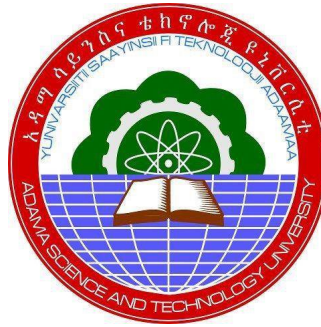


ANALYSIS OF DIRECT BOUNDARY-DOMAIN INTEGRAL
EQUATIONS FOR VARIABLE COEFFICIENT NEUMANN BVP IN
PLANE



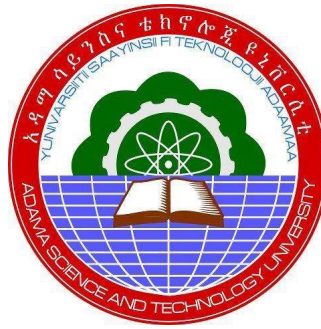
By
Getachew Tesfaye Feyisa

A Thesis Submitted to the Program of Applied Mathematics
School of Applied Natural Science
Presented in Partial Fulfillment of the Requirement for the Degree of
Master's of Science in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

Adama, Ethiopia
September, 2017

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Advisor:
Tamirat Temesgen (PhD)

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Approval Page

This is to certify that the thesis prepared by Getachew Tesfaye entitled “**Analysis of Direct Boundary-Domain Integral Equations for Variable Coefficient Neumann BVPs in Plane**” submitted in partial fulfillment of the requirement for the degree of Master of Science in **Applied Mathematics** with the regulation of the University and meets the accepted standards with respect to originality.

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Abstract

In this thesis, analysis of direct boundary-domain integral equations for the scalar second order “stationary heat transfer” elliptic partial differential equation with variable coefficient subjected to Neumann boundary condition in plane were investigated. Using an appropriate parametrix in Green’s identity, the considered Neumann boundary value problem is reduced to two different systems of Boundary-Domain Integral Equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the theory of BDIEs in plane need a special consideration due to their different equivalence properties. Consequently, we need to set conditions on the domain or on the spaces to insure the invertibility of the corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the obtained BDIE systems to the original BVPs, BDIE systems solvability, solution uniqueness/non-uniqueness, as well as Fredholm property and invertibility of the BDIE operators for the Neumann BVP were analyzed.

Keywords: Partial differential equation, variable coefficients, Sobolev spaces, parametrix, integral equations, equivalence, invertibility.

Acknowledgements

The first and foremost gratitude and praise goes to the Lord Jesus, who is the source of my knowledge and helping me in every aspect of my life.

secondly, my heartfelt incalculable gratitude goes to my advisor Dr.Tamirat Temesgen for his valuable guidance, giving different materials and constructive comments to make this thesis paper valuable.

I would like also to thank my family for their support. Though my parents were far from me during my graduate education at ASTU, their support and loves were always with me.

Last but not least, I want to thank all the colleagues, friends and staffs here in Mathematics Department of Adama Science and Technology University.

Adama Science and Technology University
September, 2017

Getachew Tesfaye

Table of Contents

Abstract	i
Acknowledgements	ii
Table of Contents	iii
1 Introduction	1
1.1 Background of the study	1
1.2 Statement of the problem	3
1.3 Objectives of the study	3
1.3.1 General Objective	3
1.3.2 Specific Objectives	3
1.4 Significance of the study	4
1.5 Thesis Design	4
2 Review of the Literature	5
3 Mathematical Preliminaries	8
3.1 Basic Notations and Spaces	8
3.2 Compact and Fredholm Operators	12
4 Direct Boundary-Domain Integral Equations for the Neumann BVP	14
4.1 Formulation of the Neumann Boundary Value Problem	14
4.1.1 First Green's Identity	14
4.1.2 Second Green's Identity	17
4.2 Parametrix and Parametrix-Based Potential Operators	19
4.2.1 Volume Potentials	21
4.2.2 Surface Potentials	22
4.2.3 Invertibility of the single layer potential operator in 2D.	25
4.2.4 The Third Green's Identity	26
4.2.5 Invertibility of the Hypersingular Operator in 2D	29
4.3 BDIE systems for the Neumann BVP	31
4.3.1 Formulation of BDIE system (N1)	31

4.3.2	Formulation of BDIE system (N2)	32
4.4	Equivalence and Invertibility Theorems	33
5	Conclusion	36
	Bibliography	37

Chapter 1

Introduction

1.1 Background of the study

Partial Differential Equations (PDEs) with variable coefficients arise naturally in mathematical modeling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermoconductivity, fluid flows through porous media, and other areas of physics and engineering (see [3, 4] and the references therein).

The Boundary Integral Equation Method/Boundary Element Method (BIEM/BEM), which is a well established tool for solution of boundary value problems (BVPs) with constant coefficients, has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing a BVP for a scalar partial differential equation (PDE) in a domain to an integral equation on the boundary of the domain and then use the latter for further numerical analysis. This approach diminishes the problem dimensionality by one which is very important for construction of various numerical algorithms requiring small computer resources. The main ingredient for reducing a BVP for a PDE to a BIE is a fundamental solution to the original PDE. After the fundamental solution is used in the corresponding Greens formulae, as a first step, one can reduce the BVP to a BIE. The next essential step is to

establish the equivalence of the original BVP and the corresponding BIE, and to show the invertibility of the BIE operator [2].

However, such a fundamental solution is generally not available if the coefficients of the original BVP are not constant. The BVPs of heat transfer with variable heat conductivity coefficients (inhomogeneous material) belong to this category [1]. Using a parametrix as a substitute of a fundamental solution in the Green formulae, it is possible however to reduce such a BVP to some systems of boundary-domain integral equations where the Dirichlet, Neumann and Robin problems for some PDEs were reduced to BDIE systems [2, 7, 9, 11]. This is a function which contains the same singularity as the fundamental solution and correctly describes the main part of the fundamental solution but is not required to satisfy the original differential equations apart from the singular point. This allows a reduction of the boundary value problem not to Boundary Integral Equation (BIE) but to a coupled system of Boundary-Domain Integral Equations (see [1] and the references therein).

1.2 Statement of the problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$. We shall derive and investigate the BDIE systems for the following Neumann BVP: Find a function $u \in H^1(\Omega)$ satisfying

$$Au(x) = \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] = f(x) \quad \text{in } \Omega, \quad (1.1)$$

$$T^+u(x) = \sum_{i=1}^2 n_i(x) \gamma^+ \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] = \psi_0(x) \quad \text{on } \partial\Omega, \quad (1.2)$$

where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$ are given functions.

So, the thesis is intended to answer the following basic questions:

- i. What is the necessary condition for the solvability of this Neumann BVP?
- ii. Is the BVP uniquely solvable?
- iii. How to reduce this Neumann BVP to BDIE systems?
- iv. Are the obtained BDIE systems equivalent to the original BVP?
- v. Are the BDIE operators for the Neumann BVP invertible?

1.3 Objectives of the study

1.3.1 General Objective

The general objective of the thesis is to analyze direct boundary-domain integral equations for variable coefficient Neumann boundary value problem in 2D.

1.3.2 Specific Objectives

The specific objectives of the thesis are:

- i. To investigate solvability of the Neumann BVP.

- ii. To show that the Neumann BVP with variable coefficient in interior domain can be reduced to some systems of BDIEs.
- iii. To investigate the equivalence of the corresponding BDIE systems to the original Neumann BVP.
- iv. To give a suitable condition for the existence and uniqueness of BDIEs.
- v. To investigate invertibility of the boundary-domain integral operators.

1.4 Significance of the study

The study of this problem will have the following importance:

- It improves our understanding of the BDIEs so that we can reduce and analyse the given Neumann BVPs for variable coefficients.
- It may serve as references for those who needs to conduct research on this area.

1.5 Thesis Design

The thesis is designed as:

- ✓ Formulation of the BVP;
- ✓ Reduction of the BVP to two BDIE sytems using the method of an explicit parametrix and parametrix-based third Green's identity;
- ✓ Investigation of the equivalence BDIE systems to the original BVP;

Chapter 2

Review of the Literature

In different time; many researchers give different analysis of Boundary-Domain Integral Equations for “stationary heat transfer” elliptic partial differential equation with variable coefficient subjected to some boundary condition. Many questions about their equivalence to the original BVP, solvability, solution uniqueness and invertibility of corresponding integral operator remained open for rather long time [9]. In this section, we reviewed some of these analyses developed by different researchers to study the analysis of BDIEs that are directly related to the objectives of this study.

In [2], analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient in three-dimension was considered. In the paper, an explicit parametrix and the parametrix-based third Green’s identity has been used to reduce the mixed boundary value problem for the “heat transfer” partial differential equation with a variable coefficient (heat conductivity) to four different versions of direct segregated Boundary-Domain Integral Equation Systems (BDIES). The obtained BDIE systems contain integral operators defined on the domain under consideration as well as potential-type and pseudo-differential operators defined on open submanifolds of the boundary. It has been shown that the BDIE systems were equivalent to the original mixed BVP, and the operators of the BDIE systems were invertible in appropriate Sobolev spaces.

In [4], analysis of some direct segregated localized boundary-domain integral equation (LBDIE) systems associated with the Dirichlet and Neumann boundary value problems for a scalar “stationary heat transfer” elliptic PDE with variable coefficient were formulated and analyzed. The parametrix was localized by multiplication with a radial localizing function. Mapping and jump properties of surface and volume integral potentials based on a localized parametrix and constituting the LBDIE systems were studied in a scale of Sobolev (Bessel potential) spaces. The main results established in the paper were the LBDIEs equivalence to the original variable-coefficient BVPs and the invertibility of the LBDIE operators in the corresponding Sobolev spaces.

In [3], analysis of direct segregated boundary-domain integral equations for variable-coefficient corresponding to mixed BVPs in exterior domains was presented . In the paper, direct segregated systems of boundary-domain integral equations were formulated for the mixed boundary value problems for a scalar second-order divergent elliptic partial differential equation with a variable coefficient in an exterior three dimensional domain. The boundary-domain integral equation system equivalence to the original boundary value problems and the Fredholm properties and invertibility of the corresponding boundary-domain integral operators were analyzed in weighted Sobolev spaces suitable for infinite domains. The analysis was based on the corresponding properties of the BVPs in weighted Sobolev spaces that were proved as well.

Analysis of segregated boundary-domain integral equations (BDIEs) based on a parametrix and associated with the Dirichlet and Neumann boundary value problems for the linear stationary diffusion partial differential equation with a variable coefficient in three-dimension were presented in [9]. Based on an explicit parametrix and the parametrix-based third Green’s identity these BVPs were reduced to four different versions of Direct Segregated Boundary-Domain Integral Equation (BDIE) systems. Equivalence of the

BDIEs to the original BVP, BDIE solvability, solution uniqueness/non-uniqueness, and invertibility of the BDIE operators were analysed in Sobolev (Bessel potential) spaces. It has been shown that the BDIE operators for the Neumann BVP were not invertible.

The analysis of Boundary-Domain Integral Equations for the second order “stationary heat transfer” elliptic partial differential equation with Variable-Coefficient Dirichlet BVP in two-dimension was presented in [7]. In the paper, the interior Dirichlet problem for variable coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega)$ has been considered. The BVP has been reduced to two systems of Boundary-Domain Integral Equations and their equivalence to the original BVP has been shown. The invertibility of the associated operators in the corresponding Sobolev spaces has been also proved.

In this thesis, the BDIE formulations and analysis for the second order “stationary heat transfer” elliptic partial differential equation with variable coefficients based on explicit parametrix and parametrix-based third Green’s identity proposed in [7] by Mikhailov has been extended to the treatment of Neumann BVPs in two-dimension.

Chapter 3

Mathematical Preliminaries

3.1 Basic Notations and Spaces

We will start by considering some basic notations, spaces and some mathematical concepts which are required for the development and understanding of the Boundary-Domain Integral Equations. Throughout this work the term domain and the symbol Ω shall be reserved for an open connected set in $\mathbb{R}^2$ bounded by a simple smooth curve $\partial\Omega$, $\bar{\Omega}$ be closure of Ω and $n(x)$ be the exterior unit normal vector defined on $\partial\Omega$.

Definition 3.1. (Support of the function, [8])

For a function $u(x)$ defined for $x \in \Omega \subset \mathbb{R}^2$ we denote

$$\text{supp } u = \overline{\{x \in \Omega : u(x) \neq 0\}}$$

to be the support of the function u .

Definition 3.2. (Space of test functions, [6])

Let $\Omega \subset \mathbb{R}^2$. The class of test functions $\mathcal{D}(\Omega)$ consists of all functions $u(x)$ defined in Ω , vanishing outside a bounded subset of Ω that stays away from the boundary of Ω , and such that all partial derivatives of all orders of φ are continuous and is denoted as

$$\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega) = \{u \in \mathcal{C}^\infty(\Omega) : \text{supp } u \subset \Omega\}.$$

Definition 3.3. (Distributions, [14])

A complex valued continuous linear functional T acting on $\mathcal{D}(\Omega)$ is called a distribution. T is continuous on $\mathcal{D}(\Omega)$, if $\varphi_k \rightarrow \varphi$ in $\mathcal{D}(\Omega)$ always implies $\langle T, \varphi_k \rangle \rightarrow \langle T, \varphi \rangle$ as $n \rightarrow \infty$. The set of all distributions on $\mathcal{D}(\Omega)$ is denoted by $\mathcal{D}'(\Omega)$.

Definition 3.4. (Schwartz Space, [14])

The Schwartz space, $\mathcal{S}(\mathbb{R}^2)$ is the space of functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying

$$\|\varphi\|_{k,l} = \sup_{x \in \mathbb{R}^2} (|x|^k + 1) \sum_{|\alpha| \leq l} |\partial^\alpha \varphi(x)| < \infty, \quad k, l \in \mathbb{N}_0.$$

In particular, the function φ and all of its derivatives decreases faster than any polynomial.

Definition 3.5. (Tempered Distribution, [14])

As in definition 3.3 we can introduce the space $\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions as the space of complex valued linear forms T over $\mathcal{S}(\mathbb{R}^2)$.

Definition 3.6. (The Sobolev space $W^{m,p}(\Omega)$, [8])

Let Ω be a non-empty open subset of $\mathbb{R}^2$, m be a non-negative integer and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ of order m based on $L^p(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega): \partial^\alpha u \in L^p(\Omega) \text{ for any } \alpha \in \mathbb{N}^2 \cup \{0\}, |\alpha| \leq m\}.$$

Here, of course $\partial^\alpha u$ is viewed as a distribution on Ω , so the condition $\partial^\alpha u \in L^p(\Omega)$ means that there exists a function $v \in L^p(\Omega)$ such that

$$\langle u, \partial^\alpha \phi \rangle_\Omega = (-1)^{|\alpha|} \langle v, \phi \rangle_\Omega \text{ for all } \phi \in \mathcal{D}(\Omega)$$

Such a function v is often described as a weak partial derivative of u . Notice that $W^{0,p}(\Omega) = L^p(\Omega)$. For $p = 2$, the symbol $W^{m,2}(\Omega)$ is generally replaced by $H^m(\Omega)$. The completeness of $L^p(\Omega)$ implies that $W^{m,p}(\Omega)$ becomes a Banach space on putting

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_\Omega |\partial^\alpha u(x)|^p dx \right)^{\frac{1}{p}}.$$

In particular, for $m = 1$ and $p = 2$, the space $W^{1,2}(\Omega) = H^1(\Omega)$ and expressed as

$$H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega) \text{ for } i=1,2\}.$$

And the inner product in $H^1(\Omega)$ is given by

$$\begin{aligned} \langle u, v \rangle_{H^1(\Omega)} &= \sum_{|\alpha| \leq 1} \int_\Omega \overline{\partial^\alpha u(x)} \partial^\alpha v(x) dx \\ &= \int_\Omega \overline{u(x)} v(x) dx + \sum_{i=1}^2 \int_\Omega \overline{\frac{\partial u}{\partial x_i}} \frac{\partial v}{\partial x_i} dx \\ &= \int_\Omega \overline{u(x)} v(x) dx + \int_\Omega \nabla \overline{u} \cdot \nabla v dx \end{aligned}$$

while the norm is given by

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2.$$

If $m_1 \geq m > 0$, then we can see that, $u \in H^{m_1}(\Omega)$ implies that $u \in H^m(\Omega)$. We have the inclusions $H^{m_1}(\Omega) \subset H^m(\Omega) \subset L^2(\Omega) = H^0(\Omega)$, with continuous injection, i.e., $\|u\|_{H^m(\Omega)} \leq \|u\|_{H^{m_1}(\Omega)}$. For all $m \in \mathbb{N}$ we have the following inclusions:

$$\mathcal{D}(\Omega) \subset H^m(\Omega) \subset H^{m-1}(\Omega) \subset \dots \subset H^1(\Omega) \subset L_2(\Omega) \subset \mathcal{D}'(\Omega).$$

Definition 3.7. (Bessel potential operator, [8]). For $s \in \mathbb{R}$, we define a continuous linear operator $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$, called the Bessel potential of order s , by

$$\mathcal{J}^s u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \exp(i\xi \cdot x) d\xi \text{ for } x \in \mathbb{R}^2.$$

In this way,

$$\mathcal{F}_{x \rightarrow \xi} \{ \mathcal{J}^s u(x) \} = (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) \quad (3.1)$$

Notice that $\forall s, t \in \mathbb{R}$,

$$\mathcal{J}^{s+t} = \mathcal{J}^s \mathcal{J}^t, (\mathcal{J}^s)^{-1} = \mathcal{J}^{-s}, \mathcal{J}^0 = \text{identity operator}.$$

It follows from (3.1) that

$$\langle \mathcal{J}^s u, v \rangle = \langle u, \mathcal{J}^s v \rangle \quad \text{and} \quad (\mathcal{J}^s u, v) = (u, \mathcal{J}^s v) \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n),$$

giving a natural extension of the Bessel potential to a linear operator $\mathcal{J}^s : \mathcal{S}^*(\mathbb{R}^2) \rightarrow \mathcal{S}^*(\mathbb{R}^2)$ on the space of tempered distributions.

Definition 3.8. (Bessel potential space, [8]). For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^2)$, the Sobolev space of order s on $\mathbb{R}^2$, by

$$H^s(\mathbb{R}^2) = \{ u \in \mathcal{S}^*(\mathbb{R}^2) : \mathcal{J}^s u \in L^2(\mathbb{R}^2) \},$$

and equip this space with the inner product

$$(u, v)_{H^s(\mathbb{R}^2)} = (\mathcal{J}^s u, \mathcal{J}^s v)$$

and the induced norm

$$\|u\|_{H^s(\mathbb{R}^2)} = \sqrt{(u, u)_{H^s(\mathbb{R}^2)}} = \|\mathcal{J}^s u\|_{L^2(\mathbb{R}^2)}.$$

The space $H^{-s}(\mathbb{R}^2)$ is the dual space of $H^s(\mathbb{R}^2)$ equipped with the norm

$$\|u\|_{H^{-s}(\mathbb{R}^2)} = \sup_{0 \neq v \in H^s(\mathbb{R}^2)} \frac{|\langle u, v \rangle|}{\|v\|_{H^s(\mathbb{R}^2)}} \quad \text{for } u \in H^{-s}(\mathbb{R}^2).$$

We will denote $\tilde{H}^s(\Omega)$, the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^2)$, which can be characterized as

$$\tilde{H}^s(\Omega) = \{ u : u \in H^s(\mathbb{R}^2) : \text{supp } u \subset \bar{\Omega} \},$$

and for $s \in (\frac{-1}{2}, \frac{1}{2})$, the space $H^s(\Omega)$ can be identified with $\tilde{H}^s(\Omega)$ (see e.g., [8], Theorem 3.29]) whereas for any non-empty open set $\Omega \subseteq \mathbb{R}^2$, the space $H^s(\Omega)$ consists of restrictions on Ω of distributions from $H^s(\mathbb{R}^2)$,

$$H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = v|_{\Omega}, \text{ for some } v \in H^s(\mathbb{R}^2)\},$$

where $|$ denotes the restriction operator on Ω . We recall that $H^1(\Omega)$ coincides with the sobolev space $W^{1,2}(\Omega)$, with equivalent norms.

Notice that $\tilde{H}^s(\Omega)$ is a closed subspace of $H^s(\mathbb{R}^2)$, and is therefore a Hilbert space when equipped with the restriction of the inner product of $H^s(\mathbb{R}^2)$.

Theorem 3.1. (Trace Theorem, [[6], Theorem 4.24])

Let Ω be a bounded C^∞ domain in $\mathbb{R}^2$ and let $\partial\Omega$ its boundary.

- (i) Let $s > \frac{1}{2}$. Then the trace operator γ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{1}{2}}(\partial\Omega)$ where

$$\gamma(H^s(\Omega)) = H^{s-\frac{1}{2}}(\partial\Omega)$$

- (ii) Let $s > \frac{3}{2}$. Then the normal derivative operator $\gamma_1 = \gamma(\frac{\partial}{\partial n})$ is a linear and bounded map of $H^s(\Omega)$ onto $H^{s-\frac{3}{2}}(\partial\Omega)$ where

$$\gamma_1(H^s(\Omega)) = H^{s-\frac{3}{2}}(\partial\Omega)$$

In this thesis, we need to use the traces of the $H^1(\Omega)$ functions, that form the space $H^{\frac{1}{2}}(\partial\Omega)$, i.e., $H^{\frac{1}{2}}(\partial\Omega) = \gamma(H^1(\Omega))$ and we can use the norm for $H^{\frac{1}{2}}(\partial\Omega)$ as

$$\|g\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{v \in H^1(\Omega), \gamma v = g} \|v\|_{H^1(\Omega)}.$$

The trace theorem enables us to define unambiguously the interior trace of u , γ^+u or $u|_{\partial\Omega}$, provided that u is smooth enough to be in $H^m(\Omega)$. i.e., for any $u \in H^m(\Omega)$, we have $\partial^\alpha u \in H^1(\Omega)$ for $|\alpha| \leq m-1$. Thus by the trace theorem the values of $\partial^\alpha u$ on the boundary is well-defined and belongs to $L^p(\Omega)$ which implies $\gamma^+ \partial^\alpha u \in L^p(\Omega)$.

Definition 3.9. (Ellipticity, [14]). Consider the scalar partial differential equation

$$Au(x) = \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} [a_{ij}(x) \frac{\partial u(x)}{\partial x_j}] = f(x) \quad \text{in } \Omega, \quad (3.2)$$

where A is self-adjoint partial differential operator of second order which is applied to a scalar real valued function u , the coefficient functions $a_{ij}(x)$ are assumed to be sufficient smooth satisfying $a_{ij}(x) = a_{ji}(x) \forall i, j = 1, 2, x \in \Omega$ (i.e., symmetric condition).

For a classification of equation (3.2), we consider the real eigenvalues $\lambda_k(x)$ of the symmetric matrix

$$M(x) = (a_{ij}(x))_{i,j=1}^2, x \in \Omega.$$

Then, the partial differential operator A is called elliptic at $x \in \Omega$ if and only if $\lambda_k(x) > 0$ is satisfied $\forall k = 1, 2$. If this condition is satisfied $\forall x \in \Omega$, then A is elliptic in Ω .

3.2 Compact and Fredholm Operators

Definition 3.10. (Compact Operator, [8]). Suppose that X and Y are normed spaces. A linear operator $A : X \mapsto Y$ is said to be *compact* (or *completely continuous*) if it maps every bounded subset of X to a relatively compact subset of Y .

Remark 3.1. Every compact operator is bounded. Also, any linear operator with a finite-dimensional image is compact, because in a finite-dimensional normed space every bounded set is totally bounded.

Suppose now that X and Y are Banach spaces.

Definition 3.11. (Fredholm Operator, [8]). A bounded linear operator $A : X \mapsto Y$ is said to be *Fredholm* if

- (i) the subspace $\text{im}A$ is closed in Y ;
- (ii) the subspaces $\ker A$ and $Y/\text{im}A$ are finite-dimensional.

In this case, the index of A is integer defined as,

$$\text{index}(A) = \dim(\ker A) - \dim(Y/\text{im}A).$$

The $\dim(Y/\text{im}A)$ also denoted by $\dim(\text{coker}(A))$.

Theorem 3.2. [8]. If $A : X \mapsto Y$ is Fredholm operator, and if $K : X \mapsto Y$ is compact operator, then their sum $A + K : X \mapsto Y$ is Fredholm, and $\text{index}(A + K) = \text{index}(A)$.

Theorem 3.3. [8]. Assume $A : X \mapsto Y$ is Fredholm with $\text{index}(A)=0$ and A is injective, then for each $f \in Y$, the inhomogeneous equation $Au = f$ has a unique solution $u \in X$.

Definition 3.12. (Adjoint Operator, [8]). Consider a linear map $A : X \mapsto Y$. If X and Y are each equipped with a conjugation, then we define the adjoint operator $A^* : Y^* \mapsto X^*$ by

$$(A^*v, u) = (v, Au) \text{ for } v \in Y^* \text{ and } u \in X.$$

Theorem 3.4. (Fredholm Alternative, [8]). Assume $A : X \mapsto Y$ is Fredholm with $\text{index}(A)=0$. There are two, mutually exclusive possibilities:

- (i) The homogeneous equation $Au = 0$ has only the trivial solution $u = 0$. In this case,
 - (a) for each $y \in Y$, the inhomogeneous equation $Au = f$ has a unique solution $u \in X$.

- (b) for each $g \in X^*$, the adjoint equation $A^*v = g$ has a unique solution $v \in Y^*$.
- (ii) The homogeneous equation $Au = 0$ has exactly p linearly independent solutions $u_1, \dots, u_p$ for some finite $p \geq 1$. In this case,
- (a) the homogeneous adjoint equation $A^*v = 0$ has exactly p linearly independent solutions $v_1, \dots, v_p$;
 - (b) the inhomogeneous equation $Au = f$ is solvable if and only if the right hand side f satisfies $(v_j, f) = 0$ for $j=1, \dots, p$;
 - (c) the inhomogeneous adjoint equation $A^*v = g$ is solvable if and only if the right hand side g satisfies $(g, u_j) = 0$ for $j=1, \dots, p$.

Chapter 4

Direct Boundary-Domain Integral Equations for the Neumann BVP

4.1 Formulation of the Neumann Boundary Value Problem

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, and $n(x)$ is the exterior unit normal vector which is defined for all $x \in \partial\Omega$. We shall consider the following scalar elliptic partial differential equation:

$$Au(x) = \sum_{i=1}^2 \frac{\partial}{\partial x_i} [a(x) \frac{\partial u(x)}{\partial x_i}] = f(x) \quad \text{in } \Omega, \quad (4.1)$$

where u is unknown function and f is a given function in Ω . We assume that $a(x) \in C^\infty(\mathbb{R}^2)$, $a(x) > 0$. For $\phi, \psi \in \mathcal{D}(\Omega)$, we can see that $(A\phi, \psi)_{L_2(\Omega)} = (\phi, A\psi)_{L_2(\Omega)}$, which implies that the operator A is formally self-adjoint.

4.1.1 First Green's Identity

The starting point for what follows is the well known theorem of Gauss and Ostrogradski, i.e., if $h \in C_0^1(\overline{\Omega})$, then

$$\int_{\Omega} \frac{\partial}{\partial x_i} h(x) dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) ds_x, \quad (i = 1, 2) \quad (4.2)$$

where,

$$\gamma^+ h(x) = \lim_{\Omega \ni y \rightarrow x \in \partial\Omega} h(y) \quad \text{for } x \in \partial\Omega$$

is the interior boundary trace of $h(x)$. Using the results for density and trace theorem (see, e.g., [[8], Theorem 3.29, 3.38]), the integral relation (4.2) holds for any $h \in H^1(\Omega)$. Indeed, since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$ we have a sequence $h_n \in C^1(\overline{\Omega})$ such that $\|h_n - h\|_{H^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Since $h_n \in C^1(\overline{\Omega})$, the integral relation (4.2) implies

$$\int_{\Omega} \frac{\partial}{\partial x_i} h_n(x) dx = \int_{\partial\Omega} \gamma^+ h_n(x) n_i(x) ds_x, (i = 1, 2). \quad (4.3)$$

Taking the limit as $n \rightarrow \infty$ in the equation (4.3), we have

$$\left| \int_{\Omega} \frac{\partial h_n}{\partial x_i} dx - \int_{\Omega} \frac{\partial h}{\partial x_i} dx \right| \leq \int_{\Omega} \left| \frac{\partial h_n}{\partial x_i} - \frac{\partial h}{\partial x_i} \right| dx \leq \|h_n - h\|_{H^1(\Omega)} \rightarrow 0.$$

Hence,

$$\int_{\Omega} \frac{\partial h_n}{\partial x_i} dx - \int_{\Omega} \frac{\partial h}{\partial x_i} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For the right side of equation (4.3),

$$\left| \int_{\partial\Omega} \gamma^+ h_n n_i ds_x - \int_{\partial\Omega} \gamma^+ h n_i ds_x \right| \leq \int_{\partial\Omega} |\gamma^+(h_n - h)| n_i ds_x \leq \|\gamma^+(h_n - h)\|_{L^2(\partial\Omega)},$$

the continuity of trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ and hence to $L^2(\partial\Omega)$ implies,

$\|\gamma^+(h_n - h)\|_{L^2(\partial\Omega)} \leq c \|\gamma^+(h_n - h)\|_{H^1(\partial\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$\int_{\partial\Omega} \gamma^+ h_n n_i ds_x - \int_{\partial\Omega} \gamma^+ h n_i ds_x \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ if we put $h(x) = a(x) \frac{\partial u(x)}{\partial x_i} v(x)$ and applying the Gauss-Ostrogradski theorem 4.2, we obtain

$$\int_{\Omega} \frac{\partial}{\partial x_j} (a(x) \frac{\partial u(x)}{\partial x_j} v(x)) dx = \int_{\partial\Omega} n_j \gamma^+ (a(x) \frac{\partial u(x)}{\partial x_j} v(x)) ds_x.$$

Summing over j and noting the relation

$$\frac{\partial}{\partial x_j} (a(x) \frac{\partial u}{\partial x_j} v(x)) = \frac{\partial}{\partial x_j} (a(x) \frac{\partial u}{\partial x_j}) v(x) + a(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j},$$

we obtain the following *Green's first identity*

$$\mathcal{E}(u, v) = - \int_{\Omega} (Au)(x) v(x) dx + \int_{\partial\Omega} T^+ u(x) \gamma^+ v ds_x \quad (4.4)$$

where,

$$\mathcal{E}(u, v) = \sum_{i=1}^2 \int_{\Omega} a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx, \quad (4.5)$$

is the symmetric bilinear form, and

$$T^+u(x) = \lim_{\Omega \ni y \rightarrow x \in \partial\Omega} \left[\sum_{i=1}^2 n_i(x) \gamma^+ \left[a(x) \frac{\partial u(x)}{\partial x_i} \right] \right] \quad \text{for } x \in \partial\Omega, \quad (4.6)$$

is the interior co-normal derivative. Thus we have proved the following Lemma.

Lemma 4.1. For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$,

$$\mathcal{E}(u, v) = -(Au, v)_{\Omega} + (T^+u, \gamma^+v)_{\partial\Omega}.$$

Remark 4.1. For $v \in \mathcal{D}(\Omega)$, $\gamma^+v = 0$. If $u \in H^1(\Omega)$, then we can define Au as a distribution on Ω by, $(Au, v) = -\mathcal{E}(u, v)$ for $v \in \mathcal{D}(\Omega)$.

For the operator A from (4.1), following [2, 7], we define the subspace

$$H^{1,0}(\Omega; A) = \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\},$$

endowed with the norm

$$\|g\|_{H^{1,0}(\Omega; A)}^2 = \|g\|_{H^1(\Omega)}^2 + \|Ag\|_{L_2(\Omega)}^2.$$

For $u \in H^1(\Omega)$ the classical co-normal derivative (4.6) is not well defined, but for $u \in H^{1,0}(\Omega; A)$, there exists the following continuous extension of this definition hinted by the first Green's identity (4.4)(see [5, 12] and reference therein).

Definition 4.1. [3, 7] For $u \in H^{1,0}(\Omega; A)$ the (canonical) co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined in the following weak form,

$$\langle T^+u, w \rangle_{\Omega} = \mathcal{E}(u, \gamma_{-1}^+w) + \int_{\Omega} (Au) \gamma_{-1}^+w dx \quad \text{for all } w \in H^{\frac{1}{2}}(\partial\Omega), \quad (4.7)$$

where $\gamma_{-1}^+ : H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega)$ is a continuous right inverse of the interior trace operator γ^+ , which maps $H^1(\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega)$, while $\langle \cdot, \cdot \rangle_{(\partial\Omega)}$ denote the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product, and $\mathcal{E}(u, v) = \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x)$ is the symmetric bilinear form. The operator $T^+ : H^{1,0}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and gives the continuous extension on $H^{1,0}(\Omega; A)$ of the classical co-normal derivative operator. For $u \in H^s(\Omega)$, $s > \frac{3}{2}$, the canonical co-normal derivative defined by (4.7) coincides with the classical one, defined in the trace sense, i.e., $T^+u(x) = a(x) \gamma^+ \left(\frac{\partial u(x)}{\partial n(x)} \right)$ where $n(x)$ is the interior unit normal vector.

Remark 4.2. The first Green identity (4.4) also holds for $u \in H^{1,0}(\Omega; A)$ and $v \in H^1(\Omega)$ (see, [5, 12]), i.e.,

$$\mathcal{E}(u, v) = -(Au, v)_\Omega + (T^+u, \gamma^+v)_{\partial\Omega}. \quad (4.8)$$

4.1.2 Second Green's Identity

In the first Green's identity, by interchanging the role of u and v and subtracting the result, we obtain the *second Green's identity*. For $u, v \in H^{1,0}(\Omega; A)$, the second Green's identity is

$$\int_{\Omega} (vAu - uAv) dx = \langle T^+u, \gamma^+v \rangle_{\partial\Omega} - \langle T^+v, \gamma^+u \rangle_{\partial\Omega}. \quad (4.9)$$

The boundary value problem is to find a scalar function $u \in H^{1,0}(\Omega; A)$ satisfying the partial differential equation

$$Au = f \quad \text{in} \quad \Omega, \quad (4.10)$$

the Neumann boundary conditions

$$T^+u = \psi_0 \quad \text{on} \quad \partial\Omega \quad (4.11)$$

where $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$ are some given functions. The boundary value problem (4.10) and (4.11) is called a Neumann boundary value problem. Here equation (4.10) is understood in distributional sense as in Remark 4.1 and equation (4.11) is understood in functional sense (4.7).

In application, the BVP (4.10)-(4.11) may describe a stationary heat transfer boundary value problem in isotropic inhomogeneous plane body Ω , where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\psi_0(x)$ is known heat flux on the boundary.

A function $v_0(x) = 1$ for $x \in \Omega$ is a solution of the homogeneous Neumann boundary

value problem

$$Av_0 = 0 \quad \text{in} \quad \Omega, \quad (4.12)$$

$$T^+v_0 = 0 \quad \text{on} \quad \partial\Omega. \quad (4.13)$$

Applying the second Green's identity to equations (4.12)-(4.13) we then obtain the orthogonality

$$\int_{\Omega} Au(x)dx - \int_{\partial\Omega} T^+u(x) = 0. \quad (4.14)$$

When considering the Neumann boundary value problem (4.10)-(4.11), and using the orthogonality (4.14) for the given data f and ψ_0 , we have to assume the solvability condition

$$\langle 1, f \rangle_{\Omega} = \langle \psi_0, 1 \rangle_{\partial\Omega} \quad (4.15)$$

Since there exists a nontrivial solution $v_0(x) = 1$ for $x \in \Omega$ of the homogeneous Neumann boundary value problem (4.12)-(4.13), we conclude that the solution of the Neumann boundary value problem (4.10)-(4.11) is only unique up to an additive constant.

Theorem 4.2. [9] a) *The homogeneous problem corresponding to the equation (4.10)-(4.11), admits solutions in $H^1(\Omega)$ spanned by $u^0 = 1$.*
b) *The non-homogeneous problem (4.10)-(4.11) is solvable if and only if*

$$\langle f, u^0 \rangle_{\Omega} - \langle \psi_0, \gamma^+ u^0 \rangle_{\partial\Omega} = 0. \quad (4.16)$$

Hence, the solution of the Neumann boundary value problem (4.10)-(4.11) is only unique up to an additive constant α . The constant $\alpha \in \mathbb{R}$ is uniquely determined when requiring an additional scaling condition on the solution u of (4.10)-(4.11), e.g.,

$$\int_{\Omega} u(x)dx = 0, \quad \text{or} \quad \int_{\partial\Omega} \gamma^+ u(x)dx = 0.$$

For this we define

$$H_*^1(\Omega) = \{v \in H^1(\Omega) : \int_{\Omega} v(x)dx = 0\}. \quad (4.17)$$

Using Green's first identity (4.4) we obtain the variational formulation of the Neumann boundary value problem (4.10)-(4.11) to find $u \in H_*^1(\Omega)$ such that

$$\mathcal{E}(u, v) = -\langle f, v \rangle_\Omega + \langle \psi_0, \gamma^+ v \rangle_{\partial\Omega} \quad (4.18)$$

is satisfied for all $v \in H_*^1(\Omega)$ [14].

Theorem 4.3. ([14], Theorem 4.9). *Let $f \in L_2(\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be given satisfying the solvability condition (4.16). Then there exists a unique solution $u \in H_*^1(\Omega)$ of the variational problem (4.18) satisfying*

$$\|u\|_{H^1(\Omega)} \leq \frac{1}{C_1} \{ \|f\|_{L_2(\Omega)} + C_T \|\psi_0\|_{H^{-\frac{1}{2}}(\partial\Omega)} \}$$

Since, the solution of the Neumann boundary value problem (4.10)-(4.11) is not unique, we can add an arbitrary constant $\alpha \in \mathbb{R}$ to the solution $u \in H_*^1(\Omega)$ to obtain the general solution $\tilde{u} = u + \alpha \in H^1(\Omega)$.

4.2 Parametrix and Parametrix-Based Potential Operators

Definition 4.2. A function $P(x, y)$ is a parametrix (Levi function) for the operator $A(x, \partial x)$ if

$$A(x, \partial x)P(x, y) = \delta(x - y) + R(x, y), \quad (4.19)$$

where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

There are different method of obtaining a parametrix. The method of frozen coefficient is among these. For the operator A , to every point y associate “frozen coefficient” $a(y)$ and obtain

$$A_y = \sum_{i=1}^2 \frac{\partial}{\partial x_i} [a(y) \frac{\partial}{\partial x_i}] = a(y) \Delta.$$

In this case, it is an operator with constant coefficient and the fundamental solution is given by

$$P(x, y) = \frac{\log |x - y|}{2\pi a(y)}, \quad x, y \in \mathbb{R}^2. \quad (4.20)$$

In particular, see [7], equation (4.20) is a parametrix for the operator A with singularity at y and after substituting (4.20) in (4.19) the corresponding remainder is calculated as

$$R(x, y) = \sum_{i=1}^2 \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2. \quad (4.21)$$

To show $P(x, y)$ is the parametrix (Levi-function) for the operator $A(x, \partial x)$ we have to show:

$$\begin{aligned} A(x, \partial x)P(x, y) &= \delta(x - y) + R(x, y). \\ A(x, \partial x)P(x, y) &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} [a(x) \frac{\partial P(x, y)}{\partial x_i}] \\ &= \sum_{i=1}^2 [a(x) \frac{\partial}{\partial x_i} (\frac{\partial P(x, y)}{\partial x_i}) + \frac{\partial a(x)}{\partial x_i} \frac{\partial P(x, y)}{\partial x_i}] \\ &= \sum_{i=1}^2 [a(x) \frac{\partial}{\partial x_i} (\frac{\partial}{\partial x_i} (\frac{P_\Delta(x, y)}{a(y)})) + \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} (\frac{P_\Delta(x, y)}{a(y)})] \\ &= \sum_{i=1}^2 [a(x) \frac{\partial^2}{\partial x_i^2} (\frac{P_\Delta(x, y)}{a(y)})] + \sum_{i=1}^2 [\frac{\partial}{\partial x_i} (\frac{P_\Delta(x, y)}{a(y)}) \frac{\partial a(x)}{\partial x_i}] \\ &= \sum_{i=1}^2 [a(x) \frac{\partial^2}{\partial x_i^2} (\frac{\log |x - y|}{2\pi a(y)})] + \sum_{i=1}^2 [\frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}] \\ &= \delta(x - y) + R(x, y) \end{aligned}$$

(since $\frac{\log |x - y|}{2\pi}$ is the fundamental solution of the Laplace operator Δ).

Here, we use $\sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} (\frac{\log |x - y|}{2\pi}) = \Delta_x (\frac{\log |x - y|}{2\pi}) = \delta(x - y)$ that followed from the definition of the fundamental solution of the Laplace operator. Also recall that the multiplication of a smooth function $a(x)$ with δ , $a(x)\delta(x - y) = a(y)\delta(x - y)$.

Observe that, $A_x P(x, y) = \delta(x - y) + \nabla a(x) \cdot \nabla P(x, y)$ and we can write the remainder as $R(x, y) = \nabla a(x) \cdot \nabla P(x, y)$. And for constant coefficient $a(x) = 1$, the operator A becomes the Laplace operator, Δ , and the parametrix $P(x, y)$ becomes its fundamental solution, $P_\Delta(x, y)$, and $R(x, y) \equiv 0$.

If $u \in H^{1,0}(\Omega; A)$, then substituting $v(x)$ by $P(x, y)$ in the second Green's identity (4.9) for $\Omega \setminus B(y, \varepsilon)$, where $B(y, \varepsilon)$ is a disk of radius ε centered at y , and taking the limit $\varepsilon \rightarrow 0$, we arrive at the parametrix-based third Green identity for $y \in \Omega$ (see [2, 11]) in 3D:

$$u(y) = \int_{\partial\Omega} [\gamma^+ u(x) T^+ P(x, y) - P(x, y) T^+ u(x)] ds_x - \int_{\Omega} R(x, y) u(x) dx + \int_{\Omega} P(x, y) f(x) dx \quad (4.22)$$

To derive appropriate boundary-domain integral equation to find complete Cauchy data $[\gamma^+ u(y), T^+ u(y)]$ for $y \in \partial\Omega$, we shall study the properties of volume potentials, single and double layer potentials that appears in the representation formula (4.22).

4.2.1 Volume Potentials

Similar to [2, 11], we define the parametrix-based logarithmic and remainder potential operators in 3D case as,

$$\mathcal{P}g(y) = \int_{\Omega} P(x, y) g(x) dx, \quad (4.23)$$

$$\mathcal{R}g(y) = \int_{\Omega} R(x, y) g(x) dx, \quad y \in \mathbb{R}^3. \quad (4.24)$$

Let P_{Δ} denotes the fundamental solution of the Laplace operator Δ , and the corresponding logarithmic potential by, $\mathcal{P}_{\Delta}$. Then,

$$\mathcal{P}g = \frac{1}{a} \mathcal{P}_{\Delta} g, \quad \mathcal{R}g = -\frac{1}{a(y)} \sum_{i=1}^2 \partial_i \mathcal{P}_{\Delta} [g(\partial_i a)]. \quad (4.25)$$

To see this,

$$\begin{aligned}
\mathcal{R}g(y) &= \int_{\Omega} R(x, y)g(x)dx \\
&= \int_{\Omega} \sum_{i=1}^2 \frac{x_i - y_i}{2\pi a(y) |x - y|^2} \frac{\partial a(x)}{\partial x_i} g(x)dx \\
&= \frac{1}{a(y)} \sum_{i=1}^2 \int_{\Omega} \frac{x_i - y_i}{2\pi |x - y|^2} \frac{\partial a(x)}{\partial x_i} g(x)dx \\
&= -\frac{1}{a(y)} \sum_{i=1}^2 \frac{\partial}{\partial y_i} \int_{\Omega} \frac{\log |x - y|}{2\pi} \frac{\partial a(x)}{\partial x_i} g(x)dx \\
&= -\frac{1}{a(y)} \sum_{i=1}^2 \frac{\partial}{\partial y_i} \mathcal{P}_{\Delta} \left(\frac{\partial a(x)}{\partial x_i} g(x) \right).
\end{aligned}$$

4.2.2 Surface Potentials

The single and double layer potential operators corresponding to the parametrix $P(x, y)$, are defined for $y \notin \partial\Omega$ as

$$Vg(y) = - \int_{\partial\Omega} P(x, y)g(x)ds_x, \quad (4.26)$$

$$Wg(y) = - \int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x, \quad (4.27)$$

where g is some scalar density function [2, 7].

As in [2, 7], let us introduce the following boundary integral (Pseudo-differential) operators of the direct values and co-normal derivatives of the single and double layer potentials for $y \in \partial\Omega$:

$$\mathcal{V}g(y) = - \int_{\partial\Omega} P(x, y)g(x)ds_x, \quad (4.28)$$

$$\mathcal{W}g(y) = - \int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x, \quad (4.29)$$

$$\mathcal{W}'g(y) = - \int_{\partial\Omega} T_y^+ P(x, y)g(x)ds_x, \quad (4.30)$$

$$\mathcal{L}^+g(y) = T_y^+ Wg(y) \quad (4.31)$$

Let $V_\Delta, W_\Delta, \mathcal{V}_\Delta, \mathcal{W}_\Delta, \mathcal{L}_\Delta^+$ denote potentials corresponding to the Laplace operator $A = \Delta$.

Then the following relations hold (see [2] for 3D case),

$$Vg = \frac{1}{a}V_\Delta g, \quad (4.32)$$

$$Wg = \frac{1}{a}W_\Delta(ag), \quad (4.33)$$

$$\mathcal{V}g = \frac{1}{a}\mathcal{V}_\Delta g, \quad (4.34)$$

$$\mathcal{W}g = \frac{1}{a}\mathcal{W}_\Delta(ag), \quad (4.35)$$

$$\mathcal{W}'g = \mathcal{W}'_\Delta g + [a \frac{\partial}{\partial n}(\frac{1}{a})]\mathcal{V}_\Delta g, \quad (4.36)$$

$$\mathcal{L}^+g = \mathcal{L}_\Delta^+(ag) + [a \frac{\partial}{\partial n}(\frac{1}{a})]W_\Delta^+(ag). \quad (4.37)$$

Proof. Considering $P_\Delta = aP$ and $T_\Delta^+ = \frac{1}{a}T^+$:

For $y \notin \partial\Omega$,

$$\begin{aligned} Vg(y) &= - \int_{\partial\Omega} P(x, y)g(x)ds_x \\ &= - \int_{\partial\Omega} \frac{\log|x-y|}{2\pi a(y)}g(x)ds_x \\ &= - \frac{1}{a(y)} \int_{\partial\Omega} \frac{\log|x-y|}{2\pi}g(x)ds_x \\ &= - \frac{1}{a(y)} \int_{\partial\Omega} P_\Delta(x, y)g(x)ds_x \\ &= \frac{1}{a(y)}V_\Delta g(y) \end{aligned}$$

Therefore, $Vg = \frac{1}{a}V_\Delta g$.

$$\begin{aligned} Wg(y) &= - \int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x \\ &= - \int_{\partial\Omega} [\sum_{i=1}^2 a(x)n_i(x)\frac{\partial}{\partial x_i}P(x, y)]g(x)ds_x \\ &= - \int_{\partial\Omega} [\sum_{i=1}^2 n_i(x)\frac{\partial}{\partial x_i}\frac{P_\Delta(x, y)}{a(y)}](ag)(x)ds_x \\ &= - \frac{1}{a(y)} \int_{\partial\Omega} [\sum_{i=1}^2 n_i(x)\frac{\partial}{\partial x_i}P_\Delta(x, y)](ag)(x)ds_x \\ &= \frac{1}{a(y)}W_\Delta(ag)(y) \end{aligned}$$

Therefore, $Wg = \frac{1}{a(y)}W_\Delta(ag)$.

Similarly, one can prove relations (4.34) and (4.35).

For $y \in \partial\Omega$,

$$\begin{aligned}
\mathcal{W}'g(y) &= - \int_{\partial\Omega} T^+(y, n(y), \partial y) P(x, y) g(x) dx \\
&= - \int_{\partial\Omega} [a(y) T_\Delta^+(y, n(y), \partial y) \frac{P_\Delta(x, y)}{a(y)}] g(x) dx \\
&= - \int_{\partial\Omega} [a(y) \frac{\partial}{\partial n_y} (\frac{P_\Delta(x, y)}{a(y)})] g(x) dx \\
&= - \int_{\partial\Omega} [a(y) (a(y) \frac{\partial P_\Delta(x, y)}{\partial n_y} - P_\Delta(x, y) \frac{\partial a(y)}{\partial n_y}) \frac{1}{a(y)^2}] g(x) dx \\
&= - \int_{\partial\Omega} \frac{\partial P_\Delta(x, y)}{\partial n_y} g(x) dx + \int_{\partial\Omega} \frac{1}{a(y)} P_\Delta(x, y) \frac{\partial a(y)}{\partial n_y} g(x) dx \\
&= - \int_{\partial\Omega} \frac{\partial P_\Delta(x, y)}{\partial n_y} g(x) dx + \frac{1}{a(y)} \frac{\partial a(y)}{\partial n_y} \int_{\partial\Omega} P_\Delta(x, y) g(x) dx \\
&= - \int_{\partial\Omega} \frac{\partial P_\Delta(x, y)}{\partial n_y} g(x) dx - a(y) \frac{\partial}{\partial n_y} (\frac{1}{a}) \int_{\partial\Omega} P_\Delta(x, y) g(x) dx \\
&= \mathcal{W}'_\Delta g + [a \frac{\partial}{\partial n} (\frac{1}{a})] \mathcal{V}_\Delta g
\end{aligned}$$

Hence, the relation (4.36) holds.

$$\begin{aligned}
\mathcal{L}^+g(y) &= T^+(y, n(y), \partial y) Wg(y) \\
&= a(y) \frac{\partial}{\partial n_y} (Wg(y)) \\
&= a(y) \frac{\partial}{\partial n_y} (\frac{1}{a(y)} W_\Delta(ag)) \\
&= a(y) [\frac{1}{a(y)} \frac{\partial}{\partial n_y} (W_\Delta(ag)) + \frac{\partial}{\partial n_y} (\frac{1}{a(y)}) W_\Delta(ag)] \\
&= \mathcal{L}_\Delta^+(ag) + [a \frac{\partial}{\partial n} (\frac{1}{a})] W_\Delta^+(ag).
\end{aligned}$$

□

Theorem 4.4. *For $s \in \mathbb{R}$, the following operators are continuous*

$$V : H^s(\partial\Omega) \mapsto H^{s+\frac{3}{2}}(\Omega)$$

$$W : H^s(\partial\Omega) \mapsto H^{s+\frac{1}{2}}(\Omega)$$

$$\mathcal{V} : H^s(\partial\Omega) \mapsto H^{s+1}(\partial\Omega)$$

$$\mathcal{W}, \mathcal{W}': H^s(\partial\Omega) \mapsto H^{s+1}(\partial\Omega)$$

$$\mathcal{L}^+ : H^s(\partial\Omega) \mapsto H^{s-1}(\partial\Omega)$$

Proof. We have the corresponding mappings for the corresponding constant-coefficient operators (see [5, 8, 14]). Then, the relations (4.32)-(4.36) imply the theorem claim. $\square$

Theorem 4.5. (*The jump relations*)

Let $u \in H^{-\frac{1}{2}}(\partial\Omega)$ and $v \in H^{\frac{1}{2}}(\partial\Omega)$. Then the following jump relation hold on $\partial\Omega$.

$$\gamma^+ V u(y) = \mathcal{V} u(y), \quad (4.38)$$

$$\gamma^+ W v(y) = -\frac{1}{2}v(y) + \mathcal{W} v(y), \quad (4.39)$$

$$T^+ V u(y) = \frac{1}{2}u(y) + \mathcal{W}' u(y), \quad (4.40)$$

where $y \in \partial\Omega$.

Proof. For the constant coefficient case, this theorem is well known (see [8, Theorem 6.11 and Equation (7.5)]). Then taking into account the relations (4.32)-(4.36), we can prove the theorem hold true for the variable coefficient $a \in C^\infty(\mathbb{R}^2)$. $\square$

4.2.3 Invertibility of the single layer potential operator in 2D.

The boundary integral operator, $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator of index zero ([8], Theorem 7.6). Thus the relation (4.34) leads to the same result for the single layer potential $\mathcal{V}$. For the 3D case, the following result holds [2], Lemma 4.2(i)].

Theorem 4.6. *If $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\mathcal{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof. The theorem holds for the operator $\mathcal{V}_\Delta$ (see [8, Corollary 8.11(ii)]), which by (4.34) implies it for the operator $\mathcal{V}$ as well. Hence, the theorem implies invertibility of the single layer potential operator mapping from $H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$. $\square$

But this is not true for 2D case. It is well-known (see [14, proof of Theorem 6.22]) that for some 2D domains the kernel of the operator $\mathcal{V}_\Delta$ is non-zero, which by (4.34) also implies that $\ker \mathcal{V} \neq \{0\}$ for the same domains.

In order to have invertibility for the single layer potential operator in 2D, we define the following subspace of the space $H^{-\frac{1}{2}}(\partial\Omega)$, (see [14, equation(6.30)])

$$H_*^{-\frac{1}{2}}(\partial\Omega) = \{\phi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle \phi, 1 \rangle_{\partial\Omega} = 0\}$$

where the norm in $H_*^{-\frac{1}{2}}(\partial\Omega)$ is induced by the norm in $H^{-\frac{1}{2}}(\partial\Omega)$ so that the theorem holds true for 2D.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^2$ have the diameter $\text{diam}(\Omega) < 1$. Then the single layer potential $\mathcal{V} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof. By [9, Theorem 6.23], for $\text{diam}(\Omega) < 1$ the operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ is $H^{-\frac{1}{2}}(\partial\Omega)$ -elliptic and since it is also bounded, c.f. Theorem 4.4 for $s = -\frac{1}{2}$, the Lax-Milgram theorem implies its invertibility. Then by the relation in (4.36) the invertibility of the operator $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$ also follows. $\square$

4.2.4 The Third Green's Identity

For $u \in H^{1,0}(\Omega; A)$ and using the notations for the potential operators (4.19), (4.20), (4.23), (4.24), (4.26) and (4.27) the third Green's identity (4.22) can be written as (cf.e.g., [2, 11, 13]),

$$u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}Au \quad \text{in } \Omega. \quad (4.41)$$

If $u \in H^1(\Omega)$, then the third Green identity below holds [9],

$$u + \mathcal{R}u + W\gamma^+u\mathcal{P}Au \quad \text{in } \Omega. \quad (4.42)$$

Applying the trace operator to equation (4.41) and using the jump relation theorem, we have

$$\begin{aligned} \gamma^+u + \gamma^+\mathcal{R}u - \gamma^+VT^+u + \gamma^+W\gamma^+u &= \gamma^+\mathcal{P}Au \\ \gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u - \frac{1}{2}\gamma^+u + \mathcal{W}\gamma^+u &= \gamma^+\mathcal{P}Au \\ \frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u &= \gamma^+\mathcal{P}Au \quad \text{on } \partial\Omega. \end{aligned} \quad (4.43)$$

Similarly, applying co-normal derivative operator to equation (4.41), and using again the jump relation, we have

$$\begin{aligned} T^+u + T^+\mathcal{R}u - T^+VT^+u + W\gamma^+u &= T^+\mathcal{P}Au \\ T^+u + T^+\mathcal{R}u - \frac{1}{2}T^+u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u &= T^+\mathcal{P}Au \\ \frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u &= T^+\mathcal{P}Au, \quad \text{on } \partial\Omega. \end{aligned} \quad (4.44)$$

For arbitrary functions u, f, Ψ, Φ , let us consider a more general “indirect” integral relation, associated with the equation (4.41),

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in } \Omega. \quad (4.45)$$

Lemma 4.8. *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$, satisfy equation (4.45). Then u belongs to $H^{1,0}(\Omega; A)$ and is a solution of PDE (4.10), i.e., $Au = f$ in Ω and*

$$V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega.$$

Proof. The proof follow in the similar way as in the corresponding proof in 3D case in [2, Lemma 4.1]. First of all, let us prove that $u \in H^{1,0}(\Omega; A)$. Observe that,

$$\begin{aligned} \Delta(au) &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} (au) \right) \\ &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(\frac{\partial a}{\partial x_i} u + a \frac{\partial u}{\partial x_i} \right) \\ &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(\frac{\partial a}{\partial x_i} u \right) + \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(a \frac{\partial u}{\partial x_i} \right) \\ &= \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(\frac{\partial a}{\partial x_i} u \right) + Au. \end{aligned}$$

Therefore we have, $Au = \Delta(au) - \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left(u(x) \frac{\partial a(x)}{\partial x_i} \right)$ and the last term belongs to $L_2(\Omega)$. We need only to show that $\Delta(au) \in L_2(\Omega)$. Rewriting equation (4.45) as a function of u , we have

$$u = \mathcal{P}f - \mathcal{R}u + V\Psi - W\Phi,$$

then multiplying by $a(y)$ gives

$$\begin{aligned} au &= a\mathcal{P}f - a\mathcal{R}u + aV\Psi - aW\Phi \\ &= \mathcal{P}_\Delta f - a\mathcal{R}u + V_\Delta\Psi - W_\Delta(a\Phi) \end{aligned}$$

Then taking the Laplacian of the last equation gives

$$\Delta(au) = \Delta\mathcal{P}_\Delta f - \Delta(a\mathcal{R}u) + \Delta(V_\Delta\Psi) - \Delta(W_\Delta(a\Phi)).$$

The last two terms in the right hand side are harmonic and also $\mathcal{R}u \in H^2(\Omega)$ for $u \in H^1(\Omega)$

$$\Delta(\mathcal{P}_\Delta f) = f \in L_2(\Omega)$$

implies

$$\Delta(au) \in L_2(\Omega)$$

Hence $Au \in L_2(\Omega)$, $u \in L_2(\Omega)$. Subtracting (4.45) from (4.42)

$$\text{i.e., } \begin{cases} u + \mathcal{R}u + W\gamma^+u = \mathcal{P}Au \\ u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \end{cases}, \text{ we have}$$

$$\begin{aligned} -V(T^+u - \Psi) - W(\Phi - \gamma^+u) &= \mathcal{P}(Au - f) \text{ in } \Omega \\ \Leftrightarrow -V\Psi^* - W\Phi^* &= \mathcal{P}(Au - f) \text{ in } \Omega, \end{aligned} \quad (4.46)$$

where $\Psi^* = T^+u - \Psi$ and $\Phi^* = \Phi - \gamma^+u$

Multiplying equality (4.46) by $a(y)$ we get

$$-a(y)V\Psi^* - a(y)W\Phi^* = a(y)\mathcal{P}(Au - f) \text{ in } \Omega,$$

or

$$-V_\Delta\Psi^* - W_\Delta(a\Phi^*) = \mathcal{P}_\Delta(Au - f) \text{ in } \Omega. \quad (4.47)$$

Applying the Laplace operator Δ to the last equation (4.47) and taking in to consideration that both functions in the left hand-side are harmonic surface potentials, while the right-hand side functions is the classical Logarithmic volume potential, we arrive at equation

$$Au = f \text{ in } \Omega.$$

This shows that u solves the differential equation (4.10). Substituting $Au = f$ back in to (4.46) leads to

$$V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega.$$

□

Lemma 4.9. [7, Lemma 2] (i) Let either $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\text{diam}(\Omega) < 1$, or $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$. If $V\Psi^* = 0$ in Ω , then $\Psi^* = 0$ on $\partial\Omega$.

(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W\Phi^* = 0$ in Ω , then $\Phi^* = 0$ on $\partial\Omega$.

Proof. (i) Taking the trace of equation in $V\Psi^* = 0$ on $\partial\Omega$, by the jump relation (4.39) we have $\mathcal{V}\Psi^* = 0$ on $\partial\Omega$. If $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\text{diam}(\Omega) < 1$, then the result follows from the invertibility of the single layer potential given by Theorem 4.7. On the other hand, if $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 4.6.

(ii) Let us take the trace of equation in Lemma 4.9(ii) on $\partial\Omega$, and use the jump relation (4.40) to obtain,

$$-\frac{1}{2}\Phi^* + \mathcal{W}\phi^* = 0 \text{ on } \Omega.$$

Multiplying this equation by $a(y)$, denoting $\hat{\Phi}^* = a\Phi^*$ and using the relation in (4.35), we obtain equation

$$\hat{\Phi}^* - [\frac{1}{2}I + \mathcal{W}_\Delta]\hat{\phi}^* = 0 \text{ on } \partial\Omega.$$

It is well known that this equation has only the trivial solution. It is particularly due to the contraction property of the operator $\frac{1}{2}I + \mathcal{W}_\Delta$ [15, Theorem 3.1]. Since $a(y) \neq 0$, the result follows. □

4.2.5 Invertibility of the Hypersingular Operator in 2D

The kernel of the co-normal derivative of the double layer potential $\mathcal{L}_\Delta^+$ consists of non-zero functions. Infact, it is the same for 3D case. This can be seen by $u^0(y) \equiv 1$ in Ω and inserting in to the integral equation (4.44) for the case $a \equiv 1$, we obtain $\mathcal{L}_\Delta^+ \gamma^+ u^0 = 0$ on $\partial\Omega$. In order to have invertibility for the hypersingular Operator, define the subspace of the space $H^{\frac{1}{2}}(\partial\Omega)$, (see [14, p.147]),

$$H_{**}^{\frac{1}{2}}(\partial\Omega) = \{g \in H^{\frac{1}{2}}(\partial\Omega) : \langle g, 1 \rangle_{(\partial\Omega)} = 0\}.$$

Let us introduce the operator,

$$\hat{\mathcal{L}}g = [\mathcal{L}^+ + \frac{\partial a}{\partial n}(-\frac{1}{2}I + \mathcal{W})]g = \mathcal{L}_\Delta^+(ag) \text{ on } \partial\Omega.$$

Then the following result follows.

Theorem 4.10. *Let $\partial\Omega$ be an infinitely smooth boundary curve.*

(i) *The pseudo-differential operator*

$$\hat{\mathcal{L}} : H_{**}^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega), \quad (4.48)$$

is invertible.

(ii) *The operator*

$$\mathcal{L}^+ - \hat{\mathcal{L}} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega), \quad (4.49)$$

is bounded and compact.

Proof. For $g \in H^{\frac{1}{2}}(\partial\Omega)$ using the jump relation (4.39), one can obtain the relation, $\mathcal{L}^+g = \mathcal{L}_\Delta(ag) - \frac{\partial a}{\partial n}(\frac{-1}{2}I + \mathcal{W})g$, or $\hat{\mathcal{L}}g = \mathcal{L}_\Delta(ag) = \mathcal{L}^+g + \frac{\partial a}{\partial n}(-\frac{1}{2}I + \mathcal{W})g$. Theorem 4.4, where $s = \frac{1}{2}$ implies the hypersingular boundary integral operator $\mathcal{L}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is bounded. Moreover, it is $H_{**}^{\frac{1}{2}}(\partial\Omega)$ -elliptic [14, equation(6.38)]. Then, applying the Lax-Milgram lemma we conclude the $H_{**}^{\frac{1}{2}}(\partial\Omega)$ -invertibility of $\mathcal{L}_\Delta$. Hence the invertibility of (4.48). The operator $\mathcal{W} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{3}{2}}(\partial\Omega)$ is continuous (Theorem 4.4 for $s = \frac{1}{2}$), and since $H^{\frac{3}{2}}(\partial\Omega)$ is continuously embedded in $H^{\frac{1}{2}}(\partial\Omega)$, using the relation $\mathcal{L}^+ - \hat{\mathcal{L}} = -\frac{\partial a}{\partial n}(\frac{1}{2}I - \mathcal{W})$, we obtain continuity of the operator $\mathcal{L}^+ - \hat{\mathcal{L}} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$. The embedding $H^{\frac{1}{2}}(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$ is compact, which implies that the operator $\mathcal{L}^+ - \hat{\mathcal{L}} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is compact. $\square$

Corollary 4.11. *The operator $\mathcal{L}^+ : H^{\frac{1}{2}}(\partial\Omega) \mapsto H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm operator index zero.*

Proof. The operator $\mathcal{L}_\Delta : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm of index zero (see [8, Theorem 7.8]). Thus the operator $\hat{\mathcal{L}} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ is Fredholm of index zero. Since $(\mathcal{L}^+ - \hat{\mathcal{L}}) + \hat{\mathcal{L}}$, it is the sum of a compact operator and a Fredholm operator of index zero and hence is also a Fredholm operator of index zero (cf. [[8], Theorem 2.26]). $\square$

4.3 BDIE systems for the Neumann BVP

In this section, we shall derive and investigate the BDIE systems for the Neumann boundary value problem (4.10)-(4.11).

To reduce the variable coefficient Neumann BVP (4.10)-(4.11) to a segregated boundary-domain integral equation system, let us denote the unknown co-normal derivative as $\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$, and further consider φ as formally independent of u .

Assuming that the function u satisfies PDE $Au = f$, by substituting the Neumann condition into the third Green identity (4.41) and either into its trace (4.43) or either into its co-normal derivative (4.44) on $\partial\Omega$, we can reduce the BVP (4.10)-(4.11) to two different systems of BDIEs for the unknown function $u \in H^{1,0}(\Omega; A)$ and $\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega)$.

4.3.1 Formulation of BDIE system (N1)

From the third Green's identity (4.41) in Ω we get

$$\begin{aligned} u + \mathcal{R}u - VT^+u + W\gamma^+u &= \mathcal{P}f \quad \text{in } \Omega \\ u + \mathcal{R}u - V\psi_0 + W\varphi &= \mathcal{P}f \\ u + \mathcal{R}u + W\varphi &= \mathcal{P}f + V\psi_0 \\ u + \mathcal{R}u + W\varphi &= G_0 \quad \text{in } \Omega, \end{aligned} \tag{4.50}$$

where $G_0 = \mathcal{P}f + V\psi_0$ in Ω .

Taking the co-normal derivative (4.44) of the third Green's identity (4.41) we obtain

$$\begin{aligned} \frac{1}{2}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+\gamma^+u &= T^+\mathcal{P}f \quad \text{on } \partial\Omega, \\ \frac{1}{2}\psi_0 + T^+\mathcal{R}u - \mathcal{W}'\psi_0 + \mathcal{L}^+\varphi &= T^+\mathcal{P}f, \\ T^+\mathcal{R}u + \mathcal{L}^+\varphi &= T^+\mathcal{P}f + \mathcal{W}'\psi_0 - \frac{1}{2}\psi_0, \\ T^+\mathcal{R}u + \mathcal{L}^+\varphi &= T^+G_0 - \psi_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.51}$$

where $T^+G_0 = T^+(\mathcal{P}f + V\psi_0) = T^+\mathcal{P}f + T^+V\psi_0 = T^+\mathcal{P}f + \frac{1}{2}\psi_0 + \mathcal{W}'\psi_0 \Leftrightarrow T^+G_0 - \psi_0 = T^+\mathcal{P}f + \mathcal{W}'\psi_0 - \frac{1}{2}\psi_0 \Leftrightarrow T^+G_0 = T^+\mathcal{P}f + \mathcal{W}'\psi_0 + \frac{1}{2}\psi_0$ (from jump relation Theorem 4.5).

Hence, the above BDIEs (4.50)-(4.51) together

$$\begin{aligned} u + \mathcal{R}u + W\varphi &= G_0 \quad \text{in } \Omega, \\ T^+\mathcal{R}u + \mathcal{L}^+\varphi &= T^+G_0 - \psi_0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.52}$$

System (4.52) can be written in matrix operator form as:

$$\mathcal{B}^1 \mathcal{U} = \mathcal{G}^1$$

where $\mathcal{U} = [u, \varphi]^T \in H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega)$; the superscript T denotes transposition and

$$\mathcal{B}^1 = \begin{bmatrix} I + \mathcal{R} & W \\ T^+\mathcal{R} & \mathcal{L}^+ \end{bmatrix}, \quad \mathcal{G}^1 = \begin{bmatrix} G_0 \\ T^+G_0 - \psi_0 \end{bmatrix} \tag{4.53}$$

From the mapping properties of the operators V in Theorem 4.4 and $\mathcal{P}$ in [7, Theorem 3], we get the inclusion $G_0 \in H^{1,0}(\Omega; A)$, and definition 4.1 implies $T^+G_0 \in H^{-\frac{1}{2}}(\partial\Omega)$.

Therefore, $\mathcal{G}^1$ belongs to $H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$. Due to the mapping properties of the operators involved in $\mathcal{B}^1$, the operator $\mathcal{B}^1 : H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$, is bounded.

4.3.2 Formulation of BDIE system (N2)

From the third Green's identity (4.41) in Ω we get

$$\begin{aligned} u + \mathcal{R}u - VT^+u + W\gamma^+u &= \mathcal{P}f \quad \text{in } \Omega \\ u + \mathcal{R}u - V\psi_0 + W\varphi &= \mathcal{P}f \\ u + \mathcal{R}u + W\varphi &= \mathcal{P}f + V\psi_0 \\ u + \mathcal{R}u + W\varphi &= G_0 \quad \text{in } \Omega, \end{aligned} \tag{4.54}$$

where $G_0 = \mathcal{P}f + V\psi_0$ in Ω .

Taking the trace (4.43) of the third Green's identity (4.41), we get

$$\begin{aligned} \frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u &= \gamma^+\mathcal{P}f \quad \text{on } \partial\Omega, \\ \frac{1}{2}\varphi + \gamma^+\mathcal{R}u - \mathcal{V}\psi_0 + \mathcal{W}\varphi &= \gamma^+\mathcal{P}f, \\ \gamma^+\mathcal{R}u + \frac{1}{2}\varphi + \mathcal{W}\varphi &= \gamma^+G_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.55}$$

where $\gamma^+ G_0 = \gamma^+ \mathcal{P}f + \gamma^+ \mathcal{V}\psi_0$.

Combining (4.54) and (4.55) we have the BDIE system

$$\begin{aligned} u + \mathcal{R}u + W\varphi &= G_0 \quad \text{in } \Omega, \\ \gamma^+ \mathcal{R}u + \frac{1}{2}\varphi + \mathcal{W}\varphi &= \gamma^+ G_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.56}$$

In a matrix form, (4.56) can be represented as

$$\mathcal{B}^2 \mathcal{U} = \mathcal{G}^2$$

where

$$\mathcal{B}^2 = \begin{bmatrix} I + \mathcal{R} & W \\ \gamma^+ \mathcal{R} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \quad \mathcal{G}^2 = \begin{bmatrix} G_0 \\ \gamma^+ G_0 \end{bmatrix} \tag{4.57}$$

By the trace theorem $\gamma^+ G_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathcal{G}^2 \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. The mapping properties of operators involved in $\mathcal{B}^2$ implies the operator $\mathcal{B}^2 : H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded.

4.4 Equivalence and Invertibility Theorems

In the following theorem we shall see the equivalence of the original boundary value problem to the obtained boundary-domain integral equation systems.

Theorem 4.12. *Let $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $f \in L_2(\Omega)$ satisfy the solvability condition (4.16).*

(i) *If a function $u \in H^1(\Omega)$ solves the Neumann BVP (4.10)- (4.11), then the couple (u, φ) , where*

$$\varphi = \gamma^+ u \in H^{\frac{1}{2}}(\partial\Omega), \tag{4.58}$$

solves the BDIE systems (4.52) and (4.56).

(ii) *Vice versa, if a pair $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves one of the BDIE systems, (4.52) or (4.56), then the pair solves the other one BDIE system and u solves the Neumann BVP (4.10)- (4.11), and φ satisfies (4.58).*

(iii) *The homogeneous BDIE systems (4.52) and (4.56) have unique linearly independent solutions spanned by $\mathcal{U}^0 = (u^0, \varphi^0)^T = (1, 1)^T$ in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$. Condition (4.16) is necessary and sufficient for solvability of the non-homogeneous BDIE systems (4.52) and (4.56) in $H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.*

Proof. (i) Let $u \in H^1(\Omega)$ be solution of the Neumann BVP (4.10)- (4.11). Since $f \in L_2(\partial)$, then $u \in H^{1,0}(\Omega; A)$. Setting $\varphi = \gamma^+ u$, and recalling how BDIE systems (4.52) and (4.56) were constructed, we obtain that (u, φ) solves them .

(ii) Let now a couple $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the system (4.52) or (4.56). Due the first equations in the BDIE systems, the hypotheses of Lemma 4.8 are satisfied implying that u belongs to $H^{1,0}(\Omega; A)$ and solves PDE (4.10) in Ω , while the following equation also holds,

$$V(\psi_0 - T^+ u)(y) - W(\varphi - \gamma^+ u)(y) = 0, y \in \Omega \quad (4.59)$$

If a pair $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the system (4.52) then taking the co-normal derivatives of the first equation in (4.52) and subtracting the second from it, we get $T^+ u = \psi_0$ on $\partial\Omega$. Thus the Neumann condition is satisfied, and using it in (4.59) we get $W(\varphi - \gamma^+ u)(y) = 0$ on $y \in \partial\Omega$. Lemma 4.9(ii) implies $\varphi = \gamma^+ u$ on $\partial\Omega$.

If a pair $(u, \varphi) \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ solves the system (4.56) then taking the trace of the first equation in (4.56) and subtracting the second from it, we get $\varphi = \gamma^+ u$ on $\partial\Omega$. Then inserting it in (4.59) gives $V(\psi_0 - T^+ u)(y) = 0$ on $y \in \partial\Omega$. Lemma 4.9(i) implies $\psi_0 = T^+ u$ on $\partial\Omega$. Hence the Neumann condition is satisfied.

(iii) Theorem 4.2 along with item (i)-(ii) imply the claims of item (iii). $\square$

Note that Theorem 4.12(iii) implies that the operators

$$\mathcal{B}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$$

and

$$\mathcal{B}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

are not injective.

Theorem 4.13. *The operator*

$$\mathcal{B}^1 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega), \quad (4.60)$$

is bounded Fredholm operator with zero index.

Proof. By corollary 4.11, the operator $\hat{\mathcal{L}} : H^{\frac{1}{2}}(\partial\Omega) \mapsto H^{-\frac{1}{2}}(\partial\Omega)$ is a Fredholm operator with zero index. Therefore, the operator

$$\mathcal{B}_0^1 = \begin{bmatrix} I & W \\ 0 & \hat{\mathcal{L}} \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

is also Fredholm with zero index. By the properties of operators $\mathcal{R}$ and $T^+\mathcal{R}$ (see, e.g., [[7], corollary 2]) and statement (4.49) of Theorem 4.10, the operator

$$\mathcal{B}^1 - \mathcal{B}_0^1 = \begin{bmatrix} \mathcal{R} & 0 \\ T^+\mathcal{R} & \mathcal{L}^+ - \hat{\mathcal{L}} \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega),$$

is compact, implying that operator (4.60) is Fredholm with zero index ([8], Theorem 2.26). $\square$

Theorem 4.14. *The operator*

$$\mathcal{B}^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \quad (4.61)$$

is Fredholm operator with zero index.

Proof. Let us consider the operator

$$\mathcal{B}_0^2 = \begin{bmatrix} I & W \\ 0 & \frac{1}{2}I \end{bmatrix}$$

Then, $\mathcal{B}_0^2 : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators, $I : H^1(\Omega) \mapsto H^1(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega) \mapsto H^{\frac{1}{2}}(\partial\Omega)$. Due to the compactness properties of $\mathcal{R}$, $\gamma^+\mathcal{R}$ and $\mathcal{W}$ (see, e.g., [[7], corollary 1 and 2]), the operator

$$\mathcal{B}^2 - \mathcal{B}_0^2 = \begin{bmatrix} \mathcal{R} & 0 \\ \gamma^+\mathcal{R} & \mathcal{W} \end{bmatrix} : H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \mapsto H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega),$$

is compact. This implies that operator (4.61) is a Fredholm operator with zero index. $\square$

Moreover, the kernels (null-spaces) of these operators are spanned by the element $(u^0, \varphi^0) = (1, 1)$ and thus the kernels and co-kernels of the operators are one dimensional. This follows from Theorem 4.12(iii).

Chapter 5

Conclusion

In this thesis work, we have considered the interior Neumann boundary value problem for a variable-coefficient PDE in a 2D domain, where the right hand side function is from $L_2(\Omega)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial\Omega)$. Unique solvability of the Neumann BVP have been investigated. Mapping and jump properties of surface and volume integral potentials based on a generalized parametrix of this BVP were presented. The BVP have been reduced to two systems of Boundary-Domain Integral Equations and equivalence of these Boundary-Domain Integral Equation systems to the original BVP were proved.

The null-spaces of the corresponding BDIE systems are not trivial. Moreover, the BDIE systems are neither uniquely nor unconditionally solvable. The BDIE operators for the Neumann BVP are bounded but only Fredholm with zero index. Moreover, the kernels of these operators are spanned by the element $(u^0, \varphi^0) = (1, 1)$. Thus, the BDIE operators $\mathcal{B}^1$ and $\mathcal{B}^2$ have non-zero kernels and are not invertible.

Bibliography

- [1] Andreas Pomp: The Boundary-Domain Integral Method for Elliptic Systems With Applications in Shells, Lecture Notes in Mathematics, vol. 1683, Springer, Berlin Heidelberg, 1998.
- [2] Chkadua, O., Mikhailov, S.E., and D. Natroshvili: Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, I: Equivalence and invertibility. J. Integral Equations and Appl., Vol.21(4), 499-543, 2009.
- [3] Chkadua, O., Mikhailov, S.E., and D.Natroshvili: Analysis of direct segregated boundary-domain integral equations for variable-coefficient mixed BVPs in exterior domains. Analysis and Appl., vol.11, 1-33, 2013.
- [4] Chkadua, O., Mikhailov, S.E., and D. Natroshvili: Analysis of some localized boundary-domain integral equations. J. Integral Equations and Appl., Vol.21, 405-445, 2009.
- [5] Costabel, M.: Boundary integral operators on Lipschitz domains: elementary results. SIAM J. Mathematical Anal. 19, 613-626, 1988.
- [6] Dorothee D. Haroske Hans Triebel. Distributions, Sobolev Spaces, Elliptic Equations. EMS Textbooks in Mathematics, 2008.

- [7] Dufera, T.T. and Mikhailov, S.E.: Analysis of Boundary-Domain Integral Equations with variable-coefficient Dirichlet BVP in 2D: Integral Methods in science and Engineering: Computational and Analytic Aspects (Edited by C. Constanda and A. Kirch), Birkhauser, Boston: 163-175, 2015.
- [8] McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge, 2000.
- [9] Mikhailov, S.E.: Analysis of Segregated Boundary-Domain Integral Equations for Variable-Coefficient Dirichlet and Neumann Problems with General Data: 1-32, 2015.
- [10] Mikhailov, S.E.: Finite-dimensional perturbations of linear operators and some applications to boundary integral equations. Eng. Anal. Bound. Elem., Vol. 23, 805-813, 1999.
- [11] Mikhailov, S.E.: Localized boundary-domain integral formulations for problems with variable coefficients. Int. J. Engineering Analysis with Boundary Elements, 26, 681-690, 2002.
- [12] Mikhailov, S.E.: Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. Mathematical Analysis and Appl. 378, 324-342, 2011.
- [13] Miranda, C.: Partial Differential Equations of Elliptic Type. Berlin-Heidelberg-New York, Springer, 1970.
- [14] Steinbach, O.: Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements. Springer, 2007.
- [15] Steinbach, O., Wendland, W.L.: On C. Neumann's method for second order elliptic systems in domains with non-smooth boundaries. Mathematical Analysis and Appl. Vol. 262, 733-748, 2001.