SUMS OF EVEN POWERS IN RATIONAL FUNCTION FIELDS OVER THE FIELD OF REAL NUMBERS: THE CASE OF FOURTH POWERS

Lemi Hailu Demessa

A Thesis Submitted to the Department of Applied Mathematics

College of Applied Natural Science

Presented in Partial Fulfillment of the requirement for the Degree of Master's in Mathematics(Algebra)

Office of Graduate Studies

Adama Science and Technology University

June, 2025

Adama, Ethiopia

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Lemi Hailu Demessa

Major Advisor: Natea Hunde (PhD)

Co-advisor: Dinkayehu Mengesha (PhD)

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Declaration

I hereby declare that this master thesis entitled ***"Sums of even powers in rational function fields over the field of real numbers: The case of fourth powers"*** is my own work and has not been submitted to any University for similar purpose. The references used in this thesis are dualy recognized by proper citations.

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We, the major advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled ***"Sums of even powers in rational function fields over the field of real numbers: The case of fourth Powers"*** prepared under my guidance by **Lemi Hailu Demessa** submitted in partial fulfillment of the requirements for the degree of Master's of Science in Applied Mathematics. Therefore, we recommend the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled ***"Sums of even powers in rational function fields over the field of real numbers: The case of fourth powers"*** and developed by **Lemi Hailu Demessa**, hereby certify that the recommendation and suggestions made by the board of examiners are appropriately incorporated into the final version of the thesis.

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We, the undersigned, members of the Board of Examiners of the thesis by **Lemi Hailu Demessa** have read and evaluated the thesis entitled ***"Sums of even powers in rational function fields over the field of real numbers: The case of fourth powers"*** and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics(Algebra).

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Abstract

This study focus on sums of even powers in field of rational function over the field of real numbers; specifically sums of fourth powers. The main idea of the study extended from Hilbert's seventeenth problem. Hilbert's Seventeenth problem state that "Is it possible to express a multivariate polynomial that takes only non-negative values over the reals as a sum of squares of rational functions"?. In 1927, Artin demonstrated that a real polynomial, which is positive semidefinite, can indeed be represented as a sum of squares of rational functions, thereby resolving Hilbert's 17th problem. The natural extension of Hilbert's seventeenth problem to include sums of even powers has been explored by several mathematicians during the mid-20th century. In 1981, Becker established conditions that allows non-negative rational functions to be expressed as sums of 2m-th powers, where m *is a positive integer and for a real function field* F *in one variable over a real closed field, he established that* $P_4(F) \leq 36$*. In 1996, Choi, Lam and Reznick showed that for any formally real field* $\mathbb{R}$*, any sums of fourth power in* $\mathbb{R}(x)$ *can be represented as at most 6 fourth powers of rational functions using geometric argument. This research aims to investigate the criteria for expressing rational functions or polynomials as sums of fourth powers and to offer an alternative proof for the effective upper bound of such sums.*

keywords: Hilbert's Seventeenth Problem; Holomorphy Ring; Positive Semidefinite; Sum of Square; Totally Positive Units; Valuations.

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Chapter One
Introduction

1.1 Background of the study

David Hilbert was the best influential German mathematician and philosopher of mathematics at his time. Hilbert made significant contributions by discovering and developing a wide collection of essential concepts, such as invariant theory, the calculus of variations, commutative algebra, algebraic number theory, the foundations of geometry, spectral theory of operators and its use in integral equations, mathematical physics, and the foundations of mathematics (Weyl, 1944). At the International Congress of Mathematicians in Paris in 1900, David Hilbert introduced 23 important and productive problems that laid the groundwork for mathematical investigation in the 20th century. They have a profound impact on the expansion of mathematics in the 20th century, inspiring countless researchers and shaping the direction of many mathematical fields(Pfister, 1995).

Thus Hilbert's problems are a collection of 23 mathematical problems posed by the German mathematician David Hilbert(Hilbert, 2021) in 1900. Hilbert's problems form a foundational and historically important framework in the field of mathematics. Speaking on August 8 at the Sorbonne, Hilbert introduced 10 of the problems: 1, 2, 6, 7, 8, 13, 16, 19, 21, and 22 during the International Congress of Mathematicians conference in Paris. Out of the 23 problems, the mathematical community has reached a consensus on the solutions for problems numbered 3, 7, 10, 14, 17, 18, 19, and 20. Solutions for problems 1, 2, 5, 6, 9, 11, 12, 15, 21, and 22 have been partially accepted, yet there is some debate about whether they truly address the problems(Hilbert et al., 2000). Most Hilbert problems have been solved, but a few are still open (e.g. Hilbert 8th problem called the Riemann hypothesis (problem of prime numbers), 13th and 16th problem are unresolved)(Pfister, 1976).

Hilbert's problems represent a monumental collection of challenges that have shaped the landscape of mathematics. They continue to inspire new research and remain a testament to the power and beauty of mathematics(Kosinski, 2003). Two of Hilbert's problems belong to the area of real algebraic geometry, namely problem number 16 (still largely open and again included in his list of problems) and problem 17 (expression of definite forms by square)(Plaumann, 2023). This research examines the expansion of Hilbert's Seventeenth Problem to include sums of even powers within the rational function field over the real number field. Hilbert's 17th problem asks "Is it possible to express a multivariate polynomial, which only take non-negative values over the real numbers, as a sum of squares of rational functions?"(Devanur et al., 2005).

The 17th problem is described by Hilbert as follows: "A rational integral function or form with real coefficients, involving any number of variables, is deemed definite if it never takes on negative values for any real values of these variables. The set of all definite forms remains consistent under the operations of addition and multiplication. However, if the ratio of two definite forms results in an integral function of the variables, it is also considered a definite form. It is evident that the square of any form is invariably a definite form. Nevertheless, as I have shown, not every definite form can be constructed by summing squares of forms. This raises the question, which I have affirmatively resolved for ternary forms, of whether every definite form can be represented as a quotient of sums of squares of forms. It is crucial, for certain inquiries regarding the feasibility of specific geometrical constructions, to ascertain whether the coefficients of the forms used in the expression can always be selected from the rational domain defined by the coefficients of the represented form"Devanur et al. (2005).

Hilbert's seventeenth problem is a profound consequence in the theory of polynomials and their representations. The study of the relationship between sum of squares and positivity goes back at least to Lagrange who proved that all positive integer is a sum of 4 squares of integers(Delzell, 1980). The fundamental framework for researching positivity in algebraic fields is provided by Hilbert's 17th problem, which highlights the intricate interactions between positive polynomials, sums of squares, and quadratic forms(Alexander & Delzell, 2013). Some of the tools and techniques developed to address H17, such as quadratic forms, real holomorphy rings, and valuations, have direct applications in the study of sums of even powers of non negative rational functions that result from them.

Hilbert's 17th problem finds positive semidefinite elements in fields that are associated with completely positive elements. These components are essential for researching even power amounts(Schmid, 1994b). The notion of positivity is associated with formally real field orderings. Research on completely positive elements and units in real holomorphy rings is a fundamental aspect of contemporary algebraic and arithmetic studies. Understanding these structures is crucial for comprehending the characteristics of formally real fields, specially quadratic forms and associated invariant. Examining the sums of even powers can be regarded as an extension of Hilbert's seventeenth problem, to powers of a higher degree. Hilbert's 17th problem is concerned with expressing non-negative elements as sums of squares(B. A. Reznick, 1992). Sums of even powers deal with expressing non-negative polynomial or rational functions as sums of higher even powers. It is crucial for understanding the structure of formally real fields and the algebra of positivity. This study mainly focus on sums of even power in rational function field $\mathbb{R}(x)$, particularly the case of fourth powers .

1.2 Preliminaries

1.2.1 Ordered field and real closed field

Consider a field K; an ordering of K is defined as a total order relation $\leq$ on the set K that fulfills

$$x \leq y \Longrightarrow x + z \leq y + z \quad \& \quad x \leq y, 0 \leq z \Longrightarrow xz \leq yz$$

for every $x, y, z \in K$ (Scheiderer, 2024). An ordered field is defined by a couple $(K, \leqslant)$, where K signifies the field and $\leqslant$ represents the ordering within K. This ordering is a total order on the set K, compatible with both addition and multiplication, like to the ordering of real numbers. Thus, an ordered fields are essentially a field that incorporates a total order of their elements, which aligns with the field's operations. An ordered field must have characteristic 0, because finite fields cannot be ordered. Finite fields cannot be ordered because they have positive characteristic. A field K is considered real if it can be ordered in at least one way. A field is considered as formally real if -1 is not a sum of squares in it(Noquez, 2008). A formally real field is a kind of field that is essential to ordered fields, real algebra, and real algebraic geometry. Every field that is ordered is formally real, and every field that is formally real is also ordered(Schwarzweller, 2017). Real algebraic numbers, the field of real numbers, and the field of rational numbers are, examples of formally real fields. The field of real numbers ($\mathbb{R}$) are the subject of this investigation. Since -1 can be expressed as sum of square in $\mathbb{C}$ is the complex field $\mathbb{C}$ is not ordered field.

Real closed fields are those fields that possess the same algebraic characteristics as the field of real numbers(Basu, 2014). It is a type of field that has properties similar to the real numbers, particularly in terms of order and the existence of roots for polynomials. A field K is termed real closed if it is formally real and does not possess any proper algebraic extension field that is also formally real(Noquez, 2008). An algebraic extension L of an ordered field K is considered a real-closure if it is real closed and its (unique) ordering is an extension of the ordering defined on K(Safar, 1959). Real closed fields are crucial in logic, model theory, and real algebraic geometry. They provide an ideal setting for studying polynomial equations and inequalities over ordered fields. In general, a field is considered real closed if it is ordered, all positive element has a square root, and all polynomial with odd degree has a root. Real closed fields form a special category of ordered fields that play a key role in solving Hilbert's seventeenth problem(Fernando & Gamboa, 2012).

1.2.2 Positive semidefinite Polynomials

Positive Semidefinite (PSD) polynomials , are a class of polynomials that take non-negative values for all inputs from a certain domain, typically for all real numbers or for specific regions in the Euclidean space. The concept of positive semidefiniteness is important in various fields, including optimization, control theory, and statistics. This is a key result from real algebraic geometry. A polynomial $f \in \mathbb{R}[x]$ is considered positive semidefinite if it satisfies the condition $f(x) > 0$ for every $x \in \mathbb{R}^n$(Goel, 2014). The property of being positive (or non-negative) is a significant aspect in mathematical proofs and derivations. For example, a polynomial $f(x) = ax^2 + bx + c$, considered as Positive Semidefinite if its leading coefficient $(a > 0)$and its discriminant $(b^2 - 4ac \leq 0)$. A quartic polynomial in two variables, denoted as $(P(x, y))$, is considered positive semidefinite if it remains non-negative for every real value of (x) and (y). This means that $(P(x, y) > 0)$ for all points $((x, y) \in \mathbb{R}^2)$. Positive semidefinite polynomials are widely used in fields such as optimization, real algebraic geometry, group theory, and nonlinear analysis(Berner, 2018).

1.2.3 Sum of Squares

A Sum of Squares (SOS) refers to expressing a polynomial as the sum of the squares of other polynomials or rational functions. A polynomial $f \in \mathbb{R}[x]$ is a sum of square, if there are $g_1, g_2, ..., g_k \in \mathbb{R}[X]$ such that $f = g_1^2 + g_2^2 + ... + g_k^2$ (Powers, 2021). Expressing a PSD polynomial f as a sum of squares of polynomials or rational functions serves as a proof of its non-negativity, confirming that f is indeed positive semidefinite. Every positive semidefinite quadratic form, in any number of variables, is a sum of squares(Davidek, 2010).
Consider a polynomial $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}, a \neq 0$. Completing the square or diagonalization gives explicit sum of square forms(Powers, 2011). Assume that $a = 1$, then completing the square we have:

$$f(x) = \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4}$$

We know that squares of real numbers are always non-negative. Thus we conclude that $f(x)$ has positive evaluations if and only if $4c - b^2$ is non-negative. Therefore, if $4c - b^2 \geq 0$,

$$f(x) = \left(x + \frac{b}{2}\right)^2 + \left(\sqrt{\frac{4c - b^2}{4}}\right)^2$$

shows that f is a sum of squares of polynomials. Any sum of squares of rational functions, which are not necessarily polynomials, is a positive semidefinite polynomial. If a real polynomial f in n variables can be written as a sum of squares of real polynomials, then clearly f must take only non-negative values in $\mathbb{R}^n$(Powers, 2015).

1.3 Statement of Problem

The classical mathematical problem of representing non negative elements in a field as sums of specific powers is closely linked to Hilbert's 17th problem and the theory of quadratic forms. Although there has been considerable advancement in understanding sums of squares in real fields, including $\mathbb{R}(x)$, the exploration of sums of higher even powers, like fourth powers, is still relatively underdeveloped. This prompts several important questions, such as whether all positive semidefinite rational functions can be represented as sums of fourth powers of rational functions, how many fourth powers are required to represent any positive element , and how these properties relate to or differ from sums of squares.

Several experts investigate sums of even powers in various field and find the effective bounds on sum of fourth powers in a formally real field K. In 1981, Becker demonstrated that for a formally real field K, the sets $P_{2n}(K)$ are finite if $P_2(K)$ is finite. When considering a real function field F in one variable over a real closed field, he established that $P_4(F) \leq 36$. This finding was later refined by the author in 1988, reducing the bound to $P_4(F) \leq 24$. In 1994, Schmid showed that for a formally real field K, where 3 is a square in K and $P_2(K) = 2$, the inequality $P_4(K) \leq 6$ is satisfied, using units from the real holomorphy ring. In 1996, Choi, Lam, Prestel, and Reznick showed that, for any real closed field $\mathbb{R}$, $P_4(\mathbb{R}(x)) \leq 6$ through geometric argument. This research aims to offer an alternative proof that, for any real closed field $\mathbb{R}$, every non-negative rational function can be expressed as a sum of at most 6 fourth powers, using totally positive units in the real holomorphy ring.

1.4 Objectives

1.4.1 General Objectives

The general objective of this study is to investigate the sums of fourth powers of $\mathbb{R}(x)$.

1.4.2 Specific Objectives

The specific objectives of this study are to:

- analyze conditions for $\mathbb{R}(x)$ fourth powers to be positive semi definite.
- examine the structure of totally positive units in the real holomorphy rings of formally real field.
- give alternative proof for effective bounds of sums of fourth powers of $\mathbb{R}(x)$ using totally positive units of holomorphy ring.

1.4.3 Significance of the Study

This study may:

- adopted as a reference material for researchers who works on this area.
- motivate other researchers to find elementary proof of Hilbert's seventeenth problem.
- motivate other researcher to work on sums of even powers of higher order.

1.4.4 Expected outcome of the study

The expected outcome of this study is to contribute new theoretical insights and tools for analyzing sums of even powers.

Chapter Two
Literature Review

2.1 Hilbert's Seventeenth Problem

Hilbert's effort on sums of squares and positive polynomials paved the path for contemporary real algebra, but some aspects existed before to his time. In his Inaugural Dissertation on quadratic forms in July 1885, Minkowski was the first to suggest that a positive semi definite form might not be necessarily sum of squares. Hilbert, who was one of his official "opponents," commented that Minkowski's arguments had persuaded him that this thesis could be valid for ternary forms (Prestel et al., 2001). Hilbert was supposed to criticize this dissertation during its public defense, but he concluded by saying that he "was convinced by Minkowski's exposition that already for $n = 3$ there may well be such remarkable forms, which are so stubborn as to remain positive without allowing themselves to submit to a representation as sums of squares of forms"(Safar, 1959).

Minkowski's "conjecture," which states that a polynomial in a PSD real variable cannot be expressed as a sum of squares of polynomials, was demonstrated by Hilbert in 1888(Alexander & Delzell, 2013). In 1888, Hilbert also studied further and considered the problem of representing any Positive semidefinite, $f \in \mathbb{R}[x_1, ..., x_n]$ as sum of element of $\mathbb{R}(x_1, ..., x_n)$. He proved that this does happen for $n = 2$ (Safar, 1959). He state that let $f \in \mathbb{R}[x_1, x_2]$ be positive semidefinite. Then, f is a sum of 4 squares in $\mathbb{R}(x_1, x_2)$. In the same publication, he showed that any positive semidefinite ternary quartic homogeneous polynomial of degree 4 in three variables can be expressed as a sum of squares. Hilbert managed to establish that for $n = 2$, every positive semidefinite form is a sum of squares of rational functions, but he could not prove this for n greater than 2.

This challenge became the 17th problem on his renowned list of 23 mathematical problems, which he presented at the 1900 International Congress of Mathematicians in Berlin(Scheiderer, 2009). Hilbert's 17th Problem explores the connection between positivity, a geometric trait, and sums of squares, which are algebraic expressions. Regardless of the exact definition of positivity, it is evident that sums of squares are inherently positive. Consequently, it is logical to inquire under what conditions these sums are the sole positive elements within a ring of functions. A pertinent question arising from Hilbert's seventeenth problem is why the focus is on expressing semidefinite polynomials as sums of squares of rational functions rather than as sums of squares of polynomials(Fernando & Gamboa, 2012).

The exploration of the connection between non-negativity and sums of squares was first conducted in Hilbert's 1888 fundamental paper. In 1888 Hilbert also described there is PSD polynomials that are not sums of squares, but there was no concrete example until the late 1960s when Motzkin published one(Schabert, 2019). In real algebraic geometry, the study of non-negativity and its relationship to sums of squares is a classic topic that began with Hilbert's work, which categorized the cases where non-negative polynomials are always SOS of polynomials according to degree and number of variables(Blekherman, 2020).

Hilbert demonstrated that all non-negative polynomials are sums of squares of polynomials only in three situations: univariate polynomials, quadratic polynomials and bivariate polynomials of degree 4(Blekherman, 2012). For univariate polynomials, Hilbert's seventeenth problem confirms that any non-negative polynomial may be written as the sum of squares of other polynomials(Iliman, 2014). This result is foundational in the field of real algebraic geometry. For univariate polynomials, every non-negative polynomial can indeed be expressed as a sum of two squares of polynomials.

By combining algebraic methods with polynomial properties, we can demonstrate that each non-negative univariate polynomial may be represented as a sums of two squares of polynomials. Since the PSD polynomial's graph is whole above or touches the x-axis, its roots are either real with even multiplicity or complex in conjugate pairs. Thus the roots have the form $a_j \pm ib_j$ and we can write such polynomial as

$$f(x) = [(x - [a_1 + ib_1]) \ldots (x - [a_n + ib_n])] \ldots [(x - [a_1 - ib_1]) \ldots (x - [a_n - ib_n])]$$

$$= [q_1(x) + iq_2(x)] \, . \, [q_1(x) - iq_2(x)] \quad = \quad q_1(x)^2 + q_2(x)^2.$$

Obtaining this decomposition is challenging, as we typically cannot determine the precise form of the roots(Davidek, 2010).

A degree two, homogeneous polynomial with several variables is called a quadratic form. In other words, it's a mathematical expression where each term is a product of two variables, multiplied by a constant coefficient. One can express a quadratic polynomial with n variables as follows:

$$f(x_1, x_2, ..., x_n) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + ... + a_{nn}x_n^2 = \sum_{i,j=1}^{n} a_{ij}x_ix_j$$

where $x_1, x_2, ..., x_n$ are variable and a_{ij} are the constant coefficient and $a_{ij} = a_{ji}$ shows the matrix of coefficients is symmetric. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be symmetric matrix, if $A^T = A$. If $f(x_1, x_2, ..., x_n) \geq 0$ for all real values of $x_1, x_2, ..., x_n$, then $f(x_1, x_2, ..., x_n)$ can be represented as a sum of squares of linear forms with rational coefficients. The proof relies on the concept of Sylvester's law of inertia. Sylvester's law of inertia states that for a symmetric matrix A, the number of positive, negative, and zero eigenvalues is independent of the choice of basis(Roman et al., 2005). This law is the key to understanding when a quadratic polynomial can be expressed as a sum of squares. We can represent a quadratic form more compactly using a symmetric matrix:

$$f(x) = x^T A x$$

Where x is a column vector containing the variables: $x = [x_1, x_2, ..., x_n]^T$, A is the symmetric coefficient matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & & \\ . & . & & \\ . & . & & \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$$

and x^T is transpose of x. For $x \in \mathbb{R}^n$, $||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, denote the norm of x. When doing matrix computations, we will always view x as a column vector(Marshall, 2008). This means,

$$x^T x = \begin{pmatrix} x_1 & x_2 & .... & x_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + ... + x_n^2 = ||x||^2.$$

A quadratic form can be expressed as a sum of squares of linear forms if and only if it's corresponding matrix has no negative eigenvalues. The matrix $A \in \mathbb{R}^{n \times n}$ is PSD if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$(Zhou, 2022). This means all eigenvalues of A are non negative. Suppose $f \geq 0$ on $\mathbb{R}^n$, then

$$f(x) = x^T A x = x^T (U^T U) x = (Ux)^T (Ux) = ||Ux||^2 \geq 0$$

with x viewed as a column vector, f is a sum of squares of linear forms. $A = U^T U$ has a special name Cholesky decomposition. Cholesky decomposition state that for any positive definite symmetric matrix A, there exists a unique upper triangular matrix U (with positive diagonal elements) such that: $A = U^T U$ (Higham, 1990). Hilbert then showed that every bivariate non-negative polynomial is a sum of squares of rational functions(Blekherman, 2012). In all other cases Hilbert showed the existence of non-negative polynomials that are not sums of squares of polynomials. It is obvious that f must only take non-negative values in $\mathbb{R}^n$, if a real polynomial f in n variables can be expressed as a sum of squares of real polynomials. If f is a SOS of polynomials, then f is PSD (Powers, 2011).

The proof of Hilbert's 17th problem is complex and involves advanced concepts from algebraic geometry, real algebra, and the theory of polynomials Bochnak et al. (2013). The

17th problem was solved by E.Artin in 1926 in the affirmative (Niculescu & Stephan, 2012). He proved that a polynomial with n variables and real coefficients, where all evaluations are non-negative, can be expressed as a sum of squares of rational functions in n variables with real coefficients. He demonstrated this as an existence theorem. His extraordinary proof initiated the new field of model theory, and today, his argument is considered a specific instance of Tarski's transfer principle. The proof also highlighted the real spectrum of rings like $R[x_1, ..., x_n]$ and laid the groundwork for the field of real algebraic geometry. The proof is applicable to more general 'real closed' fields, which were examined by Artin and Schreier in a paper within the same volume where Artin's resolution of Hilbert's seventeenth problem was published(Safar, 1959).

Emil Artin answered Hilbert's 17th problem by applying the Artin-Schreier theory of real closed fields. Artin Schreier theory state that a field K is formally real if and only if K has an order(Jacobson, 1964). A non negative polynomial over a real closed field is a sum of squares of rational functions(Bochnak et al., 2013). If we choose $\mathbb{C}$ instead of $\mathbb{R}$, we can see that each polynomial in $\mathbb{C}[x1, x2, ..., xn]$ is a sum of squares of rational functions(Scheiderer, 2016). Since a polynomial with an odd degree changes its sign, and the homogenization of a polynomial takes only non-negative values, Hilbert's question is applicable solely to homogeneous polynomials of even degree. According to Artin's resolution of Hilbert's seventeenth problem, a multivariate polynomial f is non-negative over the real numbers if and only if it can be expressed as a SOS of rational functions(Prestel & Delzell, 2013).

In 1927 Artin proof a theorem "for every non-negative polynomial $f \in \mathbb{R}[x_1, ..., x_2]$, there exist rational functions $g_1, g_2, ..., g_s \in \mathbb{R}(x_1, ..., x_n)$ such that $f^2 = g_1^2 + g_2^2 + ... + g_s^2$ (Devanur et al., 2005)". Artin's demonstration of this theorem played a pivotal role in advancing real algebra. Working alongside Schreier and considering Hilbert's 17th problem, he developed the theory of ordered fields. Their research led to the conclusion that an element in a field K can be expressed as a sum of squares in K if and only if it is non-negative with respect to all orderings of K that align with the field's structure. It is necessary to demonstrate that if f is negative concerning some ordering of $\mathbb{R}(x_1, ..., x_n)$, then its value at a certain point $(x_1, ..., x_n) \in \mathbb{R}^n$ is also negative.

Artin's proof is composed of several elements: a specialization argument, Sturm's Theorem, and the ordered fields theory developed by Artin and Schreier, which is specifically applied to address H17. In 1953, Lang revisited Artin's proof and proposed a similar solution, substituting the specialization argument with the theory of real places(Fernando & Gamboa, 2012). In 1955, Robinson introduced a completely new solution using model theoretical techniques(Robinson, 1955). Actually, he showed what is known as Artin-Lang's Theorem, which is a result of Tarski's 1949 discovery of the quantifier elimination theorem for real closed fields(Fernando & Gamboa, 2012). The ideas and mathematical instruments employed in the different solutions of Hilbert's 17th problem became essential to the expansion of a new branch of mathematics that is now known as Real Algebra and Real Geometry. It should be noted that Hilbert did not give an explicit non-negative polynomial that is not a sum of squares of polynomials(Blekherman, 2012). In 1967 Motzkin puplished his first example of a psd polynomial which is not a sum of squares of other polynomials(Scheiderer, 2009). This polynomial, now known as the Motzkin polynomial, is given by $M(x,y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$. Specifically, it shows that not all PSD polynomials can be expressed as sums of squares of other polynomials. To show $M(x,y)$ is positive semidefinete or non-negative we use Arithmetic-Geometric Mean Inequality. In the field of algebra, the AM-GM inequality, formally referred to as the Inequality of Arithmetic and Geometric means or informally as AM-GM , asserts that the arithmetic mean of any set of non-negative real numbers is always greater than or equal to their geometric mean(Israel et al., 2016). The inequality states that for any real numbers, $x_1, ...x_n \geq 0$

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}$$

where the equality holds if and only if all the x_i are equal(Toussaint, 2005). In addition, the two means are identical if and only if every number in the set is identical. Then by the Arithmetic-Geometric Mean inequality we have

$$\frac{x^4y^2 + x^2y^4 + 1}{3} \geq \sqrt[3]{x^4y^2 \cdot x^2y^4 \cdot 1} = x^2y^2,$$

hence $x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geq 0$, which implies $M(x,y)$ is non-negative for all $(x,y) \in \mathbb{R}^2$.

To show that Motzkin Polynomial is cannot be represented as sum of squares, we need to make the following observation(Zhou, 2022): "suppose $M(x,y) = 1 + x^2y^2(x^2+y^2-3) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 = \sum_{i=1}^{m} h_i(x,y)^2$ can be expressed as a SOS,then $h_i(x,y) = a_ix^2y + b_ixy^2 + c_ixy + d$ with $a_i, b_i, c_i \in \mathbb{R}$. To make this observation, we notice that for any two terms $a = x^{i_2}y^{j_2}$ and $a = x^{i_1}y^{j_1}$,$ab = x^{i_1+i_2}y^{j_1+j_2}$ Hence, we know h_i would be expressive enough if $M(x,y)$ can be written in a sum of squares, as we can see every term x^iy^j in $M(x,y)$, there exists a term $x^{\frac{i}{2}}y^{\frac{j}{2}}$ in $h_i(x,y)$. Now, we can see that the coefficients of the term x^2y^2 in $\sum_{i=1}^{m} h_i(x,y)^2$ equals to $\sum_{i=1}^{m} c_i^2$ yet in Motzkin's polynomial the coefficient of the term is -3, a contradiction. Hence, we conclude that Motzkin's polynomial is not a SOS".

Motzkin's polynomial is the best-known example of a polynomial that is positive on $\mathbb{R}^2$ but cannot be written as a sum of squares of polynomials(Motzkin, 1967). Every PSD polynomial is a sum of squares of rational functions according to Artin's theorem. Therefore, the Motzkin polynomial can be expressed as:

$$M(x,y) = \frac{x^2y^2(x^2+y^2-2)^2(x^2+y^2+1) + (x^2-y^2)^2}{(x^2+y^2)^2}$$

$$= \frac{x^4y^2(x^2+y^2-2)^2}{(x^2+y^2)^2} + \frac{x^2y^4(x^2+y^2-2)^2}{(x^2+y^2)^2} + \frac{(x^2+y^2-2)^2}{(x^2+y^2)^2} + \frac{(x^2-y^2)^2}{(x^2+y^2)^2}$$

$$= \left(\frac{x^2y(x^2+y^2-2)}{(x^2+y^2)}\right)^2 + \left(\frac{xy^2(x^2+y^2-2)}{(x^2+y^2)}\right)^2 + \left(\frac{(x^2+y^2-2)}{(x^2+y^2)}\right)^2 + \left(\frac{(x^2-y^2)}{(x^2+y^2)}\right)^2.$$

.

This show that M is a sum of 4 squares of rational functions with real coefficients(Marshall, 2008).

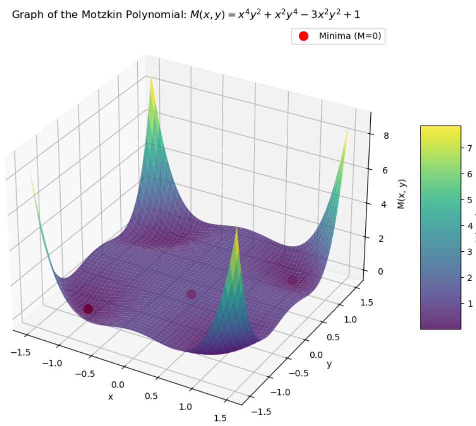

Figure 2.1

The 7 regions demonstrates how non negativity can be achieved without SOS decomposition. Following the identification of Motzkin's polynomials, numerous efforts have been made to discover additional non-negative polynomials that cannot be expressed as a sum of squares(B. Reznick, 2007). For example, In 1977, Choi and Lam offered two examples of non-negative polynomials that cannot be written in the form of sum of squares(B. Reznick, 2007). We call those polynomial Choi-Lam polynomial:

$$P(x, y, z) = x^2y^2 + y^2z^2 + z^2x^2 + 1 - 4xyz$$

$$p(x, y, z, w) = x^2y^2 + y^2z^2 + z^2x^2 + w^4 - 4wxyz$$

are positive semidefinte polynomials but cannot be represented as sum of square of other polynomials.

The exploration of the quantitative aspect of Hilbert's 17th problem concerns the number of squares needed(Bochnak et al., 2013). Hilbert's theorem from 1888 states that every positive semidefinite (PSD) ternary quartic, which is a homogeneous polynomial of degree 4 in three variables, can be expressed as the sum of three squares of quadratic forms. However, Hilbert's proof is non-constructive, meaning it does not provide answers to the following questions: How can one identify three such quadratic forms for a given PSD ternary quartic?, how many fundamentally distinct methods exist for achieving this?. Powers and Reznick outline techniques for identifying and enumerating representations of a PSD ternary quartic, and they fully address these studies for certain particular cases(Powers, 2011).

In 1967, Albrecht Pfister demonstrated that a positive semidefinite polynomial involving (n) variables can be represented as a sum of (2^n) squares(Devanur et al., 2005). For examples Motzkin polynomials is the sum of $2^2 = 4$ square of rational function. Pfister's proof is a powerful demonstration that connects algebraic forms with geometric interpretations. Finding that any positive semidefinite form involving n variables can be represented as a sum of 2^n squares is not just a theoretical invention; it also has significant applications in fields like optimization, algebraic geometry, and real algebraic geometry.

In 1981, Eberhard Becker expanded the Artin-Schreier theory of fields, extending the connection between sums of squares and the orderings of a real field to encompass sums of 2m-th powers. Becker's shows that the simple polynomial $p_0 = y^4 + ty^2 + 1$ viewed as a one parameter family of elements in $\mathbb{R}[y]$, reveals that for each fixed positive real t, this is an element of $\mathbb{R}[y]$ which is a sum of fourth powers of rational functions(Stengle, 1994). In 1981, Becker also established necessary and sufficient criteria for a rational function f over a formally real field K, to be a sum of 2m-th powers of rational functions. Thus Becker extends Hilbert's 17th problem to sum of even powers (Gondard et al., 1994). He also find the effective upper bound of sums of fourth power.

Chapter Three
Valuations and Holomorphy Rings

In this chapter, we present the fundamental ideas of valuations and real holomorphy rings which are important to express non negative rational functions or polynomials as sums of even powers in $\mathbb{R}(x)$.

3.1 Valuations and Its properties

3.1.1 Valuations

(Salvador, 2006) An ordered abelian group is an abelian group $(G, +, 0)$, together with a binary relation $\leq$ on G, satisfying the following axioms: for all $\gamma, \delta, \lambda \in G$.

- $\gamma \leq \gamma$
- $\gamma \leq \delta, \delta \leq \gamma \Rightarrow \gamma = \delta$
- $\gamma \leq \delta, \delta \leq \lambda \Rightarrow \gamma \leq \lambda$
- $\gamma \leq \delta$ or $\delta \leq \gamma$
- $\gamma \leq \delta \Rightarrow \gamma + \lambda \leq \delta + \lambda$

With the sum and the standard order, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are the most evident examples of ordered abelian groups. Let G be an ordered abelian group and K be a field, then a valuation V on a field K is a surjective map $V : K \longrightarrow G \cup \{\infty\}$ satisfying the following axioms: for all $x, y, \in K$

$$(1) \qquad V(x) = \infty \Rightarrow x = 0$$

$$(2) \qquad V(xy) = V(x) + V(y)$$

$$(3) \qquad V(x+y) \geq min\,\{V(x), V(y)\}$$

where ∞ is a symbol(the value of 0) that satisfies $\infty = \infty + \infty = \gamma + \infty = \infty + \gamma$, for all $\gamma \in \mathbb{R}$ (Salvador, 2006). The inclusion of the symbol ∞ serves the sole purpose of defining $V(0)$ in a manner that also fulfills conditions (2) and (3) of the definition. If $G = \{0\}$, we call V the trivial valuation(Engler & Prestel, 2005). When V maps onto $G = \mathbb{Z}$ in a surjective way, it is considered a discrete valuation. In order to compare elements based on their sizes, valuations offer a consistent method of assigning sizes to a field's elements(Chaiser, 2017). A field that includes a valuation is known as a valued field(Kuhlmann, 2004).

Definition 1 (Real valuations). *(Gondard-Cozette, 2023) A valuation V on a field K is said to be real if and only if the residue field K_v is real (meaning $-1 \notin \sum K_v^2$).*

Real valuations are functions from fields to real numbers that give field members a "size" or "multiplicity" while meeting specific criteria. They are a generalization of absolute values for real numbers. A valuation V on K is real if and only if for any $r \geq 1$ and $x_1, ..., x_r \in K$ the equality $V(x_1^2 + ... + x_r^2) = 2min(V(x_1), ..., V(x_r))$ holds (Becher & Leep, 2013). There are many types of real valuation on a field K. some of them are trivial valuation, p-adic valuation and degree valuation.

Definition 2 (P-adic valuation). *(Salvador, 2006) If $K = \mathbb{Q}$ and $G = \mathbb{Z}$, for a prime number p and $x \in \mathbb{Q}$ a valuation $V_p : \mathbb{Q} \longrightarrow \mathbb{Z}$ defined by $V_p(x) = n$, where $x = p^n \frac{a}{b}$, $a, b, n, \in \mathbb{Z}$, and $p \nmid a, p \nmid b$ is called P-adic valuation.*

In this expression p is prime number or irreducible polynomials. The p-adic valuation is a function that measures the divisibility of a rational number by a prime p. Any non-trivial valuation on the set of rational numbers $\mathbb{Q}$ corresponds to a p-adic valuation for some prime number p. Therefore valuation $V(x)$ can be thought of as measuring how many times a prime element divides x.

Definition 3 (Degree Valuation). *(Engler & Prestel, 2005) The degree valuation is a valuation defined on the field of rational functions $K(x)$, where K is a field. Let $K(x)$ be the field of rational functions over a field K. The degree valuation $V : K \longrightarrow \mathbb{Z} \cup \{\infty\}$ is defined as follows:*
For a rational function $\frac{f(x)}{g(x)}$, where $f(x), g(x) \in K[x]$ and $g(x) \neq 0$, the degree valuation is defined as: $V_\infty \left(\frac{f(x)}{g(x)} \right) = deg(f) - deg(g)$. This implies that $V_\infty = \left\{ \frac{f}{g} | deg(f) \leq deg(g) \right\}$.

Degree valuation measures the "degree" of a rational function, which is related to the behavior of the function at infinity. It is a specific type of valuation that is often used in the background of polynomial rings or function fields. It is strongly associated with the idea of a polynomial's degree.

Properties of Valuations

(Salvador, 2006) Let K represent a field and V represent a valuation. Then, for every $x, y \in K$:

1). $V(1) = 0$.

Proof. Let $x \in K$. Then $V(1) = (V(1) + V(x)) - V(x) = V(1x) - V(x) = V(x) - V(x) = 0$. □

2). $V(x^{-1}) = -V(x)$.

Proof. Let $x \in K$. Consider $V(x) + V(x^{-1}) = V(xx^{-1}) = V(1) = 0$. Thus $V(x^{-1}) = -V(x))$. □

3). $V(x^k) = kV(x)$, for $k \in \mathbb{Z}$.

Proof. For $k = 1$, we have $V(x^1) = V(x))$ which is true by the definition of $x^1 = x$. For $k > 1$ assume $V(x^n) = nV(x)$ holds for some $n \geq 0$. Then for $k = n+1$, $V(x^{n+1}) = V(x^n.x) = V(x^n) + V(x)$. Using the inductive hypothesis $V(x^n) = nV(x)$: $V(x^{n+1}) = nV(x) + V(x) = (n+1)V(x)$. Hence $V(x^k) = kV(x)$ holds for all $k \geq 1$. For $k < 0$, let $k = -m$, then $V(x^k) = V(x^{-m}) = V((x^m)^{-1}) = -V(x^m) = -mV(x) = kV(x)$. □

4). $V(x) = V(-x)$.

Proof. From the definition for $x, y \in K$, $V(x+y) \geq \min(V(x), V(y))$. Let $y = -x$. Then $V(x + (-x)) \geq \min(V(x), V(-x)) \implies V(0) \geq \min(V(x), V(-x))$. Substituting $V(0) = \infty$ we get: $\infty \geq \min(V(x), V(-x))$. since $V(x)$ AND $V(-x)$ are finite for $x \neq 0$ this implies: $V(x) = V(-x)$. □

5). $V(x) < V(y) \Rightarrow V(x+y) = V(x)$.

Proof. since $V(x+y) \geq \min\ (V(x), V(y))$ and $V(x) < V(y)$, then $V(x+y) = V(x)$. □

3.1.2 Valuation Ring

Gathmann (2018) Let K be a field, and let V be a valuation on K. The valuation ring associated with V, denoted by R_v, is the set:

$$R_v = \{x \in K : V(x) \geq 0\}$$

Observe that it is indeed a ring: we have $V(0) = \infty \geq 0$ and $V(1) = 0$, and for any $x, y \in K$, it holds that $V(x) \geq 0$ and $V(y) \geq 0$. Consequently, $V(x+y) \geq \min(V(x), V(y)) \geq \min(0, 0) \geq 0$, which implies $x + y \in R_v$. Additionally, $V(xy) = V(x) + V(y) \geq 0$, so $xy \in R_v$. The additive inverse of $x \in K$ is $-x$, and since $V(-x) = V(x) \geq 0$, it follows that $-x \in R_v$. Given that $R_v \subseteq K$, the properties of associativity and distributivity are carried out.

If V represents a discrete valuation, then the ring associated with this valuation is referred to as a discrete valuation ring. Any non-trivial valuation on $K(x)$ that remains trivial on K is either the degree valuation V_∞ or a p-adic valuation corresponding to some irreducible polynomial, $p \in \mathbb{R}[x]$.

Properties of Valuation Rings

(Chaiser, 2017) Valuation rings have many interesting properties. some of them are:

1). For every $x \in K$, either $x \in R_v$ or $x^{-1} \in R_v$.

Proof. . Given that $V(x) + V(x^{-1}) = V(xx^{-1}) = V(1) = 0$, it implies that $V(x) \geq 0$ or $V(x^{-1}) \geq 0$. Consequently, this means $x \in R_v$ or $x^{-1} \in R_v$. $\square$

2). Valuation ring (R_v) is an integral domain.

Proof. Consider $x, y \in R_v$ such that $xy = 0$. This implies $\infty = V(xy) = V(x) + V(y)$. Therefore, it must be that either $V(x) = \infty$ or $V(y) = \infty$, leading to the conclusion that either $x = 0$ or $y = 0$. $\square$

3). The group of units in R_v is defined as $U = \{x \in R_v : V(x) = 0\}$.

Proof. Consider $x \in R_v$ where $x \neq 0$. If x belongs to U, then by definition, x^{-1} must also be in U, implying $x^{-1} \in R_v$. Consequently, $V(x^{-1}) = -V(x) \geq 0$, which can only occur if $V(x) = -V(x) = 0$. $\square$

4). A valuation ring (R_v) has a unique maximal ideal $M_v = \{a \in R_v : V(x) > 0\}$.

Proof. Clearly, $0 \in M_v$, indicating that M_v is not empty. Consider $x, y \in M_v$ and $r \in \mathbb{R}$. Then, $V(x + y) \geq \min\{V(x), V(y)\} > 0$ and $V(xy) = V(x) + V(y) > 0$. Therefore, M_v forms a subring of R_v. Additionally, $V(rx) = V(r) + V(x) > 0$, confirming that M_v is an ideal of R_v. Now, assume I_v is an ideal of R_v such that $M_v \subset I_v$. Consequently, I_v includes an element with valuation 0, which is a unit by 2, leading to $I_v = R_v$. Hence, M_v is maximal. The remaining task is to show that M_v is unique. Notice that M_v comprises all non-units of R_v. Suppose there exists another maximal ideal J_v of R_v. Then J_v must exclude units; otherwise, $J_v = R_V$, implying $J_v \subseteq M_v$. Since both M_v and J_v are maximal, it follows that $J_v = M_v$. $\square$

5). All valuation rings are local ring.

Proof. A ring R is considered local if it possesses exactly one maximal ideal, denoted as M_v. According to property 4, R_v has a singular maximal ideal. Consequently, a valuation ring is indeed a local ring, signifying it has a unique maximal ideal. $\square$

6). A valuation ring is integrally closed.

Proof. Let $x \in K$ and assume that $x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$ is an integral equation for x over integrally closed domain. If x^{-1} belongs to R_v , then, multiplying the integral equation by $(x^{-1})^{n-1}$ and re arranging terms we get $x = -\left(a_{n-1} + a_{n-2}x^{-1} + ... + a_0 (x^{-1})^{n-1}\right)$. So x belongs to R_v(Raghavan, 1995). $\square$

Definition 4 (Real valuation ring)**.** *(Chaiser, 2017) Let R_v is valuation ring and M_v is the maximal ideal, then $R_v/M_v = \overline{K}_v$ is called residue field. Then if R_v/M_v is formally real field, V is real valuation and R_v is real valuation ring. Therefore, real valuation ring is valuation rings with formally real residue class field*

A real valuation ring is a valuation ring associated with real-valued valuation, where the valuation function takes values in real numbers(Engler & Prestel, 2005). The irreducible polynomials over the field of real number are linear and quadratic. Then for linear polynomial; $V_{(x-\alpha)} = V_\alpha = \left\{(x-\alpha)^n \frac{f}{g} | (x-\alpha) \nmid f, (x-\alpha) \nmid g, g(\alpha) \neq 0\right\}$. Then $R_{v_\alpha}/M_v \cong \mathbb{R}(x)$. Then R_{v_α}/M_v formally real field.
For quadratic polynomial; $V_{(ax^2+bx+c)} = V_q = \left\{(ax^2+bx+c)^n \frac{f}{g} | (ax^2+bx+c) \nmid f, (ax^2+bx+c) \nmid g\right\}$. Then $R_{v_q}/M_v \cong \mathbb{C}$. There is no order in $\mathbb{C}$. Thus R_{v_q}/M_v is not formally real field.
Another intriguing valuation on $K[x]$ is the degree valuation (V_∞). The value group for degree valuation is $\mathbb{Z}$, which is a subgroup of $\mathbb{R}$(Engler & Prestel, 2005). Since its value group is a subgroup of the real numbers V_∞ is real valuation. Then R_{v_∞} is real valuation ring.

3.2 Holomorphy Rings

A holomorphy ring is formed by the intersection of valuation rings that share a common quotient field. A holomorphy ring is a unique kind of ring that is linked to a number field, which is a finite extension of the rational numbers $\mathbb{Q}$. A unique kind of holomorphy ring connected to function fields over formally real (orderable) fields is called a real holomorphy ring(Becker & Gondard, 1992). Holomorphy ring is used extensively in the study of formally real fields, especially for quadratic forms and sums of 2n-th powers, and in real algebraic geometry(Powers & Becker, 1996). Let K be a formally real field (-1 is not a sum of squares in K). The real holomorphy ring of K be denoted by $H(K)$ is the intersection of all valuation rings of K with a formally real residue class field(Craven, 1990). Then it can be defined as:

$$H(K) = \bigcap R_v$$

where R_v is valuation ring. Real holomorphy rings play a significant role in addressing the issue of expressing elements of K as a sum of a finite number of squares or, more broadly, even powers within K(Buchner & Kucharz, 1989). In his foundational paper, Becker demonstrated that the holomorphy ring H (K) is essential in examining sums of 2m-th powers within fields, for a positive integer m(Berr, 1995).

It was known that the orders of formally real fields are related with valuation ring, from the very beginning of the theory such a field(Becker, 1979). Choi et al. (1996) “Let K be a formally real field. Then $f \in K$ is a sum of 2m-th powers of elements of K if and only if f is a sum of squares in K and $2m$ divides $V(f)$ in G for every valuation $V : K^{*} \to G$, where G is an ordered abelian group and the residue fied K_v of V is formally real”. When examining formally real fields and their orderings, holomorphy rings become important objects in real algebraic geometry and value theory. The concept of rings of “well-behaved”functions is generalized by these rings, which are crucial in positivity-related problems like sums of squares and Hilbert’s seventeenth problem.

A real holomorphy ring is real, meaning that -1 is not a sum of squares in it. All prime ideals in a holomorphy ring are real. When p is a prime ideal of a holomorphy ring, the corresponding local ring becomes a real valuation ring of K. Since valuation rings are integral domain, holomorphy rings are integral domains. Since valuation rings are integrally closed, holomorphy rings are integrally closed. All finitely generated nonzero ideals are invertible since holomorphy rings are Prüfer domains.

3.2.1 Totally Positive Units in Real Holomorphy Rings

The concept of totally positive units is fundamental to understanding how orderings, values, and algebraic structures interact when one studies real holomorphy rings. Totally positive units are used to define and study the order structure of real holomorphy rings. It has a crucial role in the study of quadratic forms and sums of squares over real function fields. A non-negative polynomial $f \in [x_1, x_2, ..., x_n]$ can be viewed as a totally positive element in the real holomorphy ring of $\mathbb{R}$.

During the Oberwolfach conference on "real algebraic geometry" in 1987, H.-W. Schfilting posed the following question: “Suppose $f, g \in \mathbb{R}[x]$ are of identical degree without real zeroes and assume that $\frac{f}{g}$ is positive definite. Are there $f_i, g_i \in \mathbb{R}[x](1 < i < n)$ without real zeroes with $deg(f_i) = deg(g_i)$ $(1 < i < n)$ such that

$$\frac{f}{g} = \sum_{i=1}^{n} \left(\frac{f_i}{g_i}\right)^{2m}$$

for a certain natural number n?”. Schmid has positively answered Schiilting’s question whether a totally positive unit in the real holomorphy ring $H(K)$ of a formally real field K is a sum of squares of totally positive units of $H(K)$. Schmid’s result applies in particular to algebraic function fields in one variable over a real closed field $\mathbb{R}$(Choi et al., 1996). Using this concept this study give alternative proof for effective bounds of sums of fourth powers of $\mathbb{R}(x)$ using totally positive units of holomorphy ring.

Chapter Four
Methodology of the study

This chapter presents the study design, source of information and procedures used to achieve the objectives of the study.

4.1 Study Design

This study analytical methods such as understanding positivity in $\mathbb{R}(x)$, relating sums of fourth powers to positivity and establishing effective upper bound for sums of fourth powers in $\mathbb{R}(x)$.

4.2 Source of Information

The relevant sources of information from both published and unpublished document such as books, journals, and related studies found on internet were used to obtain the desired knowledge.

4.3 Procedures of the study

The study followed the following procedures in order to achieve the specified particular objectives.

- exploring the structure of positive elements in $\mathbb{R}(x)$ by examining their representation as sums of even powers and the connection to real holomorphy ring.
- examining the sums of fourth powers in $\mathbb{R}(x)$ through the framework of Hilbert's seventeenth problem.
- drawing the graph of the Motzkin polynomial.
- providing generalizations and theoretical proofs of effective upper bounds on sums of fourth powers in $\mathbb{R}(x)$.

Chapter Five
Result and Discussion

5.1 Sums of Even Powers of rational functions over $\mathbb{R}$

This section illustrates when a polynomials or rational functions can be expressed as the sum of even powers of rational functions. A polynomial $f \in \mathbb{R}[x]$ is sums of even powers if it can be represented as:

$$f(x) = \sum_{i=1}^{k} \left(\frac{g_i(x)}{h_i(x)} \right)^{2m}$$

where each $g_i, h_i \in \mathbb{R}[x]$ and m is positive integer. If $m = 1$, these are sums of squares and if $m = 2$, these are sums of fourth powers. Any sum of fourth powers of rational functions can be represented as a sum of squares of rational functions. Specifically, a single-variable polynomial might be a sum of fourth powers of rational functions, but not a sum of fourth powers of polynomials(Kowalczyk & Vill, 2023). Becker's provides a necessary and sufficient criteria for determining whether a polynomial or rational function can be expressed as the sum of 2m-th powers of rational functionsB. Reznick (2000). The criteria is called Becker's valuation criteria. It improves Hilbert's seventeenth Problem for greater powers and has its roots in valuation theory.

(Choi et al., 1996) According to Becker's criterion, consider $f \in \mathbb{R}[x]$ as a polynomial in a single variable over the real closed field $\mathbb{R}$. The polynomial f can be expressed as

$$f = \sum_{i=1}^{n} \left(\frac{g_i(x)}{h(x)} \right)^{2m}$$

with $g_i, h \in \mathbb{R}[x]$ if and only if the following conditions are met:

1). $f(a) \geq 0$ for every $a \in \mathbb{R}$ (indicating that f is positive semidefinite).

2). $2m$ is a divisor of the degree of f.

3). For every $a \in \mathbb{R}$, $2m$ divides the multiplicity of the factor $x - a$ in f for every $a \in \mathbb{R}$ (ensuring no real roots have odd multiplicity).

Consequently, for $m = 2$, f can be written as $f = \sum_{i=1}^{n} \left(\frac{g_i(x)}{h(x)} \right)^4$ if and only if $f \geq 0$, 4 divides the degree of f, and 4 divides the multiplicity of $x - a$ in f for every $a \in \mathbb{R}$.

5.2 The Case of Sums of 4-th Powers in $\mathbb{R}(x)$

The main result of this section will be $P_4(\mathbb{R}(x)) \leq 6$ for any real closed field $\mathbb{R}$. And also we see a concrete case in sums of 4-th powers in $\mathbb{R}(x)$. We note that a polynomial $f \in \sum \mathbb{R}(x)^4$ must a degree which is divisible by 4. Consider a polynomial of degree 4:

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$$

Identifying the precise conditions for a quartic to be represented as a sum of fourth powers is a complex task that requires a deeper investigation into real algebraic geometry. When a quartic polynomial is written as a sum of fourth powers and its leading coefficient is positive, its discriminant must be non-negative. For $f(x)$ to be expressed as the sum of fourth powers, it must remain non-negative for all real values of x, the leading coefficient a must be positive, and the discriminant should be positive. Because if $a < 0$, $f(x) \to 0$ as $x \to \pm\infty$. This means if $a < 0$, $f(x)$ would eventually become negative, making it impossible to represent as such a sum. The discriminant of $ax^4 + bx^3 + cx^2 + dx + e$ is $256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2$ (Klaška, 2022). For instance; consider the quartic polynomial $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$. Since $1 > 0$ and $f(x) \geq 0$, for all $x \in \mathbb{R}$, $f(x)$ is positive semidefinite. Then, $f(x)$ can be expressed as a sums of fourth powers of linear function and zero polynomial.

The polynomial $ax^4 + bx^3 + cx^2 + dx + e$ can be simplified and possibly expressed as a sum of fourth powers of rational functions by eliminating the cubic term (bx^3). This is because it makes factorization and manipulation of the polynomial easier. By normalizing we may assume that $a = 1$. Then to write $f = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$ as $f = x^4 + \alpha x^2 + \beta x + \gamma :$ we eliminate term x^3 and normalize the coefficient of x^4 to 1. By dividing the entire polynomial for a we get:

$$f = x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a}$$

To eliminate the term x^3 perform a substitution. let $x = t - \frac{b}{4a}$. Then by substitution we get:

$$\left(t - \frac{b}{4a}\right)^4 + \frac{b}{a}\left(t - \frac{b}{4a}\right)^3 + \frac{c}{a}\left(t - \frac{b}{4a}\right)^2 + \frac{d}{a}\left(t - \frac{b}{4a}\right) + \frac{e}{a} =$$

$$\left(t^4 - \frac{b}{a}t^3 + \frac{6b^2t^2}{16a^2} - \frac{4b^3t}{64a^3} + \frac{b^4}{256a^4}\right) + \frac{b}{a}\left(t^3 - \frac{3bt^2}{4a} + \frac{3b^2t}{16a^2} - \frac{b^3}{64a^3}\right) +$$

$$\frac{c}{a}\left(t^2 - \frac{bt}{2a} + \frac{b^2}{16a^2}\right) + \frac{d}{a}\left(t - \frac{b}{4a}\right) + \frac{e}{a} =$$

$$t^4 - \frac{b}{a}t^3 + \frac{6b^2t^2}{16a^2} - \frac{4b^3t}{64a^3} + \frac{b^4}{256a^4} + \frac{b}{a}t^3 - \frac{3b^2t^2}{4a^2} + \frac{3b^3t}{16a^3} - \frac{b^4}{64a^4} + \frac{ct^2}{a} - \frac{bct}{2a^2} + \frac{b^2c}{16a^3} + \frac{dt}{a} - \frac{bd}{4a^2} + \frac{e}{a}$$

collecting like terms we get:

$$t^4+\left(-\frac{3b^2}{8a^2}+\frac{c}{a}\right)t^2+\left(\frac{b^3}{8a^3}-\frac{bc}{2a^2}+\frac{d}{a}\right)t-\frac{3b^4}{256a^4}+\frac{b^2c}{16a^3}-\frac{bd}{4a^2}+\frac{e}{a}$$

By renaming the coefficients of t as: $\alpha=-\frac{3b^2}{8a^2}+\frac{c}{a}, \beta=\frac{b^3}{8a^3}-\frac{bc}{2a^2}+\frac{d}{a}, \gamma=\frac{3b^4}{256a^4}+\frac{b^2c}{16a^3}-\frac{bd}{4a^2}+\frac{e}{a}$, we get

$$t^4+\alpha t^2+\beta t+\gamma$$

Since $t=x+\frac{b}{4a}$, we can repeat t with x to express the polynomial in terms of x: $x^4+\alpha x^2+\beta x+\gamma$, where α,β, and γ are constants determined by the substitutions and simplification process. After some argument about sums of 4-th powers we concerned with the following situation

$$f(x)=x^4+\alpha x^2+\beta x+\gamma$$

no zero in $\mathbb{R}$. Such a polynomial is a sum of fourth powers in $\mathbb{R}(x)$ by valuation criterion. Now if $f(x)>0$ for all $x\in\mathbb{R}$, necessarily $\gamma>0$. Because when $x=0$, $f(x)=\gamma$. The new polynomial $f(x)$ is positive semi definite if $f(x)>0$, for all $x\in\mathbb{R}$. The discriminant of $x^4+\alpha x^2+\beta x+\gamma$ is $256\gamma^3-128\alpha^2\gamma^2+144\alpha\beta^2\gamma-27\beta^4+16\alpha^4\gamma-4\alpha^3\beta^2$.

5.2.1 Structure of totally Positive Units in Real Holomorphy Rings

This research explores the role of the real holomorphy ring of a formally real field $\mathbb{R}(x)$ in analyzing the sums of fourth powers within $\mathbb{R}(x)$. The holomorphy ring $H(\mathbb{R}(x))$ is characterized as the common intersection of all valuation rings of $\mathbb{R}(x)$ that possess a residue field which is formally real. Since there are two types of non-trivial real valuation on a field $\mathbb{R}(x)$ we have; $H(\mathbb{R}(x))=\bigcap$ real valuation rings. This suggests that

$$H(\mathbb{R}(x))=V_\alpha\cap V_\infty=\left\{(x-\alpha)^r\frac{f}{g}|g(\alpha)\neq 0,(x-\alpha)\nmid f,(x-\alpha)\nmid g\right\}\bigcap\left\{\frac{f}{g}|deg(f)\leq deg(g)\right\}$$

$$=\left\{\frac{f}{g}|deg(f)\leq deg(g), g(\alpha)\neq 0\right\}.$$

Suppose $deg(f)=deg(g)$, then $\frac{f}{g}$ is unit of $H(\mathbb{R}(x))$. Let $E=\left\{\frac{f}{g}|deg(f)=deg(g)\right\}$ is a unit of real holomorphy ring, then $E_+=\left\{\frac{f}{g}|deg(f)=deg(g), f,g>0\right\}$ is totally positive units of real holomorphy rings. If $\frac{f}{g}$ is totally positive unit $deg(f)=deg(g)=2t, t\in\mathbb{Z}_{\geq 0}$ which means deg(f) and deg(g) are even positive numbers(Engler & Prestel, 2005).
For example, $\frac{f(x)}{g(x)}=\frac{1+x^2}{2+x^2}$ is totally positive unit because $deg(f)=deg(g)$ and $f,g>0$ for all $x\in\mathbb{R}(x)$.

Lemma 5.1. *(Schmid, 1994a) Let $H(\mathbb{R}(x))$ be real holomorphy ring and E_+ be totally positive units of $H(\mathbb{R}(x))$. Then:*
(1) for $a_1, ..., a_n \in \mathbb{R}(x)$ we have $\sum_{i=1}^{n} a_i^2 \in H(\mathbb{R}(x)) \Leftrightarrow a_1, ..., a_n \in H(\mathbb{R}(x))$
(2) let $x \in H(\mathbb{R}(x)), a, y \in E_+$ and assume that $a = x^2 + y^2$. Then there are $u, v \in H(\mathbb{R}(x))$ such that $a = u^2 + v^2$.

Theorem 5.1. *(Schmid, 1994a) Let $\mathbb{R}$ be real closed field and $f \in \mathbb{R}[x]$ is monic and strictly positive. Then $f \in \sum \mathbb{R}(x)^4$ if and only there exist strictly positive polynomials, $g_1, g_2, h \in \mathbb{R}[x]$ such that $h^2 f = g_1^2 + g_2^2$ and $deg(g_1) = deg(g_2)$.*

Theorem 5.2. *For a formally real field $\mathbb{R}(x)$, every non negative rational function in $\mathbb{R}(x)$ can be written as a sum of at most 6 fourth powers.*

Proof. Let $\varepsilon \in E_+$, and $E_+ = \left\{\varepsilon = \frac{f}{g} | deg(f) = deg(g), f, g > 0\right\}$ is totally positive units in real holomorphy ring. Then $\varepsilon = \frac{f}{g} = \frac{fg}{g^2}$ is totally positive units. Therefore $fg \in \sum \mathbb{R}(x)^4$ and $4 \mid deg(fg)$. Since $fg \in \sum \mathbb{R}(x)^4$, by theorem 5.1 there exist strictly positive polynomial $f_1, f_2, h \in \mathbb{R}(x)$ such that

$$h^2 fg = f_1^2 + f_2^2$$

$$\frac{f}{g} = \left(\frac{f_1}{hg}\right)^2 + \left(\frac{f_2}{hg}\right)^2$$

and $deg(\frac{f_1}{gh}) = deg(\frac{f_2}{gh})$. In fact, the rational functions $\frac{f_1}{hg}$ and $\frac{f_2}{hg}$ being positive definite are in $\mathbb{R}(x)^2 + \mathbb{R}(x)^2$.
For any fourth powers

$$\sum (x_i)^4 = \left(\frac{\sum (x_i)^4}{\sum (x_i^2)^2}\right)\left(\sum (x_i^2)^2\right)$$

Every totally positive unit of the real holomorphy ring $H(\mathbb{R}(x))$ of a formally real field $\mathbb{R}(x)$ can be expressed as sum of squares of totally positive units(Schmid, 1994a). Then

$$\begin{aligned}
\sum (x_i)^4 &= \left(\frac{\sum (x_i)^4}{\sum (x_i^2)^2}\right)\left(\sum (x_i^2)^2\right) \\
&= \left(\eta^2 + \omega^2\right) q^2; (\eta, \omega \in E_+, q > 0) \\
&= (\eta q)^2 + (\omega q)^2 ; (\eta q, \omega q > 0) \\
&= \left(a^2 + b^2\right)^2 + \left(c^2 + d^2\right)^2
\end{aligned}$$

Because every totally positive units can be expressed as a sums of squares.
Then using the identity: for $x, y \in \mathbb{R}$, $(x^2 + y^2)^2 = \frac{1}{2}\left(x + \frac{1}{\sqrt{3}}y\right)^4 + \frac{1}{2}\left(x - \frac{1}{\sqrt{3}}y\right)^4 + \frac{8}{9}y^4$:

$$\sum (x_i)^4 = \frac{1}{2}\left(a + \frac{1}{\sqrt{3}}b\right)^4 + \frac{1}{2}\left(a - \frac{1}{\sqrt{3}}b\right)^4 + \frac{8}{9}b^4 + \frac{1}{2}\left(c + \frac{1}{\sqrt{3}}d\right)^4 + \frac{1}{2}\left(c - \frac{1}{\sqrt{3}}d\right)^4 + \frac{8}{9}d^4$$

where $a, b, c, d \in \mathbb{R}(x)$.

Thus $P_4(\mathbb{R}(x)) \leq 6$. This means every non negative rational function in $\mathbb{R}(x)$ can be written as a sum of at most 6 fourth powers. $\square$

5.3 Discussion

Not every polynomial or rational function can be expressed as sum of even powers of polynomials or rational functions. A fundamental question in real algebraic geometry is determining when a polynomial can be represented in this way. It is a central question in real algebraic geometry when a polynomial can be written as a sums of even powers of polynomials or rational functions. A polynomial can be expressed as the sum of even powers of rational functions if and only if it's a non-negative polynomial. In other words, a polynomial can be expressed as the sum of even powers (like squares) of rational functions or other polynomials if and only if it's always greater than or equal to zero for all values of the variable. For every even integer $2m$, any positive semidefinite rational function in $\mathbb{R}(x)$ can be represented as a sum of $2m$-th powers of rational functions. Now if m= 2 these are sums of fourth powers. Becker put a criteria when a polynomial or rational functions can be written as sums of fourth powers and he also find effective upper bounds to this summands. In this work, we used a totally positive unit in the real holomorphy ring to provide alternative bound on the number of summands needed to represent polynomials or rational functions as sums of fourth powers. This study also provide conditions quartic polynomials to be positive semidefinte.

Chapter Six
Conclusion and Recommendation

Conclusion

This thesis has systematically investigated the representation of elements in the rational function field over the field of real numbers as sums of even powers, with a focus on both theoretical existence and explicit construction. The origin of this study is Hilbert's seventeenth problem. Hilbert's seventeenth problem is a profound result in the theory of polynomials and their representations. Many researchers extends the idea of Hilbert's seventeenth problem to sum of even powers. Every positive semidefinite rational function $f \in \mathbb{R}(x)$ can be written as a sum of even powers of rational functions or others polynomials (extensions of Hilbert's seventeenth Problem). Becker's provide a necessary and sufficient conditions for representations of rational functions or polynomials as a sum of higher powers. Numerous researchers have shown that, for a natural integer m, every totally positive unit of $H(\mathbb{R}(x))$ is a sum of 2m-th powers of totally positive units of $H(\mathbb{R}(x))$, based on certain polynomial identities that Hilbert established. The case of $m = 2$ is the main idea of this study. In general in this study we showed that every non negative or positive semidefinite rational function $\mathbb{R}(x)$ can be written as a sum of at most 6 fourth powers using totally positive unit in real holomorphy ring.

Recommendation

This thesis specifically focuses on sums of fourth powers of rational function field over the field of real numbers. We use totally positive units in real holomorphy ring to find effective upper bound of sums of fourth powers in $\mathbb{R}(x)$. The results suggest that while sums of squares are well-understood, higher even powers introduce rich complexity, inviting further study at the intersection of algebra, analysis, and computation. Using the idea of this study researchers may extends the results to sums of 6th powers. And also researchers may reduce the effective upper bound of representing every non negative rational functions as sum of fourth powers.

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