MODES COUPLING SEISMIC WAVES AND VIBRATING BUILDINGS

BY

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This is to certify that the thesis prepared by Abebe Mekuria entitled "MODES COUPLING SEISMIC WAVES AND VIBRATING BUILDINGS"submitted in partial fulfillment of the requirements for MSc.degree in Applied Mathematics, and has been carried out by Abebe Mekuria under my supervision. Therefore, I recommend that the student has fulfilled the requirements and hence hereby he can submit the thesis to the department.

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This is to certify that the thesis prepared by Abebe Mekuria, entitled “”and submitted in partial fulfillment of the requirements for the degree of Master’s of Science in Applied Mathematics(Specialization in Differential).

As a member of the board of examiners of the MSc.Thesis open defense examination, we certify that we have read and evaluated the thesis prepared by Abebe Mekuria and examined the candidate. We recommended that the thesis be accepted as fulfilling the thesis requirement for the degree of Master’s of Science in Applied Mathematics.

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Abstiract

We study a model for the so-called city-effect in which an earthquake can be locally enhanced by the collective response of tall buildings in a large city. We use a set of equations coupling vibrations in buildings to motion under the ground. These equations were previously studied by different author and we able to show the well posedness of this system.Furthermore we show how solving for the wavenumber in a non-linear equation and we are able to find resonant frequencies coupling seismic waves and vibrating tall buildings which ensure the existence seismic risk due to building.

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Chapter 1

Introduction

1.1 Background of the study

Seismic waves are the waves of energy caused by the sudden breaking of rock within the earth or seismic waves are the waves of energy produced by an earthquake,see,e.g[7,8]. They are the energy that travel through the earth and recorded on seismograph.An earthquake is a trembling of Earth caused by sudden release of stored energy, usually along faults. It has been observed that when an earthquake strikes a large city, the seismic activity is altered by the collective response of the buildings of the city,see,e.g[14]. This phenomenon is called the city-effect",see,e.g[9] and these seismic waves primarily of two types called the body waves and surface waves together cause shaking of the ground (surface of the Earth) on which the buildings are founded. The characteristics of the ground shaking control earthquake response of buildings, in addition to the building characteristics. The ground motion can be measured in the form of acceleration, velocity or displacement. Earth scientists are interested in capturing the size and origin of earthquakes worldwide and measure feeble ground displacements even at great distances from the epicenter of the earthquakes. Instruments that measure these low level displacements are called Seismographs. In the vicinity of the epicenters of large earthquakes the ground shaking is violent,see,e.g[2]. Seismographs get saturated as their design is such that they get saturated under large displacement shaking and become ineffective in capturing the displacement of the ground. And on the other hand, engineers are interested in studying levels of ground shaking at which buildings are damaged and are conversant with forces (as part of the design process of building). Hence, this motivated the development of instruments called Accelerographs that record during the earthquake shaking acceleration as a function of time of the location where the instrument is placed. These instruments successfully capture the ground shaking even in the near field of the earthquake faults, where the shaking is violent,see,e.g[4].

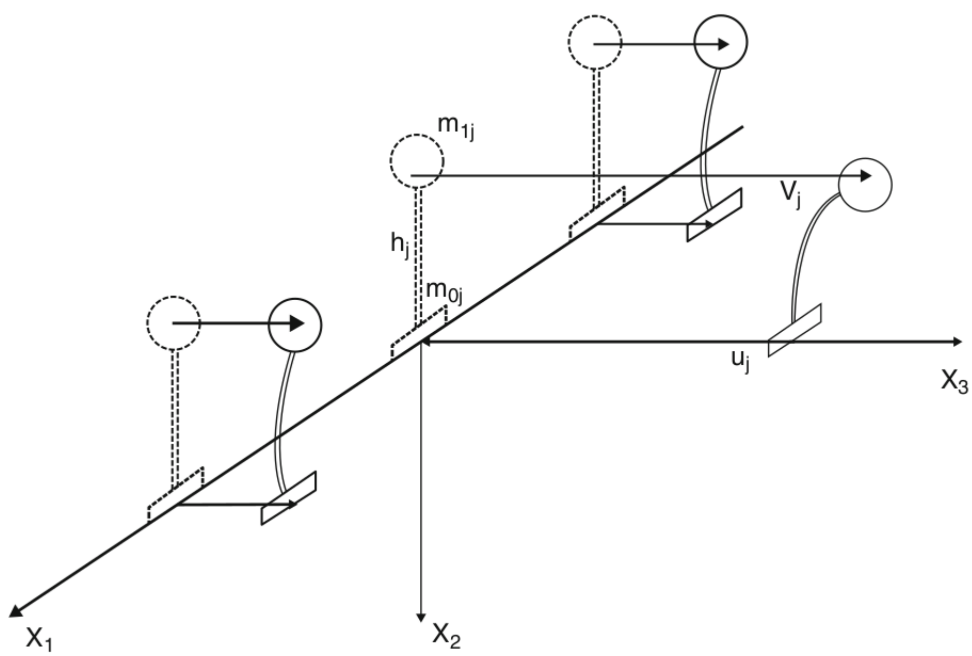
The two most important variables affecting earthquake damage are; the intensity of ground shaking caused by the quake coupled and the quality of the engineering of structures in the region. The level of shaking in turn, is controlled by the proximity of the earthquake source to the affected region and the types of rocks that seismic waves pass through en route (particularly those at or near the ground surface). Generally, the bigger and closer the earthquake, the stronger the shaking. But there have been large earthquakes with very little damage either because they caused little shaking or because the buildings were built to withstand that kind of shaking. In other cases, moderate earthquakes have caused significant damage either because the shaking was locally amplified or more likely because the structures were poorly engineered,see,e.g[11]. During an earthquake, buildings oscillate. But, not all buildings respond to an earthquake equally. Small building are more affected or shaken by high frequency waves (short and frequent). For example; a small boat sailing in the ocean will not be greatly affected by a large swell. On the other hand several small waves in quick succession can overturn or capsize the boat. In much the same way a small building experiences more shaking by high-frequency earthquake waves. Large structures or high rise buildings are more affected by long period or slow shaking. For instance;an ocean liner will experience little disturbance by short waves in quick succession. However; a large swell will significantly affect the ship. Similarly; a skyscraper will sustain greater shaking by long-period earthquake waves than by the shorter waves,see,e.g[16].

1.2 Problem formulation

The traditional approach to evaluating seismic risk in urban area is to consider seismic waves in the underground as the only cause for motion at the top. Accordingly, in earlier studies seismic wave propagation was evaluated in a separate step and then impacts on man-made structures above ground were calculated. But, is traditional approach and earlier study show the existence of seismic wave around tall building? The aim of this paper is to answer this question.However; observational evidence has since then demonstrated that structures built on the earth surface may in turn impact seismic waves. That idea was supported by several other technical and computational studies. More recently; Ghergu and Ionescu have derived a model for the city effect based on the equations of solid mechanics and appropriate coupling of the different elements involved in the physical set-up of the problem.

1.3 Physical model

We model the underground to be an elastic half-space in $x_1x_2x_3$ the space. We denote by Ω the half plane and hence $x_2 > 0$. All displacements occur in the x_3 direction and are independent of x_3 therefore $x_3 = 0$. Consider N buildings with width $2l_j$ and height h_j standing on the x_1 axis. In this study N is either a finite number or it is infinite. The index j will range from 1 to N. The rigid building foundations are all located along the x_1 -axis and denoted by $\Gamma_j = [a_j, b_j] \times \{0\}$ in Ω. The anti-plane shearing displacements in $x_1x_2x_3$ space are sketched in the following Fig. Here $N = 3$. Building foundations are represented



by rectangles and tops are represented by circles. Initial positions are sketched in dashed lines, displaced positions are sketched in solid lines.

Where;

- m_1j is mass of the top of the building,
- m_oj is mass of the foundations of the building,
- vj is the displacements of the tops j building,
- u_j is the displacement of the foundation of the j building,
- w is all displacement occur in x_3 direction,
- The shear rigidity s, the density ρ and the shear velocity $\beta = \sqrt{s/\rho}$.

Let us introduce the following mathematical expression : the displacements of the foundations u_j and the displacements of the tops v_j of the buildings in a city.All displacements occur in the x_3 direction and are independent of x_3.The displacement is therefore simply denoted by $w = w(t, x_1, x_2)$ and satisfies the wave equation.

$$\rho w_{tt} = s\Delta w \;\; in \;\; \Omega \tag{1.1}$$

The foundations will be supposed rigid. That is, they have a much larger stiffness than the soil. If we denote by $u_j \in R$ the displacement of the foundation Γ_j the boundary condition for the soil displacements w reads.

$$w(t, x_1, 0) = u_j(t) \;\; for \;\; all \;\; (x_1, 0) \in \Gamma_j; \frac{\partial w}{\partial x_2}(t, x_1, 0) = 0 \;\; for \;\; all \;\; (x_1, 0) \in \Gamma_{free} \tag{1.2}$$

Each building j is modeled as an elastic spring (of elastic modulus k) which relates two concentrated masses.That is, mass of the foundation $m_{0,j}$ and mass of the top $m_{1,j}$ of the building (see the above Fig). If we denote by v_j the displacement of the top of the building with respect to the foundation, then the elastic force acting on the top is $E_j = -k(v_j - u_j)$.(-)indicate restoring force. Therefore, the equation corresponding to the concentrated mass $m_{1,j}$ on the top of the building reads:

$$m_{1,j}v_{tt} = -k(v_j - u_j) \tag{1.3}$$

On the foundation Γ_j in addition to the force E_j there acts the soil force, denoted by

$$R_j(w) = \int_{\Gamma_j} s\frac{\partial w}{\partial x_2}(t,0)dt. \tag{1.4}$$

In this way we associate at each soil displacement $w : \Omega \to R$ the N-dimensional vector of soil forces R_j.

The equation corresponding to the concentrated mass $m_{0,j}$ on the foundation can now be written as

$$m_{0,j}u_{tt} = \int_{\Gamma_j} s\frac{\partial w}{\partial x_2}(t,0)dt + k(v_j - u_j) \tag{1.5}$$

Finally, the mathematical formulation of the seismic soil- structure interaction model (or city-effect problem) considered above consists in finding the soil displacement w and the buildings displacements (foundations and tops)u, v solution of(1.1)-(1.5) satisfying some initial conditions.

1.4 The associated spectral problem

Let the time harmonic solution for the system of equation (1.1)-(1.5) is as follows,since our aim is to finding the soil displacement w and hence we set by

$$w(t, x_1, x_2) = \Phi(x_1, x_2)e^{-i\mu t} \tag{1.6}$$

Here $\Phi; \Omega \longrightarrow \Re$ represents the soil displacement and $\mu > 0$ is the frequency. Now let $\alpha = u_j, \eta = v_j$ where u and v are the building displacement of foundation and the top of the buildings respectively.

Now since $w(t, x_1, x_2) = \Phi(x_1, x_2)e^{-i\mu t}$ this implies $w_t = -i\mu\Phi(x_1, x_2)e^{-i\mu t}$ and $w_{tt} = -\mu^2\Phi(x_1, x_2)e^{-i\mu t}$.

Now substituting $w_t t$ in system(1.1),(1.2),(1.3)and(1.5) we obtain the following spectral(linear) problem to the evolution problem (1.1)-(1.5). And again since Φ is represents the soil displacement equation (1.9) obtained.

$$-s\Delta\Phi = p\mu^2\Phi \ \ in \ \Omega \tag{1.7}$$

$$k(\eta - \alpha) = \mu^2 m_1\eta, R(\Phi) - k(\eta - \alpha) = \mu^2 m_0\alpha \tag{1.8}$$

$$\Phi = \alpha, \ on \ \Gamma_j; \ \frac{\partial\Phi}{\partial x_2} = 0 \ on \ \Gamma_{free} \tag{1.9}$$

where $R(\Phi)$ is the soil resistance associated to Φ, defined in (1.4). In order to give a non-dimensional formulation of the spectral problem (1.7)-(1.9), we introduce the characteristic length l, the non-dimensional spatial coordinates (x_1, x_2) and the frequency (eigenvalue) ξ defined as:

$$x_1^{'} = \frac{x_1}{l}, x_2^{'} = \frac{x_2}{l}, \xi = \mu\frac{l}{\beta}$$

To analyze the eigenvalues (city's frequencies) with respect to the principal non dimensional characteristic values of the buildings we consider different city parameters. The non-dimensional city parameters are:

- $\gamma_b = \frac{m_1}{m_0}$ is the ratio between the top and foundations masses of the building,
- $f_b = \frac{l_b}{h}$ is the aspect ratio of the building (i.e, the ratio between the half-width l_b and the height h of the building),
- $2c_b = \frac{l_b}{l}$ is the non-dimensional length of the building,
- $\Upsilon = \frac{\rho_b}{\rho}$is the ratio between the densities of the building ρ_b and of the soil ρ,
- $b = \frac{\beta_b}{\beta}$ is the ratio between the shear wave velocities in the building $\beta_b = \sqrt{s_b/\rho_b}$ and in the soil β,
- The scale parameter $\varepsilon = \dfrac{l_b}{l_v}$ is the ratio between the width $2l_b$ of one building and the width of the city $2v_b$.

Following these notations we can write the elastic modulus k of the building and the top building mass $m_{1,j}$

$$k = \frac{2s_b l_b}{h}, m_1 = 2l_b h\rho_b$$

If we eliminate from (1.8) the expressions of α and η and we make use of the above non-dimensional constants then the(linear) spectral problem (1.7)-(1.9) can be written as a nonlinear eigenvalue problem in the soil. The only unknowns are the normalized frequency

μ^2 and the soil displacement $\Phi;\Omega \rightarrow R$ solution of the following boundary value problem:

$$\Delta\Phi + k^2\Phi = 0 \;\; in \;\; \Omega \tag{1.10}$$

$$\frac{\partial\Phi}{\partial x_2} = 0 \;\; on \;\; \Gamma_{free} \tag{1.11}$$

$$q(k^2)\Phi(x_1,0) = p(k^2)\int_{\Gamma_j}\frac{\partial\Phi}{\partial x_2}(t,0)dt \;\; for \;\; (x_1,0) \;\; in \;\Gamma_j \;\; and \;\; j \;\; in \;\; [1,N] \tag{1.12}$$

where,$\rho(k^2) = c_b^2k^2 - b^2f_b^2, \; q(k^2) = 2\frac{\Upsilon c_b^2k^2}{f_b}(c_b^2k^2 - \frac{\gamma_b+1}{\gamma_b}\rho(k^2))$
we can calculate $\alpha, \; \eta$ as $\alpha_j = \Phi(x_1,0) \;\; for \;\; (x_1,0) \;\; in \;\Gamma_j, \;\; \eta_j = -\frac{b^2f_b^2\alpha_j}{\rho(k^2)}$ and for $k = \mu$ We call the above system of Eqn.(1.10)-(1.12) equation modelingcity-effect",In this thesis we answer the following questions;

1. Is the equations modeling "city-effect" are solvable?
2. Is the equations modeling "city-effect" has unique solution?If solvable.
3. Is the associated frequencies that will lead to the resonance between vibrating building and underground seismic waves should exists? '

1.5 Objective of the Study

1.5.1 General Objectives

The general objective of this study is to analyze the equations modeling the "city-effect".

1.5.2 Specific Objectives

The specific objectives of the study are to:

(i) Prove that the equations modeling the "city-effect" is solvable.

(ii) Show that the solution for equation modeling the "city-effect" is uniqueness.

(iii) Prove the existence of the coupling frequency between vibrating building and underground force.

1.6 Significance

The study of the existence for modes coupling of seismic wave and vibrating building will have the following importance.

- It contributes knowledge to the seismic wave around the building and the effect of building for vibrating.
- The process of the research will have contribution for engineering in case of building around city.
- It will contribute that when an earthquake strikes a large city the seismic activity is altered by the collectivity response of the building of the city.

1.7 Scope of study

This study deal with a problem of seismic waves to the building and consider coupling frequencies should exist in urban area. This why we call city-effect" problem.That is, it does not deal with in which people,animals and plants live.(or environmental area).

Chapter 2

Literature Review

Now days the existence of preferred frequencies for Coupling of Seismic waves and vibrating tall buildings (seismic activity) is well studied,[14].The starting point of the city-effect" problem is studied by Ghergu and Ionescu with a stronger theoretical and mathematical flavor.When earthquake strikes a large city,the seismic activity is altered the collective of buildings of the city,[11]. and they proposed a model derived from the equation of physics. Ghergu and Ionescu were able to compute a city frequency constant for fundamental solution to the city-effect model.which given by

$$w(t, x, y) = \Phi(x, y)e^{-i\mu t} \tag{2.1}$$

in other words $\mu > 0$ is the associated frequency of a time harmonic vibration.that is given the geometry and specific physical constant of an idealized two dimensional city. They computed a frequency that will lead to the resonance between vibrating building and underground seismic waves,see,e.g[9]. They were obtained by increasing the number of building at the expense of solving large and larger systems. This computation, were limited to the case of a finite set of identical equally spaced buildings. Ghergu and Ionescu's results are giving a lot of useful information(quite instructive)for sergey Zheltukhin. Since they point to a collective response of buildings to seismic waves a phenomenon that authors have named city-effect,see,e.g[9]. On this Sergey Zheltukhin contribution to extend these work and provide a mathematical analysis for proving the existence of preferred frequency coupling vibrating of buildings to underground seismic waves. Using a model involving vibrating tall buildings and harmonic elastic (anti plane) displacements of the ground; Sergey Zheltukhin have introduced in his paper methods for computing frequencies that achieve the coupling of these two vibration. Sergey Zheltukhin noted that; the method for finding coupling frequencies introduced by

Ghergu and Ionescu becomes quickly computationally expensive as the number of buildings grows large and may be hard to extend to a fully three dimensional setting. Given that weakness;Sergey Zheltukhin have resorted to the use of periodic formulations. They first obtained results that coincide with those of Ghergu and Ionescu's at a much lower computational cost. From there Zheltukhin have shown how their method can be extended to repeated sets of non-identical buildings. More realistic(excepted in sensible way) city layouts(arranged) can thus be modeled.

Interestingly;in the case of non-identical buildings theirs simulations indicate that the response to this coupling phenomenon may differ drastic-cally from one building to another,see,e.g[15].

Chapter 3

Research Methodology

3.1 Study procedure

To show the equations modeling the 'city-effect' problem introduced in chapter 4 is well-posed and the existence of this frequency coupling seismic wave, the step to be followed;

- We acknowledge that these equation were carefully derived following the low of solid mechanics combined with the knowledge of relevant dominant effects causing this phenomenon.
- We provide the equation we introduced in section (1.3) in non-dimensional form.
- We deal with an overview of results on Hankel function relevant to our work.
- Given building with positive height,mass and elastic modulus, the set of coupling equation given in chapter 4 defining frequency coupling that building and the ground beneath have at least one solution.
- We have to carefully study the Boundary Dirichlet to Neumann operator T_k for Helmholtz problems outside the disk in R^2,where k is wave numbers
- We show that T_k is (strongly) real analytic in k and we need to determine the strong limit of T_k as k tends to zero.
- We deals with the much more delicate equation of high frequency asymptotic

Chapter 4

The analysis of the "city-effect" model

We describe here mathematical approach and we will develop to solve the nonlinear eigenvalue problem (1.10)-(1.12). The (linear) spectral problem (1.7)-(1.9) or equivalently the nonlinear eigenvalue problem (1.10)-(1.12) have an infinity of solutions. The outgoing Sommerfeld radiation condition at infinity involved in the Helmholtz problem to insures the uniqueness of the solution of Eqn (1.10)-(1.11). Finally, we get the following non-linear eigenvalue problem;

$$\Delta\Phi + k^2\Phi = 0 \;\; in \;\; \Re^{2+} \tag{4.1}$$

$$\Phi = 1 \;\; on \; \Gamma \tag{4.2}$$

$$\frac{\partial\Phi}{\partial x_2} = 0 \;\; (x_2 = 0)\backslash\Gamma \tag{4.3}$$

$$\frac{\partial\Phi}{\partial r} - ik\Phi = o(r^{-1}), uniformlyas \;\; r \;\; \to \infty \tag{4.4}$$

$$q(k^2) = p(k^2) Re \int_{\Gamma_j} \frac{\partial\phi_k}{\partial x_2}(t,0)dt \tag{4.5}$$

where

$$p(t) = c_1 t - c_2, \quad q(t) = t(c_3 t - c_4) \tag{4.6}$$

Our main objective is to find values of the non-dimensionalized frequency k for which system (4.1)-(4.5) will be solvable. Assuming that the building has width 1 and is standing on the x_1 axis, so that its foundation may be assumed to be the line segment $\Gamma = [\frac{-1}{2}, \frac{1}{2}]$. Here $k > 0$ is the wavenumber, $r = \sqrt{x_1^2 + x_2^2}$ the physical displacement is $Re\Phi e^{-ikt}$ and the constants c_1, c_2, c_3, c_4, are determined by the physical properties of the underground and the building. Note that system (4.1)-(4.6) is non-linear in the unknown wavenumber k.Simply the goal of this paper is to show the following theorem.

Theorem 4.0.1. *For any value of the constant c_1, c_2, c_3, c_4 the system ((4.1))- ((4.6)) has at least one solution in k,that is, there exists a positive k and a function Φ in R^2 such that Eqs. ((4.1))- ((4.6)) are satisfied. Moreover, this system of equations has at most a finite number of solutions in k.*

Proof. To proof the theorem the following must hold

$$F(k) = q(k^2) - p(k^2) Re \int_{\Gamma_j} \frac{\partial \phi_k}{\partial x_2}(t,0)dt \;\; for \;\; k \;\; in \;\; (0,\infty) \tag{4.7}$$

where Φ_k solves ((4.1)) through ((4.4)). We will first show that F is real analytic in k. Then we will perform a low frequency and a high frequency analysis of Φ_k.The low frequency analysis will show that F must be negative in $(0,\alpha)$ for some positive number . The high frequency analysis will prove that the $\lim_{k\to\infty} F(k) = \infty$ is concluding the proof of Theorem (4.0.1). □

4.1 The boundary Dirichlet to Neumann operator: analyticity with regard to the wavenumber

We denote by D the open unit disk of R^2 centered at the origin. Our first lemma is certainly well known to most readers, but we chose to include it since it is helpful in the detailed study of the related Dirichlet to Neumann operator relevant to this study.

By Theorem (4.0.1)

Lemma 4.1 Let $K > 0$ be a wave number and f be a function in the sobolev space $H^{\frac{1}{2}}(\partial D)$.The problem

$$\Delta v + k^2 v = 0 \;\; in \, R^2 \backslash D_e \tag{4.8}$$

$$u = f \;\; \partial D \tag{4.9}$$

$$\frac{\partial u}{\partial r} - iku = o(r^{-1}) \;\; uniformly \;\; as \;\; r \;\; \to \infty \tag{4.10}$$

has unique solution.written $f = \sum_{n=\infty}^{\infty} a_n e^{-in\theta}$,we have

$$u = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta} \frac{H_n(kr)}{H_n(k)} \tag{4.11}$$

This series and all its derivatives are uniformly convergent on any subset on R^2 in the form $r \geq A$ where $A > 1$.

Proof. When we are dealing with an interior problem we seak of progressive wave solutions.Considering the fact plane waves are solution to the impose additional boundary condition at infinity to equation of Dirichlet and Neumann problem in order to guarante uniqueness.

Thus,we add the extra sommerfeld radiation condition or outgoing wave condition.that is

$$\mid \frac{\partial u}{\partial r} - iku \mid \leq \frac{c}{r^2} \tag{4.12}$$

It is clear that a change of convention in equation (4.12) yield a radiation condition with the opposite sign.

$$\mid \frac{\partial u}{\partial r} + iku \mid \leq \frac{c}{r^2} \tag{4.13}$$

We expect that this ingoing sommerfeld condition will as well lead to a well-posed problem which we are going to prove.

$$\Delta u + k^2 u = 0 \ \ in \ \ \Omega_e \tag{4.14}$$

$$u \mid_\Gamma = u_d \tag{4.15}$$

$$\mid \frac{\partial u}{\partial r} - iku \mid \leq \frac{c}{r^2} \tag{4.16}$$

where,u_d is Dirichlet data
Thus solution of equation (4.14)-(4.16) is not unique in this space(sphere). We need to add the radiation condition.A form of this condition is $\mid \frac{\partial u}{\partial r} - iku \mid \leq \frac{c}{r^2}$
A weaker for is

$$\int_\Omega \mid \frac{\partial u}{\partial r} - iku \mid ds \leq c \tag{4.17}$$

□

Theorem 4.1.1. *(uniqueness Theorem)*
The exerior Dirichlet and Neumann problem (4.14)-(4.16) admit at most one solution in the Hilbert space.

Proof. The difference u between two solution u_1 and u_2 has a zero boundary condition.
That is satisfies $u \mid_\Gamma = 0$ or $(\frac{\partial u}{\partial r}) \mid_\Gamma = 0$, we multiply this equation by $\overline{u}$ and integrate by parts to obtain

$$\int_{\Omega n B_R} (\mid \nabla u \mid^2 - k^2 \mid u \mid)^2 dx - \int_{S_R} \frac{\partial u}{\partial r} \overline{u} d\sigma = 0 \tag{4.18}$$

At the exterior of the ball B_R,we expand the solution in the spherical harmonics, It holds that:

$$u(r, \theta, \varphi) = \sum_{L=0}^{\infty} \sum_{m=t}^{L} (\alpha_L^m \frac{h_l^{(1)}(kr)}{h_l^{(1)}(kR)} + \beta_l^{(m)} \alpha_L^m \frac{h_1^{(2)}(kr)}{h_1^{(1)}(kR)}) Y_l^{(m)}(\theta, \varphi) \tag{4.19}$$

and $\dfrac{\partial u}{\partial r} - iku$ has the expansion

$$\frac{\partial u}{\partial r} - iku = \sum_{L=0}^{\infty}\sum_{m=t}^{L}\left(\frac{k\alpha_L^m}{h_l^{(1)}(kR)}\left(\frac{d}{dr}h_l^{(1)}(kr) - ih_{(l)}^1(kr)\right) + \frac{k\beta_l^{(m)}}{h_l^{(2)}(kr)}\left(\frac{d}{dr}h_l^{(2)}(kr) - ih_l^{(2)}(kr)\right)Y_l^{(m)}(\theta,\varphi)\right) \tag{4.20}$$

The orthogonality of the spherical harmonic implies that the integral on the complement of B_R of a function of the form

$$u = \sum_{L=0}^{\infty}\sum_{m=-l}^{L}\varphi_l^{(m)}(r)Y_l^{(m)}(\theta,\varphi) \quad for \ \ r > R, \ \ is \tag{4.21}$$

$$\int_{\beta_R^c} \mid u \mid^2 dx = \sum_{L=0}^{\infty}\sum_{m=-l}^{L}\int_R \mid \varphi_l^m(r) \mid^2 r^2 dr \tag{4.22}$$

The dominant term in the expression $\dfrac{\partial u}{\partial r} - iku$ is the one coming form $h_L^{(2)}$.It behaves as $-2i(e^{-ikr}/r)(\beta_L^m/h^{(2)}(kR))$. Thus, the convergence of the associated series is possible only when all the coefficients;

$$\beta_l^m = 0 \tag{4.23}$$

We have proved that the solution contains only the terms $h_l^{(1)}$. The imaginary part in the expression (4.18) reduces to

$$\int_{S_R} (T_R u, \overline{u})d\sigma = 0 \tag{4.24}$$

where T_R is the capacity operator on the ball with radius R. From the expression of this operator follows

$$\alpha_l^m = 0 \tag{4.25}$$

From which we infer that;

$$U/S_R = 0 \tag{4.26}$$

$$\frac{\partial u}{\partial n}/S_R = 0 \tag{4.27}$$

and thus the solution vanishes on the exterior of the ball β_R Moreover, the Helmholtz equation is an elliptic operator, from which it follows that the solution is analytic in the domain Ω_e . Thus, it is zero.

Remarks There exists a weaker form for the radiation condition which is

$$\lim_{R\to\infty}\int_{S_R} \mid \frac{\partial u}{\partial r} - iku \mid^2 d\sigma = 0 \tag{4.28}$$

In the same way as above, it implies $\beta_l^m = 0$ and so the uniqueness of the solution.

Note that:The fredholm alternative,we know that uniqueness implies existence. □

To show the uniformly convergence of the series $U = \sum_{n=-\infty} a_n e^{-in\theta} \frac{H_n(kr)}{H_n(k)}$,we need to consider the usefulness of Hankel function.

lemma 4.2;The following equivalence as $n \to \infty$ is uniform for all Z in a compact set of $(0, \infty)$.

$$H_n(z) \sim \frac{-i}{\pi}(\frac{1}{2}z)^{-n}(n-1)! \tag{4.29}$$

proof,since for any positive z

$$\mid J_n(z)n!(\frac{1}{2}z)^{-n} - 1 \mid = \frac{1}{n+1} \mid \sum_{k=1}^{\infty}(\frac{-1}{4})^k \frac{z^{2k}(n+1)!}{k!(n+k)!} \mid \leq \frac{1}{n+1}\sum_{k=1}^{\infty}(\frac{1}{4})^k \frac{z^{2k}}{k!} \tag{4.30}$$

It is clear that $J_n \sim (\frac{1}{2}z)^n \frac{1}{n!}$ is uniformly for all z in a compact set of $(0, \infty)$.we can show similarly that $Y_n \sim \frac{-1}{\pi}(\frac{1}{2}z)^{-n}(n-1)!$ uniformly for all z in a compact set of $(0, \infty)$.We conclude that (4.29) must hold. Now from (4.29) we have
$H_n(k) \sim \frac{-i}{\pi}(\frac{1}{2}z)^{-n}(n-1)!$ and $H_n(kr) \sim \frac{-i}{\pi}(\frac{1}{2}z)^{-n}(n-1)!$
which implies $\frac{H_n(kr)}{H_n(k)} \sim \frac{1}{r^n}$
$\therefore \frac{H_n(kr)}{H_n(k)} \sim \frac{1}{r^n}$ uniformly in r as long as r remains in a compact set of $(0, \infty)$.Let A and B be two real numbers such that $1 < A < B$,set $M = sup \mid a_n \mid$
$\Rightarrow \mid a_n e^{in\theta\theta} \frac{H_n(kr)}{H_n(k)} \mid \leq \frac{2M}{r^n} = \frac{2M}{A^n}$
This implies $\mid a_n e^{in\theta\theta} \frac{H_n(kr)}{H_n(k)} \mid \leq \frac{2M}{A^n}$ which is bounded and closed.
Therefore for all n large enough,uniformly for all r in [A,B],so that above series is uniformly convergent on any compact subset of $R^2/\overline{D}$.

Lemma 4.3 For $z > 0$,the following limit as $z \longrightarrow 0$ is uniform for all integer n different from 0.

$$\lim_{z \to 0^+} -\frac{z}{\mid n \mid} \frac{H_n'(z)}{H_n(z)} = 1 \tag{4.31}$$

For the special case $n = 0$,we have
$\lim_{z \to 0^+} z \frac{H_n'(z)}{H_n(z)} = 0$
proof We observe that for $n \geq 2$

$(H_n(z)(\frac{-i}{\pi}(\frac{1}{2}z)^{-n}(n-1)!)^{-1}-1)(n-1)$
can be bounded by a function in z which is continuous on $[0,\infty)$ and independent of n, therefore

$$H_n(k) \sim \frac{-1}{\pi}(\frac{1}{2}k)^{-n}(n-1)! \tag{4.32}$$

as $z \to 0^+$ uniformly on [0, b]for a fixed $b > 0$. Using the formula

$$H_n^{'} = -H_{n+1}(z) + \frac{n}{z}H_n(z) \tag{4.33}$$

We obtain $-\frac{z}{n}\frac{H_n^{'}(z)}{H_n(z)} \sim 1$ as $z \to 0^+$,uniformly on [0.b] for a fixed $b > 0$. for $n \leq -2$,we can apply the formula

$$H_{-n}(z) = (-1)^n H_n(z) \tag{4.34}$$

Next,Let we show all its derivatives are uniformly convergent on any subset of R^2 in the form $r \geq A$ where $A > 1$. Thus, $u_r = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} k\frac{H_n^{'}(kr)}{H_n(k)}$
Then we simply $k\frac{H_n^{'}(kr)}{H_n(k)}$ to do this,let use derivatives of Hankel function.
That is

$$H_n^{'}(z) = -H_{n+1}(z) + \frac{n}{z}H_n(z) \tag{4.35}$$

$\Rightarrow H_n^{'}(kr) = -H_{n+1}(kr) + \frac{n}{kr}H_n(kr)$
$\Rightarrow -k\frac{H_n^{'}(kr)}{H_n(k)} \sim -k\frac{H_{n+1}^{'}(kr)}{H_n(k)} + \frac{n}{r}k\frac{H_n(kr)}{H_n(k)}$
Thus,the system $-k\frac{H_{n+1}(kr)}{H_n(k)} \sim \frac{-2n}{r^{n+1}}$ and the term $\frac{n}{r}k\frac{H_n(kr)}{H_n(k)} \sim \frac{n}{r^{n+1}}$ then
$k\frac{H_{n+1}(kr)}{H_n(k)} \sim \frac{-2n}{r^{n+1}} + \frac{n}{r^{n+1}} = \frac{-n}{r^{n+1}}$
$\therefore k\frac{H_{n+1}(kr)}{H_n(k)} \sim \frac{-n}{r^{n+1}}$,set $M = supp \mid a_n \mid$,it follows that $\mid a_n e^{in\theta} k\frac{H_{n+1}(kr)}{H_n(k)} \mid \leq \frac{2M}{A^{n+1}}$
for $e^{in\theta} \leq 2$

Therefore,at this stage we can conclude that the r derivatives of the given series
$u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}\frac{H_n(kr)}{H_n(k)}$ is uniformly convergent on any compact subset of R^2
Next we show that the convergence properties of a θ derivatives of the series
$u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}\frac{H_n(kr)}{H_n(k)}$

that is u_θ is a multiplicative of a series u by 'in' that is $u_\theta = \sum_{n=-\infty}^{\infty} ina_n e^{in\theta} \dfrac{H_n(kr)}{H_n(k)}$ and $\mid u_\theta = \sum_{n=-\infty}^{\infty} ina_n e^{in\theta} \dfrac{H_n(kr)}{H_n(k)} \mid \sim \dfrac{2Min}{A^n}$ when $M = sup \mid a_n \mid$ and $\dfrac{H_n(kr)}{H_n(k)} \sim \dfrac{1}{r^n}$ Thus,u_θ is closed and bounded series .This implies u_θ is uniformly converges.

Lemma 4.4 for any n in N, $\mid H_n(z) \mid$ is a decreasing function of z on $(0, \infty)$

case-2 for $r \geq 2$,we conclude that form this lemma and $H_n(k) \sim -\dfrac{i}{\pi}(\dfrac{1}{2}z)^{-n}(n-1)i$ we have $\mid \dfrac{H_n(kr)}{H_n(k)} \mid \leq \mid \dfrac{H_n(2k)}{H_n(k)} \mid$. Then using $H_n(2k) \sim -\dfrac{i}{\pi}(\dfrac{1}{2}(2k))^{-n}(n-1)!$ and $H_n(k) \sim -\dfrac{i}{\pi}(\dfrac{1}{2}2)^{-n}(n-1)!$ implies $\dfrac{H_n(2k)}{H_n(k)} \sim 2^{-n}, \Rightarrow \mid \dfrac{H_n(kr)}{H_n(k)} \mid \leq \mid \dfrac{H_n(2k)}{H_n(k)} \mid \leq 2^{-n+1}$ for all n greater than some N for all $r \geq 2$

Similarly $\mid k\dfrac{H_n'(kr)}{H_n(k)} \mid \leq \mid k\dfrac{H_{n+1}(2k)}{H_n(k)} \mid + \mid \dfrac{n}{r}\dfrac{H_n(kr)}{H_n(k)} \mid \leq n2^{-n+1}$ for all n greater than some N,for all $r \geq 2$ given that a_n is bounded ,it follows that the series $u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \frac{H_n(kr)}{H_n(k)}$ and its r derivatives are uniformly convergence for all r in $[2, \infty]$. Finally,we need to show that the uniform convergent of u_{rr} and $u_{\theta\theta}$ for $r > 1$.

Remark Similar argument can be carried out for all second derivatives.

Definition;The linear operator T_k which maps $H^{\frac{1}{2}}(\partial D)$ into $H^{\frac{-1}{2}}(\partial D)$ by the formula

$$T_k(f) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \frac{H_n'(kr)}{H_n(k)} \tag{4.36}$$

Where $f = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. T_k is continuous since formula (4.35) combined with formula (4.34) implies $k\dfrac{H_n'(kr)}{H_n(k)} \sim -n, n \to \infty$.

Lemma 4.5 T_k is real analytic in k for k in $(0, \infty)$.

proof let K_0 and b be two real number such that $0 < b < k_0$, define $D_b(k_0)$the closed disk in the complex plane centered at k_0 and with radius b.(9)

To show that T_k real analytic in k for k in $(0, \infty)$ in operator norm, it suffices to fix f and g in $H_n^{\frac{1}{2}}$ and to show that $Re\langle T_k(f), g\rangle$ is an analytic function of k in $D_b(k_0)$

writing $f=\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, $g=\sum_{n=-\infty}^{\infty} b_n e^{in\theta}$,we have

$$\langle T_k(f),g\rangle = 2\pi \sum_{n=-\infty}^{\infty} a_n e^{in\theta}\overline{b_n e^{in\theta}} k\frac{H_n^{'}(k)}{H_n(k)} \tag{4.37}$$

since $k\dfrac{H_n^{'}(k)}{H_n(k)} \sim k\dfrac{-H_{n+1}^{'}(k)}{H_n(k)} + \dfrac{n}{k}\dfrac{H_n(k)}{H_n(k)} = -k\dfrac{-H_{n+1}^{'}(k)}{H_n(k)} + n = -k\dfrac{2n}{k} + n = -n$
$\therefore k\dfrac{H_n^{'}(k)}{H_n(k)} \sim -n \Longrightarrow \mid \sum_{n=-\infty}^{\infty} a_n b_n k \dfrac{H_n^{'}(k)}{H_n(k)} \mid = \sum_{n=-\infty}^{\infty} \mid a_n \overline{b_n} \dfrac{H_n^{'}(k)}{H_n(k)} \mid$

$= \sum_{n=-\infty}^{\infty} n \mid a_n\overline{b_n} \mid \leq \infty$ and since $k\dfrac{H_n^{'}(k)}{H_n(k)} \sim -\mid n \mid, \mid n \mid \to \infty$,uniformly for all k in $D_b(k_0)$,see lemma(4.3)

The series in(4.37)is a uniformly convergent sum of analytic function of k thus,$\langle T_k(f),g\rangle$ is analytic in $D_b(k_0)$.

4.2 The limit of the boundary operator T_k as approaches zero

Lemma 4.6 The operator T_k converges strongly to the operator T_0 which maps $H^{\frac{1}{2}}(\partial D)$ into $H^{\frac{-1}{2}}(\partial D)$ and is defined by the formula

$$T_o(f) = \sum_{n=-\infty}^{\infty} - \mid n \mid a_n e^{in\theta} \tag{4.38}$$

where $f=\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$

4.3 An equivalent problem in the plane R^2 minus a line segment.Low wavenumber asymtotics

We use the following notation in this section:Γ is the line segment $[\frac{-1}{2},\frac{1}{2}]\times\{0\}$, Ω is the open set $\{x\in R^2 :\mid x \mid < 1\}/\Gamma$. ∂D will denote the boundary as it known. On Γ an upper and lower trace for all functions in $H^1(\Omega)$ can be defined:even though Γ is an interior boundary of Ω,since $\Gamma$is C^1,an adequate version of the trace theorem on Γ holds,

We denote by $H^1_{0,\Gamma}(\Omega)$ the closed subspace of $H^1(\Omega)$ consisting of function whose upper and lower traces on Γ are zero.

Lemma 4.7 Let L be a continuous linear function on $H^1_{o,\Gamma}(\Omega)$,The following variational problem has a unique solution.Find u in $H^1_{o,\Gamma}(\Omega)$
such that

$$\int_{\Omega} \nabla u.\nabla v - \int_{\partial D} (T_0 u)v = L(v) \ \ for \ \ all \ \ v \ \ in \ \ H_{o,\Gamma}(\Omega) \tag{4.39}$$

proof We observe that due to the definition (4.38) of $T_0, \langle T_0(f), f\rangle$ is real for all f in $H^{\frac{1}{2}}(\partial D)$ and $\langle T_0(f), f\rangle \leq 0$,
We conclude that problem (4.39) is uniquely solvable and that the solution u depends continuously on L. Let φ be a smooth compactly supported function in D, which is equal to 1 on Γ and such that $\varphi(x_1, -x_2) = \varphi(x_1, x_2)$ for $k \geq 0$ we set $\tilde{u_k}$ in $H^1_{o,\Gamma}(\Omega)$ to be the solution to

$$\int_{\Omega} \nabla \tilde{u_k}.\nabla v - k^2 \tilde{u_k} v - \int_{\partial D} (T_k \tilde{u_k})v = \int_{\partial D} (\triangle \ \varphi + k^2 \varphi)v, \ \ for \ \ all \ \ v \ \ in \ \ H^1_{o,\Gamma}(\Omega) \tag{4.40}$$

and we set $u_k = \tilde{u_k} + \varphi$

Lemma 4.8 For $k > 0, u_k$ satisfies the following properties

(i) u_k is in $H^{'}(\Omega)$

(ii) The upper and lower traces on Γ of u_k are both equal to the constant 1.

(iii) u_k can be extended to a function in R^2/Γ such that if we still denote by u_k that extension

$$(\Delta + k^2)u_k = 0 in R^2/\Gamma$$
$$\frac{\partial u_k}{\partial r} - iku_k = o(r^{-1}), \text{uniformly as } r \to \infty$$
$$u_k(x_1, -x_2) = u_k(x_1, x_2) \quad \text{for all } (x_1, x_2) \ \ in \ \ R^2/\Gamma$$
$$\frac{\partial u_k}{\partial r}(x_1, 0) = 0 \ \ if \ \ (0.x) \notin \ \ \Gamma$$

(iv) Denote by $\dfrac{\partial u_k}{\partial x_2^-}$ the lower trace of $\dfrac{\partial u_k}{\partial x_2}$ on Γ

$$Im \int_\Gamma \frac{\partial u_k}{\partial x_2^-} < 0 \tag{4.41}$$

proof properties (i) and (ii) are clear.The first two items of property (iii) hold simply because we can write $u/_{\partial D} = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ and then set $u = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \dfrac{H_n(kr)}{H_n(k)}$ in $R^2/\overline{\Omega}$
Then use lemma(4.1) in combination to the fact that variational problem (4.40) implies that $T_k u_k$ is the limit of $\dfrac{\partial u_k}{\partial r}$ as $r \to 1^-$
To show the third item in (iii)
We set $\tilde{u_k}(x_1, x_2) = \tilde{u_k}(x_1, -x_2)$ and for any arbitrary v in $H^1_{0,\Gamma}(\Omega)$.

$\underline{v}(x_1, x_2) = v(x_1, -x_2)$.

Next we observe that

$\int_\Omega \bigtriangledown \underline{\tilde{u_k}}. \bigtriangledown v - k^2 \underline{\tilde{u_k}} v = \int_\Omega \bigtriangledown \tilde{u_k}. \bigtriangledown \underline{v} - k^2 \tilde{u_k} \underline{v}$
In polar coordinates we have the relations
$\tilde{u_k}(r, \theta) = \tilde{u_k}(r, -\theta)$ and $\underline{v}(r, \theta) = v(r, -\theta)$

so $\int_{\partial D} (T_k \underline{\tilde{u_k}}) v = \int_{\partial D} (T_k \tilde{u_k}) \underline{v}$
finally φ is even in $x_2 \Longrightarrow \int_\Omega (\Delta\varphi + k^2\varphi)\underline{v} = \int_\Omega (\Delta\varphi + k^2\varphi) v$
since the solution to the problem (4.38) is unique we must have $\underline{\tilde{u_k}} = \tilde{u_k}$,proving the third item in(iii).since u_k is even in x_2,it follows that $\dfrac{\partial u_k}{\partial x_2}$ is zero on the line $x_2 = 0$ minus the segment Γ,
proving the last item in (iii).
since $u = 1$ on Γ, $Im \int_\Gamma \dfrac{\partial u_k}{\partial x_2^-} = Im \int_\Gamma \dfrac{\partial u_k}{\partial x_2^-} \bar{u_k} = -Im \int_{\partial D} \dfrac{\partial u_k}{\partial r} \bar{u_k}$
Where $\partial \overline{D}$ is the intersection of circle ∂D and the lower half plane $x_2 < 0$.We use parity one more time to argue that
$Im \int_D \dfrac{\partial u_k}{\partial r} \bar{u_k} = \dfrac{1}{2} Im \int_{\partial D^-} \dfrac{\partial u_k}{\partial r} \bar{u_k}$
Knowing that u_k is not zero everywhere and using Rellichi's lemma we can claim that $Im \int_{\partial D} \dfrac{\partial u_k}{\partial r} \bar{u_k} > 0$.
lemma 4.9 Let $\tilde{u_k}$ be solution to (4.40) for all $k \geq 0$, and set $u_k = \tilde{u_k} + \varphi$.

(i) u_k is analytic in k for $k > 0$.

(ii) u_k converge strongly to u_0 in $H^1_{0,\Gamma}(\Omega)$.More precisely there is a constant C such that

$$\| u_k - u_0 \|^1_H (\Omega) \leq C(k^2 + \| T_k - T_0 \|) \tag{4.42}$$

lemma 4.10 Denote $\frac{\partial u_k}{\partial x_2^{\pm}}$ the upper and lower traces of $\frac{\partial u_k}{\partial x_2}$ on Γ.Then

(i)

$$\frac{\partial u_k}{\partial x_2^+} = -\frac{\partial u_k}{\partial x_2^-} \tag{4.43}$$

(i) Denote $G_k(x, y) = \frac{i}{4} H_0(k \mid x - y \mid)$. For all x in Ω

$$u_k(x) = 2 \int_\Gamma G_k(x, y) \frac{\partial u_k}{\partial x_2^-}(y) dy \tag{4.44}$$

proof

Denote $\Omega^+ = (x_1, x_2) \in \Omega : x_2 > 0$ and $\Omega^- = (x_1, x_2) \in \Omega : x_2 < 0$.It is well know from potential theory that if x in Ω^+

$$u_k(x) = \int_{\partial\Omega^+} G_k(x, y) \frac{\partial u_k}{\partial n}(y) - G_k(x, y) \frac{\partial u_k}{\partial n(y)} u_k(y) dy \tag{4.45}$$

and

$$0 = \int_{\partial\Omega^-} G_k(x, y) \frac{\partial u_k}{\partial n}(y) - G_k(x, y) \frac{\partial u_k}{\partial n(y)} u_k(y) dy \tag{4.46}$$

Where n is the exterior normal vector in each case. If y is in $\partial\Omega^-$ and is such that $y_2 = 0$ it is clear that $\frac{\partial G_k(x, y)}{\partial n(y)} = 0$. we also use that $u_k$is even in x_2 so that $\frac{\partial u_k}{\partial x_2}(x) = 0$ if $x_2 = 0$ and x not in Γ,since u_k is even in x_2. combining (4.45) and (4.46) we find that for x inΩ^+

$$u_k(x) = \int_{\partial D} G_k(x, y) \frac{\partial u_k}{\partial n}(y) - G_k(x, y) \frac{\partial u_k}{\partial n(y)} u_k(y) dy - \int_\Gamma G_k(x, y) (\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-})(y) dy \tag{4.47}$$

But, due to (iii)and lemma(4.10),since $u_k = \tilde{u_k}$ on ∂D,
$\int_{\partial D} G_k(x, y) \frac{\partial u_k}{\partial n}(y) - G_k(x, y) \frac{\partial u_k}{\partial n(y)} u_k(y) dy = 0$ for all x in Ω^+,thus

$$u_k(x) = \int_\Gamma G_k(x, y) (\frac{\partial u_k}{\partial x_2^+} - \frac{\partial u_k}{\partial x_2^-})(y) dy \tag{4.48}$$

for all x in Ω^-.Now take the x_2 derivative of (4.48) for x in Ω^+ approaching Γ.Due to the condition for normal derivatives of single layer potentials,we find that
$\frac{\partial u_k}{\partial x_2^-})(y) = -\frac{1}{2}(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial U_k}{\partial x_2^-})(x) - \int_\Gamma \frac{\partial G_k(x,y)}{\partial x_2}(\frac{\partial u_k}{\partial x_2^+} - \frac{\partial U_k}{\partial x_2^-})(y)dy$.We observe that for x and y on Γ, $\frac{\partial G_k(x,y)}{\partial x_2} = 0$.It follows that $\frac{\partial u_k}{\partial x_2^-}(x) = -\frac{\partial u_k}{\partial x_2^+})(x)$ for x on Γ.

Theorem 4.2 The following estimate as k approaches 0^+ holds

$$\int_\Gamma \frac{\partial u_k}{\partial x_2^-} \sim \pi k \frac{H_1(k)}{H_0(k)} \tag{4.49}$$

consequently,$Re \int_\Gamma \frac{\partial u_k}{\partial x_2^-}$ must be strictly positive for small values of $K > 0$.

proof. Set $v = 1 - \varphi$ in variational problem (4.40) (note that the trace of v is zero on Γ, as required in the space $H^1_{(0.\Gamma)}(\Omega)$,) to obtain

$$-\int_\Omega \nabla\tilde{u_k}.\nabla\varphi - \int_\Omega k^2\tilde{u_k}(1-\varphi) - \int_{\partial D} T_k\tilde{u_k}(1-\varphi) = \int_\Omega (\Delta\Delta\ \varphi + k^2\varphi)(1-\varphi). \tag{4.50}$$

We first observe that $\int_\Omega k^2\varphi(1-\varphi) = 0(k^2)$ and due to lemma(4.9) $\int_\Omega k^2\tilde{u_k}(1-\varphi) = 0(k^2)$. As have found that

$$-\int_\Omega \nabla\tilde{u_k}.\nabla\varphi + \int_\Omega \Delta\varphi\varphi = \int_{\partial D} T_k\tilde{u_k} + 0(k^2) \tag{4.51}$$

Next using Green's theorem,

$$2\int_\Gamma \frac{\partial\bar{u_k}}{\partial x_2^-} = 2\int_\Gamma \frac{\partial\bar{u_k}}{\partial x_2^-}\varphi = \int_\Omega \nabla\bar{u_k}.\nabla\varphi + \int_\Omega \Delta\bar{u_k}\varphi \tag{4.52}$$

since in $\Omega, \Delta\bar{u_k} = -\Delta\varphi - k^2\bar{u_k} - k^2\varphi$ combining (4.51) and (4.51) yields

$$2\int_\Gamma \frac{\partial\tilde{u_k}}{\partial x_2^-} = -\int_{\partial D} T_k\tilde{u_k} + 0(k^2) \tag{4.53}$$

But $\int_{\partial D} T_k\tilde{u_k} = -k\frac{H_1(k)}{H_0(k)}2\pi a_0(k)$
where $a_k = \frac{1}{2\pi}\int_{\partial D}\tilde{u_k}$ so using again that $\tilde{u_k}$ is strongly convergent to $1-\varphi$ in $H^1(\Omega), a_0(k)$ tends to 1 as $k \to 0$ thus we claim that

$$\int_{\partial D} T_k\tilde{u_k} \sim -2\pi k\frac{H_1(k)}{H_0(k)} \tag{4.54}$$

as$k \to 0$. Going back to the definition of of Hankel functions it is easy to see that $k\frac{H_{1(k)}}{H_0(k)} \sim -(lnk)^{-1}$ as $k \to 0$, from where we conclude that

$$\int_\Gamma \frac{\partial u_k}{\partial x_2^-} \sim -\pi(lnk)^{-1} \tag{4.55}$$

so$Re \int_\Gamma \frac{u_k}{\partial x_2^-}$ must be strictly positive for all $k > 1$ small enough.

4.4 Asymptotics for high wavenumber

We provide in this section a derivation of an equivalent for $\int_\Gamma \frac{\partial u_k}{\partial x_2^-}$ as $k \to \infty$, where $u_k = \tilde{u_k} + \varphi$,and $\tilde{u_k}$ solve variational problem (4.40).We will prove the following theorem.

Theorem 4.4.1. *Let $\tilde{u_k}$be the solution to(4.40),and set $u_k = \tilde{u_k} + \varphi$.The following estimates hold as k approaches infinity,*

$$\int_\Gamma \frac{\partial u_k}{\partial x_2^-} = -ik + o(k^{3/4}). \tag{4.56}$$

What is the asymptotic behavior of $\frac{\partial u_k}{\partial x_2^-}$ as the wave number k approaches infinity? High frequency approximation for the wave equation is a vast subject which has been extensively studied over time. Kirchho may have been the first one to write specific equations and asymptotic formulas for high frequency wave phenomena. However, his derivation was informal.

An informal derivation of estimate(4.56)

This informal derivation will be helpful since it will give us an idea of what the asymptotic behavior of $\int_\Gamma \frac{\partial u_k}{\partial x_2^-}$ should be.Denote by $f_k(y_1)$ the value of $\frac{\partial u_k}{\partial x_2^-}(y_1, 0)$ for y_1 in I. According to (4.44),f_k satisfies the integral equation.$S_k f_k = \frac{1}{2}$ or

$$\int_I \frac{i}{4} H_0(k|x_1 - y_1|) f_k(y_1) dy_1 = \frac{1}{2}, x_1 \in I \tag{4.57}$$

We multiply Eq.(4.57)by $-2ik$ and we integrate in x_1 over I to obtain

$$\int_I \int_I \frac{k}{2} H_0(k|x_1 - y_1|) f_k(y_1) dy_1 dx_1 = -ik \tag{4.58}$$

If possible to interchange order of integration in the left hand side of Eq. (4.58). then

$$\int_I \int_I \frac{k}{2} H_0(k|x_1 - y_1|) dx_1 f_k(y_1) dy_1 = -ik \tag{4.59}$$

Now for every x_1 in I,setting $v = k(x_1 - y_1)$ and using the following lemma,we can find that

$$\lim_{k\to\infty} \int_I \frac{k}{2} H_0(k|x_{_1} - y_1|)dx_1 = 1 \tag{4.60}$$

so we led to believe that $\int_I f_k(y_1))dy_1 \sim -ik$, which we set out to prove in the next section.

Lemma 4.14 Denote by St_n the struve function of order n as defined in the following formula hold for any $t > o$

$$\int_0^t H_0(z)dz = tH_0(t) + \frac{\pi}{2}t(St_0 H_1(t) - St_1 H_0(t)) \tag{4.61}$$

It follows that the semi-convergent integral $\int_0^\infty H_0(z)dz$ is exactly equal to 1.

4.5 Proof of Theorem 4.1

The case $C_1, C_2, C_3, C_4 > 0$

In this section we cover the case where the constants C_1, C_2, C_3, C_4 are all positive, which was assumed in Theorem 4.0.1. We explained in lemma(4.8) how the function Φ defined by (**??**)-(4.4) relates to u_k where $u_k = \tilde{u}_k + \varphi$ and $\tilde{u}_k$ solves (4.40). If u_k is extended to R^2 as indicated by Lemma (4.8) then u_k and Φ are equal on $\overline{R^2}$. Accordingly, recalling the definition of the function F given in (4.7) and formula (4.43), F can also be expressed as follows,

$$F(k) = q(k^2) + p(k^2)Re \int_\Gamma \frac{\partial u_k}{\partial x_2^-} \tag{4.62}$$

We know from lemma(4.9) that F is analytic in $(0, \infty)$ recalling the definition of p and q in (**??**) and using the estimate (4.55) we have that

$$F(k) \sim C_2\pi(lnk)^{-1}, k \to 0^+ \tag{4.63}$$

Recalling estimate (4.56) we claim that

$$F(k) \sim C_3 k^4, k \to \infty \tag{4.64}$$

It follows from estimate (4.63) that there is a positive α such that $F(k) < 0$ if k is in $(0, \alpha)$,and from estimate (4.64) we infer that $\lim_{k\to\infty} F(k) = \infty$.Since F is continuous in $(0, \infty)$,we conclude that F must achieve the value zero on that interval.We claim that the zeros of F are isolated since F is an analytic function.Due to (4.62) these zeros occur in some interval [A,B] where A and B are two positive constants. in particular the equation $F(k) = 0$ has at most a finite number of solutions.

Use of estimate in the case where $C_3 < 0$

We first claim that for any positive value of C_1, C_2, C_4 there exist negative values of C_3 such that the equation F(k) =0 has no solutions. According to estimate (4.55) there is a positive α such that for all k in $(0, \alpha)$

$$-\frac{1}{2}\pi(lnk)^{-1} \leq Re \int_\Gamma \frac{\partial u_k}{\partial x_2^-} \leq -\frac{3}{2}\pi(lnk)^{-1} \tag{4.65}$$

consequently for k in $(0, \alpha)$,

$$F(k) \leq C_4k^2 + C_1(-\frac{3}{2}\pi(ink)^{-1}) + C_2\frac{1}{2}\pi(lnk)^{-1} \tag{4.66}$$

so if α is small enough,$F(k) < 0$ if k is in $(0, \alpha)$.According to (4.56) there is negative A_1 and a positive A_2 such that for all k in $[0, \infty)$,

$$A_1k^{3/4} \leq Re \int_\Gamma \frac{\partial u_k}{\partial x_2^-} \leq A_2k^{3/4} \tag{4.67}$$

so for all k in $[\alpha, \infty)$,

$$F(k) \leq C_3k^4 + C_4k^2 + C_1A_2k^{2+3/4} - C_2A_1k^{3/4} \tag{4.68}$$

so if we choose C_3 less than some negative constant,$F(k) < 0$ for all k in $[\alpha, \infty)$.We conclude that for that choice of C_3, $F(k) < 0$ for all positive k,so the equation $F(k) = 0$ has no solution in $(0, \infty)$.

Next we show the following claim:for any value of the positive constants C_1, C_2 and any value of the negative constant,the equation $F(k) = 0$ has at least one solution.We may assume that α defined above is less than 1.for all k in $[\alpha, \infty)$

$$F(k) \geq C_3k^4 + C_4k^2 + C_1A_2k^{2+3/4} - C_2A_2k^{3/4} \tag{4.69}$$

thus $F(k) > 0$ for any C_4 greater than some constant.As $C_3 < 0$, thanks to (4.68) ,we have that,$\lim_{k\to\infty} = -\infty$, so we conclude that the equation $F(k) = 0$ has at least one solution in $(0, \infty)$.

Chapter 5

Conclustion and perspectives

We have proved in this paper the existence of frequency modes coupling seismic waves and vibrating tall buildings. Although the well-posedness of a set of equations modeling city-effect.To the best of our knowledge, up to now, there was no formal proof of existence of frequncy modes coupling seismic wave and vibrating of tall building.

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