{n}{l})^2 = 0$$
$$\frac{\delta(k-1-2\delta)}{k(1+\delta)} - (1+d)(\frac{n}{l})^2 = 0$$
$$\delta(k-1-2\delta) - k(1+d)(\frac{n}{l})^2 = 0$$

Then we get quadratic formula

$$2\delta^2 + (1-k+(1+d)(\frac{n}{l})^2)\delta + k(1+d)(\frac{n}{l})^2 = 0$$

Therefore

$$\delta = \delta_n(l) = \frac{1}{4}\left(-\left(1-k+k(1+d)(\frac{n}{l})^2\right) \pm \sqrt{\left(1-k+k(1+d)(\frac{n}{l})^2\right)^2 - 8\left(k(1+d)(\frac{n}{l})^2\right)}\right).$$

To consider Hopf bifurcations for diffusive system, we assume that for some δ_0, the following condition holds:

(H_1) There exists a neighborhood O of δ_0 such that for $\delta \epsilon o$, $L(\delta)$ has a pair of complex, simple, conjugate eigenvalues $\alpha(\delta) \pm i\omega(\delta)$, continuously differentiable in δ,with $\alpha(\delta_0) = 0$, $\omega(\delta_0) = \omega_0 > 0$, and $\alpha'(\delta_0) \neq 0$; all other eigenvalues of $L(\delta)$ have non-zero real parts for $\delta \ \epsilon \ O$.

Due to the stability in Theorem (3.3.5), and in order to concentrate on the investigation of Hopf bifurcation, we always assume that $k > 1$, and we consider δ in the range of $0 < \delta < k-1$. We shall identify Hopf bifurcation values δ_0 which satisfy the condition (H_1), which takes the following form there exists $n \ \epsilon \ N_0$ such that

$$T_n(\delta_0) = 0 \ \ and \ \ J_n(\delta_0) > 0$$

and for the unique pair of complex eigenvalues near the imaginary axis $\alpha(\delta) \pm i\omega(\delta)$, $\alpha'(\delta_0) \neq 0$. From equations (3.10) and (3.11) $T_n(\delta_0) < 0 \ \ and \ \ J_n(\delta_0) > 0$
for $\delta \ \epsilon \ ((k-1)/2, k-1)$, which implies that the trivial steady state (δ, v_δ) is locally asymptotically stable. Hence any potential bifurcation point δ_0 must be in the interval $(0, (k-1)/2]$. For any Hopf bifurcation point δ_0 in $(0, (k-1)/2]$, $\alpha(\delta) \pm i\omega(\delta)$ are the

eigenvalues of $L_n(\delta)$,so

$$\alpha(\delta) = \frac{A(\delta)}{2} - \frac{(d+1)n^2}{2l^2} \quad , \quad \omega(\delta) = \sqrt{J_n(\delta) - \alpha^2(\delta)}$$

and

$$\alpha'(\delta_0) = \left(\frac{A'(\delta)}{2}\right)_{\delta=\delta_0}$$

$$= \left(\frac{k-1-4\delta-2\delta^2}{2k(1+\delta)^2}\right)_{\delta=\delta_0} \begin{cases} > 0 & if \quad 0 < \delta_0 < \delta_* \\ < 0 & if \quad \delta_* < \delta_0 \leq (k-1)/2 \end{cases}$$

where

$$\delta_* = \sqrt{\frac{k+1}{2}} - 1\epsilon \left(0, \frac{k-1}{2}\right)$$

Hence condition $\alpha'(\delta_0) \neq 0$ is always satisfied as long as $\delta_0 \neq \delta_*$.

Moreover when $\delta_* < \delta_0 < (k-1)/2$, the real part of one pair of complex eigenvalues of $L(\delta)$ becomes positive when δ decreases crossing δ_0, and when $0 < \delta_0 < \delta_*$, the real part of one pair of complex eigenvalues of $L(\delta)$ becomes negative when δ decreases crossing δ_0. That is, in $(\delta_0, (k-1)/2)$, the constant steady state loses stability when δ decreases across a bifurcation point, but in $(0, \delta_*)$ it regains the stability when δ decreases across a bifurcation point. From discussions above, the determination of Hopf bifurcation points reduces to describing the set

$$\Lambda_1 := \{\delta\epsilon(0, (k-1)/2] \setminus \delta_*\} \quad for \quad some \quad n\epsilon N\}$$

when a set of parameters (l, d, θ, k) is given. We fix $d, \theta > 0$ and $k > 1$ but choose l appropriately. First $\delta_0^H = (k-1)/2$ is always an element of Λ_1 for any $l > 0$ This corresponds to the Hopf bifurcation which has been known from the studies of non-diffusive model. Hence in the following we look for Hopf bifurcation for $n \geq 1$. We notice that $A(0) = A(\delta_o^H) = 0$ and $A(\delta) > 0$ in $(0, \delta_o^H)$, and $A(\delta)$ has a $A(\delta_*) = 2\delta_*^2/k = M_* > 0$. Define

$$l_n = n\sqrt{\frac{d+1}{M_*}} \quad , \; n\epsilon N, \quad where \quad M_* = \frac{(\sqrt{k+1} - \sqrt{2})^2}{k} > 0$$

Then for $l_n < l < l_{n+1}$, and $1 \leq j \leq n$, we define $\delta^H_{j,-}$ and $\delta^H_{j,+}$ be the roots of $A(\delta) = \frac{(d+1)j^2}{l^2}$ satisfying $0 < \delta^H_{j,-} < \delta_* < \delta^H_{j,+} < \delta^H_0$. These points satisfy

$$0 < \delta^H_{1,-} < \delta^H_{2,-} < .... < \delta^H_{n,-} < \delta^H_* < \delta^H_{n,+} < ....\delta^H_{2,+} < \delta^H_{1,+} < \delta^H_0$$

Clearly $T_j(\delta^H_{j,\pm}) = 0$ and $T_i(\delta^H_{j,\pm}) \neq 0$ for $i \neq j$. now we only need to whether $J_i(\delta^H_\pm) \neq 0$ for all $i \epsilon N_0$, and in particular, $J_j(\delta^H_\pm) > 0$. Indeed if $\delta \epsilon [0, \delta^H_0]$, then

$$J_i(\delta) \geq \frac{\theta}{k} - M_* \frac{i^2}{l^2} + d\frac{i^4}{l^4} := g(\frac{i^2}{l^2}).$$

The function $g(y) = \frac{\theta}{k} - M_* y + dy^2$ is positive for all $y \epsilon R$ if

$$\frac{\theta}{k} > \frac{M_*^2}{4d}.$$

Therefore for $d > kM_*/4\theta$ there exists $j + 1$ Hopf bifurcation points.

Chapter 4

Numerical simulation

Theoretical studies always remain incomplete without numerical verification of the results. Here, we present numerical simulation to illustrate the results obtained in previous sections. The numerical simulation was implemented in Matlab. In this chapter, we present some numerical simulations to illustrate the theoretical analysis, using ODE45 for non-diffusive and PDEPE for diffusive system to plot graphs. Our theoretical results are verified by numerical simulations for different parameter values and initial conditions. We take the values of parameters randomly which satisfies the conditions for stability and occurrence of Hopf bifurcation.

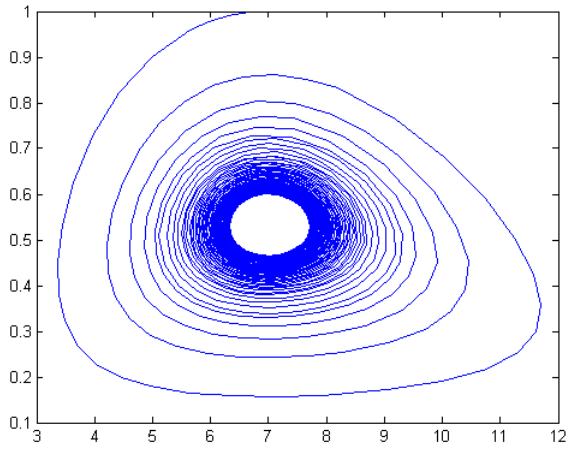

Figure 4.1: Numerical simulation of corresponding non-diffusive system of equation (3.2) for $k = 15$ $m = 8$, and $\theta = 7$. The positive equilibrium $E_*(7, 0.53)$, since $\delta = 7$ is bifurcation point in which Hopf bifurcation occur. The initial values are $u(x, 0) = 7.01$ and $v(x, 0) = 1$

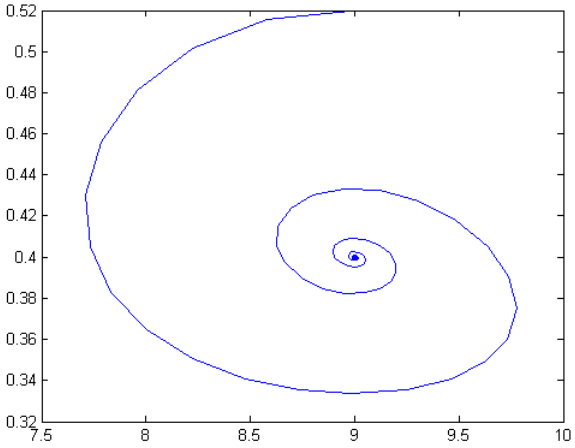

Figure 4.2: Numerical simulation of corresponding non-diffusive system of equation (3.2) for $k = 15$ $m = 10$, and $\theta = 9$. The positive equilibrium $E_*(9, 0.5)$ is stable , since $\delta = 9$ is on the interval $(\frac{k-1}{2}, k)$. The initial values are $u(x, 0) = 9.01$ and $v(x, 0) = 0.52$

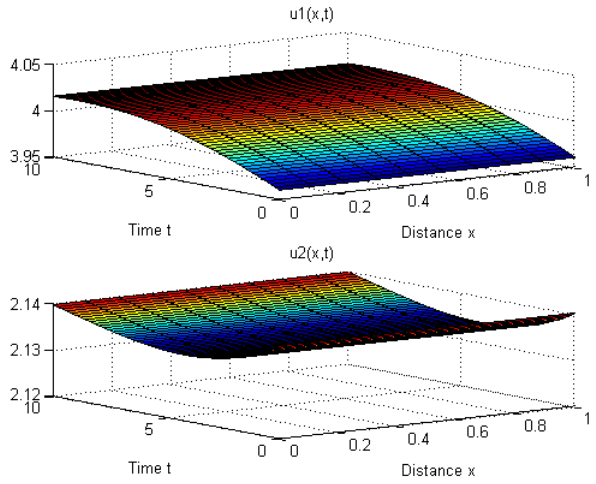

Figure 4.3: Numerical simulation of system (3.2) for $d = 0.3$, $m = 1$ $k = 7$, and $\theta = 0.8$. The positive equilibrium $E_*(4, 2.15)$ is asymptotically stable since $\delta = 4$ which is in the interval $(\frac{k-1}{2}, k)$. and also $(\frac{\theta}{k}) = 0.12$ and $\frac{M_*^2}{4d} = 0.067$ which implies the condition $\frac{\theta}{k} > \frac{M_*^2}{4d}$ is satisfied thus the set of Hopf bifurcation occur. The initial values are $u(x, 0) = u_* - 0.04$ and $v(x, 0) = v_* - 0.01c$

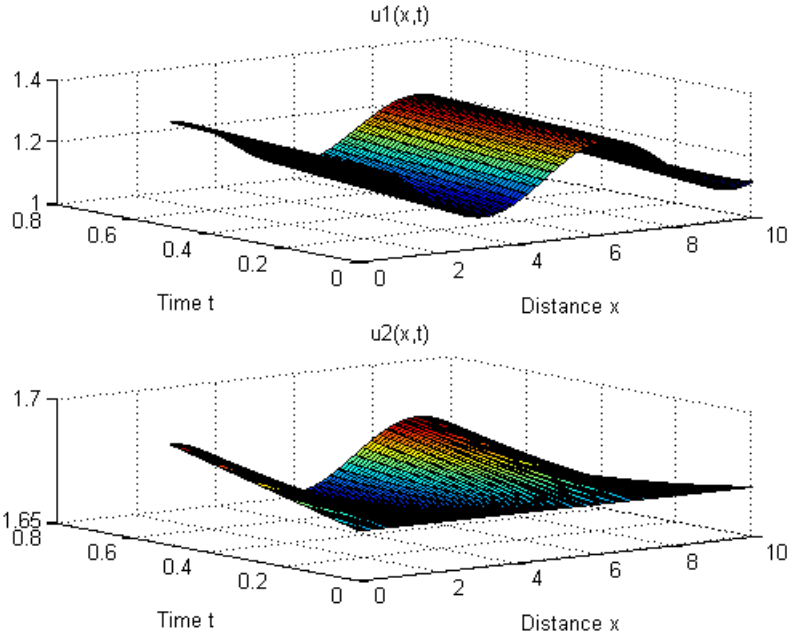

Figure 4.4: Numerical simulation of system (3.2) for $d = 0.3$, $m = 1$ $k = 7$, and $\theta = 0.54$. The positive equilibrium $E_*(1.2, 1.65)$ is unstable since $\delta = 1.2$ which is not in the interval $(\frac{k-1}{2}, k)$. The initial values are $u(x, 0) = u_* + 0.01cos(x)$ and $v(x, 0) = v_* + 0.02$

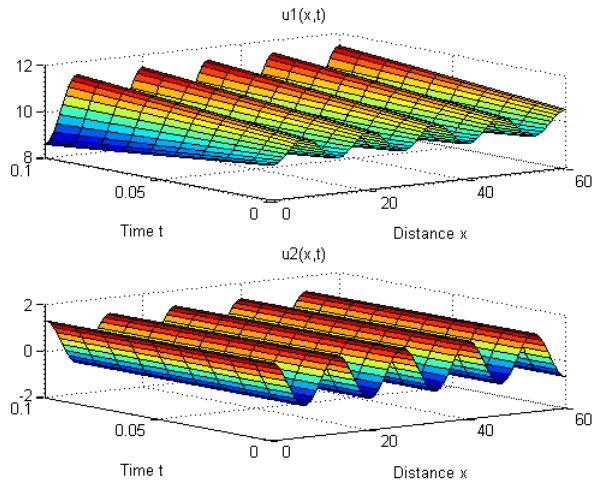

Figure 4.5: Numerical simulation of system (3.2) for $d = 0.26$,$k = 15$ $m = 11$, and $\theta = 10$. The positive equilibrium $E_*(10, 0.34)$ is asymptotically stable, since $\delta = 10$ The initial values are $u(x, 0) = 10 - 0.5cos((9/19)x)$ and $v(x, 0) = 0.34 + 0.02cos((9/19)x)$

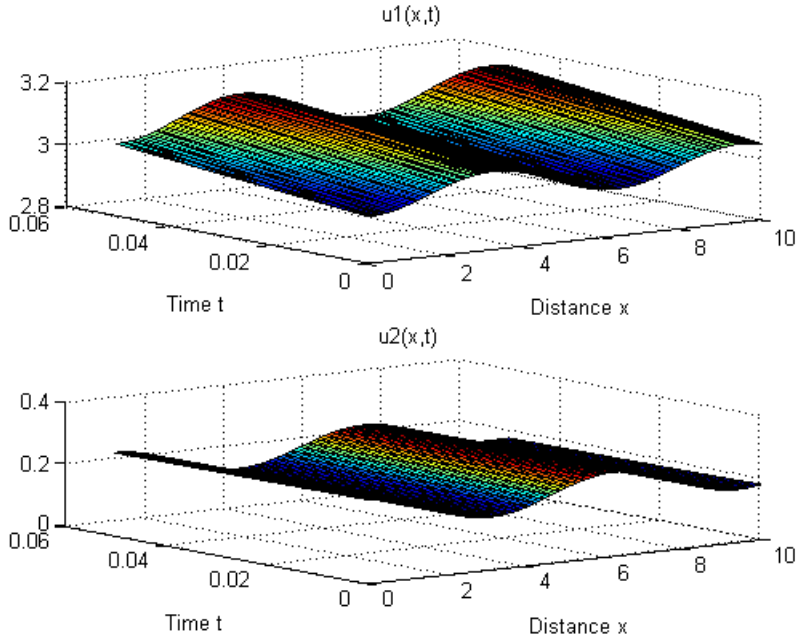

Figure 4.6: Numerical simulation of system (3.2) for $d = 0.26$,$k = 15$ $m = 4$, and $\theta = 3$. The positive equilibrium $E_*(3, 0.22)$ is unstable, since $\delta = 3$ The initial values are $u(x, 0) = 3 - 0.05cos(x)$ and $v(x, 0) = 0.22 + 0.05cos(x)$

Chapter 5

Conclusion

In this thesis, the diffusive predator-prey system with Holling type-II functional response was studied. The effects of the rates of death, biomass conversion, and the diffusion on the stability of the system and diffusion-driven instability have been investigated in detail. For both diffusive and non-diffusive model we also investigated the existence of Hopf bifurcation points.

Finally to verify our theoretical results we simulate our equation numerically and we got expected results. And from our numerical simulation results we observed stability and instability of positive equilibrium with different values of given parameters and our numerical results match the analytic result as expected.

Further bifurcation analysis of equation (3.2) remains challenging problem. Our results show that steady state solutions exist, and they form loops. The stability of these patterned solutions are also not known except near the bifurcation points and left for future work.

Bibliography

[1] Ajraldi V, Pittavino M, Venturino E. Modeling herd behavior in population systems.Nonlinear Analysis: Real World Applications 2011; vol 12:2319-2338.

[2] Baek H. Bifurcation analysis of predator-prey system with self- and cross-diffusion and constant harvesting rate. vol 29 , 1-14 :2014

[3] Baurmann M, Gross T, Feudel U. Instabilities in spatially extended predatorprey systems: spatio-temporal patterns in the neighborhood of Turing- Hopf bifurcations.Journal of Theoretical Biology 2007; vol 245:220-229.

[4] Beddington JR. Mutual interference between parasites or predators and its effect on searching efficiency.Journal of Animal Ecology 1975 vol 44:331-340.

[5] Braza PA. Predatorprey dynamics with square root functional responses.Nonlinear Analysis: Real World Applications 2012; vol 13:1837-1843

[6] Brentnall SJ, Richards KJ, Brindley J, Murphy E. Plankton patchiness and its effect on larger-scale productivity.Journal of Plankton Research 2003; vol 25:121-140.

[7] Celik C. Karaaslanl ,Bifurcation Analysis and Its Applications 2012; 1-32

[8] Edwin A.: Modeling and analysis of a two prey-one predator system with harvesting, Holling Type II and ratio-dependent responses; 2010, vol 1:1-14

[9] Hurkova Jana ;prey-predator model with ratio dependent food processing response : university of Minnesota Duluth 2013

[10] Kooij RE, Zegeling A. A predatorprey model with Ivlevs functional response.Journal of Mathematical Analysis and Applications 1996; vol 198:473-489.

[11] Murray JD. Mathematical Biology I an introduction third edition 2001 .

[12] Sandro Salsa Partial Differential Equations in Action From Modelling to Theory , Verlag Italia, Milano 2008

[13] Sambath M., Balachandran K. Bifurcation analysis of a diffusive predator-prey model with Beddington-DeAngelis response:2015; vol 22, 263-280

[14] Song YL, Zou XF. Bifurcation analysis of a diffusive ratio-dependent predatorprey model. Nonlinear Dynamics 2014: vol 78:49-70.

[15] Souza M.O.; A derivation of Hollings type I, II and III functional responses in predator-prey systems 2011;

[16] Tanner JT. The stability and the intrinsic growth rates of prey and predator populations.Ecology 1975;vol 56:855-867.

[17] Venturino E, Petrovskii S. Spatiotemporal behavior of a prey-predator system with a group defense for prey.Ecological Complexity 2013; vol 14:37-47.

[18] Walter G. Kelley and Allan C. Peterson, The Theory of Differential Equation: 2010.

[19] Wang J. , Junie Wei, Junping Shi .Global bifurcation analysis and pattern formation in homogeneous diffusive predator-prey systems: Journal of differential equation. 2016; vol 260 3495-3523

[20] Xu Z. and Song Y. Bifurcation analysis of a diffusive predatorprey system with a herd behavior and quadratic mortality. 2015; vol 38:2994-3006

[21] Zhang JF, Li WT, Yan XP. Hopf bifurcation and turing instability in spatial homogeneous and inhomogeneous predator-prey models.Applied Mathematics and Computation 2011; vol 218:1883-1893.

[22] Yi FQ, Wei JJ, Shi JP. Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system.Journal of Differential Equations 2009; vol 246:1944-1977.

[23] Yuan SL, Xu CQ, Zhang TH. Spatial dynamics in a predatorprey model with herd behavior.2013; vol 23: 0331021-03310210.

[24] Zhou J. Bifurcation analysis of a diffusive predator-prey model with ratio-dependent Holling type III functional response. Non linear dynamics 2015; vol 81: 1535-1552.

[25] Zuo WJ, Wei JJ. Stability and Hopf bifurcation in a diffusive predatoryprey system with delay effect.Nonlinear Analysis: Real World Applications 2011;
vol 12:1998-2011.