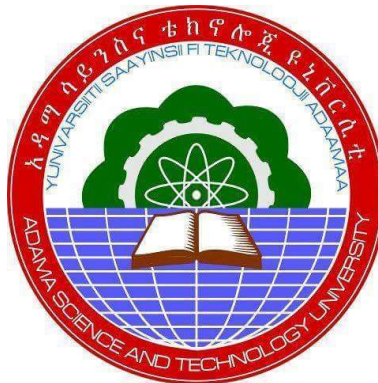


MATHEMATICAL MODELING OF PEST MANAGEMENT WITH OPTIMAL CONTROL



Teklebirhan Abraha

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Science

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy
in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

September, 2021
Adama, Ethiopia

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Main Supervisor: Prof. Delfim F.M. Torres

Co-Supervisor: Dr. Legesse Lemecha

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Declaration

I hereby declare that this dissertation entitled "**Mathematical Modeling of pest management with optimal control**" is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials used for this thesis have been duly acknowledged through appropriate citations.

Teklebirhan Abraha

Name of Student

Signature

Date

Recommendation

We, the supervisors of this dissertation, hereby certify that we have read and revised the dissertation entitled "**Mathematical Modeling of pest management with optimal control**" prepared under our guidance by **Teklebirhan Abraha** submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics. Therefore, we recommend the submission of the dissertation to the department for further review and defense.

Prof. Delfim F.M. Torres
Major Supervisor

Signature

Date

Dr. Legesse Lemecha
Co-supervisor

Signature

Date

Approval page

We hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**Mathematical Modeling of pest management with optimal control**" by Teklebirhan Abraha.

Prof. Delfim F.M. Torres
Major Supervisor

Signature

Date

Dr. Legesse Lemecha
Co-supervisor

Signature

Date

We, the undersigned, members of the Board of Examiners of the dissertation open defense by **Teklebirhan Abraha** have read and evaluated the dissertation entitled "**Mathematical Modeling of pest management with optimal control**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in Applied Mathematics.

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Finally, approval and acceptance of the dissertation is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the candidate's Department Graduate Council (DGC) and School Graduate Committee (SGC).

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Abstract

Pests are damaging organisms that affect both human activities and crops production. Agricultural pests are the species that cause damage to crops or interfere with crops. One of the main goals of pest control is to maintain the density of the pest population at the equilibrium level below economic damages. This dissertation developed mathematical models for the control and management of crop pests with environmentally safe strategies to minimize crop damage from pests with optimal awareness levels. We proposed and analyzed mathematical models in crop pest management, considering crop biomass, pest and the effect of farming awareness and Holling type-II functional response. The desired goals are to identify trade-offs between costs, impacts, and outcomes using biological control, chemical control, and farming awareness level. We proposed pest control models of a farming awareness-based integrated approach employing systems of nonlinear ordinary and delay differential equations to build the models. We further assumed that there may be some time delay in measuring the healthy pests in the crop field. Thus, we considered a mathematical model incorporating two-time delays. Qualitative analyses were applied to the equilibrium points to study the stability analysis. The existence and the stability criteria of the equilibria are studied. The local stability of equilibria was analyzed using the Routh-Hurwitz criterion. Moreover, we have used optimal control theory to provide the cost-effective outline of bio-pesticides and a global awareness campaign. Pontryagin's minimum principle is employed to establish the necessary conditions for the existence and characterization of the optimal controls. A Pontryagin-type minimum principle for retarded optimal control problems applied for the cost-effectiveness of the delayed system. We use Runge-Kutta's forward-backward sweep method to solve the optimal control systems. The forward and backward Euler is used to approximate the optimal control problem with delays. Time delays destabilize the system, but they have no effect when global awareness is high or when the consumption rate of the crop by the pest is low. If the density of awareness campaigns increases, the crop density increases, and as a result pest prevalence decline. Stability switches occur through Hopf-bifurcation when bifurcation parameters cross critical values. The negative effect of time delays is seen minimized using the optimal control approach.

Keywords: Pest control; Mathematical model; Time delays; Awareness campaign; Stability; Holling type-II functional response; Hopf-bifurcation; optimal control; Numerical simulations.

Original Results

This PhD dissertation reports original results that contribute to the expansion of the frontiers of knowledge and scientific development of Mathematics. The original contributions of this PhD dissertation were already submitted to international peer-review, being contained in the following original papers:

- P1) T. Abraha, F. Al Basir, L. L. Obsu and D. F. M. Torres, Pest control using farming awareness: impact of time delays and optimal use of biopesticides, *Chaos Solitons Fractals* 146 (2021), Art. 110869, 11 pp. DOI:10.1016/j.chaos.2021.110869;
- P2) T. Abraha, F. Al Basir, L. L. Obsu and D. F. M. Torres, Farming awareness based optimum interventions for crop pest control, *Math. Biosci. Eng.* 18 (2021), no. 5, 5364–5391. DOI:10.3934/mbe.2021272;
- P3) T. Abraha, F. Al Basir, L. L. Obsu and D. F. M. Torres, Controlling crop pest with a farming awareness based integrated approach and optimal control, *Computational and Mathematical Methods*, 2021, CMM41194. DOI: 10.1002/cmm4.1194

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Acronyms

CBB		Coffee Berry Bore
DDE		Delay Differential Equations
ET		Economic Threshold
IPM		Integrated Pest Management
IVP		Initial Value Problem
OCP		Optimal Control Problem
ODE		Ordinary Differential Equation
PMP		Pontryagin Maximum Principle
RK4		Runge Kutta Fourth order
ROCP		Retarded Optimal Control Problem

CHAPTER 1

INTRODUCTION

1.1 Background of the study

Food losses due to pests and plant diseases are nowadays one of the major threats to food security, particularly in large parts of the developing world. Pests are organisms that cause physical damage to man, animals, and crops. The damage caused by pests is one of the primary factors for reduced crop production. In this chapter we give a brief discussion on the background of crop pests and their control mechanisms.

1.2 Basic concepts of pests

A pest is any animal or plant which has a harmful effect on humans, their food, or their living conditions. Since the beginnings of agriculture about 10,000 years ago, growers have had to compete with harmful organisms, animal pests (insects, mites, nematodes, rodents, slugs and snails, birds) plant pathogens (viruses, bacteria, fungi), and weeds, collectively called pests (Oerke, 2006). Animals such as codling moth on apples, or boll weevil on cotton are called pests when they cause damage to agriculture by feeding on crops or parasitism livestock (Nandi et al., 2015). Pests may be classified as *Vertebrate pests*, *Invertebrate pests*, *Weeds and disease*. Pests that occur in high densities are shown in Figure 1.1.



Figure 1.1: Pests in densities. (Source www.rampestcontrol.com.au)

1.3 Pest prevention and control

Pest control is the regulation or management of a species defined as a pest, that impacts adversely on human activities. The human response depends on the importance of the damage done, and will range from tolerance, through deterrence and management, to attempts to completely eradicate the pest ([van den Bosch et al., 1982](#)). Control of pest attack is an important aspect in agriculture to obtain healthy crop as well as high yield. It is often difficult to predict the outcome of a particular control technique against a specific pest under given conditions. The choice of method(s) to be used depends on several factors: Degree of risk, nature of pest and disease complex, nature of the crop and agricultural system, economic factors and ecological factors ([Conway, 1984](#)). The following are the common pest control methods:

Physical or Mechanical control: is the management and control of pests using physical devices such as fences, barriers or electronic wires ([Khan et al., 2013](#)). For weeds, cultivation is a mechanical control technique that is economical, but only temporary as it must be repeated with each new emergence of weeds ([Manosathiyadevan et al., 2017](#)).

Cultural control: These are regular farm operations, that do not require the use of specialized equipment or extra skills, designed to destroy pests or to prevent them causing economic damage. In home it can be cleaning up food and keeping the area clean, in agriculture it can be crop rotation ([Kumar et al., 2012](#)).

Chemical control: is based on substances that are toxic (poisonous) to the pests involved. When chemical pesticides are applied to protect plants from pests, diseases or overgrowth by weeds, we speak of plant protection products. They are commonly grouped according to the type of pest they control ([Manosathiyadevan et al., 2017](#)):

Pesticides (chemicals to control pests), Bactericides (chemicals to control bacteria), Fungicides (chemicals to control fungi), Miticides (control to mites), Avicides (control to pest birds), Insecticides (chemicals to control insects), Rodenticides (chemicals to control rodents).

In the home, mouse bait(a rodenticide) might be used. In agriculture pesticides are sometimes used to control various pests to crops. The diagram Figure 1.2 shows an example of chemical pest control.



Figure 1.2: Chemical pest control. (Source:<http://www.canna-uk.com>)

Biological control: Biological control is a strategy where living organisms are only used to control pests. The term bio-pesticide is used for microbial biological pest control agents that are applied in a similar manner to chemical pesticides (Pathak and Maiti, 2012). Biological pest control is depicted in Figure 1.3. In nature, other than pest, there are other types of population species known as natural enemy, which predate pests. For example, there are some birds, frog, etc which feed on agricultural pests and thus those species can be treated as predator species to the pests they consume. In the home, that could be a cat that eats the mouse. So these populations can be used as one of the biological controls for the pest (Nandi et al., 2015).



Figure 1.3: Biological pest control. (Source:<http://www.canna-uk.com>)

Integrated Pest Management (IPM): IPM is the use of multiple pest control mechanisms (e.g., biological control, cultural practices, and selected chemical methods) in ways that complement one another, maintaining pest damage below acceptable economic levels, while minimizing hazards to humans, animals, plants, and the environment. The steps in using IPM include: *Identify the pest*, *Monitor pest activity*, *Choose control methods* and *Evaluate results*. Recently, IPM is gaining more attention among researchers and its application is also increasing in the crop field. Awareness campaign through radio or TV is required so

that people will gain awareness on biological control approach ([Naranjo et al., 2015](#)).

1.4 Some impacts of pests on crop production

Damage of crops by pest attack is a serious international problem in farming fields and forest ecosystems. If the pests are causing no crop loss, their presence on the plants and the damage they cause may be ignored. However, most pest populations produce some significance damage to yield loss that is difficult to estimate ([Hill, 2008](#)). Insects that cause less than 5% damage are not considered as pests. The insects which cause damage between 5–10% are called minor pests and those that cause damage above 10% are considered as major pests ([Paul, 2008](#)). The issue of minimization of crop losses due to pests and crop diseases, which has estimated to be as high as 40% of the world production has been a matter of active research for many decades. The main challenge lies in the unavoidable trade-off between pest reduction, financial costs, effects on human health, and environmental impact. ([Páez Chávez et al., 2017](#)).

There have been many reports worldwide on estimates of crop losses. The first systematic attempt to estimate crop losses due to various pests globally was made in 1967 that estimated overall annual losses in major crops (cereals, potato, vegetables, fruits, oil crops and fiber crops) to be about 34%. An analysis of crop losses in different regions showed that production losses in Europe (28.2%), North America (31.2%) and Oceania (36.2%) were below average, whereas in Africa and Asia reached almost 50% ([Sharma et al., 2017](#)). In 1994/5, CBB infested 650,000 hectares of coffee in Colombia, reduced national crop production by 1.5 million bags and cost an estimated 100 million USD and in southern Africa yield losses as high as 25% have been reported with more than 80% of coffee farms affected due to White Coffee Stem Bore ([Rutherford and Phiri, 2006](#)).

As pointed out by [Oerke \(2006\)](#), estimates on potential and actual losses despite the current crop protection practices are given for some crops for the period 2001–2003 on a regional basis as well as for the global total. Among crops, the total global potential loss due to pests varied from about 50% in wheat to more than 80% in cotton production. The responses are estimated as losses of 26–29% for soybean, wheat and cotton, and 31, 37 and 40% for maize, rice and potatoes, respectively. Overall, weeds produced the highest potential loss (34%), with animal pests and pathogens being less important (losses of 18 and 16%).

It is now estimated that approximately 42% of the world's foods supply is wasted due to pests destroying agricultural crops ([Pathak and Maiti, 2010](#)). Insect pests in agricultural systems are one of the major causes of damage to crop production and storage. In tropical countries, these pests are believed to cause losses approaching 60–70%, principally in stored products ([Manosathiyadevan et al., 2017](#)).

Based on a recent survey, 83% of 200 wheat farmers in Ethiopia perceived that post harvest losses during storage are caused mainly by insect pests. However, the perceptions of wheat

farmers remain unsubstantiated by measurement of such losses in storage (Kalsa et al., 2019).

1.5 Major important crops

Crop is a plant or plant product that can be grown and harvested for profit or subsistence. Since the establishment of agriculture, planted and stored crops have always been infested by multiple pest species. This is still the rule today for many economically important crops all over the world such as rice in Asia, corn and sorghum in Africa, wheat in Europe, corn in North America and potato in South America (Manosathiyadevan et al., 2017).

World program census of agriculture, indicative crop classification version 1.0 (2010) grouped crops as: *Cereals, Vegetables and Melons, Fruits and nuts, Oil seed crops, Root/tuber crops, Beverage crops, Leguminous crops, Sugar crops and Fiber crops.*

1.6 Major crop pests and control

1.6.1 Major cereal crop pests and control

Major pests that affect the most important cereal crops, like; maize, sorghum and rice are listed below. We adopt the description of these major crop pests and control methods from (Sishekanu et al., 2015).

Maize stalk borers: are pests of maize, sorghum and other crops throughout many countries. The pest can be *controlled* through a combination of *cultural* practices and *chemical insecticides*.

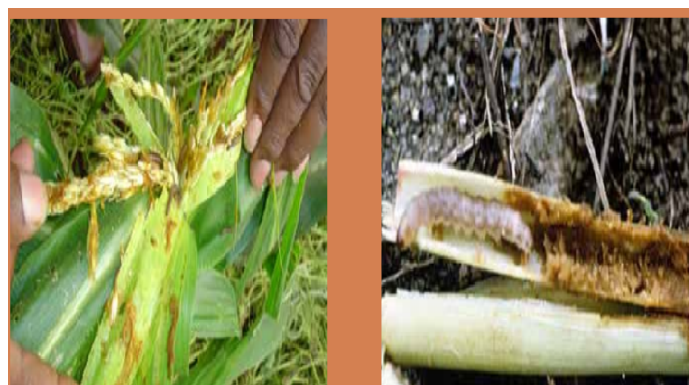


Figure 1.4: Maize stalk borer. (Source <http://bit.ly/1FEC61c>)

The Figure in 1.4 illustrates the maize stalk bore larva and inside a young maize tassel (left) and larva feeding on maize stem (right).

Larger grain borer: is a serious pest of stored maize throughout the world specially in Africa. The beetle eats its way into the grain leaving a hole and empty shell. Having a clean storage facility is important to prevent the pest from becoming unmanageable. Shelling the

grains from the cob prior to storage can minimize the damage (Sishekanu et al., 2015).

The **African army-worm**: is a pest of cereal crops in Africa. It is capable of destroying entire crops in a matter of weeks. Spraying of chemicals and biological control using a virus are the possible control methods.

Sorghum midge: is one of the most important pests of sorghum. The larvae of the midge feed on developing seeds causing malformation of the grain and empty or chaff heads as shown in Figure 1.5. Cultural controls are the best approach; chemical sprays may also be used.



Figure 1.5: Adult midge on sorghum bloom (left), Damaged sorghum heads (right). (Source <http://bit.ly/1a8PJuf>)

Red-billed quelea: They are seed-eaters, both of wild grasses and cereals such as sorghum, rice and wheat, but also eat insects, including pests of crops. Figure 1.6 illustrates partial pictures of red-billed quelea. *Control methods* include spraying fenthion, an organo-phosphate, but this also kills non-target birds and the pesticide needs careful handling. Other methods with fewer side-effects include fire bombs and dynamite.



Figure 1.6: Juvenile, female and male red-billed quelea(left), Local measures taken by farmers to scare birds(right). (Source <http://bit.ly/1Gs86sq>)

1.6.2 Major Legumes pests and control

The following are the most common pests that affect bean, groundnut and cow pea.

Bean bruchid: is a major post-harvest pest of most bean species. Storing the beans in a clean facility is the most important measure. The diagram in Figure 1.7 shows an adult bean bruchid on bean with exit hole(left), and adult, 3-4.5 mm long body grey, brown and reddishbrown, without any distinctive patterns(right).



Figure 1.7: Adult bean bruchid. (Source www.bugwood.org)

Legume pod borer: is a prominent pest of cow peas and other beans throughout East and West Africa. Control options include handpicking and destroying eggs and larvae.

Bean flower thrips: are major pest of cow-pea and other leguminous crops throughout sub-Saharan Africa. Control of the pest can be achieved through a combination of cultural practices and use of chemical pesticides.

1.6.3 Major tuber crop pests and control

Cassava green mite: is a pest of cassava in Africa that can cause significant losses in yield. The mite can easily spread from one place to another on leaves and cuttings of the plant, and by wind. Figure 1.8 depicts cassava green mite and adult egg. An integrated approach is required to control the pest and reduce the damage.



Figure 1.8: Cassava green mite adult and egg (Almoaet et al., 2019)

Sweet potato weevil: are insect pests that can cause severe damage to sweet potatoes. Use of clean planting material, planting early in season and selecting deep rooting varieties are some prevention methods.

Rats and mole rats of sweet potato: Rats and burrowing mole rats both attack sweet potato, but rats are the more serious pests. Control relies on community action and early intervention, mostly using cultural methods.

1.6.4 Major banana pest and control

Banana weevil: is a serious pest to bananas worldwide. We presented the damage caused by banana weevil in Figure 1.9. A combination of clean planting material, destruction of crop residues can be used to reduce weevil populations.



Figure 1.9: Damage caused by banana weevil (left), Adult banana weevil (right). (Source <http://bit.ly/1allhxe>)

1.6.5 Major coffee pest and control

The following are the major coffee pests and their control (Rutherford and Phiri, 2006).

Coffee Berry Borer (CBB): This beetle originated in Africa but it is now so widespread and

causes such damage that it is widely considered to be the most important pest of coffee. We illustrate adult coffee berry borer in Figure 1.10.



Figure 1.10: Side view of adult coffee berry borer ((Rutherford and Phiri, 2006))

Control of CBB has always been difficult and costly. Combining two or more approaches (e.g. cultural practices and biological control), as part of an IPM package, may have a greater impact and should therefore be considered.

1.6.6 Cotton pests and control

Cotton boll worm: is a major pest of many important food, oil and cash crops worldwide, including cereals, legumes, fruits and vegetables. Chemical control with care and important cultural controls are possible methods.



Figure 1.11: Adult moth of cotton bollworm.(left),Cotton bollworm caterpillars in two different colours.(right). (Source <http://bit.ly/1a8PJuf>)

1.7 The role of mathematical modeling in pest control

Mathematical modeling is a method of simulating real-life situations with mathematical equations to predict future behavior. One of the most important applications of mathematics in the biological sciences has been the mathematical model of population interaction, growth and dynamics.

The 1940s saw not only the discovery of modern pesticides and drugs but also the birth of operations research and systems analysis, and two decades later, a number of people started looking to the latter to find remedies for the deficiencies of the former. Concurrently, there was also a revival of the 1930s interest in both the mathematical theory of animal population dynamics and in the mathematical theory of epidemics. Finally, the growth of modern computing facilities meant that pest and disease systems could be efficiently modeled and analyzed, and in some cases mimicked in considerable detail ([Conway, 1984](#)).

Population dynamics models are important tools to support decision making for prevention and pest control. The role of mathematics in pest management is to aid in decision making. Decision making in agricultural pest management is complicated by the highly stochastic and uncertain nature of the problem ([Plant and Mangel, 1987](#)).

Mathematical models are more and more used for the study of agricultural problems, because with the use of simulation tools the system environment-pest-natural enemies can be understood better. This allows the researcher to have a general vision of the system and to accomplish computational experiments of the real data. One of the main challenges in mathematical description of ecological processes is the construction of mathematical models that provide reliable predictions and understanding of field observations in real ecosystems, such that these models can be used as decision support tools. In the particular case of agricultural pest control, the challenge consists in developing robust mathematical models able to at least qualitatively describe different pest control methods, so as to make the development of pest control strategies and policies less intuitive or empirical ([Nandi et al., 2015](#)).

The problem of pest control has a long history, but it was only in the mid seventies that mathematical models for pest management emerged. Perhaps the wide range of popularity of the so-called sterile insect release method (SIRM) was the key to attract mathematical biologists in the field of pest management ([Pathak and Maiti, 2010](#)).

Recently, some models for pest control have been proposed and studied by several researchers. More specifically, the use of the mathematical modeling, applied to the problems of biological pest control, allows a qualitative and quantitative evaluation of the impact between the pest and its natural enemies populations. Another form, in that the mathematical model can be used in the pest control, is the formulation of the optimal control strategy, through the dynamic manipulation of the control variables of the pest-natural enemy system.

1.8 Statement of the problem

Destruction of crops by pest infestation is a serious worldwide concern in agricultural sectors and forest ecosystems. The problem connected with pests has been realized since cultivation of crops began. Minimizing losses due to insect pests and insect vectors of important plant, animal, and human diseases remains an essential component of the programs in agricultural entomology. Appropriate and relevant knowledge about crop and its pests is very much

essential for people engaged in cultivation. Television, radio and mobile telephone are particularly useful media in providing information about agricultural practices and crop protection (Kumar et al., 2012; Khan et al., 2013). Differential control systems with delays in state or control variables played an important role in the modeling of real-life phenomena in various fields of applications. Bashier and Patidar (2017), described the participation of farming communities in Jatropha projects for bio-diesel production and protect the plant from mosaic disease using a mathematical model to forecast the development of renewable energy resource. Al Basir et al. (2019b) developed a mathematical model by employing two time delays to predict the effect of farming awareness via social media for the control of crop pest in agriculture. Furthermore, optimal control theory was applied to confirm the cost-effective profile of bio-pesticides and global awareness campaign. In Al Basir et al. (2019a), authors have proposed and analyzed a model for pest control and the impact of farming awareness and a time delay in local awareness. They concluded that raising awareness among people with tolerable time delay may be a proper aspect for the control of pest in crop field, reduce the serious issues that pesticides have on human health and environment. Chowdhury et al. (2019), developed a mathematical model for pest control using bio-pesticides. They have incorporated optimal control theory to minimize the cost in pest management due to bio-pesticides. Previous works on crops pest control models have separately concerned on farming awareness based biological control methods. Most reports on these models focused on developing a mathematical model by either using ordinary differential equations or employing a time delay to study the impact of awareness based crop pest management. In the previous findings, mathematical studies have not been widely undertaken to model the pest control of crops using bio-pesticides, chemical control and farming awareness integrated approach. Moreover, they lack to incorporate several optimal controls and multi-time delays. Therefore, we found it interesting to develop mathematical models with different optimal controls and multiple time-delays that take into account the effect of farming awareness in crop pest management, considering crop biomass, and level of awareness.

1.9 Objectives of the study

Based on the statement of the problem, the objectives of the study are as follows;

1.9.1 General objective

The general objective of this study is investigate crop pest management with awareness interventions through the application of mathematical modeling and optimal control.

1.9.2 Specific objectives

The specific objectives of the study are to:

1. formulate and analyze a mathematical model on the herd dynamics and control of crop pests with an optimum use of bio-pesticides.
2. model and analyze crop pest management with IPM techniques.
3. develop and analyze a mathematical model with delay for the control of pests in terms of the reproduction number, equilibrium points and stability.
4. perform both qualitative and quantitative analysis of the developed mathematical models and determine the parameters for eradication or control of the crop pests.
5. perform sensitivity analysis of the model parameters to determine the effects of each parameter on the control or eradication of the pest.

1.10 Significance of the study

Mathematical modeling on farming awareness to control plant pests and disease to study the impact of awareness program and the time delay in pest management are vital. This research is important in many ways. The mathematical models designed in this dissertation are significant for the following reasons:

1. It will use as a reference for future studies to monitor and control biological models with time delays.
2. Since previous mathematical study are insignificant in developing countries like Ethiopia on the most effective control measure of the pest, the results on the control measures will provide relevant guidance for decision makers on which intervention to focus on and the most effective control measure to use in crop pest managements.
3. It will also help to improve control strategies for the occurrence of pest outbreak in a community.
4. This study will help the health care sector to optimize controls and minimize the cost of control strategies in order to reduce the spread of crop damaging pests in the ecosystem.
5. Furthermore, the agricultural sectors would see the need for organising seminars, workshops or training programmes to educate the general public about the dangers and control strategies of pests.

1.11 Organization of the dissertation

After this brief introductory chapter, the dissertation is organized as follows. In chapter 2, relevant related literature is presented. Chapter 3 deals with the methodology used in the dissertation. In chapter 4, a deterministic model is developed with a system of ordinary differential equations to study the force of farming awareness-based optimum interventions for crop pest control. Stability and Hopf bifurcation analysis of the dynamics have been studied for the developed model. Optimal control techniques have been formulated and explained analytically and numerically. The obtained analytical results are assured by computing numerical simulations. Chapter 5 deals with another mathematical model that is extended from the model in chapter 4 and detail analysis of the model with the inclusion of three time dependent optimal controls is studied. Chapter 6 presents a basic deterministic model that describes the dynamics of crop pest control for a system governed by delay differential equations. The deterministic model is carefully analyzed in qualitative as well as quantitative techniques. The developed model is further extended into an optimal control problem with the intervention of continuous controls with the goal to minimize the cost of bio-pesticides. Finally, we end the dissertation with conclusions and recommendations.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter we presented a short overview on the mathematical models developed by many researches to study the dynamics of pest controls that are directly related to the objectives of this study. The discussion is mainly guided by two criteria: model type and control, in a mathematical sense, and the underlying entomological phenomena in the framework of agricultural practices. Recently mathematical models are becoming critical tools to predict and solve real world problems in the life sciences.

Many researchers utilized mathematical model for the pest control to study the different aspects of the pest management policies with probable outcomes for improved applications using the system's analysis within the mathematical paradigm. Most of them suggest using chemical pesticides ([Ghosh et al., 2007](#); [Kar et al., 2012](#)). However, it was recorded that chemical pesticides have resulted in pest resurgence, acute and chronic health problems, and environmental pollution. Thus, to resolve this type of problem, the concept of IPM is becoming more popular among researchers and increasing application in the field by marginal farmers.

Mathematical models may be classified a descriptive or a predictive. Traditionally, analytical models have been used for the former role and model simulation for the latter. To be effective in an applied context such as agricultural pest control, modeling must be done at both level. The role of mathematical modeling in strategy selection in the IPM context is to aid in determining the optimal strategy for a given situation ([Plant and Mangel, 1987](#)).

2.2 Crop pest control models

A number of studies have been conducted to highlight the control of crop pests and its dynamic by developing different type of mathematical models. In this dissertation special attention is paid to those models related to farming awareness, chemical and bio-pesticides techniques.

The first major application of systems analysis to crop pest control was the use of operations research in the insecticidal control of locusts between 1940 and 1960. It is only in the last few years that interest has grown dramatically, with major efforts devoted to the analysis and modeling of the pest complexes on cotton, rice, alfalfa, apples, and olives and of specific

pests such as the mountain pine beetle, the cereal leaf beetle, the sugar cane froghopper, and the cattle tick ([Conway, 1984](#)).

Much of the mathematical models developed for biological systems may be used in crop pest control. According to [Ding et al. \(2010\)](#) an SI epidemic model with stage structure was studied. In their model, impulsive biological control which release infected pest to the field at a fixed time periodically was considered, and they obtained the sufficient conditions for the global attractivity of pest-extinction periodic solution and permanence of the system. Their results provided a reliable tactic basis for the practice of pest management. [Bhattacharyya and Bhattacharya \(2006\)](#) proposed a mathematical model for viral infection of pest in agriculture and forest ecosystem. In their study, they have tried to show the usefulness of a simple mathematical model for understanding the evolution of virulence in the context of the biological control of pests.

[Rafikov and de Holanda Limeira \(2012\)](#) considered a mathematical model of interaction between the sugarcane borer (*Diatraea saccharalis*) and its egg parasitoid *Trichogramma galloi*. In their model, the sugarcane borer was represented by the egg and the larval stages, and the parasitoid considered in terms of the parasitized eggs. They proposed linear feedback control strategy to indicate how the natural enemies could be introduced into the environment.

Farmers, in their own awareness, can keep the crop under observation and, therefore, if correctly instructed, they will spray bio-pesticides or incorporate fertile to make the pest susceptible to their bio-agents. For instance, [Le Bellec et al. \(2012\)](#) studied how allowing surroundings for connections among farmers, researchers, and other factors can provide advantages to reduce existing problems related to the crop. Their result have shown how an enabling environment for interactions between farmers, researchers, and other factors, can contribute to reduce current problems associated with crop. Besides, farmers learn the use and dangers of pesticides mainly by oral communication. Self-aware farmers employ considerably improved agronomic practices, safeguarding health and reducing environmental hazards ([Yang et al., 2014](#)). Therefore, awareness is important in crop pest management. Quite a few models have been formulated and analyzed to explain the dynamics of crop pest control with farmers awareness.

In [Roy et al. \(2015\)](#), bio-pesticide were applied in impulsive periodic manner. Here, authors derived a mathematical model for *Jatropha curcas* pests and remarked that an integrated pest management with optimal control approach leads to a cost effective and rapid reduction of the crop pest. Later in [Chowdhury et al. \(2016\)](#), a mathematical model for pest control using bio-pesticides was considered. Here they described the participation of farming communities in *Jatropha* projects for biodiesel construction and guard of plants from mosaic disease, using a mathematical model to predict the growth of renewable power resources. Moreover, the authors tried to incorporate optimal control theory to minimize the cost in pest management due to bio-pesticides. Then [Bashier and Patidar \(2017\)](#), considered an extended mathematical model described in [Chowdhury et al. \(2016\)](#) to incorporate time delay and

optimal control. Accordingly authors recommended that the participation of farming communities in *Jatropha* projects to protect the plant from mosaic virus was effective.

It is now well known that the effect of time-delay is to be taken into account. Time-delays occur so often in almost every situation that to ignore them is to ignore reality ([Pathak and Maiti, 2010](#)).

Most of the previous studies on pest control focus long term dynamics of the system behavior governed by ordinary differential equations. crop pest control modeling with time delays and optimal control are not rich in the literature. However, considering both the time delay and multi optimal control behaviors of the system plays an important role in understanding the complete picture of the ecosystem dynamics.

Due to complexity and numerical intensity of the problem, a black box solver was developed which gives better solution though with long computational time compared to existing algorithms. [Cai et al. \(2003\)](#) developed a computational method for solving optimal control problems governed by switched dynamical system with time delay. They derived the required gradient of the cost function which is obtained via solving a number of delay differential equations forward in time, in which the resulting control problem can be solved as a mathematical programming problem.

[Göllmann et al. \(2009\)](#) developed a general method for solving optimal control problems with delay in state and control variables subject to mixed constraints. In the paper, a Pontryagin-type minimum principle was derived for retarded optimal control problems with delays in the state and control variable when the control system is subject to a mixed control–state constraint. Authors presented a discretization method (Euler’s method) whereby the control problem was transcribed into a high-dimensional nonlinear programming problem.

Later, [Silva et al. \(2017\)](#), considered optimal control problems for a delayed non-linear cost functional subject to a delayed differential system that is linear with respect to state, but not with respect to control, and derived a sufficient optimality condition for this type of problems. Also in 2019, the same authors obtained a sufficient optimality condition for non-linear optimal control problems with delays in both state and control variables.

[Al Basir et al. \(2018\)](#) developed a mathematical model to control *Jatropha* sp. mosaic disease in the *Jatropha* carcus plantations considering awareness as a controlling measure. The model considered, healthy and infected plant biomass, infected vectors and the cumulative density of aware people. Besides, they explicitly include time delay from the moment the information becomes available up to the time farmers comply with it and change their behavior. The main result shown was how time delays in accepting the awareness campaign could affect the ultimate dynamics of the mosaic diseases. Moreover, the authors showed aware farmers can keep the crop under surveillance. Later [Al Basir and Ray \(2020\)](#), have extended the model presented in [Al Basir et al. \(2018\)](#) by making additional assumptions and introducing control parameters. Here, a mathematical model has been derived using saturated response functions

(nonlinear infection rate) for studying the dynamics of mosaic disease with farming awareness based roguing (removal of infected plants) and insecticide spraying. Their main results suggested that the awareness campaigns through radio, TV advertisement are important for eradication of the disease awareness campaign. Also roguing and spraying should be incorporated with optimal level for cost effective control of mosaic disease, authors underlined. In [Pathak and Maiti \(2012\)](#), authors studied dynamical behaviours of a model for biocontrol of pests, where it is assumed that the pest is affected by a virus. Boundedness and stability of the model were studied, the effect of time-delay investigated. They considered the mutual relations going on an ecosystem where a virus influences a pest population nosh on a plant, the final creature unaltered by the virus. It was observed that the time-delay had a regulatory impact on the system. They further indicated that when the mortality rate of the infected pest due to virus is high then the system will be pest free. [Al Basir et al. \(2019b\)](#), proposed a prey-predator type mathematical model for studying the effect of time delay in controlling crop pest using only farming awareness based interventions. Their model assumed that the use of biological pesticides is proportional with the density of pest present in the plantation. According to their result, raising farming awareness based bio-controls with tolerable time delay lead to eradication of pests in crop fields.

According to [Al Basir et al. \(2019a\)](#) a prey-predator type mathematical model has been proposed and analyzed for crop pest control focusing on the impact of farming awareness in agricultural practice. In the paper, authors assumed that aware people adopts biological control such as integrated pest management as it is ecofriendly and less harmful for human health. Another important factor, authors considered was the time delay that might occur in implementing awareness activities. It was established that raising awareness among people with tolerable time delay may be a proper aspect for the control of pest in crop field as large delay causes periodic oscillation in the system. [Chowdhury et al. \(2019\)](#) presented and analyzed a mathematical model for *Jatropha curcas* plantation with a view to control its natural pests using application of integrated pesticides. Their model dealt about the control of pests using integrated approach i.e. using combination of bio-pesticides and chemical pesticides. They have incorporated optimal control theory to minimize the cost in pest management due to integrated pesticides. According to the study, application of chemical pesticide minimizes effect of oscillation and makes the system stable in shorter time. Besides the control policy was effective in pest eradication, stabilizing and maintaining the system at a level for which the *Jatropha curcas* production was enhanced.

Another important thing in mathematical modeling of life sciences is the application of optimal control theory.

To the best of our knowledge, there are few articles in which optimal control approach has been applied in delayed systems. Mathematical models with multiple delays for pest

control using bio-pesticide and farming awareness with the application of optimal control problems with constant delays in state variables are not reach in the literature.

Inspired by the works of [Al Basir et al. \(2019b,a\)](#) and [Chowdhury et al. \(2019\)](#), we proposed mathematical models to protect crops from pest damage through biological and chemical control methods with accounts to awareness campaign considering time delays that may occur due to the measure of pest in the field.

CHAPTER 3

RESEARCH METHODOLOGY

This dissertation is devoted to describe derivation and analysis of two non-delayed mathematical models and a delayed with the effect of farming awareness in crop pest management considering plant biomass, pest and level of awareness that are discussed in the study in detail.

3.1 Method of analysis

The parameter values supporting the deterministic models were taken from previously published articles and they have been duly cited in this dissertation. The parameters obtained have been used as a baseline parameter value that is used in the numerical simulations. The study involved both qualitative and quantitative analysis. Thus, the study was analyzed using the software MATLAB and the results are displayed in the form of figures.

3.2 Model assumptions and formulation

To achieve the objective of this study, we employed analytical and numerical methods. First we formulated mathematical models with four densities that describe crop pest control dynamics using systems of delay and ordinary differential equations. The following assumptions were taken to formulate all the desired three mathematical models. The models divided the total densities at time t into four compartments (X, S, I, A) , crop biomass, susceptible pest, infected pest, and level of awareness.

Crop biomass $(X(t))$ includes the total plant biomass(leaf,stem etc).

Susceptible pest $(S(t))$ includes these pest populations, that attack the crop causing considerable crop reduction.

Infected pest $(I(t))$ includes these pests that are once deactivated due to human interactions and activity such as the use of bio-pesticides.

Level of awareness $(A(t))$ this class includes aware farmers that adopt biological pesticides for the control of crop pest.

3.2.1 Mathematical procedures

We formulated and analyzed three deterministic mathematical models for the management of crop pests. Then the well-posedness of the formulated models were determined both in the

mathematical and biological sense and presented. We also performed the qualitative analysis of the models and, basic reproduction number and sensitivity analysis of the delayed model. Local stability and Hopf bifurcations were analyzed using stability theory. We have shown that the incorporated time delays can reduce the pest attack in the crop biomass. The effect of discrete-time delays on the stability of the coexistence equilibrium points was investigated. Then we extended the proposed models into optimal control problems. The existence and characterization of the formulated optimal control problems being studied using Pontryagin's minimum principle. To supplement the analytical solution of the mathematical models, we performed numerical simulations using MATLAB software.

CHAPTER 4

AWARENESS BASED OPTIMAL INTERVENTIONS FOR CROP PEST CONTROL

This chapter is devoted to the formulation of a deterministic model for the impact of farming awareness in pest control and further dealt with the qualitative and numerical analysis for the system of ordinary differential equations representing the model. We found two critical parameters that characterizes the stability behavior of the pest free equilibrium point. Two continuous control parameters are incorporated to the developed model and completely analyzed. Numerical simulations are carried out.

4.1 Model formulation and description

We consider four densities into our mathematical model, explicitly the crop biomass $X(t)$, the susceptible pests $S(t)$, the infected pests $I(t)$ and the awareness level $A(t)$. The following assumptions have been made to formulate the mathematical model.

- Influence of bio-pesticide, healthy pest population becomes infected. Infected pest can attack the crop but the rate is very lesser than susceptible pest and hence we assume that infected pests consume less amount of crop compared to susceptible pests.
- Due to the finite size of crop field, we assume logistic growth for the density of crop biomass, with net growth rate r and carrying capacity K .
- Crop–pest interaction can be studied with the Michaelis-Menten type (or Holling type II) functional response.
- Susceptible pests consume the crop thereby causing considerable crop reduction. If we infect the susceptible pest with pesticides, then the pest attack can be reduced as an infected pest is lesser harmful. Here we assume that aware farmers will adopt biological pesticides for the control of crop pest as it has less side effects and also environment friendly. Biopesticides are used to infect the healthy pest. Infected pest has additional mortality due to infection.
- We assume that α is the consumption rate of pest. We again assume that there is a pest infection rate, λ , because of human awareness connections and movement such as use of bio-pesticides, modeled through the usual mass action term λAS . If d is the natural

death rate of pest and δ is the extra death rate of infected pest due to aware people activity.

- The level of awareness $A(t)$, among the farmers increases due to general public awareness and media campaigns. Also, it is assumed that rate of local awareness (general public awareness) is proportional to the visibility of the susceptible pests and increases with a rate σ . Also, awareness campaign through the global factors such as radio, TV, etc. can increase the level of awareness at a rate γ . In addition, there could be fading of interest in this exploitation, at a rate η .

Once susceptible pests feed on plants that have been treated by this or similar viruses, they will ingest virus particles, which will make them infected. In this model, we consider a scenario, where plants are treated (using spraying or soaking) with such bio-pesticides that only affect insect pests causing in them a persistent and fatal infection.

susceptible pests if once infected are assumed lesser harmful for crop biomass. The aware people infect the pest population and the infected population does not recover or become immune. The infection rate is governed by the mass-action incidence.

Here we realize a huge number of pests for the crop field (saturation occur). Type-II functional response is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food.

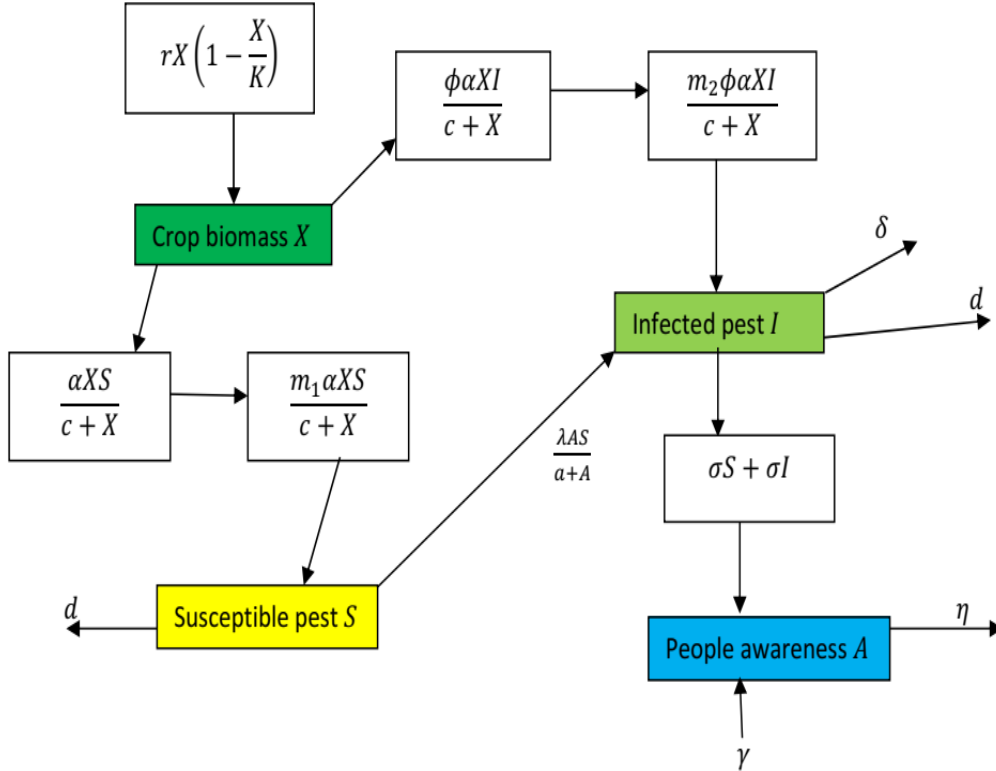


Figure 4.1: Schematic diagram for the crop pest control dynamics

Based on the above assumptions, we have the following mathematical model governed by ordinary differential equations:

$$\begin{cases} \frac{dX}{dt} = rX \left[1 - \frac{X}{K} \right] - \frac{\alpha XS}{c+X} - \frac{\phi \alpha XI}{c+X}, \\ \frac{dS}{dt} = \frac{m_1 \alpha XS}{c+X} - \frac{\lambda AS}{a+A} - dS, \\ \frac{dI}{dt} = \frac{m_2 \phi \alpha XI}{c+X} + \frac{\lambda AS}{a+A} - (d + \delta)I, \\ \frac{dA}{dt} = \gamma + \sigma(S+I) - \eta A, \end{cases} \quad (4.1)$$

with initial conditions $X(0) > 0, S(0) > 0, I(0) > 0, A(0) > 0$.

Here, α is the consumption rate of pests on crops, the infected pest may also attack the crop but at a lesser rate, $\phi \alpha$, with $\phi < 1$, a and c are the half-saturation constants, m_1 , m_2 are the "conversion efficiency" of the susceptible and the infected pest respectively i.e. how proficiently can the pests utilize plant supply. As pesticide affecting pests have lesser efficiency, $m_1 > m_2$, and γ denotes the increase of level from a global advertisement by radio, TV, etc. It is clear to assume that all the parameters are non-negative.

4.2 Model analysis

In this part, some essential properties of solutions of the system (4.1) are justified. In reality, we show non-negative in-variance and limits of results. From a biological point of view, positivity entails the continued existence of the populations. Therefore, some basic properties such as positive in-variance and limits of the solutions of the system (4.1) are demonstrated.

4.2.1 Positivity of solutions and the invariant region

Feasibility and positivity of the solutions are the basic properties of equation (4.1). The practical result explains the region in which the solution of the equations is biologically meaningful. The positivity of the solutions illustrates the non-negativity of the equation. All state variables remain non-negative since they represent crop and pest population with their feasible region given by Theorem 4.2.2. To show that all the solutions of our system (4.1) are positive we use the following lemma from (Kar et al., 2013).

Lemma 4.2.1. *Any solution of the differential equation $\frac{dX}{dt} = X\psi(X, Y)$ is always positive if $\psi(X, Y)$ is integrable.*

Proof. The differential equation of the form $\frac{dX}{dt} = X\psi(X, Y)$ can be written as $\frac{dX}{X} = \psi(X, Y)dt$. Integrating we can write $\ln X = C_0 + \int \psi(X, Y)dt$ i.e., we have,

$$X = C_1 \exp\left(\int \psi(X, Y)dt\right) > 0 \quad \text{for } C_1 > 0.$$

Hence the proof. Now we prove the positivity of the solutions of the system (4.1). $\square$

Theorem 4.2.1. *The solutions of system (4.1) together with the initial conditions remain non-negative for all $t > 0$.*

Proof. It follows from the first equation of system (4.1) that

$$\frac{dX}{dt} = X \left[r \left(1 - \frac{X}{K} \right) - \frac{\alpha S + \phi \alpha I}{c + X} \right],$$

separating variables and integrating from 0 to t , we obtain

$$\ln X = D_0 + \int_0^t \left[r \left(1 - \frac{X}{K} \right) - \frac{\alpha S + \phi \alpha I}{c + X} \right] dt.$$

Solving,

$$X(t) = D_1 \exp \left(\int_0^t \left[r \left(1 - \frac{X}{K} \right) - \frac{\alpha S + \phi \alpha I}{c + X} \right] dt \right) > 0 \quad (\because D_1 = e^{D_0}).$$

From the second equation of system (4.1) we have

$$\frac{dS}{dt} = S \left[\frac{m_1 \alpha X}{c + X} - \frac{\lambda A}{a + A} - d \right].$$

By the same approach, we get

$$\ln S = K_0 + \int_0^t \left(\frac{m_1 \alpha X}{c + X} - \frac{\lambda A}{a + A} - d \right) dt$$

So that

$$S(t) = K_1 \exp \left(\int_0^t \left[\frac{m_1 \alpha X}{c + X} - \frac{\lambda A}{a + A} - d \right] dt \right) > 0.$$

To show that I and A are non-negative, consider the sub-system of (4.1)

$$\begin{cases} \frac{dI}{dt} = \frac{m_2 \phi \alpha X I}{c + X} + \frac{\lambda A S}{a + A} - (d + \delta) I, \\ \frac{dA}{dt} = \gamma + \sigma (S + I) - \eta A. \end{cases} \quad (4.2)$$

In this case use proof by contradiction to show positivity of $I(t)$. Suppose there exists $t_0 \in (0, T)$ such that $I(t_0) = 0$, $I'(t_0) \leq 0$ and $I(t) > 0$, for $t \in [0, t_0)$. Then $A_0 > 0$, for $t \in [0, t_0)$. If this not be the case then there exists $t_1 \in [0, t_0)$ such that $A(t_1) = 0$, $A'(t_1) \leq 0$ and $A(t) > 0$,

for $t \in [0, t_0)$. Integrating the third equation of system(4.1) gives

$$\begin{aligned} I(t) = & I(0) \exp \left[m_2 \phi \alpha \int_0^t \left(\frac{X(t)}{c+X(t)} dt \right) - (d+\delta)t \right] \\ & + \left[\exp \left(m_2 \phi \alpha \int_0^t \left(\frac{X(t)}{c+X(t)} dt \right) - (d+\delta)t \right) \right] \\ & \times \left[\int_0^t \frac{\lambda A(t)S(t)}{a+A(t)} dt \exp \left((d+\delta)t - m_2 \phi \alpha \int_0^t \left(\frac{X(t)}{c+X(t)} dt \right) \right) \right] > 0, \quad \text{for } t \in [0, t_1] \end{aligned}$$

This implies $A'(t_1) = \gamma + \sigma(S(t_1) + I(t_1)) > 0$. This is a contradiction. Hence $I(t) > 0, \forall t \in [0, t_0)$.

Finally from the second equation of the subsystem (4.2) we have

$$\frac{dA}{dt} = \gamma + \sigma(S+I) - \eta A.$$

By similar technique

$$A(t) = A(0) \exp(\eta t) + \exp(\eta t) \int_0^t (\gamma + \sigma(S(\tau) + I(\tau))) \exp(-\eta \tau) d\tau > 0.$$

That is, $A(t) > 0$, for all $t \in (0, T)$. □

Theorem 4.2.2. Every solution of system (4.1) are uniformly bounded, in the region Θ given by

$$\Theta = \left\{ (X, S, I, A) \in \mathbb{R}_+^4 : 0 \leq X + S + I \leq \frac{\Omega}{d}, 0 \leq A \leq \frac{\gamma d + \sigma \Omega}{\eta d} \right\}, \quad \text{with } \Omega = \frac{K(r+d)^2}{4r}.$$

Proof. In our assumption on the model formulation since pesticides affecting pest have lesser efficiency we assumed, $m_1 > m_2$. Now at any time t let $W = X + \frac{1}{m_1}(S + I)$, then the time derivative of W along the solution of system (4.1) is given by

$$\begin{aligned} \frac{dW(t)}{dt} = & rX \left[1 - \frac{X}{K} \right] - \frac{\alpha XS}{c+X} - \frac{\phi \alpha XI}{a+X} + \frac{\alpha XS}{c+X} + \frac{m_2 \phi \alpha XI}{m_1 a+X} - \frac{d}{m_1} S - \left(\frac{d+\delta}{m_1} \right) I, \\ = & rX \left[1 - \frac{X}{K} \right] - \left(\frac{m_1 - m_2}{m_1} \right) \frac{\phi \alpha XI}{c+X} - \left(\frac{d+\delta}{m_1} \right) I - \frac{d}{m_1} S, \\ \leq & rX \left[1 - \frac{X}{K} \right] - \frac{d}{m_1} S - \frac{d}{m_1} I = rX \left(1 - \frac{X}{K} \right) - \frac{d}{m_1} (S+I) - dX + dX, \\ = & rX \left(1 - \frac{X}{K} \right) - \left(\frac{d}{m_1} S + \frac{d}{m_1} I + dX \right) + dX. \end{aligned}$$

Then from the above we have

$$\frac{dW}{dt} \leq rX \left(1 - \frac{X}{K} \right) - dW + dX.$$

That is

$$\frac{dW}{dt} + dW \leq rX \left(1 - \frac{X}{K}\right) + dX = (r+d)X - \frac{rX^2}{K} = \Phi(X).$$

Hence $\Phi(X) = (r+d)X - \frac{rX^2}{K}$ is a quadratic equation whose maximum value equals

$$\frac{1}{4r}K(r+d)^2 = \Omega \quad (\text{say}).$$

Thus we have a constant $\Omega = \frac{K(r+d)^2}{4r}$ such that $\frac{dW}{dt} + dW \leq \Omega$. Now we can apply the theory of differential inequality as follows;

$$e^{dt} \left(\frac{dW}{dt} + dW \right) \leq e^{dt} \Omega \Rightarrow \frac{d}{dt} (We^{dt}) \leq e^{dt} \Omega$$

Integrating gives

$$We^{dt} \leq \int (\Omega e^{dt}) dt + C \quad \text{or} \quad W(X, S, I) \leq e^{-dt} \left[d\Omega e^{dt} \right] + Ce^{-dt}$$

At initial conditions

$$\begin{aligned} W(X(0), S(0), I(0)) &\leq \frac{\Omega}{d} + C \\ \therefore W(X, S, I) &\leq \frac{\Omega}{d} (1 - e^{-dt}) + W(X(0), S(0), I(0))e^{-dt}. \end{aligned}$$

So we get

$$0 < W(X, S, I) \leq \frac{\Omega}{d} (1 - e^{-dt}) + W(X(0), S(0), I(0))e^{-dt}.$$

Thus for $t \rightarrow \infty$, we obtain $0 < X + S + I \leq \frac{\Omega}{d}$. Finally from last equation of system (4.1) we have

$$\frac{dA}{dt} = \gamma + \sigma(S + I) - \eta A \leq \gamma + \sigma\left(\frac{\Omega}{d}\right) - \eta A = \frac{d\gamma + \sigma\Omega}{d} - \eta A.$$

Equivalently,

$$\frac{dA}{dt} + \eta A \leq \frac{d\gamma + \sigma\Omega}{d}.$$

Similarly from differential inequality we have

$$e^{\eta t} \left(\frac{dA}{dt} + \eta A \right) \leq e^{\eta t} \left(\frac{d\gamma + \sigma\Omega}{d} \right) \Rightarrow \frac{d}{dt} (Ae^{\eta t}) \leq e^{\eta t} \left(\frac{d\gamma + \sigma\Omega}{d} \right).$$

Integrating and simplifying we obtain that

$$\begin{aligned} Ae^{\eta t} &\leq \int \left(\frac{d\gamma + \sigma\Omega}{d} \right) e^{\eta t} dt + C \Rightarrow A(t) \leq \left(\frac{d\gamma + \sigma\Omega}{\eta d} \right) + Ce^{-\eta t}, \\ \Rightarrow \limsup_{t \rightarrow \infty} A(t) &\leq \frac{\gamma d + \Omega \sigma}{\eta d}. \end{aligned}$$

Therefore, all solutions of system (4.1) are attracted to the region of attraction Θ . $\square$

4.2.2 Equilibria and their existence criteria

To get equilibrium points, we put the right hand sides of system (4.1) to 0:

$$\begin{cases} rX \left[1 - \frac{X}{K} \right] - \frac{\alpha XS}{c+X} - \frac{\phi \alpha XI}{c+X} = 0, \\ \frac{m_1 \alpha XS}{c+X} - \frac{\lambda AS}{a+A} - dS = 0, \\ \frac{m_2 \phi \alpha XI}{c+X} + \frac{\lambda AS}{a+A} - (d + \delta)I = 0, \\ \gamma + \sigma(S + I) - \eta A = 0. \end{cases} \quad (4.3)$$

The system (4.1) exhibits four equilibrium points.

- i. The axial equilibrium point $E_0(0, 0, 0, \frac{\gamma}{\eta})$ always exists,
- ii. The pest free equilibrium point $E_1\left(K, 0, 0, \frac{\gamma}{\eta}\right)$ always exists,
- iii. The healthy pest free equilibrium point $E_2\left(\bar{X}, 0, \bar{I}, \bar{A}\right)$.

If we set $S = 0$ in system (4.3), then we get

$$\begin{cases} rX \left[1 - \frac{X}{K} \right] - \frac{\phi \alpha XI}{c+X} = 0, \\ \frac{m_2 \phi \alpha XI}{c+X} - (d + \delta)I = 0, \\ \gamma + \sigma I - \eta A = 0. \end{cases} \quad (4.4)$$

From the second equation of system (4.4) we obtain

$$\frac{m_2 \phi \alpha X}{c+X} - (d + \delta) = 0, \frac{m_2 \phi \alpha X}{c+X} = (d + \delta).$$

This implies

$$m_2 \phi \alpha X = (d + \delta)(c + X), m_2 \phi \alpha X = (d + \delta)X + (d + \delta)c.$$

Solving this we obtain

$$\bar{X} = \frac{(d + \delta)c}{m_2 \phi \alpha - (d + \delta)}.$$

Further from the first equation of (4.4) we have

$$r \left[1 - \frac{X}{K} \right] - \frac{\phi \alpha I}{c+X} = 0, \frac{\phi \alpha I}{c+\bar{X}} = \frac{r(K - \bar{X})}{K}.$$

Again solving for I we have

$$\bar{I} = \frac{r(c + \bar{X})(K - \bar{X})}{\phi \alpha K}.$$

From the third equation of system (4.4) we get

$$\bar{A} = \frac{\gamma + \sigma \bar{I}}{\eta}.$$

Hence, the healthy pest free equilibrium point is $(\bar{X}, 0, \bar{I}, \bar{A})$ where

$$\begin{aligned}\bar{X} &= \frac{c(d + \delta)}{m_2 \phi \alpha - (d + \delta)}, \\ \bar{I} &= \frac{r(c + \bar{X})(K - \bar{X})}{\phi \alpha K} = \frac{K \alpha \phi m_2 - (K + c)(d + \delta)}{K(\alpha \phi m_2 - d - \delta)^2} c r m_2, \\ \bar{A} &= \frac{\gamma + \sigma \bar{I}}{\eta} = \frac{K \gamma (\phi \alpha m_2 - (d + \delta))^2 + K \phi \alpha \sigma m_2^2 c r - (K + c)(d + \delta) c r m_2 \sigma}{K \eta (\phi \alpha m_2 - (d + \delta))^2}.\end{aligned}$$

E_2 exists if

$$d + \delta < \frac{m_2 \phi \alpha}{c + K}.$$

This indicates that if the death rate of the infected pest is low, then E_2 exists. Equivalently it means that if the conversion factor m_2 of the infected pest is high, then E_2 exists.

iv. The coexistence equilibrium

In the presence of pest, $X(t) > 0, S(t) > 0, I(t) > 0, A(t) > 0$. Our model system (4.1) has an equilibrium point called coexistence or endemic equilibrium point which is denoted by $E^*(X^*, S^*, I^*, A^*) \neq 0$. E^* is the steady state solution where pest persist in the crop biomass population. It is obtained by solving (4.3).

From the second equation of system (4.3) we get

$$\left(\frac{m_1 \alpha X}{c + X} - \frac{\lambda A}{a + A} - d \right) S = 0 \Rightarrow \frac{m_1 \alpha X}{c + X} - \frac{\lambda A}{a + A} - d = 0 \text{ from which we obtain}$$

$$A^* = \frac{a((\alpha m_1 - d)X^* - cd)}{(-\alpha m_1 + d + \lambda)X^* + c(d + \lambda)}.$$

From the first equation of system (4.3), we have

$$r \left(1 - \frac{X}{K} \right) - \frac{\alpha S}{c + X} - \frac{\phi \alpha I}{c + X} = 0, \alpha S + \phi \alpha I = \frac{r(K - X)(c + X)}{K}. \quad (4.5)$$

From the last equation of system (4.3) we have

$$\gamma + \sigma(S + I) - \eta A = 0. \quad (4.6)$$

Solving equations (4.5) and (4.6) simultaneously, we get

$$\begin{cases} S^* = \frac{\alpha K \phi (\eta A - \gamma) - \sigma r (K - X^*) (c + X^*)}{\sigma \alpha K (\phi - 1)}, \\ I^* = \frac{r \sigma (K - X^*) (c + X^*) - \alpha K (\eta A^* - \gamma)}{\sigma \alpha K (\phi - 1)}. \end{cases}$$

Therefore $E^* = (X^*, S^*, I^*, A^*)$ is the coexistence steady state, where

$$\begin{aligned} A^* &= \frac{a((\alpha m_1 - d)X^* - cd)}{(-\alpha m_1 + d + \lambda)X^* + c(d + \lambda)}, \\ S^* &= \frac{\alpha K \phi (\eta A^* - \gamma) - \sigma r (K - X^*) (c + X^*)}{\sigma \alpha K (\phi - 1)}, \\ I^* &= \frac{r \sigma (K - X^*) (c + X^*) - \alpha K (\eta A^* - \gamma)}{\sigma \alpha K (\phi - 1)}, \end{aligned}$$

and X^* is the positive root of the following equation

$$f(X) = a_0 X^4 + a_1 X^3 + a_2 X^2 + a_3 X + a_4 = 0. \quad (4.7)$$

whose coefficients are given by (we used maple18)

$$\begin{aligned} a_0 &= ((-\phi m_2 + m_1) \alpha + \delta) (-\alpha m_1 + d + \lambda) r \sigma, \\ a_1 &= \alpha \sigma \phi m_2 r (-m_1 (K - c) \alpha + (d + \lambda) (K - 2c)) \\ &\quad + (\alpha^2 (K - c) m_1^2 - \alpha (d - \delta + \lambda) (K - 2c) m_1 - \delta (d + \lambda) (K - 3c)) \sigma r \\ a_2 &= m_1 (m_1 - m_2) (a \eta + \gamma) \phi K \alpha^3 + (\sigma m_1^2 c r + ((d + \lambda) \gamma + a d \eta) m_2 \phi) K \alpha^2 \\ &\quad - ((m_2 c r \sigma + (\lambda + 2d) \gamma + 2a d \eta) \phi - (d + \delta) (a \eta + \gamma)) m_1 K \alpha^2 \\ &\quad + \left(2 r c \sigma ((-d + \delta - \lambda) m_1 + m_2 \phi (d + \lambda)) \left(K - \frac{c}{2} \right) \right) \alpha \\ &\quad + (((a \eta + \gamma) d + \lambda \gamma) ((\phi - 1) d - \delta) K \alpha - 3 c \delta r \sigma (d + \lambda) (K - c)) \\ a_3 &= ((d + \lambda) \gamma + a d \eta) m_2 \phi c \alpha^2 K \\ &\quad - (((\lambda + 2d) \phi - d - \delta) \gamma + a \eta (2d \phi - d - \delta)) m_1 c \alpha^2 K \\ &\quad + r ((-d + \delta - \lambda) m_1 + m_2 \phi (d + \lambda)) \sigma c^2 \alpha K \\ &\quad + 2 ((\phi - 1) d - \delta) ((a \eta + \gamma) d + \lambda \gamma) c \alpha K \\ &\quad - 3 \sigma (K - c/3) c^2 r (d + \lambda) \delta \\ a_4 &= c^2 ((a \eta + \gamma) d + \lambda \gamma) ((\phi - 1) d - \delta) K \alpha - K c^3 \delta r \sigma (d + \lambda) \end{aligned}$$

The nature of the roots for (4.7) is determined by noting the sign of its discriminant. Therefore, a sufficient condition that guarantees that (4.7) has at least one positive root is $a_4 < 0$ which leads to $((a \eta + \gamma) d + \lambda \gamma) ((\phi - 1) d - \delta) \alpha < c \delta r \sigma (d + \lambda)$. Since we assume that $\phi < 1$ a unique coexistence equilibrium point E^* always exists. For a

feasible equilibrium(positive S^* and I^*) we also need

$$\frac{r(c+X^*)(K-X^*)\sigma + K\alpha\gamma}{K\alpha\eta} < A^* < \frac{r(c+X^*)(K-X^*)\sigma + K\alpha\gamma\phi}{K\phi\alpha\eta}.$$

4.2.3 Stability of the equilibria

The stability analysis for system (4.1) is studied by linearization of the non-linear system (4.1) to the Jacobian matrix J at the fixed points or equilibrium point of the system and computing the characteristic equation. Hence stability of the equilibrium point is studied depending on the eigenvalues of the corresponding Jacobian, which are functions of the model structures(parameters).

The Jacobian matrix for the system (4.1) at any point is given by

$$J(X, S, I, A) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial A} \\ \frac{\partial f_4}{\partial X} & \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial A} \end{bmatrix}.$$

At any equilibrium point it becomes

$$\begin{bmatrix} r \left[1 - \frac{2X}{K} \right] - \frac{\alpha c S}{(c+X)^2} - \frac{\phi \alpha c I}{(c+X)^2} & -\frac{\alpha X}{c+X} & -\frac{\phi \alpha X}{c+X} & 0 \\ \frac{m_1 \alpha c S}{(c+X)^2} & \frac{m_1 \alpha X}{c+X} - \frac{\lambda A}{a+A} - d & 0 & -\frac{\lambda a S}{(a+A)^2} \\ \frac{m_2 \phi \alpha c I}{(c+X)^2} & \frac{\lambda A}{a+A} & \frac{m_2 \phi \alpha X}{c+X} - d - \delta & \frac{\lambda a S}{(a+A)^2} \\ 0 & \sigma & \sigma & -\eta \end{bmatrix}$$

where

$$\begin{aligned} f_1 &= rX \left[1 - \frac{X}{K} \right] - \frac{\alpha XS}{c+X} - \frac{\phi \alpha XI}{c+X}, \quad f_2 = \frac{m_1 \alpha XS}{c+X} - \frac{\lambda AS}{a+A} - dS, \\ f_3 &= \frac{m_2 \phi \alpha XI}{c+X} + \frac{\lambda AS}{a+A} - (d+\delta)I, \quad f_4 = \gamma + \sigma(S+I) - \eta A. \end{aligned}$$

Theorem 4.2.3. *The crop–pest free equilibrium point E_0 is unstable.*

Proof. We linearize the system (4.1) using the Jacobian technique. At the axial equilibrium E_0 , the Jacobian matrix is

$$J \left(0, 0, 0, \frac{\gamma}{\eta} \right) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\frac{\lambda \gamma}{a\eta + \gamma} - d & 0 & 0 \\ 0 & \frac{\lambda \gamma}{a\eta + \gamma} & -d - \delta & 0 \\ 0 & \sigma & \sigma & -\eta \end{bmatrix} \quad (4.8)$$

whose characteristic equation is given by

$$|\rho - J(E_0)| = \begin{vmatrix} \rho - r & 0 & 0 & 0 \\ 0 & \rho - \left(\frac{\lambda\gamma}{a\eta + \gamma} + d\right) & 0 & 0 \\ 0 & -\frac{\lambda\gamma}{a\eta + \gamma} & \rho - (d + \delta) & 0 \\ 0 & -\sigma & -\sigma & \rho + \eta \end{vmatrix} = 0. \quad (4.9)$$

This implies

$$(\rho - r) \left(\rho + \frac{\lambda\gamma}{a\eta + \gamma} + d \right) (\rho + d + \delta) (\rho + \eta) = 0.$$

The corresponding eigenvalues are

$$\rho_1 = r > 0, \rho_2 = -\eta < 0, \rho_3 = -d - \delta < 0, \rho_4 = \frac{-\lambda\gamma}{a\eta + \gamma} - d < 0.$$

Since one eigenvalue $\rho_1 = r > 0$, the axial equilibrium E_0 is always unstable. $\square$

Theorem 4.2.4. *The pest free equilibrium point E_1 is locally asymptotically stable if*

$$R_{01} = \frac{m_1 \alpha K (a\eta + \gamma)}{(\lambda\gamma + d(a\eta + \gamma))(c + K)} < 1 \quad \text{and} \quad R_{02} = \frac{m_2 \phi \alpha K}{(c + K)(d + \delta)} < 1.$$

And unstable if

$$R_{01} > 1 \quad \text{or} \quad R_{02} > 1.$$

Proof. We linearize the system (4.1) using the Jacobian technique. At the pest free equilibrium point $E_1(K, 0, 0, \frac{\gamma}{\eta})$, the Jacobian matrix is

$$J(K, 0, 0, \frac{\gamma}{\eta}) = \begin{bmatrix} -r & -\frac{\alpha K}{c+K} & -\frac{\phi \alpha K}{c+K} & 0 \\ 0 & \frac{m_1 \alpha K}{c+K} - \frac{\lambda\gamma}{a\eta + \gamma} - d & 0 & 0 \\ 0 & \frac{\lambda\gamma}{a\eta + \gamma} & \frac{m_2 \phi \alpha K}{c+K} - d - \delta & 0 \\ 0 & \sigma & \sigma & -\eta \end{bmatrix}. \quad (4.10)$$

It is clear from equation (4.10) that the first and second eigenvalues are $\rho_1 = -r$ and $\rho_2 = -\eta$. Therefore the matrix (4.10) reduces to a 2×2 matrix shown in matrix below.

$$J(K, 0, 0, \frac{\gamma}{\eta}) = \begin{bmatrix} \frac{m_1 \alpha K}{c+K} - \frac{\lambda\gamma}{a\eta + \gamma} - d & 0 \\ \frac{\lambda\gamma}{a\eta + \gamma} & \frac{m_2 \phi \alpha K}{c+K} - d - \delta \end{bmatrix}. \quad (4.11)$$

The characteristic equation of (4.11) is found from $\det(\rho I - J(E_1)) = 0$, that is

$$|\rho I - J(E_1)| = \begin{vmatrix} \rho - \left(\frac{m_1 \alpha K}{c+K} - \frac{\lambda\gamma}{a\eta + \gamma} - d\right) & 0 \\ -\frac{\lambda\gamma}{a\eta + \gamma} & \rho - \left(\frac{m_2 \phi \alpha K}{c+K} - d - \delta\right) \end{vmatrix} = 0,$$

which gives

$$\left(\rho - \frac{m_1 \alpha K}{c+K} + \frac{\lambda \gamma}{a\eta + \gamma} + d\right) \left(\rho - \frac{m_2 \phi \alpha K}{c+K} + d + \delta\right) = 0.$$

Thus, the eigenvalues are $-r$, $\frac{m_1 \alpha K}{c+K} - \frac{\lambda \gamma}{a\eta + \gamma} - d$, $\frac{m_2 \phi \alpha K}{c+K} - d - \delta$ and $-\eta$.

Clearly, $J(E_1)$ has two negative values namely, $-r$ and $-\eta$.

Therefore, E_1 is stable if and only if

$\frac{m_1 \alpha K}{c+K} < \frac{\lambda \gamma}{a\eta + \gamma} + d$ and $\frac{m_2 \phi \alpha K}{c+K} < d + \delta$. That is

$$R_{01} = \frac{m_1 \alpha K (a\eta + \gamma)}{(c+K)(\lambda \gamma + d(a\eta + \gamma))} < 1,$$

and

$$R_{02} = \frac{m_2 \phi \alpha K}{(c+K)(d + \delta)} < 1.$$

Otherwise, E_1 is unstable if

$$R_{01} > 1 \quad \text{or} \quad R_{02} > 1.$$

□

The conditions in the stability of the pest free equilibrium point E_1 indicates that if the death rates of the pest population are high, then the system may stabilize to the pest free steady state. Further, it can be noticed that existence of E_2 destabilizes E_1 .

Theorem 4.2.5. *The healthy pest free equilibrium E_2 is locally asymptotically stable if*

$$\frac{m_1(d + \delta)(a + \bar{A})}{(a + \bar{A})(\lambda + d) - \lambda a} < m_2 \quad \text{and} \quad R_{02} > 1.$$

Proof. At the healthy pest free equilibrium point $E_2(\bar{X}, 0, \bar{I}, \bar{A})$, the Jacobian matrix $J(E_2)$ is given by

$$J(E_2) = \begin{bmatrix} F_{11} & -\frac{\alpha \bar{X}}{c + \bar{X}} & -\frac{\phi \alpha \bar{X}}{c + \bar{X}} & 0 \\ 0 & F_{22} & 0 & 0 \\ \frac{m_2 \phi \alpha c \bar{I}}{(c + \bar{X})^2} & \frac{\lambda \bar{A}}{a + \bar{A}} & F_{33} & 0 \\ 0 & \sigma & \sigma & -\eta \end{bmatrix}$$

where

$$\begin{aligned} F_{11} &= r \left(1 - \frac{2\bar{X}}{K} \right) - \frac{\phi \alpha c \bar{I}}{(c + \bar{X})^2}, \\ F_{22} &= \frac{m_1 \alpha \bar{X}}{c + \bar{X}} - \frac{\lambda \bar{A}}{a + \bar{A}} - d, \\ F_{33} &= \frac{m_2 \phi \alpha \bar{X}}{c + \bar{X}} - d - \delta, \end{aligned}$$

with the characteristic equation in ρ given by

$$|\rho I - J(E_2)| = \begin{vmatrix} \rho - F_{11} & \frac{\alpha \bar{X}}{c + \bar{X}} & \frac{\phi \alpha \bar{X}}{c + \bar{X}} & 0 \\ 0 & \rho - F_{22} & 0 & 0 \\ -\frac{m_2 \phi \alpha c \bar{I}}{(c + \bar{X})^2} & -\frac{\lambda \bar{A}}{a + \bar{A}} & \rho - F_{33} & 0 \\ 0 & -\sigma & -\sigma & \rho + \eta \end{vmatrix} = 0.$$

This gives

$$(\rho + \eta)(\rho - F_{22}) \left[\rho^2 - (F_{11} + F_{33})\rho + F_{11}F_{33} + \frac{m_2 \phi \alpha c \bar{I} \phi \alpha \bar{X}}{(c + \bar{X})^3} \right] = 0,$$

Equivalently

$$(\rho + \eta)(\rho - F_{22}) \left[\rho^2 - (F_{11})\rho + \frac{m_2 \phi \alpha c \bar{I} \phi \alpha \bar{X}}{(c + \bar{X})^3} \right] = 0,$$

$$(\rho + \eta)(\rho - F_{22}) \left[\rho^2 + (-F_{11})\rho + \frac{m_2 \phi \alpha c \bar{I} \phi \alpha \bar{X}}{(c + \bar{X})^3} \right] = 0,$$

or

$$(\rho^2 + B\rho + C)(\rho + \eta)(\rho - F_{22}) = 0,$$

where $B = -F_{11}$, and $C = \frac{m_2 \phi \alpha c \bar{I} \phi \alpha \bar{X}}{(c + \bar{X})^3} > 0$.

The corresponding eigenvalues are

$$\rho_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}, \rho_3 = F_{22} \text{ and } \rho_4 = -\eta.$$

From the Routh-Hurwitz criteria, with a second degree polynomial, i.e, $\rho^2 + a_1\rho + a_2 = 0$, the necessary and sufficient condition for the local stability of the system is, that all eigenvalues must have negative real part(the condition $a_1 > 0, a_2 > 0$ must hold). Therefore, in our case if $B > 0$, then ρ_1 and ρ_2 are negative(since $C > 0$).

This implies that E_2 is locally asymptotically stable if $F_{11} < 0$ and $F_{22} < 0$. That is

$$\begin{aligned} r \left(1 - \frac{2\bar{X}}{K} \right) - \frac{\phi \alpha c \bar{I}}{(c + \bar{X})^2} &< 0, \\ 1 + \frac{c(d + \delta)}{(-m_2 \phi \alpha + d + \delta)K} + \frac{((-m_2 \phi \alpha + d + \delta)K + c(d + \delta))r\sigma}{K\alpha \phi m_2} &< 0, \\ \frac{(-K\alpha \phi m_2 + (c + K)(d + \delta))(-\phi \alpha (r\sigma - 1)m_2 + r\sigma(d + \delta))}{(-\phi \alpha m_2 + d + \delta)K\alpha \phi m_2} &< 0. \end{aligned}$$

This means $R_{02} > 1$, and

$$F_{22} < 0 \Leftrightarrow \frac{m_1 \alpha \bar{X}}{c + \bar{X}} - \frac{\lambda \bar{A}}{a + \bar{A}} - d < 0, \quad \frac{m_1 \alpha \bar{X}}{c + \bar{X}} < \frac{\lambda \bar{A}}{a + \bar{A}}, \quad \frac{m_1(d + \delta)(a + \bar{A})}{(a + \bar{A})(\lambda + d) - \lambda a} < m_2 \phi.$$

Hence E_2 is locally asymptotically stable provided that $\frac{m_1(d+\delta)(a+\bar{A})}{(a+\bar{A})(\lambda+d)-\lambda a} < m_2\phi$ and $R_{02} > 1$ where we can substitute $\bar{A}$ with $\frac{K\gamma(\phi\alpha m_2-(d+\delta))^2+K\phi\alpha\sigma m_2^2cr-(K+c)(d+\delta)crm_2\sigma}{(\phi\alpha m_2-(d+\delta))^2}$ to get the expression in parameter values only. $\square$

From this relation we can summarize that when the conversion rate(m_2) of the pest governs a moderate value and the pest infection rate (λ) is high, then it is expected that the system will stabilize at the steady state when all the pest become infected.

Theorem 4.2.6. *The coexistence equilibrium point E^* is locally asymptotically stable if the conditions*

$$\begin{aligned} C_1 &> 0, C_2 > 0, C_3 > 0, C_4 > 0, \\ C_1 C_2 - C_3 &> 0 \\ C_1 C_2 C_3 - C_3^2 - C_1^2 C_4 &> 0, \end{aligned} \tag{4.12}$$

are satisfied, where C_i , $i = 1, 2, 3, 4$ are given below.

Proof. The Jacobian matrix $J(E^*)$, at the coexistence equilibrium point E^* is computed as

$$J(E^*) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & 0 & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix}.$$

The characteristic equation in ρ for the Jacobian matrix $J(E^*)$ is given by

$$\rho^4 + C_1\rho^3 + C_2\rho^2 + C_3\rho + C_4 = 0, \tag{4.13}$$

where

$$\begin{aligned} C_1 &= -(m_{11} + m_{22} + m_{33} + m_{44}). \\ C_2 &= m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} - m_{12}m_{21} - m_{13}m_{31} + m_{22}m_{33} \\ &\quad + m_{22}m_{44} - m_{24}m_{42} + m_{33}m_{44} - m_{34}m_{43}. \\ C_3 &= -m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} + m_{11}m_{24}m_{42} - m_{11}m_{33}m_{44} + m_{11}m_{34}m_{43} + m_{12}m_{21}m_{33} \\ &\quad + m_{12}m_{21}m_{44} - m_{13}m_{21}m_{32} + m_{13}m_{22}m_{31} + m_{13}m_{31}m_{44} \\ &\quad - m_{22}m_{33}m_{44} + m_{22}m_{34}m_{43} - m_{24}m_{32}m_{43} + m_{24}m_{33}m_{42}. \\ C_4 &= m_{11}m_{22}m_{33}m_{44} - m_{11}m_{22}m_{34}m_{43} + m_{11}m_{24}m_{32}m_{43} - m_{11}m_{24}m_{33}m_{42} \\ &\quad - m_{12}m_{21}m_{33}m_{44} + m_{12}m_{21}m_{34}m_{43} - m_{12}m_{24}m_{31}m_{43} + m_{13}m_{21}m_{32}m_{44} \\ &\quad - m_{13}m_{21}m_{34}m_{42} - m_{13}m_{22}m_{31}m_{44} + m_{13}m_{24}m_{31}m_{42}. \end{aligned}$$

and

$$\begin{aligned}
m_{11} &= r \left(1 - \frac{2X^*}{K} \right) - \frac{\alpha c S^*}{(c + X^*)^2} - \frac{\phi \alpha c I^*}{(c + X^*)^2}, \quad m_{12} = -\frac{\alpha X^*}{c + X^*}, \quad m_{13} = -\frac{\phi \alpha X^*}{c + X^*}. \\
m_{21} &= \frac{m_1 \alpha c S^*}{(c + X^*)^2}, \quad m_{22} = \frac{m_1 \alpha X^*}{c + X^*} - \frac{\lambda A^*}{a + A^*} - d, \quad m_{24} = \frac{\lambda a S^*}{(a + A^*)^2}. \\
m_{31} &= \frac{m_2 \phi \alpha c I^*}{(c + X^*)^2}, \quad m_{32} = \frac{\lambda A^*}{a + A^*}, \quad m_{33} = \frac{m_2 \phi \alpha X^*}{c + X^*} - d - \delta, \quad m_{34} = -\frac{\lambda a S^*}{(a + A^*)^2}. \\
m_{42} &= \sigma, \quad m_{43} = \sigma, \quad m_{44} = -\eta.
\end{aligned}$$

According to the Routh-Hurwitz criterion, the characteristic equation (4.13) will have negative roots or imaginary roots with negative real part if the conditions mentioned in Theorem 4.2.6 are satisfied. Thus, the coexistence equilibrium E^* of the system (4.1) will be locally asymptotically stable if the conditions in (4.12) are satisfied.

Due to the complex form of all the $C_i, i = 1, 2, 3$, it is quite unlikely to find an explicit parametric condition for which all those conditions are satisfied. But through simulation works the result is obtained that there exists some threshold of α , say α^* such that for $0 < \alpha < \alpha^*$ all of $C_i, C_1 C_2 C_3 - C_3^2 - C_1^2 C_4$ are positive and as α passes through α^* then $C_1 C_2 C_3 - C_3^2 - C_1^2 C_4$ becomes negative. Hence it is concluded that at $0 < \alpha < (>) \alpha^*, E^*$ is locally asymptotically stable (unstable) and at $\alpha = \alpha^*, C_1 C_2 C_3 - C_3^2 - C_1^2 C_4$ vanishes and so E^* has a pair of purely imaginary eigenvalues for $\alpha = \alpha^*$. Further we verified that at that critical value α^* , the system (4.1) undergoes a Hopf bifurcation as the transversality conditions hold good (more detail is given below). $\square$

We focus on the pest consumption rate α which we considered as the most biologically significant parameter. Hopf-bifurcation of the coexistence steady state E^* may happen if the auxiliary equation (4.13) has a pair of purely imaginary eigenvalues for $\alpha = \alpha^* \in (0, \infty)$ with all other eigenvalues containing negative real parts. For the Hopf bifurcation to normally appear, the transversality condition,

$$\left. \frac{d \operatorname{Re}[\rho(\alpha)]}{d\alpha} \right|_{\alpha^*} \neq 0$$

must be satisfied. Let us define a continuously differentiable function $\Psi : (0, \infty) \rightarrow \mathbb{R}$ of α as below

$$\Psi(\alpha) := C_1(\alpha)C_2(\alpha)C_3(\alpha) - C_3^2(\alpha) - C_4(\alpha)C_1^2(\alpha).$$

Theorem 4.2.7. *The coexistence equilibrium E^* of the system (4.1) enters into a Hopf bifurcation at $\alpha = \alpha^* \in (0, \infty)$ if and only if*

$$C_2(\alpha^*) > 0, C_3(\alpha^*) > 0, C_4(\alpha^*) > 0, C_1(\alpha^*)C_2(\alpha^*) - C_3(\alpha^*) > 0, \Psi(\alpha^*) = 0$$

and

$$C_1^3 C_2' C_3 (C_1 - 3C_3) > 2(C_2 C_1^2 - 2C_3^2)(C_3' C_1^2 - C_1' C_3^2).$$

Moreover, at $\alpha = \alpha^*$, two characteristic eigenvalues $\rho(\alpha)$ are purely imaginary, and the remaining two with negative real parts. Here, primes indicate derivatives with respect to parameter α .

Proof. By the condition $\Psi(\alpha^*) = 0$, the characteristic equation (4.13) can be written as $(\rho^2 + \frac{C_3}{C_1})(\rho^2 + C_1\rho + \frac{C_1 C_4}{C_3}) = 0$.

If it has four roots, say $\rho_i, (i = 1, 2, 3, 4)$ with the pair of purely imaginary roots at $\alpha = \alpha^*$ as $\rho_1 = \bar{\rho}_2$, so, we get

$$\begin{aligned} \rho_3 + \rho_4 &= -C_1, \\ \omega_0^2 + \rho_3 \rho_4 &= C_2, \\ \omega_0^2(\rho_3 + \rho_4) &= -C_3, \\ \omega_0^2 \rho_3 \rho_4 &= C_4, \end{aligned} \tag{4.14}$$

where $\omega_0 = \text{Im}\rho(\alpha^*)$. From the above we have $\omega_0 = \sqrt{\frac{C_3}{C_1}}$. Now, if ρ_3 and ρ_4 are complex conjugate, then from (4.14), it follows that $2\text{Re}(\rho_3) = -C_1$; if they are real roots, then by (4.13) and (4.14) $\rho_3 < 0$ and $\rho_4 < 0$. Now let us verify the transversality condition. As $\Psi(\alpha^*)$ is a continuous function of all its roots, so there exists an open interval $\alpha \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$ where ρ_1 and ρ_2 are complex conjugate for α . Suppose, their general forms in this neighborhood are

$$\rho_1(\alpha) = \chi(\alpha) + iv(\alpha), \quad \rho_2(\alpha) = \chi(\alpha) - iv(\alpha).$$

Now, we will verify the transversality condition

$$\left. \frac{d\text{Re}[\rho_j(\alpha)]}{d\alpha} \right|_{\alpha=\alpha^*} \neq 0, j = 1, 2.$$

Substituting $\rho_j(\alpha) = \chi(\alpha) \pm iv(\alpha)$, into (4.13) and calculating the derivative, we have

$$\begin{aligned} A(\alpha)\chi' - B(\alpha)v'(\alpha) + C(\alpha) &= 0, \\ B(\alpha)\chi' + A(\alpha)v'(\alpha) + D(\alpha) &= 0, \end{aligned}$$

where

$$\begin{aligned} A(\alpha) &= 4\chi^3 - 12\chi v^2 + 3C_1(\chi^2 - v^2) + 2C_2\chi + C_3, \\ B(\alpha) &= 12\chi^2 v + 6C_1\chi v - 4\chi^3 + 2C\chi, \\ C(\alpha) &= (\chi^3 - 3\chi v^2)C_1' + (\chi^2 - v^2)C_2' + \chi C_3' + C_4', \\ D(\alpha) &= (3\chi^2 v - v^3)C_1' + 2\chi v C_2' + v C_3'. \end{aligned}$$

Hence, solving for $\chi'(\alpha^*)$ we get

$$\begin{aligned} \left. \frac{dRe[\rho(\alpha)]}{d\alpha} \right|_{\alpha=\alpha^*} &= \chi'(\alpha)|_{\alpha=\alpha^*}, \\ &= -\frac{B(\alpha^*)D(\alpha^*) + A(\alpha^*)C(\alpha^*)}{A^2(\alpha^*) + B^2(\alpha^*)}, \\ &= \frac{C_1^3 C_2' C_3 (C_1 - 3C_3) - 2(C_2 C_1^2 - 2C_3^2)(C_3' C_1^2 - C_1' C_3^2)}{C_1^4 (C_1 - 3C_3)^2 + 4(C_2 C_1^2 - 2C_3^2)^2} > 0, \end{aligned}$$

This tell us that

$$C_1^3 C_2' C_3 (C_1 - 3C_3) - 2(C_2 C_1^2 - 2C_3^2)(C_3' C_1^2 - C_1' C_3^2) > 0,$$

which means

$$C_1^3 C_2' C_3 (C_1 - 3C_3) > 2(C_2 C_1^2 - 2C_3^2)(C_3' C_1^2 - C_1' C_3^2).$$

Thus, the transversality conditions hold, and hence, Hopf bifurcation occurs at α^* . Numerically the critical number α^* is found to be approximately 0.09. $\square$

4.3 Optimal control

In this part we extend the model system (4.1) by incorporating two time dependent controls $u_1(t)$ and $u_2(t)$. With the first control $u_1(t)$ as an admissible control representing the efficiency of pesticide that can be used and $u_2(t)$ characterizes the cost of awareness campaign. The objective here is to reduce the price of announcement for farming awareness via radio, TV, telephony and other social media. And the price regarding to control measures. Our target here is to find optimal parameters $u_1^*(t)$ and $u_2^*(t)$ respectively using the minimum principle given in (Fleming and Lions, 2012).

Therefore our system (4.1) is modified to the control induced state nonlinear dynamics given by

$$\begin{cases} \frac{dX}{dt} = rX \left[1 - \frac{X}{K}\right] - \frac{\alpha XS}{c+X} - \frac{\phi \alpha XI}{c+X}, \\ \frac{dS}{dt} = \frac{m_1 \alpha XS}{c+X} - u_1 \frac{\lambda AS}{a+A} - dS, \\ \frac{dI}{dt} = \frac{m_2 \phi \alpha XI}{c+X} + u_1 \frac{\lambda AS}{a+A} - (d + \delta)I, \\ \frac{dA}{dt} = u_2 \gamma + \sigma(S + I) - \eta A, \end{cases} \quad (4.15)$$

with $X(0) = X_0, S(0) = S_0, I(0) = I_0$ and $A(0) = A_0$, as initial conditions.

At this point we need to reduce the number of pests and also the price of pest administration by reducing the cost of pesticides and exploiting the stage of awareness. Hence, we describe the price function for the minimization problem as,

$$J(u_1(t), u_2(t)) = \int_0^{t_f} [A_1 S^2(t) - A_2 A^2(t) + \frac{1}{2} B_1 u_1^2(t) + \frac{1}{2} B_2 u_2^2(t)] dt \quad (4.16)$$

issued to the persuaded state system ((4.15)). Here parameters A_1, A_2 represent the penalty multiplier on benefit of the cost and B_1, B_2 stand for weighting constants on the profit of the price of making. The terms $\frac{1}{2}B_1u_1^2(t)$ and $\frac{1}{2}B_2u_2^2(t)$ stand for the cost linked with pest managing and level of awareness. We prefer a quadratic cost functional on the controls as an approximation for nonlinear function depending on the supposition that the cost take nonlinear form and also to prevent the bang bang or singular optimal control cases. The target here is to find the optimal control (u_1^*, u_2^*) such that

$$J(u_1^*(t), u_2^*(t)) = \min(J(u_1(t), u_2(t)) | (u_1, u_2) \in U), \quad (4.17)$$

where

$$U = \left\{ (u_1(t), u_2(t)) | u_i(t) \text{ is Lebesgu measurable and } 0 \leq u_i(t) \leq 1, t \in [0, t_f] \right\}$$

is a control set.

4.3.1 Existence of the optimal control pair

The existence of the optimal control pair can be guaranteed by using the results in the book of (Fleming and Lions, 2012). All the state variables involved in the model are continuously differentiable. Therefore, we need to verify the following combined conditions from (Lenhart and Workman, 2007; Fleming and Lions, 2012).

Theorem 4.3.1. *Given the objective functional*

$$J(u_1(t), u_2(t)) = \int_0^{t_f} [A_1S^2(t) - A_2A^2(t) + \frac{1}{2}B_1u_1^2(t) + \frac{1}{2}B_2u_2^2(t)]dt$$

where

$$U = (u_1(t), u_2(t)) : u_i(t) \text{ is Lebesgue mesearable and } 0 \leq u_i(t) \leq 1, t \in [0, t_f]$$

subject to the control system (4.15) with the initial condition $(X_0, S_0, I_0, A_0) \geq (0, 0, 0, 0)$.

Then there exists an optimal control pair $(u_1^*(t), u_2^*(t))$ and corresponding state solution X^*, S^*, I^*, A^* such that

$J(u_1^*(t), u_2^*(t)) = \min_U J(u_1(t), u_2(t))$ if the following conditions are met:

- (i) the set of solutions to the system (4.15) with control variables in (4.3) are non-empty,
- (ii) The control set U is convex and closed,
- (iii) Each right hand side of the state system(4.15) is continuous, bounded above and can be written as a linear function of u with coefficients depending on time and the state,
- (iv) The integrand $g(t, Y, u) = A_1S^2 - A_2A^2 + \frac{1}{2}B_1u_1^2 + \frac{1}{2}B_2u_2^2$ of the objective functional (4.16) is convex,

(v) There exist positive numbers ℓ_1, ℓ_2, ℓ_3 and a constant $\ell > 1$ such that

$$g(t, S, A, u_1, u_2) \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell$$

Proof. We now justify each of the items stated above to complete the proof.

(i) Since U is a nonempty set of real valued measurable functions on the finite time interval $0 \leq t_f$, the system (4.15) has bounded coefficients and hence any solutions are bounded on $[0, t_f]$ (see the proof on the boundedness). It follows that the corresponding solutions for the system (4.15) exist.

(ii) To prove the convexity of the control set U , let us consider

$$U = \{u \in \mathbb{R}^2 : \|u\| \leq 1\}.$$

Let $u_1, u_2 \in U$ such that $\|u_1\| \leq 1$ and $\|u_2\| \leq 1$. Then for an arbitrary $\lambda \in [0, 1]$, we have

$$\begin{aligned} \|\lambda u_1 + (1 - \lambda) u_2\| &\leq \|\lambda u_1\| + \|(1 - \lambda) u_2\| \\ &= |\lambda| \|u_1\| + (1 - \lambda) \|u_2\| \\ &= \lambda \|u_1\| + (1 - \lambda) \|u_2\| \\ &\leq \lambda + (1 - \lambda) = 1. \end{aligned}$$

Hence $\|\lambda u_1 + (1 - \lambda) u_2\| \leq 1$. This implies that the control set U is convex. The control space $U = \{(u_1, u_2) : u_1, u_2\}$ is measurable, $0 \leq u_{1\min} \leq u_1(t) \leq u_{1\max} \leq 1, 0 \leq u_{2\min} \leq u_2(t) \leq u_{2\max} \leq 1$ is closed by definition. Therefore U is convex and closed set.

iii All the right-hand sides of equations of system (4.15) are continuous, all variables X, S, I, A and u are bounded on $[0, t_f]$, and can be written as a linear function of u_1 and u_2 with coefficients depending on the time and state.

(iv) Let $\lambda \in [0, 1]$ and $a = (a_1, a_2), b = (b_1, b_2) \in U$, then we have

$$\begin{aligned}
& g(t, Y, (1 - \lambda)a + \lambda b) - ((1 - \lambda)g(t, Y, a) + \lambda g(t, Y, b)) \\
&= \frac{B_1}{2} \left[(1 - \lambda)^2 a_1^2 + 2\lambda(1 - \lambda)a_1 b_1 + \lambda^2 b_1^2 \right] \\
&+ \frac{B_2}{2} \left[(1 - \lambda)^2 a_2^2 + 2\lambda(1 - \lambda)a_2 b_2 + \lambda^2 b_2^2 \right] \\
&- \left[(1 - \lambda) \left(\frac{B_1}{2} a_1^2 + \frac{B_2}{2} a_2^2 \right) \right] \\
&- \lambda \left(\frac{B_1}{2} b_1^2 + \frac{B_2}{2} b_2^2 \right) \\
&= (\lambda^2 - \lambda) \left(\frac{B_1}{2} (a_1 - b_1)^2 + \frac{B_2}{2} (a_2 - b_2)^2 \right) \\
&= \frac{(\lambda^2 - \lambda)}{2} (a - b)^2 \\
&= \frac{1}{2} \lambda \underbrace{(\lambda - 1)}_{\leq 0} (a - b)^2 \leq 0
\end{aligned}$$

Hence $g(t, Y, (1 - \lambda)a + \lambda b) \leq (1 - \lambda)g(t, Y, a) + \lambda g(t, Y, b)$. Therefore $g(t, Y, u)$ is convex with regard to u_1 and u_2 .

(v) Finally it remains to show that there exists a constant $\ell > 1$ and positive constants ℓ_1, ℓ_2 and ℓ_3 such that

$$\frac{B_1 u_1^2(t)}{2} + \frac{B_2 u_2^2(t)}{2} + A_1 S^2 - A_2 A^2 \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell.$$

In 4.1, we already show that the state variables are bounded;

let $\ell_1 = \sup (A_1 S^2 - A_2 A^2)$, $\ell_2 = B_1$, $\ell_3 = B_2$ and $\ell = 2$; then it follows that

$$\frac{B_1 u_1^2(t)}{2} + \frac{B_2 u_2^2(t)}{2} + A_1 S^2 - A_2 A^2 \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell.$$

Then, from [Fleming and Lions \(2012\)](#), there exists an optimal control pair.

□

4.3.2 Characterization of the optimal control

Since there exists an optimal control pair for minimizing the functional,

$$J(u_1(t), u_2(t)) = \int_0^{t_f} [A_1 S^2(t) - A_2 A^2(t) + \frac{1}{2} B_1 u_1^2(t) + \frac{1}{2} B_2 u_2^2(t)] dt$$

subject to the system (4.15), we derive necessary conditions on the optimal control pair using the Pontryagin's Minimum Principle ([Fleming and Lions, 2012](#)).

Theorem 4.3.2. Let $u_1^*(t)$ and $u_2^*(t)$ be an optimal control pair that minimizes

$$J(u_1(t), u_2(t)) = \int_0^{t_f} [A_1 S^2(t) - A_2 A^2(t) + \frac{1}{2} B_1 u_1^2(t) + \frac{1}{2} B_2 u_2^2(t)] dt$$

over the control set U and the corresponding solution (X^*, S^*, I^*, A^*) subject to the system (4.15), then there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying the following system of equations

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1 \left(\frac{\alpha c S}{(c+X)^2} + \frac{\phi \alpha c I}{(c+X)^2} - r \left(1 - \frac{2X}{K} \right) \right) - \lambda_2 \left(\frac{m_1 \alpha c S}{(c+X)^2} \right) - \lambda_3 \left(\frac{m_2 \phi \alpha c I}{(c+X)^2} \right), \\ \frac{d\lambda_2}{dt} = -2A_1 S + \lambda_1 \left(\frac{\alpha X}{c+X} \right) + \lambda_2 \left(\frac{u_1 \lambda A}{(a+A)} - \frac{m_1 \alpha X}{c+X} + d \right) - \lambda_3 \left(\frac{u_1 \lambda A}{a+A} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_3}{dt} = \lambda_1 \left(\frac{\phi \alpha X}{c+X} \right) + \lambda_3 \left(d + \delta - \frac{m_2 \phi \alpha X}{c+X} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_4}{dt} = 2A_2 A + \lambda_2 \left(\frac{u_1 \lambda a S}{(a+A)^2} \right) - \lambda_3 \left(\frac{u_1 \lambda a S}{(a+A)^2} \right) + \lambda_4 \eta, \end{cases} \quad (4.18)$$

with transversality conditions $\lambda_i(t_f) = 0$ for $i = 1, 2, 3, 4$.

Furthermore we obtain the optimal control set (u_1^*, u_2^*) characterized by

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3) \lambda A S}{B_1 (a+A)} \right\} \right\}, \\ u_2^* &= \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \gamma}{B_2} \right\} \right\}. \end{aligned}$$

Proof. Suppose (u_1^*, u_2^*) is an optimal control and X, S, I and A are the state reply to the optimal control. To establish the adjoint equations and the transversality suitaitions we first describe the Hamiltonian function connected with the system that can be given by

$$\begin{aligned} H(t, u, X, S, I, A, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= A_1 S^2 - A_2 A^2 + \frac{1}{2} B_1 u_1^2 + \frac{1}{2} B_2 u_2^2 \\ &\quad + \lambda_1 \frac{dX}{dt} + \lambda_2 \frac{dS}{dt} + \lambda_3 \frac{dI}{dt} + \lambda_4 \frac{dA}{dt} \end{aligned}$$

where, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are adjoint functions to be determined appropriately.

$$\begin{aligned} H &= A_1 S^2 - A_2 A^2 + \frac{1}{2} B_1 u_1^2 + \frac{1}{2} B_2 u_2^2 \\ &\quad + \lambda_1 \left(rX \left[1 - \frac{X}{K} \right] - \frac{\alpha X S}{c+X} - \frac{\phi \alpha X I}{c+X} \right) \\ &\quad + \lambda_2 \left(\frac{m_1 \alpha X S}{c+X} - \frac{u_1 \lambda A S}{a+A} - dS \right) \\ &\quad + \lambda_3 \left(\frac{m_2 \phi \alpha X I}{c+X} + \frac{u_1 \lambda A S}{a+A} - (d + \delta)I \right) \\ &\quad + \lambda_4 \left(u_2 \gamma + \sigma(S + I) - \eta A \right). \end{aligned}$$

Pontryagin Minimum Principle provides necessary conditions for an optimal control problem. This principle converts optimality system into a problem of minimizing the Hamiltonian, $H(t)$. Substituting $X(t) = X^*(t)$, $S(t) = S^*(t)$, $I(t) = I^*(t)$ and $A(t) = A^*(t)$ and taking derivative of the Hamiltonian with respect to X, S, I and A , we get the following result

$$\begin{cases} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial X} = \lambda_1 \left(\frac{\alpha c S}{(c+X)^2} + \frac{\phi \alpha c I}{(c+X)^2} - r \left(1 - \frac{2X}{K} \right) \right) - \lambda_2 \left(\frac{m_1 \alpha c S}{(c+X)^2} \right) - \lambda_3 \left(\frac{m_2 \phi \alpha c I}{(c+X)^2} \right), \\ \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial S} = -2A_1 S + \lambda_1 \left(\frac{\alpha X}{c+X} \right) + \lambda_2 \left(\frac{u_1 \lambda A}{(a+A)} - \frac{m_1 \alpha X}{c+X} + d \right) - \lambda_3 \left(\frac{u_1 \lambda A}{a+A} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial I} = \lambda_1 \left(\frac{\phi \alpha X}{c+X} \right) + \lambda_3 \left(d + \delta - \frac{m_2 \phi \alpha X}{c+X} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial A} = 2A_2 A + \lambda_2 \left(\frac{u_1 \lambda a S}{(a+A)^2} \right) - \lambda_3 \left(\frac{u_1 \lambda a S}{(a+A)^2} \right) + \lambda_4 \eta, \end{cases} \quad (4.19)$$

with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0. \quad (4.20)$$

Based on Fleming and Rishel ([Fleming and Lions, 2012](#)), the optimal controls u_1^* and u_2^* satisfy

$$\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0$$

which implies

$$\begin{aligned} \frac{\partial H}{\partial u_1^*}(t) &= B_1 u_1^* - \lambda_2 \frac{\lambda A S}{a+A} + \lambda_3 \frac{\lambda A S}{a+A} = 0, \\ \frac{\partial H}{\partial u_2^*}(t) &= B_2 u_2^* + \lambda_4 \gamma = 0. \end{aligned}$$

Solving for u_1 and u_2 we obtain that

$$u_1^* = \frac{(\lambda_2 - \lambda_3) \lambda A S}{B_1 (a+A)}, \quad u_2^* = -\frac{\lambda_4 \gamma}{B_2}$$

Using the boundedness of the standard controls $u_1(t), u_2(t)$ we get

$$u_1^*(t) = \begin{cases} 0, & \text{if } \frac{(\lambda_2 - \lambda_3)\lambda AS}{B_1(a+A)} \leq 0, \\ \frac{(\lambda_2 - \lambda_3)\lambda AS}{B_1(a+A)}, & \text{if } 0 < \frac{(\lambda_2 - \lambda_3)\lambda AS}{B_1(a+A)} < 1, \\ 1, & \text{if } \frac{(\lambda_2 - \lambda_3)\lambda AS}{B_1(a+A)} \geq 1. \end{cases} \quad (4.21)$$

$$u_2^*(t) = \begin{cases} 0, & \text{if } -\frac{\lambda_4 \gamma}{B_2} \leq 0, \\ -\frac{\lambda_4 \gamma}{B_2}, & \text{if } 0 < -\frac{\lambda_4 \gamma}{B_2} < 1, \\ 1, & \text{if } -\frac{\lambda_4 \gamma}{B_2} \geq 1. \end{cases} \quad (4.22)$$

In compact form we get

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3)\lambda AS}{B_1(a+A)} \right\} \right\} \text{ and} \\ u_2^* &= \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \gamma}{B_2} \right\} \right\}. \end{aligned} \quad (4.23)$$

□

In order to confirm that the optimal control is really a minimizer we need to check the Hessian matrix of the Hamiltonian H . The Hessian matrix for H with respect the control U is given by

$$\frac{\partial^2 H}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1^2} & \frac{\partial^2 H}{\partial u_1 \partial u_2} \\ \frac{\partial^2 H}{\partial u_2 \partial u_1} & \frac{\partial^2 H}{\partial u_2^2} \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

This matrix is a positive definite matrix as a consequence of the positive weights B_1 and B_2 . This confirm that Hamiltonian function H is convex and as a result of this the optimal control $u = (u_1, u_2)$ is really a minimizer.

4.3.3 The optimality system

The optimal controls and the states are found by solving the optimality system, that consists the state system (4.15) together with their initial conditions, the adjoint system (4.19), the boundary conditions (4.20) and, the characterization of the optimal controls (4.23). In

combination we represent it as

$$\left\{ \begin{array}{l} \frac{dX^*}{dt} = rX^* \left[1 - \frac{X^*}{K} \right] - \frac{\alpha X^* S^*}{c+X^*} - \frac{\phi \alpha X^* I^*}{c+X^*}, \\ \frac{dS^*}{dt} = \frac{m_1 \alpha X^* S^*}{c+X^*} - \frac{u_1^* \lambda A^* S^*}{a+A^*} - dS^*, \\ \frac{dI^*}{dt} = \frac{m_2 \phi \alpha X^* I^*}{c+X^*} + \frac{u_1^* \lambda A^* S^*}{a+A^*} - (d + \delta) I^*, \\ \frac{dA^*}{dt} = u_2^* \gamma + \sigma(S^* + I^*) - \eta A^*, \\ X(0) \geq 0, S(0) \geq 0, I(0) \geq 0, A(0) \geq 0, \\ \frac{d\lambda_1}{dt} = \lambda_1 \left(\frac{\alpha c S^*}{(c+X^*)^2} + \frac{\phi \alpha c I^*}{(c+X^*)^2} - r \left(1 - \frac{2X^*}{K} \right) \right) - \lambda_2 \left(\frac{m_1 \alpha c S^*}{(c+X^*)^2} \right) - \lambda_3 \left(\frac{m_2 \phi \alpha c I^*}{(c+X^*)^2} \right), \\ \frac{d\lambda_2}{dt} = -2A_1 S^* + \lambda_1 \left(\frac{\alpha X^*}{c+X^*} \right) + \lambda_2 \left(\frac{u_1^* \lambda A^*}{(a+A^*)} - \frac{m_1 \alpha X^*}{c+X^*} + d \right) - \lambda_3 \left(\frac{u_1^* \lambda A^*}{a+A^*} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_3}{dt} = \lambda_1 \left(\frac{\phi \alpha X^*}{c+X^*} \right) + \lambda_3 \left(d + \delta - \frac{m_2 \phi \alpha X^*}{c+X^*} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_4}{dt} = 2A_2 A^* + \lambda_2 \left(\frac{u_1 \lambda a S^*}{(a+A^*)^2} \right) - \lambda_3 \left(\frac{u_1 \lambda a S^*}{(a+A^*)^2} \right) + \lambda_4 \eta, \\ u_1^* = \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3) \lambda A S}{B_1 (a+A)} \right\} \right\}, \\ u_2^* = \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \gamma}{B_2} \right\} \right\}, \\ \lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0. \end{array} \right. \quad (4.24)$$

Here, we notice that the adjoint system (4.19) is also linear in λ_i for $i = 1, 2, 3, 4$ with bounded coefficients. Thus, there exists a positive real number M such that $|\lambda_i| \leq M$ on $t \in [0, t_f]$. Hence for sufficiently small time t_f the solution to the optimality system (4.24) is unique.

The need for the smallness of the time span is due to the opposite time direction/orientations of the optimality system; the state problem has initial values and the adjoint problem has terminal values. Applying Pontryagin Minimum Principle to solve this problem analytically is not practical. Consequently we use numerical algorithms to find approximate optimal solutions Φ^* and u^* .

4.4 Numerical simulations

In this section we demonstrate numerical simulation for the analytical solutions of the system (4.1). From realistic point of view numerical solutions are very important beside analytical results.

Table 4.1: Parameter values used in numerical simulations.

Parameters	Description	Value	Source
r	growth rate of crop biomass	0.05	(Abraha et al., 2021b)
K	maximum density of crop biomass	1	(Al Basir et al., 2019a)
λ	Aware people activity rate	0.06	Al Basir et al. (2019a)
d	natural mortality of pest	0.01	(Chowdhury et al., 2019)
m_1	conversion efficacy of susceptible pests	0.8	Assumed
m_2	conversion efficiency of infected pest	0.6	(Abraha et al., 2021b)
δ	Disease related mortality rate	0.1	(Al Basir et al., 2019a)
a	half saturation constant	0.5	(Jana and Kar, 2013)
α	Attack rate of pest	0.025	(Al Basir et al., 2019a)
σ	Aware people growth rate	0.015	Assumed
η	fading of memory of aware people	0.015	(Al Basir et al., 2019a)
γ	rate of awareness from global source	0.003	(Abraha et al., 2021b)
c	half saturation constant	1	(Pathak and Maiti, 2012)

Numerical simulations are carried out with a set of parameter values given in Table 4.1. First, we analyze the system without optimal control and then the system with optimal control. We examine the impact of optimal control strategies by applying Runge-Kutta fourth-order scheme on the optimality system. The optimality system is obtained by taking the state system together with the adjoint system, the optimal control, and the transversality conditions. The dynamical behavior of the models in relation to two controls is also studied. The optimal strategy is achieved by finding a solution to the state system (4.15) and co-state system (4.19). An iterative design is explored and used to decide the solution for the optimality system. The numerical method utilized is the forward-backward sweep method that includes iterative Runge-Kutta fourth-order progressive-regressive schemes. The progressive scheme is used in obtaining the solutions of the state ODEs (4.15) with the initial conditions, while the regressive scheme is applied in obtaining the solutions of the adjoint system given by (4.19) with transversality conditions (4.20).

Following this, using a convex grouping of the previous controls and the value from the characterization values, the controls are updated. This procedure continues and the iterative values are repeated if the values of the unknowns at the preceding iteration are not considerably near to the ones at the present iteration. It is a two point boundary-value problem, with divided boundary conditions at $t_0 = 0$ and $t = T$. For our numerical experiments, we take $T = 2000$ days. Besides, the values of the weight function are taken as $A_1 = 1015, A_2 = 1010, B_1 = 1.6, B_2 = 1$ and as initial state variables $X(0) = 0.2, S(0) = 0.07, I(0) = 0.05$, and $A(0) = 0.5$, on the numerical simulations.

In Figure 4.2, time series solution of the model system (4.1) is sketched. It is also observed that our model variables $X(t), S(t), I(t)$ and $A(t)$ become oscillating as the consumption rate α increases. Also, steady state value of both pest population (when exists) are increased as α rises. Oscillating solution is seen for great value of α ($\alpha = 0.1$). Bifurca-

tion illustration is shown in Figure 4.3 taking α as the bifurcating parameter. Decisive value depends on many parameters such as recruitment rate of awareness program (see figure 4.4).

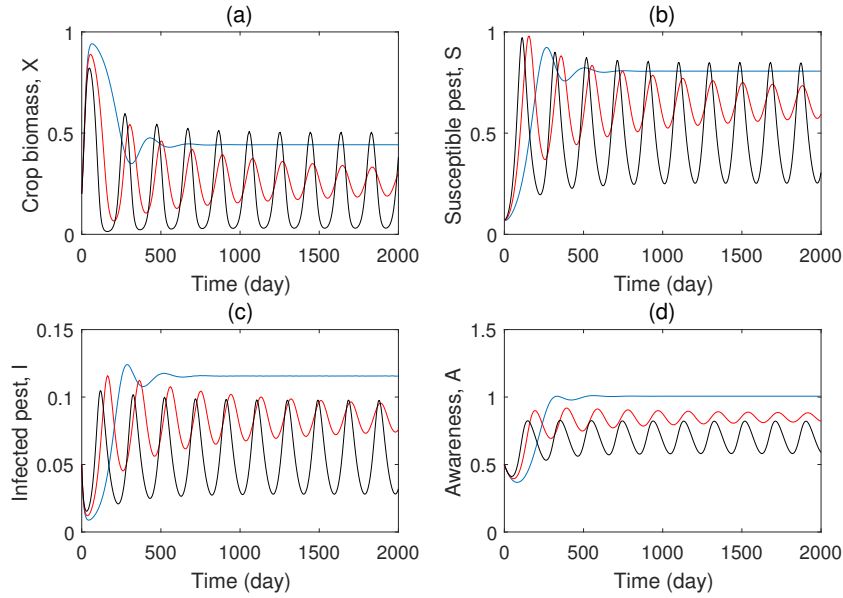


Figure 4.2: Numerical solution of system (4.1) for different values of α (without control): $\alpha = 0.0621$ (blue line), $\alpha = 0.0882$ (red line), $\alpha = 0.1$ (black line). Other parameters are taken from Table 4.1.

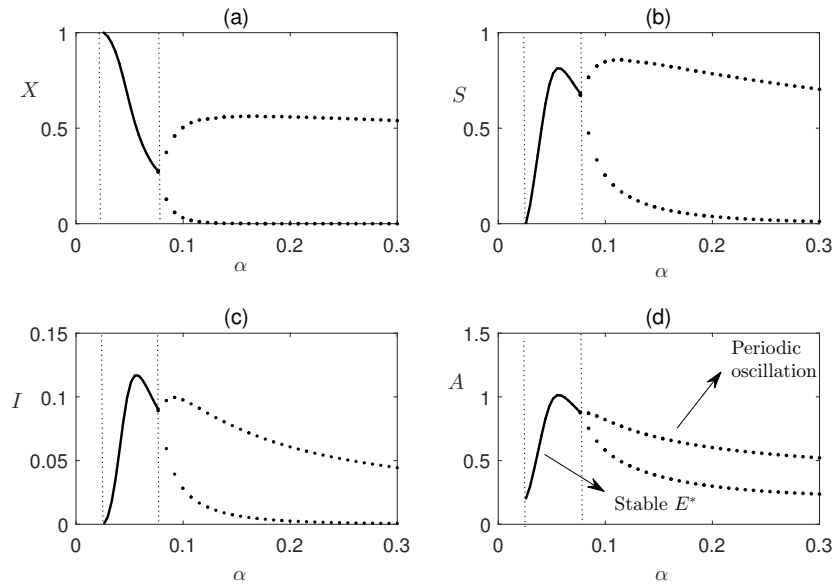


Figure 4.3: Bifurcation diagram of coexisting equilibrium E^* (when exists) of the system (4.1) for the value of α (without control). Parameters values are same as in Figure 4.2.

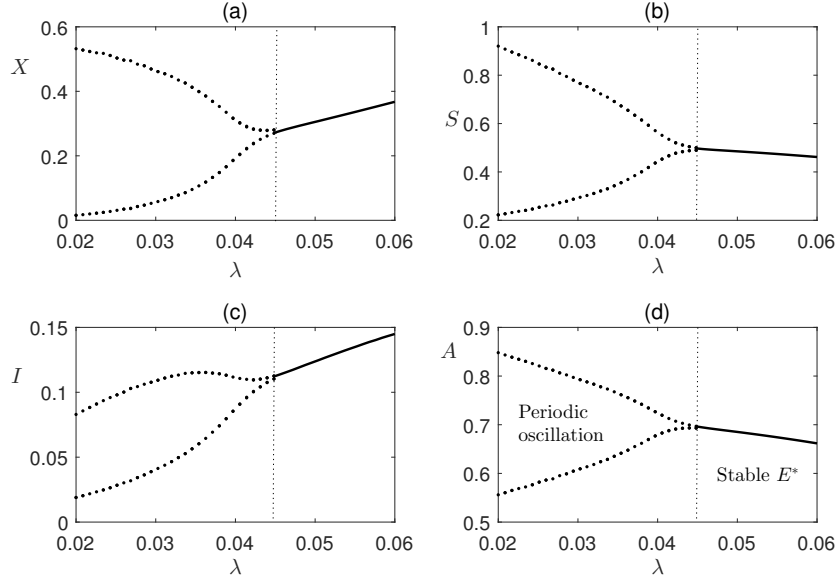


Figure 4.4: Bifurcation diagram of coexisting equilibrium E^* (when exists) of the system 4.1 for the value of λ (without control). Parameters values are same as in Figure 4.2.

We now solve the optimal control system with numerical method in Figures 4.5, 4.6 and 4.7 and the impact of optimal pair is shown. In Figure 4.7 we illustrated our dynamics without optimal control and with optimal control. We operate our control for the first 100 days. Oscillation is not occurred with the optimal control strategy. Crop biomass increases and the population of pest reduced significantly with the influence of the optimal profiles of global awareness (i.e. $u_2^* \gamma$) and awareness based control activity, $u_1^* \lambda$ as exposed in Figure 4.7. It is also seen that the healthy pest population decreases due to the effect of optimal control. As a result, the effect of optimal control by means of awareness based bio-control has a large input in controlling the pest problem in the crop field.

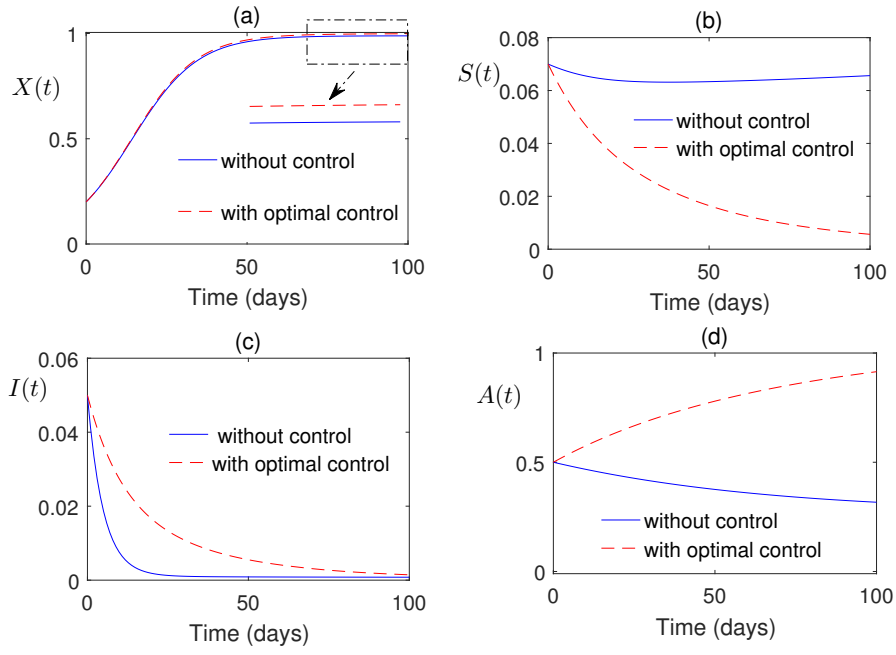


Figure 4.5: Numerical solution of optimally controlled system ($u_1 = 0, u_2 \neq 0$)

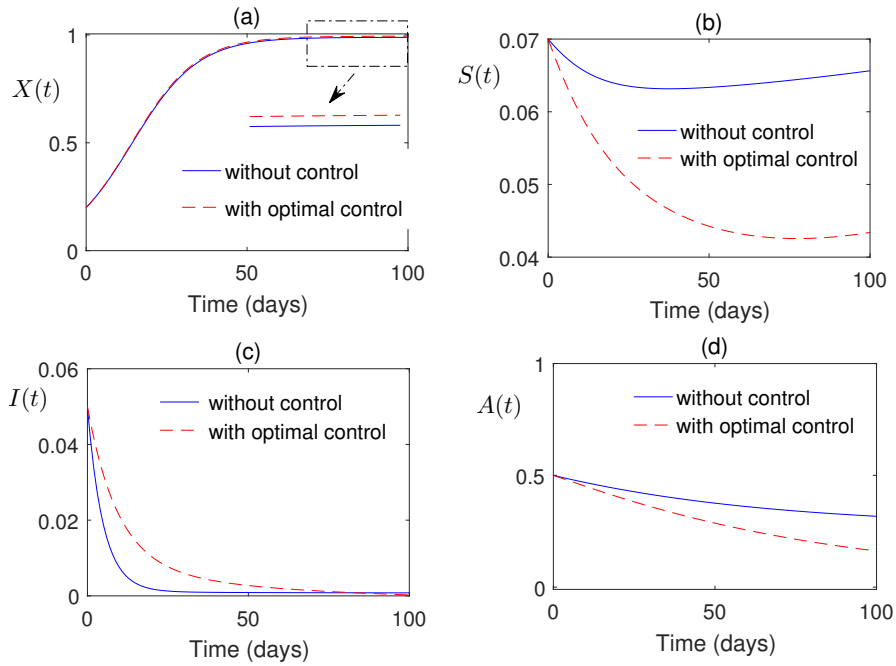


Figure 4.6: Numerical solution of optimally controlled system ($u_1 \neq 0, u_2 = 0$)

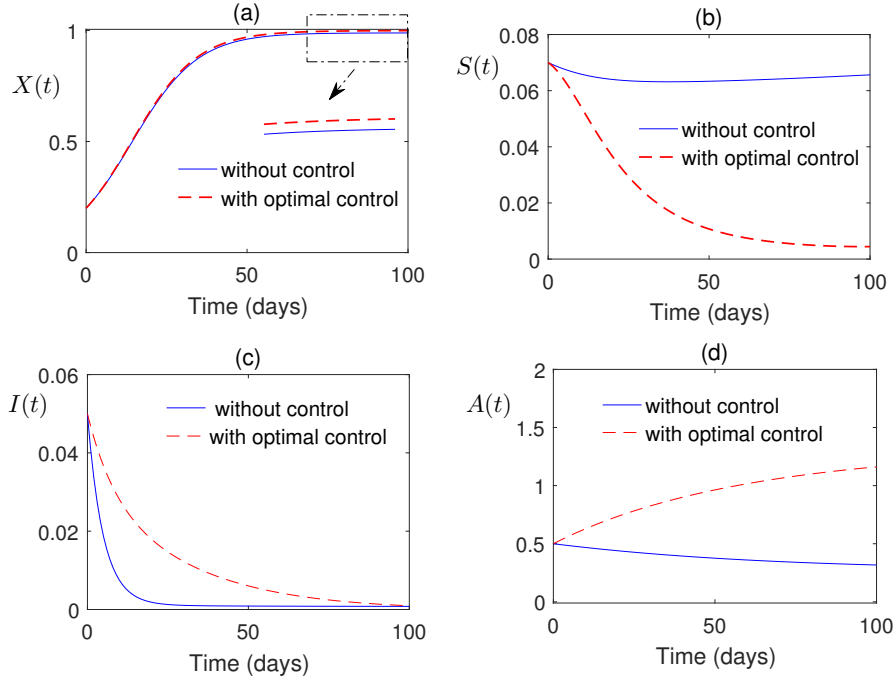


Figure 4.7: Numerical solution of optimally controlled and uncontrolled system

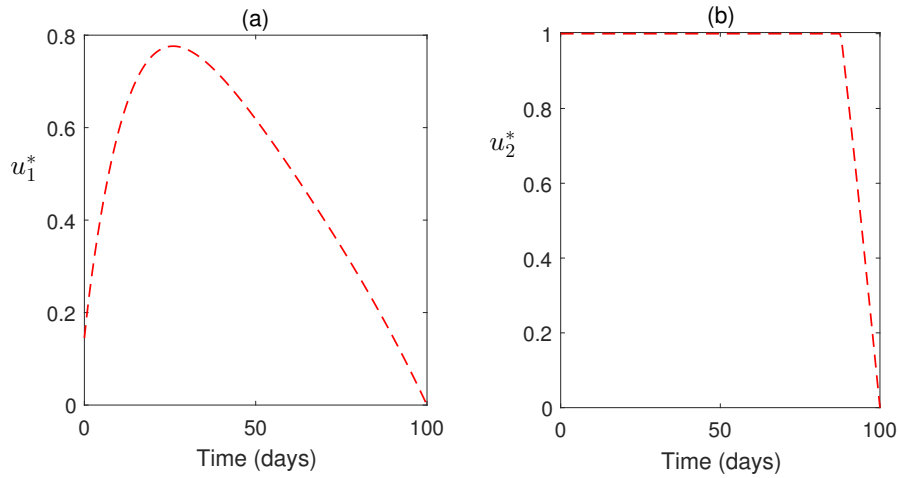


Figure 4.8: Optimal control u_1^* and u_2^* are plotted as function of time, taking parameters values as in Table 4.1

4.5 Results and discussion

In this chapter, a mathematical model described by a system of ordinary differential equations has been proposed and analyzed to study the effect of awareness programs on the

control of pests in agriculture. Our model contains four densities, specifically, density of crop-biomass, density of susceptible and infected pest, and the awareness level.

Our model assume that aware people adopt bio-control measures, like an integrated pest management, since it is less harmful to human health. Again, we assume infected pests cannot damage the crop or they are less harmful. The proposed model exhibits four steady state points: (i) the axial fixed point, which is unstable at all time, (ii) the pest free fixed point, which is stable when two critical values R_{01} and R_{02} are less than unity and unstable for R_{01} and R_{02} are bigger than unity, (iii) susceptible pest free fixed point, which may exist when conditionally, and (iv) the coexistence fixed point.

From our analytical and numerical investigations, we saw that the most ecological noteworthy parameters in the system are the consumption rate α and the awareness activity rate λ . If the density of awareness campaigns increases, the density of crop increases, and as a result pest prevalence declines. Our study illustrates that if the level of awareness movements increases, pest occurrence decreases. Consequently, the density of crop-biomass rises. The system changes its dynamics asymptotically to a cyclic oscillation. However, global awareness through radio, TV, and social media can control the oscillations and makes the system stable. Our study suggests that with an increase in awareness level γ , pest outbreak may be declined to a lower level even with moderately high pest consumption rate α . Moreover, we have used optimal control theory to provide the cost effective outline of bio-pesticides and a global awareness movements. Awareness between people with optimal outline of bio-pesticides can be a correct feature for the control of pest in the crop field, dropping the matters that synthetic pesticides have on side effects on individual and the environment.

CHAPTER 5

CONTROLLING CROP PESTS WITH FARMING AWARENESS BASED INTEGRATED APPROACH

This chapter is devoted to the formulation and mathematical analysis of a crop pest control model. The analysis of this model shows that if two critical parameters R_0 and R_1 are less than unity, then crop pests are to be minimized and the crop may get pleasure. Then this model is analyzed for local stability and Hopf bifurcation using stability analysis theory. Numerical simulations show that depending on some critical parameters, the solutions approach either the coexistence steady state or a periodic orbit. For the support of theoretical results numerical simulation are given.

5.1 Model formulation and description

We consider four population into the mathematical model, namely plants biomass $X(t)$, susceptible pest $S(t)$, infected pest $I(t)$ and level of awareness $A(t)$. The following assumptions are made to formulate the mathematical model:

- Healthy pest population becomes infected using bio-pesticide. Infected pest can attack the plant but the rate is very lesser than susceptible pest. We assume that infected pest can consume the plant biomass following Holling type II responses, where as susceptible pest consume following the usual mass action function.
- Due to the finite size of crop field, we assume logistic growth for the density of crop biomass, with net growth rate r and carrying capacity K .
- Susceptible pests attack the crop, thereby causing considerable crop reduction. If we infect the susceptible pest by pesticides, then the attack by pest can be controlled. Here we assume that aware farmers will adopt biological pesticides for the control of crop pest, as it has less side effects and is also environment friendly. Bio-pesticides are used to infect the healthy pest. Infected pest has an additional mortality due to infection.
- Let α be the consumption rate of pests. There is a pest infection rate, λ , because of aware human interactions and activity such as use of biopesticides (e.g. NPV virus), modeled via the usual mass action term λAS . We take d as the natural mortality rate of pest and δ is the additional mortality rate of infected pest due to aware people activity.

- The level of awareness $A(t)$, among the farmers increases due to general public awareness and media campaigns. Also, it is assumed that rate of local awareness (general public awareness) is proportional to the visibility of the susceptible pests and increases with a rate σ . Also, awareness campaign through the global factors such as radio, TV, etc. can increase the level of awareness at a rate ω . In addition, there could be fading of interest in this exploitation, at a rate η .
- To speed up the pest control process, chemical synthetic pesticides are introduced. It causes additional death to both susceptible and infected pest populations. We model the situation by the terms $\frac{\gamma SA}{1+A}$ and $\frac{\gamma IA}{1+A}$ respectively.

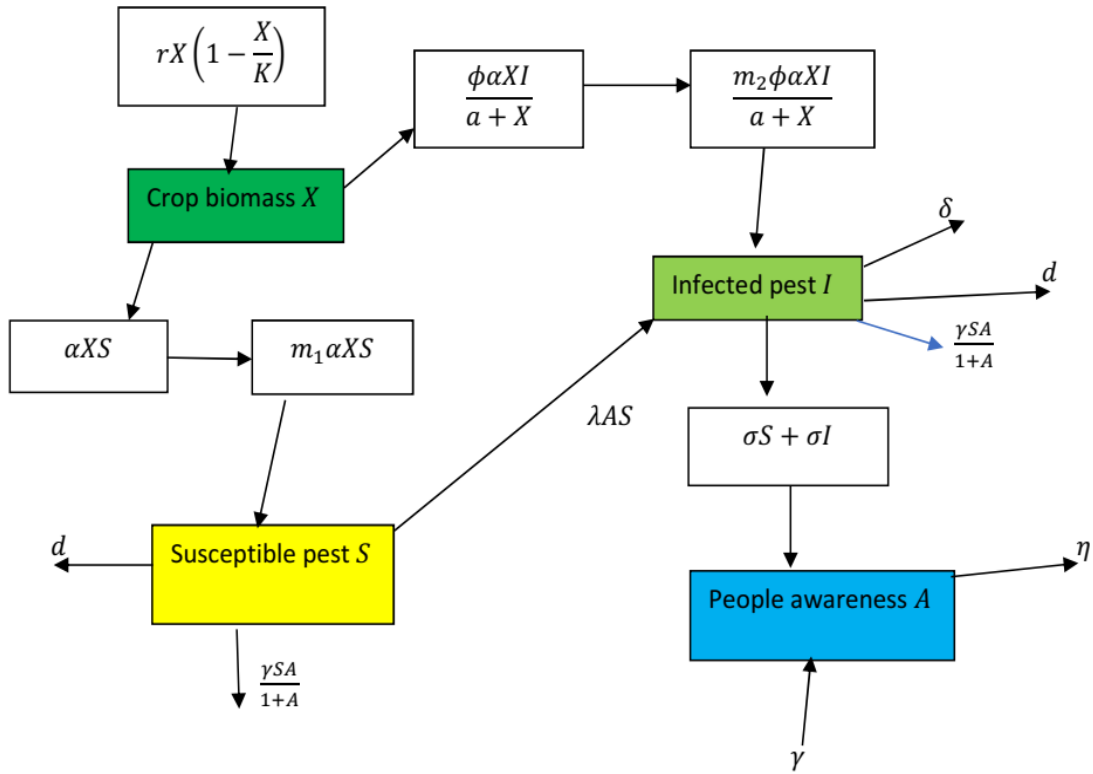


Figure 5.1: Schematic diagram for the crop pest control dynamics

Based on the above assumptions, we have the following mathematical model:

$$\begin{cases} \frac{dX}{dt} = rX \left(1 - \frac{X}{K} \right) - \alpha XS - \frac{\phi \alpha XI}{a + X}, \\ \frac{dS}{dt} = m_1 \alpha XS - \lambda AS - dS - \frac{\gamma SA}{1 + A}, \\ \frac{dI}{dt} = \frac{m_2 \phi \alpha XI}{a + X} + \lambda AS - (d + \delta)I - \frac{\gamma IA}{1 + A}, \\ \frac{dA}{dt} = \omega + \sigma(S + I) - \eta A, \end{cases} \quad (5.1)$$

subject to the initial conditions

$$X(0) \geq 0, S(0) \geq 0, I(0) \geq 0, A(0) \geq 0. \quad (5.2)$$

Here α is the attack rate of pests on crop, Infected pests can also attack the crop but with lower rate, $\phi\alpha$, with $\phi < 1$; a is the half saturation constants, m_1, m_2 are the “conversion efficiency” of susceptible and infected pests respectively i.e. how efficiently can the pests utilize plant resource. Since pesticide affected pests have lowered efficiency, $m_1 > m_2$, and γ denotes the increase of level from global advertisement by radio, TV, etc. It is obvious to assume that all the parameters are positive.

5.1.1 Positivity of solutions and the invariant region

Existence and positivity of the solutions are the main properties that should be verified. The possible result describes the range in which the solution of the equations is biologically important, while the positivity of the solutions illustrates the non-negativity of the solutions of the equation.

Theorem 5.1.1. *The solutions of system (5.1) together with the initial conditions remain non-negative for all $t \geq 0$.*

Proof. Let $(X(t), S(t), I(t), A(t))$ be a solution of the system (5.1) with their initial conditions. We use Lemma (4.2.1) to proof the positivity of the first two equations in the system. Let us consider $X(t)$ for $t \in [0, T]$. We obtain from the first equation of system (5.1) that

$$\frac{dX}{dt} = X \left[r \left(1 - \frac{X}{K} \right) - \alpha S - \frac{\phi \alpha I}{a + X} \right], \quad \frac{dX}{X} = \left[r \left(1 - \frac{X}{K} \right) - \frac{\alpha S + \phi \alpha I}{c + X} \right] dt.$$

Integrating from 0 to t and simplifying gives

$$\ln X = D_0 + \int_0^t \left[r \left(1 - \frac{X}{K} \right) - \alpha S - \frac{\phi \alpha I}{a + X} \right] dt$$

where D_0 is an arbitrary constant of integration. Hence

$$X(t) = D_1 e^{\int_0^t \left[r \left(1 - \frac{X}{K} \right) - \alpha S - \frac{\phi \alpha I}{a + X} \right] dt} > 0,$$

where $D_1 = e^{D_0}$.

From the second equation of system (5.1) we have

$$\frac{dS}{dt} = S \left[m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1 + A} \right], \quad \frac{dS}{S} = \left[m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1 + A} \right] dt.$$

Similarly we can obtain

$$\ln S = K_0 + \int_0^t \left(m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1+A} \right) dt.$$

So that

$$S(t) = K_1 e^{\int_0^t (m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1+A}) dt} > 0,$$

where $K_1 = e^{K_0}$ for K_0 is constant of integration. To show that I and A are non-negative, consider the sub-system of (5.1)

$$\begin{aligned} \frac{dI}{dt} &= \frac{m_2 \phi \alpha X I}{a+X} + \lambda A S - (d+\delta)I - \frac{\gamma I A}{1+A}, \\ \frac{dA}{dt} &= \omega + \sigma(S+I) - \eta A. \end{aligned} \quad (5.3)$$

To show the positivity of $I(t)$, we do the proof by contradiction. Suppose there exists $t_0 \in (0, T)$ such that $I(t_0) = 0$, $I'(t_0) \leq 0$ and $I(t) > 0$ for $t \in [0, t_0)$. Then, $A_0 > 0$ for $t \in [0, t_0)$. If this is not to be the case, then there exists $t_1 \in [0, t_0)$ such that $A(t_1) = 0$, $A'(t_1) \leq 0$ and $A(t) > 0$ for $t \in [0, t_0)$. Integrating the third equation of the system (5.1) gives

$$\begin{aligned} I(t) &= I_0 \exp \left(\int_0^t \left(\frac{m_2 \phi \alpha X(t)}{a+X(t)} - \frac{\gamma A(t)}{1+A(t)} \right) dt - (d+\delta)t \right) \\ &\quad + \left[\exp \left(\int_0^t \left(\frac{m_2 \phi \alpha X(t)}{a+X(t)} - \frac{\gamma A(t)}{1+A(t)} \right) dt - (d+\delta)t \right) \right] \\ &\quad \times \left[\int_0^t \lambda A(t) S(t) dt \exp \left((d+\delta)t - \int_0^t \left(\frac{m_2 \phi \alpha X(t)}{a+X(t)} - \frac{\gamma A(t)}{1+A(t)} \right) dt \right) \right] > 0, t \in [0, t_1]. \end{aligned}$$

Then, $A'(t_1) = \gamma + \sigma(S(t_1) + I(t_1)) > 0$. This is a contradiction. Hence, $I(t) > 0$ for all $t \in [0, t_0)$. Finally, from the second equation of subsystem (4.2), we have

$$\frac{dA}{dt} = \omega + \sigma(S+I) - \eta A.$$

Integration gives

$$A(t) = A(0) e^{\eta t} + e^{\eta t} \int_0^t (\omega + \sigma(S(t) + I(t))) e^{-\eta t} dt > 0,$$

that is, $A(t) > 0$ for all $t \in (0, T)$. □

Theorem 5.1.2. *Every solution of system (5.1) are uniformly bounded, in the region $\mathcal{V}$ defined by*

$$\mathcal{V} = \left\{ (X, S, I, A) \in \mathbb{R}_+^4 : 0 < X + S + I \leq \frac{L}{d}, 0 < A \leq \frac{\omega d + \sigma L}{\eta d} \right\}, \quad \text{with} \quad L = \frac{K(r+d)^2}{4r}.$$

Proof. At any time t , let $W = X + \frac{1}{m_1}S + \frac{1}{m_1}I$, then the time derivative of W along the solution of system (5.1) is given by

$$\begin{aligned}
\frac{dW(t)}{dt} &= rX \left[1 - \frac{X}{K} \right] - \alpha XS - \frac{\phi \alpha XI}{a+X} + \alpha XS - \frac{\lambda}{m_1}AS + \frac{\lambda}{m_1}AS \\
&\quad + \frac{m_2}{m_1} \frac{\phi \alpha XI}{a+X} - \frac{d}{m_1}S - \left(\frac{d+\delta}{m_1} \right) I - (S+I) \frac{\gamma}{m_1} \frac{A}{1+A} \\
&= rX \left[1 - \frac{X}{K} \right] - \left(\frac{m_1 - m_2}{m_1} \right) \frac{\phi \alpha XI}{a+X} - \left(\frac{d+\delta}{m_1} \right) I - \frac{d}{m_1}S - (S+I) \frac{\gamma}{m_1} \frac{A}{1+A} \\
&\leq rX \left[1 - \frac{X}{K} \right] - \frac{d}{m_1}S - \frac{d}{m_1}I = rX \left(1 - \frac{X}{K} \right) - \frac{d}{m_1}(S+I) - dX + dX \\
&= rX \left(1 - \frac{X}{K} \right) - \left(\frac{d}{m_1}S + \frac{d}{m_1}I + dX \right) + dX.
\end{aligned}$$

Then we have from the above

$$\frac{dW}{dt} \leq rX \left(1 - \frac{X}{K} \right) - dW + dX$$

That is

$$\frac{dW}{dt} + dW \leq rX \left(1 - \frac{X}{K} \right) + dX = (r+d)X - \frac{rX^2}{K} = \Phi(X) \quad (\text{say})$$

Here, $\Phi(X)$ is a concave parabola for which its maximum value is attained at the vertex whose abscissa is $X_v = \frac{K(r+d)}{2r}$.

It follows therefore,

$$\Phi(X) \leq \Phi(X_v) = \frac{1}{4r}K(r+d)^2 = \Phi^*.$$

Thus we have a constant $L = \frac{K(r+d)^2}{4r}$ such that

$$\frac{dW}{dt} + dW \leq L.$$

To solve this we apply the theory of differential inequality

$$e^{dt} \left(\frac{dW}{dt} + dW \right) \leq e^{dt} L \frac{d}{dt} (We^{dt}) \leq e^{dt} L.$$

$$\text{This implies, } We^{dt} \leq \int (Le^{dt})dt + CW(X, S, I) \leq e^{-dt} \left[dLe^{dt} \right] + Ce^{-dt}.$$

$$W(X(0), S(0), I(0)) \leq \frac{L}{d} + C$$

$$\text{That is } W(X, S, I) \leq \frac{L}{d} (1 - e^{-dt}) + W(X(0), S(0), I(0))e^{-dt}.$$

Hence we get

$$0 < W(X, S, I) \leq \frac{L}{d} (1 - e^{-dt}) + W(X(0), S(0), I(0))e^{-dt}.$$

For large value of t , $0 < X + S + I \leq \frac{L}{d}$.

Again from the fourth equation of system (5.1) we know that

$$\begin{aligned}\frac{dA}{dt} &= \omega + \sigma(S + I) - \eta A \leq \omega + \sigma\left(\frac{Lm_1}{d}\right) - \eta A \\ &= \frac{d\omega + \sigma L}{d} - \eta A, \quad \frac{dA}{dt} + \eta A \leq \frac{d\omega + \sigma L}{d}.\end{aligned}$$

Again by the same differential inequality

$$\begin{aligned}e^{\eta t} \left(\frac{dA}{dt} + \eta A \right) &\leq e^{\eta t} \left(\frac{d\omega + \sigma L}{d} \right), \quad \frac{d}{dt} (Ae^{\eta t}) \leq e^{\eta t} \left(\frac{d\omega + \sigma L}{d} \right). \\ \text{That is } Ae^{\eta t} &\leq \int \left(\frac{d\omega + \sigma L}{d} \right) e^{\eta t} dt + C, \quad A(t) \leq \left(\frac{d\omega + \sigma L}{\eta d} \right) + Ce^{-\eta t}. \\ A(0) &\leq \frac{\omega d + L\sigma}{\eta d} + C.\end{aligned}$$

This results in

$$0 < A \leq \frac{\omega d + \sigma L}{\eta d} + Ce^{-\eta t}.$$

Thus for $t \rightarrow \infty$, we obtain $0 < A \leq \frac{\omega d + \sigma L}{\eta d}$. Hence all the solutions of (5.1) originating in $\mathbb{R}_+^4$ are confined in the region

$$\mathcal{V} = \left\{ (X, S, I, A) \in \mathbb{R}_+^4 : 0 < X + S + I \leq \frac{L}{d} + \varepsilon, 0 < A \leq \frac{\omega d + \sigma L}{\eta d} \right\}$$

for any $\varepsilon > 0$ and for $t \rightarrow \infty$. Thus the system (5.1) is always uniformly bounded. □

5.1.2 Equilibria and their existence

To get the equilibrium points, we put the right hand sides of system (5.1) to 0:

$$\begin{cases} rX \left(1 - \frac{X}{K} \right) - \alpha XS - \frac{\phi \alpha XI}{a+X} = 0, \\ m_1 \alpha XS - \lambda AS - dS - \frac{\gamma SA}{1+A} = 0, \\ \frac{m_2 \phi \alpha XI}{a+X} + \lambda AS - (d + \delta)I - \frac{\gamma IA}{1+A} = 0, \\ \omega + \sigma(S + I) - \eta A = 0. \end{cases} \quad (5.4)$$

The system (5.1) always have four feasible equilibrium points.

- i. The axial equilibrium point $E_0 \left(0, 0, 0, \frac{\omega}{\eta} \right)$. This equilibrium point always exists,
- ii. The pest free equilibrium point $E_1 \left(K, 0, 0, \frac{\omega}{\eta} \right)$ again always exists.
- iii. The healthy pest free equilibrium point $E_2 (\tilde{X}, 0, \tilde{I}, \tilde{A})$,
where $\tilde{X}, \tilde{I}, \tilde{A}$ are computed as follows:

If we set $S = 0$ in system (5.4), then we get

$$\begin{cases} rX \left[1 - \frac{X}{K}\right] - \frac{\phi \alpha X I}{a+X} = 0, \\ \frac{m_2 \phi \alpha X I}{a+X} - (d + \delta + \frac{\gamma A}{1+A})I = 0, \\ \omega + \sigma I - \eta A = 0. \end{cases} \quad (5.5)$$

Solving I from the first equation of system (5.5) we obtain

$$\tilde{I} = \frac{r(a + \tilde{X})(K - \tilde{X})}{\phi \alpha K}.$$

And from the third equation of system (5.5) solving for A we get

$$\tilde{A} = \frac{\omega + \sigma \tilde{I}}{\eta} = \frac{r(a + \tilde{X})(K - \tilde{X})\sigma + \omega K \alpha \phi}{K \alpha \eta \phi}.$$

Finally solving for X from the second equation of system (5.5), we see that $\tilde{X}$ is the positive root of the equation

$$X^3 + a_1 X^2 + a_2 X + a_3 = 0, \quad (5.6)$$

where

$$\begin{aligned} a_1 &= \frac{(\alpha \phi m_2 - d - \delta - \gamma)K - a(\alpha \phi m_2 - 2d - 2\delta - 2\gamma)}{-\alpha \phi m_2 + d + \delta + \gamma}, \\ a_2 &= \frac{((\alpha \phi m_2 - 2d - 2\delta - 2\gamma)K + a(d + \delta + \gamma))a}{-\alpha \phi m_2 + d + \delta + \gamma}, \\ &\quad - \frac{K((-\alpha \phi m_2 + d + \delta + \gamma)\omega + \eta(-\alpha \phi m_2 + d + \delta))\phi \alpha}{r\sigma(-\alpha \phi m_2 + d + \delta + \gamma)}, \\ a_3 &= - \frac{Ka^2 r(d + \delta + \gamma)\sigma + K\phi a \alpha ((d + \delta + \gamma)\omega + \eta(d + \delta))}{r\sigma(-\alpha \phi m_2 + d + \delta + \gamma)}, \end{aligned}$$

The model system (5.1) may have one or more healthy pest free equilibrium points E_2 depending on the positive solutions of equation (5.6). The healthy pest free equilibrium point E_2 exists only if the equation (5.6) has positive root of X .

Let $\Delta = 18a_1 a_2 a_3 - 4a_1^3 a_3 + a_1^2 a_2^2 - 4a_2^2 - 27a_3^2$ be the discriminant of (5.6). Again, let $\Delta > 0$, then (i) if $a_1 < 0, a_2 < 0, a_3 < 0$, then (5.6) has a single positive root, (ii) if either $a_1 < 0, a_2 < 0, a_3 > 0$ or $a_1 > 0, a_2 < 0, a_3 > 0$ then (5.6) may have two different positive roots; (iii) if $a_1 < 0, a_2 > 0, a_3 < 0$, then (5.6) may have three different positive roots.

iv. The interior equilibrium point $E^*(X^*, S^*, I^*, A^*)$.

In the presence of pest, $X(t) \geq 0, S(t) \geq 0, I(t) \geq 0, A(t) \geq 0$. Our model system (5.1) has an equilibrium point called coexistence or endemic equilibrium point which is

dented by $E^*(X^*, S^*, I^*, A^*) \neq 0$. E^* is the steady state solution where pest persist in the crop biomass population. It is obtained by setting each equation of the system (5.1) equal to zero. That is

$$\frac{dX}{dt} = \frac{dS}{dt} = \frac{dI}{dt} = \frac{dA}{dt} = 0.$$

From the second equation of system (5.4) we get

$$\left(m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1+A}\right) S = 0 \Rightarrow m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1+A} = 0 \text{ from which we obtain}$$

$$X^* = \frac{\lambda A^2 + (d + \lambda + \gamma)A + d}{m_1 \alpha (1 + A)}.$$

From the first equation of system (5.4), we have

$$r \left(1 - \frac{X}{K}\right) - \alpha S - \frac{\phi \alpha I}{a + X} = 0$$

Simplifying this we obtain

$$\alpha(a + X)S + \phi \alpha I = \frac{r(K - X)(a + X)}{K} \quad (5.7)$$

and from the last equation of system (5.4) we have

$$\sigma S + \sigma I = \frac{\eta A - \omega}{\sigma}. \quad (5.8)$$

Solving equations (5.7) and (5.8) simultaneously, we get

$$S^* = \frac{r(a + X)(K - X)\sigma - K\phi\alpha(\eta A - \omega)}{K\alpha\sigma(X + a - \phi)}$$

$$I^* = \frac{(a + X)(AK\alpha\eta - K\alpha\omega - Kr\sigma + \sigma rX)}{K\alpha\sigma(X + a - \phi)}$$

Therefore $E^* = (X^*, S^*, I^*, A^*)$ is the coexistence steady state, where

$$X^* = \frac{\lambda A^2 + (d + \lambda + \gamma)A + d}{m_1 \alpha (1 + A)},$$

$$S^* = \frac{r(a + X)(K - X)\sigma - K\phi\alpha(A\eta - \omega)}{K\sigma\alpha(X + a - \phi)},$$

$$I^* = \frac{(a + X)((A\eta - \omega)\alpha - \sigma r)K + \sigma rX}{K\sigma\alpha(X + a - \phi)},$$

and A^* is the positive roots of the following equation

$$f(A) = A^6 + a_1 A^5 + a_2 A^4 + a_3 A^3 + a_4 A^2 + a_5 A + a_6 = 0, \quad (5.9)$$

whose coefficients are given by (we used maple18 to compute)

$$\begin{aligned}
a_1 &= \frac{3\lambda^2 r \sigma - r \sigma ((K-a)m_1 + m_2 \phi) \alpha - 3d - \delta - 3\gamma) \lambda}{\lambda^2 r \sigma}, \\
&+ \frac{m_1 \eta (\phi (m_1 - m_2) \alpha + d + \delta + \gamma) \alpha^2}{\lambda^2 r \sigma}, \\
a_2 &= \frac{3\sigma \lambda^3 r - 3\sigma r ((K-a)m_1 + m_2 \phi) \alpha - 3d - \delta - 2\gamma) \lambda^2}{\sigma \lambda^3 r} \\
&- \frac{m_1 K (\phi (m_1 - m_2) (\omega - 3\eta) \alpha + \omega (d + \delta + \gamma) + \eta + \sigma r (m_1 a - m_2 \phi)) \alpha^2}{\lambda^2 r \sigma} \\
&+ \frac{(-2(K-a)(d + \delta/2 + \gamma)m_1 - 2m_2 \phi (d + \gamma)) \alpha + 3(d + \gamma)(d + \delta + \gamma)}{\lambda^2} \\
&+ \frac{K \eta ((a(d + \delta + \gamma)m_1 - m_2 \phi (d + \gamma)) \alpha + (d + \gamma)(d + \delta + \gamma)) m_1 \alpha^2}{\sigma \lambda^3 r}, \\
a_3 &= - \frac{3m_1^2 K \alpha^3 (\phi (\omega - \eta) \lambda + 1/3 a (\omega (d + \delta + \gamma) - 3\eta (d + \delta + 2/3 \gamma)))}{\sigma \lambda^3 r} \\
&+ \frac{3m_1 K \alpha^3 \phi ((\lambda + d/3 + \gamma/3) \omega - (\lambda + d + 2/3 \gamma) \eta) m_2}{\sigma \lambda^3 r} \\
&+ \frac{3m_1 K (\sigma ar (\lambda + d/3 + \delta/3 + \gamma/3) m_1 - \phi m_2 r \sigma + (\omega/3 - \eta/3) \gamma^2) \alpha^2}{\sigma \lambda^3 r} \\
&- \frac{3m_1 K (((2/3 \omega - 4/3 \eta) d + \lambda + 1/3 \delta (\omega - 2\eta)) \gamma + (\omega/3 - \eta) d) \alpha^2}{\sigma \lambda^3 r} \\
&- \frac{(K-a)(d^2 + (\delta + 2\gamma + 6\lambda) d + \gamma (\delta + 3\gamma + 4\lambda)) m_1 + \phi (d^2 + 6\lambda d + 3\lambda^2) m_2}{3\lambda^3 r} \\
&+ \frac{\lambda^3 + (9d + 3\delta + 3\gamma) \lambda^2 + (3\gamma^2 + (12d + 4\delta) \gamma + 9d^2 + 6d\delta) \lambda + (d + \gamma)^2}{\lambda^3}, \\
a_4 &= \frac{((a(\omega - \eta) d + \phi (\omega - \eta/3) \lambda + a\gamma + \delta (\omega - \eta)) m_1 - \phi m_2 \lambda) K \alpha^3}{\sigma \lambda^3 r} \\
&- \frac{K (\sigma ar (\lambda + d + \delta) m_1^2 - (\lambda + \gamma) \phi m_2 r \sigma + \delta (\omega - \eta) + \delta (\omega - \eta) \lambda) \alpha^2}{\sigma \lambda^3 r} \\
&- \frac{(3(K-a)(\lambda + 3\delta + 4\gamma) d + (\lambda^2 + (3\delta + 2\gamma) \lambda + 2\delta \gamma + \gamma^2) m_1 + \phi m_2) \alpha}{\lambda^3} \\
&- \frac{\sigma (d^3 + (\lambda + \delta + 2\gamma) d^2 + (\lambda^2 + (2\delta + 2\gamma) \lambda + \gamma^2) d + \delta (\lambda + \gamma)^2)}{\lambda^3}, \\
a_5 &= - \frac{K (((\lambda \phi + (d + \delta + \gamma) a) \omega - a\eta (d + \delta)) m_1 - m_2 \phi \lambda) \alpha^3 m_1}{\sigma \lambda^3 r} \\
&- \frac{m_1 (\sigma ar (\lambda + 3\delta + \gamma) m_1 - m_2 r \phi (\lambda + 3d) \sigma + (\delta + \omega (\lambda + 2\gamma)) d) K \alpha^2}{\sigma \lambda^3 r} \\
&- \frac{((K-a)(3d^2 + (\lambda + \delta + \gamma) d + \delta (\lambda + \gamma)) m_1 + m_2 \phi d (\lambda + 3/2 d + \gamma)) \alpha}{\lambda^3} \\
&+ \frac{(d^2 + (\lambda + \delta + \gamma) d + 2/3 \delta (\lambda + \gamma)) d}{\lambda^3}, \\
a_6 &= \frac{m_2 d \alpha (K \alpha^2 \omega m_1 + K \alpha r \sigma m_1 - dr \sigma) \phi}{\sigma \lambda^3 r} \\
&- \frac{((adm_1 + a\delta m_1) \alpha + d(d + \delta)) (K \alpha^2 \omega m_1 + K \alpha r \sigma m_1 - dr \sigma)}{\sigma \lambda^3 r}.
\end{aligned}$$

The coexistence equilibrium point E^* exists only if the characteristic equation (5.9)

has positive root in A . For a feasible equilibrium (positive S^* and I^*) we also need

$$\frac{\alpha \omega K + \sigma r(K - X^*)}{K \alpha \eta} < A^* < \frac{\sigma r(a + X^*)(K - X^*) + \phi \alpha \omega K}{K \phi \alpha \eta}.$$

5.1.3 Stability of the equilibria

The stability analysis for system (5.1) is studied by linearization of the non-linear system (5.1) at the equilibrium points of the system and computing the characteristic equation. Hence stability of the equilibrium point is studied depending on the eigenvalues of the corresponding Jacobian, which are functions of the model parameters.

The Jacobian matrix for the system (5.1) at any steady state $J(X, S, I, A)$ is given by

$$\begin{bmatrix} F_{11} & -\alpha X & -\frac{\phi \alpha X}{a+X} & 0 \\ m_1 \alpha S & F_{22} & 0 & -\lambda S - \frac{\gamma S}{(1+A)^2} \\ \frac{m_2 \phi \alpha a I}{(a+X)^2} & \lambda A & F_{33} & -\lambda S - \frac{\gamma I}{(1+A)^2} \\ 0 & \sigma & \sigma & -\eta \end{bmatrix},$$

where

$$\begin{aligned} F_{11} &= r \left(1 - \frac{2X}{K} \right) - \alpha S - \frac{\phi \alpha a I}{(a+X)^2}, \\ F_{22} &= m_1 \alpha X - \lambda A - d - \frac{\gamma A}{1+A}, \\ F_{33} &= \frac{m_2 \phi \alpha X}{a+X} - d - \delta - \frac{\gamma A}{1+A}. \end{aligned}$$

Theorem 5.1.3. *The system is always unstable around the crop-pest free equilibrium point E_0 .*

Proof. The Jacobian matrix at the crop-pest free equilibrium E_0 is given as

$$J(0, 0, 0, \frac{\omega}{\eta}) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\frac{\lambda \omega}{\eta} - d - \frac{\gamma \omega}{\eta + \omega} & 0 & 0 \\ 0 & \frac{\lambda \omega}{\eta} & -d - \delta - \frac{\gamma \omega}{\eta + \omega} & 0 \\ 0 & \sigma & \sigma & -\eta \end{bmatrix},$$

whose characteristic equation in ρ , $|\rho - J(E_0)|$ is given by

$$\begin{vmatrix} \rho - r & 0 & 0 & 0 \\ 0 & \rho + \left(\frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega}\right) & 0 & 0 \\ 0 & -\frac{\lambda \omega}{\eta} & \rho + \left(d + \delta + \frac{\gamma \omega}{\eta + \omega}\right) & 0 \\ 0 & -\sigma & -\sigma & \rho + \eta \end{vmatrix} = 0.$$

The characteristic equation equals

$$(\rho - r) \left(\rho + \frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega} \right) \left(\rho + d + \delta + \frac{\gamma \omega}{\eta + \omega} \right) (\rho + \eta) = 0.$$

Here the corresponding eigenvalues are

$\rho_1 = r > 0$, $\rho_2 = -\eta < 0$, $\rho_3 = -\left(\frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega}\right) < 0$, and $\rho_4 = -\left(d + \delta + \frac{\gamma \omega}{\eta + \omega}\right) < 0$. Since one eigenvalue $\rho_1 = r > 0$, the axial equilibrium E_0 is always unstable. $\square$

Theorem 5.1.4. *The pest free equilibrium point E_1 is locally asymptotically stable if there exist two critical parameters R_0 and R_1 such that $R_0 < 1$ and $R_1 < 1$, where*

$$R_0 = \frac{m_1 \alpha K \eta (\eta + \omega)}{\lambda \omega (\eta + \omega) + \eta \gamma \omega + d \eta (\eta + \omega)} \quad \text{and} \quad R_1 = \frac{m_2 \phi \alpha K (\eta + \omega)}{(a + K) (d + \delta) (\eta + \omega) + (a + K) \gamma \omega}$$

and unstable otherwise.

Proof. The Jacobian matrix $J(E_1)$ at the pest free equilibrium point $E_1(K, 0, 0, \frac{\omega}{\eta})$ is given by

$$J(K, 0, 0, \frac{\omega}{\eta}) = \begin{bmatrix} -r & -K\alpha & -\frac{\phi \alpha K}{a+K} & 0 \\ 0 & m_1 \alpha K - \frac{\lambda \omega}{\eta} - d - \frac{\gamma \omega}{\eta + \omega} & 0 & 0 \\ 0 & \frac{\lambda \omega}{\eta} & \frac{m_2 \phi \alpha K}{a+K} - d - \delta - \frac{\gamma \omega}{\eta + \omega} & 0 \\ 0 & \sigma & \sigma & -\eta \end{bmatrix}.$$

The characteristic equation in ρ at E_1 , $|\rho I - J(E_1)|$ is

$$\begin{vmatrix} \rho + r & K\alpha & \frac{\phi \alpha K}{a+K} & 0 \\ 0 & \rho - m_1 \alpha K + \frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega} & 0 & 0 \\ 0 & -\frac{\lambda \omega}{\eta} & \rho - \frac{m_2 \phi \alpha K}{a+K} + d + \delta + \frac{\gamma \omega}{\eta + \omega} & 0 \\ 0 & -\sigma & -\sigma & \rho + \eta \end{vmatrix} = 0,$$

which gives

$$(\rho + r) \left(\rho - m_1 \alpha K + \frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega} \right) \left(\rho - \frac{m_2 \phi \alpha K}{a + K} + d + \delta + \frac{\gamma \omega}{\eta + \omega} \right) (\rho + \eta) = 0.$$

Thus, the eigenvalues are $\rho_1 = -r$, $\rho_2 = m_1 \alpha K - \frac{\lambda \omega}{\eta} - d - \frac{\gamma \omega}{\eta + \omega}$, $\rho_3 = \frac{m_2 \phi \alpha K}{a + K} - d - \delta - \frac{\gamma \omega}{\eta + \omega}$ and $\rho_4 = -\eta$.

E_1 is locally asymptotically stable if all the four eigenvalues ρ_1, ρ_2, ρ_3 and ρ_4 are negative. Here it is clearly seen that $\rho_1 = -r < 0$, $\rho_4 = -\eta < 0$ since $r > 0$ and $\eta > 0$. So for E_1 to be stable, we should have $\rho_2 < 0$ and $\rho_3 < 0$. That is

$$\begin{aligned} m_1 \alpha K - \frac{\lambda \omega}{\eta} - d - \frac{\gamma \omega}{\eta + \omega} &< 0 \quad \text{and} \quad \frac{m_2 \phi \alpha K}{a + K} - d - \delta - \frac{\gamma \omega}{\eta + \omega} < 0 \\ m_1 \alpha K &< \frac{\lambda \omega}{\eta} + d + \frac{\gamma \omega}{\eta + \omega} \quad \text{and} \quad \frac{m_2 \phi \alpha K}{a + K} < d + \delta + \frac{\gamma \omega}{\eta + \omega} \\ \Rightarrow m_1 \alpha K &< \frac{\lambda \omega (\eta + \omega) + \eta \gamma \omega + d \eta (\eta + \omega)}{\eta (\eta + \omega)} \quad \text{and} \quad \frac{m_2 \phi \alpha K}{a + K} < \frac{(d + \delta) (\eta + \omega) + \gamma \omega}{\eta + \omega} \\ m_1 \alpha K \eta (\eta + \omega) &< \lambda \omega (\eta + \omega) + \eta \gamma \omega + d \eta (\eta + \omega) \quad \text{and} \\ m_2 \phi \alpha K (\eta + \omega) &< (a + K) (d + \delta) (\eta + \omega) + (a + K) \gamma \omega \\ \text{i.e. } \frac{m_1 \alpha K \eta (\eta + \omega)}{\lambda \omega (\eta + \omega) + \eta \gamma \omega + d \eta (\eta + \omega)} &< 1 \quad \text{and} \quad \frac{m_2 \phi \alpha K (\eta + \omega)}{(a + K) (d + \delta) (\eta + \omega) + (a + K) \gamma \omega} < 1 \\ \text{i.e. } R_0 &< 1 \quad \text{and} \quad R_1 < 1 \end{aligned}$$

where $R_0 = \frac{m_1 \alpha K \eta (\eta + \omega)}{\lambda \omega (\eta + \omega) + \eta \gamma \omega + d \eta (\eta + \omega)}$ and $R_1 = \frac{m_2 \phi \alpha K (\eta + \omega)}{(a + K) (d + \delta) (\eta + \omega) + (a + K) \gamma \omega}$ are known as critical parameters. So E_1 is stable if $R_0 < 1$ and $R_1 < 1$ and unstable otherwise. $\square$

The conditions in the stability of the pest free equilibrium point E_1 indicates that if the attack rate of the pest population α is low, then the system may stabilize to the pest free equilibrium point.

Theorem 5.1.5. *The healthy pest free equilibrium $E_3 = (\tilde{X}, 0, \tilde{I}, \tilde{A})$ is locally asymptotically stable if*

$l_i > 0$, for $i = 1, 2, 3, 4$, $l_1 l_2 - l_3 > 0$ and $l_1 l_2 l_3 - l_3^2 - l_1^2 l_4 > 0$ and unstable otherwise, where l_i are defined as in the proof.

Proof. At the healthy pest free fixed point $E_3 (\tilde{X}, 0, \tilde{I}, \tilde{A})$, the Jacobian matrix $J(E_3)$ is given by

$$J(E_3) = \begin{bmatrix} F_{11} & -\alpha \tilde{X} & -\frac{\phi \alpha \tilde{X}}{a + \tilde{X}} & 0 \\ 0 & F_{22} & 0 & 0 \\ \frac{m_2 \phi \alpha a \tilde{I}}{(a + \tilde{X})^2} & \lambda \tilde{A} & F_{33} & -\frac{\gamma \tilde{I}}{(1 + \tilde{A})^2} \\ 0 & \sigma & \sigma & -\eta \end{bmatrix},$$

with the characteristic equation in ρ given by

$$\rho^4 + l_1 \rho^3 + l_2 \rho^2 + l_3 \rho + l_4 = 0, \quad (5.10)$$

where

$$\begin{aligned} F_{11} &= r \left(1 - \frac{2\bar{X}}{K} \right) - \frac{\phi \alpha a \tilde{I}}{(a + \tilde{X})^2}, \\ F_{22} &= m_1 \alpha \bar{X} - \lambda \tilde{A} - d - \frac{\gamma \tilde{A}}{1 + \tilde{A}}, \\ F_{33} &= \frac{m_2 \phi \alpha \tilde{X}}{a + \tilde{X}} - d - \delta - \frac{\gamma \tilde{A}}{1 + \tilde{A}}, \\ C_1 &= -F_{11} - F_{33} + \eta, \\ C_2 &= F_{11} F_{33} - \eta (F_{11} + F_{22}) + \frac{\tilde{I} a \tilde{X} \phi^2 \alpha^2}{(\tilde{X} + a)^3} + \frac{\tilde{I} \gamma \sigma}{(\tilde{A} + 1)^2}, \\ C_3 &= \frac{F_{11} (\eta F_{33} (\tilde{A} + 1)^2 - \tilde{I} \gamma \sigma)}{(\tilde{A} + 1)^2} + \frac{m_2 a \alpha^2 \phi^2 \tilde{I}}{(\tilde{X} + a)^3}, \\ l_1 &= C_1 - F_{22}, \\ l_2 &= C_2 - C_1 F_{22}, \\ l_3 &= C_3 - C_2 F_{22}, \\ l_4 &= -C_3 F_{22}. \end{aligned}$$

From the Routh-Hurwitz criteria, with a fourth degree polynomial, the necessary and sufficient condition for $\Re e(\rho)$ to have negative value is that the conditions

- (i) $l_1 > 0, l_2 > 0, l_3 > 0, l_4 > 0$
- (ii) $l_1 l_2 - l_3 > 0, l_1 l_2 l_3 - l_3^2 - l_1^2 l_4 > 0$

must hold. Hence the proof □

Theorem 5.1.6. *The system (5.1) at the interior equilibrium point E^* is locally asymptotically stable if*

- (i) $y_1 > 0, y_2 > 0, y_3 > 0, y_4 > 0, y_1 y_2 - y_3 > 0$ and
- (ii) $(y_1 y_2 - y_3) y_3 - y_1^2 y_4 > 0$, with each y_i given in the proof.

Proof. The Jacobian matrix $J(E^*)$ at the coexistence equilibrium point E^* is given by

$$J(E^*) = \begin{bmatrix} F_{11} & -\alpha X^* & -\frac{\phi \alpha X^*}{a + X^*} & 0 \\ m_1 \alpha S^* & F_{22} & 0 & -\lambda S^* - \frac{\gamma S^*}{(1 + A^*)^2} \\ \frac{m_2 \phi \alpha a I^*}{(a + X^*)^2} & \lambda A^* & F_{33} & \lambda S^* - \frac{\gamma I^*}{(1 + A^*)^2} \\ 0 & \sigma & \sigma & -\eta \end{bmatrix}.$$

where

$$\begin{aligned}
F_{11} &= r \left(1 - \frac{2X^*}{K} \right) - \alpha \bar{S} - \frac{\phi \alpha a I^*}{(a + X^*)^2}, \\
F_{22} &= m_1 \alpha X^* - \lambda A^* - d - \frac{\gamma A^*}{1 + A^*}, \\
F_{33} &= \frac{m_2 \phi \alpha X^*}{a + X^*} - d - \delta - \frac{\gamma A^*}{1 + A^*}.
\end{aligned}$$

The characteristic equation in ρ for the Jacobian matrix $J(E^*)$ is given by

$$|\rho I - J(E^*)| = \begin{vmatrix} \rho - F_{11} & \alpha X^* & \frac{\phi \alpha X^*}{a + X^*} & 0 \\ -m_1 \alpha S^* & \rho - F_{22} & 0 & \lambda S^* + \frac{\gamma S^*}{(1 + A^*)^2} \\ -\frac{m_2 \phi \alpha a I^*}{(a + X^*)^2} & -\lambda A^* & \rho - F_{33} & -\lambda S^* + \frac{\gamma I^*}{(1 + A^*)^2} \\ 0 & -\sigma & -\sigma & \rho + \eta \end{vmatrix} = 0,$$

which gives

$$\rho^4 + y_1 \rho^3 + y_2 \rho^2 + y_3 \rho + y_4 = 0, \tag{5.11}$$

where

$$\begin{aligned}
y_1 &= -(F_{11} + F_{22} + F_{33}) + \eta > 0, \\
y_2 &= (F_{22} + F_{33} - \eta) F_{11} + (F_{33} - \eta) F_{22} - \eta F_{33} + S^* X^* \alpha^2 m_1, \\
&\quad - \frac{\gamma (I^* + S^*) \sigma}{(1 + A^*)^2 (X^* + a)} - \frac{m_2 \phi^2 \alpha^2 a I^* X^*}{(X^* + a)^3}, \\
y_3 &= \left((-F_{33} + \eta) F_{22} + \eta F_{33} + \frac{\gamma (I^* + S^*) \sigma}{(1 + A^*)^2} \right) F_{11} \\
&\quad + \left(\eta F_{33} + \left(-\lambda S^* + \frac{\gamma I^*}{(1 + A^*)^2} \right) \sigma + \frac{m_2 \phi^2 \alpha^2 a I^* X^*}{(X^* + a)^3} \right) F_{22} \\
&\quad + \left(-S^* X^* \alpha^2 m_1 + \sigma \left(\lambda S^* + \frac{\gamma S^*}{(1 + A^*)^2} \right) \right) F_{33} \\
&\quad + S \alpha^2 \left(\eta + \frac{A^* \lambda \phi}{X^* + a} \right) X^* m_1 - A^* \left(\lambda S^* + \frac{\gamma S^*}{(1 + A^*)^2} \right) \sigma \lambda - \frac{m_2 \phi^2 \alpha^2 a I^* X^* \eta}{(X^* + a)^3}, \\
y_4 &= \left(\left(\lambda S^* - \frac{\gamma I^*}{(1 + A^*)^2} \right) \sigma - \eta F_{33} \right) F_{22} - \sigma \left(\lambda S^* + \frac{\gamma S^*}{(1 + A^*)^2} \right) F_{33} \\
&\quad + A^* \left(\lambda S^* + \frac{\gamma S^*}{(1 + A^*)^2} \right) \sigma \lambda F_{11} + \frac{\alpha^2 \phi^2 \eta X^* F_{22} m_2 a I}{(X^* + a)^3} - S^* X^* \alpha^2 \eta m_1 F_{33} \\
&\quad - \frac{(X^* + a - \phi) m_2 I^* \sigma a \alpha^2 (\lambda (1 + A^*)^2 + \gamma) \phi X^* S}{(1 + A^*)^2 (X^* + a)^3} \\
&\quad + \frac{\left((S^* (X^* + a - \phi) \sigma + A^* \eta \phi) (1 + A^*)^2 \lambda - I^* \gamma \sigma (X^* + a - \phi) \right) S^* X^* \alpha^2 m_1}{(1 + A^*)^2 (X^* + a)}
\end{aligned}$$

Applying the Routh-Hurwitz criterion, the interior equilibrium E^* is locally asymptotically stable if

$$\begin{aligned}
y_1 > 0, y_2 > 0, y_3 > 0, y_4 > 0, y_1 y_2 - y_3 > 0, \\
y_1 y_2 y_3 - y_3^2 - y_1^2 y_4 > 0
\end{aligned}$$

and unstable otherwise. $\square$

Now, we shall find out the conditions for which the system enters into Hopf bifurcation around the interior equilibrium E^* . We focus on the pest consumption rate α which is considered as the most biologically significant parameter.

Routh-Hurwitz Criterion and Hopf-bifurcation: Let us define a continuously differentiable function $\Psi : (0, \infty) \rightarrow \mathbb{R}$ of α as below

$$\Psi(\alpha) := y_1(\alpha) y_2(\alpha) y_3(\alpha) - y_3^2(\alpha) - y_4(\alpha) y_1^2(\alpha).$$

The conditions for the occurrence of Hopf-bifurcation require that the spectrum $\sigma(\alpha) =$

$\{\rho : D(\rho) = 0\}$ of the characteristic equation should satisfy the following conditions:

- (i) there exists $\alpha^* \in (0, \infty)$ at which a pair of complex eigenvalues $\rho(\alpha^*), \bar{\rho}(\alpha^*) \in \sigma(\alpha)$ are such that

$$\Re(\rho(\alpha^*)) = 0, \Im(\rho(\alpha^*)) = \omega_0 > 0$$

with transversality conditions as

$$\left. \frac{d\Re[\rho(\alpha)]}{d\alpha} \right|_{\alpha^*} \neq 0;$$

- (ii) All other elements of $\sigma(\alpha)$ has negative real parts.

An equivalent criteria for the occurrence of the Hopf bifurcation can be stated as:

A Hopf bifurcation of the system at the coexistence equilibrium E^* occurs at $\alpha = \alpha^* \in (0, \infty)$ if and only if

$$\Psi(\alpha^*) = 0, \left. \frac{d\Re[\rho(\alpha)]}{d\alpha} \right|_{\alpha^*} \neq 0 \quad (5.12)$$

and all other eigenvalues have negative real parts. Where $\rho(\alpha)$ is purely imaginary at $\alpha = \alpha^*$.

Theorem 5.1.7. *The system (5.1) around the coexistence equilibrium E^* enters into a Hopf bifurcation at $\alpha = \alpha^* \in (0, \infty)$ if and only if*

$$(1) \Psi(\alpha^*) = 0 \text{ and}$$

$$(2) A(\alpha^*)C(\alpha^*) + B(\alpha^*)D(\alpha^*) \neq 0.$$

and all other eigenvalues are of negative real parts, where $\rho(\alpha)$ is purely imaginary at $\alpha = \alpha^*$.

Proof. The critical value α^* is obtained from the equation $\Psi(\alpha) = 0$. For $\alpha = \alpha^*$ we have

$$\begin{aligned} \Psi(\alpha) = 0 &= y_1(\alpha)y_2(\alpha)y_3(\alpha) - y_3^2(\alpha) - y_4(\alpha)y_1^2(\alpha) \\ &= y_1(\alpha)y_2(\alpha)y_3(\alpha) = y_3^2(\alpha) + y_4(\alpha)y_1^2(\alpha) \\ \text{i.e. } y_1y_2y_3 &= y_3^2 + y_4y_1^2 \end{aligned}$$

from which we get

$$y_2 = \frac{y_3}{y_1} + \frac{y_4y_1}{y_3}.$$

Then the characteristic equation (5.10) becomes

$$\begin{aligned}
0 &= \rho^4 + y_1 \rho^3 + \left(\frac{y_3}{y_1} + \frac{y_1 y_4}{y_3} \right) \rho^2 + y_3 \rho + y_4, \\
&= \rho^4 + y_1 \rho^3 + \frac{y_1 y_4}{y_3} \rho^2 + \frac{y_3}{y_1} \rho^2 + y_3 \rho + y_4, \\
&= \rho^2 \left(\rho^2 + y_1 \rho + \frac{y_1 y_4}{y_3} \right) + \frac{y_3}{y_1} \left(\rho^2 + y_1 \rho + \frac{y_1 y_4}{y_3} \right), \\
&\left(\rho^2 + \frac{y_3}{y_1} \right) \left(\rho^2 + y_1 \rho + \frac{y_1 y_4}{y_3} \right).
\end{aligned} \tag{5.13}$$

Suppose equation (5.13) has four roots, say $\rho_i, (i = 1, 2, 3, 4)$ with the pair of purely imaginary roots at $\alpha = \alpha^*$ as $\rho_1 = \bar{\rho}_2$, so, we get

$$\begin{aligned}
\rho_3 + \rho_4 &= -y_1, \\
\omega_0^2 + \rho_3 \rho_4 &= y_2, \\
\omega_0^2 (\rho_3 + \rho_4) &= -y_3, \\
\omega_0^2 \rho_3 \rho_4 &= y_4,
\end{aligned} \tag{5.14}$$

where $\omega_0 = \text{Im} \rho_1(\alpha^*)$.

From the above relations we obtain that $\omega_0^2 = \frac{y_3}{y_1}$.

Now, if ρ_3 and ρ_4 are complex conjugate. Then from the first part of (5.14), it follows that $2\Re(\rho_3) = -y_1$; if they are real roots, then by the first and last parts of (5.14), $\rho_3 < 0$ and $\rho_4 < 0$. Further as $\psi(\alpha^*)$ is a continuous function of all its roots, there exists an open interval $\alpha \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon)$. ρ_1 and ρ_2 are complex conjugate for α .

Suppose, their general forms in this neighborhood are

$$\begin{aligned}
\rho_1(\alpha) &= \beta_1(\alpha) + i\beta_2(\alpha), \\
\rho_2(\alpha) &= \beta_1(\alpha) - i\beta_2(\alpha),
\end{aligned}$$

Now, we will verify the transversality condition

$$\left. \frac{d\Re[\rho_j(\alpha)]}{d\alpha} \right|_{\alpha=\alpha^*} \neq 0, j = 1, 2. \tag{5.15}$$

Substituting $\rho_j(\alpha) = \beta_1(\alpha) \pm i\beta_2(\alpha)$, into (4.13) we get the following equation

$$(\beta_1(\alpha) + i\beta_2(\alpha))^4 + y_1 (\beta_1(\alpha) + i\beta_2(\alpha))^3 + y_2 (\beta_1(\alpha) + i\beta_2(\alpha))^2 + y_3 (\beta_1(\alpha) + i\beta_2(\alpha)) + y_4 = 0$$

and differentiating with respect to α , we have

$$\begin{aligned} & 4(\beta_1(\alpha) + i\beta_2(\alpha))^3 (\beta_1'(\alpha) + i\beta_2'(\alpha)) + y_1'(\beta_1(\alpha) + i\beta_2(\alpha))^3 \\ & + 3y_1(\beta_1(\alpha) + i\beta_2(\alpha))^2 (\beta_1'(\alpha) + i\beta_2'(\alpha)) + y_2'(\beta_1(\alpha) + i\beta_2(\alpha))^2 \\ & + 2y_2(\beta_1(\alpha) + i\beta_2(\alpha)) (\beta_1'(\alpha) + i\beta_2'(\alpha)) \\ & + y_3'(\beta_1(\alpha) + i\beta_2(\alpha)) + y_3(\beta_1'(\alpha) + i\beta_2'(\alpha)) + y_4' = 0. \end{aligned}$$

That is

$$\begin{aligned} & \underbrace{4i\beta_1^3\beta_2' + 12i\beta_1^2\beta_2\beta_1' + 3i\beta_1^2\beta_2y_1' + 3i\beta_1^2\beta_2'y_1 - 12i\beta_1\beta_2^2\beta_2' + 6i\beta_1\beta_2\beta_1'y_1}_{-4i\beta_2^3\beta_1' - i\beta_2^3y_1' - 3i\beta_2^2\beta_2'y_1 + 2i\beta_1\beta_2y_2' + 2i\beta_1\beta_2'y_2 + 2i\beta_2\beta_1'y_2} \\ & + 4\beta_1^3\beta_1' + \beta_1^3y_1' - 12\beta_1^2\beta_2\beta_2' + 3\beta_1^2\beta_1'y_1 - 12\beta_1\beta_2^2\beta_1' - 3\beta_1\beta_2^2y_1' - 6\beta_1\beta_2\beta_2'y_1 + 4\beta_2^3\beta_2' \\ & - 3\beta_2^2\beta_1'y_1 + i\beta_2y_3' + i\beta_2'y_3 + \beta_1^2y_2' + 2\beta_1\beta_1'y_2 - \beta_2^2y_2' - 2\beta_2\beta_2'y_2 + \beta_1y_3' + \beta_1'y_3 + y_4' = 0. \end{aligned}$$

Comparing real and imaginary parts we get

$$\begin{aligned} & \left(4\beta_1^3 - 12\beta_1\beta_2^2 + 3y_1(\beta_1^2 - \beta_2^2) + 2\beta_1y_2 + y_3\right)\beta_1' \\ & + \left(-12\beta_1^2\beta_2 - 6\beta_1\beta_2y_1 + 4\beta_2^3 - 2\beta_2y_2\right)\beta_2' \\ & + \left(\beta_1^3 - 3\beta_1\beta_2^2\right)y_1' + \left(\beta_1^2 - \beta_2^2\right)y_2' + \beta_1y_3' + y_4' = 0 \end{aligned} \quad (5.16)$$

and

$$\begin{aligned} & \left(12\beta_1^2\beta_2 + 6\beta_1\beta_2y_1 - 4\beta_2^3 + 2\beta_2y_2\right)\beta_1' \\ & + \left(4\beta_1^3 - 12\beta_1\beta_2^2 + 3y_1(\beta_1^2 - \beta_2^2) + 2\beta_1y_2 + y_3\right)\beta_2' \\ & + \left(3\beta_1^2\beta_2 - \beta_2^3\right)y_1' + 2\beta_1\beta_2y_2' + \beta_2y_3' = 0. \end{aligned} \quad (5.17)$$

Equivalently we can write (5.16) and (5.17) in compact form as

$$\begin{aligned} A(\alpha)\beta_1'(\alpha) - B(\alpha)\beta_2'(\alpha) + C(\alpha) &= 0, \\ B(\alpha)\beta_1'(\alpha) + A(\alpha)\beta_2'(\alpha) + D(\alpha) &= 0, \end{aligned} \quad (5.18)$$

where

$$\begin{aligned} A(\alpha) &= 4\beta_1^3 - 12\beta_1\beta_2^2 + 3y_1(\beta_1^2 - \beta_2^2) + 2y_2\beta_1 + y_3, \\ B(\alpha) &= 12\beta_1^2\beta_2 + 6y_1\beta_1\beta_2 - 4\beta_2^3 + 2y_2\beta_2, \\ C(\alpha) &= (\beta_1^3 - 3\beta_1\beta_2^2)y_1' + (\beta_1^2 - \beta_2^2)y_2' + \beta_1y_3' + y_4', \\ D(\alpha) &= (3\beta_1^2\beta_2 - \beta_2^3)y_1' + 2\beta_1\beta_2y_2' + \beta_2y_3'. \end{aligned}$$

Hence, solving the system (5.18) for $\beta'_1(\alpha^*)$ we get

$$\begin{aligned} \left. \frac{d\Re[\rho_j(\alpha)]}{d\alpha} \right|_{\alpha=\alpha^*} &= \beta'_1(\alpha)|_{\alpha=\alpha^*} \\ &= -\frac{A(\alpha^*)C(\alpha^*) + B(\alpha^*)D(\alpha^*)}{A^2(\alpha^*) + B^2(\alpha^*)} \\ &\neq 0 \Leftrightarrow A(\alpha^*)C(\alpha^*) + B(\alpha^*)D(\alpha^*) \neq 0. \end{aligned}$$

Thus, the transversality conditions hold, and hence, Hopf bifurcation occurs at $\alpha = \alpha^*$. $\square$

5.2 The optimal control problem

In this section we introduce an optimal control problem to minimize the negative effects of chemical pesticides and to minimize cost of pest management.

We extend the model system (5.1) by incorporating three time dependent controls $u_1(t)$, $u_2(t)$ and $u_3(t)$, with the first control $u_1(t)$ is for controlling chemical pesticides uses, the second control $u_2(t)$ is for bio-pesticides, and the third control $u_3(t)$ is for advertisement cost. The objective here is to reduce the price of announcement for farming awareness via radio, TV, telephony and other social media. And the price regarding to control measures. Our target here is to find optimal parameters $u_1^*(t)$, $u_2^*(t)$ and $u_3^*(t)$ using the Pontryagin minimum principle approaches given in (Fleming and Lions, 2012). Therefore our system (5.1) is modified to the induced state nonlinear dynamics given by:

$$\begin{cases} \frac{dX}{dt} = rX \left(1 - \frac{X}{K}\right) - \alpha XS - \frac{\phi \alpha XI}{a + X}, \\ \frac{dS}{dt} = m_1 \alpha XS - u_2 \lambda AS - dS - \frac{u_1 \gamma SA}{1 + A}, \\ \frac{dI}{dt} = \frac{m_2 \phi \alpha XI}{a + X} + u_2 \lambda AS - (d + \delta)I - \frac{u_1 \gamma IA}{1 + A}, \\ \frac{dA}{dt} = u_3 \omega + \sigma(S + I) - \eta A, \end{cases} \quad (5.19)$$

with the initial conditions

$$X(0) = X_0, S(0) = S_0, I(0) = I_0 \quad \text{and} \quad A(0) = A_0. \quad (5.20)$$

At this point we need to reduce the number of pests and also the price of pest administration by reducing the cost of pesticides and exploiting the stage of awareness. The objective cost function for the minimization problem is denoted as $J(u_1, u_2, u_3)$ and is defined by

$$J(u_1(t), u_2(t), u_3(t)) = \int_0^{t_f} g(t, \Phi, u) dt = \int_0^{t_f} \left[\frac{P_1 u_1^2(t)}{2} + \frac{P_2 u_2^2(t)}{2} + \frac{P_3 u_3^2(t)}{2} + QS^2 - RA^2 \right] dt \quad (5.21)$$

where $\Phi = (X, S, I, A)$ solves to the induced state system (5.19) for the specific control u . Where the amounts $\frac{P_1}{2}$, $\frac{P_2}{2}$ and $\frac{P_3}{2}$ are the positive weight constants on the benefit of the cost, where as the terms Q and R are the penalty multipliers. We prefer a quadratic cost functional on the controls as an approximation for nonlinear function depending on the assumption that the cost take nonlinear form and also to prevent the bang bang or singular optimal control cases. The control set is defined on $[t_0, t_f]$ subject to the condition $0 < u_i(t) < 1, i = 1, 2, 3$, where t_0 and t_f are starting and final time of giving control, respectively. The aim here is to find the optimal profile of $u_1(t), u_2(t)$ and $u_3(t)$, denoted respectively as $u_i^*(t), i = 1, 2, 3$, so that $J(u_1, u_2, u_3)$ is minimum, that means,

$$J(u_1^*(t), u_2^*(t), u_3^*(t)) = \min(J(u_1(t), u_2(t), u_3(t)) : (u_1, u_2, u_3) \in \mathcal{U}), \quad (5.22)$$

subject to the state system (5.19), where,

$$\mathcal{U} = \left\{ u = (u_1, u_2, u_3) / 0 \leq u_{1\min} \leq u_1(t) \leq u_{1\max} \leq 1, 0 \leq u_{2\min} \leq u_2(t) \leq u_{2\max} \leq 1, \right. \\ \left. 0 \leq u_{3\min} \leq u_3(t) \leq u_{3\max} \leq 1, t \in [0, t_f] \right\} \quad (5.23)$$

is an admissible control set.

Pontryagin Minimum Principle Pontryagin (2018) is used to find the optimal control triplet $u^*(t) = (u_1^*, u_2^*, u_3^*)$ that includes the Hamiltonian function defined as

$$\mathcal{H} = \left[\frac{P_1 u_1^2}{2} + \frac{P_2 u_2^2}{2} + \frac{P_3 u_3^2}{2} + QS^2 - RA^2 \right] + \sum_{i=1}^4 \lambda_i(t) f_i(X, S, I, A) \quad (5.24)$$

where, $\lambda_i, i = 1, 2, \dots, 4$ are adjoint variables and $f_i, i = 1, 2, 3, 4$ are the right hand sides of system (5.19) at the i^{th} state.

5.2.1 Existence of the optimal control triple

The existence of the optimal control triple can be guaranteed by using the results in the book by (Fleming and Lions, 2012). All the state variables involved in the model are continuously differentiable. Therefore, we need to verify the following combined conditions from (Lenhart and Workman, 2007; Fleming and Lions, 2012; Pontryagin, 2018).

Theorem 5.2.1. *Given the objective functional*

$$J(u_1(t), u_2(t), u_3(t)) = \int_0^{t_f} \left[\frac{P_1 u_1^2}{2} + \frac{P_2 u_2^2}{2} + \frac{P_3 u_3^2}{2} + QS^2 - RA^2 \right] dt$$

where

$$\mathcal{U} = \{(u_1(t), u_2(t), u_3(t)) : u_i(t) \text{ is Lebesgue measurable and } 0 \leq u_i(t) \leq 1, \quad t \in [0, t_f]\}$$

subject to the system (5.19) with the initial condition $(X_0, S_0, I_0, A_0) \geq (0, 0, 0, 0)$. Then there exists an optimal control triple $(u_1^*(t), u_2^*(t), u_3^*(t))$ and corresponding state solution X^*, S^*, I^*, A^* such that

$J(u_1^*(t), u_2^*(t), u_3^*(t)) = \min_{\mathcal{U}} J(u_1(t), u_2(t), u_3(t))$ if the following conditions are met

- (i) the set of solutions to the system (5.19)–(5.20) with control variables in (5.23) are non-empty.
- (ii) The control set $\mathcal{U}$ is convex and closed
- (iii) Each right hand side of the state system (5.19) is continuous, is bounded above by a sum of the bounded control and the state, and can be written as a linear function of u with coefficients depending on time and the state.
- (iv) The integrand $g(t, \Phi, u)$ of the objective functional (5.21) is convex on $\mathcal{U}$
- (v) There exist positive numbers $\ell_1, \ell_2, \ell_3, \ell_4$ and a constant $\ell > 1$ such that

$$g(t, \Phi, u) \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell + \ell_4 |u_3|^\ell$$

Proof. We now justify each of the items stated above to complete the proof.

- (i) Since $\mathcal{U}$ is a nonempty set of real valued measurable functions on the finite time interval $0 \leq t_f$, the system (5.19) has bounded coefficients and hence any solutions are bounded on $[0, t_f]$ (see (5.1.2)). It follows that the corresponding solutions for the system (5.19) exist.

- (ii) To prove the convexity of the control set $\mathcal{U}$, let us consider

$$\mathcal{U} = \{u \in \mathbf{R}^3 : \|u\| \leq 1 - \varepsilon\} \quad \text{for small } 0 < \varepsilon < 1$$

Let $u_1, u_2 \in \mathcal{U}$ such that $\|u_1\| \leq 1 - \varepsilon$ and $\|u_2\| \leq 1 - \varepsilon$. Then for an arbitrary $\lambda \in$

$[0, 1]$, we have

$$\begin{aligned}
\|\lambda u_1 + (1 - \lambda) u_2\| &\leq \|\lambda u_1\| + \|(1 - \lambda) u_2\| \\
&= |\lambda| \|u_1\| + |(1 - \lambda)| \|u_2\| \\
&= \lambda \|u_1\| + (1 - \lambda) \|u_2\| \\
&\leq \lambda (1 - \varepsilon) + (1 - \lambda) (1 - \varepsilon) \\
&= \lambda (1 - \varepsilon) + 1 - \varepsilon - \lambda (1 - \varepsilon) \\
&= 1 - \varepsilon
\end{aligned}$$

Hence $\|\lambda u_1 + (1 - \lambda) u_2\| \leq 1 - \varepsilon$. This implies that the control set $\mathcal{U}$ is convex. $0 \leq u_{1\min} \leq u_1(t) \leq u_{1\max} \leq 1$, $0 \leq u_{2\min} \leq u_2(t) \leq u_{2\max} \leq 1$, and $0 \leq u_{3\min} \leq u_3(t) \leq u_{3\max} \leq 1$ is closed by definition. Therefore $\mathcal{U}$ is convex and closed set.

iii All the right-hand sides of equations of system (5.19) are continuous, all variables X, S, I, A and u are bounded on $[0, t_f]$, and can be written as a linear function of u_1, u_2 , and u_3 with coefficients depending on the time and state.

(iv) Let $\lambda \in [0, 1]$ and $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathcal{U}$, then we have

$$\begin{aligned}
&g(t, \Phi, (1 - \lambda)a + \lambda b) - ((1 - \lambda)g(t, \Phi, a) + \lambda g(t, \Phi, b)) \\
&= \frac{P_1}{2} \left[(1 - \lambda)^2 a_1^2 + 2\lambda(1 - \lambda)a_1 b_1 + \lambda^2 b_1^2 \right] \\
&+ \frac{P_2}{2} \left[(1 - \lambda)^2 a_2^2 + 2\lambda(1 - \lambda)a_2 b_2 + \lambda^2 b_2^2 \right] \\
&+ \frac{P_3}{2} \left[(1 - \lambda)^2 a_3^2 + 2\lambda(1 - \lambda)a_3 b_3 + \lambda^2 b_3^2 \right] \\
&- \left[(1 - \lambda) \left(\frac{P_1}{2} a_1^2 + \frac{P_2}{2} a_2^2 + \frac{P_3}{2} a_3^2 \right) \right] \\
&- \lambda \left(\frac{P_1}{2} b_1^2 + \frac{P_2}{2} b_2^2 + \frac{P_3}{2} b_3^2 \right) \\
&= (\lambda^2 - \lambda) \left(\frac{P_1}{2} (a_1 - b_1)^2 + \frac{P_2}{2} (a_2 - b_2)^2 + \frac{P_3}{2} (a_3 - b_3)^2 \right) \\
&= \frac{(\lambda^2 - \lambda)}{2} (a - b)^2 \\
&= \frac{1}{2} \lambda \underbrace{(\lambda - 1)}_{\leq 0} (a - b)^2 \leq 0
\end{aligned}$$

Hence $g(t, \Phi, (1 - \lambda)a + \lambda b) \leq (1 - \lambda)g(t, \Phi, a) + \lambda g(t, \Phi, b)$. Therefore $g(t, \Phi, u)$ is convex in the control set $\mathcal{U}$.

(v) Finally it remains to show that there exists a constant $\ell^* > 1$ and positive constants

ℓ_1, ℓ_2, ℓ_3 and ℓ_4 such that

$$\frac{P_1 u_1^2(t)}{2} + \frac{P_2 u_2^2(t)}{2} + \frac{P_3 u_3^2(t)}{2} + QS^2 - RA^2 \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell + \ell_4 |u_3|^\ell$$

In section 5.1.1, we already show that the state variables are bounded;

let $\ell_1 = \sup (QS^2 - RA^2)$, $\ell_2 = P_1$, $\ell_3 = P_2$, $\ell_4 = P_3$ and $\ell = 2$; then it follows that

$$\frac{P_1 u_1^2(t)}{2} + \frac{P_2 u_2^2(t)}{2} + \frac{P_3 u_3^2(t)}{2} + QS^2 - RA^2 \geq -\ell_1 + \ell_2 |u_1|^\ell + \ell_3 |u_2|^\ell + \ell_4 |u_3|^\ell$$

Then, from Fleming and Lions (2012), we conclude that there exists an optimal control triple.

□

5.2.2 Characterization of the optimal control

Since there exists an optimal control triple for minimizing the functional,

$$J(u_1(t), u_2(t), u_3(t)) = \int_0^{t_f} \left[\frac{P_1 u_1^2(t)}{2} + \frac{P_2 u_2^2(t)}{2} + \frac{P_3 u_3^2(t)}{2} + QS^2 - RA^2 \right] dt$$

subject to the controlled system (5.19), we derive necessary conditions on the optimal control triple using the Pontryagin's Minimum Principle (Fleming and Lions, 2012). The necessary conditions include the optimality solutions and the adjoint equations that an optimal control must satisfy which come from Pontryagin's maximum principle (Pontryagin, 2018). This principle converts systems (5.19) and (5.21) into a problem of minimizing point-wise Hamiltonian function $\mathcal{H}$, which is formed by allowing each of the adjoint variables that correspond to each of the state variables accordingly and combining the results with the objective functional. The Hamiltonian associated to our problem at time t is explicitly given by

$$\begin{aligned} \mathcal{H}(t, \Phi, u, \lambda) = & \frac{P_1 u_1^2(t)}{2} + \frac{P_2 u_2^2(t)}{2} + \frac{P_3 u_3^2(t)}{2} + QS^2 - RA^2 \\ & + \lambda_1 \left(rX \left(1 - \frac{X}{K} \right) - \alpha XS - \frac{\phi \alpha XI}{a + X} \right) \\ & + \lambda_2 \left(m_1 \alpha XS - u_2 \lambda AS - dS - \frac{u_1 \gamma SA}{1 + A} \right) \\ & + \lambda_3 \left(\frac{m_2 \phi \alpha XI}{a + X} + u_2 \lambda AS - (d + \delta)I - \frac{u_1 \gamma IA}{1 + A} \right) \\ & + \lambda_4 (u_3 \omega + \sigma(S + I) - \eta A). \end{aligned}$$

If the optimal control $u^* = (u_1^*, u_2^*, u_3^*)$ and the corresponding state $\Phi^* = (X^*, S^*, I^*, A^*)$ are an optimal couple, then necessarily there exists a non-trivial adjoint vector $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$

satisfying the following equality

$$\begin{cases} \frac{d\Phi}{dt} = \frac{\partial \mathcal{H}(t, \Phi, u, \lambda)}{\partial \lambda}, \\ \frac{d\lambda}{dt} = -\frac{\partial \mathcal{H}(t, \Phi, u, \lambda)}{\partial \Phi}, \\ \frac{\partial \mathcal{H}(t, \Phi, u, \lambda)}{\partial u} = 0. \end{cases}$$

By the theory given for bounded controls of several variables in [Lenhart and Workman \(2007\)](#) we obtain the optimal solution for the controls as

$$\begin{cases} u_i^* = 0, & \text{if } \frac{\partial \mathcal{H}}{\partial u_i} < 0, \\ 0 \leq u_i^* \leq u_{i\max}, & \text{if } \frac{\partial \mathcal{H}}{\partial u_i} = 0, \\ u_i^* = u_{i\max}, & \text{if } \frac{\partial \mathcal{H}}{\partial u_i} > 0. \end{cases} \quad (5.25)$$

Theorem 5.2.2. *Given the optimal controls (u_1^*, u_2^*, u_3^*) and the solutions (X^*, S^*, I^*, A^*) of the corresponding state system (5.19), then there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying the following system of equations*

$$\begin{cases} \frac{d\lambda_1}{dt} = \lambda_1 \left(\alpha S + \frac{\phi \alpha a I}{(a+X)^2} - r \left(1 - \frac{2X}{K} \right) \right) - \lambda_2 m_1 \alpha S - \lambda_3 \frac{m_2 \phi \alpha a I}{(a+X)^2}, \\ \frac{d\lambda_2}{dt} = -2QS + \lambda_1 \alpha X + \lambda_2 \left(\frac{u_1 \gamma A}{1+A} + u_2 \lambda A + d - m_1 \alpha X \right) - \lambda_3 u_2 \lambda A - \lambda_4 \sigma, \\ \frac{d\lambda_3}{dt} = \lambda_1 \frac{\phi \alpha X}{a+X} + \lambda_3 \left(\frac{u_1 \gamma A}{1+A} + d + \delta - \frac{m_2 \phi \alpha X}{a+X} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_4}{dt} = 2RA + \lambda_2 \left(\frac{u_1 \gamma S}{(1+A)^2} + u_2 \lambda S \right) + \lambda_3 \left(\frac{u_1 \gamma I}{(1+A)^2} - u_2 \lambda S \right) + \lambda_4 \eta, \end{cases} \quad (5.26)$$

with transversality conditions $\lambda_i(t_f) = 0$ for $i = 1, 2, 3, 4$.

Furthermore for $t \in [0, t_f]$, the optimal controls u_1^*, u_2^* and u_3^* are characterized by

$$\begin{aligned} u_1^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} \right\} \right\}, \\ u_2^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} \right\} \right\}, \\ u_3^* &= \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \omega}{P_3} \right\} \right\}. \end{aligned} \quad (5.27)$$

Proof. Suppose (u_1^*, u_2^*, u_3^*) is an optimal control and X^*, S^*, I^* and A^* are the state solutions to the optimal control. To establish the adjoint equations and the transversality condi-

tions we describe the Hamiltonian function connected with the system as

$$\begin{aligned}
\mathcal{H}(t, \Phi, u, \lambda) = & \frac{1}{2}P_1 u_1^2 + \frac{1}{2}P_2 u_2^2 + \frac{1}{2}P_3 u_3^2 + QS^2 - RA^2 \\
& + \lambda_1 \left(rX \left(1 - \frac{X}{K} \right) - \alpha XS - \frac{\phi \alpha XI}{a+X} \right) \\
& + \lambda_2 \left(m_1 \alpha XS - u_2 \lambda AS - dS - \frac{u_1 \gamma SA}{1+A} \right) \\
& + \lambda_3 \left(\frac{m_2 \phi \alpha XI}{a+X} + u_2 \lambda AS - (d + \delta)I - \frac{u_1 \gamma IA}{1+A} \right) \\
& + \lambda_4 (u_3 \omega + \sigma(S+I) - \eta A).
\end{aligned} \tag{5.28}$$

where, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are adjoint functions to be determined appropriately.

For $t \in [0, t_f]$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's minimum principle. Substituting $X(t) = X^*(t), S(t) = S^*(t), I(t) = I^*(t)$ and $A(t) = A^*(t)$ and taking derivative of the Hamiltonian with respect to X, S, I and A , we get the following result

$$\begin{cases}
\frac{d\lambda_1}{dt} = -\frac{\partial \mathcal{H}}{\partial X} = \lambda_1 \left(\alpha S + \frac{\phi \alpha a I}{(a+X)^2} - r \left(1 - \frac{2X}{K} \right) \right) - \lambda_2 m_1 \alpha S - \lambda_3 \frac{m_2 \phi \alpha a I}{(a+X)^2}, \\
\frac{d\lambda_2}{dt} = -\frac{\partial \mathcal{H}}{\partial S} = -2QS + \lambda_1 \alpha X + \lambda_2 \left(\frac{u_1 \gamma A}{(1+A)} + u_2 \lambda A + d - m_1 \alpha X \right) - \lambda_3 u_2 \lambda A - \lambda_4 \sigma, \\
\frac{d\lambda_3}{dt} = -\frac{\partial \mathcal{H}}{\partial I} = \lambda_1 \frac{\phi \alpha X}{a+X} + \lambda_3 \left(\frac{u_1 \gamma A}{1+A} + d + \delta - \frac{m_2 \phi \alpha X}{a+X} \right) - \lambda_4 \sigma, \\
\frac{d\lambda_4}{dt} = -\frac{\partial \mathcal{H}}{\partial A} = 2RA + \lambda_2 \left(\frac{u_1 \gamma S}{(1+A)^2} + u_2 \lambda S \right) + \lambda_3 \left(\frac{u_1 \gamma I}{(1+A)^2} - u_2 \lambda S \right) + \lambda_4 \eta.
\end{cases} \tag{5.29}$$

Assuming that the final states $X(t_f), S(t_f), I(t_f), A(t_f)$ are free we get the transversality(boundary) conditions for the adjoint variables

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0. \tag{5.30}$$

For $t \in [0, t_f]$, the optimal controls u_1^*, u_2^* and u_3^* can be solved from the optimality conditions and the properties of the control set $\mathcal{U}$. The optimality condition of the Hamiltonian in the interior of the control set are given as

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial u_1} &= 0, \\
\frac{\partial \mathcal{H}}{\partial u_2} &= 0, \\
\frac{\partial \mathcal{H}}{\partial u_3} &= 0.
\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial u_1} &= P_1 u_1 - \frac{\lambda_2 \gamma A S}{1+A} - \frac{\lambda_3 \gamma I A}{1+A} = 0, \\ \frac{\partial \mathcal{H}}{\partial u_2} &= P_2 u_2 - \lambda_2 \lambda A S + \lambda_3 \lambda A S = 0, \\ \frac{\partial \mathcal{H}}{\partial u_3} &= P_3 u_3 + \lambda_4 \omega = 0.\end{aligned}\tag{5.31}$$

Hence solving for $u_i, i = 1, 2, 3$ we have

$$\begin{aligned}u_1(t) &= \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)}, \\ u_2(t) &= \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2}, \\ u_3(t) &= -\frac{\lambda_4 \omega}{P_3}.\end{aligned}\tag{5.32}$$

Using the boundedness of the standard controls u_1, u_2 and u_3 in $\mathcal{U}$ we obtain

$$u_1^* = \begin{cases} 0, & \text{if } \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} \leq 0, \\ \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)}, & \text{if } 0 < \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} < 1, \\ 1, & \text{if } \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} \geq 1. \end{cases}\tag{5.33}$$

$$u_2^* = \begin{cases} 0, & \text{if } \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} \leq 0, \\ \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2}, & \text{if } 0 < \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} < 1, \\ 1, & \text{if } \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} \geq 1. \end{cases}\tag{5.34}$$

$$u_3^* = \begin{cases} 0, & \text{if } -\frac{\lambda_4 \omega}{P_3} \leq 0, \\ -\frac{\lambda_4 \omega}{P_3}, & \text{if } 0 < -\frac{\lambda_4 \omega}{P_3} < 1, \\ 1, & \text{if } -\frac{\lambda_4 \omega}{P_3} \geq 1. \end{cases}\tag{5.35}$$

In compact notation we can write the optimal solutions for the controls as

$$\begin{aligned}u_1^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} \right\} \right\}, \\ u_2^* &= \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} \right\} \right\}, \\ u_3^* &= \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \omega}{P_3} \right\} \right\}.\end{aligned}\tag{5.36}$$

□

In order to confirm that the optimal control is really a minimizer we need to check the Hessian matrix of the Hamiltonian $\mathcal{H}$. Now the Hessian matrix for $\mathcal{H}$ with respect the control $\mathcal{U}$ is given by

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = \begin{bmatrix} \frac{\partial^2 \mathcal{H}}{\partial u_1^2} & \frac{\partial^2 \mathcal{H}}{\partial u_1 \partial u_2} & \frac{\partial^2 \mathcal{H}}{\partial u_1 \partial u_3} \\ \frac{\partial^2 \mathcal{H}}{\partial u_2 \partial u_1} & \frac{\partial^2 \mathcal{H}}{\partial u_2^2} & \frac{\partial^2 \mathcal{H}}{\partial u_2 \partial u_3} \\ \frac{\partial^2 \mathcal{H}}{\partial u_3 \partial u_1} & \frac{\partial^2 \mathcal{H}}{\partial u_3 \partial u_2} & \frac{\partial^2 \mathcal{H}}{\partial u_3^2} \end{bmatrix} = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}$$

This matrix is found to be a positive definite matrix as a consequence of the positive weights P_1, P_2, P_3 . This results in the convexity of the corresponding Hamiltonian function $\mathcal{H}$ and as a result of this the optimal control $u = (u_1, u_2, u_3)$ is really a minimizer.

5.2.3 The optimality system

The optimal controls and the states are found by solving a system of dynamics called **optimality system**, and consisting of

- (i) the state system (5.19) together with their initial conditions(5.20),
- (ii) the adjoint system (5.29),
- (iii) the boundary conditions (5.30) and,
- (iv) the characterization of the optimal controls (5.36).

In combination we represent it as

$$\left\{ \begin{array}{l} \frac{dX^*}{dt} = rX^* \left(1 - \frac{X^*}{K}\right) - \alpha X^* S^* - \frac{\phi \alpha X^* I^*}{a+X^*}, \\ \frac{dS^*}{dt} = m_1 \alpha X^* S^* - u_2^* \lambda A^* S^* - dS^* - \frac{u_1^* \gamma S^* A^*}{1+A^*}, \\ \frac{dI^*}{dt} = \frac{m_2 \phi \alpha X^* I^*}{a+X^*} + u_2^* \lambda A^* S^* - (d + \delta) I^* - \frac{u_1^* \gamma I^* A^*}{1+A^*}, \\ \frac{dA^*}{dt} = u_3^* \omega + \sigma(S^* + I^*) - \eta A^*, \\ X(0) \geq 0, S(0) \geq 0, I(0) \geq 0, A(0) \geq 0, \\ \frac{d\lambda_1}{dt} = \lambda_1 \left(\alpha S^* + \frac{\phi \alpha a I^*}{(a+X^*)^2} - r \left(1 - \frac{2X^*}{K}\right) \right) - \lambda_2 m_1 \alpha S^* - \lambda_3 \frac{m_2 \phi \alpha a I^*}{(a+X^*)^2}, \\ \frac{d\lambda_2}{dt} = -2Q S^* + \lambda_1 \alpha X^* + \lambda_2 \left(\frac{u_1^* \gamma A^*}{(1+A^*)} + u_2^* \lambda A^* + d - m_1 \alpha X^* \right) - \lambda_3 u_2^* \lambda A^* - \lambda_4 \sigma, \\ \frac{d\lambda_3}{dt} = \lambda_1 \frac{\phi \alpha X^*}{a+X^*} + \lambda_3 \left(\frac{u_1^* \gamma A^*}{1+A^*} + d + \delta - \frac{m_2 \phi \alpha X^*}{a+X^*} \right) - \lambda_4 \sigma, \\ \frac{d\lambda_4}{dt} = 2R A^* + \lambda_2 \left(\frac{u_1^* \gamma S^*}{(1+A^*)^2} + u_2^* \lambda S^* \right) + \lambda_3 \left(\frac{u_1^* \gamma I^*}{(1+A^*)^2} - u_2^* \lambda S^* \right) + \lambda_4 \eta, \\ u_1^* = \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 S + \lambda_3 I) \gamma A}{P_1 (1+A)} \right\} \right\}, \\ u_2^* = \max \left\{ 0, \min \left\{ 1, \frac{(\lambda_2 - \lambda_3) \lambda A S}{P_2} \right\} \right\}, \\ u_3^* = \max \left\{ 0, \min \left\{ 1, -\frac{\lambda_4 \omega}{P_3} \right\} \right\}, \\ \lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = 0. \end{array} \right. \quad (5.37)$$

Here, we do notice that the adjoint system (5.29) is also linear in λ_i for $i = 1, 2, 3, 4$ with bounded coefficients. Thus there exists a positive real number M such that $|\lambda_i| \leq M$ on $t \in [0, t_f]$. Hence with all the privileges previously discussed in the existence and uniqueness properties for our system (5.19) and the control set (5.23), we clearly come with the conclusion that for sufficiently small time t_f the solution to the optimality system (5.37) is unique. The need for the littleness limit of the time span is due to the opposite time direction/orientations of the optimality system; the state problem has initial values and the adjoint problem has terminal values. Following Pontryagin Minimum Principle to solve this problem analytically is not practical. Consequently we use numerical algorithms to find approximate optimal solutions Φ^* and u^* .

5.3 Numerical simulations

In this section, we solve the model systems numerically and observe the behavior of some solutions as time varies. Since the analytical solutions of systems (5.1) are not practical to analyze, the numerical results play a great role in characterizing the dynamics. Our numerical solutions show how realistic our results are and illustrate well the predicted analytical behavior. We begin by analyzing the system (5.1), without controls, then our model (5.19) subject to the optimal controls, as characterized by Pontryagin Minimum Principle (PMP). Our numerical simulations are acquired with a set of parameter values as given in Table 5.1.

For our numerical experiments, we take $t_f = 600$ days. Besides, on the numerical simulations the values of the weight function are taken as $P_1 = 0.8$, $P_2 = 0.5$, $P_3 = 0.5$, $Q = 10$, and $R = 10$, and the initial state variables as $X(0) = 0.2$, $S(0) = 0.07$, $I(0) = 0.05$, $A(0) = 0.5$. In Figure ?? and Figure 5.3, the time series solution of model system (5.1) are sketched with different values of the parameters α and γ . It is observed that our model variables $X(t)$, $S(t)$, $I(t)$ and $A(t)$ become oscillating as the values of the rates (i.e., α and γ) get larger and finally become stable. Also, steady state value of both pest population (when exists) are decreased as α and γ rise. A bifurcation illustration is shown in Figure 5.4, taking α as the main parameter. Critical values depend on many parameters, such as the conversion rates m_1 and m_2 , rate of the awareness program σ , the recruitment rate of global awareness ω , the chemical pesticide control u_1 etc.

Table 5.1: Parameter values used in our numerical simulations.

Parameters	Description	Value	Source
r	Growth rate of crop biomass	0.05	(Al Basir and Ray, 2020)
K	Maximum density of crop biomass	1	(Abraha et al., 2021b)
λ	Aware people activity rate	0.025	(Al Basir et al., 2015)
d	Natural mortality of pest	0.01	(Abraha et al., 2021b)
m_1	Conversion efficacy of susceptible pests	0.8	(Al Basir et al., 2015)
m_2	Conversion efficiency of infected pest	0.6	(Al Basir et al., 2019a)
δ	Disease related mortality rate	0.1	(Abraha et al., 2021b)
a	Half saturation constant	0.2	(Jana et al., 2017)
α	Attack rate of pest	0.025	(Al Basir et al., 2019a)
σ	Local rate of increase of awareness	0.015	(Abraha et al., 2021a)
γ	Increase level from global advertisement	0.025	(Al Basir and Ray, 2020)
η	Fading of memory of aware people	0.015	(Al Basir and Ray, 2020)
ω	Rate of global awareness (via TV, radio)	0.003	(Abraha et al., 2021b)

We examine the impact of optimal control profiles by implementing a Runge–Kutta fourth-order scheme on the optimality system(5.37). The dynamical behavior of the model, in relation to the triple controls is presented. The optimal policy is achieved by finding a solution to the state system (5.1) and costate system (5.19). The control problem is numerically solved by the forward-backward sweep method Lenhart and Workman (2007) in combination with the well-known fourth-order Runge–Kutta algorithm. It needs to solve four-state equations and four adjoint equations. For that, first we solve system (5.19) with an initial guess for the controls forward in time and then using the transversality conditions as initial values; the adjoint system (5.29) is solved backward in time using the current iteration solution of the state system. The controls are updated by using a convex combination of the previous controls and the values from (5.36). The process continues until the solution of the state equations at the present is very close to the previous iteration values. For our numerical computations we use the following algorithm. It is a two point boundary-value problem, with divided boundary conditions at $t_0 = 0$ and $t = t_f$.

Algorithm 1 Forward-Backward Sweep Method

- 1: Make an initial guess for $\vec{u}$ over the time interval ($\vec{u} \equiv 0$ is almost always sufficient)
 - 2: Using the initial condition $x_1 = x(t_0) = a$ and the values for $\vec{u}$, solve $\vec{x}$ **forward** in time in compliance with its differential equation in the optimality system (using, e.g., RK4).
 - 3: Using the transversality condition $\lambda_{N+1} = \lambda(t_f)$ and the values for $\vec{u}$ and $\vec{x}$, solve $\vec{\lambda}$ **backward** in time according to its differential equation in the optimality system (using, e.g., RK4).
 - 4: Update $\vec{u}$ using the new values for $\vec{x}$ and $\vec{\lambda}$ into the characterization of the optimal control.
 - 5: Check convergence: if the variables are sufficiently close to the corresponding in the previous iteration, then output the current values as solutions, else return to Step 2.
-

If we take We take the parameters of the system as $r = 0.1, K = 1, \lambda = 0.08, d = 0.01, m_1 = 0.8, m_2 = 0.6, \delta = 0.2, \sigma = 0.015, a = 0.5, \eta = 0.015, \alpha = 0.012, \gamma = 0.08, \phi = 0.5$ and $(X(0), S(0), I(0), A(0)) = (0.5, 0.5, 0.5, 0.5)$. Then $E_1(K, 0, 0, \frac{\gamma}{\eta}) = E_1(1, 0, 0, 2)$. Figure 5.2 shows that X, Y, S, A approach to their steady state values $E_1(K, 0, 0, \frac{\gamma}{\eta})$ in finite time.

If we take We take the parameters of the system as $r = 0.1, K = 1, \lambda = 0.008, d = 0.01, m_1 = 0.8, m_2 = 0.9, \delta = 0.2, \sigma = 0.015, a = 1, \eta = 0.015, \alpha = 0.05, \gamma = 0.008, \phi = 0.9$ and $(X(0), S(0), I(0), A(0)) = (0.4, 0.3, 0.2, 0.7)$. Then $E_1(K, 0, 0, \frac{\gamma}{\eta}) = E_2(1, 0, 0, 2)$. Figure 5.3 shows that X, Y, S, A approach to their steady state values $E_2(0.649, 0, 1.285, 3.285)$ in finite time.

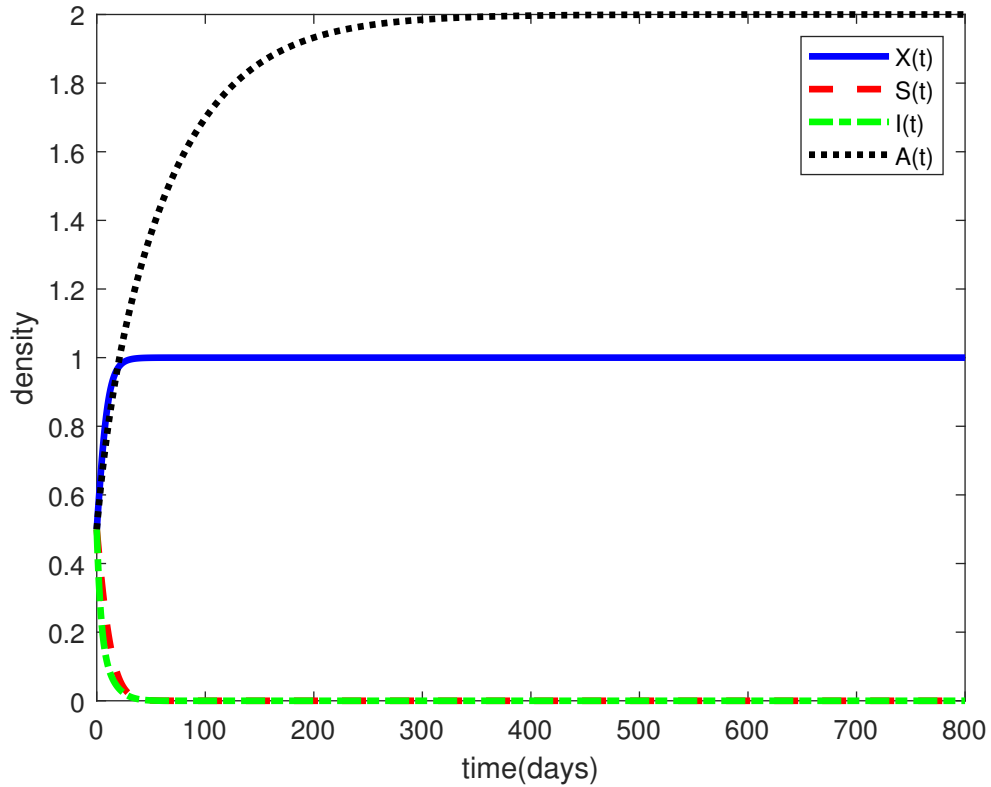


Figure 5.2: X, S, I, A approach to their equilibrium value $=E_2(1, 0, 0, 2)$ in finite time.

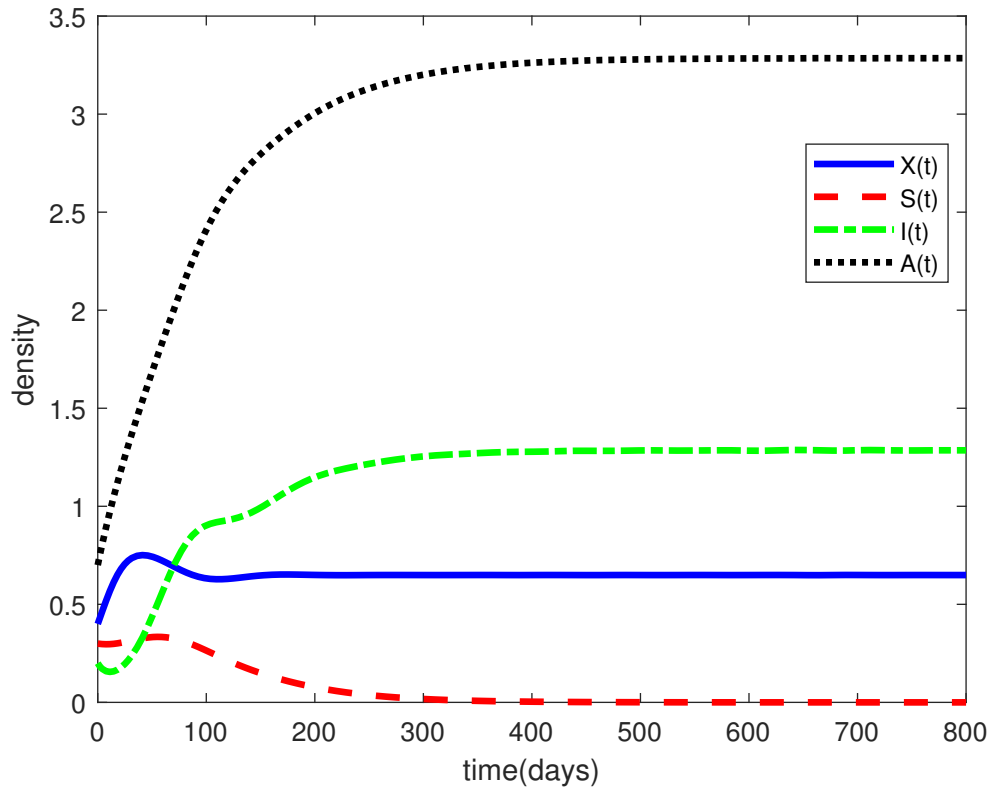


Figure 5.3: X, S, I, A approach to their equilibrium value $E_2(0.649, 0, 1.285, 3.285)$ in finite time.

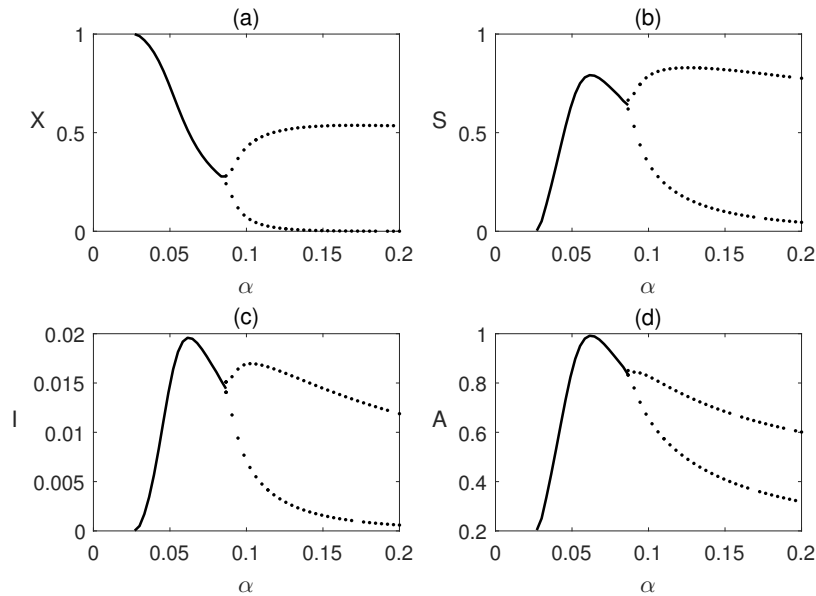


Figure 5.4: Bifurcation diagram of the coexisting equilibrium E^* (when exists) of system (5.1) with respect to the attack rate α of pest. Solid line indicates stable endemic equilibrium.

The numerical solution of optimal control problem on the system is given in Figure 5.5 showing the impact of optimal control. We apply the control through chemical pesticide

effects, bio-pesticides and cost of advertisements for a time period of 60 days. In this figure, we notice that due to effort of optimal controls u_1^*, u_2^*, u_3^* , crop biomass population obtains its maximum value in 60 days, susceptible pest minimizes and infected pest is also minimized and reduced to 0 in the first 20 days. The population of pest reduced radically with an influence of the best frameworks of universal awareness (i.e. $u_2^* \lambda$) and chemical pesticides control movement, $u_1^* \gamma$. It is also seen that the susceptible pest population goes to devastation inside the earliest 50 days, due to the effort of the extremal controls, which are shown.

Thus, optimal control policy by means of chemical pesticides, biological control, and global farming awareness has great influences in making the system free of pest and maintaining the stable nature in the remaining time period. Figure 5.9 show that optimal chemical pesticides and biological control is needed to control environmental crop biomass and to minimize the cost of cultivation with optimal awareness through global Media. Figures 5.6, 5.7 and 5.8 are plotted for the dynamics of the system with the influence of pairwise controls, $u_1^*, u_2^*, u_1^*, u_3^*$ and u_2^*, u_3^* respectively. Comparing with Figure 5.5, we observe that when one control parameter is absent, other controls are required but with higher efficacy in order to get maximum yield i.e. with a maximum density of crop biomass X as well as minimum density of susceptible pest S .

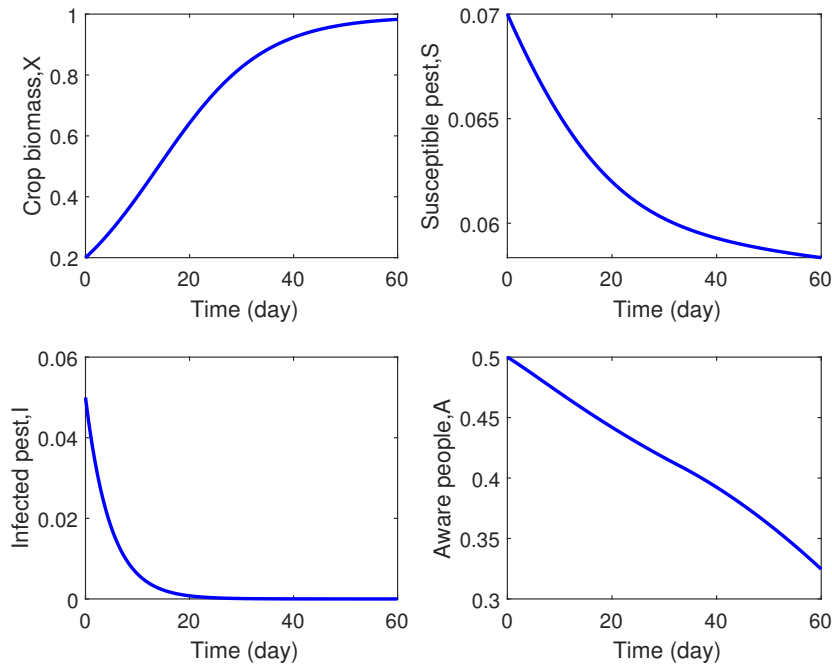


Figure 5.5: Numerical solutions of the optimal control system model (5.19) with $u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$ taking parameters as in Table 5.1.

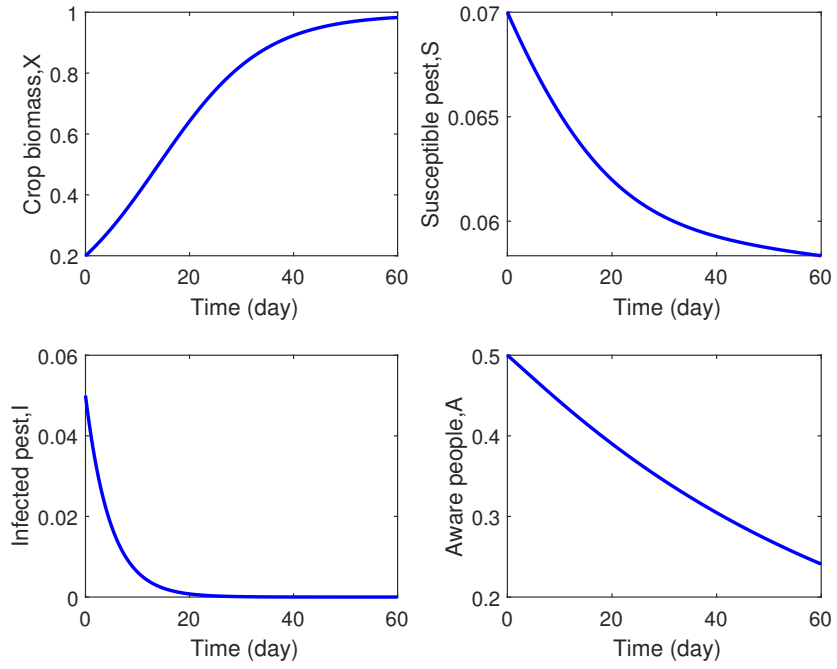


Figure 5.6: Numerical solutions of the optimal control system model (5.19) with $u_1 \neq 0, u_2 \neq 0$ and $u_3 \equiv 0$ using parameter values as as in Figure 5.5.

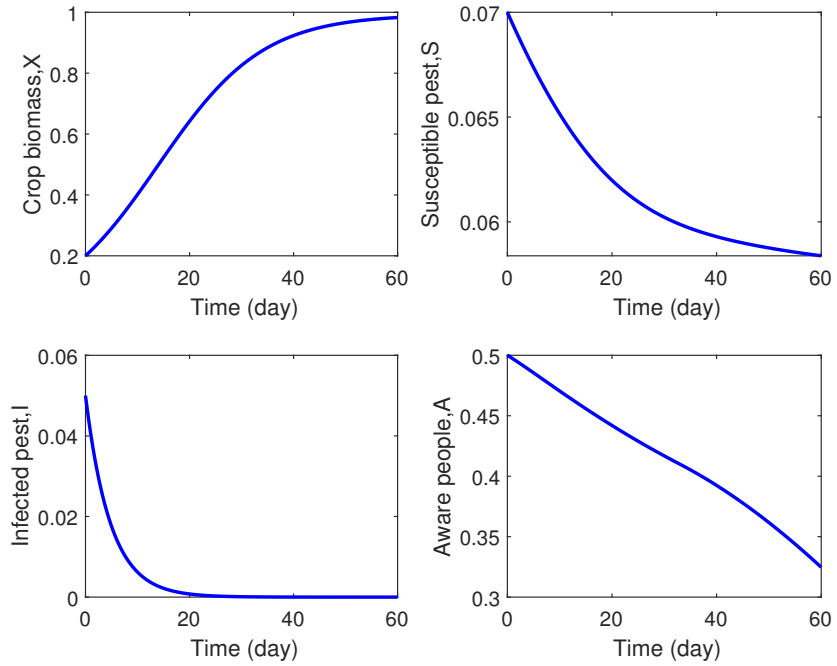


Figure 5.7: Numerical solutions of the optimal control system model (5.19) with $u_1 \neq 0, u_3 \neq 0$ and $u_2 \equiv 0$ using parameter values as as in Figure 5.5.

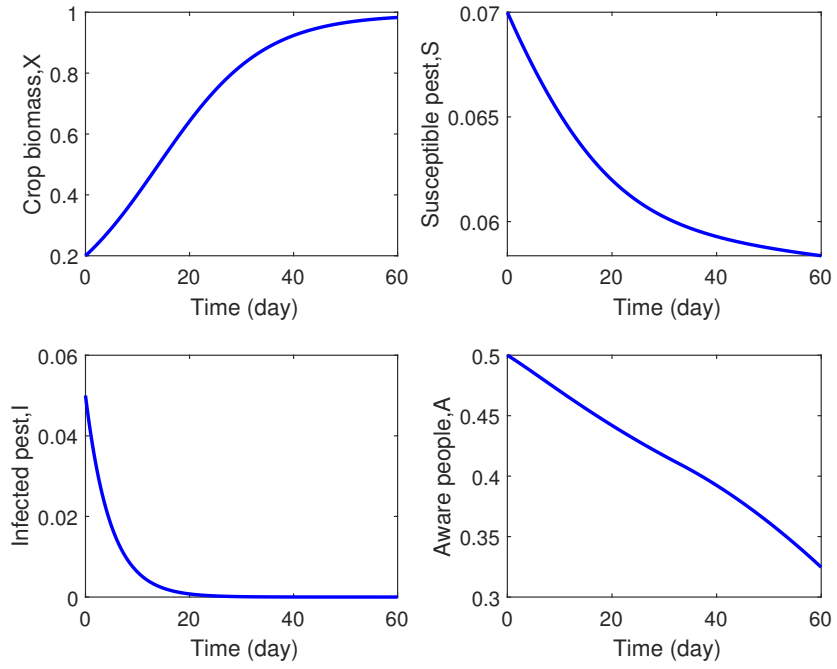


Figure 5.8: Numerical solutions of the optimal control system model (5.19) with $u_2 \neq 0, u_3 \neq 0$ and $u_1 \equiv 0$ using parameter values as as in Figure 5.5.

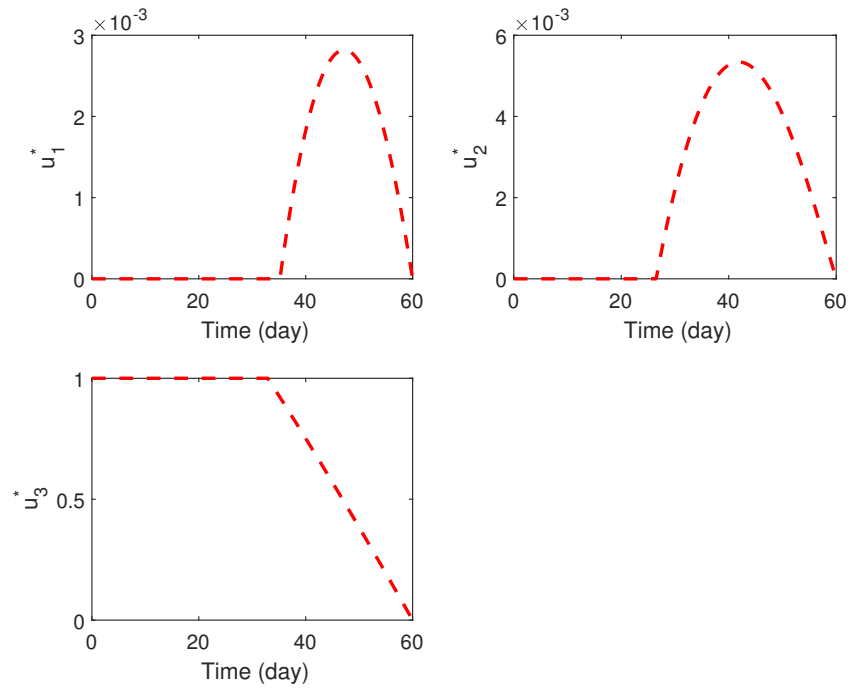


Figure 5.9: Pontryagin extremal controls u_1^*, u_2^* and u_3^* plotted as functions of time.

5.4 Results and Discussion

In this chapter, a mathematical model described by a system of ordinary differential equations has been developed and analyzed to study the effect of awareness based chemical and biological control of pests in farming environment. Our model contains four densities, specifically, density of crop-biomass, density of susceptible pests, infected pests, and level of awareness. The model under consideration exhibits four feasible equilibrium points: The crop-pest free equilibrium point, which is unstable for all parameter values, the pest free equilibrium point, the susceptible pest free equilibrium point, which may exist conditionally, and the interior equilibrium point. local stability of the positive interior equilibrium point E^* and local Hopf-bifurcation around it have been studied.

We have shown how the dynamics changes with the parameter value α (consumption rate of pest to crops) of the system. The dynamical behavior of the system is investigated using stability theory, optimal control theory and numerical simulations. We assumed that aware farmers use bio-control, as the integrated pest management, as it is eco-friendly and is less harmful to human health and environments. Local awareness campaign may be enough as comparative to the density of susceptible pest available in the crop biomass. We expect that the global issues, disseminated by radio, TV, telephone, internet, etc., enlarge the stage of awareness.

Moreover, we have used optimal control theory to provide the price effective outline of bio-pesticides, chemical pesticide costs and a universal alertness movement. We observed the dynamical behavior of the controlled system and the effects of the three controls by testing them one in the absence of others and including all at the same time.

CHAPTER 6

PEST CONTROL USING FARMING AWARENESS AND OPTIMAL USE OF BIO-PESTICIDES: IMPACT OF TIME DELAYS

This chapter is dedicated to the formulation and mathematical analysis of pest control model with time delays. The delayed model consists of four densities, crop biomass, susceptible pest, infected pest and level of awareness. This model is analyzed for local stability and hopf bifurcation and it is shown that the incorporated time delays can reduce the pest attach in the crop biomass. The effect of discrete time delays on the stability of the coexistence steady states is investigated. Criteria are given to ensure that these steady states are asymptotically stable for all time delays. The system is extended to include two continuous controls and is analyzed using optimal control theory techniques. Numerical representation is also given.

6.1 Model formulation and description

In this section, we presented a deterministic pest control model for crop loss management. The model includes crop biomass, $X(t)$; healthy pest, $S(t)$; infected pest, $I(t)$; and level of awareness, $A(t)$. Due to the finite size of the crop field, we assume logistic growth for the density of crop biomass, with net growth rate r and carrying capacity K . The susceptible pests attack the crop, thereby causing considerable crop reduction. If we infect the susceptible pest by pesticides, then the attack by pest can be controlled. Here we assumed that self-aware farmers will adopt biological pesticides for the control of crop pest as it has fewer side effects and are also environment-friendly. Bio-pesticides are used to infect susceptible pests. The infected pest has additional mortality due to infection.

Let λ be the consumption rate of pests with conversion rate m . There is a pest infection rate, h , because of self-aware human interactions and activity such as the use of bio-pesticides, modeled via the saturated incidence rate $\frac{hAS}{1+A}$. We assume, d as the natural mortality rate of pest and α is the additional mortality rate of infected pest due to self-aware people activity. We assumed that the level of awareness will increase at a rate proportional to the number of susceptible pests per crop noticed in the farming system. A fading of interest in this exploitation is possible, and we denoted it by η as the rate of fading of awareness level.

There might be a delay in observing the number or activity of pests in a field. Usually, this estimate is generated by observing previous cases of pest occurrence and thus the intensity of awareness and the level of implication of preventive countermeasures varies. A

delay $\tau_1 > 0$ is expected in the execution of such measures. The intensity of the awareness programs, being executed at time t , will be in accordance with the number of pests at the time $t - \tau_1 > 0$. Also, after an advertisement, farmers take some time to become aware of the technologies/pesticide to use and their management. We assume $\tau_2 > 0$ as the time delay parameter taken for organizing an awareness campaign and the farmers to become aware. The above assumptions can be summarized into the flow diagram given in figure 6.1

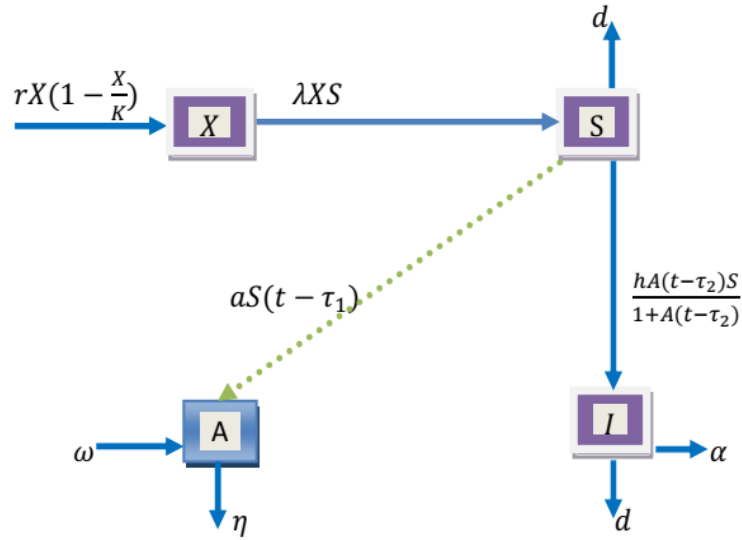


Figure 6.1: Schematic diagram for the pest control dynamics with time delays

Based on the above assumptions and diagram 6.1, we obtain the following mathematical model with nonlinear delay differential equations:

$$\begin{cases} \frac{dX}{dt} = rX \left[1 - \frac{X}{K} \right] - \lambda XS, \\ \frac{dS}{dt} = m\lambda XS - \frac{hA(t - \tau_2)S}{1 + A(t - \tau_2)} - dS, \\ \frac{dI}{dt} = \frac{hA(t - \tau_2)S}{1 + A(t - \tau_2)} - (d + \alpha)I, \\ \frac{dA}{dt} = \omega + aS(t - \tau_1) - \eta A. \end{cases} \quad (6.1)$$

In order to study the system of the delayed differential equation (6.1), the initial functions need to be first stated and the functional framework needs to be specified.

Let ϕ denote the Banach space of continuous functions $\phi : [-\tau, 0] \rightarrow \mathbb{R}^4$ equipped with the sup-norm

$$\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \{|\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|, |\phi_4(\theta)|\}.$$

For biological reasons, populations always have non-negative values, therefore, the initial

functions for our model (6.1) is given as below:

$$\begin{cases} X(\theta) = \phi_1(\theta), & S(\theta) = \phi_2(\theta), & I(\theta) = \phi_3(\theta), & A(\theta) = \phi_4(\theta), \\ \phi_i(\theta) \geq 0, & \theta \in [-\tau, 0], & \phi_i(0) > 0, & \phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in ([-\tau, 0], \mathbb{R}_+^4), \end{cases} \quad (6.2)$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $\mathbb{R}_+^4 = \{(x_1, x_2, x_3, x_4) \mid x_i \geq 0, i = 1, 2, 3, 4\}$..

The description of the parameters of model (6.1), together with the values that later will be used in our numerical simulations is summarized in Table 6.1.

Table 6.1: Meaning of the parameters of model (6.1) with values used in Section 6.5 for numerical simulations.

Parameters	Description	Value
r	growth rate of crop biomass	0.1
K	maximum density of crop biomass	100
λ	consumption rate of pest	0.05
d	natural mortality of pest	0.02
m	conversion efficacy	0.6
h	aware people activity rate	0.025
a	aware people growth rate	0.012
α	additional mortality rate	0.025
ω	rate of awareness from global source	0.003
η	fading of memory of aware people	0.015

6.2 Model Analysis

In this section, some basic properties such as non-negativity, boundedness of solutions and possible equilibria of the delayed system (6.1) are analyzed. In brief, we show positive in-variance, boundedness, existence, and stability of equilibria of the system (6.1).

6.2.1 Positivity and boundedness of the solutions

It is important to show the positivity and the boundedness of the solution of the system (6.1) as they represent populations.

Theorem 6.2.1 (Positivity of the solutions). *All solutions of (6.1) with given initial conditions (6.2) remain non-negative for all $t > 0$.*

We use 4.2.1 to proof the positivity of the first two equations.

Proof. It follows from the first equation of system (6.1) that

$$\frac{dX}{dt} = X \left[r \left(1 - \frac{X}{K} - \lambda S \right) \right].$$

Separating variables and integrating we can easily obtain

$$\ln X = C_0 + \int_0^t \left(r \left(1 - \frac{X}{K} \right) - \lambda S \right) dt.$$

Simplifying we conclude that

$$X(t) = C_1 \exp \left[\int_0^t \left(r \left(1 - \frac{X(t)}{K} \right) - \lambda S(t) \right) dt \right] > 0,$$

where $C_1 = e^{C_0}$. From the second equation of system (6.1) we have,

$$\frac{dS}{dt} = S \left[m\lambda X - \frac{hA(t-\tau)}{1+A(t-\tau)} - d \right], \quad \frac{dS}{S} = \left[m\lambda X - \frac{hA(t-\tau)}{1+A(t-\tau)} - d \right] dt.$$

By similar argument,

$$\ln S = C_0 + \int_0^t \left(m\lambda X - \frac{hA(t-\tau)}{1+A(t-\tau)} - d \right) dt.$$

So that

$$S(t) = C_1 \exp \left[\int_0^t \left(m\lambda X(t) - \frac{hA(t-\tau)}{1+A(t-\tau)} - d \right) dt \right] > 0.$$

To show positivity of the third and the fourth equations of system (6.1) we use the linear differential equations approach.

Again it follows from the fourth equation of system (6.1) that

$$\frac{dA}{dt} + \eta A = \omega + aS(t-\tau), \quad (6.3)$$

which is linear in A with integrating factor $u(t) = e^{\int(\eta)dt} = e^{\eta t}$. Then multiplying both sides of equation (6.3) by the integrating factor gives

$$e^{\eta t} \left(\frac{dA}{dt} + \eta A \right) = (\omega + aS(t-\tau))e^{\eta t},$$

which is equivalent to

$$\frac{d}{dt} (A(t)e^{\eta t}) = (\omega + aS(t-\tau))e^{\eta t}.$$

Again integrating from 0 to t yields,

$$A(t)e^{\eta t} - A(0) = \int_0^t (\omega + aS(t-\tau))e^{\eta t} dt.$$

This implies that,

$$A(t) = A(0)e^{-\eta t} + e^{-\eta t} \left[\int_0^t (\omega + aS(t-\tau))e^{\eta t} dt \right] > 0.$$

Finally from the third equation of system (6.1) we show the positivity of $I(t)$. That is,

$$\frac{dI}{dt} + (d + \alpha)I = \frac{hA(t - \tau)}{1 + A(t - \tau)}, \quad (6.4)$$

which is linear in I and integrating factor $\mu(t) = e^{(d+\alpha)t}$. Then multiplying both sides of (6.4) by $\mu(t)$ and after some mathematical manipulation we obtain,

$$I(T) = I(0)e^{(d+\alpha)T} + e^{-(d+\alpha)T} \left[\int_0^T \left(\frac{hA(T - \tau)}{1 + A(T - \tau)} \right) e^{-(d+\alpha)\tau} d\tau \right] > 0.$$

□

For biological reasons, the variables and parameters are assumed positive. The analysis of the system (6.1) is biologically meaningful in the region $\mathcal{D}$.

Theorem 6.2.2 (Boundedness of solutions). *Every solution of system (6.1) are uniformly bounded, in the region $\mathcal{D}$ given by*

$$\mathcal{D} := \left\{ (X, S, I, A) \in \mathbb{R}_+^4 : 0 \leq X \leq K, 0 \leq S + I \leq \frac{Km(r+d)^2}{4rd}, 0 \leq A \leq \frac{4r\omega d + Kma(r+d)^2}{4r\eta d} \right\}.$$

Proof. Considering the first equation of system (6.1), we have,

$$\begin{aligned} \frac{dX}{dt} &= rX \left[1 - \frac{X}{K} \right] - \lambda XS \leq rX \left[1 - \frac{X}{K} \right], \\ \frac{dX}{dt} &\leq rX \left(1 - \frac{X}{K} \right), \quad \text{separable ordinary differential inequality,} \\ X(t) &\leq \frac{KX_0}{X_0 + (K - X_0)e^{-rt}}, \\ \limsup_{t \rightarrow \infty} X(t) &\leq K. \end{aligned} \quad (6.5)$$

Letting $W = mX + S + I$ at any time t , it follows using (6.5) and the fact $rX \left[1 - \frac{X}{K} \right] + mX$ is quadratic in X with maximum value $\frac{Km(r+d)^2}{4r}$ that:

$$\begin{aligned} \frac{dW}{dt} &= rX \left[1 - \frac{X}{K} \right] - dS(d + \alpha)I \\ &= rX \left[1 - \frac{X}{K} \right] - dS - dI - \alpha I \\ &\leq rX \left(1 - \frac{X}{K} \right) - dS - dI \\ &= rX \left(1 - \frac{X}{K} \right) - dS - dI + mX - mX \\ &= rX \left(1 - \frac{X}{K} \right) - dW + mX, \end{aligned}$$

$$\begin{aligned}
\frac{dW}{dt} + dW &\leq rmX \left(1 - \frac{X}{K}\right) + mdX, \\
\frac{dW}{dt} + dW &\leq \frac{Km(r+d)^2}{4r} \\
W(t) &\leq \frac{Km(r+d)^2}{4rd} + Ce^{-dt}, \\
\limsup_{t \rightarrow \infty} W(t) &\leq \frac{Km(r+d)^2}{4rd}.
\end{aligned} \tag{6.6}$$

Hence, $0 \leq S(t) + I(t) \leq \frac{Km(r+d)^2}{4rd}$.

Finally, from the last equation of system (6.1) we can write

$$\begin{aligned}
\frac{dA}{dt} &= \omega + aS(t - \tau_1) - \eta A \leq \omega + \frac{Kma(r+d)^2}{4rd} - \eta A, \\
\frac{dA}{dt} &\leq \frac{4r\omega d + Kma(r+d)^2}{4rd} - \eta A, \quad \frac{dA}{dt} + \eta A \leq \frac{4r\omega d + Kma(r+d)^2}{4rd}, \\
A(t) &\leq \frac{4r\omega d + Kma(r+d)^2}{4r\eta d} + Ce^{-\eta t}, \\
\limsup_{t \rightarrow \infty} A(t) &\leq \frac{4r\omega d + Kma(r+d)^2}{4r\eta d}.
\end{aligned}$$

Thus, the region of attraction given by the set $\mathcal{D}$ is positively invariant, attracting all solutions initiating inside the interior of the positive octant. $\square$

Consequently, the model is mathematically and biologically well posed and is sufficient to consider the dynamics of the system (6.1).

6.2.2 Equilibria and their existence criteria

First of all we try to find all possible non-negative equilibria of the system (6.1) and then analyze the stability criterion around those equilibria. The pest is endemic whenever persist in the crop biomass and the crop biomass is free if and only if no pest persist in it.

The equilibrium points $E^*(X^*, S^*, I^*, A^*)$ of the model system in (6.1) is obtained by setting the system equal to zero and thus solving for the state variables.

Since we have mathematical model with time delays, we set time derivatives of system (6.1) equal to zero and look for equilibrium points X^*, S^*, I^*, A^* such that $S(t) = S(t - \tau_1) = S^*$, $A(t) = A(t - \tau_2) = A^*$. We obtain the equilibria by solving the equations for all time derivatives equal to zero. Therefore, we have the following nonlinear algebraic system

$$\begin{cases}
rX^* \left(1 - \frac{X^*}{K}\right) - \lambda X^* S^* = 0 \\
m\lambda X^* S^* - \frac{hA^* S^*}{1+A^*} - dS^* = 0 \\
\frac{hA^* S^*}{1+A^*} - (d + \alpha)I^* = 0 \\
\omega + aS^* - \eta A^* = 0
\end{cases} \tag{6.7}$$

(i) Setting $X = S = I = 0$ in the system (6.7), we get from the last equation

$$\omega + aS^* - \eta A^* = 0 \Rightarrow \omega - \eta A^* = 0 \Rightarrow A^* = \frac{\omega}{\eta}.$$

Hence one equilibrium point is the crop-pest free steady state $E_0 = \left(0, 0, 0, \frac{\omega}{\eta}\right)$ which is always feasible.

(ii) The pest free equilibrium in which only the crop density thrives,

If we $S = 0$ then $I = 0$ and from system (6.7), we get

$$\begin{cases} rX(t) \left[1 - \frac{X(t)}{K}\right] - \lambda X(t)S(t) = 0, \\ \omega + aS(t - \tau_1) - \eta A(t) = 0, \end{cases} \quad (6.8)$$

which results in the solution $X^* = K, A^* = \frac{\omega}{\eta}$. So, $E_1 = \left(K, 0, 0, \frac{\omega}{\eta}\right)$, is the pest free steady state, which is always feasible, and

(iii) The coexistence equilibrium ($X \neq 0, S \neq 0, I \neq 0, A \neq 0$).

From the second equation of system (6.7), we have

$$\begin{aligned} \left(m\lambda X^* - \frac{hA^*}{1 + A^* - d}\right)S^* &= 0 \\ X^* &= \frac{hA^* + d(1 + A^*)}{m\lambda(1 + A^*)} \end{aligned}$$

and from the third and the last equations of the system (6.7), we obtain

$$S^* = \frac{\eta A^* - \omega}{a}, \quad I^* = \frac{hA^* S^*}{(1 + A^*)(d + \alpha)}. \quad (6.9)$$

Then we have the coexistence equilibrium point $E^* = (X^*, S^*, I^*, A^*)$ with

$$X^* = \frac{hA^* + d(1 + A^*)}{m\lambda(1 + A^*)}, \quad S^* = \frac{\eta A^* - \omega}{a}, \quad I^* = \frac{hA^* S^*}{(1 + A^*)(d + \alpha)}, \quad (6.10)$$

and putting X^*, S^*, I^* in the first equation of system (6.7), we found that A^* satisfies the quadratic equation

$$L_1 A^2 + L_2 A + L_3 = 0, \quad (6.11)$$

whose coefficients are given by,

$$\begin{aligned} L_1 &= -Km\eta\lambda^2 < 0, \\ L_2 &= Km\lambda^2[\omega - \eta] + ar[Km\lambda - d - h], \\ L_3 &= ra[m\lambda K - d] + Km\omega\lambda^2. \end{aligned}$$

Our next result characterizes the feasibility of the coexistence equilibrium E^* .

Proposition 6.2.1 (Feasibility of the coexistence equilibrium). *Let $E^* = (X^*, S^*, I^*, A^*)$ be as (6.10) and (6.11).*

- (i) *If $L_3 > 0$, then there is a unique coexisting equilibrium E^* .*
- (ii) *If $L_3 = 0$ and $L_2 > 0$, then there is a unique coexisting equilibrium E^* with $A^* = -\frac{L_2}{L_1}$.*
- (iii) *If $L_3 < 0$ and $L_2 \leq 0$, then there is no positive coexisting equilibrium of the system.*
- (iv) *Assuming that $L_3 < 0$ and $L_2 > 0$,*
 - *if $D = L_2^2 - 4L_1L_3 > 0$, then there exist two coexisting equilibria E^* with $A^* = \frac{-L_2 \pm \sqrt{D}}{2L_1}$;*
 - *if $D = 0$, then there exists a unique coexisting equilibrium E^* ;*
 - *if $D < 0$, then there exists no positive coexisting equilibrium E^* .*

Proof. The result follows applying Descartes' rule of signs to equation (6.11). □

Remark 6.2.1. *From (6.10), the additional condition for feasible endemic(coexistence) equilibrium E^* is $A^* > \frac{\omega}{\eta}$.*

6.2.3 Characteristic equation

In this section, the local asymptotically stability of the three equilibrium points of the delayed system (6.1) is studied. Before we go to the detail properties of our model, let us introduce some basic ideas on the delayed differential equations as a motivation.

Consider a nonlinear system of the delay differential equations with constant delays $\tau_j \geq 0, j = 1, 2, \dots, m$ of the form

$$\dot{x}(t) = f(x(t), x(t - \tau_1), x(t - \tau_2), \dots, x(t - \tau_m), \eta) \quad (6.12)$$

where, $x(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^{(m+1)n+p} \rightarrow \mathbb{R}^n$ is a nonlinear smooth function depending on a number of (time independent) parameters $\eta \in \mathbb{R}^p$.

The (local) stability of the steady state X^* is determined by the stability of (the zero solution) of the linearized equation

$$\frac{dY(t)}{dt} = A_0 Y(t) + \sum_{j=1}^m A_j Y(t - \tau_j) \quad (6.13)$$

where $A_j \in \mathbb{R}^{n \times n}$ denotes the partial derivative of f with respect to its $(j+1)^{th}$ argument, evaluated at the steady state solution X^* .

The linearized equation (6.13) is asymptotically locally stable if all its roots λ of the characteristic equation

$$\det \left(\lambda I - A_0 - \sum_{j=1}^m A_j e^{-\lambda \tau_j} \right) = 0$$

lie in the open half-plane (i.e., $\Re(e(\lambda)) < 0$).

If we linearize our system (6.1) about any steady state $E(X, S, I, A)$, we obtain that

$$\frac{dY}{dt} = FY(t) + GY(t - \tau_1) + HY(t - \tau_2) \quad (6.14)$$

Here F , G and H are 4×4 matrices given as below

$$F = \begin{bmatrix} F_{11} & -\lambda X & 0 & 0 \\ m\lambda S & F_{22} & 0 & 0 \\ 0 & \frac{hA}{1+A} & -d - \alpha & 0 \\ 0 & 0 & 0 & -\eta \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{hS}{(1+A)^2} \\ 0 & 0 & 0 & \frac{hS}{(1+A)^2} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{bmatrix}$$

where

$$F_{11} = r(1 - \frac{2X}{K}) - \lambda S, \quad F_{22} = m\lambda X - \frac{hA}{1+A} - d.$$

The characteristic equation of the delayed system (6.1) then becomes

$$\Delta(\xi, \tau_1, \tau_2) = |\xi I - F - e^{-\xi \tau_1} G - e^{-\xi \tau_2} H| = 0.$$

This implies

$$\begin{vmatrix} \xi - F_{11} & \lambda X & 0 & 0 \\ -m\lambda S & \xi - F_{22} & 0 & \frac{hS}{(1+A)^2} e^{-\xi \tau_1} \\ 0 & -\frac{hA}{1+A} & \xi + d + \alpha & -\frac{hS}{(1+A)^2} e^{-\xi \tau_1} \\ 0 & -ae^{-\xi \tau_2} & 0 & \xi + \eta \end{vmatrix} = 0.$$

Equivalently,

$$(\xi + d + \alpha) \begin{vmatrix} \xi - F_{11} & \lambda X & 0 \\ -m\lambda S & \xi - F_{22} & \frac{hS}{(1+A)^2} e^{-\xi \tau_1} \\ 0 & -ae^{-\xi \tau_2} & \xi + \eta \end{vmatrix} = 0.$$

Or

$$(\xi + d + \alpha) \left[(\xi - F_{11}) \left((\xi - F_{22})(\xi + \eta) + \frac{ahS}{(1+A)^2} e^{-\xi(\tau_1 + \tau_2)} \right) + m\lambda^2 X^2 \xi + m\lambda^2 X^2 \eta \right] = 0.$$

This gives,

$$(\xi + d + \alpha) \left(\xi^3 + a_1 \xi^2 + a_2 \xi + (a_3 + a_4 \xi) e^{-\xi(\tau_1 + \tau_2)} \xi + a_5 \right) = 0, \quad (6.15)$$

with one eigenvalue being $\xi = -(d + \alpha) < 0$, and the rest of the spectrum being given by the roots of the transcendental equation

$$\psi(\xi, \tau_1, \tau_2) = \xi^3 + a_1 \xi^2 + a_2 \xi + [a_3 + a_4 \xi] e^{-\xi(\tau_1 + \tau_2)} + a_5 = 0, \quad (6.16)$$

where

$$\begin{aligned} a_1 &:= \eta - \left(r \left(1 - \frac{2X}{K} \right) - \lambda S \right) - \left(m\lambda X - \frac{hA}{1+A} - d \right), \\ a_2 &:= m\lambda^2 X^2 - \eta \left(r \left(1 - \frac{2X}{K} \right) - \lambda S + m\lambda X - \frac{hA}{1+A} - d \right) \\ &\quad + \left(r \left(1 - \frac{2X}{K} \right) - \lambda S \right) \left(m\lambda X - \frac{hA}{1+A} - d \right), \\ a_3 &:= - \frac{ahS \left(r \left(1 - \frac{2X}{K} \right) - \lambda S \right)}{(1+A)^2}, \quad a_4 := \frac{ahS}{(1+A)^2}, \\ a_5 &:= \eta \left(m\lambda^2 X^2 + \left(r \left(1 - \frac{2X}{K} \right) - \lambda S \right) \left(m\lambda X - \frac{hA}{1+A} - d \right) \right). \end{aligned}$$

Remark 6.2.2. Characteristic equation (6.15) has negative root, $-d - \alpha$. Thus for the distribution, it is sufficient to analyze the roots of the equation (6.16).

6.2.4 The threshold number

Threshold number in general denoted by R_0 and is often considered as the threshold quantity that determines the dynamic behavior of the model (Jana et al., 2017). To compute the basic threshold number in our model system we use the next-generation matrix method studied in (Van den Driessche and Watmough, 2002). System (6.1) has the pest free equilibrium $E_1 = (K, 0, 0, \frac{\omega}{\eta})$. To obtain the basic threshold number we write down the system (6.1) as follows:

$$\frac{dY}{dt} = \phi(Y) - \psi(Y),$$

where

$$Y = (S, I, A, X)^T, \phi(Y) = \begin{bmatrix} m\lambda XS \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \psi(Y) = \begin{bmatrix} \frac{hAS}{1+A} + dS \\ (d + \alpha)I - \frac{hAS}{1+A} \\ -\omega - aS + \eta A \\ \lambda XS - rX \left(1 - \frac{X}{K} \right) \end{bmatrix}.$$

The Jacobian matrices of $\phi(Y)$ and $\psi(Y)$, at the pest free equilibrium E_1 , are given respectively as

$$D\phi(E_1) = \begin{bmatrix} m\lambda K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D\psi(E_1) = \begin{bmatrix} \frac{h\omega}{\eta+\omega} + d & 0 & 0 & 0 \\ -\frac{h\omega}{\eta+\omega} & d + \alpha & 0 & 0 \\ -a & 0 & \eta & 0 \\ \lambda K & 0 & 0 & r \end{bmatrix}.$$

$$\text{Let: } \mathcal{F} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} V & 0 \\ J_1 & J_2 \end{bmatrix},$$

$$\text{where: } F = \begin{bmatrix} m\lambda K & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \frac{h\omega}{\eta+\omega} + d & 0 \\ -\frac{h\omega}{\eta+\omega} & d + \alpha \end{bmatrix}.$$

The basic threshold number R_0 of system ((6.1)) is defined by the spectral radius of the matrix FV^{-1} , and is given by

$$R_0 = \rho(FV^{-1}) = \begin{pmatrix} m\lambda X K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\eta+\omega}{d(\eta+\omega)+h\omega} & 0 \\ \frac{h\omega}{(d(\eta+\omega)+h\omega)(d+\alpha)} & \frac{1}{d+\alpha} \end{pmatrix} = \begin{pmatrix} \frac{m\lambda K(\eta+\omega)}{d(\eta+\omega)+h\omega} & 0 \\ 0 & 0 \end{pmatrix}. \quad (6.17)$$

Hence, the expression of R_0 becomes,

$$R_0 = \frac{m\lambda K(\eta + \omega)}{d(\eta + \omega) + h\omega}.$$

6.2.5 Stability of the equilibria without delays ($\tau_1 = \tau_2 = 0$)

For the stability of equilibria, we need the nature of the roots of the characteristic equation of the Jacobian matrix at any equilibrium point $E(X, S, I, A)$. In this section we consider $\tau_1 = \tau_2 = 0$, investigating the direction and stability of the bifurcating periodic solution. More precisely, we focus our analysis on the consumption rate λ of pest, which is one of the biologically most important parameters of the model.

Theorem 6.2.3. *The crop pest free equilibrium point E_0 is unstable.*

Proof. We linearize the system (6.1) using the Jacobian technique. The Jacobian matrix of the system (6.1) at any equilibrium point $E(X, S, I, A)$ without delays is given by

$$J(X, S, I, A) = \begin{bmatrix} r\left(1 - \frac{2X}{K}\right) - \lambda S & -\lambda X & 0 & 0 \\ m\lambda S & m\lambda X - \frac{hA}{1+A} - d & 0 & -\frac{hS}{(1+A)^2} \\ 0 & \frac{hA}{1+A} & -d - \alpha & \frac{hS}{(1+A)^2} \\ 0 & a & 0 & -\eta \end{bmatrix}. \quad (6.18)$$

Then from (6.18) the Jacobian matrix at the crop-pest free equilibrium point E_0 for $\tau_1 = \tau_2 = 0$ is given by

$$J\left(0, 0, 0, \frac{\omega}{\eta}\right) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\frac{h\omega}{\eta+\omega} - d & 0 & 0 \\ 0 & \frac{h\omega}{\eta+\omega} & -d - \alpha & 0 \\ 0 & a & 0 & -\eta \end{bmatrix},$$

whose characteristic equation in ξ is given by

$$|\xi I - J(E_0)| = \begin{vmatrix} \xi - r & 0 & 0 & 0 \\ 0 & \xi + \left(\frac{h\omega}{\eta+\omega} + d\right) & 0 & 0 \\ 0 & \frac{h\omega}{\eta+\omega} & \xi + d + \alpha & 0 \\ 0 & a & 0 & \xi + \eta \end{vmatrix} = 0.$$

From this we obtain

$$(\xi - r) \left(\xi + \frac{h\omega}{\eta + \omega} + d \right) (\xi + d + \alpha) (\xi + \eta) = 0.$$

Hence, the eigenvalues are, $\xi_1 = r > 0$, $\xi_2 = -\left(\frac{h\omega}{\eta} + d\right) < 0$, $\xi_3 = -(d + \alpha) < 0$ and $\xi_4 = -\eta < 0$. Thus one eigenvalues is always positive and consequently the axial equilibrium point, E_0 is always unstable. $\square$

Theorem 6.2.4 (Local stability of the pest-free equilibrium). *The pest free equilibrium point E_1 is locally asymptotically stable for $R_0 < 1$, and unstable for $R_0 > 1$.*

Proof. Using (6.18), the Jacobian matrix $J(E_1)$, at the pest free equilibrium point $E_1(K, 0, 0, \frac{\omega}{\eta})$ is

$$J(K, 0, 0, \frac{\omega}{\eta}) = \begin{bmatrix} -r & -\lambda K & 0 & 0 \\ 0 & m\lambda K - \frac{h\omega}{\eta+\omega} - d & 0 & 0 \\ 0 & \frac{h\omega}{\eta+\omega} & -d - \alpha & 0 \\ 0 & a & 0 & -\eta \end{bmatrix}.$$

The characteristic equation in ξ at E_1 is then given by

$$|\xi I - J(E_1)| = \begin{vmatrix} \xi + r & \lambda K & 0 & 0 \\ 0 & \xi - m\lambda K + \frac{h\omega}{\eta+\omega} + d & 0 & 0 \\ 0 & -\frac{h\omega}{\eta+\omega} & \xi + d + \alpha & 0 \\ 0 & -a & 0 & \xi + \eta \end{vmatrix} = 0.$$

That is,

$$(\xi + r)(\xi + \eta) \left(\xi - m\lambda K + \frac{h\omega}{\eta + \omega} + d \right) (\xi + d + \alpha) = 0.$$

Solving the eigenvalues, we obtain $\xi_1 = -r$, $\xi_2 = -\eta$, $\xi_3 = -(d + \alpha)$, and $\xi_4 = m\lambda K - \frac{h\omega}{\eta + \omega} - d$.

Clearly, three eigenvalues ξ_1, ξ_2 , and ξ_3 are negative, and the remaining eigenvalue ξ_4 will be negative if

$$m\lambda K - \frac{h\omega}{\eta + \omega} - d < 0 \Leftrightarrow m\lambda K < \frac{h\omega}{\eta + \omega} + d,$$

This is equivalent to

$$m\lambda K < \frac{h\omega + d(\eta + \omega)}{\eta + \omega}, \underbrace{\frac{m\lambda K(\eta + \omega)}{h\omega + d(\eta + \omega)}}_{=R_0} < 1.$$

Therefore, E_1 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. $\square$

Remark 6.2.3. *It is important to note that when pest-free equilibrium is stable, co-existing steady state is not feasible. It is also good to notice that R_0 does not depend on the local awareness level a . For this reason, irrespective of how efficiently farmers become aware of disease due to observation of pests in a crop plant, this by itself is not enough result to control the pests. Furthermore, R_0 is monotonically decreasing with an increase of ω . It suggests that eradication of crop pests is only possible if $R_0 < 1$. The possible means to achieve this goal is to increase the global awareness rate ω .*

Theorem 6.2.5 (Local stability of the coexistence equilibrium). *Let us consider system (6.1) with $\tau_1 = \tau_2 = 0$ and let $E^* = (X^*, S^*, I^*, A^*)$ be its coexistence equilibrium. Then E^* is locally asymptotically stable if and only if,*

$$\eta > \max \left\{ -\frac{ah(KS^*\lambda - Kr + 2rX^*)}{mX^*(1 + A^*)^2 \lambda^2 K}, \eta^* \right\}. \quad (6.19)$$

Proof. The Jacobian matrix of the system (6.1) without delays at the coexistence state E^* is

$$J(E^*) = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix},$$

where:

$$\begin{aligned}
b_{11} &= r\left(1 - \frac{2X^*}{K}\right) - \lambda S^*, b_{12} = -\lambda X < 0, b_{13} = 0, b_{14} = 0, \\
b_{21} &= m\lambda X^* > 0, b_{22} = m\lambda X^* - \frac{hA^*}{1+A^*} - d = 0, b_{23} = 0, b_{24} = -\frac{hS^*}{(1+A^*)^2} < 0, \\
b_{31} &= 0, b_{32} = \frac{hA^*}{1+A^*} > 0, b_{33} = -(d + \alpha) < 0, b_{34} = \frac{hS^*}{(1+A^*)^2} > 0, \\
b_{41} &= 0, b_{42} = a > 0, b_{43} = 0, b_{44} = -\eta < 0
\end{aligned}$$

Then the characteristic equation in ξ for the Jacobian matrix $J(E^*)$ can be written as

$$|\xi I - J(E^*)| = 0.$$

That is,

$$\begin{vmatrix}
\xi - b_{11} & -b_{12} & -b_{13} & -b_{14} \\
-b_{21} & \xi - b_{22} & -b_{23} & -b_{24} \\
-b_{31} & -b_{32} & \xi - b_{33} & -b_{34} \\
-b_{41} & -b_{42} & -b_{43} & \xi - b_{44}
\end{vmatrix} = 0,$$

which gives

$$(\xi + d + \alpha) \left(\xi^3 + \sigma_1 \xi^2 + \sigma_2 \xi + \sigma_3 \right) = 0, \quad (6.20)$$

with

$$\begin{aligned}
\sigma_1 &= -(b_{11} + b_{44}), \\
\sigma_2 &= b_{11}b_{44} - b_{21}b_{12} - b_{42}b_{24}, \\
\sigma_3 &= b_{11}b_{24}b_{42} + b_{12}b_{21}b_{44}.
\end{aligned} \quad (6.21)$$

Further

$$\sigma_1 \sigma_2 - \sigma_3 = -b_{44}b_{11}^2 + (b_{21}b_{12} - b_{44}^2) b_{11} + b_{24}b_{42}b_{44}.$$

According to the elements of $J(E^*)$ we have

$$\begin{aligned}
\sigma_1 &= \frac{\lambda^2 (\eta A^* - \omega) m (1 + A^*) K - Kam (r - \eta) (1 + A^*) \lambda + 2 (r (d + h) A^* + d) a}{am\lambda (1 + A^*) K} \\
\sigma_3 &= \frac{\left((m\eta X^* (A^* + 1)^2 \lambda^2 + ah\lambda S^* - ahr) K + 2ahrX^* \right) S^*}{K (A^* + 1)^2} \\
\sigma_1 \sigma_2 - \sigma_3 &= \frac{((\lambda S^* + \eta - r) K + 2rX^*) \left(\left((A^* + 1)^2 (\lambda S^* - r) \eta + S^* (mX^* (A^* + 1)^2 \lambda^2 + ah) \right) K \right)}{K^2 (A^* + 1)^2} \\
&\quad + \frac{2r\eta X^*}{K^2 (A^* + 1)^2} - \frac{\left((m\eta X^* (A^* + 1)^2 \lambda^2 + ah\lambda S^* - ahr) K + 2ahrX^* \right) S^*}{(A^* + 1)^2 K}
\end{aligned}$$

According to the existence condition for E^* , $\sigma_1 > 0$.

Now $\sigma_1 \sigma_2 - \sigma_3 = 0$ gives

$$\eta^2 + Q_1 \eta + Q_2 = 0, \quad (6.22)$$

where

$$Q_1 = b_{11} + \frac{hSa}{(1+A)^2 b_{11}},$$

$$Q_2 = m\lambda^2 X S.$$

Now $\sigma_3 > 0$ implies $\eta > -\frac{ah(KS^*\lambda - Kr + 2rX^*)}{mX^*(1+A^*)^2 \lambda^2 K}$ and $\sigma_1 \sigma_2 - \sigma_3 > 0$ implies the positive root of (6.22) (assuming that it has one positive and one negative root) less than η i.e., $\eta > \left\{ -Q_1 + \sqrt{Q_1^2 - 4Q_2} \right\} / 2 = \eta^*$, say. Therefore, we have $\sigma_1, \sigma_2, \sigma_3$, and $\sigma_1 \sigma_2 - \sigma_3$ all are positive for

$$\eta > \max \left\{ -\frac{ah(KS^*\lambda - Kr + 2rX^*)}{mX^*(1+A^*)^2 \lambda^2 K}, \eta^* \right\} \quad (6.23)$$

By the Routh–Hurwitz theorem, it follows that the coexistence equilibrium E^* is locally asymptotically stable if and only if, $\sigma_1 > 0$, $\sigma_3 > 0$, and $\sigma_1 \sigma_2 - \sigma_3 > 0$. So the stability condition can be given by (6.19). $\square$

6.2.6 Hopf bifurcation analysis

We assume that the system (6.1) is stable around the interior equilibrium point E^* under the conditions in (6.19). For Hopf bifurcation, we take the consumption rate λ of pest as a bifurcation parameter, since it is one of the biologically most important parameters. Hopf bifurcation of the endemic steady state can occur if the characteristic equation (6.20) has a pair of purely imaginary eigenvalues for some $\lambda = \lambda^* \in (0, \infty)$ and, additionally, all other eigenvalues have negative real parts. To proof the existence of Hopf bifurcation, we apply the method of bifurcation following the theorem described in (Basir et al., 2017). Let $\phi : (0, \infty) \rightarrow \mathbb{R}^+$ be a positive real valued continuously differential function defined by

$$\phi(\lambda) = \sigma_1(\lambda) \sigma_2(\lambda) - \sigma_3(\lambda). \quad (6.24)$$

The characteristic equation (6.20) has one negative root, namely, $-(d + \alpha)$. Thus Hopf bifurcation will occur according to the following theorem

Theorem 6.2.6 (Hopf bifurcation). *Let σ_i , $i = 1, 2, 3$, be given by (6.21). System (6.1) with $\tau_1 = \tau_2 = 0$, at the endemic equilibrium point E^* , undergoes Hopf bifurcation at $\lambda = \lambda^*$ if and only if:*

- (1) $\phi(\lambda^*) = 0$ and
- (2) $\sigma_3' - (\sigma_1' \sigma_2 + \sigma_1 \sigma_2') \neq 0$.

where we use the 'prime' to denote differentiation with respect to λ .

Proof. The Routh–Hurwitz criterion implies that the coexistence equilibrium point E^* of the

system (6.1) is locally asymptotically stable if $\phi(\lambda) > 0$.

The necessary and sufficient conditions for existence of Hopf bifurcation of (6.1) with respect to λ occurred at $\lambda = \lambda^*$ are

$$(i) \quad \sigma_1(\lambda^*) > 0, \sigma_3(\lambda^*) > 0 \text{ and } \phi(\lambda^*) = 0,$$

$$(ii) \quad \left. \frac{d\Re(\xi)}{d\lambda} \right|_{\lambda=\lambda^*} \neq 0.$$

The critical value λ^* is obtained from the equation $\phi(\lambda) = 0$. For $\lambda = \lambda^*$ we have

$$\phi(\lambda) = 0, \sigma_1(\lambda)\sigma_2(\lambda) - \sigma_3(\lambda) = 0, \sigma_1\sigma_2 = \sigma_3.$$

Then the right side bracketed expression in the characteristic equation (6.20) becomes

$$\left(\xi^3 + \sigma_1\xi^2 + \sigma_2\xi + \sigma_1\sigma_2 \right) = 0.$$

That is

$$\left(\xi^2 + \sigma_2 \right) \left(\xi + \sigma_1 \right) = 0$$

which has three roots: $\xi_1 = i\sqrt{\sigma_2}$, $\xi_2 = -i\sqrt{\sigma_2}$, and $\xi_3 = -\sigma_1$. Therefore, a pair of purely imaginary eigenvalues exists for $\sigma_1\sigma_2 - \sigma_3 = 0$. For all λ , the roots of the characteristic equation are of the form $\xi_1(\lambda) = u(\lambda) + iv(\lambda)$, $\xi_2(\lambda) = u(\lambda) - iv(\lambda)$ and $\xi_3(\lambda) = -u(\lambda)$. Now, we verify the transversality condition.

$$\left[\frac{d(\Re(\xi_j(\lambda)))}{d\lambda} \right]_{\lambda=\lambda^*} \neq 0, j = 1, 2.$$

Substituting $\xi(\lambda) = u(\lambda) + iv(\lambda)$ in (6.20), we get

$$\begin{aligned} (u + iv)^3 + \sigma_1(u + iv)^2 + \sigma_2(u + iv) + \sigma_3 &= 0 \\ -3uv^2 + 3iu^2v - iv^3 + u^3 + 2iuv\sigma_1 + u^2\sigma_1 - v^2\sigma_1 + \sigma_2iv + u\sigma_2 + \sigma_3 &= 0. \end{aligned}$$

Separating real and imaginary parts, we get

$$u^3 - 3uv^2 + \sigma_1(u^2 - v^2) + \sigma_2u + \sigma_3 = 0, \quad (6.25)$$

$$3u^2v - v^3 + 2\sigma_1uv + \sigma_2v = 0. \quad (6.26)$$

From (6.26), as $v \neq 0$. Let

$$f(u) = v^2 = 3u^2 + 2\sigma_1u + \sigma_2. \quad (6.27)$$

Now putting the value of v^2 in equation (6.25), we get

$$u^3 - 3uf(u) + \sigma_1(u^2 - f(u)) + \sigma_2u + \sigma_3 = 0. \quad (6.28)$$

Hence, differentiating with respect to λ , we have

$$\begin{aligned} 3u^2 \frac{du}{d\lambda} - 3uf'(u) \frac{du}{d\lambda} - 3f(u) \frac{du}{d\lambda} + \dot{\sigma}_1(u^2 - f(u)) \\ + \sigma_1(2u - f'(u)) \frac{du}{d\lambda} + u\dot{\sigma}_2 + \sigma_2 \frac{du}{d\lambda} + \dot{\sigma}_3 = 0. \end{aligned}$$

Now $u(\lambda^*) = 0$, so from (6.27) we get $f(0) = \sigma_2$, $f'(0) = 2\sigma_1$.

Therefore,

$$\begin{aligned} \left. \frac{d\Re(\xi)}{d\lambda} \right|_{\lambda=\lambda^*} &= \frac{\sigma_2(\sigma_1' \sigma_2 + \sigma_1 \sigma_2' - \sigma_3')}{-2\sigma_2(\sigma_1^2 + \sigma_2)} \\ &= \frac{\sigma_3' - (\sigma_1' \sigma_2 + \sigma_1 \sigma_2')}{2(\sigma_2 + \sigma_1^2)} \\ &\neq 0 \Leftrightarrow \sigma_3' - (\sigma_1' \sigma_2 + \sigma_1 \sigma_2') \neq 0. \end{aligned}$$

Hence, interior equilibrium point E^* is unstable, when $\lambda < \lambda^*$ and stable when $\lambda > \lambda^*$. Thus, Hopf bifurcation occurs at $\lambda = \lambda^*$. $\square$

6.2.7 Stability and Hopf bifurcation of the delayed system

In this section, we investigate the local stability and Hopf bifurcation for the delayed system (6.1) around the coexistence equilibrium only. Without loss of generality, it is assumed that $E^* = (X^*, S^*, I^*, A^*)$ is the interior equilibrium point of the system. The expressions for X^* , S^* , I^* and A^* are given in (6.10) and (6.11). For the stability of endemic equilibrium, the distribution of the roots of (6.15) will be analyzed by letting $\tau := \tau_1 + \tau_2$.

Theorem 6.2.7 (Local stability of the coexistence equilibrium—delayed case). *Let a_i , $i = 1, \dots, 5$, be given by (6.16). Define $S_1 := a_1^2 - 2a_2$, $S_2 := a_2^2 - 2a_1a_5 - a_4^2$, and $S_3 := a_5^2 - a_3^2$. If the conditions*

$$(i) \ a_1 > 0, \ a_3 + a_5 > 0, \text{ and } a_1(a_2 + a_4) - (a_3 + a_5) > 0;$$

$$(ii) \ S_1 \geq 0, \ S_3 \geq 0, \text{ and } S_2 > 0;$$

are satisfied, then the coexistence steady state E^* is locally asymptotically stable for all $\tau \geq 0$.

Proof. For the stability of the endemic equilibrium, the distribution of the roots of (6.16) needs to be analyzed. The characteristic equation (6.16) becomes

$$\psi(\xi, \tau) = \xi^3 + a_1\xi^2 + a_2\xi + a_5 + [a_3 + a_4\xi]e^{-\xi\tau} = 0. \quad (6.29)$$

Equation (6.29) is a transcendental equation in ξ with infinitely many roots. It is known that the coexistence equilibrium E^* is locally stable (or unstable) if all the roots of the corre-

sponding characteristic equation have negative real parts (or have positive real parts). Suppose $\xi = iw(\tau)$ is a root of equation (6.29). Then,

$$-iw^3 - a_1w^2 + ia_2w + (a_4iw + a_3)(\cos w\tau - i\sin w\tau) + a_5 = 0. \quad (6.30)$$

Separating the real and imaginary parts, we obtain the following equations:

$$w^3 - a_2w = a_4w \cos w\tau - a_3 \sin w\tau, \quad (6.31)$$

$$a_1w^2 - a_5 = a_3 \cos w\tau + a_4w \sin w\tau. \quad (6.32)$$

Squaring and adding the real and imaginary parts, we get

$$w^6 + (a_1^2 - 2a_2)w^4 + (a_2^2 - 2a_1a_5 - a_4^2)w^2 + (a_5^2 - a_3^2) = 0. \quad (6.33)$$

Let $\theta = w^2$. Then, equation (6.33) becomes

$$H(\theta) = \theta^3 + S_1\theta^2 + S_2\theta + S_3 = 0. \quad (6.34)$$

Now, if the following conditions

$$S_1 = a_1^2 - 2a_2 \geq 0, \quad S_2 = a_2^2 - 2a_1a_5 - a_4^2 > 0, \quad S_3 = a_5^2 - a_3^2 \geq 0$$

are satisfied, then equation (6.34) has no positive real roots. We get from (6.34) that

$$\frac{dH(\theta)}{d\theta} = 3\theta^2 + 2S_1\theta + S_2 = 0, \quad (6.35)$$

which has two roots

$$\theta_1 = \frac{-S_1 + \sqrt{S_1^2 - 3S_2}}{3}, \quad \theta_2 = \frac{-S_1 - \sqrt{S_1^2 - 3S_2}}{3}. \quad (6.36)$$

Since we assume $S_2 > 0$, we have $\sqrt{S_1^2 - 3S_2} < S_1$ and hence neither θ_1 nor θ_2 is positive. Thus, equation (6.35) does not have any positive roots. Since $H(0) = S_3 \geq 0$, we conclude that equation (6.34) has no positive roots. $\square$

Remark 6.2.4. We can say that if the parameters, in Table 6.1 satisfy the conditions of Theorem 6.2.7, then the coexistence equilibrium point E^* of the delayed model (6.1) is asymptotically stable for all $\tau > 0$, i.e., the stability of the system at the coexistence equilibrium is independent of the delay τ .

A Hopf-bifurcating periodic solution appears when purely imaginary roots exist. We shall now check the possible occurrence of Hopf-bifurcation. If $S_3 < 0$, then there exists

a positive root θ_0 for which the characteristic equation (6.16) has pair of purely imaginary roots $\pm iw_0$. Equation (6.34) satisfies $H(0) < 0$ and $\lim_{\theta \rightarrow \infty} H(\theta) = \infty$. Thus, equation (6.34) has at least one positive root, θ_0 . Again, if $S_2 < 0$, then $\sqrt{S_1^2 - 3S_2} > S_1$ and hence $\theta_1 > 0$. This implies that equation (6.33) possesses purely imaginary roots $\pm iw_0$. For $w(\tau_0) = w_0$, equations (6.31) and (6.32) give

$$\tau_n = \frac{1}{w_0} \arccos \left[\frac{(a_4 w_0^4) - (a_2 a_4 - a_1 a_3) w_0^2 - a_3 a_5}{a_4^2 w_0^2 + a_3^2} \right] + \frac{2n\pi}{w_0}, \quad n = 0, 1, 2, \dots, \quad (6.37)$$

where

$$w_0^2 = \frac{-S_1 + \sqrt{S_1^2 - 3S_2}}{3}. \quad (6.38)$$

Therefore, the following result holds.

Theorem 6.2.8 (Hopf bifurcation—delayed case). *Let a_i , $i = 1, \dots, 5$, and S_j , $j = 1, \dots, 3$, be as in 6.2.7. Suppose that the interior equilibrium point E^* exists and $a_1 > 0$, $a_3 + a_5 > 0$, and $a_1(a_2 + a_4) - (a_3 + a_5) > 0$. Define*

$$\tau_0 := \frac{1}{w_0} \arccos \left[\frac{(a_4 w_0^4) - (a_2 a_4 - a_1 a_3) w_0^2 - a_3 a_5}{a_4^2 w_0^2 + a_3^2} \right], \quad (6.39)$$

where w_0 is given by (6.38). *If either $S_3 < 0$ or $S_3 \geq 0$ and $S_2 < 0$, then E^* is asymptotically stable when $\tau < \tau_0$ and unstable when $\tau > \tau_0$. When $\tau = \tau_0$, a Hopf bifurcation occurs: a family of periodic solutions bifurcates at E^* as τ passes through the critical value τ_0 , provided the transversality condition*

$$P(w_0)R(w_0) - Q(w_0)S(w_0) > 0 \quad (6.40)$$

is satisfied with

$$\begin{aligned} P(w) &= -3w^2 + a_1 + a_4 \cos w\tau - \tau(a_3 \cos w\tau + a_4 w \sin w\tau), \\ Q(w) &= -ca_1 w + a_4 \sin w\tau - \tau(a_3 \sin w\tau - a_4 w \cos w\tau), \\ R(w) &= a_3 w \sin w\tau - a_4 w^2 \cos w\tau, \\ S(w) &= a_3 w \cos w\tau + a_4 w^2 \sin w\tau. \end{aligned}$$

Proof. Differentiating (6.31) and (6.32) with respect to τ , we have

$$\frac{d}{d\tau} [Re\{\lambda(\tau)\}]_{\tau=\tau_0, w=w_0} = \frac{P(w)R(w) - Q(w)S(w)}{P^2(w) + Q^2(w)} \Big|_{\tau=\tau_0, w=w_0}. \quad (6.41)$$

Therefore, $\frac{d}{d\tau} [Re\{\lambda(\tau)\}]_{\tau=\tau_0, w=w_0} > 0$ if $P(w_0)R(w_0) - Q(w_0)S(w_0) > 0$.

Thus, the transversality condition is satisfied and hence a Hopf bifurcation occurs at $\tau = \tau_0$

(Freedman and Rao, 1983). □

Remark 6.2.5. *It follows from the form of the characteristic equation (6.1), where time delays come in as τ a sum $\tau_1 + \tau_2$, so the result of Theorem [Theorem 6.2.7](#) can automatically be applied to the cases when $\tau_1 > 0$, $\tau_2 = 0$ or $\tau_2 > 0$, $\tau_1 = 0$ or where both time delays are positive. When $\tau_1 > 0$, $\tau_2 = 0$, the critical value is denoted as τ_{10} , and when $\tau_2 > 0$, $\tau_1 = 0$, the critical value is τ_{20} .*

The full characterization of the direction and stability of the bifurcating periodic solution for the delayed system (6.1) is cumbersome but can be done following the approach of (Al Basir et al., 2019b; Riad et al., 2019). In Section 6.5 we illustrate the nature of Hopf bifurcation through numerical simulations.

6.3 Local sensitivity analysis of model parameters

We performed sensitivity analysis, on the basic parameters to help us to know the parameters that have high impact on the threshold number R_0 . The sensitivity of the reproduction number to the given parameters is used to determine the robustness of system predictions to parameter values (since there are usually errors in data collection and presumed parameter values). Hence, studying its sensitivity is important in system dynamics. A small variation in highly sensitive parameter will lead to large quantitative changes (Berhe et al., 2018). The sensitivity of a variable with respect to system parameters is usually measured by sensitivity index.

Definition 6.3.1. *The normalized forward sensitivity index of all the basic parameters is defined as*

$$\Pi_x^{R_0} = \frac{x}{R_0} \times \frac{\partial R_0}{\partial x}$$

for x representing all the basic parameters.

Notice that $\Pi_x^{R_0}$ has a maximum value of magnitude 1.

The sensitivity index of R_0 with respect to the given parameters are computed as follows:

$$\begin{aligned}\Pi_K^{R_0} &= \frac{K}{R_0} \times \frac{\partial R_0}{\partial K} = 1 \\ \Pi_\lambda^{R_0} &= \frac{\lambda}{R_0} \times \frac{\partial R_0}{\partial \lambda} = 1 \\ \Pi_m^{R_0} &= \frac{m}{R_0} \times \frac{\partial R_0}{\partial m} = 1 \\ \Pi_h^{R_0} &= \frac{h}{R_0} \times \frac{\partial R_0}{\partial h} = -\frac{h\omega}{d(\eta + \omega) + h\omega} \\ \Pi_d^{R_0} &= \frac{d}{R_0} \times \frac{\partial R_0}{\partial d} = -\frac{d(\eta + \omega)}{d(\eta + \omega) + h\omega} \\ \Pi_\omega^{R_0} &= \frac{\omega}{R_0} \times \frac{\partial R_0}{\partial \omega} = -\frac{h\omega\eta}{(\eta + \omega)(d(\eta + \omega) + h\omega)} \\ \Pi_\eta^{R_0} &= \frac{\eta}{R_0} \times \frac{\partial R_0}{\partial \eta} = \frac{h\omega\eta}{(\eta + \omega)(d(\eta + \omega) + h\omega)}\end{aligned}$$

The normalized sensitivity of the basic reproduction number with respect to the given parameters is given in Table 6.2

Table 6.2: The sensitivity of indices.

Parameter symbol	Value	Sensitivity index
K	100	1
λ	0.05	1
d	0.02	-0.827
m	0.6	1
h	0.025	-0.2083
ω	0.003	-0.15
η	0.015	0.15

6.4 The optimal control problem with state delays

This section is devoted to some preliminaries of optimal control problems with delay, which is necessary in the next subsections. For more details, one can see (Göllmann et al., 2009).

6.4.1 The optimization problem

Pontryagin maximum principle(PMP): The Pontryagin maximum (or minimum) principle (PMP) is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls.

A quite general optimal control problem governed by a delay differential system can be

formulated in the following form (Göllmann et al., 2009).

$$\text{minimize } L(x, u) = \int_0^T G(t, x(t), u(t), x(t - \tau)) dt + \varphi(x(T)), \quad (6.42)$$

subject to $u \in U \subset L^\infty(0, T; \mathbb{R}^m)$ ($T > 0$) where x is the solution to

$$\begin{cases} x'(t) = f(t, u(t), x(t), x(t - \tau)), & t \in (0, T) \\ x(0) = x_0 \end{cases} \quad (6.43)$$

Here

$$G : [0, T] \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R},$$

$$\varphi : \mathbb{R}^N \rightarrow \mathbb{R},$$

$$f : [0, T] \times \mathbb{R}^m \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$x_0 \in \mathbb{R}^N, m, N \in \mathbb{N}$ and $U \subset L^\infty(0, T; \mathbb{R}^m)$ is a convex closed subset.

We assume here that for any $u \in L^\infty(0, T; \mathbb{R}^m)$, problem (6.43) admits a unique solution denoted by x . Equation (6.43) is called the state problem.

1. $u \in U$ is called the control (or controller). This is a constrained control because $u \in U$ and U is a subset of $L^\infty(0, T; \mathbb{R}^m)$.
2. x is the state corresponding to the control u and the mapping.
3. $u \mapsto L(u, x) = \phi(u)$ is the cost functional (the function that should be minimized).

We say that $u^* \in U$ is an optimal control for Problem (6.42) if

$$L(x^*, u^*) \leq L(x, u)$$

for any $u \in U$. The pair (x^*, u^*) is called an optimal pair and $L(x^*, u^*)$ is the optimal value of the cost functional.

Pontryagin function with delay is given by:

$$H(t, u(t), x(t), x(t - \tau), p(t)) = G(t, u(t), x(t), x(t - \tau)) + f(t, u(t), x(t), x(t - \tau))p(t)$$

This function is Hamiltonian, if it satisfies the following relations.

$$x'(t) = H_p$$

and

$$p'(t) = -H_x$$

$p(t)$ is defined as the adjoint function. Since we are going to use optimal control, we state the following theorem which one can find the proof in (Göllmann et al., 2009).

Theorem 6.4.1. (minimum principle for the retarded optimal control problem (ROCP)). If $u^*(t)$ and $x^*(t)$ are optimal for problem (6.42) with delay in the state, then there exists a piece-wise differentiable co-state (ad-joint) function $p(t)$ such that

$$H(t, x^*(t), x^*(t - \tau), u^*(t), p(t)) \leq H(t, x^*(t), x^*(t - \tau), u(t), p(t)) \quad (6.44)$$

for all controls u at each time t , where H is the Hamiltonian previously defined and

$$\begin{aligned} p'(t) = & -\frac{\partial H}{\partial x}(t, x(t), x(t - \tau), u(t), p(t)) \\ & -\chi_{[0, t_f - \tau]}(t) \frac{\partial H}{\partial x_\tau}(t + \tau, x(t + \tau), x(t), u(t + \tau), p(t + \tau)) \end{aligned} \quad (6.45)$$

where $x_\tau = x(t - \tau)$

$$p(t_f) = 0 \text{ (transversality condition)} \quad (6.46)$$

the OCP must satisfy (optimality condition):

$$\frac{\partial H}{\partial u} = 0$$

i.e., the minimization is over all admissible controls.

6.4.2 Application of optimal control problem to the pest control model

Here, we consider an optimal control problem governed by a system of delay differential equations with multiple time delays in the state constraints. Then we derive first-order necessary conditions for existence of the optimal control and develop a numerical method for solving them. Optimal control with delays is a subject under strong current research [Fred-erico and Torres \(2012\)](#); [Benharrat and Torres \(2014\)](#); [Lemos-Paião et al. \(2019\)](#), in particular with respect to applications in biological control systems ([Silva et al., 2017](#); [Allali et al., 2018](#)). Our main goal here is to find the optimal profile $u_1(t)$ and $u_2(t)$ to minimize the cost of bio-pesticides used in pest control. We introduce the time dependent parameters $u_1(t)$ and $u_2(t)$ as control parameters: $u_1(t)$ is an admissible control representing the efficiency of pesticide being used and $u_2(t)$ characterizes the cost of the awareness campaign. The aim is to find the optimal profiles $u_1(t)$ and $u_2(t)$, denoted respectively by $u_1^*(t)$ and $u_2^*(t)$ using the minimum principle in ([Göllmann et al., 2009](#)). In order to state the optimization problem, we first consider that u_1 and u_2 vary with time. The control induced state system is given as

follows:

$$\begin{cases} \frac{dX}{dt} = rX \left[1 - \frac{X}{K}\right] - \lambda XS, \\ \frac{dS}{dt} = m\lambda XS - (1 - u_1) \frac{hA(t-\tau_2)S}{1+A(t-\tau_2)} - dS, \\ \frac{dI}{dt} = (1 - u_1) \frac{hA(t-\tau_2)S}{1+A(t-\tau_2)} - (d + \alpha)I, \\ \frac{dA}{dt} = u_2 \omega + aS(t - \tau_1) - \eta A. \end{cases} \quad (6.47)$$

subject to initial conditions

$$X(\theta) = \phi_1(\theta), \quad S(\theta) = \phi_2(\theta), \quad I(\theta) = \phi_3(\theta), \quad A(\theta) = \phi_4(\theta) \quad (6.48)$$

Here, we want to minimize the pest and also the cost of pest management by minimizing the use of pesticides and maximizing the level of awareness. To do this, we construct the following objective functional

$$J(u_1, u_2) = \int_0^{t_f} \left[CS^2(t) + B_1 u_1^2(t) + B_2 u_2^2(t) \right] dt, \quad (6.49)$$

subject to the control system (6.47). Where $[0, t_f]$ is the entire time interval over which the two control strategies are applied. The constants B_1, B_2 and C are positive weighted constants to make the terms of integrand keep balance in objective functional J . B_1 and B_2 represent the benefits of the cost of production, while C is the penalty multiplier. In other words, we are seeking optimal control pair (u_1^*, u_2^*) such that

$$J(u_1^*, u_2^*) = \min(J(u_1, u_2) : (u_1, u_2) \in U), \quad (6.50)$$

where U is the control set defined by

$$U = \left\{ u(t) = (u_1(t), u_2(t)) \mid u_1, u_2 \text{ are Lebesgue measurable, } \right. \\ \left. 0 \leq u_i(t) \leq u_{i\max}, t \in [0, t_f], \quad i = 1, 2 \right\}. \quad (6.51)$$

6.4.3 Existence of the optimal control pair

In this subsection, we study the existence of the optimal control pair in finite time for the system (6.47)–(6.49). Using Shamsara et al. (2016) in relation to our problem, we show the existence of the solutions for system (6.47) in the entire interval $[-\tau, t_f]$.

We assume that $\tau_1 = \tau_2 = \tau > 0$, and we introduce the following notations. For $x = (X, S, I, A) \in \mathbb{R}^4$, the state vector of (6.47) in compact form can be written in DDE as

$$\begin{aligned} x'(t) &= f(t, x(t), x(t - \tau), u_1(t), u_2(t)), t \in [0, t_f] \\ x(t) &= \phi(t), t \in [-\tau, 0]. \end{aligned} \quad (6.52)$$

$u_1(t)$ and $u_2(t)$ satisfy:

$$0 \leq u_1(t) \leq u_{1max} < 1, \text{ and } 0 \leq u_2(t) \leq u_{2max} < 1, t \in [0, t_f] \quad (6.53)$$

$$(u_1(t), u_2(t)) \in L^\infty([0, t_f], \mathbb{R}^2) \rightarrow \min J(u_1(t), u_2(t))$$

$$J(u_1, u_2) = \int_0^{t_f} [B_1 u_1^2(t) + B_2 u_2^2(t) + CS^2(t)] dt, \quad (6.54)$$

subject to (6.52) and (6.53). The set of all admissible controls is called the admissible set and we refer to it as

$$U = \{(u_1, u_2) \in L^\infty([0, t_f], \mathbb{R}^2) | (u_1(t), u_2(t)) \text{ satisfy (6.53)}\}. \quad (6.55)$$

Theorem 6.4.2. *There exists an optimal solution $(x^*, u_1^*, u_2^*) \in U^{1,\infty}([0, t_f], \mathbb{R}^4) \times L^\infty([0, t_f], \mathbb{R}^2)$ for the optimal control problem (6.52)–(6.54) such that*

$$J(u_1^*, u_2^*) = \min_{u_1, u_2 \in U} J(u_1, u_2), \quad (6.56)$$

if the following conditions are satisfied:

1. The set of admissible states is nonempty
2. The admissible control set U (6.55) is nonempty, convex and closed.
3. The right-hand side of the state system (6.47) is bounded by a linear combination of the state and control variables.
4. The integrand $L(S, u_1, u_2) = (B_1 u_1^2(t) + B_2 u_2^2(t) + CS^2(t))$ is convex on U
5. There exist constants $\ell_1, \ell_2 > 0$, and $\beta > 1$ such that

$$L(S, u_1, u_2) \geq \ell_1 + \ell_2 \left(|u_1|^2 + |u_2|^2 \right)^{\frac{\beta}{2}}$$

Proof. To proof the existence theorem we show conditions above

1. Since $\frac{hA(t)}{1+A(t)} < h$, neglecting the negative terms in (6.47) we have,

$$\begin{cases} X'(t) \leq rX(t) \left(1 - \frac{X(t)}{K} \right), \\ S'(t) \leq m\lambda X(t) S(t), \\ I'(t) \leq hS, \\ A'(t) \leq u_2 \omega + aS(t - \tau). \end{cases} \quad (6.57)$$

System (6.57) can be rewritten in the vector form as:

$$\begin{pmatrix} X(t) \\ S(t) \\ I(t) \\ A(t) \end{pmatrix}' \leq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X(t) \\ S(t) \\ I(t) \\ A(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \begin{pmatrix} X(t-\tau) \\ S(t-\tau) \\ I(t-\tau) \\ A(t-\tau) \end{pmatrix} + \begin{pmatrix} \frac{rK}{4} \\ Q \\ 0 \\ u_2\omega \end{pmatrix}$$

- This system is linear in finite time with bounded coefficients.
- Then the solutions of this linear system are uniformly bounded.
- Therefore, the solutions of the nonlinear system (6.47) are bounded and exist (Lukes and DL, 1982).
- Hence, condition one is satisfied.

2. Condition 2 is satisfied by the definition of U .

3. System (6.47) is bilinear in the control variables u_1, u_2 and cab be rewritten as

$$f(t, x(t), x(t-\tau), u_1, u_2) = \alpha(t, x) + \beta_1(t, x(t-\tau)) + \frac{rK}{4} + Q + u_1 + u_2$$

$x(t) = [X(t), S(t), I(t), A(t)]^T$, $x(t-\tau) = [X(t-\tau), S(t-\tau), I(t-\tau), A(t-\tau)]^T$, α and β_1 are the vector valued functions of $x(t)$ and $x(t-\tau)$, respectively.

The solutions are bounded, and we have $|f(t, x(t), x(t-\tau), u_1, u_2)| \leq |M_1 x(t)| + |M_2 x(t-\tau)| + |M_3| + |M_4| + |M_5| \leq h_1 |x| + \frac{rK}{4} + |Q| + |u_1| + |u_2|$ h_1 depends on the coefficients of the system, and

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} \frac{rK}{4} \\ 0 \\ 0 \\ 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 \\ Q \\ 0 \\ 0 \end{pmatrix}, M_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_2\omega \end{pmatrix}$$

4. The integrand $(B_1 u_1^2(t) + B_2 u_2^2(t) + CS^2(t))$ is clearly convex on U .

5. The state variables being bounded, let $\ell_1 = \inf(S(t))$, $\ell_2 = \inf(B_1, B_2)$ and $\beta_1 = 2$ then

$$CS^2 + B_1 u_1^2 + u_2^2 \geq \ell_1 + \ell_2 \left(|u_1|^2 + |u_2|^2 \right)^{\frac{\beta}{2}}$$

Therefore, from Fleming and Rishel (2012), there exists an optimal control pair u_1, u_2 . $\square$

6.4.4 Characterization of optimal control

In this subsection, we derive the first order necessary conditions to characterize the optimal control pair, by constructing the Hamiltonian H and using Pontryagin's Minimum Principle with delay in state (Göllmann et al., 2009; Rodrigues et al., 2018). This Principle converts system (6.47)–(6.50) in to a problem of minimizing the Hamiltonian H .

Before characterizing the optimal control pair, we first define the Hamiltonian H for the control problem (6.47)–(6.49).

The function

$$\begin{aligned} H &= H\left(X^*(t), S^*(t), I^*(t), A^*(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t), u_1^*(t), u_2^*(t)\right) \\ &= \left(rX\left(1 - \frac{X}{K}\right) - \lambda XS\right)\xi_1(t) \\ &\quad + \left(m\lambda XS - (1 - u_1)\frac{hA(t - \tau_2)S}{1 + A(t - \tau_2)} - dS\right)\xi_2(t) \\ &\quad + \left((1 - u_1)\frac{hA(t - \tau_2)S}{1 + A(t - \tau_2)} - (d + \alpha)I\right)\xi_3(t) \\ &\quad + \left(u_2\omega + aS(t - \tau_1) - \eta A\right)\xi_4(t) \\ &\quad + [B_1u_1^2 + B_2u_2^2 + CS^2], \end{aligned}$$

with the optimality condition

$$\frac{\partial H}{\partial u^*} = 0 \Rightarrow \begin{cases} \frac{\partial H}{\partial u_1^*} = 0, \\ \frac{\partial H}{\partial u_2^*} = 0, \end{cases} \quad (6.58)$$

and the adjoint equation

$$\begin{cases} \dot{\xi}_1 = -\frac{\partial H}{\partial X}, \\ \dot{\xi}_2 = -\frac{\partial H}{\partial S} - \chi_{[0, t_f - \tau]}(t) \frac{\partial H}{\partial S_\tau}(t + \tau), \\ \dot{\xi}_3 = -\frac{\partial H}{\partial I}, \\ \dot{\xi}_4 = -\frac{\partial H}{\partial A} - \chi_{[0, t_f - \tau]}(t) \frac{\partial H}{\partial A_\tau}(t + \tau), \end{cases} \quad (6.59)$$

with transversality conditions $\xi_i(t_f) = 0, i = 1, 2, 3, 4$ is a Hamiltonian function. Now we apply the necessary conditions to the Hamiltonian function H in (6.58).

For the sake of simplicity, we first define a characteristic function $\chi_{[a, b]}(t)$ as

$$\chi_{[a, b]}(t) = \begin{cases} 1, & \text{if } t \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 6.4.3 (Necessary optimality conditions for the delayed optimal control problem).

Let u_1^* and u_2^* be optimal control variables and X^* , S^* , I^* , and A^* be the corresponding (optimal) state variables of the control system (6.47)–(6.50). Then there exists adjoint variable $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)) \in \mathbb{R}^4$ that satisfy the following adjoint equations

$$\begin{cases} \frac{d\xi_1(t)}{dt} = -\xi_1(t) \left[r \left(1 - \frac{2X^*}{K} \right) - \lambda S^* \right] - \xi_2(t) (m\lambda S^*), \\ \frac{d\xi_2(t)}{dt} = -2CS^* + \lambda X^* \xi_1(t) + \left((1 - u_1) \frac{hA^*(t-\tau)}{1+A^*(t-\tau)} - m\lambda X^* + d \right) \xi_2(t) \\ \quad - \left((1 - u_1) \frac{hA^*(t-\tau)}{1+A^*(t-\tau)} \right) \xi_3(t) + \chi_{[0, t_f - \tau]}(t) a \xi_4(t)(t + \tau), \\ \frac{d\xi_3(t)}{dt} = (d + \alpha) \xi_3(t), \\ \frac{d\xi_4(t)}{dt} = \xi_4(t) \eta + \chi_{[0, t_f - \tau]}(t) (1 - u_1) \frac{hS^*(t+\tau)}{(1+A^*)^2} [\xi_2(t)(t + \tau) - \xi_3(t + \tau)], \end{cases} \quad (6.60)$$

with transversality conditions

$$\xi_i(t_f) = 0, i = 1, 2, 3, 4 \quad \text{and} \quad \chi_{[0, t_f - \tau]} = \begin{cases} 1, & \text{if } t \in [0, t_f - \tau], \\ 0, & \text{otherwise.} \end{cases} \quad (6.61)$$

Furthermore, the corresponding optimal controls are given by:

$$\begin{cases} u_1^* = \max \left(\min \left(\frac{hA^*(t-\tau)S^*(t)(\xi_3(t) - \xi_2(t))}{2B_1(1+A^*(t-\tau))}, 1 \right), 0 \right), \\ u_2^* = \max \left(\min \left(-\frac{\xi_4(t)\omega}{2B_2}, 1 \right), 0 \right). \end{cases} \quad (6.62)$$

Proof. Using the Pontryagin's Minimum principle with delay in state [Göllmann et al. \(2009\)](#), we obtain the adjoint equations and transversality conditions such that:

$$\begin{aligned} \frac{d\xi_1(t)}{dt} &= -\frac{\partial H}{\partial X}, & \xi_1(t_f) &= 0, \\ \frac{d\xi_2(t)}{dt} &= -\frac{\partial H}{\partial S} - \chi_{[0, t_f - \tau]} \frac{\partial H}{\partial S_\tau}(t + \tau), & \xi_2(t_f) &= 0, \\ \frac{d\xi_3(t)}{dt} &= -\frac{\partial H}{\partial I}, & \xi_3(t_f) &= 0, \\ \frac{d\xi_4(t)}{dt} &= -\frac{\partial H}{\partial A} - \chi_{[0, t_f - \tau]} \frac{\partial H}{\partial A_\tau}(t + \tau), & \xi_4(t_f) &= 0. \end{aligned} \quad (6.63)$$

Now by the optimality condition (6.58), we can have

$$\begin{cases} \frac{\partial H}{\partial u_1^*} = 2B_1 u_1^* + \frac{hA_\tau S}{1+A_\tau} \xi_2 - \frac{hA_\tau S}{1+A_\tau} \xi_3 = 0, \\ \frac{\partial H}{\partial u_2^*} = 2B_2 u_2^* + \omega \xi_4 = 0. \end{cases} \Rightarrow \begin{cases} u_1^* = \frac{(\xi_3(t) - \xi_2(t))hA^*(t-\tau)S^*(t)}{2B_1(1+A^*(t-\tau))}, \\ u_2^* = \frac{-\xi_4(t)\omega}{2B_2}. \end{cases} \quad (6.64)$$

Using the boundedness of the controls, i.e., the fact that admissible controls take values such that $0 \leq u_i(t) \leq 1, i = 1, 2$, it follows from the minimality condition of Pontryagin's Principle

that control u_1^*, u_2^* satisfies the relationships

$$u_1^* = \begin{cases} 0 & \text{if } \frac{(\xi_3(t) - \xi_2(t))hA^*(t-\tau)S^*(t)}{2B_1(1+A^*(t-\tau))} \leq 0, \\ \frac{(\xi_3(t) - \xi_2(t))hA^*(t-\tau)S^*(t)}{2B_1(1+A^*(t-\tau))} & \text{if } 0 < \frac{(\xi_3(t) - \xi_2(t))hA^*(t-\tau)S^*(t)}{2B_1(1+A^*(t-\tau))} < 1, \\ 1 & \text{if } \frac{(\xi_3(t) - \xi_2(t))hA^*(t-\tau)S^*(t)}{2B_1(1+A^*(t-\tau))} \geq 1, \end{cases} \quad (6.65)$$

$$u_2^* = \begin{cases} 0 & \text{if } -\frac{\xi_4(t)\omega}{2B_2} \leq 0, \\ -\frac{\xi_4(t)\omega}{2B_2} & \text{if } 0 < -\frac{\xi_4(t)\omega}{2B_2} < 1, \\ 1 & \text{if } -\frac{\xi_4(t)\omega}{2B_2} \geq 1. \end{cases}$$

Since $(\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t))$ is a solution of (6.60), then the optimal control pair $(u_1^*(t), u_2^*(t))$ can be characterized by (6.62). $\square$

6.4.5 Optimality system

The optimal control pair and the state are found by solving the following optimality system, which consists of the state system (6.47), the adjoint system (6.60), boundary conditions (6.61), and the characterization of the optimal control pair (u_1^*, u_2^*) (6.62):

$$\left\{ \begin{array}{l} \frac{dX^*(t)}{dt} = rX^* \left[1 - \frac{X^*}{K} \right] - \lambda X^* S^*, \\ \frac{dS^*(t)}{dt} = m\lambda X^* S^* - \frac{(1-u_1^*)hA^*(t-\tau)S^*}{1+A^*(t-\tau)} - dS^*, \\ \frac{dI^*(t)}{dt} = \frac{(1-u_1^*)hA^*(t-\tau)S^*}{1+A^*(t-\tau)} - (d+\alpha)I^*, \\ \frac{dA^*(t)}{dt} = u_2^*\omega + aS^*(t-\tau) - \eta A^*, \\ \frac{d\xi_1(t)}{dt} = -\xi_1(t) \left[r \left(1 - \frac{2X^*}{K} \right) - \lambda S^* \right] - \xi_2(t) (m\lambda S^*), \\ \frac{d\xi_2(t)}{dt} = -2CS^* + \lambda X^* \xi_1(t) + \left(\frac{(1-u_1^*)hA^*(t-\tau)}{1+A^*(t-\tau)} - m\lambda X^* + d \right) \xi_2(t) \\ \quad - \left(\frac{(1-u_1^*)hA^*(t-\tau)}{1+A^*(t-\tau)} \right) \xi_3(t) + \chi_{[0, t_f - \tau]}(t) a \xi_4(t)(t+\tau), \\ \frac{d\xi_3(t)}{dt} = (d+\alpha)\xi_3(t), \\ \frac{d\xi_4(t)}{dt} = \xi_4(t)\eta + \chi_{[0, t_f - \tau]}(t) \frac{(1-u_1^*)hS^*(t+\tau)}{(1+A^*)^2} \left[\xi_2(t)(t+\tau) - \xi_3(t+\tau) \right], \\ u_1^* = \max \left(\min \left(\frac{hA^*(t-\tau)S^*(t)(\xi_3(t) - \xi_2(t))}{2B_1(1+A^*(t-\tau))}, 1 \right), 0 \right), \\ u_2^* = \max \left(\min \left(-\frac{\xi_4(t)\omega}{2B_2}, 1 \right), 0 \right), \\ \xi_1(t_f) = 0, \xi_2(t_f) = 0, \xi_3(t_f) = 0, \xi_4(t_f) = 0, \\ X^*(\theta) = \phi_1(\theta), S^*(\theta) = \phi_2(\theta), I^*(\theta) = \phi_3(\theta), A^*(\theta) = \phi_4(\theta), \theta \in [-\tau, 0]. \end{array} \right. \quad (6.66)$$

6.5 Numerical simulations

In this section, we provide numerical simulations obtained from the application of our analytical results, as given in previous sections. To illustrate the behavior of model (6.1), we did numerical simulations with the set of parameter values in Table 6.1. We begin by simulating the system without delays, then with delays.

6.5.1 Numerical simulations of the stability of equilibria

The parameters of model (6.1) used in the numerical simulations are given in Table 6.1. The initial values are chosen as

$$(X(\theta), S(\theta), I(\theta), A(\theta)) = (0.02, 0.12, 0.02, 0.35), \theta \in [-\tau, 0], \tau = \max\{\tau_1, \tau_2\}. \quad (6.67)$$

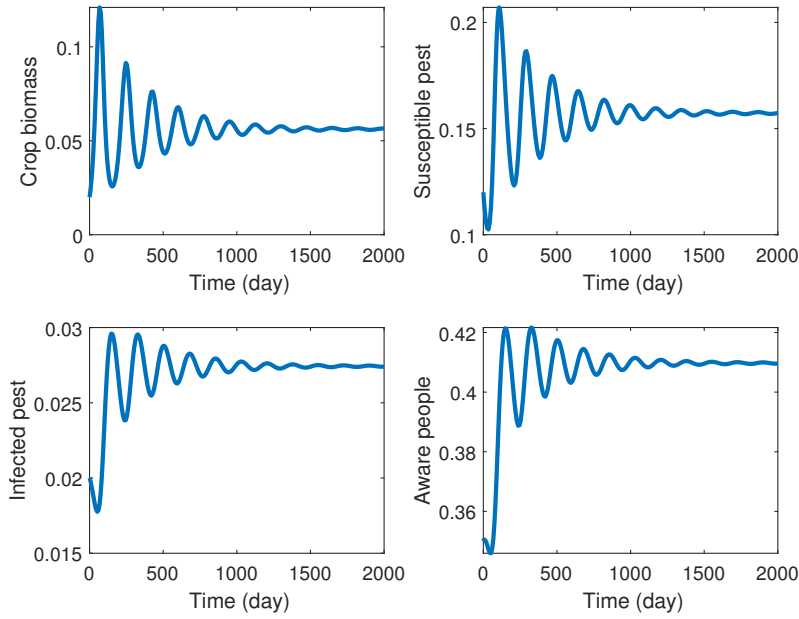


Figure 6.2: Stability behavior of the endemic state for the non-delayed population system (6.1), i.e., when $\tau_1 = 0$ and $\tau_2 = 0$.

When we choose $\tau_1 = 0$ and $\tau_2 = 0$, the behavioral patterns of the densities of susceptible pest, infected pest, crop biomass, and aware population are presented taking the parameters from Table 6.1. All the system populations oscillate initially and finally become asymptotically stable and converge to the endemic state value (see Figure 6.2). Here, it is to be noted that all conditions of Theorem 6.2.5 are satisfied and thus the coexistence equilibrium E^* is stable.

Figure 6.3 contains the contour plot of the basic reproduction number R_0 , given by equation(6.17), as a function of the awareness active rate h and global awareness rate ω . The

region where E^* exists and E_1 is stable are identified. We see that if both h and ω are large, then the coexistence equilibria E^* will not be feasible but the pest-free equilibrium E_1 is feasible and stable as $R_0 < 1$ there. Consequently, a transcritical bifurcation occurs at $R_0 = 1$. This is also clear from Figure 6.4.

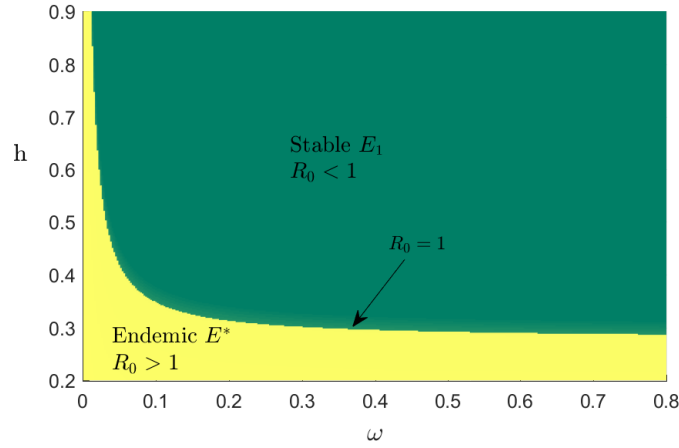


Figure 6.3: Plot of the basic reproduction number R_0 as a function of h and ω . All other parameters are as in Table 6.1.

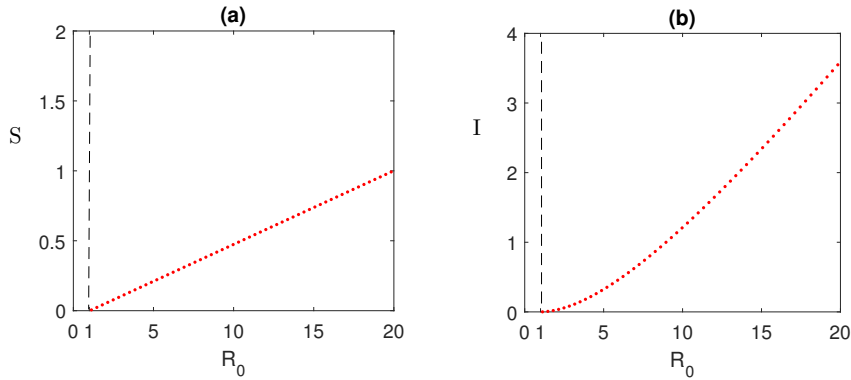


Figure 6.4: Forward bifurcation. Plot of the basic reproduction number R_0 as a function of S and I . All other parameters are as in Table 6.1. Coexisting equilibrium E^* exists for $R_0 > 1$ and for $R_0 < 1$ it is not feasible but pest-free equilibrium E_1 is stable.

Figure 6.5 displays the effect of local awareness rate a on the steady state values of the model populations (without delay). As expected, we see that awareness of population is increased as the value of a increases and the uninfected pest population decreases.

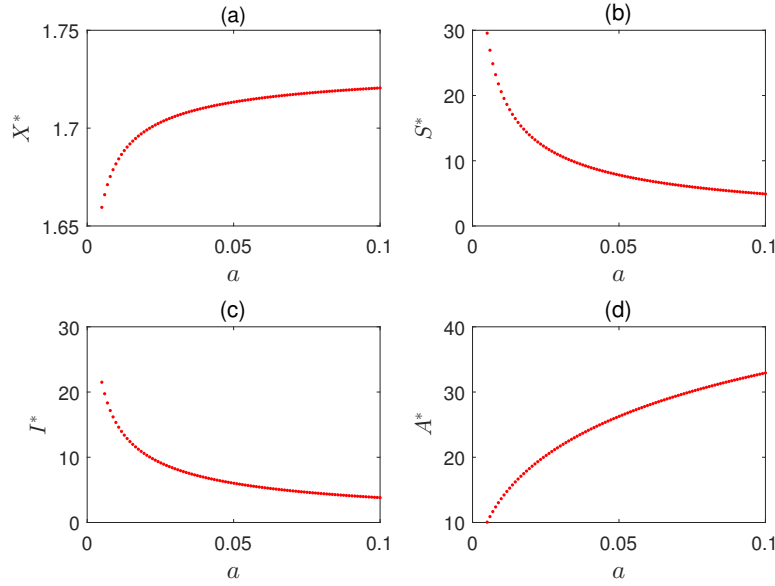


Figure 6.5: Effect of local awareness, a , on the nondelayed system (6.1) with values of the other parameters given in Table 6.1.

6.5.2 Numerical simulations with different time delays

In this subsection, we examine the influence of the time delays on the number of infected pest by performing some numerical simulations. When the time delay $\tau_2 = 0$ is fixed and τ_1 takes values $\tau_1 = 6$ and $\tau_1 = 16$, respectively, we observe that oscillation increases as τ_1 increases and periodic oscillation is seen at $\tau_1 = 16$ days (see Figure 6.6). Stability switch occurs through Hopf bifurcation. In Figure 6.7, the time delay $\tau_1 = 0$ is fixed and τ_2 takes value $\tau_2 = 16$. We see a similar result as the one of Figure 6.6. For any combination of τ_1 and τ_2 , when $\tau_1 + \tau_2 = 16$, periodic oscillation (Hopf bifurcation) can be seen. Figure 6.8 confirms that the Hopf bifurcation is of supercritical type.

Figures 6.6, 6.7 and 6.8 indicate that the populations always oscillate initially and ultimately approach towards their equilibrium values when $\tau_1 < \tau_0 = 16$. This indicates that the number of pest will be high and sometimes low. If the value of delay, τ_i , exceeds its critical value, $\tau_0 = 16$ days approximately, the population becomes periodic. In this situation, it is very difficult to make the prediction about the size of the epidemic. Hence, these figures indicate that the endemic equilibrium E^* of model system (6.1) with delay is stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. For $\tau > \tau_0$, bifurcating periodic solutions are observed. This is in agreement with Theorem 6.2.8. In Figure 6.9, the region of stability in the λ - τ_2 parameter plane is shown. The critical value of τ_2 , for which Hopf bifurcation occurs, depends on the consumption rate λ . The critical value of τ_2 decreases as λ increases.

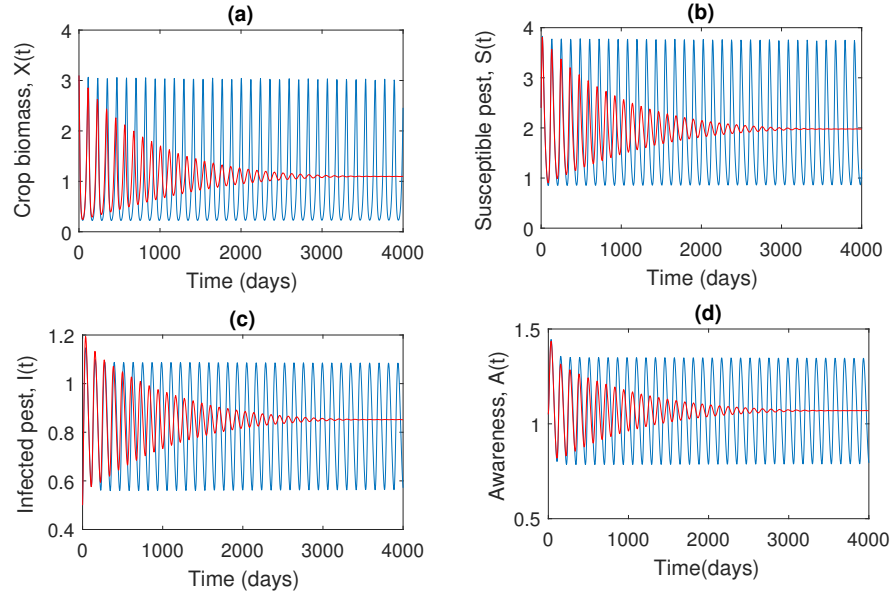


Figure 6.6: The behavior of the model population for different values of τ_1 and fixed $\tau_2 = 0$. Red line represents $\tau_1 = 6$ and blue line represents $\tau_1 = 16$. Other parameters are as in Figure 6.2.

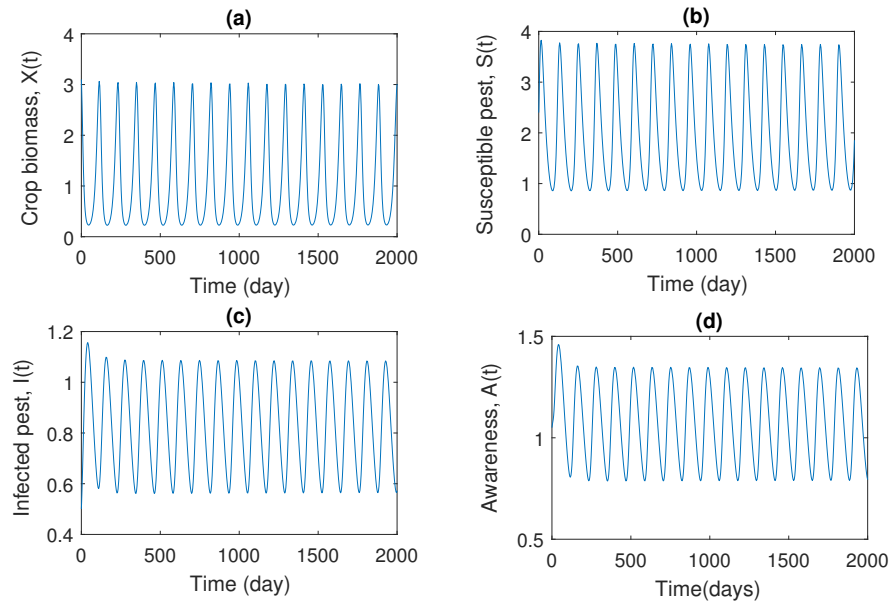


Figure 6.7: The behavior of the model population for $\tau_2 = 16$ and fixed $\tau_1 = 0$.

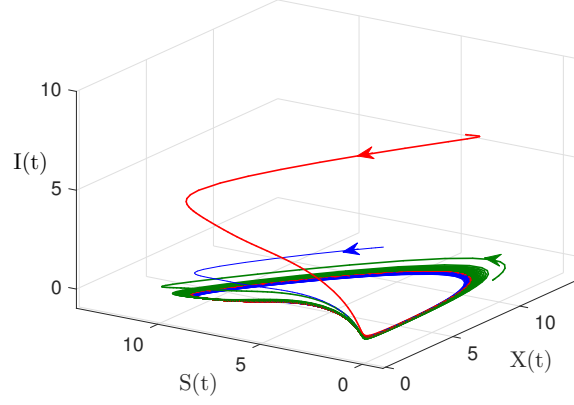


Figure 6.8: Supercritical bifurcation. The behavior of the solution for different initial conditions and $\tau_1 = 16$. Phase portrait converges to the same limit cycle.

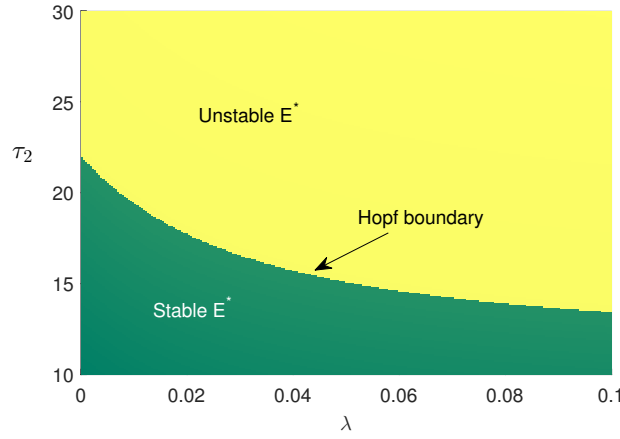


Figure 6.9: Region of stability in the λ - τ_2 parameter plane. Other parameters are as in Table 6.1.

6.5.3 Numerical simulations of optimal control

In this subsection, we solve numerically the optimality system (6.66) and we present the results found. We note that the optimality system is a two-point boundary value problem, with separated boundary conditions at times $t = 0$ and $t = t_f$ and we display the results found graphically. Solving the system (6.66) requires an iterative scheme developed by (Hattaf and Yousfi, 2012). This involves use of an appropriate algorithm.

Numerical algorithm

Direct and indirect approaches are usually used to solve the OCPs. In the direct approach, the OCP is transformed into a nonlinear programming problem. The algorithm of solving the above OCP is roughly based on the following steps:

- Step 1.** Provide the initial guess for the control parameters u_{10} and u_{20} over the interval, and declare the parameters.
- Step 2.** Set the initial conditions for the state variables with the stored values of the optimal controls and solve the state system forward in time, using any DDEs solver.
- Step 3.** Use the transversality condition $\xi(t_f) = 0$, the stored values u_{10} and u_{20} and $X(t), S(t), I(t), A(t)$ and solve the adjoint system backward in time.
- Step 4.** Check the control by entering the new values of the state and the adjoint state into the characterization of the optimal control.
- Step 5.** Verify for convergence. If values of the variables in this iteration and the latest iteration are not negligibly small, output the current values as solutions. If values are not small, return to Step

Let $h > 0$ be the discretization step size and let us consider integers $(n, m) \in \mathbb{N}^2$ with $\tau = mh$ and $t_f = nh$. In our programming we consider m knots to the left of 0 and to the right of t_f , and we obtain the following partition:

$$\Delta = (t_{-m} = -\tau < \dots < t_{-1} < 0 < t_1 < \dots < t_n = t_f < \dots < t_{n+m}).$$

Then, we have $t_i = ih (-m \leq i \leq n+m)$. Next we define the state and adjoint variables $X(t), S(t), I(t), A(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)$, and controls $u_1(t), u_2(t)$ in terms of the nodal points $X_i, S_i, I_i, A_i, \xi_1^i, \xi_2^i, \xi_3^i, \xi_4^i$. Now, a combination of forward and backward difference approximations is applied to obtain the solutions.

For the simulations, we use the parameter values given in Table 6.1 and the initial values (6.67). The numerical approximations of the ROCP are carried out using forward and backward Euler methods. Starting with an initial guess for the value of the controls on the time interval $[0, t_f]$, we solve the state system with control variables (6.47) using forward Euler's scheme. Meanwhile, the adjoint system is solved using the solutions of the state system and the transversality conditions backward in time.

The optimal control function $u^* = (u_1^*, u_2^*)$ is intended in such a way that it minimizes the cost functional given by (6.49). In Figure 6.10, we present the numerical solutions of the population class with control. We operate the control through insecticide spraying up to 100 days. Crop biomass increased and pest population decreased significantly with the influence of the optimal profiles of global awareness and awareness based control activity, as shown in Figure 6.10. It is also observed that susceptible pest population goes to extinction within first 90 days due to effort of optimal control. Thus, optimal control by means of awareness-based biocontrol is of great value in controlling a pest problem in a crop field.

The Pontryagin controls are shown in Figure 6.11. We see they are bang-bang controls.

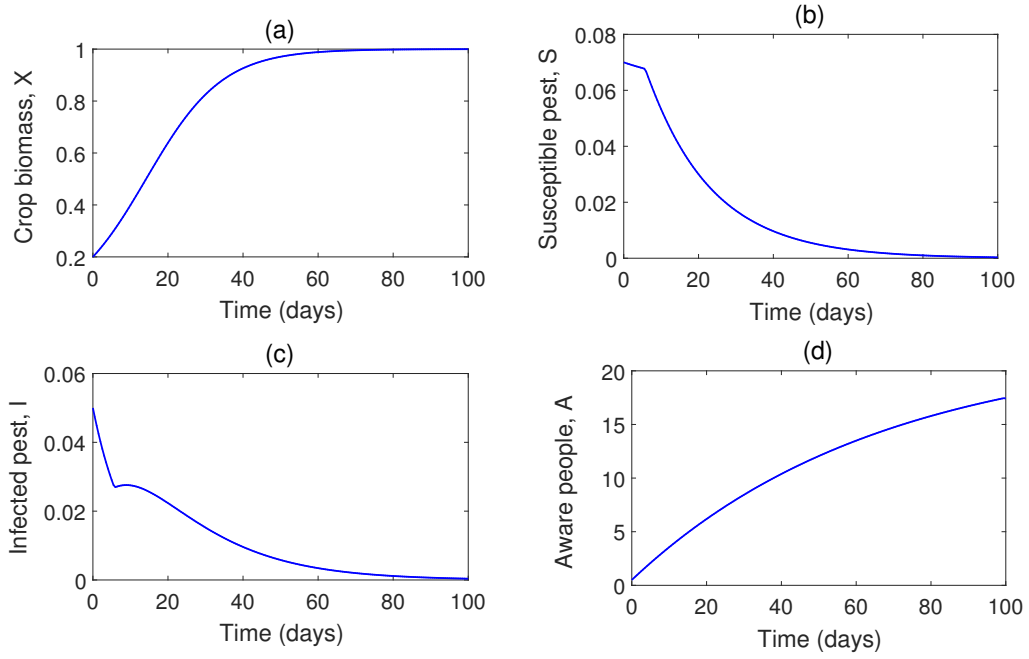


Figure 6.10: Numerical solution of the optimal control problem with $\tau_1 = 0$, $\tau_2 = 40$, and remaining parameter values from Table 6.1.

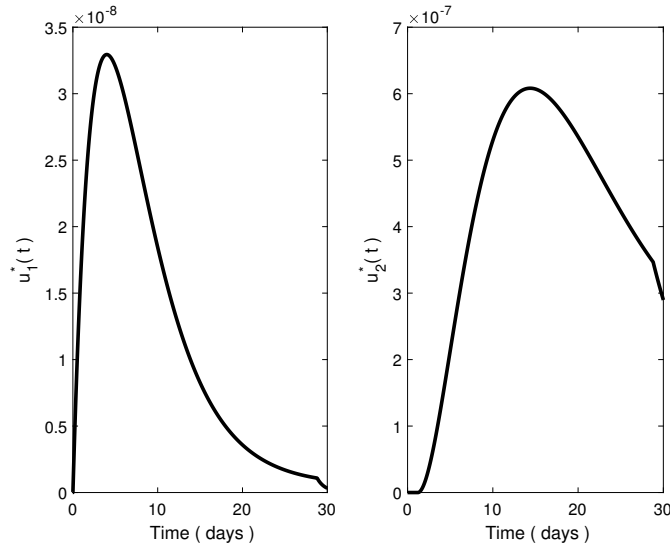


Figure 6.11: Numerical approximation of the Pontryagin extremal controls u_1^* and u_2^* .

6.6 Results and Discussion

In this chapter, a mathematical model for effect of farming through social media on the crop pest has been derived and analyzed. The model contains the crop biomass, susceptible pests, infected pests and the level of awareness. It is assumed that aware people adopts biological control such as IPM as it is eco-friendly and less harmful for human health and the environment. Using bio-pesticides, susceptible pests are made infected ones as infected pest is less harmful for crops, and ultimately they die for their higher mortality rate due to induced viral

infection. Local awareness campaigns are taken as proportional to the density of susceptible pest present in the crop field. We have assumed that the global factors such as radio, TV, etc. increase the level of awareness. From this study, we observed that if the impact of awareness campaigns increases, the density of crop increases and as a consequence, pest prevalence decreases.

Next, we have also investigated the effect of time-delay that can be regarded as delay between advertisement made and action of a pest control on crop field. A rigorous analysis on the stability and occurrence of bifurcation of the coexistence (interior) equilibrium point has been presented. Our analysis indicates that the value of delay in certain specified range could ensure the stable coexistence of the pest and also the delay could drive the system to an unstable state. Thus the time-delay has a regulatory impact on the whole system. The system changes its stability asymptotically to a limit cycle behavior (i.e. periodic oscillation occurs). But global awareness ω through Radio, TV, mobile telephony can control the oscillations and makes the system stable. Effects of both delay τ_1, τ_2 are seen to destabilizing the system. But when global awareness is high or consumption rate of crop by pest is lower, then delay has no effect on the stability of endemic equilibrium.

On the other hand, optimal control theory gives the cost-effective profile of bio-pesticides and global awareness campaign. This approach reduced the cost of management as well as increased the yield. Also, negative effects of delays can be minimized using optimal control approach.

CONCLUSION AND RECOMMENDATIONS

This part presents summary of the findings and conclusions of the study, and recommendations for further studies are also given. The list of adverse effects of chemical pesticides is quite long and this emphasized the need to search for an alternative method, which would be a safer and environmentally acceptable alternative to chemical pesticides. The method of microbial pest control has emerged as a result of this kind of realization.

This dissertation aims to provide strong theoretical support to microbial, chemical and awareness based integrated pest control methods. We have tried to formulate several sophisticated mathematical models with the help of up-dated mathematical machineries. The details of the construction of the model is presented in chapters 4,5 and 6.

Conclusions

In this dissertation, deterministic models with ordinary and delay differential equations for crop pest control managements have been developed. The proposed autonomous models were first analyzed and then extended to optimal control problems. Appropriate and relevant knowledge about crop and its pests is very much essential for people engaged in agriculture. Also, there might be time lag in measuring the density of susceptible pest or in implementing awareness based interventions. Thus, we incorporated two time delay parameters, τ_1 and τ_2 , in the modeling process.

A mathematical model for effect of farming through social media on the crop pest has been derived and analyzed. Here it is observed that, increasing impact of awareness campaign, increases, the crop density and as a consequence, pest prevalence decreases. Effects of both delay times τ_1, τ_2 are seen to destabilizing the system. But when global awareness is high or consumption rate of crop by pest is lower, then delay has no effect on the stability of endemic equilibrium.

Applying optimal control theory gives the cost-effective profile of bio-pesticides and reduced the cost of management as well as increased the yield. Also, negative effects of delays were seen to be minimized using optimal control approach.

Another mathematical model described by a system of ordinary differential equations has been projected and analyzed. The model contains four densities, specifically, density of crop-biomass, density of susceptible and infected pests, and population awareness. The proposed model exhibits four feasible steady state points. From our analytical and numerical investigations, we saw that the most ecological noteworthy parameters in the system are the consumption rate α and the awareness activity rate λ . We proved that the interior equi-

librium is locally asymptotically stable under some critical parametric conditions and the system enters into the Hopf bifurcation around the interior equilibrium while the pest attack rate passes through the threshold value α^* . We further, confirmed that the global awareness, circulated by radio, TV, telephone, internet, etc., increase the level of population awareness to crop fields. Moreover, we have used optimal control theory to provide the cost effective outline of bio-pesticides and people awareness campaigns. It is found that, increased awareness between people with optimal use of bio-pesticides is a correct recommendation for the control of pest in crop field.

An extended mathematical model with a system of ordinary differential equations has been developed and analyzed to study control crop pests with farming awareness integrated approach. The model under consideration exhibits four feasible equilibrium points: A crop-pest free equilibrium point, a pest free equilibrium point, a susceptible pest free equilibrium point, and an interior equilibrium point. It is shown that the dynamics changes with the parameter value α (consumption rate of pest to crops). Awareness induced integrated approach can stabilize the endemic equilibrium and also can make the system free from crop pests. Pest eradication may not be attained by increasing the rate of local awareness that arises from observation of their fields. It can be achieved by increasing the rate of global awareness through media such as TV, radio etc. which increases the awareness level about the pest, and relates optimal level of using integrated approach. Finally, optimal control theory has been adopted in order to make the system cost effective. Optimal control profiles of bio-pesticides, chemical pesticide and global advertisement (i.e. ω) have been determined which minimize the cost function as well as the pest number.

The hypothetical sets of parameter values are used to verify the analytical findings obtained in this work. In future work, it would be interesting to expand on simulations by using realistic data to estimate the parameters.

From the analysis of this thesis (from our three models) it can be concluded that raising awareness among people with optimal profile of IPM may be a proper aspect for the control of pest in crop field reducing the serious issues that synthetic pesticides have on human health and environment.

Recommendations

Recently, the entire globe is concerned about the menace of harmful pests and means to find pathways to control them. Pesticides have a great role to destroy pests but the detrimental consequences of pesticide use are numerous and include concerns over the worker and food safety, environmental contamination, and decline of biodiversity in agro-ecosystems. IPM is an approach that seeks to minimize and rationalize the use of chemical pesticides as well as

promote the use of safer alternatives to pesticides(e.g., bio-pesticides, cultural practices)

According to the recommendations of the National Farmer Policy 2007, promotion of bio-pesticides for increasing agricultural production while at the same time sustaining the health of farmers and environment is of utmost important.

Awareness campaigns related to farming have helped in spreading the education among farmers about the risks of using pesticides on both human health and also the different related environmental hazards. These campaigns encourage the proper and moderate use of such pesticides so as to minimize the harmful effects.

In this dissertation mathematical theory is presented for designing optimal pest control programs. The pest population is governed by a susceptible and infected concentration and optimal control is considered for a finite time period. An optimal program is one which minimizes the cost of administering the program and the loss due to crop damage, and causes minimum harm to the environment. A mathematical model which describes the relationship of the crop biomass, pest, and people awareness level is presented. The controlling action is assumed to have significant effects and this is built into the model. The techniques of mathematical optimization are applied to work out the existence, characterization, and optimality for an optimal control policy.

In the light of the above, the researcher wishes to make some recommendations, which, if taken into consideration, might bring some positive changes to the minimization of crop pests and increase in crop production.

- Biological pest control methodologies must be prioritized and people in general and agriculturists, in particular, must be well informed and educated about the management and use of such control measures.
- policy makers in agriculture should focus on raising awareness among people with an optimal profile of IPM as they are proper aspect for the control of pests in crop fields reducing the serious issues that synthetic pesticides have on human health and the environment.
- Biological control using insects, pathogens, or other organisms has an important role as a management tool as an integrated pest management strategy.
- Mathematical models are important tools in pest controls using biological, chemical, genetic, to answer the questions as, when, how frequently, and with what intensity the controlling action should be effected.
- Ecologists should make efforts, in informing the people engaged in agriculture to increase their global awareness through state media such as Radio and TV, which increases the awareness level about the pest and relates the optimal level of using an integrated approach.

- Optimal controls must be employed in crop pest management, as it reduces the cost of management and increases the crop yield. Besides, it minimizes the negative effects of time delays.
- An increase in awareness level is essential as pest outbreaks may decline to a lower level even with a moderately high pest consumption rate.
- The theoretical investigation performed in this work may help the experimental ecologists to do some experimental studies and as a result, the theoretical ecology may be developed to some extent.

Future work

The three deterministic models that were formulated in this dissertation can be extended in divers ways. In this research, several interesting mathematical questions are still open for further study. The following are recommended for future research; As a first, it remains open the question of how to define a Lyapunov function to prove global stability for the proposed delayed model. In this direction, the techniques of different approaches in the literature should be helpful. This is under investigation and will be addressed elsewhere. Second, the works on the mathematical models in chapter 5 and chapter 6 can be extended in several ways, for example by introducing time delays in the awareness level of farmers attitude towards observation of fields and in becoming aware of their farm after campaigns made. Third, consideration of the crop population as infected and uninfected cases is also another possible extension of the problem to develop a valid mathematical model of pest control mechanism. Moreover, use of real data and estimation of parameters to analyze the developed mathematical models of crop pests may be an important part of research to be considered.

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APPENDIX A

PRELIMINARY MATHEMATICAL BACKGROUND

In this Appendix, we present some essential tools in the form of definitions, lemmas, propositions and theorems which are adapted from different literature. Since those are available in existing literature, we do not present any proof in most cases. We use those in our analysis as helping tools as some of those have significant roles in our developments.

A.1 Basics of dynamical systems

A system of ordinary differential equations of the form

$$\frac{dX}{dt} = \dot{X} = f(X(t)), X(t_0) = X_0 \quad (\text{A.1})$$

where X takes values in n -dimensional Euclidean space and t is usually time is said to be *autonomous* or time-invariant if f does not depend explicitly on t . Otherwise they are *non-autonomous*.

A.1.1 Equilibrium points and stability

Equilibrium point

Definition A.1.1. A point $X^* \in \mathbb{R}^n$ is called an *equilibrium point* (also *critical point*, *stationary point*, *steady state solution*, etc) of the system (A.1), if $f(X^*(t)) = 0$ for all $t \geq 0$.

We consider the dynamical system $\dot{X} = f(X(t)); f : D \rightarrow \mathbb{R}^n, D : \text{an open and connected subset of } \mathbb{R}^n$. Let X^* be the equilibrium point.

- The equilibrium point X^* is said to be *stable* if, for each $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that $\|X(0) - X^*\| < \delta(\varepsilon) \Rightarrow \|X(t) - X^*\| < \varepsilon \forall t \geq t_0$
- The equilibrium point is said to be *unstable* if it is not stable
- The equilibrium point X^* is said to be *convergent equilibrium* if $\exists \delta$ such that $\|X(0) - X^*\| < \delta \Rightarrow \lim_{t \rightarrow \infty} X(t) = X^*$
- The equilibrium point X^* is said to be *asymptotically stable* if it is stable and convergent.

Jacobian Matrix

We considered $\dot{X} = F(X(t))$, $X(t_0) = X_0$ where

$$F = [f_1, f_2, \dots, f_n], X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The variational matrix or Jacobian matrix of $F(x_1, x_2, \dots, x_n)$ denoted by J is defined as follows:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Invariance Principle

Definition A.1.2. : A set M is invariant if and only if for all $x \in M$, $\phi(x, t) \in M$ for all t . A set is positively (negatively) invariant if for all $x \in M$, $\phi(x, t) \in M$ for all $t > 0$ ($t < 0$).

Theorem A.1.1. Let $x_i = F_i(t, x_1, \dots, x_n)$ for $i \in [1, n]$ be a system of n differential equations with initial conditions $x_i(t_0) = x_i^0$ for $i \in [1, n]$. If each of the functions $F_1, \dots, F_n$ and the partial derivatives $\frac{\partial F_1}{\partial x_1}, \dots, \frac{\partial F_1}{\partial x_n}, \dots, \frac{\partial F_n}{\partial x_1}, \frac{\partial F_n}{\partial x_n}$ are continuous in $\mathbb{R}^{n+1}$ space, then there exists a unique solution $x_1 = \sigma_1(t), \dots, x_n = \sigma_n(t)$ that satisfies the initial conditions.

A.1.2 Characteristic Polynomials and Routh-Hurwitz

Definition A.1.3. A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz or asymptotically stable, if and only if $\Re(\lambda_i) < 0; \forall i = 1, 2, \dots, n$. where λ_i 's are the eigenvalues of the matrix A .

If the system is of n th order, the characteristic polynomial can be taken in the general form

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$$

where $a_i, i = 0, 1, \dots, n$ are all real and $a_n \neq 0$. We require conditions on the $a_i, i = 0, 1, \dots, n$ such that the zeros of $P(\lambda)$ have $\Re \lambda < 0$. The necessary and sufficient conditions for this to hold are the Routh-Hurwitz conditions. There are various equivalent forms of these, one of which is, together with $a_n > 0$,

$$D_1 = a_1 > 0, D_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0,$$

$$D_k = \begin{vmatrix} a_1 & a_3 & \cdot & \cdot & \cdot & \cdot \\ 1 & a_2 & a_4 & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot \\ 0 & 1 & a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & a_k \end{vmatrix} > 0, k = 1, 2, \dots, n.$$

As an example, for the cubic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

the conditions for $\Re \lambda < 0$ are $a_1 > 0, a_3 > 0; a_1a_2 - a_3 > 0$. The Routh-Hurwitz Criteria for $n = 2, 3, 4$, and 5 is summarized as follows:

$n = 2$: $a_1 > 0$ and $a_2 > 0$.

$n = 3$: $a_1 > 0, a_3 > 0$ and $a_1a_2 > a_3$.

$n = 4$: $a_1 > 0$ and $a_2 > 0, a_4 > 0$ and $a_1a_2a_3 > a_2^2 + a_1^2a_4$.

$n = 5$: $a_i > 0, i = 1, 2, 3, 4, 5, a_1a_2a_3 > a_3^2 + a_1^2a_4$ and $(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) > a_5(a_1a_2 - a_3)^2 + a_1a_2a_5$.

A.1.3 Delay differential equations

Delay differential equations (DDEs) are differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. A general form of the DDE for $y(t) \in \mathbb{R}^n$ is

$$y'(t) = f(t, y(t), y(t - \tau_1), y(t - \tau_2), \dots, y(t - \tau_k))$$

Constant Delay DDE: $\dot{x}(t) = f(t, x(t), x(t - \tau)), x(t) \in \mathbb{R}^d$ where delay $\tau > 0$ is constant.

Variable Delays: DDEs with Time-Dependent Delay $\dot{x}(t) = f(t, x(t), u(t - \tau(t))), x(t) \in \mathbb{R}^d$ where delay $\tau(t) \geq 0$ is a given function.

State-Dependent Delays: DDEs with State-Dependent Delay $\dot{x}(t) = f(t, x(t), x(t - \tau(t, x(t))))$, $x(t) \in \mathbb{R}^d$ where delay $\tau(t, x(t)) \geq 0$ depends on solution.

A.2 Optimal control problems (OCP)

Optimal Control is the process of determining control and state trajectories for a dynamic system over a period of time in order to minimize/maximize a performance index.

A.2.1 Lagrange, Mayer and Bolza formulations

There are three well known equivalent formulations to describe the OCP, which are Lagrange, Mayer and Bolza forms.

Definition A.2.1. (*Lagrange formulation:*) An OCP in Lagrange form is given by

$$\begin{aligned} \max_u J(x(t), u(t)) &= \int_{t_0}^{t_f} (f(t, x(t), u(t))) dt \\ \text{S.t. } x'(t) &= g(t, x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \quad (\text{A.2})$$

Definition A.2.2. (*Bolza formulation:*) The Bolza formulation of the OCP can be defined as

$$\begin{aligned} \max_u J(x(t), u(t)) &= \phi(t_0, x_0(t_0, t_f, x(t_f))) + \int_{t_0}^{t_f} (f(t, x(t), u(t))) dt \\ \text{S.t. } x'(t) &= g(t, x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \quad (\text{A.3})$$

where ϕ is a continuously differentiable function.

Definition A.2.3. (*Mayer formulation:*) The Mayer formulation of the OCP can be defined as

$$\begin{aligned} \max_u J(x(t), u(t)) &= \phi(t_0, x_0(t_0, t_f, x(t_f))) \\ \text{S.t. } x'(t) &= g(t, x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \quad (\text{A.4})$$

A.2.2 Pontryagin's maximum principle (PMP)

The necessary first order optimality conditions were developed by Lev Pontryagin and his coworkers. Pontryagin introduced the idea of adjoint functions to append the differential equation to the objective functional. Adjoint functions have a similar purpose as Lagrange multipliers in multivariate calculus, which append constraints to the function of several variables to be maximized or minimized.

Without loss of generality we consider the OC problem in Lagrange form (A.2).

Definition A.2.4. (*Hamiltonian.*) The function

$$H(t, x(t), u(t), \lambda(t)) = f(t, x(t), u(t)) + \lambda(t)g(t, x(t), u(t)) \quad (\text{A.5})$$

is called the Hamiltonian function, and λ is the adjoint variable. Now we are formulate the PMP for problem for the Lagrange formulation (A.2).

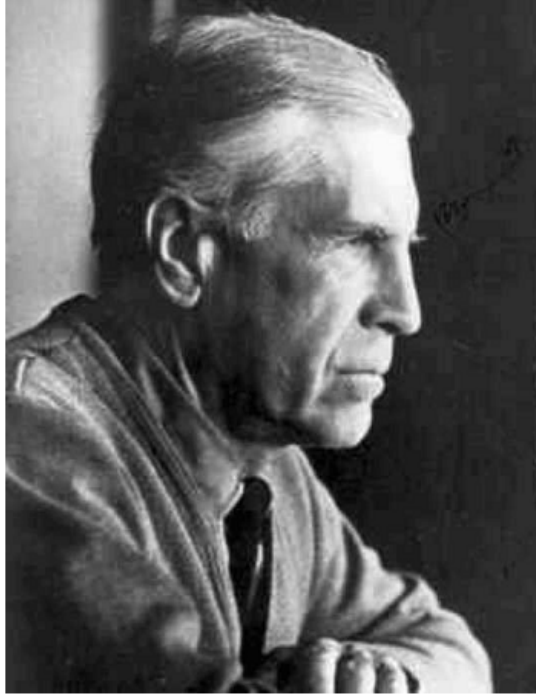


Figure A.1: Lev Semyonovich Pontryagin(1908-1988)

Theorem A.2.1 (PMP). *If u^* and x^* are optimal for problem (A.2), then there exists a piece wise differentiable ad-joint variable $\lambda(t)$ such that*

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t))$$

for all controls u at each time t , where the Hamiltonian H is defined by

$$H = f(t, x, u) + \lambda g(t, x, u)$$

and

$$\begin{aligned} \lambda'(t) &= -\frac{\partial}{\partial x} H(t, x^*, u^*, \lambda) \\ \lambda(t_f) &= 0 \end{aligned} \tag{A.6}$$

The last condition of [Theorem A.2.1](#), $\lambda(t_f) = 0$, is called the transversality condition, and is only used when the OC problem does not have the terminal value in the state variable, i.e., $x(t_f)$ is free. [Theorem A.2.1](#) converts the problem of finding a control which maximizes the objective functional subject to the state ODE and initial condition, into the problem of optimizing the Hamiltonian point-wise. As a consequence, we have

$$\frac{\partial H}{\partial u} = 0, \tag{A.7}$$

where u is an open set, at u^* for each t , that is, the Hamiltonian has a critical point at u^* . Usually this is called the optimality condition.

Remark A.2.1. If the Hamiltonian is linear in the control variable u , it can be difficult to calculate u^* from the optimality equation, since $\frac{\partial H}{\partial u}$ would not contain u .

In addition, the following conditions hold:

- If $\frac{\partial^2 H}{\partial u^2} < 0$ at u^* , for each $t \in [0, t_f]$, then the OC problem is of the maximization type.
- If $\frac{\partial^2 H}{\partial u^2} > 0$, for each $t \in [0, t_f]$, then the OC problem is of the minimization type.

Theorem A.2.2. Let the set of controls for problem (A.2) be Lebesgue integrable functions on $t_0 \leq t \leq t_f$ in $\mathbb{R}$. Suppose that $f(t, x, u)$ is concave in u , and there exist constants $C_1, C_2, C_3 > 0, C_4$, and $\beta > 1$ such that

$$\begin{aligned} g(t, x(t), u(t)) &= \alpha(t, x(t)) + \beta(t, x(t))u(t), \\ \|g(t, x(t), u(t))\| &\leq C_1(1 + |x(t)| + |u(t)|), \\ |g(t, \hat{x}(t), u(t)) - g(t, x(t), u(t))| &\leq C_2|\hat{x}(t) - x(t)|(1 + |u(t)|), \\ f(t, x(t), u(t)) &\leq C_3|u|^\beta - C_4 \end{aligned}$$

for all t with $t_0 \leq t \leq t_f$ in $\mathbb{R}$. Then there exists an optimal pair (x^*, u^*) maximizing J , with $J(x^*, u^*)$ finite.

Optimal control with bounded controls

Definition A.2.5. (OCPs with bounded controls). An OCP with bounded control, in Lagrange form, is:

$$\begin{aligned} \max_u J(x(\cdot), u(\cdot)) &= \int_0^{t_f} (f(t, x(\cdot), u(\cdot))) dt, \\ \text{S.t. } x'(t) &= g(t, x(t), u(t)), \\ x(t_0) &= x_0, \\ a &\leq u(t) \leq b \end{aligned} \tag{A.8}$$

where a and b are fixed real constants with $a < b$.

Proposition A.2.1. (Necessary conditions). If $u^*(t)$ and $x^*(t)$ are optimal for problem (A.8), then there exists a piece wise differentiable adjoint variable $\lambda(t)$ such that

$$H(t, x^*(t), u(t), \lambda(t)) \leq H(t, x^*(t), u^*(t), \lambda(t))$$

for all admissible controls u at each time t , where H is the Hamiltonian (A.5), and

$$\begin{aligned} \lambda'(t) &= -\frac{\partial H(t, x^*(t), u^*(t), \lambda(t))}{\partial x}, \quad (\text{adjoint function}) \\ \lambda(t_f) &= 0 \quad (\text{transversality condition}) \end{aligned} \tag{A.9}$$

We assume that H is a nonlinear differentiable function of u . We should adjust the optimal control as follows:

$$\begin{aligned} f_u + \lambda g_u &\leq 0 \quad \text{at} \quad u^*(t) \Rightarrow u^*(t) = a \\ f_u + \lambda g_u &= 0 \quad \text{at} \quad u^*(t) \Rightarrow a \leq u^*(t) \leq b \quad f_u + \lambda g_u \geq 0 \quad \text{at} \quad u^*(t) \Rightarrow u^*(t) = b \end{aligned} \quad (\text{A.10})$$

The other necessary conditions for the optimal control are:

$$\begin{aligned} -\frac{\partial H}{\partial x} &= \lambda' \quad \text{i.e.,} \quad \lambda'(t) = -(f_x + g_x) \quad \text{adjoint equation} \\ \frac{\partial H}{\partial \lambda} &= x' \quad \text{i.e.,} \quad x'(t) = g \quad \text{state equation.} \end{aligned} \quad (\text{A.11})$$

By an adaptation of the PMP, the OCP must satisfy the optimality condition:

$$u^*(t) = \begin{cases} a & \text{if } \frac{\partial H}{\partial u} < 0 \\ a < \bar{u} < b & \text{if } \frac{\partial H}{\partial u} = 0 \\ b & \text{if } \frac{\partial H}{\partial u} > 0 \end{cases} \quad (\text{A.12})$$

i.e., the maximization is over all admissible controls, and u^* is obtained by the expression $\frac{\partial H}{\partial u} = 0$. In particular, the optimal control u^* maximizes H point wise with respect to $a \leq u \leq b$.

Remark A.2.2. If we have a minimization problem instead of maximization, then u^* is instead chosen to minimize H point-wise. This has the effect of reversing $<$ and $>$ in the first and third lines of the optimality condition (A.12).

Remark A.2.3.

$$u^*(t) = \min \left(a, \max \left(b, \frac{\partial H}{\partial u} \right) \right)$$

Optimal control of several variables

Definition A.2.6. (OC with several variables and several controls). An OC with n state variables, m control variables can be written in the form

$$\begin{aligned} \int_{t_0}^{t_f} f(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) dt &\rightarrow \max_{u, \dots, u_m} \\ S.t \quad x_i'(t) &= g_i(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ x_i(t_0) &= x_{i0}, i = 1, 2, \dots, n \end{aligned} \quad (\text{A.13})$$

where the functions f, g_i are continuously differentiable in all variables.

To simplify the notation, we denote

$$X(t) = [x_1(t), \dots, x_n(t)], U(t) = [u_1(t), \dots, u_m(t)], X_0 = [x_{10}, \dots, x_{n0}]$$

So, the problem (A.13) can be rewritten in a compact way as

$$\begin{aligned} & \max_U \int_{t_0}^{t_f} f(t, X(t), U(t)) \\ & S.t \quad X'(t) = g_i(t, X(t), U(t)) \\ & X(t_0) = X_0, i = 1, 2, \dots, n \end{aligned} \quad (A.14)$$

Using the same approach of the control with one variable, it is possible to derive generalized necessary conditions.

Proposition A.2.2. (Necessary conditions). *Let U^* be an optimal control functions and X^* be the vector of corresponding optimal state variables. With n states, we will need n adjoints, one for each state. There is a piece-wise differentiable vector-valued function $\lambda(t) = \lambda_1(t), \dots, \lambda_n(t)$, where each λ_i is the adjoint variable corresponding to x_i , and the Hamiltonian is*

$$H(t, X, U) = f(t, X, U) + \sum_{i=1}^n \lambda_i(t) g_i(t, X, U).$$

It is possible to find the variables satisfying optimality, adjoint and transversality conditions in each vector component. Namely, X^* maximizes $H(t, X, U, \lambda)$ with respect to U at each t , and U^* , X^* and λ satisfy

$$\begin{aligned} \lambda'_j(t) &= -\frac{\partial H}{\partial x_j} \quad \text{for } j = 1, 2, \dots, n \quad (\text{adjoint conditions}) \\ \lambda_j(t_f) &= 0 \quad \text{for } j = 1, \dots, n \quad (\text{transversality conditions}) \\ \frac{\partial H}{\partial u_k} &= 0 \quad \text{at } u_k^* \quad \text{for } k = 1, \dots, n \quad (\text{optimality condition}) \end{aligned}$$

Remark A.2.4. *Similarly to the previous section, if bounds are placed on a control variable, $a_k \leq u_k \leq b_k$ (for $k = 1, \dots, m$), then the optimality condition is changed from $\frac{\partial H}{\partial u_k} = 0$ to*

$$u_k^*(t) = \begin{cases} a_k & \text{if } \frac{\partial H}{\partial u_k} < 0 \\ a_k < \frac{\partial H}{\partial u_k} < b_k & \text{if } \frac{\partial H}{\partial u_k} = 0 \\ b_k & \text{if } \frac{\partial H}{\partial u_k} > 0 \end{cases}$$

The following is an outline of steps in formulating optimal system of an optimal control problem.

(1) Formulate the Hamiltonian for the problem

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$$

(2) Write the adjoint differential equation, transversality boundary condition and optimality

condition. Now there are three unknowns, u^* , x^* and λ .

$$\lambda' = -\frac{\partial H}{\partial x} \Rightarrow \lambda' = -(f_x + \lambda g_x) \quad (\text{adjoint condition})$$

$$\lambda(t_f) = 0 \quad (\text{transversality condition})$$

$$\frac{\partial H}{\partial u} = 0 \quad \text{at} \quad u = u^* \Rightarrow f_u(t, x, u) + \lambda g_u(t, x, u) = 0 \quad (\text{optimality condition})$$

(3) Solve for u^* in terms of x^* and λ .

(4) After finding the optimal states and adjoint, solve for the optimal control.

A.2.3 Optimal control with delay differential equation constraints

We consider the delayed differential equation of the cost functional problem with respect to the control:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-r), u(t), u(t-s)), & t \in [a, b] \\ x(t) = x_0(t), & t \in [a-r, a] \\ u(t) = u_0(t), & t \in [a-s, a] \end{cases} \quad (\text{A.15})$$

where $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the delayed dynamics, with delays r and s respectively in the state and control variables, $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the control belonging to the class of admissible controls

$$\mathcal{U} = \{u : [a-s, b] \rightarrow \mathbb{U} \subseteq \mathbb{R}^m, u \text{ measurable}\}.$$

Moreover, $x_0(t) \in \mathbb{R}^n$ and $u_0(t) \in \mathbb{R}^m$ are given initial functions. The optimal control problem consists in minimizing the functional

$$J(u) = \int_a^b (L(t, x(t), x(t-r), u(t), u(t-s))) dt + g(x(b)), \quad (\text{A.16})$$

among all controls $u \in \mathcal{U}$ subject to the delayed differential equation (A.15), where $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is the running cost and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is the final cost.

Basic assumptions:

- The functions $f(x, y, u, v)$ and $L(x, y, u, v)$ are twice continuously differentiable with respect to all arguments, and f is Lipschitz continuous w.r.t. x and y ;
- The delays r, s satisfy $r, s \geq 0$, $(r, s) \neq (0, 0)$ and $\frac{r}{s} \in \mathbb{Q}$ for $s > 0$ or $\frac{s}{r} \in \mathbb{Q}$ for $r > 0$; $g \in C^1(\mathbb{R}^n)$;
- Either U is a bounded convex set, or L is convex and coercive w.r.t. u and v .

Theorem A.2.3. *Let the basic assumptions hold, and let $(\hat{x}, \hat{u})$ be local optimal for the functional J in (A.16) subject to the delayed differential equation (A.15). Then, there exists an*

adjoint state function $\hat{\lambda} \in W^{1,\infty}([a,b], \mathbb{R}^n)$ such that the following conditions hold a.e. for $t \in [a,b]$:

1. Adjoint equation

$$\hat{\lambda}^T(t) = \begin{cases} -\hat{H}_x(t) - \hat{H}_y(t+r), & t \in [a, b-r] \\ -\hat{H}_x(t), & t \in [b-r, b] \end{cases}$$

where $\hat{H}_x, \hat{H}_y$ denote the partial derivatives of H computed on the optimal triple $(\hat{x}, \hat{u}, \hat{\lambda})$ 2. Minimum condition for Hamiltonian: for any $u \in U$,

$$\begin{aligned} \hat{H}(t) + \mathcal{X}_{[a,b-s]}(t)\hat{H}(t+s) &= \hat{H}(t, \hat{x}(t), \hat{x}(t-r), \hat{u}(t), \hat{u}(t-s), \hat{\lambda}(t)) \\ &\quad + \mathcal{X}_{[a,b-s]}(t)H(t+s, \hat{x}(t+s), \hat{x}(t+s-r), \\ &\quad \hat{u}(t+s-r), \hat{u}(t+s), \hat{u}(t), \hat{\lambda}(t+s)) \\ &\leq \hat{H}(t, \hat{x}(t), \hat{x}(t-s), u(t), \hat{u}(t-s), \hat{\lambda}(t)) \\ &\quad + \mathcal{X}_{[a,b-s]}(t)H(t+s, \hat{x}(t+s), \hat{x}(t+s-r), \\ &\quad \hat{u}(t+s-r), \hat{u}(t+s), u, \hat{\lambda}(t+s)); \end{aligned}$$

3. The transversability condition:

$$\hat{\lambda}^T(b) = g_x(\hat{x}(b)). \quad (\text{A.17})$$

A.3 Runge-Kutta Forth order Scheme

The fourth order Runge-Kutta (RK4) method is a numerical approach used to solve ordinary differential equation and the optimality system.

$$\frac{dy}{dt} = F(x, y) \text{ at } y(0) = y_0 \quad (\text{A.18})$$

The RK4 method is based on the following general equation.

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h \quad (\text{A.19})$$

where from knowing the value of $y = y_i$ at the point x_i , we can find the value of $y = y_{(i+1)}$ at the point x_{i+1} , and $h = x_{i+1} - x_i$ is the step size. The equation (A.19) gives the first five terms of the Taylor's series.

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f^3(x_i, y_i)h^4 \quad (\text{A.20})$$

Based on the equation (A.20), we have,

$$y_{i+1} = y_i + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f\left(x_i + \frac{1}{2}, y_i + \frac{1}{2}k_1h\right), \\ k_3 &= f\left(x_i + \frac{1}{2}, y_i + \frac{1}{2}k_2h\right), \\ k_4 &= f\left(x_i + 1, y_i + k_3h\right) \end{aligned}$$

APPENDIX B

NUMERICAL ANALYSIS CODE

B.1 Model DDE code

```
%This function solves the delay pest control model using dde23 and then
%produces graphs of the solutions in different window
%Author Teklebirhan A.
%First paper for PhD thesis
%Adama Science and Technology University,Department of Applied Mathematics
%Optimization group, 2020/21 A.Y
function sol = pestcontrolwithdelay2
```

```
%y(1) is X(plant biomass)
%y(2) is S(healthy pest)
%y(3) is I(infected pest)
%y(4) is A(level of awareness)
global tau1 tau2
tau1=8;tau2=30;
sol = dde23(@ddes, [tau1,tau2], [0.2;0.07;0.05;0.5], [0, 2000]);
clf reset
subplot(221)
plot(sol.x,sol.y(1,:), 'LineWidth',2.5)
xlabel('Time(days)', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
ylabel('Plant biomass', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
%title('model with delays', 'fontsize',14)
%legend('Plant biomass')
subplot(222)
plot(sol.x,sol.y(2,:), 'b', 'LineWidth',2.5)
xlabel('Time(days)', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
ylabel('Susceptible pest', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
%title('model with delays', 'fontsize',14)
%legend( 'S(t)')
subplot(223)
plot(sol.x,sol.y(3,:), 'r', 'LineWidth',2.5)
xlabel('Time(days)', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
ylabel('Infected pest', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
%title('model with delays', 'fontsize',14)
%legend( 'I(t)')
subplot(224)
plot(sol.x,sol.y(4,:), 'k', 'LineWidth',2.5)
xlabel('Time(days)', 'FontAngle', 'italic', 'FontName', 'Times New Roman', 'fontsize',14)
```

B.2 Optimality system code

```
%This function solves the pest control model with two controls using the
%forward backward sweep method and produces graphs of the solutions
%Teklebirhan A.
%A second paper on pest control using farming awareness and biopesticides
%Adama Science and Technology University,Department of Applied Mathematics
%Optimization group, 2020/21 A.Y
function y=code_FBSMethod_twocontrol(t,y)
test=-1;
delta=0.001;N=2000;
t=linspace(0,100,N+1);
A1=1015;A2=1010;B1=0.8;B2=0.6;r =0.05;K =1;lambda=0.06;d=0.01;m1=0.8;
m2=0.6;deltas=0.1;sigma=0.015;a=0.5;c=1;etha=0.015;alpha=0.025;gamma=0.003;
phi=0.05;t0=0;tf=100;h=(tf-t0)/N;h2=h/2;
u1=zeros(1,N+1);
u2=zeros(1,N+1);
X=zeros(1,N+1);
S=zeros(1,N+1);
I=zeros(1,N+1);
A=zeros(1,N+1);
X(0)=0.2; S(0)=0.07;I(0)=0.05;A(0)=0.5;% initial conditions assigned to x(0)
lambda1=zeros(1,N+1);
lambda2=zeros(1,N+1);
lambda3=zeros(1,N+1);
lambda4=zeros(1,N+1);
lambda1(N+1)=0;
lambda2(N+1)=0;
lambda3(N+1)=0;
lambda4(N+1)=0;
%J=zeros(1,N+1);
while(test <0) % while tolerance is attained,repeat the process
oldu1=u1;
oldu2=u2;
oldX=X;
oldS=S;
oldI=I;
oldA=A;
oldlambda1=lambda1;
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oldlambda2=lambda2;
oldlambda3=lambda3;
oldlambda4=lambda4;
for i=1:N % forward
KX1=r*X(i)*(1-X(i)/K)-(alpha*X(i)*S(i))./(c+X(i))-(phi*alpha*X(i)*I(i))./(c+X(i));
KS1=(m1*alpha*X(i)*S(i))./(c+X(i))-u1(i)*(lambda*A(i)*S(i))./(a+A(i))-d*S(i);
KI1=(m2*phi*alpha*X(i)*I(i))./(c+X(i))+u1(i)*(lambda*A(i)*S(i))./(a+A(i))-(d+deltas)*I(i);
KA1=u2(i)*gamma+sigma*(S(i)+I(i))-etha*A(i);
KX2=r*(X(i)+h2*KX1)*(1-(X(i)+h2*KX1)/K)-(alpha*(X(i)+h2*KX1)*(S(i)+h2*KS1))...
./(c+X(i)+h2*KX1)-(phi*alpha*(X(i)+h2*KX1)*(I(i)+h2*KI1))./(c+X(i)+h2*KX1);
KS2=(m1*alpha*(X(i)+h2*KX1)*(S(i)+h2*KS1))./(c+X(i)+h2*KX1)-((0.5*(u1(i)+u1(i+1)))...
*lambda*(A(i)+h2*KA1)*(S(i)+h2*KS1))./(a+(A(i)+h2*KA1))-d*(S(i)+h2*KS1);
KI2=(m2*phi*alpha*(X(i)+h2*KX1)*(I(i)+h2*KI1))./(c+X(i)+h2*KX1)+(0.5*(u1(i)+u1(i+1)))...
*lambda*(A(i)+h2*KA1)*(S(i)+h2*KS1))./(a+A(i)+h2*KA1)-(d+deltas)*(I(i)+h2*KI1);
KA2=0.5*(u2(i)+u2(i+1))*gamma+sigma*(S(i)+h2*KS1+I(i)+h2*KI1)-etha*(A(i)+h2*KA1);
KX3=r*(X(i)+h2*KX2)*(1-(X(i)+h2*KX2)/K)-(alpha*(X(i)+h2*KX2)*(S(i)+h2*KS2))...
./(c+X(i)+h2*KX2)-(phi*alpha*(X(i)+h2*KX2)*(I(i)+h2*KI2));
KS3=(m1*alpha*(X(i)+h2*KX2)*(S(i)+h2*KS2))./(c+X(i)+h2*KX2)-((0.5*(u1(i)+u1(i+1)))...
*lambda*(A(i)+h2*KA2)*(S(i)+h2*KS2))./(a+(A(i)+h2*KA2))-d*(S(i)+h2*KS2);
KI3=(m2*phi*alpha*(X(i)+h2*KX2)*(I(i)+h2*KI2))./(c+X(i)+h2*KX2)+(0.5*(u1(i)+u1(i+1)))...
*lambda*(A(i)+h2*KA2)*(S(i)+h2*KS2))./(a+A(i)+h2*KA2)-(d+deltas)*(I(i)+h2*KI2);
KA3=0.5*(u2(i)+u2(i+1))*gamma+sigma*(S(i)+h2*KS2+I(i)+h2*KI2)-etha*(A(i)+h2*KA2);
KX4=r*(X(i)+h*KX3)*(1-(X(i)+h*KX3)/K)-(alpha*(X(i)+h*KX3)*(S(i)+h*KS3))...
./(c+X(i)+h*KX3)-(phi*alpha*(X(i)+h*KX3)*(I(i)+h*KI3))./(c+X(i)+h*KX3);
KS4=m1*alpha*(X(i)+h*KX3)*(S(i)+h*KS3)-((u1(i+1))*lambda*(A(i)+h*KA3)*(S(i)+h*KS3))...
./(a+A(i)+h2*KA3)-d*(S(i)+h*KS3);
KI4=(m2*phi*alpha*(X(i)+h*KX3)*(I(i)+h*KI3))./(c+X(i)+h*KX3)+((u1(i+1))*lambda...
*(A(i)+h*KA3)*(S(i)+h*KS3))./(a+A(i)+h2*KA3)-(d+deltas)*(I(i)+h*KI3);
KA4=u2(i+1)*gamma+sigma*(S(i)+h*KS3+I(i)+h2*KI3)-etha*(A(i)+h*KA3);
X(i+1)=X(i)+(h/6)*(KX1+2*KX2+2*KX3+KX4);
S(i+1)=S(i)+(h/6)*(KS1+2*KS2+2*KS3+KS4);
I(i+1)=I(i)+(h/6)*(KI1+2*KI2+2*KI3+KI4);
A(i+1)=A(i)+(h/6)*(KA1+2*KA2+2*KA3+KA4);
end
for i=1:N;j=N+2-i; % backward
% ADJOINT FUNCTIONS
n11 = lambda1(j)*((alpha*c*S(j))./(c+X(j))^2)+(phi*alpha*c*I(j))./(c+X(j))^2-r*(1-2*X(j)...
./K))-lambda2(j)*((m1*alpha*c*S(j))./(c+X(j))^2)-lambda3(j)*((m2*phi*alpha*c*I(j))./(c+X(j))^2);

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$$n12 = -2 * A1 * S(j) + \lambda_1(j) * ((\alpha * X(j)) / (c + X(j))) + \lambda_2(j) * ((u_1(j) * \lambda * A(j)) / (a + A(j))) -$$

$$\dots$$

$$(m_1 * \alpha * X(j)) / (c + X(j)) + d - \lambda_3(j) * ((u_1(j) * \lambda * A(j)) / (a + A(j))) - \lambda_4(j) * \sigma;$$

$$n13 = \lambda_1(j) * (\phi * \alpha * X(j) / (c + X(j))) + \lambda_3(j) * (d + \delta - (m_2 * \phi * \alpha * X(j) / (c + X(j))) -$$

$$\dots$$

$$\lambda_4(j) * \sigma;$$

$$n14 = 2 * A_2 * A(j) + \lambda_2(j) * ((u_1(j) * \lambda * a * S(j)) / ((a + A(j))^2)) - \lambda_3(j) * ((u_1(j) * \lambda * a * S(j)) / ((a + A(j))^2)) + \lambda_4(j) * \theta;$$

$$n21 = (\lambda_1(j) - h_2 * n11) * ((\alpha * c * 0.5 * (S(j) + S(j-1))) / ((c + 0.5 * (X(j) + X(j-1))))^2) + \dots$$

$$(\phi * \alpha * c * 0.5 * (I(j) + I(j-1))) / ((c + 0.5 * (X(j) + X(j-1))))^2 - r * (1 - 2 * 0.5 * (X(j) + X(j-1)) / K) - \dots$$

$$(\lambda_2(j) - h_2 * n12) * ((m_1 * \alpha * c * 0.5 * (S(j) + S(j-1))) / (c + 0.5 * (X(j) + X(j-1))))^2 - (\lambda_3(j) -$$

$$\dots$$

$$h_2 * n13) * ((m_2 * \phi * \alpha * c * 0.5 * (I(j) + I(j-1))) / (c + 0.5 * (X(j) + X(j-1))))^2;$$

$$n22 = -2 * A_1 * 0.5 * (S(j) + S(j-1)) + (\lambda_1(j) - h_2 * n11) * ((\alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1)))) + \dots$$

$$(\lambda_2(j) - h_2 * n12) * ((0.5 * (u_1(j) + u_1(j-1)) * \lambda * 0.5 * (A(j) + A(j-1))) / (a + 0.5 * (A(j) + A(j-1)))) -$$

$$\dots$$

$$(m_1 * \alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1))) + d - (\lambda_3(j) - h_2 * n13) * ((0.5 * (u_1(j) + u_1(j-1))) * \lambda * 0.5 * (A(j) + A(j-1))) / (a + 0.5 * (A(j) + A(j-1)))) -$$

$$(\lambda_4(j) - h_2 * n14) * \sigma;$$

$$n23 = (\lambda_1(j) - h_2 * n11) * ((\phi * \alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1)))) + (\lambda_3(j) - h_2 * n13) * \dots$$

$$(d + \delta - (m_2 * \phi * \alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1)))) - (\lambda_4(j) - h_2 * n14) * \sigma;$$

$$n24 = 2 * A_2 * 0.5 * (A(j) + A(j-1)) + (\lambda_2(j) - h_2 * n12) * ((0.5 * (u_1(j) + u_1(j-1)) * \lambda * a * 0.5 * (S(j) + S(j-1))) / ((a + 0.5 * (A(j) + A(j-1))))^2) - (\lambda_3(j) - h_2 * n13) * ((0.5 * (u_1(j) + u_1(j-1))) * \lambda * a * 0.5 * (S(j) + S(j-1))) / ((a + 0.5 * (A(j) + A(j-1))))^2) + (\lambda_4(j) - h_2 * n14) * \theta;$$

$$n31 = (\lambda_1(j) - h_2 * n21) * ((\alpha * c * 0.5 * (S(j) + S(j-1))) / ((c + 0.5 * (X(j) + X(j-1))))^2) + \dots$$

$$(\phi * \alpha * c * 0.5 * (I(j) + I(j-1))) / ((c + 0.5 * (X(j) + X(j-1))))^2 - r * (1 - 2 * 0.5 * (X(j) + X(j-1)) / K) - \dots$$

$$(\lambda_2(j) - h_2 * n22) * ((m_1 * \alpha * c * 0.5 * (S(j) + S(j-1))) / (c + 0.5 * (X(j) + X(j-1))))^2 - (\lambda_3(j) -$$

$$\dots$$

$$h_2 * n23) * ((m_2 * \phi * \alpha * c * 0.5 * (I(j) + I(j-1))) / (c + 0.5 * (X(j) + X(j-1))))^2;$$

$$n32 = -2 * A_1 * 0.5 * (S(j) + S(j-1)) + (\lambda_1(j) - h_2 * n21) * ((\alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1)))) + (\lambda_2(j) - h_2 * n22) * ((0.5 * (u_1(j) + u_1(j-1)) * \lambda * 0.5 * (A(j) + A(j-1))) / (a + 0.5 * (A(j) + A(j-1)))) - (m_1 * \alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1))) + d - (\lambda_3(j) -$$

$$\dots$$

$$h_2 * n23) * ((0.5 * (u_1(j) + u_1(j-1)) * \lambda * 0.5 * (A(j) + A(j-1))) / (a + 0.5 * (A(j) + A(j-1)))) - \dots$$

$$(\lambda_4(j) - h_2 * n24) * \sigma;$$

$$n33 = (\lambda_1(j) - h_2 * n21) * (\phi * \alpha * 0.5 * (X(j) + X(j-1))) / (c + 0.5 * (X(j) + X(j-1))) + (\lambda_3(j) - h_2 * n23) \dots$$

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*(d+deltas-(m2*phi*alpha*0.5*(X(j)+X(j-1)))/(c+0.5*(X(j)+X(j-1))))-...
(lambda4(j)-h2*n24)*sigma;
n34=2*A2*0.5*(A(j)+A(j-1))+(lambda2(j)-h2*n22)*((0.5*(u1(j)+u1(j-1))*lambda*a*...
0.5*(S(j)+S(j-1)))/((a+0.5*(A(j)+A(j-1)))^2))-...
(lambda3(j)-h2*n23)*((0.5*(u1(j)+u1(j-1))*a*lambda*0.5*(S(j)+S(j-1)))/((a+0.5...
*(A(j)+A(j-1)))^2))+lambda4(j)-h2*n24)*etha;
n41=(lambda1(j)-h*n31)*((alpha*c*S(j-1))/(c+X(j-1))^2)+...
(phi*alpha*c*I(j-1))/(c+X(j-1))^2-r*(1-2*X(j-1)/K))-...
(lambda2(j)-h*n32)*((m1*alpha*c*S(j-1))/(c+X(j-1))^2))-...
(lambda3(j)-h*n33)*((m2*phi*alpha*c*I(j-1))/(c+X(j-1))^2);
n42=-2*A1*S(j-1)+(lambda1(j)-h*n31)*((alpha*X(j-1))/(c+0.5*(X(j)+X(j-1))))+...
(lambda2(j)-h*n32)*((u1(j-1)*lambda*A(j-1))/(a+A(j-1))-(m1*alpha*X(j-1))/(c+X(j-1))+d)-
...
(lambda3(j)-h*n33)*((u1(j-1))*lambda*A(j-1))/(a+A(j-1))-(lambda4(j)-h*n34)*sigma;
n43=(lambda1(j)-h*n31)*(phi*alpha*X(j-1))/(c+X(j-1))+lambda3(j)-h*n33)...
*(d+deltas-(m2*phi*alpha*X(j-1))/(c+X(j-1)))-(lambda4(j)-h*n34)*sigma;
n44=2*A2*A(j-1)+(lambda2(j)-h*n32)*((u1(j-1))*lambda*a*S(j-1))/((a+A(j-1))^2))-...
(lambda3(j)-h*n33)*((u1(j-1))*lambda*a*S(j-1))/((a+A(j-1))^2))+lambda4(j)-h*n34)*etha;
lambda1(j-1)=lambda1(j)-(h/6)*(n11+2*n21+2*n31+n41);
lambda2(j-1)=lambda2(j)-(h/6)*(n12+2*n22+2*n32+n42);
lambda3(j-1)=lambda3(j)-(h/6)*(n13+2*n23+2*n33+n43);
lambda4(j-1)=lambda4(j)-(h/6)*(n14+2*n24+2*n34+n44);
end
u1 =max(0,min(1,((lambda2-lambda3).*lambda.*A.*S)/(B1.*(a+A))));
u1=0.5*(u1+oldu1);
u2=max(0,min(1,(-lambda4*gamma)/(B2)));
u2=0.5*(u2+oldu2);
%J=sum((0.5*B1*u1.^2+0.5*B2*u2.^2+A1*S... .^2-A2*A.^2)*(t(i+1)-t(i)));
temp1=delta*sum(abs(u1))-sum(abs(oldu1-u1));
temp2=delta*sum(abs(u2))-sum(abs(oldu2-u2));
temp3=delta*sum(abs(X))-sum(abs(oldX-X));
temp4=delta*sum(abs(S))-sum(abs(oldS-S));
temp5=delta*sum(abs(I))-sum(abs(oldI-I));
temp6=delta*sum(abs(A))-sum(abs(oldA-A));
temp7=delta*sum(abs(lambda1))-sum(abs(olddlambda1-lambda1));
temp8=delta*sum(abs(lambda2))-sum(abs(olddlambda2-lambda2));
temp9=delta*sum(abs(lambda3))-sum(abs(olddlambda3-lambda3));
temp10=delta*sum(abs(lambda4))-sum(abs(olddlambda4-lambda4));
test= min(temp1, min(temp2,min(temp3,min(temp4,min(temp5,...

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min(temp6,min(temp7,min(temp8,min(temp9,min(temp10))))))));
end
y(1,:) = t;
y(2,:) = u1;
y(3,:) = u2;
y(4,:) = X;
y(5,:) = S;
y(6,:) = I;
y(7,:) = A;
y(8,:) = lambda1;
y(9,:) = lambda2;
y(10,:) = lambda3;
y(11,:) = lambda4;
subplot(2,2,1), plot(y(1,:),y(4,:), 'r', 'LineWidth', 2);
xlabel( ' Time (day)' ) ;ylabel('Crop biomass,X');
hold on
subplot(2,2,2), plot(y(1,:),y(5,:), 'r', 'LineWidth', 2);
xlabel( ' Time (day)' ) ;ylabel('Susceptible pest,S');
hold on;
subplot(2,2,3), plot(y(1,:),y(6,:), 'r', 'LineWidth', 2);
xlabel( ' Time (day)' ) ;ylabel('Infected pest,I');
hold on;
subplot(2,2,4), plot(y(1,:),y(7,:), 'r', 'LineWidth', 2);
xlabel( ' Time (day)' ) ;ylabel('Aware people,A');
hold on

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