CONVERGENCE ANALYSIS AND APPLICATIONS OF NEWTON'S METHOD AND ITS VARIANTS FOR NONLINEAR SYSTEM IN TWO DIMENSIONS

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(Numerical Analysis)

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List of symbols

Symbol	**meaning**
γ	Gamma
$\mathbb{R}$	Real number
$\bar{X}$	Solution (root)
H	Hessian matrix
J	Jacobian matrix
∂	Partial
FPM	Fixed point method
NM	Newton's method
FPIM	Fixed point iteration method
SNLEs	System of nonlinear equations
SHM	Super Halley method
HM	Halley method
CM	Chebyshev method
BVP	Boundary value problem

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Abstract

In this thesis, Newton's method and its variants used to solve f(x) = 0 for one dimension and two dimension. We discussed convergence and accelerating conditions in one dimension. Then we drive Newton's method for solving nonlinear system in 2-D. We also examined methods accelerating its convergence in 2-D. That are variants of Newton's method in 2-D. We prove order of convergence in 1-D and 2-D. We applied series expansion (linear and quadratic) and construction techniques for most of the proofs and derivation. Numerical experiments are provided in matlab codes. The study shows that most of the methods accelerating Newton's convergence are cubic order both in one dimension and two dimensions. We also considered some applications in nonlinear boundary value problems (BVBs).

Keywords: *System of Nonlinear Equation, Newton Method.*

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Numerical algorithms are at least as old as the Egyptian Rhind papyrus (c. 1650 BC), which describes a root-finding method for solving a simple equation (Jean-Luc Chabert, 1999). Ancient Greek mathematicians made many further advancements in numerical methods. In particular, Eudoxus of Cnidus created and Archimedes perfected the method of exhaustion for calculating lengths, areas, and volumes of geometric figures (Edwards, 1997). When used as a method to find approximations, it is in much the spirit of modern numerical integration and it was an important precursor to the development of calculus by Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716).

Numerical analysis is the area of mathematics that creates, analyzes, and implements algorithms for solving mathematical problems. Such problems originate generally from real-world applications of algebra, geometry and calculus. These problems also occur throughout the natural sciences, social sciences, engineering, medicine, and business. During the past half century, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering, and highly sophisticated methods of numerical analysis have been needed to solve these more detailed mathematical models of the world. The formal academic area of numerical analysis varies from quite theoretical mathematical studies to computer science issues (Edwards, 1997). With the growth in importance of using computers to carry out numerical procedures in solving mathematical models of the world, an area known as scientific computing or computational science has taken shape during the 1980s and 1990s (Dennis, Robert schambel, 1997). This area looks at the use of numerical analysis from computer science perspective. It is concerned with using the most powerful tools of numerical analysis, computer graphics, symbolic mathematical computations, and graphical user interfaces to make it easier for a user to set up, solve, and interpret complicate mathematical models of the real world. Therefore, numerical analysis is one of the fields of mathematics which uses different types of methods to solve complicated mathematical problems in the areas stated above. There are different kinds of challenging problems in science, engineering and other fields of the study which need to be solved. The most important problem among others problems is finding the solution for system of nonlinear equations. There are simple formulas for solving linear and quadratic equations; and there are also somewhat complicated formulas for cubic equations. Besides polynomial equations, we have many problems in scientific and engineering applications that involve the functions of transcendental nature. Numerical methods are often used to obtain approximated solution of some complicated problems because those problems are not possible to obtain exact solution by usual algebraic processes. The solution of system of equations have a well-developed mathematical and computational theory when solving linear system, or single nonlinear equation. The situation becomes complicated when the equations in the system do not exhibit nice linear or polynomial properties. In this case, both the mathematical theory and computational practices are far from complete understanding of the solution process. So system of nonlinear equations are one of such types of problems. They arise in various domains of practical importance such as engineering, medicines, chemistry, and robotics. They appear also in many geometric computations such as intersections, minimum distance, and when solving initial or boundary value problems in ordinary or partial differential equations. As it is mentioned above, solving system of nonlinear equations is more difficult when compared with the linear counterpart. So finding solutions of these problems is one of the most important parts of the studies in numerical analysis and because of this reason different mathematicians developed different iterative methods to solve problems of system of nonlinear equations SNEs. Halley method, chebyshev method, fixed point iteration method, newton's method, broyden's method, adomain decomposition method, steepest descent method, homotopy perturbation method, predictor methods, are some of the methods used to solve system of nonlinear equations(SNEs). This thesis focused on Newton method and its variants for solving system of nonlinear equations. Historically Newton's method was developed by Sir Isac Newton around 1600. Most experts agree that the road for the Newton's method was opened by Francois vieta 1600 (Deuflhard, 2004). Isaac Newton was using this method to approximate numerical solutions of scalar nonlinear equations. One of his first applications of this was on the numerical solution of a cubic polynomial equation.

In computational mathematics, an iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems, in which the n-th approximation is derived from the previous ones. An iterative method is called convergent if the corresponding sequence converges for given initial approximations. A mathematically rigorous convergence analysis of an iterative method is usually performed; however, heuristic-based iterative methods are also common. In the problems of finding the root of an equation (or a solution of a system of equations), an iterative method uses an initial guess to generate successive approximations to a solution.

Fixed Point is function F: $D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ has a fixed point at $\bar{x} \in D$ if F $(\bar{x}) = \bar{x}$. Fixed point method allows us to solve nonlinear equations and system of equation, (Rheinboldt, 1974), (Dennis and Schambel, 1996), (Chapra, 2010). We build an iterative method, using a sequence which converges to a fixed point. This fixed point is the exact solution of

$$F(x) = 0.$$

The fixed point iteration (FPI) method plays a significant as well as remarkable role in the study of nonlinear phenomena occurring in engineering, physics, economics, life sciences, and medical sciences. The design of fixed-point iterative methods for solving nonlinear problems, in particular nonlinear equations or systems, has gained a spectacular development in the last two decades. Nevertheless, the existence of recent and extensive literature on these iterative schemes reveals that this topic is still a dynamic branch of the applied mathematics with interesting and promising applications and researchable area.

Many researcher studied on fixed point method (FPM) for solving system of nonlinear equations (SNEs). (Chapra, 2010) has considered fixed point iteration method (FPIM). The study of the dynamical behavior of the rational operator associated to an iterative method has also become a rapidly growing area of research, since the dynamical properties of the rational operator give us important information about the convergence, efficiency, and reliability of the iterative method.

1.2. Statement of the Problem

System of nonlinear equations is a problem that arises in different areas of studies such as in Engineering, medicine, chemistry and robotics. Finding solutions for such problems are not easy when compared with other problems such as solving single nonlinear equations, linear equations, quadratic equations, polynomial equations, equations of transcendental nature, system of linear equations, and so on. Because of this, a number of mathematicians tried to find different iterative methods to solve problems of System of nonlinear equations. For example (Ostrowski, 1966) used some iterative and other method to solve system of nonlinear equation. (Chapra, 2010) used Newton and fixed point iterative method to solve system of nonlinear equation. (Batman, 1938) used Halley method (Candela, 1990) used Chebyshev method to solve system of nonlinear equation. In recent years many researcher consider to solve system of nonlinear equation using Newton's method and its variants (Cordero and Torregrosa, 2006) and (Darvishi and Barati, 2007). Because there was no detail analysis, we are interested to study system of nonlinear equation in two dimension by Newton's method and its variants.

Let F: $\mathbb{R}^2 \to \mathbb{R}^2$ be a nonlinear mapping from the 2-dimensional real linear space $\mathbb{R}^2$ into itself. Our interest is in the method for the compitation of solutions of the systems of 2 equation in 2 variables.

$$F(X) = 0, \text{ Where } X = (x_1, x_2) \in \mathbb{R}^2 \text{ and } F = (f_1, f_2).$$

We need to answer the following problem for solving the equation above:

- To what extent the convergence is accelerate?
- Which method is faster convergent?
- To what extent the solution converges?

1.3 Objective of the study

1.3.1 General objective of the study

The general objective of this study is to Analysis and applications of Newton's method its variants for two dimensions.

1.3.2. Specific objective of the study

Specific objective of the study intends to

- Show Convergence of solution Newton's method and its variants.
- Compare Newton's method with its variants based on the convergence rate.
- Accelerate convergence Newton's method.
- Prove order of convergence for Newton's method and its variants.

1.4. Importance of the study

The outcome of the study

- Be used as a reference material for students, teachers and anyone who works on this area.
- Provide significant contribution for scientific investigation in the area of root finding.
- Application of root finding in science and engineering.

1.5. Delimitation of the study.

System of nonlinear problem are perhaps great important mathematical equation in applied mathematics, and in various branches of science and engineering. Though system of nonlinear problem are vast topics and have many application in the real world. So this study delimited to the class of system of nonlinear equations in two dimension by Newton's method and its variants.

The equation of the problem are given by:

$$F(X) = 0, \text{ where } F:\mathbb{R}^2 \rightarrow \mathbb{R}^2.$$

CHAPTER TWO

LITERATURE REVIEW

Newton's method is known as classical fixed point method developed around 1669, by Isaac Newton (1643-1727). He gave a new algorithm (Newton, 1664-1671) to solve a polynomial equation and it was illustrated on the example y^3-2y-5=0. To find an accurate root of this equation, first one must guess a starting value, here y ≈ 2. Then just write y=2+p and after the substitution the equation becomes.

$$p^3 + 6p^2 + 10p - 1 = 0.$$

A few years later, in 1690, a new step was made by Joseph Raphson (1678-1715) who proposed a method (Raphson, 1690) which avoided the substitutions in Newton's approach. He illustrated his algorithm on the equation $x^3 - bx + c = 0$, and starting with an approximation of this equation x≈ g, a better approximation is given by $x \approx g + \frac{c+g^3-bg}{b-3g^2}$

The method was then studied and generalized by other mathematicians like Simpson (1710-1761), Mourraille (1720-1808), Cauchy (1789-1857), and Kantorovich (1912-1986). The important question of the choice of the starting point was first approached by Mourraille in 1768 and the difficulty to make this choice is the main drawback of the algorithm (Qiu, 1993). Newton's method is one of the popular numerical methods, and is even referred by (Burden and Faires, 2005) as the most powerful method that is used to solve the equation f(x) = 0.

Nowadays, Newton's method (NM) is a generalized process to find an approximate root of a system (or a single) equations f(x) =0.

A popular method is attributed to Edmond Halley (1656-1742) is called Halley's method. (Bateman, 1938) was the first to point out that Halley's method for a function *f*(*x*) =0 is obtained by applying Newton's method.

Pafnuty Chebyshev's (1821 –1894) a Russian mathematician who develops the Chebyshev's method .Chebyshev submitted a paper on the calculation of roots of equations in which he solved the equation *y* = *f* (*x*) by using a series expansion for the inverse function of *f*. The Chebyshev method for solving a nonlinear equation in banach space is third order method. Third order method are not considered by many authors because of their high computational cost, however, in some cases, the rise in the velocity of convergence can justify their use.

From the linear perturbation approach, local quadratic convergence will be clearly expected for the scalar case. For the general case of operator equations $F(x) = 0$, the convergence of the generalized Newton scheme has first been proved by two Russian mathematicians: IN 1939, Kantorovich was merely able to show local linear convergence Kantorovich and Akilov, (1982), which he improved in 1948/49 to local quadratic convergence. Also in 1949, Mysovshikh gave a much simpler independent proof of local quadratic convergence under slightly different theoretical assumptions, which are exploited in modern Newton algorithms. Among later convergence theorems the one due to (Ortega and Rheinboldt, 1970).

Generally the use of Newton's method and its variants for solving system nonlinear equations for last two decades developed rapidly. (Cordero and Torregrosa, 2006) Variants of Newton's method for functions of several variables, which used for solving system of nonlinear equations. (Darvishi and Barati, 2007) third-order Newton-type method to solve systems of nonlinear equations, but both of these recent researchers did not detail analysis convergence of Newton's method and its variants. We are interested in solving system of nonlinear equations in two dimensions using Newton's method and its variants with detail analysis.

CHAPTER THREE

RESEARCH METHODOLOGY

This section consists of methods and materials used to undertake the study. These are study design and period, source of information, and procedure of the study.

3.1 Study Design

This study used mixed design i.e. documentary review and experimental review (matlab).

3.2. Procedures of the study

To attain the objective of the study, the following procedures are undertaken:

- Deriving of variants of Newton's method in one dimension and two dimensions.
- Convergence analysis.
- Defining the problem.
- Find the derivative of the problem.
- Initial guess for the problem.
- Writing a code for the problem by using matlab language.
- Validating the schemes using numerical examples.
- Discussing the schemes with the graphs and tables.

3.3. Instrumentation

Solving the system of nonlinear equation using matlab codes, tables and graphs.

CHAPTER FOUR

4.1. Preliminaries on nonlinear problem solving

Definition 4.1. A system of nonlinear equations is a set of equations given by

$$f_1\ (x_1, x_2, \ldots, x_n) = 0,$$

$$f_2\ (x_1, x_2, \ldots, x_n) = 0,$$

$$\vdots$$

$$f_n\ (x_1, x_2, \ldots, x_n) = 0,$$

where$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and each f_i is a nonlinear real function, i = 1*;* 2… n. (Courtney Remani, 2013). Where $f_i: \mathbb{R}^n \to \mathbb{R}$

Definition 4.2. Jacobian matrix: - A jacobian matrix is an nxn matrix of first derivatives of component function of system of nonlinear equations (SNEs) that can be written as

$$F'(X) = J(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \text{ where } \mathbf{X}=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

Definition 4.3. Hessian matrix: - a Hessian matrix for 2x2 matrix of second order partial derivative of components function of f (x_1, x_2) =0, f: $\mathbb{R}^2 \to \mathbb{R}$ written as

$$F''\ (x_1, x_2\ ) = H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Definition 4.4. A function f: $\mathbb{R}^n \to \mathbb{R}$ is defined as being nonlinear when it does not satisfy the superposition principle that is (Courtney Remani, 2013).

$$F\ (x_1 + x_2 + \ldots) \neq f\ (x_1) + f\ (x_2) + \ldots \text{ and } F\ (xy) \neq f(x)\ f(x)$$

Definition 4.5. Normed spaced; a normed space X is a vector space with a norm defined on it. Defined by (X,$\|.\|$) that satisfies the following properties

$$1, \|X\| \geq 0$$

$$2, \|X\| = 0 \Leftrightarrow X = 0$$

$$3, \|\alpha X\| = |\alpha|\|X\|$$

$$4, \|X + Y\| \leq \|X\| + \|Y\| \text{ triangle inequality;}$$

Where x and y are arbitrary vectors in X and α is any scalar.

Definition 4.6. Banach space is a complete, normed, vector space.

Definition 4.7. Let D $\subset \mathbb{R}^n$ and let F: D $\rightarrow \mathbb{R}^n$. F is Lipschitz continuous on D with Lipschitz constant γ if

$$\|F(x) - F(y)\| \leq \gamma\|x - y\|$$

For all x, y∈D

Definition 4.8. Let D $\subset \mathbb{R}^n$. F: D $\rightarrow \mathbb{R}^n$ is a contraction mapping on D if F is Lipchitz continuous on D with Lipschitz constant γ < 1.

Definition 4.9. Let $\{x_n\}$ C $\mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$. Then

• x_n —> $\bar{x}$ q-quadratically if x_n —> $\bar{x}$ and there is K > 0 such that

$$\|x_{n+1-\bar{x}}\| \leq K\|x_n - \bar{x}\|^2.$$

• x_n —> $\bar{x}$ q-order 3 if $x_n \rightarrow \bar{x}$ and there is K > 0 such that

$$\|x_{n+1-\bar{x}}\| \leq K\|x_n - \bar{x}\|^3.$$

• $x_n \rightarrow \bar{x}$ q-linearly with q-factor δ ∈(0, 1) if

$$\|x_{n+1} - \bar{x}\| \leq \delta \|x_n - \bar{x}\| \text{ for n sufficiently large.}$$

Definition 4.10. Boundary value problem: a boundary value problem is a differential equation together with a set of additional constraints, called the **boundary** conditions. A solution to a **boundary value problem** is a solution to the differential equation which also satisfies the **boundary** conditions.
Example consider the second order boundary value problem

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx}\ a_0xy = g(x)$$

$$Y\ (a) = y_0 \qquad y\ (b) = y_1$$

4.2. Solving System of nonlinear equations in two variables

4.2.1. Newton's method

Newton's method is one of the most popular numerical methods, and is even referred by (Burden and Faires, 2005) as the most powerful method that is used to solve a nonlinear equation $f(x) = 0$. This method originates from the Taylor's series expansion of the function $f(x)$ about the point x_1.

$$F(x) = f(x_1) + (x-x_1) f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \ldots \tag{4.1}$$

Where f, and its first and second order derivatives, $f'$and f'' are calculated at x_1. If we take the first two terms of the Taylor's series expansion we have:

$$F(x) = f(x_1) + (x-x_1) f'(x_1). \tag{4.2}$$

We then set (4.2) to zero (i.e. $F(x) = 0$) to find the root of the equation which gives us:

$$f(x_1) + (x-x_1) f'(x_1) = 0 \tag{4.3}$$

Rearranging (4.3) we obtain the next approximation to the root, giving us:

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{4.4}$$

Thus generalizing (4.4) we obtain Newton's iterative method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \; i \in N \tag{4.5}$$

Where $x_i \rightarrow \bar{x}$ (as $i \rightarrow \infty$), and $\bar{x}$ is the approximation to a root of the function $f(x)$.

However, (4.5) can only be used to solve nonlinear equations involving only a single variable. This means we have to take (4.5) and alter it, in order to use it to solve a set of nonlinear algebraic equations involving multiple variables.

We know from Linear Algebra that we can take systems of equations and express those systems in the form of matrices and vectors. With this in mind and using Definition 4.1.1, we can express the nonlinear system as a matrix with a corresponding vector. Thus, the following equation is derived:

To obtain a form of Newton's method for system of equations, we start with the Taylor series in n variables e.g., n=2, we want to obtain the values of x_1, x_2 that satisfy the 2 by 2 system.

$$F(x_1, x_2) = \begin{Bmatrix} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{Bmatrix} \tag{4.6}$$

If f_1 and f_2 are expanded in two Taylor series in two dimensions (with respect to the iteration counter k)

$$f_1\left(x_1^{k+1}, x_2^{k+1}\right) = f_1\left(x_1^k, x_2^k\right) + \frac{\partial f_1\left(x_1^k, x_2^k\right)}{\partial x_1}\left(x_1^{k+1} - x_1^k\right) + \frac{\partial f_1\left(x_1^k, x_2^k\right)}{\partial x_2}\left(x_2^{k+1} - x^k{}_2\right) + \ldots$$

$$f_2\left(x_1^{k+1}, x_2^{k+1}\right) = f_2\left(x_1^k, x_2^k\right) + \frac{\partial f_2\ (x^k{}_1, x_2^k)}{\partial x_1}(x_1^{k+1} - x^k{}_1) + \frac{\partial f_2\ (x^k{}_1, x_2^k)}{\partial x_2}(x_2^{k+1} x^k{}_2) + \ldots \qquad (4.7)$$

Truncation after the first derivative terms, with the resulting equations expressed in terms of Newton corrections

$$\Delta x_1^k = (x_1^{k+1} - x_1^k)$$

$$\Delta x_2^k = (x_2^{k+1} - x_2^k)$$

And with just the right Newton correction so that the LHS's of the equation (4.7) are zero, gives

$$\frac{\partial f_1\ (x_1^k, x_2^k)}{\partial x_1}\Delta x_1^k + \frac{\partial f_1\ (x_1^k, x_2^k)}{\partial x_2}\Delta x_2^k = -f_1\left(x_1^k, x_2^k\right)$$

$$\frac{\partial f_2\ (x_1^k, x_2^k)}{\partial x_1}\Delta x_1^k + \frac{\partial f_2\ (x_1^k, x_2^k)}{\partial x_2}\Delta x_2^k = -f_2\left(x_1^k, x_2^k\right) \qquad (4.8)$$

Eqs (4.8) is 2 by 2 linear system in the two Newton corrections, Δx_1^k, Δx_2^k they can be solved by any standard method for linear equations, e.g. Guass row reduction. The unknowns can then be updated iteratively by

$$x_1^{k+1} = \Delta x_1^k + x_1^k$$

$$x_2^{k+1} = \Delta x_2^k + x_2^k \qquad K= 1, 2, \ldots \qquad (4.9)$$

Eq. (4.8) can be generalized to the n x n case by writing it in matrix form

$$J\ \Delta X = -f \qquad (4.10)$$

Where

$$J = \begin{bmatrix} \frac{\partial f_1\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_1} & \frac{\partial f_1\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_2} & \cdots & \frac{\partial f_1\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_n} \\ \frac{\partial f_2\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_1} & \frac{\partial f_2\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_1} & \cdots & \frac{\partial f_2\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial f_n\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_1} & \frac{\partial f_n\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_2} & \cdots & \frac{\partial f_n\ (x_1^k, x_2^k, \ldots x^k{}_n)}{\partial x_n} \end{bmatrix}, \ \Delta X = \begin{bmatrix} x_1^k \\ x_2^k \\ . \\ . \\ . \\ x_n^k \end{bmatrix}$$

Eq. (4.10) is Newton's method for an n x n system. After rearranging equation (4.10) it gives

$$x^{k+1} = x^k - J(x^k)^{-1}\ F\,(x^k). \qquad (4.11)$$

Where k = 1, 2... n represents the iteration, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{F}$ is a vector function, and $J(x^k)^{-1}$ is the inverse of the Jacobian matrix .

This equation represents the procedures of Newton's method for solving nonlinear algebraic systems. However, instead of solving f(x) = 0, we are now solving the system F(x) = 0. We will now go through the equation and define each component.

Let **F** be a function which maps $\mathbb{R}^{\boldsymbol{n}}$ to $\mathbb{R}^{\boldsymbol{n}}$

$$\mathbf{F}(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1 \; (x_1, x_2, \dots, x_n) \\ f_2 \; (x_1, x_2, \dots, x_n) \\ . \\ . \\ . \\ f_n \; (x_1, x_2, \dots, x_n) \end{bmatrix}$$

Wheref_i: $\mathbb{R}^{\boldsymbol{n}} \rightarrow \mathbb{R}$

Let X $\in \mathbb{R}^{\boldsymbol{n}}$ then X represents the vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Where $X_i \in \mathbb{R}$ and i = 1, 2, . . . n.

From Definition 4.2 we know that J(**X**) is the Jacobian matrix. Thus $J(X^k)^{-1}$ is

$$J(X)^{-1} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}^{-1}$$

Now we describe the steps of newton's methods:

Step 1:

Let X^0 = $(x_1^0, x_2^0, \dots, x_n^0)$ be an initial vector.

Step 2:

Calculate J (X^0) and F (X^0)

Step 3:

We now have to calculate the vector Y^0, where

$$Y^0 = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix}$$

In order to find Y^0 we solve the linear system J (X^0) $Y^0 = -$ F (X^0)

Rearranging the system in step 3 we get that $Y^0 = -J(X^0)^{-1}$ F (X^0) the significance of this is that $Y^0 = -J(X^0)^{-1}$ F (X^0), we can replace $-J(X^0)^{-1}$ F (X^0) in our iterative formula with Y^0. This result will yield that

$$X^1 = X^0 - \mathrm{J}\ (X^0)^{-1}\ \mathrm{F}\ (X^0) = X^0 + Y^0$$

Step 4: Once Y^0 is found, we can now proceed to finish the first iteration by solving for X^1. Thus using the result from Step 3, we have that

$$X^1 = X^0 + Y^0 = \begin{bmatrix} X_1^0 \\ X_2^0 \\ . \\ . \\ . \\ X_n^0 \end{bmatrix} + \begin{bmatrix} y_1^0 \\ y_2^0 \\ . \\ . \\ . \\ y_n^0 \end{bmatrix}$$

Step 5: Once we have calculated X^1 , we repeat the process again, until $X^{(k)}$converges to $\bar{X}$. This indicates we have reached the solution to **F**(**x**) = **0**, where $\bar{X}$ is the solution to the system.

4.2.2. Variants of Newton's method

4.2.2.1. Halley's method

Edmond Halley (1656-1742) obtain Halley's method to solve nonlinear equation. Number of papers have been written about HM, a third order method for a solution of nonlinear equation. For example (Traub, 1964), for real valued functions. The derivation of Halley's method comes from Taylor's expansion of one dimension.

Let f(x) = 0 be a nonlinear equation. The Taylor's expansion of equation is given by:

$$\mathrm{f}\ (x^{k+1}) = \mathrm{f}\ (x^k) + f'(x^k)(\ x^{k+1} - x^k) + \frac{1}{2!}\ f''(x^k)(\ x^{k+1} - x^k)^2 + \ldots \tag{4.12}$$

$$f'(x^k)(\ x^{k+1} - x^k) + \frac{1}{2!}\ f''(x^k)(\ x^{k+1} - x^k)^2 = -\ \mathrm{f}\ (x^k) \tag{4.13}$$

Divide both sides by $f'(x^k) + \frac{1}{2!}\ f''(x^k)(\ x^{k+1} - x^k)$for which it gives

$$(x^{k+1} - x^k) = \frac{-\mathrm{f}\ (x^k)}{f'(x^k) + \frac{(x^{k+1} - x^k)}{2} f''(x^k)}\ , \qquad (x^{k+1} - x^k) = -\frac{\mathrm{f}\ (x^k)}{f'(x^k)} \tag{4.14}$$

After rearranging eq. (4.14) it gives

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)(1-\frac{f'(x^k)^{-2} f(x^k) f''(x^k)}{2}}, \text{ k} \geq 0 \tag{4.15}$$

Which is Halley's method in one dimension. This method is also called the method tangent hyperbola.

Steps in Halley method for nonlinear equation in one dimension.

Step 1:

Let x^0 be an initial vector.

Step 2:

Calculate $f'(x^0)$, $f''(x^0)$ and f (x^0)

Step 3:

We now have to calculate the vector z^0

In order to find z^0 we solve $f'(x^0)(1 - \frac{f(x^0) f'(x^0)^{-2} f''(x^0)}{2})\ )\ z^{(0)} = -\text{f}\ (x^0)$

Rearranging the system in step 3 we get that $z^0 = \frac{-f(x^0)}{f'(x^0)(1-\frac{f(x^0) f'(x^0)^{-2} f''(x^0)}{2})}$ the significance of this is that $z^0 = \frac{-f(x^0)}{(f'(x^0) - \frac{f(x^0) f'(x^0)^{-1} f''(x^0)}{2})}$, we can replace

$-\text{f}\ (x^0)\left(f'(x^0)(1 - \frac{f(x^0) f'(x^0)^{-2} f''(x^0)}{2})\right)^{-1}$ in our iterative formula with z^0. This result will yield that

$$x^1 = x^0 - \text{f}\ (x^0)\ ((f'(x^0)(1 - \frac{f(x^0) f'(x^0)^{-2} f''(x^0)}{2}))^{-1} = x^0 + z^0$$

Step 4: Once z^0 is found, we can now proceed to finish the first iteration by solving for x^1. Thus using the result from Step 3, we have that

$$x^1 = x^0 + z^0$$

Step 5: Once we have calculated x^1 , we repeat the process again, until x^k converges to $\bar{x}$. This indicates we have reached the solution to **F**(**x**) = **0**, where $\bar{x}$ is the solution to the system.

4.2.2.2. Super Halley method

(Gutierrez and Hernandez, 2001) derive super Halley method an acceleration of Newton's method for scalar equations.

Let f: [a, b] $\subset \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f'(x) < 0$, $f''(x) > 0$, for $x \in$[a, b] and

f (a)> 0 > f (b). Let us denote $\{x_n\}$ the Newton sequence.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ x_0 = a \tag{4.16}$$

Then, taking g(x) $= f'(\bar{x})$ (x-$\bar{x}$) we obviously obtain a sequence that converges to $\bar{x}$ faster than $\{x_n\}$. The problem is that $\bar{x}$ is unknown.

Instead of $f'(\bar{x})$ (x-$\bar{x}$) we take its Taylor approximation.

$$G(x) = f(x) - \frac{1}{2} f''(x)(x - \bar{x})^2$$

So we deduce the following approximations:

$$G(x_k) \approx f(x_k) - \frac{1}{2} f''(x_k)(x_k - x_{k+1})^2$$

$$G'(x_k) \approx f'(x_k) - f''(x_k)(x_k - x_{k+1})$$

Thus, we obtain an acceleration for Newton's method in (4.16) which is defined by

$$x_{k+1} = x_k - \frac{G(x_k)}{G'(x_k)} = x_k - [1 + \frac{f'(x_k)^{-2} f''(x_k) f(x_k)}{2(1 - f'(x_k)^{-2} f''(x_k) f(x_k))}] \frac{f(x_k)}{f'(x_k)} \tag{4.17}$$

$$b_{f(x_k)} = f(x_k) \frac{f''(x_k)}{f'(k)^2}$$

$$x_{k+1} = x_k - \frac{G(x_k)}{G'(x_k)} = x_k - [1 + \frac{b_{f(x_k)}}{2(1 - b_{f(x_k)})}] \frac{f(x_k)}{f'(x_k)} \tag{4.18}$$

Lemma 4.1. Let us write

$$G(x) = x - [1 + \frac{b_{f(x)}}{2(1 - b_{f(x)})}] \frac{f(x)}{f'(x)} \tag{4.19}$$

Then we have

$$G'(x) = \frac{b_{f(x)}^2 \left(b_{f(x)} - b_{f'(x)}\right)}{2\left(1 - b_{f(x)}\right)^2}$$

4.2.2.3. Chebyshev's method

Pafnuty Chebyshev's (1821 –1894) a Russian mathematician who develops the Chebyshev's method for solving nonlinear equation (Candela and Marquina, 1990)

The derivations of Chebyshev's method for one dimension in real space by using Taylor's expansion.

Let f(x) = 0 be a nonlinear equation. The Taylor's expansion of equation is given by:

$$f(x^{k+1}) = f(x^k) + f'(x^k)(x^{k+1} - x^k) + \frac{1}{2} f''(x^k)(x^{k+1} - x^k) + \dots \tag{4.20}$$

$$f'(x^k)(x^{k+1} - x^k) + \frac{1}{2} f''(x^k)(x^{k+1} - x^k)^2 = -f(x^k) \tag{4.21}$$

$$f'(x^k)(x^{k+1} - x^k) = -\mathrm{f}(x^k) - \frac{1}{2} f''(x^k)(x^{k+1} - x^k)^2, (x^{k+1} - x^k) = -\frac{\mathrm{f}(x^k)}{f'(x^k)} \qquad (4.22)$$

$$(x^{k+1} - x^k) = \frac{-\mathrm{f}(x^k)}{f'(x^k)} - \frac{1}{2}(x^{k+1} - x^k)^2 \frac{f''(x^k)}{f'(x^k)} \qquad (4.23)$$

After rearranging eq. (4.23) it gives

$x^{k+1} = x^k - \frac{\mathrm{f}(x^k)}{f'(x^k)}(1 + \frac{\mathrm{f}(x^k)f''(x^k)}{2\mathrm{f}'(x^k)^2})$ which is Chebyshev's method in one dimension.

Steps in Chebyshev's method for nonlinear equation in one dimension.

Step 1:

Let x^0 be an initial vector.

Step 2:

Calculate $f'(x^0)$, $f''(x^0)$ and $f(x^0)$,

Step 3:

We now have to calculate the vector $z^{(0)}$

In order to find z^0 we solve $\frac{z^0 f'(x^0)}{1 - \frac{f(x^0)f''(x^0)}{\mathrm{f}'(x^0)^2}} = -f(x^0)$

Rearranging the system in step 3 we get that $z^0 = \frac{-f(x^0)}{f'(x^0)}(1 - \frac{f(x^0)f''(x^0)}{2f'(x^0)^2}$ the significance of this is that $z^0 = \frac{-f(x^0)}{f'(x^0)}(1 - \frac{f(x^0)f''(x^0)}{2f'(x^0)^2}$, we can replace $\frac{-f(x^0)}{f'(x^0)}(1 - \frac{f(x^0)f''(x^0)}{2f'(x^0)^2}$ in our iterative formula with z^0. This result will yield that

$$x^{k+1} = x^k - \frac{f(x^0)}{f'(x^0)}(1 + \frac{f(x^0)f''(x^0)}{2f'(x^0)^2}) = x^k + z^k$$

Step 4: Once z^0 is found, we can now proceed to finish the first iteration by solving for x^1. Thus using the result from Step 3, we have that

$$x^1 = x^0 + z^0$$

Step 5: Once we have calculated x^1 , we repeat the process again, until x^k converges to $\bar{x}$. This indicates we have reached the solution to **F**(**x**) = **0**, where $\bar{x}$ is the solution to the system.

4.2.2.4. Variants of Newton's method in two variables.

However, (4.15), (4.18) and (4.23) can only be used to solve nonlinear equations involving

Only a single variable. This means we have to take (4.15), (4.18) and (4.23) and alter it, in order to use it to solve a set of nonlinear algebraic equations involving multiple variables.

We know from Linear Algebra that we can take systems of equations and express those systems in the form of matrices and vectors. With this in mind and using Definition 4.1

we can express the nonlinear system as a matrix with a corresponding vector. The formula for two variables derived as follows

Schwetlick defined a class for a real scalar β and i is the number of iteration, which we call the Schwetlick class (Schwetlick, 1979). The class is obtained by solving the following equation for

$$z_n^{(i+1)}\text{ , i= 1, 2, 3,}\ldots$$

$$\mathrm{F}\,(x_n) + F'(x_n)\,(\,z_n^{(i+1)} - x_n) + \beta F''(x_n)\Big(z_n^{(i)} - x_n\Big)(\,z_n^{(i+1)} - x_n)$$

$$+(\tfrac{1}{2} - \beta)F''(x_n)\Big(z_n^{(i)} - x_n\Big)\Big(z_n^{(i)} - x_n\Big)=0 \tag{24}$$

We want to derive the accelerated Newton's Raphson method for two variable functions. The derivations is as follows.

Let $z_n^{(0)}$=x_n and $z_n^{(2)}$=x_{n+1} and from equation (24)

$$\mathrm{F}\,(x_n) + F'(x_n)(\,z_n^{(2)} - x_n) +\beta F''(x_n)\Big(z_n^{(1)} - x_n\Big)(\,z_n^{(2)} - x_n) \tag{25}$$

$$+(\tfrac{1}{2} - \beta)F''(x_n)\Big(z_n^{(1)} - x_n\Big)\Big(z_n^{(1)} - x_n\Big)=0$$

We solve first for $z_n^{(1)}$, that means i=0 since $z_n^{(0)}$=x_n, we get

$$\mathrm{F}\,(x_n) + F'(x_n)\,\Big(z_n^{(1)} - x_n\Big)=0$$

Then $z_n^{(1)}$ is given by

$$z_n^{(1)}= x_n - F'(x_n)^{-1}\,\mathrm{F}\,(x_n) \tag{26}$$

By substituting the value of $z_n^{(1)}$ (25) into (26), we obtain

$$\mathrm{F}\,(x_n) + F'(x_n)(\,z_n^{(2)} - x_n) - \beta F''(x_n)\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)\Big(\,z_n^{(2)} - x_n\Big) + (\tfrac{1}{2} - \beta)F''(x_n) \tag{27}$$

$$\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big) =0$$

Collecting the terms with ($z_n^{(2)} - x_n$) and rearranging the equations

$$F'(x_n) - \beta F''(x_n)\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)\Big(\,z_n^{(2)} - x_n\Big)$$

$$=-\mathrm{F}\,(x_n) - ((\tfrac{1}{2} - \beta)F''(x_n))\,\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big) \tag{28}$$

Therefore, by putting x_{n+1}= $z_n^{(2)}$ and from equation (28), we have

$$F'(x_n)(\mathrm{I}-\,\beta b_{F(x_n)})\,(x_{n+1} - x_n)=-\,\mathrm{F}\,(x_n)(\tfrac{1}{2} - \beta)\,F''(x_n))\,\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)\big(F'(x_n)^{-1}\,\mathrm{F}\,(x_n)\big)$$

Multiplying by $\big(I - \beta b_{F(x_n)}\big)^{-1}$ and rearranging the last equation, we have

$$x_{n+1} = x_n - \left(I - \beta b_{F(x_n)}\right)^{-1}[I + (\tfrac{1}{2} - \beta) b_{F(x_n)} F'(x_n)^{-1} \mathrm{F}\,(x_n)] \tag{29}$$

After rearranging eqn. (29) we have

$x_{n+1} = x_n - [I + \frac{1}{2} b_{F(x_n)}((I - \beta b_{F(x_n)})^{-1})] F'(x_n)^{-1} \mathrm{F}\,(x_n)$.

For $\beta = 1$ we have

$x_{n+1} = x_n - [I + \frac{1}{2} b_{F(x_n)}((I - b_{F(x_n)})^{-1})] F'(x_n)^{-1} \mathrm{F}\,(x_n)$ Super Halley method

For $\beta = 0$ we have $x_{n+1} = x_n - [I + \frac{1}{2} b_{F(x_n)}] F'(x_n)^{-1} \mathrm{F}\,(x_n)$ Chebyshev's method

For $\beta = \frac{1}{2}$ we have $x_{n+1} = x_n - [I + \frac{1}{2} b_{F(x_n)}((I - \frac{1}{2} b_{F(x_n)})^{-1})] F'(x_n)^{-1} \mathrm{F}\,(x_n)$ Halley's method

Steps in Halley class for two function of two variables

Step 1:

Let $X^0 = (x_1^0, x_2^0)$ be an initial vector.

Step 2:

Calculate J (X^0), H (X^0) and F (X^0)

Step 3:

We now have to calculate the vector Z^0, where

$$Z^0 = \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix}$$

In order to find Z^0 we solve $\left[I + \frac{1}{2} b_{F(X^0)}\left(\left(I - \beta b_{F(X^0)}\right)^{-1}\right)\right]^{-1} \mathrm{J}\,(X^0) Z^0 = -\mathrm{F}\,\left(X^0\right)$, where

$0 \leq \beta \leq 1$.

Rearranging the system in step 3 we get that

$Z^0 = -\left[I + \frac{1}{2} b_{F(X^0)}\left(\left(I - \beta b_{F(X^0)}\right)^{-1}\right)\right] \mathrm{J}\,(X^0)^{-1} \mathrm{F}\,(X^0)$ the significance of this is

That $Z^0 = -\left[I + \frac{1}{2} b_{F(X^0)}\left(\left(I - \beta b_{F(X^0)}\right)^{-1}\right)\right] \mathrm{J}\,(X^0)^{-1} \mathrm{F}\,(X^0)$ we can replace

$$-\left[I + \frac{1}{2} b_{F(X^0)}\left(\left(I - \beta b_{F(X^0)}\right)^{-1}\right)\right] \mathrm{J}\,(X^0)^{-1} \mathrm{F}\,(X^0)$$

In our iterative formula with Z^0. This result will yield that

$X^1 = X^0 - \left[I + \frac{1}{2} b_{F(X^0)} \left(\left(I - \beta b_{F(X^0)}\right)^{-1}\right)\right]$ J $(X^0)^{-1}$F (X^0) $=X^0 + Z^0$, where I is identity matrix. Where $b_{F(X^0)} =$ J $(X^0)^{-1}$ H (X^0) J $(X^0)^{-1}$ F (X^0). From the definition of 4.2 $F'(x_n)$ of several variables are called Jacobian matrix denoted by J(x) and from the definition 4.3 $F''(x_n)$ of several variables are called Hessian matrix denoted by H(x)

Step 4: Once Z^0 is found, we can now proceed to finish the first iteration by solving for X^1. Thus using the result from Step 3, we have that

$$X^1 = X^0 + Z^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} + \begin{bmatrix} z_1^0 \\ z_2^0 \end{bmatrix}$$

Step 5: Once we have calculated X^1 , we repeat the process again, until X^kconverges to $\bar{X}$. This indicates we have reached the solution to **F(x)** = **0**, where $\bar{X}$ is the solution to the system.

4.3. Convergence Analysis

4.3.1. Convergence of Newton's method

Theorem 4.1. Let $\bar{x}$ be a fixed point of the iteration x_{n+1} = f (x_n) and suppose that $f'(\bar{x})$= 0 but $f''(\bar{x}) \neq 0$. Then the iteration will have a quadratic rate of convergence.

Proof using Taylor's F(x) = f $(\bar{x})$ + (x-$\bar{x}$)$f'(\bar{x})$ + $\frac{1}{2}(\mathrm{x} - \bar{x})^2 f''(\bar{x})$ + $\frac{1}{6}(\mathrm{x} - \bar{x})^3 f'''(\bar{x})$ +…

Substitute x_k for x and the facts that x_{k+1} = f (x_k), f $(\bar{x})$= $\bar{x}$ and $f'(\bar{x})$ = 0 to obtain.

$$x_{k+1} = \bar{x} + \frac{1}{2}(\mathrm{x_k} - \bar{x})^2 f''(\bar{x}) + \frac{1}{6}(\mathrm{x_k} - \bar{x})^3 f'''(\bar{x})$$

Subtracting x^* from both side and dividing by $(\mathrm{x} - \bar{x})^2$ gives

$$\frac{x_{k+1} - \bar{x}}{(x_k - \bar{x})^2} = \frac{1}{2} f''(\bar{x}) + \frac{1}{6}(x_k - \bar{x}) f'''(\bar{x})$$

Which as n→ ∞, gives

$$\lim_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|(x_k - \bar{x})\|^2} = \frac{\|f''(\bar{x})\|}{2}$$

That is α = 2 which shows the iteration will converge quadratically.

Theorem 4.2. Let F: $\mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function in a convex open set D of $\mathbb{R}^n$ that contains $\bar{x}$. Suppose that $J_F^{-1}(\bar{x})$ exists and that there exist positive constants R, C and L such that $J_F^{-1}(\bar{x}) \leq$ C and $\|J_F(x) - J_F(y)\| \leq L\|x - y\|$ ∀ x, y ∈ B $(\bar{x};\ R)$,

Having denoted by the same symbol $\|.\|$ two consistent vector and matrix norms. Then, there exists r >0 such that, for any $x^{(0)}$ ∈ B $(\bar{x}; r)$, the sequence is uniquely defined and converges to $\bar{x}$ with

$$\|x^{n+1} - \bar{x}\| \leq CL\|x^n - \bar{x}\|^2. \tag{4.30}$$

Proof. Proceeding by induction on n, let us check (4.30) and, moreover, that $x^{n+1} \in$ B $(\bar{x};\ r)$ when r = min(R,$\frac{1}{(2CL)}$). First, we prove that for any $x^{(0)} \in$ B $(\bar{x}; r)$, the inverse matrix $J_F^{-1}(\bar{x})$ exists. Indeed $\|J_F^{-1}(\bar{x})[J_F(x^0) - J_F(\bar{x})]\| \leq \|J_F^{-1}(\bar{x})\| \|J_F(x^0) - J_F(\bar{x})\| \leq$ CLr $\leq \frac{1}{2}$

And thus, we can conclude that $J_F^{-1}(\bar{x})$ exists, since

$$\|J_F^{-1}(x^0)\| \leq \frac{\|J_F^{-1}(\bar{x})\|}{\|1 - J_F^{-1}(\bar{x})[J_F(x^0) - J_F(\bar{x})]\|} \leq 2\|J_F^{-1}(\bar{x})\| \leq 2C.$$

As a consequence x^1 is well defined and

$$x^1 - \bar{x} = x^0 - \bar{x} - J_F^{-1}(x^0)[F(x^0) - F(\bar{x})\ ]$$

Factoring out $J_F^{-1}(x^0)$ on the right hand side and passing to the norms, we get

$$\|x^1 - \bar{x}\| \leq \|J_F^{-1}(x^0)\| \|F(x^0) - F(\bar{x})\, J_F(x^0)[\bar{x} - x^0]\| < 2C\frac{L}{2}\|\bar{x} - x^0\|^2.$$

Where the remainder of Taylor's series of F has been used. The previous relation proves (4.30) in the case n = 0; moreover, since $x^{(0)} \in$ B $(\bar{x}; r)$, we have $\|\bar{x} - x^0\| \leq \frac{1}{(2CL)}$, from which

$\|x^1 - \bar{x}\| \leq \frac{1}{2}\|\bar{x} - x^0\|$. This ensures that $x^{(1)} \in$ B $(\bar{x}; r)$

By similar prove we can check that, should (4.30) be true for certain n, then the inequality would follow also for k+1 in place of k.

4.3.2. Accelerated Convergence in Newton's Method

Newton's method itself converges quadratically. It is based on linear approximation (tangent line) to the function at the current iterate x_k.

$$F(x) \approx l(x) = f(x_k) + f'(x_k)(x - x_k).$$

The equation l (x_{k+1}) = 0 leads to familiar Newton schema

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{4.31}$$

In order to accelerate Newton's method numerical analysis texts suggest using a higher order approximation to f at x_k (Dennis and Schambel, 1983)in this section we take different approach and ask, how can we modify a given function in such a way that the order of convergence increased?

Theorem 4.3. Let f is sufficiently many times differentiable; f has simple root at x =$\bar{x}$. i.e. f($\bar{x}$) =0 and $f'(\bar{x}) \neq 0$; and that the initial approximation x_0 sufficiently close to $\bar{x}$ so that converges to $\bar{x}$ will occur; Let assume that $f''(\bar{x}) = f'''(\bar{x}) = \ldots = f^{(n-1)}(\bar{x}) = 0$ and that $f^{(n)}(\bar{x}) \neq 0$. Then Newton sequence $\{x_k\}$ defined by (4.30) yields

$$\|x_{k+1} - \bar{x}\| = C\,\|x_k - \bar{x}\|^n \tag{4.32}$$

For some constant C, (Gerlach, 1994)

Proof suppose x_k has already been computed. By Taylor's theorem together with the fact that f and many of its derivatives vanish at x = $\bar{x}$ there exist a constants a_1 and a_2 between x_k and $\bar{x}$ such that

$$f(x_k) = f'(\bar{x})(x_k - \bar{x}) + \frac{f^{(n)}(a_1)}{n!}(x_k - \bar{x})^{\mathrm{n}}$$

$$\text{And } f'(x_k) = f'(\bar{x}) + \frac{f^{(n)}(a_2)}{(n-1)!}(x_k - \bar{x})^{n-1}$$

Hold upon substitution of these expression into Newton's formula we obtain

$$x_{k+1} - \bar{x} = x_k - \bar{x} - \frac{f(x_k)}{f'(x_k)} = \frac{1}{f'(x_k)}\{f'(x_k)(x_k - \bar{x}) - f(x_k)\}$$

$$= \frac{1}{f'(x_k)}\{\frac{f^{(n)}(a_2)}{(n-1)!} - \frac{f^{(n)}(a_1)}{n!}\}(x_k - \bar{x})^{\mathrm{n}}$$

$$= \frac{nf^{(n)}(a_2) - f^{(n)}(a_1)}{n!f'(x_k)}(x_k - \bar{x})^{\mathrm{n}} .$$

Now we fix a neighborhood N= $(\bar{x} - l, \bar{x} + l)$ of a for some suitably small l, so that the inequality $\|f'(x)\| \geq c_0 > c$ and $\|f^{(n)}(x)\| \leq c_1$ hold true on N for some constant c_0 and c_1.If $x_k \in N$ then

$$\left\|\frac{nf^{(n)}(a_2) - f^{(n)}(a_1)}{n!f'(x_k)}\right\| \leq \frac{nc_1 + c_1}{c_0 m!} = \mathrm{c},$$

And thus

$$\|x_{k+1} - \bar{x}\| = \mathrm{C}\,\|x_k - \bar{x}\|^n$$

4.3.3. Convergence of Halley, super Halley and Chebyshev,s method

Theorem 4.4. Let f: $\mathbb{R} \rightarrow \mathbb{R}$ is sufficiently three times differentiable; f has simple root at x =$\bar{x}$. (i.e) f ($\bar{x}$) =0 and $f'(\bar{x}) \neq 0$; and that the initial approximation x_0 sufficiently close to $\bar{x}$ so that converges to $\bar{x}$ will occur. Then Chebyshev method converges cubic. (i.e.)

$$\|x_{k+1} - \bar{x}\| = \mathrm{C}\|x_k - \bar{x}\|^3.$$

For some constant C

Proof: suppose $f'(\bar{x}) \neq 0$, $f''(\bar{x}) = 0$ and $f'(\bar{x}) \neq 0$. our n=3 by theorem 4.4. Suppose x_k has already been computed. By Taylor's theorem together with the fact that f and many of its derivatives vanish at x = $\bar{x}$ there exist a constants a_1 and a_2 between x_k and $\bar{x}$ such that

$$f(x_k) = f'(\bar{x})(x_k - \bar{x}) + \frac{f''(a_1)}{3!}(x_k - \bar{x})^3$$

$$\text{And } f'(x_k) = f'(\bar{x}) + \frac{f^{(3)}(a_2)}{2!}(x_k - \bar{x})^2$$

Hold. Upon substitution of these expression into Newton's formula we obtain

$$x_{k+1}-\bar{x} = x_k-\bar{x}-\frac{f(x_k)}{f'(x_k)} = \frac{1}{f'(x_k)}\{f'(x_k)(x_k-\bar{x})- f(x_k)\}$$

$$= \frac{1}{f'(x_k)}\{\frac{f'''(a_2)}{2!}-\frac{f'''(a_1)}{3!}\}(x_k-\bar{x})^3$$

$$= \frac{3f'''(a_2)-f'''(a_1)}{3!f'(x_k)}(x_k-\bar{x})^3 .$$

Now we fix a neighborhood N= $(\bar{x}-l,\bar{x}+l)$ of a for some suitably small l, so that the inequality $\|f'(x)\| \geq c_0 > c$ and $\|f^{(n)}(x)\| \leq c_1$ hold true on N for some constant c_0 and c_1.If $x_k \in N$ then

$$\left\|\frac{3f'''(a_2)-f'''(a_1)}{3!f'(x_k)}\right\| \leq \frac{3c_1+c_1}{c_0 m!} = \mathrm{c},$$

And thus

$$\|x_{k+1}-\bar{x}\| = \mathrm{C}\|x_k-\bar{x}\|^3$$

This implies chebyshev method converges cubic.

Theorem 4.5. Let f: $\mathbb{R} \to \mathbb{R}$ three time differentiable in a neighborhood of the solution $\bar{x}$. Assume that $f'(\bar{x}) \neq 0$. Let the starting point x_0 close to the solution. Then Halley's method is cubically convergent to the solution.

Proof: consider $\bar{x}$ to be a solution to f, such that $f(\bar{x})$ then by Taylor's expansion for function f

$$0=f(\bar{x}) = f(x_n)+(x-\bar{x})f'(x_n)+\frac{1}{2}(x-\bar{x})^2 f''(x_n)+\frac{1}{6}(x-\bar{x})^3 f'''(a_1) \qquad \text{(i)}$$

Taylor's theorem also gives:

$$0=f(\bar{x}) = f(x_n)+(x-\bar{x})f'(x_n)+\frac{1}{2}(x-\bar{x})^2 f''(a_2) \qquad \text{(ii)}$$

Where a_1 and a_2 between $\bar{x}$ and x_n

Next we multiply the first equation with by $2f'(x_n)$

$$0 =2f'(x_n)\, f(x_n)+ 2[(x-\bar{x})f'(x_n)^2 + (x-\bar{x})^2 f''(x_n)f'(x_n)+\frac{1}{3}(x-\bar{x})^3 f'''(a_1)\, 2f'(x_n)]$$

And multiply the second equation by $f''(x_n)\,(x-\bar{x})$

$$0 = f''(x_n)\,(x-\bar{x})\, f(x_n) + (x-\bar{x})^2 f''(x_n)\, f'(x_n)+\frac{1}{2}(x-\bar{x})^2 f''(a_2)$$

Now, subtracting the first equation from the second equation, results in

$$0=2f'(x_k)\, f(x_k) + (2[f'(x_k)]^2 - f(x_n)\, f''(x_n))\,(x-\bar{x}) + (\frac{f'(x_n)f'''(a_1)}{3}-\frac{f''(x_n)f''(a_1)}{2})(x_k-\bar{x})^2$$

After rearranging the equation it becomes:

$$\mathrm{x_k}-\bar{x} = \frac{-2f'(x_k)\, f(x_k)}{2[f'(x_k)]^2- f(x_k)\, f''(x_k)}-\frac{2f'(x_k)f'''(a_1)-3f''(x_k)f''(a_1)}{6(2[f'(x_k)]^2- f(x_k)f''(x_k))}(x_k-\bar{x})^3$$

x_{k+1} is defined, the first term on the right hand side can be replaced with

$$(x_{k+1} - x_k)$$

$$x_k - \bar{x} = (x_{k+1} - x_k) - \frac{2f'(x_k)f'''(a_1) - 3f''(x_k)f''(a_1)}{6\left(2[f'(x_k)]^2 - f(x_k)f''(x_k)\right)}(x_k - \bar{x})^3$$

After rearranging again, the equation becomes

$$x_{k+1} - \bar{x} = -\frac{2f'(x_k)f'''(a_1) - 3f''(x_k)f''(a_1)}{6\left(2[f'(x_k)]^2 - f(x_k)f''(x_k)\right)}(x_k - \bar{x})^3$$

Next consider

$$c = \lim_{x_k \to \bar{x}} -\frac{2f'(x_k)f'''(a_1) - 3f''(x_k)f''(a_1)}{6\left(2[f'(x_k)]^2 - f(x_k)f''(x_k)\right)}$$

$$c = -\frac{2f'(\bar{x})f'''(\bar{x}) - 3f''(\bar{x})f''(\bar{x})}{6(2[f'(\bar{x})]^2)}$$

And

$$\|c\| = \left\|\frac{2f'(\bar{x})f'''(\bar{x}) - 3f''(\bar{x})f''(\bar{x})}{6(2[f'(\bar{x})]^2)}\right\|$$

Because $\|c\|$ fixed constant for all $C \geq c$, it is possible to say

$$\|x_{k+1} - \bar{x}\| \leq C\|(x_k - \bar{x})^3\|$$

Theorem 4.6. Let f be a quadratic polynomial with a simple zero x. The Super Halley method (4.18) for solving the equation f(x) =0 is a fourth order method.

(Hernandez and Gutierrez, 2001). Gives the proof for G be the function define by (4.19). Observe that G ($\bar{x}$) =$\bar{x}$. And using lemma 4.1

4.4. Numerical tests of one dimension and two dimension nonlinear equations using Newton's method and its variants

Example-1 [2D problem] Application for solving nonlinear BVP (ODE)

Consider the nonlinear boundary value problem.

$$u'' = u^2,\ u = u(x),\ 0 \leq x \leq 1$$

$$U(0) = 1,\ u(1) = 2.$$

Let us use second order finite difference method (FDM) with h = $\Delta x = \frac{1}{3}$ (we have 3 equally spaced intervals).

We use central difference method approximation (CDM).

$$U_i'' = \frac{u_{i+1} - 2U_i + U_{i-1}}{h^2}$$

$$\therefore U'' = U^2 \text{ becomes,}$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{1}{9}U_i^{2;}\ i = 1, 2.$$

$$i = 1: u_1^2 + 18u_1 - 9u_2 - 9 = 0,\ U_0 = 1.$$

$$i = 2: u_2^2 + 18u_2 - 9u_1 - 18 = 0,\ U3 = 2.$$

These give the system of nonlinear (quadratic) equations

$$F(U_1, U_2) = \begin{Bmatrix} U_1^2 + 18U_1 - 9U_2 - 9 \rightarrow f_1 \\ U_2^2 + 18U_2 - 9U_1 - 18 \rightarrow f_2 \end{Bmatrix}$$

With tolerance $\leq 10^{-14}$

To apply Newton's method, we need the partial derivatives for Jacobian J

$$\frac{\partial F_1}{\partial U_1} = 2U_1 + 18,\ \frac{\delta F_1}{\delta U_2} = -9$$

$$\frac{\partial F_2}{\partial U_1} = -9 \qquad ,\frac{\partial F_2}{\partial U_2} = 2U_2 + 18$$

$$\text{And } J\ (u)\ \Delta U = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} U_1^2 + 18U_1 - 9U_2 - 9 \\ U_2^2\ - 9U_1 + 18U_2 - 18 \end{pmatrix}$$

$\begin{pmatrix} U_{i+1} \\ U_{j+1} \end{pmatrix} = \begin{pmatrix} U_i \\ U_j \end{pmatrix} + \begin{pmatrix} \Delta U_i \\ \Delta U_j \end{pmatrix} \longrightarrow$ is Newton's method in 2D.

Choose $u_1^0 = 1 = U_2^0$.

Table 6.1 Newton's method for example 1

K	u_1	u_2	$\|\|F_k\|\|$
0	1	1	1
1	1.163009404388715	1.473354231974922	0.026572065919163
2	1.155545801975687	1.459450536740559	5.570536097110335e-05
3	1.155537354083055	1.459437661122766	7.136158330922626e-11
4	1.155537354074375	1.459437661111107	3.552713678800501e-15

Table 6.2 super Halley method for example 1

K	u_1	u_2	$\|F_k\|$
0	1	1	1
1	1.155545801975688	1.459450536740559	5.570536098886691e-05
2	1.155537354074375	1.459437661111107	3.552713678800501e-15

Table 6.3 Halley method for example 1

K	u_1	u_2	$\|F_k\|$
0	1	1	1
1	1.155292215837754	1.459017600929443	0.001198419310105
2	1.155537354074120	1.459437661110756	2.025046796916286e-12
3	1.155537354074375	1.459437661111107	3.552713678800501e-15

Table 6.4 Chebyshev method example 1

K	u_1	u_2	$\|F_k\|$
0	1	1	1
1	1.155021882825265	1.458556635824941	0.002540281764480
2	1.155537354069651	1.459437661104626	3.762679057217611e-11
3	1.155537354074375	1.459437661111106	5.329070518200751e-15

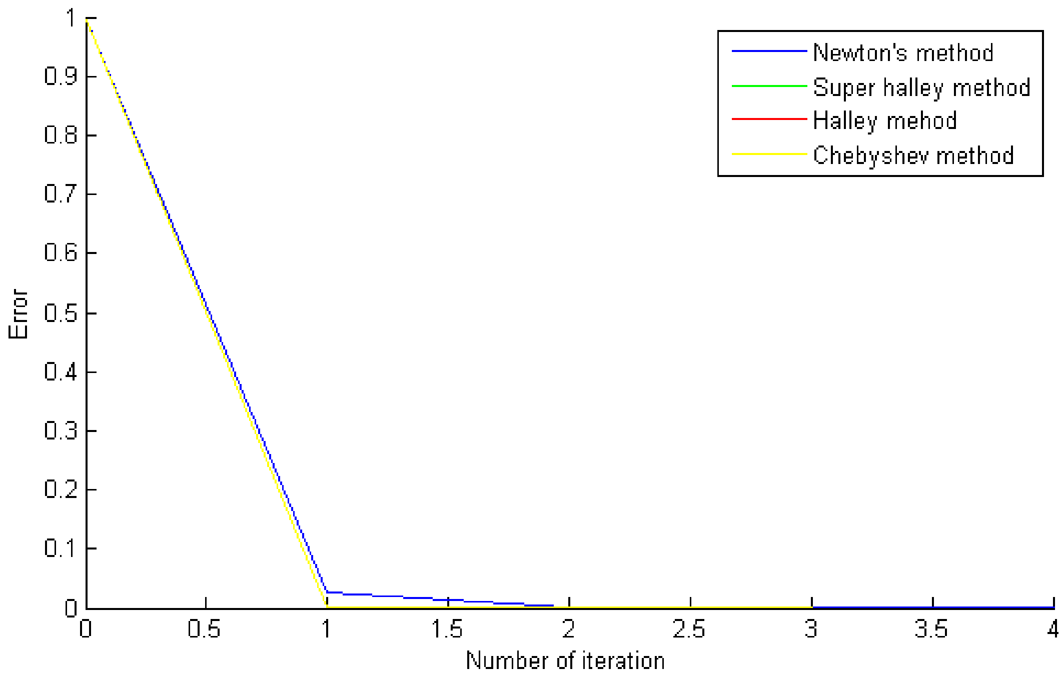

Figure 4.1 comparing the number of iteration Newton's method, Halley's method, Super Halley method and Chebyshev method of example 1

Example 2: Consider nonlinear system of equations given by

$$F(X) = \begin{bmatrix} f_1(X) \\ f_2(X) \end{bmatrix} = \begin{bmatrix} x_1^2 - 10x_1 + x_2^2 + 8 \\ x_1 x_1^2 + x_1 - 10x_2 + 8 \end{bmatrix}, \text{with tolerance} \leq 10^{-9}$$

$$\frac{\partial f_1(X)}{\partial x_1} = 2x_1 - 10 \qquad \frac{\partial f_1(X)}{\partial x_2} = 2x_2 \qquad \frac{\partial f_2(X)}{\partial x_1} = x_2^2 + 1 \qquad \frac{\partial f_2(X)}{\partial x_2} = 2x_1x_2 - 10$$

$$\frac{\partial^2 f_1(X)}{\partial x_1^2} = 2 \qquad \frac{\partial^2 f_1(X)}{\partial x_1\, \partial x_2} = 0 \qquad \frac{\partial^2 f_1(X)}{\partial x_2^2} = 2 \qquad \frac{\partial^2 f_1(X)}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial^2 f_2(X)}{\partial^2 x_1} = 0 \qquad \frac{\partial^2 f_2(X)}{\partial x_1\, \partial x_2} = 2x_2 \qquad \frac{\partial^2 f_2(X)}{\partial^2 x_2} = 2x_1 \qquad \frac{\partial^2 f_2(X)}{\partial x_2 \partial x_1} = 2x_2$$

$$H = \left(\begin{array}{cc|cc} 2 & 0 & 0 & 2x_2 \\ 0 & 2 & 2x_2 & 2x_1 \end{array}\right)$$

The following tables show the number of iteration and the function norm at each iteration.

Table 6.5 Newton's method for example 2

K	x_1	x_2	$\|F(x_k)\|$
0	0.5000	0.5000	3.500000000000000
1	0.937685459940653	0.939169139465875	0.384438094902656
2	0.998693990079859	0.998388282497394	0.007228947651134
3	0.999999200556673	0.999998951139757	4.297827868704474e-06
4	0.999999999999675	0.999999999999572	1.742606059451646e-12

Table 6.6 super Halley method for example 2

K	x_1	x_1	$\|F(x_k)\|$
0	0.5000	0.5000	3.500000000000000
1	0.995353638860719	0.983939812295173	0.005330032005544
2	0.999999958860586	0.999999838657066	6.429472065860864e-09

Table 6.7 Halley method for example 2

K	x_1	x_2	$\|F(x_k)\|$
0	0.5000	0.5000	3.500000000000000
1	0.989346729337297	0.979402036088185	0.044568005771120
2	0.999966156092192	0.999955961876188	1.826780996054822e-04
3	0.999999998849462	0.999999998482430	6.169162958258312e-09

Table 6.8 Chebyshev method for example 2

K	x_1	x_1	$\|F(x_k)\|$
0	0.5000	0.5000	3.500000000000000
1	0.984460707595296	0.975708273455008	0.076562443734622
2	0.999997067252993	0.999995329795617	1.412159770186605e-05
3	1.000000000000000	1.000000000000000	0

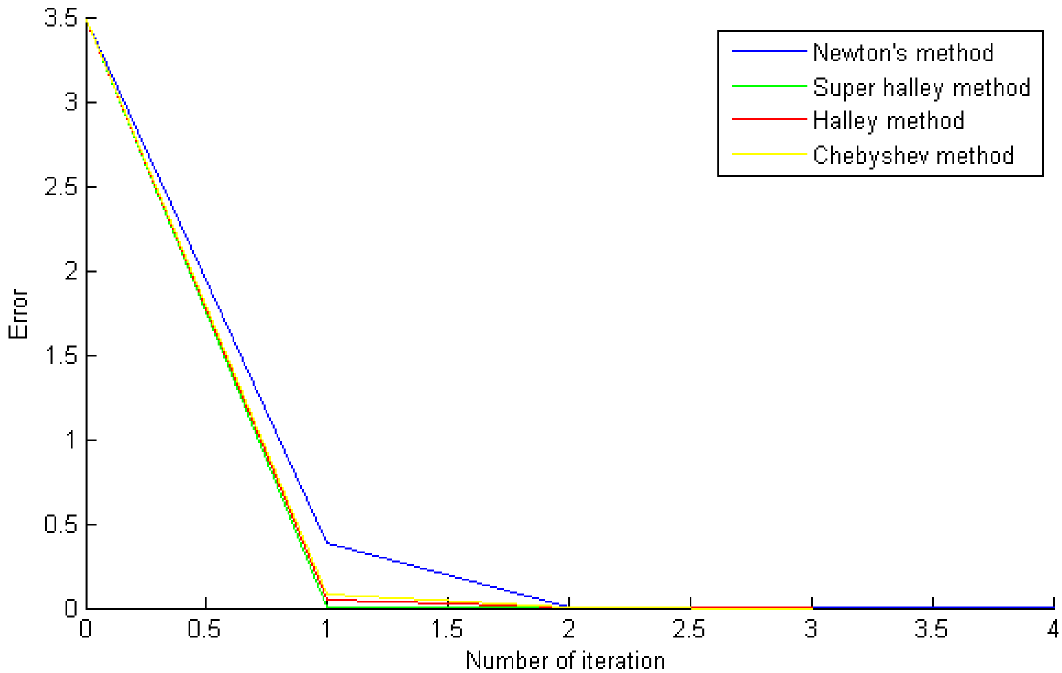

Figure 4.2 comparing the number of iteration Newton's method, Halley's method, Super Halley method and Chebyshev method of example 2

Example 3: Consider nonlinear equation in one dimension

$$F=3x^3-12$$

$$\frac{\partial f}{\partial x}=9x^2$$

$$\frac{\partial^2 f}{\partial x^2}=18x$$

$$x_0=2.11$$

Table 6.9: Newton's method for example 3

K	X	$\|F(x)\|$
0	2.1100000000	16.181792999999995
1	1.706150805237978	2.899561978136727
2	1.595474397186639	0.184024760676321
3	1.587441835427160	9.249346547601789e-04
4	1.587401053015971	2.376196839293243e-08
5	1.587401051968200	1.421085471520200e-14

Table 6.10: Halley's method for example 3

K	X	$\|F(x)\|$
0	2.11	16.181792999999995
1	1.610558919919736	0.532886513991834
2	1.587404266494148	7.290102796631004e-05
3	1.587401051968200	1.421085471520200e-14

Table 6.11: Super Halley method for example 3

K	X	$\|F(x)\|$
0	2.11	16.181792999999995
1	1.658354862578857	1.682128355311491
2	1.587401798914756	1.693969438676390e-05
3	1.587401051968199	1.243449787580175e-14

Table 6.12: Chebyshev's method for example 3

K	X	$\|F(x)\|$
0	2.11	16.181792999999995
1	1.628854989072132	0.964880612792093
2	1.587444673453297	9.893004767320690e-04
3	1.587401051968254	1.236344360222574e-12

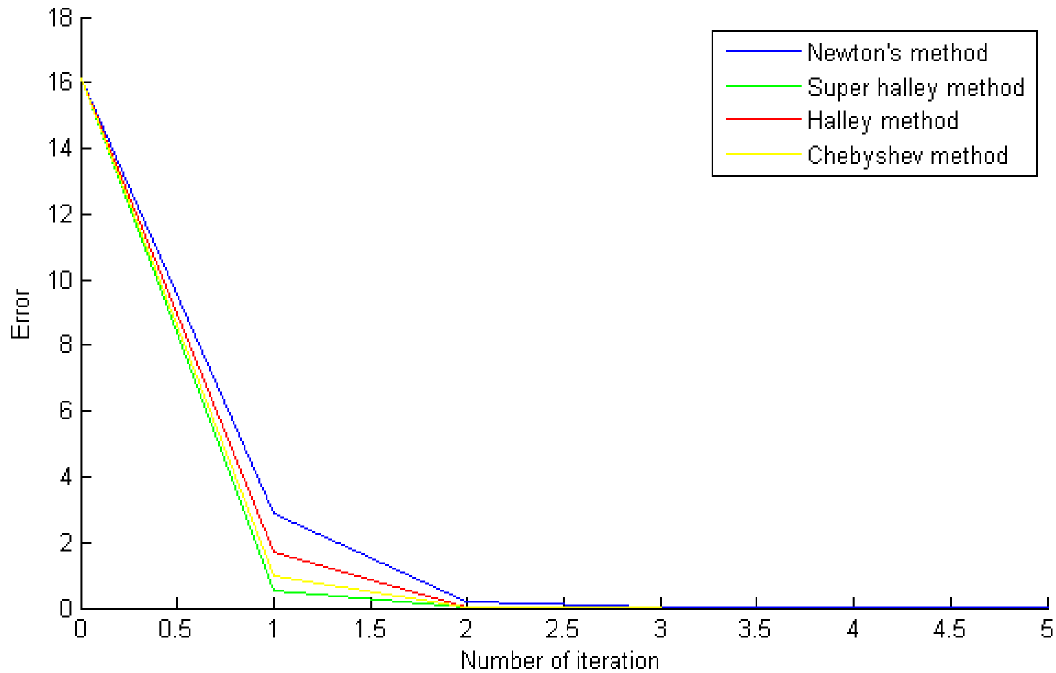

Figure 4.3 comparing the number of iteration Newton's method, Halley's method, Super Halley method and Chebyshev method of example 3

4.5. Discussion

From the tables we observed that Super Halley method converges to the solution faster than the other methods. The iteration number of Super Halley method is less than the iteration number of the other methods. As we saw before, the convergence rate of Chebyshev and Halley are similar (i.e. cubic convergence) while Super Halley is 4^{th} order. Super Halley method is faster than all. As we proved the convergence of Newton's method is slow convergence when compared to its variants (i.e. Newton's method is quadratical convergence). In the table, it also showed the iteration number of Newton's method is largest when compare to its variants. So we understood that Super Halley, Halley and Chebyshev method are faster than Newton's method for solving system of nonlinear equations.

From the example 1 we have seen that the two iterations of Newton's method is equivalent to one iteration of Super-Halley method. This shows the convergence rate of super Halley method is twice the convergence rate of Newton's method. In example two we saw the convergence of Super Halley method is the fastest of all the other methods presented.

Generally from the table of example one, example two and example three Super Halley method is the best way for solving system of nonlinear equations. Since it converges more rapidly than all of variants of Newton's method. The convergence of Newton's method is slower than all of its variants. The graphs also indicate that super Halley method is the most convergent of all. Both from the table and the graph we observed that super Halley method is the most convergent and Newton's method is the least convergent. Chebyshev's and Halley's method are 3^{rd} order while Super Halley's method is 4^{th} order.

Chapter 5

Conclusion and Recommendation

5.1. Conclusion

In this thesis, we have seen Newton's method and its variants to solve system of nonlinear equations in 1-dimension and 2- dimensions real space. Newton's method and its variants are based on quadratic model of Taylor's expansion. We also showed how to solve nonlinear boundary value problem (BVP) using Newton's method and its variants. We have shown that the convergence rate of Newton's method is quadratic while Halley's method and Chebyshev's method are cubic. Super Halley method is 4^{th} order. We have also shown that two iteration of Newton's method is one iteration of Super Halley method. We have seen that all the methods are accelerating convergence but super Halley is faster than all. There are some experiments on the rate of convergence of Chebyshev's method, Halley's method and super Halley method that were not included in the thesis regard to the limited time. We have also done matlab codes for all methods presented.

5.2 Recommendation

The suggestion for future work we state as the following

1) Solving system of nonlinear equation using higher order methods and other variants of Newton's method.

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