Boundary Integral Equation for Two-Dimensional Interface Problem in Electromagnetics

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Author's Signature

To my family

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Abstract

In this thesis we consider two-dimensional interface problem in electromagnetics. Using fundamental solution we reduce the problem to boundary integral equation. We treat the boundary integral equation with less regularity assumption in appropriate Sobolev spaces. Finally, we show the unique solvability of the integral equation.

Contents

List of Figures

Notation and Abbreviations

Ω	A simply connected bounded region in $\mathbb{R}^2$.
$\bar{\Omega}$	The closure of the set Ω
Γ	The boundary of Ω.
Ω^+	exterior of Γ(or air)
$C^0(\Gamma)$	A set of all continuous functions on Γ(or boundary).
$C^1(\Gamma)$	A set of all continuous functions and differentiable on Γ.
$C^2(\Omega)$	The set of all continuous functions and twice differentiable on Ω.
$C(\Omega)$	The space of continuous functions in Ω.
$\nabla \times \vec{F}$	curl of the function $\vec{F}$
$\nabla.\vec{F}$	Divergence of the function $\vec{F}$
$C^k(\Omega)$	The set of k-th continuously differentiable functions defined on Ω
$C^\infty(\Omega)$	The set of all infinitely differentiable function on Ω .
$C^2(\Omega^+)$	The set of twice continuously differentiable functions on Ω^+
$C_c^\infty(\Omega)$	The space of continuous functions with compact support, having continuous derivatives of every order on Ω
$H_0^{(1)}$	The Hankel function of first kind and of order zero.
$\mathbb{R}^n$	The n-dimensional Euclidean space.
Δ	Laplace's operator.
∇	The gradient operator.
$\mathscr{D}, \mathscr{A}$	The spaces of test functions.
δ	Dirac's delata function.
$\mathscr{F}$	Fourier transform operator.
$\mathscr{F}^{-1}$	Inverse Fourier transform operator.
$\mathscr{H}$	Hilbert transform operator
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
T^*	The adjoint operator of T
$W_p^k(\Omega)$	Sobolev spaces order k.

$H^k(\Omega) = W^{k,2}(\Omega)$	Hilbert-Sobolev space.
$\vert . \vert$	The Absolute value
$\Vert . \Vert$	The Euclidean Norm
$\frac{\partial u}{\partial n_q}$	The partial derivative of the function u with respect to the unit outward normal at the point q on the boundary.
D^α	$(\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, a partial differential operator of order $\vert\alpha\vert = \alpha_1 + \cdots + \alpha_n$
O	**O**rder **o**f
pv	**P**rincipal **V**alue
TM	**T**ransverse **M**agnetic
TE	**T**ransverse **E**lectric
PDE	**P**artial **D**ifferential **E**quation .
BVP	**B**oundary **V**alue **P**roblem .
BIE	**B**oundary **I**ntegral **E**quation .
Supp	**S**upport of **F**unction
Re	**R**eal **p**arts
esssup	Essential **S**upremum
MFS	**M**ethod of **F**undamental **S**olutions

Chapter 1
Introduction and Background

Boundary integral equations (BIEs) are reformulations of Boundary value problems (BVPs) for partial differential equations(PDEs). They have been present in the mathematics literature for almost two centuries, and engineers have made much use of them for solving a wide variety of real world problems. The boundary integral equation method is very well developed as a tool for the analysis and numerical solution of strongly elliptic boundary value problems in both bounded and unbounded domains, provided the boundary itself is bounded. The BIE has found application in such diverse topics as stress analysis, potential flow, fracture mechanics, acoustics and electromagnetics(the subject of this thesis). Electromagnetism is an important branch of physical science. The problem of electromagnetic scattering by various objects has always been a subject of interest for researchers in various disciplines. Many analytical and numerical methods have been proposed in order to handle the numerous electromagnetic scattering situations. For example, numerical techniques are available to solve the integral equation which arises in the formulation of the induced or polarized current on or inside objects of various shapes. As it is indicated in [N.O01], electromagnetic problems can be categorized in the case of: the solution region of the problem (interior or exterior), the nature of the equation describing the problem (differential or integral or both) or the associated boundary conditions (Drichlet or Neumann types). It has been known for a long time that boundary value problems for elliptic partial differential equations can be reformulated in terms of boundary integral equations. Elliptic PDEs are associated with steady state phenomena. An elliptic PDE usually models an interior problems and hence the solution region is usually closed or bounded.

As it is indicated in [C.A84], the first application of a boundary integral formulation for the solution of the Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0$$

was derived by Banaugh and Goldsmith. Other applications include that of Shaw also for acoustic waves, Hwang and Tuck and Lee for harbor resonance, Isaacson, Harms, and Au and Brebbia[C.A84] for diffraction of water waves. Hence, the method of solving the potential equation by transforming it into boundary integral equation is not recent. In 1900, Fredholm employed it in the potential theory to determine the unknown boundary quantities from the prescribed ones. In 1963, Jaswon and Symm presented a numerical technique to solve Fredholm boundary integral equations. Later, Katsikadelis [J.T02] formulated the boundary integral equations for the torsion problem of composite bars consisting of two or more materials. In 1983 S.I. Hariharan and E.Stephan [S.I83] solved a two-dimensional interface problem in electromagnetics using boundary integral equations with less regularity assumptions in appropriate Sobolev spaces. The essential reformulation of the PDE consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution at points in the domain.

The theory of integral equations has been an active research field for many years and is based on analysis, function theory, and functional analysis. On the other hand, integral equations are of practical interest because of the boundary integral equation method, which transforms partial differential equations on a domain into integral equations over its boundary. The integral equation method is restricted to the case of a homogeneous differential equation of the form

$$Lu = 0$$

As it is indicated in [Wol95], the BIE methods are well suited for homogeneous body scattering problems for two primary reasons. First, the discretization needs to be performed only on the bounding surface of the scatterer in contrast to volume discretization. Second, because of the nature of the kernel function used in the integral equations, the calculated scattered waves automatically satisfy the Sommerfield radiation condition without the need for further constraints.

This thesis revolves around an aim that is two fold: first, to acquire more knowledge about a boundary integral equation method for a two-dimensional interface problem and the reformulation of the problem in to integral equation in electromagnetics. Secondly, to better understand the unique solvability of the integral equation in appropriate Sobolev spaces. To do this, we discussed on some preliminary definitions and formulas that is used to transform the BVPs in to the corresponding equivalent BIE problem. Here, the use of Green's theorems and the fundamental solution of the Helmholtz equation (Green's function) are the useful and basic tools for this purpose.

The thesis is organized as follows: In chapter 2, we review some important results in electromagnetics. In chapter 3, we consider on definitions of some important terms and formulas. In chapter 4, we formulate the boundary integral equation of the interface problem from its fundamental solution and emphasis is given on the unique solvability of the boundary integral equation in appropriate Sobolev spaces and this is the main body of the thesis.Finally, in chapter 5, general conclusions and suggestions for possible further work are given.

1.1 Statement of the Problem

The development of integral methods has been advanced mainly by researchers in the electromagnetic scattering area. This is due to the fact that scattering and radiation problems involve unbounded regions,where integral methods are ideal techniques.The method for solving problems of mathematical physics has its origin in the work of G. Green. He formulated, in 1828, the integral representation of the solution for the Dirichlet and Neumann problems of the Laplace equation by introducing the so-called Green's function for these problems. In 1872, Betti presented a general method for integrating the equations of elasticity and deriving their solution in integral form. Basically, this may be regarded as a direct extension of Green's approach to the Navier equations of elasticity. In 1885, Somigliana used Betti's reciprocal theorem to derive the integral representation of the solution for the elasticity problem, including in its expression the body forces, the boundary displacements and the tractions. As the researcher read and googled various of researches on BIE, the necessary procedures for the unique solvability of the BIE in the appropriate Sobolev space are not included in some of the previous studies. In order to react on the unique solvability of the BIE in the appropriate Sobolev space arrangement of procedures are very important to understand the theory about the method for those who want to know or study on this area. Therefore, the researcher conducted the study to answer the following research questions:

1. How can the problem be transformed to Fredholm integral equation of second type?
2. Does the unique solution of the integral equation exist in appropriate Sobolev space?

1.2 Objectives of the study

1.2.1 General objectives

The general objective of this study is to solve two-dimensional interface electromagnetic problems using Boundary integral equation method.

1.2.2 Specific objectives

The specific objectives of the study are to:

- to formulate a boundary integral equation.
- to obtain the unique solvability of the integral equation in appropriate Sobolev spaces.

1.3 Significance of the study

The significance(or purpose) of the study are:

- to acquire more knowledge about a BIE method for a two-dimensional interface problem and the reformulation of the problem in to integral equation in electromagnetics.
- to better understand the unique solvability of the integral equation in appropriate Sobolev spaces.

1.4 Scope of the study

The study was carried out on the solution of the two-dimensional interface problem in electromagnetics in a less regularity assumption in appropriate Sobolev spaces.

Chapter 2
Review literature

Electromagnetic theory can be regarded as the study of Fields produced by electric charges at rest and in motion as it is indicated in [A.S07].

In [S.I83],for the problem of scattering of time harmonic electromagnetic fields by metallic obstacles, the *eddy* current problem, obtained two dimensional results using boundary integral equations in the context of classical function spaces. They suppose that if there is a wire parallel to the z-axis through $x_0 \in \Omega^+$(or air) carrying a periodic current $I(t) = Re\{I_0 e^{-i\omega t}\}$, $I_0 \in \mathbb{C}$, then the incident field E^0, H^0 due to the wire will have the form:

$$E^0(x,y,z,t) = Re\left\{I_0 \Phi_\beta(x,y)\hat{k}e^{-i\omega t}\right\}$$
$$H^0(x,y,z,t) = Re\left\{I_0\left(\frac{\partial\phi}{\partial y}\beta\hat{i} - \frac{\partial\phi}{\partial x}\beta\hat{j}\right)e^{-i\omega t}\right\}$$

with the Hankel function of first kind and of order zero

$$\Phi_\beta(X) = \frac{-i}{4}H_0^{(1)}(\beta|x-x_0|), X = (x,y)$$

The two dimensional eddy current problem for a conducting cylinder is reduced to the interface problem:

$$\begin{aligned} \Delta\phi + i\alpha^2\phi &= 0 \; in\Omega \\ \Delta\phi + \beta^2\phi &= 0 \; in\Omega^+ \end{aligned} \tag{2.1}$$

where ϕ , $\frac{\partial\phi}{\partial n}$ in $C^0(\Gamma)$ and $\phi - \phi_\beta \in C^2(\Omega^+) \cup C^0(\Gamma)$ satisfying Sommerfield's radiation condition.

They studied the interface problem 2.1 for the potential Φ of the electric field induced by a thin wire carrying a periodic current. They obtained a system of boundary integral equations whose solutions determine the solution of the interface problem 2.1. In [S.I83], an integral equation procedure for eddy current problems since β being sufficiently small, instead of 2.1 above he considered

$$
\begin{aligned}
\Delta\Psi + i\alpha^2\Psi &= 0 \; in\Omega \\
\Delta\psi &= 0 \; in\Omega^+
\end{aligned}
\tag{2.2}
$$

Where $\psi, \frac{\partial\Psi}{\partial n} \in C^0(\Gamma)$ and $\psi - \psi_0 \in C^2(\Omega^+) \cup C^0(\Gamma)$, ψ being bounded at infinity.

The problem 2.2 can be transformed in to a coupled system of integral equations on the boundary Γ by the use of simple layer potentials only[S.I83]. Thus, they observed that the solution ϕ of the reduced interface problem 2.2 was constructed by continuous densities f and g on Γ by:

$$
\psi(x) = \begin{cases} \frac{-i}{4}\int_\Gamma f(y) H_0^{(1)}(\sqrt{i}\alpha|x-y|) di_y, & \text{when } x \in \Omega, \\ \frac{1}{2\pi}\int_\Gamma g(y) \log|x-y| ds_y + \psi_0(x) + c, & \text{when } x \in \Omega^+ \end{cases} \tag{2.3}
$$

Where $\psi_0(x) = \frac{1}{2\pi}\log|x - x_0|$ and $c \in \mathbb{C}$,arbitrary.

The regularity assumptions that $\psi, \frac{\partial\psi}{\partial n} \in C^0(\Gamma)$ and $\psi(\infty)$ being bounded yield a coupled system of integral equations on Γ[S.I83]:

$$
\begin{aligned}
V_\alpha(f) - \psi_0 &= V_0(g) + c \\
-1 &= \int_\Gamma g ds \\
\frac{-1}{2} f + N_\alpha(f) &= \frac{1}{2} g + N_0(g) + \frac{\partial\psi_0}{\partial n}
\end{aligned}
\tag{2.4}
$$

As it is indicated in [S.I83], the more general system:

$$
\begin{aligned}
V_\alpha(f) - V_0(g) - c &= h_1 \\
\int_\Gamma g ds &= a \\
(\frac{-1}{2} + N_\alpha) f - (\frac{1}{2} + N_0) g &= h_2
\end{aligned}
\tag{2.5}
$$

is uniquely solvable for given $(h_1, h_2, a) \in C^1(\Gamma) \times C^0(\Gamma) \times \mathbb{R}$ with solution $(f, g, c) \in C^0(\Gamma) \times C^0(\Gamma) \times \mathbb{C}$ yielding the existence and uniqueness of the solution of 2.4.

Chapter 3
Preliminaries

Definition 3.1 (PDE, [Ioa04]). let $u = u(x_1 \cdots x_n)$ be a function of n independent variables $x_1 \cdots x_n$. A PDE is an equation that contains the independent variables $x_1 \cdots x_n$, the dependent variables or the unknown function u and its partial derivatives up to some order. It has the form:

$$F(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1 x_1}, \cdots, u_{x_i x_j}, \cdots) = 0$$

where, F is a given function and $u_{x_j} = \frac{\partial u}{\partial x_j}, u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, i, j = 1, \cdots, n$ are the partial derivatives of u.

Definition 3.2 (Linear first order PDE,[Ioa04]). A Linear first order PDE in two independent variables x, y and the dependent variable u has the form:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y) \tag{3.1}$$

where, $a, b, c, d \in c^1(\Omega), \Omega \subset \mathbb{R}^2$ and $a^2 + b^2 \neq 0$. If we consider the differential operator

$$L := a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c$$

then equation 3.1 can be written as $LU = d$; while the homogeneous equation corresponding to 3.1 is LU=0.

Definition 3.3 (Linear second order PDE, [Ioa04]). A linear second order PDE in two independent variables x, y is given by:

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y) \tag{3.2}$$

where, $a, b, c, d, e, f, g \in c^2(\Omega), \Omega \subseteq \mathbb{R}^2$ and $a^2 + b^2 \neq 0$ in Ω. If we consider the partial differential operator

$$L(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) := a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f,$$

then 3.2 is written as $LU = g$; while the homogeneous equation corresponding to 3.2 is $LU = 0$.

Definition 3.4 (Electromagnetism,[A.S07]). As its name implies, is the branch of science of electricity and magnetism. Electromagnetism is described by the electric field intensity E and magnetic field intensity H which are determined by the Maxwell's equations. It also describes the effect of charged particles on each other. Charged particles create an electric field E, whereas moving ones also create a magnetic field H. These fields are vector fields, that is vector function of space and time: $F(x,t)$ and $H(x,t)$ where $X = (x,y)$. The whole subject of electromagnetic unfolds as a logical deduction from four postulated equations, namely, Maxwell's four field equations:

$$\begin{aligned}
\frac{\partial E}{\partial t} &= \varepsilon \nabla \times H \\
\frac{\partial H}{\partial t} &= -\mu \nabla \times E \\
\nabla \cdot E &= 0 \\
\nabla \cdot H &= 0
\end{aligned} \tag{3.3}$$

where, E and H are the electric and magnetic field vectors and ε and μ are the electric permitivity and magnetic permeability in the medium respectively. In the case of vacuum we have: σ(conductivity)=0, $\mu = \mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ and $\varepsilon = \varepsilon_0 = 8.8542 \times 10^{-12} Fm^{-1}$, $c = (\mu_0 \varepsilon_0)^{\frac{-1}{2}}$, where c is the speed of light in vacuum and $c \approx 3 \times 10^8 m/s$.

Definition 3.5 (Curl,[A.S07]). It is a vector operator that describes the infinitesimal (or very small) rotation of a three dimensional vector field i.e $\nabla \times F$ is for F composed of (F_x, F_y, F_z)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

where $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors in the direction of x, y, z axes respectively. This expands as:

$$(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{\imath} - (\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z})\hat{\jmath} + (\frac{\partial Fy}{\partial x} - \frac{\partial F_x}{\partial y})\hat{k}$$

which is the same as,

$$(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{\imath} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\hat{\jmath} + (\frac{\partial Fy}{\partial x} - \frac{\partial F_x}{\partial y})\hat{k}$$

Definition 3.6 (Divergence,[A.S07]). The divergence of a function $F(x,y)$ is given by:

$$\text{div}F = \nabla .F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}$$

We call the divergence of F because it is a measure of the degree to which the vector field F diverges or flows outward from any position.

Definition 3.7 (Sommerfields radiation condition,[C.A84]). The Sommerfields radaition condition defines the condition of radiation for a scalar field satisfying the Helmholtz equation; "the sources must be sources not sinks of energy, the energy which is radiated from the sources must scatter to infinity, no energy may be radiated from infinity in to the field. It states that the scattered field consists of outgoing waves only. It is the condition at infinity for exterior problems that is given by:

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \varphi(p)}{\partial r} - ik\varphi(p)\right) = 0 \tag{3.4}$$

in two dimensions and

$$\lim_{r\to\infty} r\left(\frac{\partial \varphi(p)}{\partial r} - ik\varphi(p)\right) = 0$$

in three dimensions. Where r is the distance from a fixed origin to a general fixed point and k is the wave number where $k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0}$ [radians m^{-1}]. Solutions of the Helmholtz equation which satisfy the radiation condition are called radiating conditions or functions.

Definition 3.8 (Integral equation, [Wol95]). An integral equation in which an unknown function appears under an integral sign.

Fredholm integral equation of first type is given by

$$f(x) = \int_a^b K(x,t)\psi(t)dt \tag{3.5}$$

where ψ is unknown function, f is known function, K is another known function of two variables often called the kernel function.

Fredholm integral equation of second type is given by

$$\psi(x) = f(x) + \lambda \int_a^b K(x,t)\psi(t)dt \tag{3.6}$$

where, the parameter λ is unknown factor. Equation 3.6 can be shortened as $\psi = f + \lambda K\psi$.

Definition 3.9 (Linear functional, [Alb13]). suppose that **B** is a Banach space over $\mathbb{R}$ equipped with a norm $\|.\|$. A linear functional is a linear mapping l from

$\mathbf{B}$ to $\mathbb{R}$, that is, $l : \mathbf{B} \to \mathbb{R}$, which satisfies

$$l(\alpha f + \beta g) = \alpha l(f) + \beta l(g) \tag{3.7}$$

for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathbf{B}$

Definition 3.10 (support of the function,[Alb13]). The support of a function ϕ is the closure of the set where ϕ does not vanish; or the support of a function $\phi : \mathbb{R}^n \mapsto \mathbb{C}$ is the closure of the set of points in $\mathbb{R}^n$ at which ϕ is nonzero and it is denoted by suppϕ i.e.,

$$supp(\phi) = \{x \in \Omega : \phi(x) \neq 0\}$$

Definition 3.11 (Distribution, [Alb13]). A distribution on the open set $\Omega \subseteq \mathbb{R}^n$ is a linear functional 3.7, $\wedge : C_c^\infty(\Omega) \longrightarrow \mathbb{R}$ such that the following boundedness property holds: For every compact $K \subset \Omega$ there exists an integer $N \geq 0$ and a constant C such that

$$|\wedge(\phi)| \leq C\|\phi\|_{C^N}, \forall \phi \in C^\infty \text{ with supp}(\phi) \subseteq k$$

where, N is the order of the distribution if it is finite distribution and ϕ is any test function .

Definition 3.12 (Locally integrable function, [Alb13]). A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{C}$ is said to be locally integrable if

$$\int_\Omega |f(x)|dx$$

exists. Any continuous function is locally integrable.

Definition 3.13 (Sobolev space, [Ruz14]). Let Ω be an open set in $\mathbb{R}^n$. Let $k \in \mathbb{N} \cup \{0\}$ and let $1 \leq p \leq +\infty$. The Sobolev space $W^{k,p}(\Omega)$ (or $L_k^p(\Omega)$) is defined to be the set of all functions u defined on Ω such that for every multi-index $\alpha = (\alpha_1 \cdots \alpha_n)$ with $|\alpha| \leq k$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the mixed partial derivative

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

is both locally integrable and belongs to $L^P(\Omega)$ i.e., $\|D^\alpha u\|_{L^P} < \infty$. In short, the Sobolev space on Ω is defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : Du^\alpha \in L^p(\Omega), \forall |\alpha| \leq k\}. \tag{3.8}$$

$k \in \mathbb{N}$ is called order of Sobolev space 3.8. If $p = 2$, one often uses the notation $H^k(\Omega)$ or $L_k^2(\Omega)$. If $p = 2$ and $k = 0$, we get $H^0(\Omega) = L^2(\Omega)$.

For $u \in L_k^p(\Omega)$, we define

$$\|u\|_{L_k^p(\Omega)} = (\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^p}^p)^{\frac{1}{p}} = (\sum_{|\alpha|\leq k} \int_\Omega |\partial^\alpha u|^p dx)^{\frac{1}{p}} \text{for} 1 \leq p < \infty$$

For $p = \infty$, we define

$$\|u\|_{L_k^\infty(\Omega)} = \sum_{|\alpha|\leq k} \text{esssup}_\Omega |\partial^\alpha u|$$

i.e.,

$$\|u\|_{k,p} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty, \\ Max_{|\alpha|\leq k} \|D^\alpha u\|_\infty, & \text{for } p = \infty \end{cases}$$

These expressions are norms. Indeed,

$$\|\lambda u\|_{L_k^p} = |\lambda| \|u\|_{L_k^p}, \forall \lambda \in \mathbb{C}$$

Hence, $L_k^p(\Omega)$ is a Banach space; i.e., it is a complete normed space.

Definition 3.14 (Expansion of the kernel, [YGS00]). is simultaneously reducing the dimensionality of the data and find the non-linear mapping using an algorithm of complexity $O(zn^2)$.

Definition 3.15 (smoothing operators, [Ruz14]). Let x be an n-dimensional manifold equipped with a smooth non-vanishing measure, dx. Given $R \in C^\infty(X \times X)$, one can define an operator $T_R : C^\infty(x) \to C^\infty(x)$ by setting

$$T_R f(x) = \int R(x,y) f(y) dy \tag{3.9}$$

operators of this type are called smoothing operators.

Definition 3.16 (Fourier transform, [Mie12]). Let $f \in C_0^s(\mathbb{R}^n)$, $s = 0, 1, \cdots$. Then, we define the Fourier transform of f by

$$\hat{f}(\xi) := \mathscr{F}[f](\xi) := (2\pi)^{\frac{-n}{2}} \int_{\mathbb{R}^n} e^{-ix.\xi} f(x) dx, \tag{3.10}$$

where, $\xi \in \mathbb{R}^n$, $\xi = (\xi_1, \cdots \xi_n)$, $(x = x_1, \cdots x_n)$ and $\xi.x = \xi_1 x_1 + \cdots + \xi_n x_n = \sum_{j=1}^n \xi_j x_j$.

Definition 3.17 (Inverse Fourier transform, [Mie12]). knowing the Fourier transform $g(\xi) = \hat{f}(\xi)$, the function f can be reconstructed with the aid of the inverse Fourier transformation.

$$\mathscr{F}^{-1}[g]x := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix.\xi} g(\xi) d\xi \tag{3.11}$$

Indeed, it holds that $\mathscr{F}^{-1}[\mathscr{F}[f]] \equiv \mathscr{F}^{-1}[\hat{f}] = f$ (Inversion formula), provided the integrals on the right hand side exist.

Definition 3.18 (pseudodifferential operator [Fol95, Hel11, Mie12]). any linear differential operators with C^∞ coefficients on an open set $\Omega \subseteq \mathbb{R}^n$ can be written in the form

$$L = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha$$

The function

$$p(x,\xi) = \sum_{|\alpha| \le K} a_\alpha(x) \xi^\alpha$$

is called the symbol of p and

$$(pu)(x) := (2\pi)^{\frac{-n}{2}} \int e^{ix.\xi} p(x,\xi) \hat{u}(\xi) d\xi, \tag{3.12}$$

is said to be pseudodifferential operators; where, $u \in C_c^\infty(\Omega)$, $\hat{u}$ is a Fourier transform and $p(x,\xi)$ is a polynomial in ξ with coefficients that are C^∞ functions of $x \in \Omega$. We denote the set of pseudodifferential operators of order m on Ω by

$$\Psi^m(\Omega) = p(x,D) : p \in S^m(\Omega) \tag{3.13}$$

Definition 3.19 (principal symbol, [Fol95]). If $\Omega \subset \mathbb{R}^n$ and $p \in \psi^m(\Omega)$3.13, P is said to have a principal symbol if there is a $p^m \in C^\infty(T^0\Omega)$ such that

$$p^m(x,t\xi) = t^m p^m(x,\xi) \tag{3.14}$$

for $t > 0$ and $\sigma_p - p^m$ agrees for large ξ with an element of $S^{m-1}(\Omega)$. The restriction to large ξ is necessary since p^m is usually not C^∞ at $\xi = 0$. For example, the principal symbol of differential operator is up to factors of $2\pi i$. Also, if p is elliptic with principal symbol p^m, any parametrix for p has a principal symbol $\frac{1}{p^m}$.

Definition 3.20 (Convolution, [Fol95]). The classical definition of the convolution of two functions defined on $\mathbb{R}^n$ is:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy \tag{3.15}$$

Definition 3.21 (Singular integral equations, [Wol95]). In the general Fredholm integral equations, there arise two singular situations: In 3.6,

type 1: if $a \to -\infty$ and $b \to \infty$

type 2: if the kernel $K(x,t) = \pm\infty$ at some points in the integration limit.

Definition 3.22 (Cauchy integral, [Bk14]). Assume that γ is a rectifiable oriented and closed Jordan curve (i.e., a closed curve that does not intersect itself) in the complex plane $\mathbb{C}$ dividing the plane in two domains Ω_+ and $\Omega_- := \mathbb{C} \setminus (\Omega_+ \cup \gamma)$. Given a complex value continuous function f defined on γ, the Cauchy integral is defined in $\mathbb{C} \setminus \gamma$ by

$$C_\gamma f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)}{\tau - z} d\tau \tag{3.16}$$

and the principal value singular integral associated with $C_\gamma f_z$ is

$$S_\gamma f(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\gamma \setminus |\tau - t| \le \varepsilon} \frac{f(\tau)}{\tau - t} d\tau, t \in \gamma \tag{3.17}$$

Definition 3.23 (Hilbert transform, [Zwi97, Ksc06]). The Hilbert transform of a function $v(x)$ is defined by:

$$\mathscr{H} v(x) = \frac{-1}{\pi x} * v(x) = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{v(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(y)}{y-x} dy \tag{3.18}$$

It is the convolution of $v(x)$ with the signal $\frac{1}{\pi x}$. We observe that the integral in 3.18 is improper; the integrand has a singularity and the limits of integration are infinite. Hence, the Cauchy principal value is defined for the first integral in 3.18 as:

$$\mathscr{H} v(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \left(\int_{t-\frac{1}{\varepsilon}}^{t-\varepsilon} \frac{v(x)}{t-x} dx + \int_{t+\varepsilon}^{t+\frac{1}{\varepsilon}} \frac{v(x)}{t-x} dx \right) \tag{3.19}$$

We see that the Cauchy *pv* is obtained by considering a finite range of integration that is symmetric about the point of singularity.

Definition 3.24 (Boundary potentials, [V.I06]). The two main boundary potentials mostly used are:

Single layer boundary potential $S(x,y)$ is defined by

$$S(x,y) = \int_{\partial\Omega} \sigma(s) \log(\frac{1}{r}), \tag{3.20}$$

Double layer boundary potential $D(x,y)$ is given by

$$D(x,y) = \int_{\partial\Omega} \mu(s) \frac{\partial}{\partial n} \log(\frac{1}{r}) ds, \tag{3.21}$$

where, $\sigma(s)$ is the charge density of single layer boundary potential, $\mu(s)$ is charge density of double layer boundary potential, $\partial\Omega$ is the smooth boundary of the region Ω, n is the outward unit normal vector along $\partial\Omega$.

Definition 3.25 (Adjoint operator, [Wol95]). Let $T \in \mathscr{L}(V,W)$, where V and W are inner product spaces. An adjoint operator $T^* \in \mathscr{L}(W,V)$ is one such that

$$\langle Tv, w\rangle = \langle v, T^* w\rangle, \forall v \in V, w \in W \tag{3.22}$$

An operator $T \in \mathscr{L}(V)$ is self adjoint if $T = T^*$.

3.1 Fundamental solution of the Helmholtz equation

The fundamental solution for the scalar wave in the N-dimensional space satisfies the following Helmholtz equation,[Gra98]

$$\left(\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial x_i} + k^2\right) U_{[N]}(\mathbf{x}, \mathbf{y}, k) = -\delta(\mathbf{x} - \mathbf{y}) \tag{3.23}$$

where k is the wave number and $\delta(x - y)$ is the Dirac delta function which in two dimension is defined by (see, [J.T02])

$$\delta(\mathbf{x} - \mathbf{y}) = \begin{cases} 0, & \text{when } x \neq y, \\ \infty, & \text{when } x = y \end{cases}$$

and

$$\int_{\Omega} \delta(x - y) d\Omega_y = \int_{\Omega^*} \delta(x - y) d\Omega_y = 1$$

where, $\Omega^* \subset \Omega$.
Applying the Fourier transform 3.10 of definition 3.16 to equation 3.23, yields the solution $\tilde{U}_{[N]}(\xi, \mathbf{y}, k)$ in the transformed domain:

$$\tilde{U}_{[N]}(\xi, \mathbf{y}, k) = \frac{exp(-i\xi.\mathbf{y})}{(|\xi|^2 - (k + i\varepsilon)^2)}$$

where, ε is a small positive number to avoid the singularity appeared in the calculation of the inverse Fourier transform of $\tilde{U}_{[N]}$. Using the inverse Fourier transform 3.11, $\tilde{U}_{[N]}$ may be written as:

$$U_{[N]}(\mathbf{x}, \mathbf{y}, k) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^N} \int \frac{1}{|\xi|^2 - (k + i\varepsilon)^2} \exp(i\xi.(\mathbf{x} - \mathbf{y})) d\xi \tag{3.24}$$

Using equation 3.24 for a one dimensional problem ($N = 1$), the fundamental solution is

$$
\begin{aligned}
u_{[1]}(x,y,k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2-(k+i\varepsilon)^2}e^{i\xi(x-y)}d\xi \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2-(k+i\varepsilon)^2}(e^{i\xi(x-y)}+e^{-i\xi(x-y)})d\xi \\
&= \frac{i}{2k}e^{ik|x-y|}
\end{aligned}
\tag{3.25}
$$

For a three dimensional problem $N=3$, we have:

$$
\begin{aligned}
U_{[3]}(\mathbf{x},\mathbf{y},k) &= \frac{1}{(2\pi)^3}\int_0^{\infty}|\xi|^2 d|\xi|\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}\frac{e^{i|\xi|r\cos\theta}}{|\xi|^2-(k+i\varepsilon)^2}d\theta \\
&= \frac{-1}{8\pi^2 r}\int_{-\infty}^{\infty}\frac{s(e^{isr}-e^{-isr})}{s^2-(k+i\varepsilon)^2}ds
\end{aligned}
\tag{3.26}
$$

where $r=|\mathbf{x}-\mathbf{y}|$. Comparing equation 3.26 with equation 3.25, we find the relation between $U_{[3]}$ and $U_{[1]}$ as follows:

$$
U_{[3]} = \frac{-1}{2\pi r}\frac{du_{[1]}}{dx}\big|_{|x-y|=r}
$$

This relation leads to the explicit expression for a fundamental solution in a three dimensional Helmholtz problem in the following form:

$$
U_{[3]}(\mathbf{x},\mathbf{y},k) = \frac{e^{ikr}}{4\pi r}
$$

The fundamental solution for a two- dimensional problem can be obtained as a superposition of the three dimensional fundamental solution.

$$
\begin{aligned}
U_{[2]}(\mathbf{X},\mathbf{Y},k) &= \int_{-\infty}^{\infty}U_{[3]}(\mathbf{x},\mathbf{y},k)|_{x_3=0}dy_3 \\
&= \int_{-\infty}^{\infty}\frac{1}{4\pi\sqrt{R^2+y_3^2}}\exp(ik\sqrt{R^2+y_3^2})dy_3 \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\exp(ikR\cosh t)dt(y_3=R\sinh t) \\
&= \frac{1}{4i}H_0^{(1)}(kR) \\
&= \frac{-i}{4}H_0^{(1)}(kR) \\
&= \frac{-i}{4}H_0^{(1)}(k|x-y|)
\end{aligned}
\tag{3.27}
$$

where $\mathbf{X} = (x_1, x_2), \mathbf{Y} = (y_1, y_2), R = |\mathbf{x} - \mathbf{y}|$, $i = \sqrt{-1}$ and $H_n^{(1)}$ is the Hankel function of the first kind of the n^{th} order. Explicitly the function G is defined in the $2D$ case by (see,[CWL07]).

$$G(x, x_0) := \frac{-i}{4} H_0^{(1)}(k|x - x_0|), x, x_0 \in \mathbb{R}^2 \tag{3.28}$$

In physical terms the $2D$ solution of the Helmholtz equation given by 3.27 with G given by 3.28 is the field at x due to a coherent line source of electromagnetic wave which is perpendicular to the $2D$ plane and passes through it at the point x_0. The solution involves $H_0^{(1)}$, which denotes the Hankel function of the first kind of order zero. We term the function $G(x, x_0)$ that we have just written down a fundamental solution of the Helmholtz equation, which means that it is a solution of the Helmholtz equation appropriate to a point source excitation. We also call $G(x, x_0)$ the free-field Green's function, meaning that it is the (unique) solution to the problem of a point source in free space.

3.2 Green's identities

let $u, v \in c^2(\Omega), \Omega$ be a domain with smooth boundary $\partial\Omega, \vec{n}$ be the outward unit normal vector to $\partial\Omega$. Recall the following notations of field theory

- $\text{grad}u = \nabla u = (u_x, u_y, u_z)$
- $\text{div}\vec{F} = \nabla.\vec{F} = f_x + g_y + h_z$
- $\text{curl}\vec{F} = \nabla \times \vec{F} = (h_y - g_z, f_z - h_x, g_x - f_y)$
- $\Delta u = \text{div}(\nabla u) = \Delta^2 u = u_{xx} + u_{yy} + u_{zz}$

where $\vec{F}(f, g, h)$ is a vector field, $dv = d_x d_y d_z$, dS_y is a surface element, ds_y is an arc length element at P on $\partial\Omega$. Now the three Green's theorems are given below.

Theorem 3.1 (Green's first identity, [YGS00]). *Let $\Omega \in \mathbb{R}^2$ be a domain bounded by the closed smooth contour Γ and let n_y denote the unit normal vector to the boundary Γ directed into the exterior of Γ and corresponding to a point $y \in \Gamma$. Then, for every function u which is once continuously differentiable in the closed domain $\bar{\Omega} = \Omega \cup \Gamma$, $u \in C^1(\bar{\Omega})$ and every function v which is twice continuously differentiable in $\bar{\Omega}$,$v \in C^2(\bar{\Omega})$, then*

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v) dx = \int_\Gamma u \frac{\partial v}{\partial n_y} dsy. \tag{3.29}$$

Proof. By applying Gauss-Ostrogradiski theorem,

$$\int_\Omega \text{div}\vec{F} dx = \int_\Gamma n_y \cdot \vec{F} ds_y \tag{3.30}$$

to the vector function (vector field) $\vec{F} = (F_1, F_2) \in C^1(\bar{\Omega})$ (with the components) F_1, F_2 being once continuously differentiable in the closed domain $\bar{\Omega} = \Omega \cup \Gamma, F_i \in C^i(\bar{\Omega}), i = 1, 2)$ defined by

$$\vec{F} = u\nabla v = [u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y}]$$

the components are,

$$\vec{F_1} = u\frac{\partial v}{\partial x}$$
$$\vec{F_2} = u\frac{\partial v}{\partial y}$$

We have,

$$\begin{aligned}
\mathrm{div}\vec{F} &= \mathrm{div}(u\nabla v) \\
&= \frac{\partial}{\partial x}(u\frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(u\frac{\partial v}{\partial y}) \\
&= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} \\
&= \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
&= \nabla u \cdot \nabla v + u\Delta v
\end{aligned}$$

On the other hand, according to the definition of the normal derivative, (see,[A.S07]),

$$n_y.\vec{F} = n_y \cdot (u\nabla v) = u(n_y \cdot \nabla v) = u\frac{\partial v}{\partial n_y} \tag{3.31}$$

so that finally,

$$\int_\Omega \mathrm{div}F dx = \int_\Omega (u\Delta v + \nabla u \cdot \nabla v)dx, \tag{3.32}$$

and from equation 3.31 and using equation 3.30, we have,

$$\int_\Gamma n_y.\vec{F} ds_y = \int_\Gamma u\frac{\partial v}{\partial n_y} ds_y$$

which proves 3.29, i.e.,

$$\begin{aligned}
\int_\Gamma u\frac{\partial v}{\partial n_y} ds_y &= \int_\Omega (u\Delta v + \nabla u \cdot \nabla v)dx \\
&= \int_\Omega u\Delta v dx + \int_\Omega \nabla u \cdot \nabla v dx
\end{aligned} \tag{3.33}$$

Theorem 3.2 (Green's second identity,[YGS00]). *For $u, v \in C^2(\bar{\Omega})$, the Green's second identity is given by,*

$$\int_{\Omega}(u\Delta v - v\Delta u)dx = \int_{\Gamma}(u\frac{\partial v}{\partial n_y} - v\frac{\partial u}{\partial n_y})ds_y \tag{3.34}$$

Proof. By interchanging the role of u and v in the Green's first identity, 3.29, and subtracting the result, we get the result.

The process by which 3.34 was derived from 3.33 is an example of symmetrization . This is Green's second identity and is a basic tool in the study of Δ.

Theorem 3.3 (Green's third identity, [YGS00]). *Let $D \subset \mathbb{R}^m$ be a bounded domain of class C^1 and let $u \in C^2(\bar{D})$ be harmonic in D. Then,*

$$u(x) = \int_{\partial D}(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n})ds_y, \quad x \in D \tag{3.35}$$

where,

$$\Phi(x,y) := \begin{cases} \frac{1}{2\pi}\ln\frac{1}{|x-y|}, & when\ m = 2, \\ \frac{1}{4\pi}\frac{1}{|x-y|}, & when\ m = 3 \end{cases}$$

and n is the unit normal vector.

Proof. For $x \in D$ choose a sphere $\Omega(x;r) := \{y \in \mathbb{R}^m : |y-x| = r\}$ of radius r such that $\Omega(x;r) \subset D$ and direct the unit normal vector n to $\Omega(x;r)$ in to the interior of $\Omega(x;r)$. By applying Green's second identity 3.34 to the harmonic functions u and $\Phi(x,.)$ in the domain $\{y \in D : |y-x| > r\}$ to obtain

$$\int_{\partial D \cup \Omega(x;r)}(u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y)\Phi(x,y))ds(y) = 0$$

Now we have,

$$\int_{\Omega(x;r)}(u(y)\frac{\partial \Phi(x,y)}{\partial n(y)} - \frac{\partial u}{\partial n}(y)\Phi(x,y))ds(y) = \int_{\partial D}(\frac{\partial u}{\partial n}(y)\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial n}(y))ds(y) \tag{3.36}$$

In 3.36, for the second term since $u \in C^2(\bar{D})$, $\frac{\partial u}{\partial n}$ is bounded, say by M. Then,

$$\begin{aligned} |\int_{\Omega(x;r)}\frac{\partial u}{\partial n}(y)\Phi(x,y)ds(y)| &\leq M\int_{\Omega(x;r)}|\Phi(x,y)|ds_y \\ &= Mcr^{m-1}\Phi(r) \end{aligned}$$

Where c is the surface area of the unit sphere (2π and 4π) respectively and $\Phi(r)$ is the respective function such that $\Phi(x,y) = \Phi(|x-y|)$. This product goes to zero as $r \to 0$, since $r\ln r \to 0$. For the first term we have,

$$
\begin{aligned}
\frac{\partial \Phi(x,y)}{\partial n}(y) &= -\Phi'(r) \\
&= \frac{1}{cr^{m-1}}
\end{aligned}
$$

where the minus sign arises because the unit normal is directed in to the interior. Thus,

$$
\begin{aligned}
\lim_{r\to 0}\int_{\Omega(x;r)} u(y)\frac{\partial \Phi(x,y)}{\partial n(y)}ds(y) &= \lim_{r\to 0}\int_{\Omega(x;r)} \frac{u(y)}{cr^{m-1}}ds(y) \\
&= \lim_{r\to 0}\frac{1}{c}\int_{\Omega(x;r)} u(y)d\Omega \\
&= u(x)
\end{aligned}
\tag{3.37}
$$

This proves 3.35 where $d\Omega$ denotes integration over the solid angle, and u is continuous.

Chapter 4
Boundary Integral Equation for Interface Problem in Electromagnetics

4.1 Interface problem in electromagnetics

The problem under consideration is that of determining a function which is a solution of the Helmholtz equation in a connected bounded region and exterior to a simple closed curve. As it is explained in [C.M80], a physical prototype (or typical example) of interface problems is that of electromagnetic fields in the presence of metallic or dielectric obstacles. A case of great practical interest is that of the production of eddy currents in conductors. We shall begin with a brief description of the physical problem. Let Ω be a simply connected bounded region in $\mathbb{R}^2$ and $\Omega^+ = \mathbb{R}^2/\bar{\Omega}$. Here Ω^+ is to represent air and Ω cross section of a metallic cylinder in the $x-y$ plane (see, Figure (4.1)).

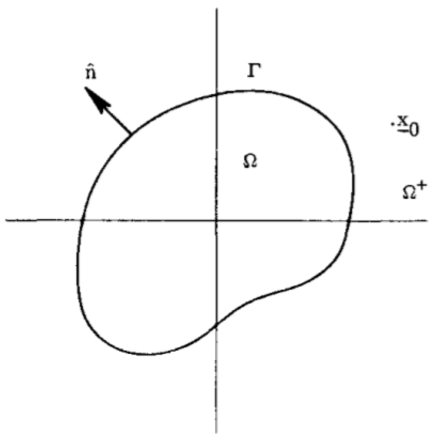

Fig. 4.1

We consider only monochromatic fields (i.e., waves of some given frequency ω) because, the Helmholtz equation stands for monochromatic waves; which means that if the time dependence is a harmonic function of circular frequency ω, we can write:

$$\phi(r,t) = Re(Ee^{-i\omega t}\psi(r)),$$

where $\psi(r)$ is some complex valued scalar function and the real part is taken, since $\phi(r,t)$ is real and $i^2 = -1$. For polychromatic waves, or sums of waves of different frequencies, we can sum up solutions with different ω. We study a special geometry in which there is a single cylindrical obstacle in the z-direction with uniform cross section Ω in the x-y plane. The outside region is air and, as usual, we assume electric conductivity of air($\sigma = 0$) there. There are two cases to consider called Transverse Magnetic(TM) and Transverse Electric(TE):

$$(TM) : E = E(x,y)\hat{k}; H = H^1(x,y)\hat{\imath} + H^2(x,y)\hat{\jmath} \text{and}$$
$$(TE) : H = H(x,y)\hat{k}; E = E^1(x,y)\hat{\imath} + E^2(x,y)\hat{\jmath}$$

But in this thesis we consider only transverse magnetic. Suppose there are incident electric and magnetic fields E^0, H^0 and that all fields E, H are TM and time harmonic with a single angular frequency ω. This means that with a proper choice of x, y, z axes:

$$E = E\hat{k}$$
$$H = H^1\hat{\imath} + H^2\hat{\jmath}$$

where E, H^1, H^2 are functions of x and y only and $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are unit vectors in the direction of x, y, z axes respectively, see, [S.I83]. Thus with appropriate scaling we have the following problem:

$$\begin{aligned} \nabla \times E = H, \nabla \times H = i\alpha^2 E \quad & \text{in } \Omega \\ \nabla \times E = H, \nabla \times H = \beta^2 E \quad & \text{in } \Omega^+ \end{aligned} \tag{4.1}$$

where n denotes the exterior normal to Ω and α, β are non dimensional constants. $n \times E, n \times H$ are continuous across Γ. E and H satisfy maxwell's equations 4.1 in Ω^+ and Ω and are subject to the interface condition that the tangential components of E and H are continuous across Γ (the boundary of Ω). As it is shown in [S.I83], if there is a wire parallel to the z-axis through $x_0 \in \Omega^+$ carrying a periodic current

$$I(t) = Re\left\{I_0 e^{-i\omega t}\right\}$$

$I_0 \in \mathbb{C}$, then the incident field E^0, H^0 due to the wire will have the form:

$$
\begin{aligned}
E^0(x,y,z,t) &= Re\left\{I_0\phi_\beta(x,y)\hat{k}e^{-i\omega t}\right\}\\
H^0(x,y,z,t) &= Re\left\{I_0\left(\frac{\partial\phi}{\partial y}\beta\hat{i}-\frac{\partial\phi}{\partial x}\beta\hat{j}\right)e^{-i\omega t}\right\}
\end{aligned}
\tag{4.2}
$$

4.1.1 Time-harmonic fields

In many practical situations, especially at low frequencies, it is sufficient to deal with only the steady state (or equilibrium) solution of electromagnetic fields when produced by sinusoidal time-varying or time-harmonic, i.e., they vary at a sinusoidal frequency ω. Consider now, the appearance of the scalar Helmholtz equation in the context of Maxwell's equations 3.3 describing propagation electromagnetic waves: taking the curl of the first equation and using the second equation we have,

$$
\begin{aligned}
\nabla\times\nabla\times E &= -\mu\frac{\partial}{\partial t}\nabla\times H\\
&= -\mu\varepsilon\frac{\partial^2 E}{\partial t^2}\\
&= \frac{-1}{c^2}\frac{\partial^2 E}{\partial t^2}
\end{aligned}
\tag{4.3}
$$

Due to the following well-known identity for the ∇ operator and the third equation in equation 3.3,

$$
\begin{aligned}
\nabla^2 E &= \nabla(\nabla.E)-\nabla\times\nabla\times E\\
&= 0-\nabla\times\nabla\times E\\
&= -\nabla\times\nabla\times E
\end{aligned}
\tag{4.4}
$$

using 4.3 and 4.4, we obtain:

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \nabla^2 E \tag{4.5}$$

Similarly,

$$\frac{1}{c^2}\frac{\partial^2 H}{\partial t^2} = \nabla^2 H \tag{4.6}$$

Thus, both the electric and magnetic field vectors satisfy the vector wave equation; which is generally given by see [N.O01, H.S11]

$$\nabla^2\psi-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{4.7}$$

Replacing the time derivative in equation 4.7 by $(i\omega)^2$, i.e.,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0,$$

putting $(i\omega)^2 = \frac{\partial^2}{\partial t^2}$, we have

$$\nabla^2\psi - \frac{1}{c^2}(i\omega)^2\psi = 0$$

This implies,

$$\nabla^2\psi - \frac{(i\omega)^2}{c^2}\psi = 0,$$

putting,

$$\frac{(i\omega)^2}{c^2} = k^2$$

which implies, $\nabla^2\psi - k^2\psi = 0$, which yields the scalar wave equation

$$\nabla^2\psi + k^2\psi = 0 \tag{4.8}$$

where k is the propagation constant (in rad/ m) given by $k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$, c is speed in vacuum. Equation 4.8 is called the Helmholtz equation and it satisfies 3.4. The general Helmholtz equation is given by

$$\nabla^2\psi + k^2\psi = g \tag{4.9}$$

We notice that 4.9 reduces to

- Poisson's equation: $\nabla^2\psi = g$ when $k = 0$
- Laplace's equation $\nabla^2\psi = 0$ when $k = 0 = g$

Thus Poisson's and Laplace's equation are special cases of the Helmholtz equation. Note that function ψ is said to be harmonic if it satisfies Laplace's equation.

The two dimensional eddy-current problems for a conducting cylinder 4.2 is reduced to the interface problem for $\phi \in c^2(\Omega) \cup c^2(\Omega^+/\{x_0\}) \cup c^0(\Gamma)$ such that

$$\begin{aligned} \Delta\phi + i\alpha^2\phi = 0 \quad & \text{in } \Omega \\ \Delta\phi + \beta^2\phi = 0 \quad & \text{in } \Omega^+ \end{aligned} \tag{4.10}$$

where ϕ, $\frac{\partial\phi}{\partial n}$ in $C^0(\Gamma)$ and $\phi - \phi_\beta \in C^2(\Omega^+) \cup C^0(\Gamma)$ satisfying 3.4. Here if we seek the total fields E, H in the same form with ϕ_β replaced by the solution ϕ of 4.10, then E, H also satisfy the remaining Max-well's equations 4.1 and are automatically divergence free and $E - E^0$, $H - H^0$ satisfy 3.4. In [S.I83], the parameters in the 4.10 α, β are real numbers, β being sufficiently small i.e.,

$\alpha^2 = O(10)$ and $\beta^2 = O(10^{-21})$; where 'O' represents order. Therefore, instead of 4.10 above we consider

$$\begin{aligned} \Delta\Psi + i\alpha^2\Psi &= 0 \quad \text{in } \Omega, \\ \Delta\psi &= 0 \quad \text{in } \Omega^+, \end{aligned} \tag{4.11}$$

where $\psi, \frac{\partial\Psi}{\partial n} \in C^0(\Gamma)$ and $\psi - \psi_0 \in C^2(\Omega^+) \cup C^0(\Gamma)$, ψ being bounded at infinity. Equation 4.11 represents equation 4.8 and satisfies 3.4.

In [Kir07] with

$$\psi_0(x) = \frac{1}{2\pi}\log|x - x_0| \tag{4.12}$$

i.e., equation 4.12 is called a fundamental solution of $\psi_0(x)$ for the operator ∇^2 in two dimension; or it is also known as the free-space Green's function. see also [J.T02]. The solution ϕ of 4.10 converges point wise to the solution ψ of 4.11 as $\beta \to 0$ at any fixed $x \in \mathbb{R}^2/\{x_0\}$. As indicated in [S.I83], the problem 4.11 can be transformed in to a coupled system of integral equations on the boundary Γ by the use of simple layer potentials 3.20 only.

4.2 Boundary integral formulation

The solution ψ of 4.11 can be constructed by continuous densities f and g on Γ using Green's function 3.28 and fundamental solution of the laplacian 4.12 i.e.,

$$\psi(x) = \begin{cases} \frac{-i}{4}\int_\Gamma f(y)H_0^{(1)}(\sqrt{i}\alpha|x-y|)di_y, & \text{when } x \in \Omega, \\ \frac{1}{2\pi}\int_\Gamma g(y)\log|x-y|ds_y + \psi_0(x) + c, & \text{when } x \in \Omega^+ \end{cases}$$

Where,

$$\psi_0(x) = \frac{1}{2\pi}\log|x - x_0|$$

and $c \in \mathbb{C}$, arbitrary, $s \subseteq \Gamma$;
and since $\Delta\phi + i\alpha^2\phi = \Delta\phi + \beta^2\phi$ we have $i\alpha^2 = \beta^2$ which gives $\beta = \sqrt{i}\alpha$. The regularity assumptions that $\psi, \frac{\partial\psi}{\partial n} \in C^0(\Gamma)$ and $\psi(\infty)$ being bounded yield a coupled system of integral equations on Γ:

$$\begin{aligned} V_\alpha(f) - \psi_0 &= V_0(g) + c \\ -1 &= \int_\Gamma g\,ds \\ \frac{-1}{2}f + N_\alpha(f) &= \tfrac{1}{2}g + N_0(g) + \tfrac{\partial\psi_0}{\partial n} \end{aligned} \tag{4.13}$$

4.2.1 Operator notation

For $x \in \Gamma$ and $f \in C^0(\Gamma)$ we define linear transformations of the operators V_α, V_0, N_α and N_0 in 4.13.see also [S.I83, C.M80]

$$\begin{aligned}
V_\alpha(f)x &:= \frac{-i}{4}\int_\Gamma f(y)H_0^{(1)}(\sqrt{i}\alpha|x-y|)ds_y \\
V_0(f)x &:= \frac{1}{2\pi}\int_\Gamma f(y)\log|x-y|ds_y \\
N_\alpha(f)x := \frac{\partial}{\partial n_x}V_\alpha f(x) &= \frac{-i}{4}\int_\Gamma f(y)\frac{\partial}{\partial n_x}H_0^{(1)}(\sqrt{i}\alpha|x-y|)ds_y \\
N_0(f)x := \frac{\partial}{\partial n_x}V_0 f(x) &= \frac{1}{2\pi}\int_\Gamma f(y)\frac{\partial}{\partial n_x}\log|x-y|ds_y
\end{aligned} \tag{4.14}$$

As it is indicated in [S.I83], the more general system

$$\begin{aligned}
V_\alpha(f) - V_0(g) - c &= h_1 \\
\int_\Gamma g ds \; &= a \\
(\frac{-1}{2}+N_\alpha)f - (\frac{1}{2}+N_0)g &= h_2
\end{aligned} \tag{4.15}$$

which is equivalent to:

$$\begin{bmatrix} V\alpha & -V_0 \\ 0 & \int_\Gamma ds \\ \frac{-1}{2}+N\alpha & \frac{-1}{2}-N_0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} C+h_1 \\ a \\ h_2 \end{bmatrix}$$

is uniquely solvable for given $(h_1, h_2, a) \in C^1(\Gamma) \times C^0(\Gamma) \times \mathbb{R}$ with solution $(f, g, c) \in C^0(\Gamma) \times C^0(\Gamma) \times \mathbb{C}$ yielding the existence and uniqueness of the solution of 4.13.

4.3 Unique solvability of the integral equation in Sobolev space

Consider the system 4.15 for data given in 3.8 on Γ. Then, the operators involved have the following mapping properties:

Lemma 4.1. *Let Γ be C^∞, boundary. Then, for any $s \in \mathbb{R}$ the operators given in 4.14 are bounded mappings as follows:*

$$\begin{aligned} V_\alpha &= H^s(\Gamma) \to H^{s+1}(\Gamma) \\ V_0 &= H^s(\Gamma) \to H^{s+1}(\Gamma) \\ N_\alpha &= H^s(\Gamma) \to H^s(\Gamma) \\ N_0 &= H^s(\Gamma) \to H^s(\Gamma) \end{aligned} \tag{4.16}$$

Proof. Assuming that $s = \frac{-1}{2}$, since $s \in \mathbb{R}$. Thus, 4.16 becomes:

$$V_\alpha : H^{\frac{-1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \tag{4.17}$$

$$V_0 : H^{\frac{-1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \tag{4.18}$$

$$N_\alpha : H^{\frac{-1}{2}}(\Gamma) \to H^{\frac{-1}{2}}(\Gamma) \tag{4.19}$$

$$N_0 : H^{\frac{-1}{2}}(\Gamma) \to H^{\frac{-1}{2}}(\Gamma) \tag{4.20}$$

to prove 4.17 we use expansion of the kernel of 3.14,[S.I83], i.e.,

$$\frac{-i}{4} H_0^{(1)}(z) = \frac{1}{2\pi} \log z + \gamma + O(z^2 \log z), \text{where}, \gamma \in \mathbb{C}$$

and $O(z^2 \log z)$ is the error introduced by truncating the series . Thus, with the smoothing operator R given by 3.9 of 3.15, we can write as

$$V_\alpha = V_0 + R$$

where, $V_\alpha = \frac{-i}{4} H_0^{(1)}(z)$, $V_0 = \frac{1}{2\pi} \log z$ and $R = O(z^2 \log z)$.
By applying the Fourier transform 3.10, V_α and V_0 have the same principal symbol 3.14i.e.,

$$\sigma(V_\alpha) = \sigma(V_0) = \frac{-1}{2} \frac{1}{|\zeta|} = F_{x \to \zeta}(\frac{1}{2\pi} \log z)$$

Hence, V_α and V_0 are pseudodifferential operators 3.12 of order -1 yielding the proposed properties 4.17, 4.18. For 4.20 we note that N_0 is just the operator adjoint to the double layer potential (i.e., the last equation of 4.14) and therefore it has a kernel $K(s,t) \in C^\infty(\Gamma \times \Gamma)$ implying the desired smoothness of operator 4.20, and the proof of 4.19 follows from 4.20.

Lemma 4.2. *For any given* $(h, a) \in H^s(\Gamma) \times \mathbb{R}, s \geq 1$, *there exists exactly one solution* $(g, c) \in H^{s-1}(\Gamma) \times \mathbb{C}$ *of*

$$V_0(g) + c = h, \int_\Gamma g ds = a \tag{4.21}$$

Proof. As it is shown in [S.I83], for $(h,a) \in C^{1+\alpha}(\Gamma) \times \mathbb{R}, \alpha > 0$, 4.21 has a unique solution in $C^0(\Gamma) \times \mathbb{C}$, provided the condition that the mapping radius of Γ is not equal to one. There exists one solution $(f_0, \gamma_0) \in C^\infty(\Gamma) \times \mathbb{R}$ satisfying

$$V_0(f_0) = \gamma_0, \int_\Gamma f_0 ds = 1 \tag{4.22}$$

In addition for every $h \in C^{1+\alpha}(\Gamma)$ there exist exactly one solution $f_p \in C^\alpha(\Gamma)$ of

$$V_0(f_p) = h + \Gamma_1(h) \tag{4.23}$$

where the functional

$$\Gamma_1(h) = \frac{1}{L}\int_\Gamma V_0(f_p)(\sigma)d\sigma - \int_\Gamma h(\sigma)d\sigma \tag{4.24}$$

where L is the length of Γ and σ is the arc length. The uniqueness and existence of $f_p \in C^\alpha(\Gamma)$ is shown by differentiating 4.23 with respect to the arc length. Then 4.23 becomes

$$Hf_p + Rf_p = \frac{dh}{ds}, Hf_p := p.v.\frac{1}{2\pi}\int_\Gamma \frac{f_p(y)}{s_x - s_y} ds_y \tag{4.25}$$

which is a singular integral equation. In definition 3.23 of equation 3.18, the integral is improper since the denominator in the integrand is zero when $y = x$. For functions $V(x)$, the integral exists as a Cauchy principal value3.19(a method for assigning values to certain improper integrals which would otherwise be undefined) i.e., it exists and is defined, via taking a symmetric limit:

$$\mathscr{H}V(x) = \frac{1}{\pi}p.v\int_{-\infty}^{\infty}\frac{V(y)}{y-x}dy = \frac{1}{\pi}\lim_{\varepsilon\to 0}(\int_{-\infty}^{x-\varepsilon}\frac{V(y)}{y-x}dy + \int_{x+\varepsilon}^{\infty}\frac{V(y)}{y-x}dy) \tag{4.26}$$

Equation 4.26 is the use of the Hilbert transform as an operator. If we apply $\mathscr{H}$ twice to a function on taking a Fourier transform,

$$\begin{aligned}\mathscr{F}(\mathscr{H}(\mathscr{H}V))(S) &= \mathscr{F}((\frac{-1}{\pi x}) * (\frac{-1}{\pi x}) * V(x)) \\ &= (isgns)(isgns)V(s) \\ &= -sgn^2 sV(s) \\ &= -V(s)\end{aligned} \tag{4.27}$$

Taking the inverse Fourier transform we see that $\mathscr{H}(\mathscr{H}V(x) = -V(x)$. Hence the Hilbert transform is invertible and its inverse is just its negative. Therefore, applying 4.26 on 4.25 becomes a Fredholm integral equation of second kind for f_p and for every $f_p \in l^2(\Gamma)$. Applying the method in 4.27,

$$\mathscr{H}(\mathscr{H}f_p) = -f_p \tag{4.28}$$

Thus, we have

$$(I+C)f_p = H\frac{dh}{ds}$$

with a compact operator C on $H^{s-1}(\Gamma)$. Now as indicated in [S.I83] using 4.26 the solution of 4.25 can be represented as

$$f_p(x) = p.v\int_\Gamma (\frac{1}{s_x - s_y} + r(x,y))\frac{dh}{ds}(y)dsy =: M(h) \tag{4.29}$$

with a smooth kernel $r(.,.)$ depending only on the regularity of Γ. Now substituting 4.29 in to 4.23 and integrating over Γ we get 4.24. For $h \in H^s(\Gamma)$ obviously $\frac{dh}{ds} \in H^{s-1}(\Gamma)$. The operator H is a singular integral operator of Cauchy type[see, 3.22] and therefore by 3.12 a pseudodifferential operator of order zero. Therefore, by density, the solution f_p of 4.29 belongs to $H^{s-1}(\Gamma)$, $s \geq 1$. Finally, with the solutions f_0, f_p of 4.22, 4.23 respectively, the solution (g,c) of 4.2 is given by:

$$g = f_p + \lambda f_0 \tag{4.30}$$

$$\lambda = a - \int_\Gamma f_p ds \tag{4.31}$$

$$c = -(\Gamma_1 h) + \lambda\gamma_0) \tag{4.32}$$

Theorem 4.1. *For $s \in \mathbb{R}, s \geq 1$, and for any given $(h_1, h_2, a) \in H^s(\Gamma) \times H^{s-1}(\Gamma) \times \mathbb{R}$ the system 4.15 has a unique solution $(f,g,c) \in H^{s-1}(\Gamma) \times H^{s-1}(\Gamma) \times \mathbb{C}$.*

Proof. In 4.15 let's set

$$h := V_\alpha(f) - h_1 \tag{4.33}$$

for every $f \in H^{s-1}(\Gamma)$ and for every $h_1 \in H^s(\Gamma)$.[S.I83]
By lemma 4.2, there exists exactly one solution (g,c) of 4.21 which coincides with the first two equations in 4.15. Due to the form 4.30 and since $f_p = M(h)$ from 4.29, $g \in H^{s-1}(\Gamma)$ is given explicitly by

$$g = \mathscr{A}(h) := M(h) + \lambda(h)f_0 \tag{4.34}$$

Therefore, with 4.33, the equation 4.34 becomes

$$g = \mathscr{D}(f) - \mathscr{A}(h_1) \tag{4.35}$$

i.e., from 4.34,

$$\mathscr{A}(h) = M(h) + \lambda(h)f_0$$

but, from 4.33,

$$h = V_\alpha(f) - h_1$$

Thus,

$$g = \mathscr{A}(V_\alpha(f) - h_1) = M(V_\alpha(f) - h_1) + \lambda(V_\alpha(f) - h_1)f_0$$

which is the same as,

$$g = \mathscr{A}(V_\alpha(f)) = M(V_\alpha(f)) + \lambda(V_\alpha(f) - h_1)f_0, h_1 \in H^s(\Gamma) \tag{4.36}$$

Hence, g & c given by 4.30 and 4.32 respectively give the solution of the first two equations of 4.15 interms of f, h_1, and a. Now by inserting 4.35 of g in to the third equation of 4.15, we can uncouple the system 4.15, i.e.,

$$(\frac{-1}{2} + N_\alpha)f - (\frac{1}{2} + N_0)g = h_2$$

is the third equation of 4.15. Which implies that,

$$\frac{-1}{2}f + N_\alpha f = \frac{1}{2}g + N_0 g + h_2 \tag{4.37}$$

then inserting

$$g = \mathscr{D}(f) - \mathscr{A}(h_1),$$

in 4.37, we have

$$\frac{-1}{2}f + N_\alpha f = \frac{1}{2}(\mathscr{D}(f) - \mathscr{A}(h_1) + N_0(\mathscr{D}f - \mathscr{A}(h_1)) + h_2$$

which becomes:

$$\frac{-1}{2}f + N_\alpha f = \frac{1}{2}(\mathscr{D}(f) - \mathscr{A}(h_1)) + N_0(\mathscr{D}f) - N_0(\mathscr{A}(h_1)) + h_2 \tag{4.38}$$

Theorem 4.2. *[Rellich- kondrachov compactness theorem or Rellich embedding theorem] [Alb13], let $\Omega \subset \mathbb{R}^n$ be a bounded open set with c^l boundary. Assume $1 \leq p < n$. Then, for each $1 \leq q < p^*$, where $p^* = \frac{np}{n-p}$ one has the compact embedding*

$$W^{1,P}(\Omega) \subset\subset L^q(\Omega) \tag{4.39}$$

Proof. Let $(u_m)_{m\geq 1}$ be a bounded sequence in $W^{1,p}(\Omega)$. Assume that all functions u_m are defined on the entire space $\mathbb{R}^n$ and vanish outside a compact set K:

$$\text{Supp}(u_m) \subseteq K \subseteq \tilde{\Omega} = B(\Omega, 1)$$

Here the right hand side denotes the open neighbor hood of radius one around the set Ω. Since $q < p^*$ and $\tilde{\Omega}$ is bounded, we have:

$$\|u_m\|_{L^q}(\mathbb{R}^n) = \|u_m\|_{L^q}(\tilde{\Omega}) \leq C\|u_m\|_{L^{p^*}(\tilde{\Omega})} \leq C'\|u_m\|W^{1,p}(\tilde{\Omega})$$

for some constants C, C'. Hence, the sequence u_m is uniformly bounded in $L^q(\mathbb{R}^n)$.

By lemma 4.1 and thanks to the Rellich embedding theorem 4.2 and using 4.39, the operators N_α and $N_0\mathscr{D}$ are compact in $H^{s-1}(\Gamma)$ since $\mathscr{D}$ defined by 4.35 is bounded in $H^{s-1}(\Gamma)$. According to Hilbert transform 3.18 and applying $\mathscr{H}$ twice as in 4.28 to 4.36 we obtain

$$\mathscr{D}(f) = f + R^+(f) + w(f) \tag{4.40}$$

where R^+ is a smooth operator and w(f) denotes terms corresponding to the asymptotic expansion of

$$H_0^{(1)}(|x-y|)$$

for small $|x-y|$, in [Sek10], the asymptotic expansion of Hankel function is given by,

$$H_\nu^{(1)(z)} = (\frac{2}{\pi z})^{\frac{1}{2}} e^{j(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2}-\nu)_m \Gamma(\nu+m+\frac{1}{2})}{m!\Gamma(\Gamma\nu+\frac{1}{2})(2jz)^m} + O(z^{-p}) \right] \tag{4.41}$$

As it is indicated in [S.I83] the kernel K(s,t) of w behaves as:

$$p.v. \int_\Gamma \frac{|x(s)-x(\delta)|}{s-t} (\frac{\partial}{\partial s}|x(s)-x(\delta)|) \log|x(s)-x(\delta)| ds \tag{4.42}$$

and therefore it behaves as

$$K(s,t) = p.v \int_\Gamma \frac{log|x(s)-x(\delta)|}{s-t} ds \tag{4.43}$$

and thus

$$w(f)(x(s)) = \int_\Gamma K(s,t) f(t) dt \tag{4.44}$$

which is the kernel of HV_0 and this composition is a pseudo differential operators of order -1; because H has the symbol $\sigma(H) = sgn\zeta$ and $\sigma(V_0) = \frac{-1}{2|\zeta|}$. Hence,

$$HV_0 f \in H^s(\Gamma)$$

for $f \in H^{s-1}(\Gamma)$. Using equation 4.40, equation 4.38 is a Fredholm equation of second kind for the unknown density $f \in H^{s-1}(\Gamma)$, i.e.,

$$(I + C_1) f = S_1(h_1, h_2) \tag{4.45}$$

where C_1 is a compact operator in $H^{s-1}(\Gamma)$.

Theorem 4.3. *Let $C : X \to X$ be a compact operator. Then, $I + C$ is Fredholm.*

Proof. as it is shown in [Alb13] first we coincide the kernel of $I+C$. Let B be a unit ball in ker(I+C). Then, $B = C(B)$ so B is image of a bounded set under a compact operator hence is precompact (i.e., if for every $\varepsilon > 0$, it can be covered by finitely many balls with radius ε). But B is closed so B is compact. Hence, ker(I+C) is finite dimensional. Thus, the operator $I+C$ is semi-Fredholm. Applying the same argument to the adjoint $I+C^*$ completes the proof.

In general, identity plus compact is Fredholm. Thus, for given $h_1 \in H^s(\Gamma)$ and $h_2 \in H^{s-1}(\Gamma)$ clearly $S_1(h_1,h_2)$ belongs to $H^{s-1}(\Gamma)$. More over, $S_1(0,0) = 0$. As it is indicated in [S.I83], by the application of potential theory $I+C_1$ is injective due to the uniqueness of the original interface problem 4.11. Thus, the inverse

$$(I+C_1)^{-1}$$

exists and is a bounded operator in $H^{s-1}(\Gamma)$. Therefore, as indicated in [S.I83], inserting the explicitly given solution $f \in H^{s-1}(\Gamma)$ of 4.45 in to the third equation of 4.15 we obtain:

$$(I+2N_0)g = S_2(h_1,h_2) \tag{4.46}$$

which again is a Fredholm equation of the second kind for $g \in H^{s-1}(\Gamma)$ due to 4.20 and theorem 4.3. Again as above, 4.46 is also uniquely solvable.

Chapter 5
Conclusion and Future work

The phenomenon of wave propagation is frequently encountered in a variety of engineering disciplines. For instance, in the design of antennas, it is important to know the interaction with electromagnetic waves:

$$\Delta^2 u(x,t) = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}, x \in \Omega$$

It was realized a long time ago that this problem can be recast in to an integral equation for the unknown potential function u. For harmonic waves (i.e., assuming a time dependent for u in the form $e^{-i\omega t}$) the problem is more conveniently described in the frequency domain where it reduces to the solution of the wave or Helmholtz equation:

$$\Delta^2 u(x) + k^2 u(x) = 0$$

The Helmholtz equation has been formulated from the scalar electromagnetic wave equation. More over, using the application of Fourier transform and its inverse, the Green's function (the fundamental solution of the Helmholtz equation) has been given. We have considered two- dimensional interface problems in electromagnetics and we have reduced these problems to their boundary integral equation form. In addition, we have treated these boundary integral equations with less regularity assumptions in appropriate Sobolev spaces. We have observed that these integral equations are Fredholm integral equations of the second type 3.6 in form. Finally, using theorems and lemmas in [S.I83] and the references there in, we have concluded that these integral equations are uniquely solvable.

We hope that this paper will inspire more work in the future in this area.

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