New Group Divisible Designs With Block Size Four and Five

Dinkayehu Mengesha Woldemariam

A Dissertation Submitted to the Department of Applied Mathematics,
School of Applied Natural Sciences

Presented in Fulfillment of the Requirement for the Degree of Doctor of Philosophy in Applied Mathematics

Office of Graduate Studies
Adama Science and Technology University

Adama, Ethiopia
July 10, 2022

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Co-Supervisor: Natea Hunde (Ph.D.)

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Declaration

I declare that this dissertation entitled "**New Group Divisible Designs with Block Size Four and Five**" is my own work and has not been submitted to any university for similar purpose. The references used in this dissertation are duly recognized by proper citations.

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Name of Student	Signature	Date

I, the main supervisor/co-supervisor of this dissertation, hereby certify that I have closely supervised the student while developing this dissertation and read the draft entitled "**New Group Divisible Designs with Block Size Four and Five**" prepared under my guidance by **Dinkayehu Mengesha Woldemariam** submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in **mathematics**. Therefore, I recommend the submission of the dissertation to the department for further review and evaluation.

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Approval page

I/we hereby certify that the recommendations and suggestions made by the board of examiners are appropriately incorporated into the final version of the dissertation entitled "**New Group Divisible Designs with Block Size Four and Five**" by **Dinkayehu Mengesha Woldemariam**.

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We, the undersigned, members of the Board of Examiners of the dissertation open defense by **Dinkayehu Mengesha Woldemariam** have read and evaluated the dissertation entitled "**New Group Divisible Designs with Block Size Four and Five**" and examined the candidate during open defense. This is, therefore, to certify that the dissertation is accepted for partial fulfillment of the requirement of the degree of Doctor of Philosophy in **Mathematics**.

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Dedication

This work is dedicated to my father, brothers and sisters.

Acknowledgement

First of all, I would like to thank God for his endless grace in my life and my accomplishments.

My sincere thanks next be expressed to my main supervisor, Prof. Dinesh G. Sarvate, for introducing me the fascinating topics of design theory and showing me the beauty behind it. His patient guidance, encouragement, incredible support and immense knowledge on the subject has helped me a lot to accomplish my research works. I appreciate and thank him for what I learnt from him during our virtual communication what research in mathematics is. His meticulous edition gave life to my scratch works and helped me much in understanding what I wrote. His edition helped me to learn writting skills in mathematics. I thank him for his continued encouragement and enthusiasm throughout the research process, along with the personal and proffessional support offered by him throughout.

I would like also to thank my co-supervisor Dr. Natea Hunde, for giving me the opportunity to join PhD program under Adama Science and Technology University and for motivation, encouragement and technical support he has provided me through out the study.

My special thanks goes to the Department of Applied Mathematics, Adama Science and Technology University, for giving me the chance to join the PhD programme and for its financial support.

I am also thankful to my fellow PhD students, Sileshi Mebrate, Yetneberk Kuma and Zebene Girma for thier valuable advice and encouragements.

Last but not the least, I am extremely grateful to thank my father, brothers and sisters for their prayer, advice, support and encouragement throughout the study.

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Acronyms

BIBD	Balanced Incomplete Block Design
GDD	Group Divisible Design
MOLS	Mutually Orthogonal Latin Square
PBIBD	Partially Balanced Incomplete Block Design
STS	Steiner Triple System
SQS	Steiner Quadruple System
3-GDD	3-Group Divisible Design
4-GDD	4-Group Divisible Design
t-GDD	t-Group Divisible Design
t-PBIBD	t-Partially Balanced Incoplete Block Design
3-PBIBD	3-Partially Balanced Incoplete Block Design
4-PBIBD	4-Partially Balanced Incoplete Block Design

List of Publications

Part of this thesis has been published in the form of the following research papers on reputable journals that are indexed by Scopus.

1. Sarvate, D. G. and Woldemariam, D. M. (2022). A note on GDD$(1,n,n,4;\lambda_1,\lambda_2)$. J. Algebra Comb. Discrete Appl., 9(2): 15–29. https://doi.org/10.13069/jacodesmath.1111720

2. Sarvate, D. G., Woldemariam, D. M., and Zhang, L. Group Divisible Designs with 3 Groups and Block size 4. J. Combin. Math. Combin. Comput., Accepted.

Abstract

Group divisible designs are classical combinatorial structures studied for their applications in construction of other combinatorial structures as well as a design of experiments. They provide an ample opportunity of developing techniques to study combinatorial structures and are inherently hard to construct, especially when the number of groups is less than the block size or when group sizes are different. The subject matter of the study is group divisible designs (GDDs) with block size 4 with 2 or 3 groups of different sizes, and with block size 5 with 2 groups of equal size. We present a general construction for GDDs with two groups of different sizes and block size k. Further, we present general constructions for such GDDs, where second associate pairs occur exactly once in the design and exactly twice in the design. We also present some interesting constructions of GDDs with block size 4 and three groups: one of size 1 and the other two of the same size n. We obtain necessary conditions for the existence of GDDs with 3 groups of sizes 1, n, n respectively and prove that the necessary conditions are sufficient for n is congruent to $0,1,4,5,8,9$ (modulo 12) when the number of occurence of pairs of elements from the same group is greater or equal to the number of occurence of pairs of elements from different groups. Then we focus on two new problems of group divisible designs with block size 4. The first problem involves GDDs with three groups of sizes 1, n and $n+2$. We also obtain several non-existence results when restrictions on block configurations are placed. The second problem deals with group sizes 3, n and $n+1$ where n is greater or equal to 3. Finally, we study on the existence of 4-group divisible designs (4-GDDs) with two groups of equal size and blocks of size 5. We obtain necessary conditions for the existence of a 4-GDDs with 2 groups of size n and block size 5 and show that the necessary conditions are sufficient for group sizes $3,4,5,6,7,9$ and 10. We apply combinatorial methods to obtain the necessary conditions and direct method to show the necessry conditions are sufficient.

Keywords: t-designs, Group divisible designs with unequal group sizes, BIBDs, t-GDDs.

Chapter 1

Introduction

1.1 Background of the study

Combinatorial design theory is an active area of mathematical research on the existence and construction of families of finite sets whose arrangements satisfy conditions of "balance". As one of the fundamental discrete structures, combinatorial designs are used in fields such as error correcting codes, statistical design of experiments, cryptography and information security, and group testing (MacWilliams and Sloane, 1977; Harwit and Sloane, 1979; Street and Street, 1986; Yener et al., 1997; Du et al., 2000; Atkinson et al., 2007).

Recreational mathematics has played a role in the development of Design theory. Many types of designs that are studied today were first considered in the context of mathematical puzzles or brain-teasers in the 18th and 19th centuries. One of the first problems in combinatorial designs is a mathematical puzzle called "Thirty-six officers problem" proposed by Leonhard Euler in 1782. The question is "Six regiments each sends six officers with different ranks. How can we arrange these 36 officers in a 6×6 array so that each regiment and each rank is represented exactly once in each row and column of this array? This problem has led to important work in combinatorics, and was the origin of the Latin squares. One of the famous modern day puzzles in the news papers, Sudokus, are just special types of Latin squares. In 1853, Jakob Steiner raised the following question: "For what integer n is it possible to arrange n elements in triplets, so that every pair of elements is contained in exactly one triplet?" (Bose, 1939). This problem was later extended to Steiner triple system. Also Thomas Kirkman (1850) asked for a walking schedule for 15 Schoolgirls. The girls are to walk in five rows of three in a day. The problem is to arrange a walking schedule so that in seven days every girl will have walked in a row of three once with every other girl. This

problem was an extension of Steiner's triple problem and was much more difficult to solve. These problems lay a foundation for the development of Combinatorial Design Theory.

The study of Design theory began in the last century mainly due to the design and analysis of Statistical experiments. Yates (1936) applied the concept of designs, specifically of balanced incomplete block designs (BIBDs) for statistical experiments. Further contributions in balanced incomplete block designs were made by (Bose, 1939), and (Fisher and Yates, 1934) concerning the structure and construction of BIBDs . As per (Bailey, 2004) BIBDs are universally optimal designs, they are not available for all numbers of treatments and block size. To overcome this difficulty group divisible designs (GDDs) were introduced and used as an auxilary designs for construction of BIBDs. So the group divisible designs became an important part of combinatorial research.

A group divisible design GDD$(n,m,k;\lambda_1,\lambda_2)$ is a set X of mn elements partitioned into m parts, called groups, of size n each together with a collection of k-subsets of X, called blocks, such that any two points within the same group are called *first associates* of each other and occur together in λ_1 blocks while any two points not in the same group are called *second associates* of each other and occur together in λ_2 blocks. Here $r = \frac{\lambda_1(n-1)+\lambda_2 n(m-1)}{k-1}$ is the replication number of each point of X and $b = \frac{mnr}{k}$ is the required number of blocks in a GDD. The numbers λ_1 and λ_2 are called indices of a GDD. As per (Mullin and Gronau, 2006) group divisible designs with $\lambda_1 = 0$ have been used in constructing other designs. But just as interesting to the statisticians are GDDs in which $\lambda_1 \neq 0$. Bose and Nair (1939) introduced a wider class of designs, viz. partially balanced incomplete block designs with m associate classes (PBIBD(m)).

Bose and Shimamoto (1952) studied PBIBDs with two associate classes. PBIBDs with two associate classes are GDDs with $\lambda_1 \neq 0$. So a GDD$(n,m,k;\lambda_1,\lambda_2)$ is a PBIBD with 2-class symmetric association scheme defined on the point set in which the point set is partitioned into m groups of size n each, such that every pair of points from the same group occurs together in λ_1 blocks whereas every pair of points not from the same group occurs in λ_2 blocks. Moreover, if each element in a PBIBD(m) has n_i i-th associate for $i = 1,2,...,m$, then the parameters of a PBIBD(m) satisfy $vr = bk$ and $\sum_{i=1}^{m} n_i\lambda_i = r(k-1)$. PBIBDs are

generalizations of BIBDs and are used in plant breeding work, in survey sampling and in group testing (Vakil and Parnes, 1994; Ghosh and Divecha, 1997; See et al., 1997; Das et al., 1998). Hanani (1975) proved that the necessary conditions are sufficient for the existence of a GDD with $k = 3$, $\lambda_1 = 0$ and $m \geq 3$. He wrote several techniques to construct GDD$(n, m, 3; 0, \lambda)$ for $m \geq 3$ by including application of the finite field of order q where q is a prime power. Similarly, (Brouwer et al., 1977) obtained the existence results for GDDs with block size 4 and $\lambda_1 = 0$.

The existence of GDD$(n, m, 3, \lambda_1, \lambda_2)$ was completely settled by (Fu and Rodger, 1998) and (Fu et al., 2000). Clatworthy (1973) has listed eleven GDDs with $k = 4$ and $m = 3$ with replication number at most 10. Henson and Sarvate (2006), generalized two of these designs. Then (Rodger and Rogers, 2010) generalized three more designs from the list. Subsequently (Rodger and Rogers, 2012) also gave generalization of another five GDDs from that list. Moreover, (Gao and Ge, 2011) gave general methods of construction of GDDs with $k = 4$ and independently generalized all the eleven designs. Hurd and Sarvate (2004) proved that necessary conditions are sufficient for even and odd GDD$(n, 2, 4; \lambda_1, \lambda_2)$. A GDD is said to be even if each block intersects every group in exactly 0 or 2 points and odd if each block intersects every group in exactly 1 or 3 points. Also, (Hurd and Sarvate, 2008) constructed GDDs with $k = 4$ using Bhaskar Rao designs. In that paper they gave necessary and sufficient conditions for such designs for $3 \leq n \leq 8$ and two groups. Henson et al. (2007) proved the necessary conditions are sufficient for the existence of GDDs with even and odd intersection patterns for block size 4 and 3 groups. Moreover, they proved that necessary conditions are sufficient for all but two families when all blocks are mixed. Then (Zhu and Ge, 2010) completely settled the case for mixed GDDs. Ndungo and Sarvate (2016) proved the existence of all but two families of GDD$(n, 2, 4; \lambda_1, \lambda_2)$ where number of even blocks is equal to the number of odd blocks.

For GDDs with $k = 5$, (Hurd et al., 2007) have given an explicit construction using MOLS of order n, with 2, 3 or 6 groups. These designs are not listed in Clatworthy table because their parameter range is beyond that of the table. Further, (Hurd et al., 2009) took into account the block-group intersection pattern to construct GDDs with $k = 5$ and $m = 2$. There

are only three intersection patterns: $(0,5)$, $(1,4)$ and $(2,3)$ with $m=2$ and $k=5$, where a block is said to be of type (a,b) or with intersection pattern (a,b) if the block contains a treatments from one of the group and b treatments from another group. They (Hurd et al., 2009) gave new necessary conditions for this family of GDDs and gave minimal or near-minimal index examples for all group sizes $n \geq 4$ except when $n = 24s+17$. Sarvate et al. (2018) concentrated on GDDs from Clatwrthy table with $k=5$ and $m \leq 5$ as well. There are 45 GDDs in Clatworthy table with $k=5$. They generalized all but one such designs listed in Clatworthy table. Also, they proved that GDD$(n,4,5;\frac{9n}{n-1},2)$ with intersection pattern $(1,4)$ does not exist for any n except $n=4$.

GDDs with $k=5$ and 4 groups with block type $(1,1,1,2)$ were studied by (Mwesigwa et al., 2016). They proved that the necessary conditions are sufficient for the existence of a GDD$(n,4,5;\lambda_1,\lambda_2)$ with block configuration $(1,1,1,2)$ except for $n \equiv 0 \pmod 6$.

For $k=6$, (Keranen and Laffin, 2012) have constructed GDDs with two groups. For block configuration $(3,3)$, they proved that the necessary conditions are sufficient for the existence of a GDD$(n,2,6;\lambda_1,\lambda_2)$. Further, for GDDs with block configuration $(1,5)$ they gave examples with minimal or near-minimal index for group sizes $n \geq 5$, except for $n = 10, 15, 160$ and 190. For configuration $(2,4)$, they provided constructions for several families of a GDD$(n,2,6;\lambda_1,\lambda_2)$.

As per (Colbourn et al., 1992) when GDDs are used to construct other combinatorial designs, the groups are preferred to have different sizes to fit in various situations. Unfortunately comparing with uniform GDDs where group sizes are equal, less is known for the construction of nonuniform GDDs. The construction of nonuniform GDDs is a challenging problem as no appropriate algebraic or geometric structures have been found which are useful for these constructions (Ge et al., 2016).

A non uniform (K,λ)-GDD of the type $n_1^{m_1}.n_2^{m_2}...n_s^{m_s}$ was used by (Mullin and Gronau, 2006) to construct other combinatorial objects like PBDs. They defined (K,λ)-GDD as follows:

Let K be a set of positive integers. A (K,λ)-GDD is a v-set X that is partitioned into m_i groups of sizes n_i for $i = 1,2,...,s$ along with a collection of blocks of sizes from K such that

a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly λ blocks, where λ is a positive integer. If $\lambda = 1$ and $K = \{k\}$, then (K,λ)-GDD can be denoted as k-GDD, but we will use (K,λ)-GDD notation to distinguish from t-GDD defined later on.

Colbourn et al. (1999) showed that the elementary necessary conditions for the existence of a group divisible design with block size three, m groups of size n, and one group of size u are sufficient for all choices of m, n and u. Specifically, they settled the existance question for $(3,1)$-GDDs of type $n^m.u^1$ for all values of m, n, and u which is denoted as a GDD$(n,n,...,n,u,3;0,1)$. Later, (Wang and Yin, 2010) investigated the existence of group divisible designs (GDDs) with block size three, group-type $n^m.u^1$ and index λ. They proved that the elementary necessary conditions for the existence of such GDD are also sufficient, which generalizes the result of Colbourn, Hoffman and Rees. Further, (Ge and Ling, 2004) investigated the spectrum for $(4,1)$-GDDs of type $g^u.m^1$ with small g (i.e., a GDD$(g,g,...,g,m,4;0,\lambda)$). They showed that, for each $g \in \{2;4;5;12;15\}$ and each admissible pair (u,m), a $(4,1)$-GDD of type $g^u.m^1$ exists with a defnite exception for $(g,u,m) = (2,6,5)$ with 5 possible exceptions when $g = 2$.

Pabhapote and Punnim (2011) studied all triples of positive integers (n_1,n_2,λ) for which a GDD$(n_1,n_2,3;\lambda,1)$ exists. Later, Pabhapote (2012) proved the existence of a GDD$(n_1,n_2,3;\lambda_1,\lambda_2)$ for all $n_1 \neq 2$ and $n_2 \neq 2$ in which $\lambda_1 \geq \lambda_2$. Lapchinda et al. (2014) gave a complete solution for the existence of a group divisible design with block size 3 and 3 groups of sizes n, n and 1. Sakda and Uiyyasathian (2017), obtained complete result on group divisible designs with block size 3 and 4 groups of sizes n, n, n and 1, namely GDD$(1,n,n,n,3;\lambda_1,\lambda_2)$. Namyalo et al. (2017) studied nonuniform GDDs with 3 groups of sizes n_1, n $(n \geq n_1)$ and $n+1$, for $n_1 = 1$ or 2 and block size 4. First, they proved that a GDD$(n_1,n,n+1,4;\lambda_1,\lambda_2)$ for $n_1 = 1$ or 2 does not exist if it is required that exactly half of the blocks have the configuration $(1,1,2)$ and the other half of the blocks have the configuration $(2,2)$ except possibly for $n = 7$ when $n_1 = 2$. Then they obtained necessary conditions for the existence of a GDD$(n_1,n,n+1,4;\lambda_1,\lambda_2)$ for $n_1 = 1$ and 2, and proved that the necessary conditions are sufficient for several families of GDDs.

So far, most of the researches that have been done on nonuniform group divisible designs were of block size 3. The previous researchers didn't study nonuniform group divisible designs with block size 4 when $\lambda_1 \neq 0$ except the work of (Namyalo et al., 2017). So motivated by the papers of (Pabhapote and Punnim, 2011; Pabhapote, 2012; Lapchinda et al., 2014; Sakda and Uiyyasathian, 2017) and (Namyalo et al., 2017), we studied nonuniform group divisible designs with two or three groups by extending the block size to 4 as they are applicable in construction of other combinatorial structures.

Another important combinatorial design is a 3-group divisible design (3-GDD). Sarvate and Bezire (2018) defined 3-GDDs with two groups and block size k by generalizing the definition of t-designs and group divisible designs (GDDs). A 3-GDD$(n,2,k,\lambda_1,\lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that every 3-subset of each group occurs in λ_1 blocks while every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks. Sarvate and Bezire (2018) obtained necessary conditions for the existence of a 3-GDD$(n,2,4;\lambda_1,\lambda_2)$. Then they and (Sarvate and Cowden, 2018) proved that those necessary conditions are sufficient for the existence of a 3-GDD$(n,2,4;\lambda_1,\lambda_2)$ except when $n \equiv 3 \pmod{12}$, $\lambda_1 \equiv 9 \pmod{12}$ and $\lambda_1 > \lambda_2$. Later (Sarvate and Shinde, 2020) defined a 3-GDD$(n,m,k;\Lambda_1,\Lambda_2)$. They obtained necessary conditions for the existence of a 3-GDD$(n,3,4;\Lambda_1,\Lambda_2)$ when n even and proved that the necessary conditions are sufficient except when $n \equiv 6 \pmod{12}$, $n \neq 6$ and $\Lambda_2 \equiv 1,4,5 \pmod{6}$. Further, they also obtained necessary conditions for the existence of a 3-GDD$(n,3,4;\Lambda_1,\Lambda_2)$ for n odd and proved that the necessary conditions are sufficient except when $n \equiv 7,11 \pmod{12}$, $\Lambda_1 \equiv 2 \pmod{12}$ and $\Lambda_2 \equiv 6 \pmod{12}$.

In fact, the concept of 3-group divisible designs (3-GDDs) was hidden in a paper of (Hanani, 1963), to construct tactical configuration $C[4,3,\lambda,v]$ which is now known as a 3-$(v,4,\lambda)$ design. He used the notation $P_n''[k,\lambda,nt]$ to represent 3-$PBIBD(n,t,k,0,0,\lambda)$ and the notation $Q_n''[k,\lambda,nt]$ to represent 3-$PBIBD(n,t,k,0,\lambda,\lambda)$ that is, 3-$GDD(n,t,k,0,\lambda)$.

But Hanani's designs are not general to construct other families of t-designs. As 3-GDDs are more general than Hanani's design, they may be used to construct other families

of t-designs. So motivated by the papers of (Hanani, 1963; Sarvate and Bezire, 2018) and (Sarvate and Shinde, 2020), we defined t-GDDs which is more general than 3-GDDs and then studied a 4-GDD with block size 5 and two groups of size n each.

In this dissertation, we applied direct methods to construct nonuniform GDDs with block size 4 and a 4-GDD with block size 5. Further, we used combinatorial argument and modular arithmetic to obtain necessary conditions.

1.2 Statement of the problem

A problem of great generality in combinatorial design is that of arranging a finite set of objects into a specified number of subsets (blocks) so that the i-th object appears r_i times in all the blocks taken together, the j-th block contains k_j objects, and so that pairs, triples, and similar groups of objects occur a specified number of times. Many of the fundamental questions in design theory are existence questions: does a design of a specified type exist? This is a very difficult problem in general, and there are many parameter sets where the answer is not yet known.

Pabhapote and Punnim (2011) studied the construction of a GDD$(n_1,n_2,3;\lambda,1)$. They obtained all triples of positive integers (n_1,n_2,λ) for which a GDD$(n_1,n_2,3;\lambda,1)$ exists. Later, (Pabhapote, 2012) studied the existence of a GDD$(n_1,n_2,3;\lambda_1,\lambda_2)$ and proved that it exists for all $n_1 \neq 2$ and $n_2 \neq 2$ in which $\lambda_1 \geq \lambda_2$. Chaiyasena et al. (2012) investigated GDDs with two association classes with blocks of size 3 and groups of unequal sizes. They also obtained some general results for a GDD$(n_1,n_2,n_3,3;\lambda_1,\lambda_2)$. Lapchinda et al. (2014) gave a complete solution for the existence of a GDD$(1,n,n,3;\lambda_1,\lambda_2)$. Sakda and Uiyyasathian (2017), obtained complete result on group divisible designs with block size 3 and 4 groups of sizes n, n, n and 1. Namyalo et al. (2017) studied nonuniform GDDs with 3 groups of sizes n_1, n $(n \geq n_1)$ and $n+1$, for $n_1 = 1$ or 2 and block size 4. Most of the results on nonuniform group divisible designs with $\lambda_1 \neq 0$ obtained by previous researchers were of block size 3. The previous researchers didn't study the existence of nonuniform GDDs with block size 4 when $\lambda_1 \neq 0$. So motivated by the papers of (Pabhapote and Punnim, 2011; Pabhapote,

2012; Chaiyasena et al., 2012; Lapchinda et al., 2014; Sakda and Uiyyasathian, 2017) and Namyalo et al. (2017), we studied nonuniform group divisible designs by enlarging the block size to 4.

Moreover, the concept of 3-group divisible designs was introduced by Hanani (1963) in order to construct 3-designs with blocks of size 4. He used the notation $P_n''[k,\lambda,v]$ to denote a 3-PBIBD$(n,t,k,0,0,\lambda)$ and $Q_n''[k,\lambda,v]$ to denote a 3-PBIBD$(n,t,k,0,\lambda,\lambda)$. But these designs do not help us to construct t-designs for $t>3$. As t-GDDs are more general than Hanani's designs, they can be applied to construct families of t-designs. Consequently, (Sarvate and Bezire, 2018) defined 3-group divisible designs by extending the definition of group divisible designs and another important block designs, called t-designs. Motivated by the paper of (Sarvate and Bezire, 2018) and (Sarvate and Shinde, 2020), we gave a general definition for t-GDD and studied 4-GDD with two groups of size n each and block size 5 which is more general to construct families of t-designs.

1.3 Objectives of the study

1.3.1 General objective

The general objective of this research is to study the existence of nonuniform group divisible designs with 2 or 3 groups with block size 4, and a uniform 4-group divisible design with 2 groups and block size 5.

1.3.2 Specific objectives

The specific objectives are to

i. present general construction for a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ and a GDD$(1,n,n,4;\lambda_1,\lambda_2)$.

ii. apply the general constructins to construct families of such GDDs.

iii. obtain necessary conditions for the existence of GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$, when $\lambda_2=1$ or 2, GDD$(1,n,n,4;\lambda_1,\lambda_2)$, GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$, GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$ and

4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$.

iv. prove that the necessary conditions are sufficient for the existence of such designs in several cases.

v. show a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ where $t \geq 0$ does not exist with equal number of blocks of block type configuration is required.

1.4 Significance of the study

The construction of nonuniform GDDs is a challenging problem as no appropriate algbraic or geometric structures have been developed which are useful for the construction. This study presents new construction techniques by employing graph decomposition, α-resolvable designs, defference families and relabeling construction to construct nonuniform GDDs with block size 4. Further, the study presents the actual spectrum for 4-$(v,5,\Lambda_1)$ and for 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ that can be used as a ready source for researchers interested in the construction of such designs. In addition, the study fills the gaps in the existing knowledge on the existence of GDDs. Thus, this study is an important contribution towards a possible solution of the existing question.

1.5 Organization of the dissertation

An important part of the reaearch on combinatorial designs concerns with the existence problem. The dissertation is divided into 8 chapters concerning existence of nonuniform group divisible designs with two or three groups and block size 4, and a uniform 4-group divisible design with two groups and block size 5. In chapter 2, we reviewed the literature relevant to our objectives of the study specifically balanced incomplete block designs, t-designs, group divisible designs, resolvable designs, large sets, f-factorization of graphs and difference families. In chapter 3, we pointed out the type of methods and techniques we employed in our study. In chapter 4, we presented the work on the problem of the existence of group divis-

ible designs (GDDs) with two groups of different sizes and with blocks of size 4. We gave a general construction technique and applied it to construct families of such GDDs when $\lambda_2 = 1$ or 2. We obtained necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$ when $\lambda_2 = 1$ or 2 and proved that the necessary conditions were sufficient in several cases. In chapter 5, we concentrated on group divisible designs with three groups of sizes 1, n and n respectively with blocks of size 4. We presented a general construction that will produce such GDDs when $\lambda_1 \geq \lambda_2$ as well as $\lambda_1 < \lambda_2$. We obtained necessary conditions for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ and proved that these necessary conditions were sufficient in several cases. In chapter 6, we focused on two new problems of group divisible designs with three groups and block size 4. The first problem involves group sizes 1, n and $n+2$ respectively. We showed that a GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ where $t > 0$ does not exist with equal number of blocks of block type configuration is required. We then concentrated our study on the case where $t = 2$ without any block configuration restriction. We obtained the necessary conditions for the existence of a GDD$(1, n, n+2, 4; \lambda_1, \lambda_2)$ and proved that these necessary conditions were sufficient in several cases. The second problem involves group sizes 3, n and $n+1$ respectively. We obtained necessary conditions for the existence of a GDD$(3, n, n+1, 4; \lambda_1, \lambda_2)$ and showed that the necessary conditions were sufficient for certain families of such GDDs. In chapter 7, we studied a t-group divisible design (t-GDD), a new approach defined by combining the definition of a t-design and a group divisible design. We focused on the existence of a 4-group divisible design (4-GDD) with two groups of equal size and blocks of size 5. We obtained necessary conditions for the existence of a 4-GDD$(n, 2, 5; \Lambda_1, \Lambda_2)$ and showed that the necessary conditions were sufficient for $n = 3, 4, 5, 6, 7, 9, 10$. In chapter 8, we concluded the dissertation by providing a detailed summary of the findings of the previous chapters, and proposed possible future work.

Chapter 2

Combinatorial designs: background and literature review

In this chapter, we reviewed results available on the topics relating to this research, namely: balanced incomplete block designs, t-designs, group divisible designs, resolvable designs, large sets, f-factorization of graphs and difference families.

2.1 Balanced Incomplete Block Designs

Among all combinatorial designs, probably the most widely studied design is a Balanced Incomplete Block Design (BIBD). As per (Bose, 1939) a BIBD is defined as follows:

Definition 2.1.1. A Balanced Incomplete Block Design BIBD(v,k,λ) is an arrangement of the elements of a v-set X into b blocks of size $(k<v)$ each, such that every element appears in exactly r blocks and every pair of distinct elements occurs together in exactly λ blocks.

A famous BIBD is the Fano plane : a BIBD$(7,3,1)$. The blocks are obtained from the triple $\{1,3,4\}$ by adding the elements of $X=\{0,1,2,3,4,5,6\}$ modulo 7. So the collection of blocks of the design is $\{\{1,3,4\},\{2,4,5\},\{3,5,6\},\{4,6,0\},\{5,0,1\},\{6,1,2\},\{0,2,3\}\}$.

The main objective of design theory is obtaining the necessary and sufficient conditions for the existence of different types of designs. The following theorem gives necessary conditions for the existence of BIBD(v,k,λ) (Stinson, 2007).

Theorem 2.1.1. *If a BIBD(v,k,λ) design $(X,\mathscr{B})$ exists, then $bk=rv$ and $r(k-1)=\lambda(v-1)$.*

A BIBD$(v,3,1)$ is called a Steiner Triple System which is denoted by STS(v) and exists if and only if $v\equiv 1,3 \pmod 6$.

A BIBD can be described by its incidence matrix and this representation is especially useful in computer programs (Stinson, 2007).

Definition 2.1.2. Let $(X, \mathscr{B})$ is a BIBD(v,k,λ) where $X = \{x_1, x_2, ..., x_v\}$ and $\mathscr{B} = \{B_1, B_2, ..., B_b\}$. The incidence matrix of $(X, \mathscr{B})$ is a $v \times b$ matrix of 0's and 1's defined by $N = (n_{ij})_{v \times b}$, where

$$n_{ij} = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{otherwise.} \end{cases}$$

The following properties are satisfied by the incidence matrix, N, of a BIBD(v,k,λ):

1. every row of N contains exactly r "1"s;
2. every column of N contains exactly k "1"s;
3. two distinct rows of N both contain "1" in exactly λ columns.

Let N be the incidence matrix of a BIBD$(7,3,1)$ design $(X, \mathscr{B})$ whose rows and columns are indexed by elements of X and $\mathscr{B}$ respectively, where $X = \{0,1,2,3,4,5,6\}$ and $\mathscr{B} = \{\{1,3,4\},\{2,4,5\},\{3,5,6\},\{4,6,0\},\{5,0,1\},\{6,1,2\},\{0,2,3\}\}$. Then

$$N = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Suppose the transpose of N donoted N^T, I_v denotes a unit matrix of order v, J_v denotes a square matrix of order n in which every entry is a "1" and u_v is a row vector of length v in

which every coordinate is a "1". Then the following theorem is a characterization of a BIBD and its Incidence Matrix (Stinson, 2007).

Theorem 2.1.2. *Let N be a $v \times b$ matrix of 0's and 1's and let $2 \leq k < v$. Then N is the incidence matrix of a BIBD(v,k,λ) if and only if $NN^T = (r-\lambda)I_v + \lambda J_v$ and $u_v N = k u_b$.*

The determinant of NN^T is denoted by $|NN^T|$ and defined as

$$|NN^T| = \begin{vmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{vmatrix}$$

To evaluate $|NN^T|$, first add row 2, row 3, ..., row v in $|NN^T|$ to row 1. Then every element of row 1 becomes $(r+\lambda(v-1)) = rk$. Next, take rk outside the determinant and add row 1 multiplied by (-1) to all rows below row 1. Then

$$|NN^T| = rk \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r \end{vmatrix} = rk \begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 1 & r & \lambda & \cdots & \lambda \\ 1 & \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda & \lambda & \cdots & r \end{vmatrix}$$

$$|NN^T| = rk \begin{vmatrix} 1 & \lambda & \lambda & \cdots & \lambda \\ 0 & r-\lambda & 0 & \cdots & 0 \\ 0 & 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r-\lambda \end{vmatrix} = rk(r-\lambda)^{v-1}.$$

Fisher (1940) gave a further necessary condition for the existence of a BIBD(v,k,λ)

which is given by the following theorem :

Theorem 2.1.3. *In any BIBD*(v,k,λ), $b \geq v$.

For example, a BIBD$(16,6,1)$ satisfies conditions of Theorem 2.1.1 but it does not exist as $b < v$.

A BIBD(v,k,λ) is called symmetric if $v = b$. The following theorem is known as the Bruck - Ryser - Chowla Theorem. It was proved for odd v by (Bruck and Ryser, 1949) and by (Chowla and Ryser, 1950). The part of the theorem for even v was first obtained by (Schützenberger, 1947).

Theorem 2.1.4. *Suppose there exists a symmetric BIBD*(v,k,λ).
(i) *If v is even, then $k-\lambda$ is the square of an integer.*
(ii) *If v is odd, then the equation $x^2 = (k-\lambda)y^2 + (-1)^{\frac{(v-1)}{2}}\lambda z^2$ has a solution in integers x,y,z not all zero.*

All necessary conditions specified above (taken together) are still not sufficient for the existence of BIBD(v,k,λ). For instance, a symmetric BIBD$(111,11,1)$ does not exist even though it satisfies the condition of Bruck - Ryser - Chowla Theorem. This is because a BIBD$(n^2+n+1,n+1,1)$ exists if n is a prime power and it was shown by (Lam et al., 1989) that a finite projective plane of order 10 does not exist. Moreover, any positive integers v,b,r,k and λ satisfying conditions of Theorem 2.1.1 may not give rise to a BIBD(v,k,λ). For instance, integers $v = 36$, $b = 42$, $r = 7$, $k = 6$ and $\lambda = 1$ satisfy conditions of Theorem 2.1.1 but do not give rise to a BIBD$(36,6,1)$ as a pair of orthogonal latin square of order 6 does not exist.

Hanani (1961) proved the sufficiency of the necessary conditions for the existence of a BIBD$(v,3,\lambda)$, a BIBD$(v,4,\lambda)$, and a BIBD$(v,5,\lambda)$ for $\lambda = 1,4$ and 20. The proof was given by induction on v for any pair of fixed values of k and λ. Specifically, he proved the following theorems.

Theorem 2.1.5. *A BIBD*$(v,3,\lambda)$ *exists if and only if*
$\lambda \equiv 1,5 \pmod 6$ *and* $v \equiv 1,3 \pmod 6$*;*

$\lambda \equiv 2,4 \pmod 6$ *and* $v \equiv 0,1 \pmod 3$*;*

$\lambda \equiv 3 \pmod 6$ *and* $v \equiv 1 \pmod 2$*;*

$\lambda \equiv 0 \pmod 6$ *and* $v \geq 3$.

Theorem 2.1.6. *A BIBD*$(v,4,\lambda)$ *exists if and only if*

$\lambda \equiv 1,5 \pmod 6$ *and* $v \equiv 1,4 \pmod{12}$*;*

$\lambda \equiv 2,4 \pmod 6$ *and* $v \equiv 1 \pmod 3$*;*

$\lambda \equiv 3 \pmod 6$ *and* $v \equiv 0,1 \pmod 4$*;*

$\lambda \equiv 0 \pmod 6$ *and* $v \geq 4$.

Theorem 2.1.7. *A BIBD*$(v,5,\lambda)$ *exists if and only if*

$\lambda \equiv 1 \pmod{20}$ *and* $v \equiv 1,5 \pmod{20}$*;*

$\lambda \equiv 4 \pmod{20}$ *and* $v \equiv 0,1 \pmod 5$*;*

$\lambda \equiv 0 \pmod{20}$ *and every v.*

2.2 t-designs

Hanani (1963) defined a tactical configuration $C[k,t,\lambda,v]$ as a system of subsets of E, having k elements each, such that every t element of E is contained in exactly λ sets of the system where E is a set of v points and k,t,λ are positive integers with $t \leq k \leq v$. But this tactical configuration $C[k,t,\lambda,v]$ is now known to be a t-design or t-(v,k,λ) design.

Definition 2.2.1. A t-(v,k,λ) design, or a t-design is a pair $(X,\mathscr{B})$ where X is a v-set of points, $\mathscr{B}$ is a collection of k-subsets (blocks) of X such that every t-subset of X is contained in λ blocks. The parameter λ is the index of the design.

As per (Laue and Khosrovshahi, 2007) a t-(v,k,λ) design is also denoted by $S_\lambda(t,k,v)$ and if $\lambda = 1$, it is simply written as $S(t,k,v)$. Repeated blocks are allowed to occur in a t-(v,k,λ) design. But if $\lambda = 1$, a t-(v,k,λ) design cannot contain repeated blocks. A t-design that does not contain repeated blocks is called a simple t-design. When $\lambda > 1$, the case that constructing simple t-designs is more difficult than constructing non simple ones. Taking λ

copies of every k-subsets of a v-set, where $k < v$ provide a t-$(v,k,\lambda\binom{v-t}{k-t})$ design. This design is called a trivial t-design. But combinatorial design theorists are interested in constructing non trivial t-designs.

Example 2.2.1. A 3-$(8,4,1)$ design with $X = \{1,2,3,4,5,6,7,8\}$. The blocks are written in columns:

1	2	3	4	5	6	7	8	1	3	1	2	1	2
2	1	2	1	4	3	2	1	2	4	4	3	8	7
4	3	4	3	6	5	6	5	5	7	6	5	3	4
8	7	6	5	8	7	8	7	6	8	7	8	6	5

Laue and Khosrovshahi (2007) gave the following properties for the existence of a t-(v,k,λ) design.

Theorem 2.2.1. *If $(X,\mathscr{B})$ is a t-(v,k,λ) design and S is any s-subset of X, with $0 \le s \le t$, then the number of blocks containing S is $\lambda_s = |\{B \in \mathscr{B} : S \subseteq B\}| = \frac{\lambda\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$. In particular the number of blocks of a t-(v,k,λ) design is $b = \lambda_0$.*

Theorem 2.2.2. *Suppose that $(X,\mathscr{B})$ is a t-(v,k,λ) design and let $A \subseteq X$, $|A| = i < t$. Then $(X \setminus A, \{B \setminus A : A \subseteq B \in \mathscr{B}\})$ is a $(t-i)$-$(v-i,k-i,\lambda)$ design.*

In view of Theorem 2.2.2, if $|A| = 1$, then a $(t-1)$-$(v-1,k-1,\lambda)$ design is called the derived design at a particular point of X. Let a t-(v,k,λ) design $(X,\mathscr{B})$ be given. As per (van Trung, 2019) the derived design at the point x is the design with the point set $X \setminus \{x\}$ and the block sets are all blocks $B \in \mathscr{B}$ containing x with x removed. For $x \in X$, the $(t-1)$-$(v-1,k-1,\lambda)$ design is the derived design at the point x. However, the design with the point set $X \setminus \{x\}$ and the block set are the blocks of $\mathscr{B}$ not containing x is called the residual design at the point x. Since all blocks containing x are removed, every $(t-1)$-subsets of $X \setminus \{x\}$ occurs in $\frac{v-k}{k-t+1}\lambda$ blocks of the residual design. Thus, $(t-1)$-$(v-1,k,\frac{v-k}{k-t+1}\lambda)$ is the residual design at the point x. For instance, the blocks in Example 2.2.1 containing 8 with 8 removed are the blocks of a 2-$(7,3,1)$ design which is the derived design at the point

8 whereas the blocks of 3-$(8,4,1)$ not containing the point 8 are the blocks of a 2-$(7,4,2)$ which is the residual design at the point 8.

In general, if a t-(v,k,λ) design exists, then it is possible to form a $(t-1)$-$(v-1,k-1,\lambda)$ design. But an important question here is whether it is possible or not to form a $(t+1)$-$(v+1,k+1,\lambda)$ design from a t-(v,k,λ) design. This is not always true but it is possible for certain classes of t-designs.

From a given t-(v,k,λ) design it may be possible to form a $(t+1)$-$(v+1,k+1,\lambda)$ design by adding a new symbol to each of the given blocks and by adding further new blocks of size $k+1$ not containing the new symbol. The new design is called an extension of the old design. Extensions of t-designs are obtained by the following theorem (Laue and Khosrovshahi, 2007).

Theorem 2.2.3. *Let $\mathscr{D} = (X, \mathscr{B})$ be a t-$(2k+1,k,\lambda)$ design and ∞ a new point not in X.*

1. *If t is even, then $(X \cup \{\infty\}, B \cup \{\infty\} : B \in \mathscr{B} \cup X \setminus B : B \in \mathscr{B})$ is a $(t+1)$-$(2k+2,k+1,\lambda)$ design.*

2. *If $\mathscr{D}$ is simple, t odd and $\lambda = \frac{\lambda_{max}}{2}$, then $(X \cup \{\infty\}, B \cup \{\infty\} : B \in \mathscr{B} \cup \{X \setminus B : B \in \binom{X}{k} \setminus \mathscr{B}\})$ is a $(t+1)$-$(2k+2,k+1,\lambda)$.*

Thus, any 2-$(2k+1,k,\lambda)$ design can be always extended to a 3-$(2k+2,k+1,\lambda)$ design by complementation; that is, by adding the same new symbol to each of the original blocks and then by taking the complement of each block with respect to the new set of symbols to form further new blocks.

Hanani (1963) proved the following theorem.

Theorem 2.2.4. *Necessary conditions are sufficient for the existence of a 3-$(v,4,\lambda)$. That is, a 3-$(v,4,\lambda)$ exists if and only if*

$\lambda \equiv 1,5,7,11 \pmod{12}$ *and* $v \equiv 2,4 \pmod{6}$;

$\lambda \equiv 2,10 \pmod{12}$ *and* $v \equiv 1,2,4,5,8,10 \pmod{12}$;

$\lambda \equiv 3,9 \pmod{12}$ *and* $v \equiv 0 \pmod{2}$;

$\lambda \equiv 4,8 \pmod{12}$ *and* $v \equiv 1,2 \pmod{3}$;

$\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0,1,2 \pmod{12}$*;*
$\lambda \equiv 0 \pmod{12}$.

A 3-$(v,4,1)$ design is said to be a Steiner Quadruple System of order v, denoted SQS(v), exists if and only if $v \equiv 2,4 \pmod{6}$. Admissible but not realizable parameter quadruples for $t = 3$ and $v \leq 30$ are 3-$(11,5,2)$, 3-$(16,6,2)$, 3-$(22,10,6)$ and 3-$(26,10,3)$ (Laue and Khosrovshahi, 2007, pp 84).

2.2.1 Resolvable t-designs and Large set of t-designs

Definition 2.2.2. A t-(v,k,λ) design $(X,\mathscr{B})$ is said to be s-resolvable, or to have an s-resolution, with $0 < s < t$, if its block set $\mathscr{B}$ can be partitioned into N classes Π_1, Π_2, ..., Π_N $(N \geq 2)$ such that each (X,Π_i) is an s-(v,k,δ) design for $i = 1,...,N$. Each Π_i is an s-resolution class or simply a resolution class and the set of N classes is called an s-resolution of the design (van Trung, 2019).

We can write (s,δ)-resolution to mean an s-(v,k,δ) design in the resolution. Thus, a $(1,1)$-resolution of a t-design is a partition of its blocks into classes of 1-$(v,k,1)$-design. As per (Abel et al., 2006) parallel classes (resolutions) for BIBDs can be defined as follows.

Definition 2.2.3. A parallel class in a BIBD(v,k,λ) is a set of disjoint blocks of the BIBD whose union is X. A partition of the collection of all blocks into r parallel classes is called a resolution.
A BIBD is said to be a resolvable BIBD, denoted RBIBD(v,k,λ) if it has a resolution.
A BIBD is called α-resolvable BIBD if its blocks can be partitioned into classes in which each point occurs α times.

Theorem 2.2.5. *A resolvable BIBD*$(v,3,1)$ *exists if and only if* $v \equiv 3 \pmod{6}$ *(Abel et al., 2006).*

A resolvable BIBD$(v,3,1)$ is called Kirkman Triple System of order v, denoted by KTS(v). There are $\frac{v-1}{2}$ parallel classes in a KTS(v).

Theorem 2.2.6. *The necessary conditions for the existence of an α-resolvable BIBD$(v,3,\lambda)$ are sufficient except for $v=6$, $\alpha=1$, and $\lambda \equiv 2 \pmod 4$ (Abel et al., 2006).*

Hence, a resolvable BIBD$(v,3,2)$ exists for $v \equiv 0 \pmod 3$ except for $v=6$.

If the trivial design k-$(v,k,1)$ is s-resolvable with N resolution classes, where each class is an s-(v,k,δ) design, then we say that there exist a *large set* of size N of s-designs. We know that a large set of an s-(v,k,δ) is a partition of all k-subsets of a v-set into $\frac{1}{\delta}\binom{v-s}{k-s}$ disjoint s-(v,k,δ) designs. As per (Ajoodani-Namini, 1996) a large set of t-(v,k,λ) design can be formally defined as follows.

Definition 2.2.4. A large set of disjoint t-(v,k,λ) design $(X,\mathscr{B})$ is a partition of all k-subsets of a v-set X into N parts of equal size, each of which is the block set of a t-(v,k,λ) design where $\lambda = \frac{1}{N}\binom{v-t}{k-t}$.

A large set of a STS(v) is a partition of all 3-subsets of a v-set into $(v-2)$ disjoint STS(v). It is known that a large set of STS(v) exists for $v \neq 7$ and a large set of 3-$(v,4,\lambda)$ exists for $v \equiv 0 \pmod 3$ (Lu, 1983; Jia-Xi, 1984; Teirlinck, 1984).

2.3 Group divisible designs

Definition 2.3.1. A group divisible design GDD$(n,m,k;\lambda_1,\lambda_2)$ is a triple $(X,\mathscr{G},\mathscr{B})$ where X is a set of $v=mn$ elements, $\mathscr{G}$ is a partition of X into m parts called groups of size n each and $\mathscr{B}$ is a collection of k-subsets of X called blocks such that every two elements from the same group occurs together in λ_1 blocks while every two elements from different groups occurs together in λ_2 blocks (Fu and Rodger, 1998).

Pair of elements occurring together in the same group are called first associates and pair of elements occurring in different groups are called second associates. For a GDD$(v,1,k;\lambda_1,\lambda_2)$, λ_2 is irrelevant and it is reduced to a BIBD(v,k,λ_1). A GDD$(n,m,k;\lambda,\lambda)$ can be collapsed to a BIBD(mn,k,λ). A BIBD(v,k,λ) can also be expressed as a GDD$(1,v,k;0,\lambda)$. Another common notation for a GDD$(n,m,k,\lambda_1=0,\lambda_2)$ is (k,λ_2)-GDD(n^m). Such kind of

GDDs were introduced and used by (Mullin and Gronau, 2006) to construct other combinatorial structures.

Definition 2.3.2. Let K and G be sets of positive integers and λ be a positive integer. A group divisible design of index λ and order v ((K,λ)-GDD) is a triple $(X,\mathscr{G},\mathscr{B})$, where X is a finite set of size v, $\mathscr{G}$ is a partition of X into parts (groups) whose sizes lie in G, and $\mathscr{B}$ is a collection of subsets of X called blocks of size from K such that every pair of distinct elements of X occurs in exactly λ blocks or one group, but not both (Mullin and Gronau, 2006).

If $v = n_1m_1 + n_2m_2 + ... + n_sm_s$, and if there are m_i groups of size n_i for $i = 1,2,...,s$, then the (K,λ)-GDD is of type $n_1^{m_1}n_2^{m_2}...n_s^{m_s}$. This is exponential notation for the group type. If $K = \{k\}$, then the (K,λ)-GDD is a (k,λ)-GDD. If all groups of a GDD have the same size, then the GDD is said to be uniform and of type n^m for some positive integer m. Note that a (k,λ)-GDD of type n^m is equivalent to the current notation GDD$(n,m,k;0,\lambda)$. So a $(k,1)$-GDD of type n^m means a GDD$(n,m,k;0,1)$.

A GDD with $\lambda_1 \neq 0$ was first introduced by (Bose and Nair, 1939) by using a symmetric association scheme defined on a point set X. In this case the GDD was considered as a Partially Balanced Incomplete Block Design with m associate classes (PBIBD(m)).

Definition 2.3.3. Let X be a v-set with a symmetric association scheme defined on it. A partially balanced incomplete block design with m associate classes (PBIBD(m)) is a design based on the v-set X with b blocks of size k each, in which each element of X appears in r blocks and every pair of points that are i-th associates occurs together in λ_i blocks such that given any two elements which are i-th associates, the number of elements common to the j-th associates of the first, and the k-th associates of the second is P^i_{jk} and is independent of the pair of elements with which we start. Also $P^i_{jk} = P^i_{kj}$. The numbers v,b,r,k,λ_i $(1 \leq i \leq m)$ are the parameters of the first kind and the numbers P^i_{jk} $(i,j,k = 1,2,...,m)$ are the parameters of the second kind, belonging to the design (Bose and Nair, 1939; Street and Street, 2006).

One can also write PBIBD(m) as a PBIBD$(v,b,r,k;\lambda_1,\lambda_2,...,\lambda_m)$.

Example 2.3.1. Consider the following 3-class symmetric association scheme on $X = \{1,2,3,4,5,6\}$.

$R_0 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$,

$R_1 = \{(1,2),(2,1),(3,4),(4,3),(5,6),(6,5)\}$,

$R_2 = \{(1,4),(1,6),(2,3),(2,5),(3,6),(4,5),(4,1),(6,1),(3,2),(5,2),(6,3),(5,4)\}$,

$R_3 = \{(1,3),(1,5),(2,4),(2,6),(3,1),(3,5),(4,2),(4,6),(5,1),(5,3),(6,2),(6,4)\}$.

A PBIBD$(6,12,6,3;4,2,2)$ with these 3 associate classes defined on X is given by the following blocks. The blocks are written as columns:

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 \\ 3 & 4 & 5 & 6 & 4 & 4 & 5 & 6 & 6 & 6 & 6 & 6 \end{array}$$

The above PBIBD can be reduced into PBIBD$(6,12,6,3;4,2)$ as $\lambda_2 = \lambda_3$. It is a group divisible association scheme in which the point set X is partitioned into 3 groups: $G_1 = \{1,2\}$, $G_2 = \{3,4\}$ and $G_3 = \{5,6\}$, such that the first associates are points from the same group and the second associates are points not from the same group. So a GDD$(n,m,k;\lambda_1,\lambda_2)$ is a PBIBD with 2-class symmetric association scheme defined on the point set in which the point set is partitioned into m groups of size n each, such that every pair of points from the same group occurs together in λ_1 blocks whereas every pair of points not from the same group occurs in λ_2 blocks. Moreover, if each element in a PBIBD(m) has n_i i-th associate for $i = 1,2,...,m$, then the parameters of a PBIBD(m) satisfy $vr = bk$ and $\sum_{i=1}^{m} n_i\lambda_i = r(k-1)$.

Bose and Shimamoto (1952) classified partially balanced incomplete block designs with two association classes into five types: group divisible designs, triangular, singly linked blocks, latin square type and cyclic. In this dissertation, we concentrate only on group divisible designs.

Group divisible designs (GDDs) can also be described by their incidence matrices. The incidence matrix of a GDD$(n,m,k;\lambda_1,\lambda_2)$ is an $mn \times b$ matrix of 0's and 1's defined by

$N = (n_{ij})_{mn \times b}$, where

$$n_{ij} = \begin{cases} 1 & \text{if } i\text{-th treatment is in the } j\text{-th block} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, as per (Beth et al., 1999) the incidence matrix N of a GDD$(n,m,k;\lambda_1,\lambda_2)$ is of the form

$$N = \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_m \end{pmatrix}$$

satisfying the following conditions

(a) $N_i N_i^T = D_i + \lambda_1 J$ for $i = 1,...,m$, where D_i is the diagonal matrix and J denotes the matrix of the appropriate size with all entries 1 and

(b) $N_i N_j^T = \lambda_2 J$ for $i, j = 1,...,m$ and $i \neq j$.

N_i^T is the transpose of N_i and N_i's are sub matrices of N each with size $n \times b$. Then

$$NN^T = \begin{pmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ B & B & A & \cdots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & B & B & \cdots & A \end{pmatrix}$$

where N^T is the transpose of the matrix N, and A and B are $n \times n$ matrices defined by

$$A = \begin{pmatrix} r & \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & r & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & \lambda_1 & r & \cdots & \lambda_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_1 & \lambda_1 & \cdots & r \end{pmatrix}, B = \begin{pmatrix} \lambda_2 & \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_2 & \lambda_2 & \lambda_2 & \cdots & \lambda_2 \end{pmatrix}.$$

Bose and Connor (1952) evaluated the determinant of NN^T and obtained value in terms of $r, k, \lambda_1, \lambda_2, m$ and n as follows:

(a) add row 2, row 3, ..., row mn in $|NN^T|$ to row 1. Then every element of row 1 now becomes $r + (n-1)\lambda_1 + n(m-1)\lambda_2 = rk$; and

(b) take rk outside the determinant, and add suitable multiple of row 1 to all rows blow row 1. Then

$$|NN^T| = rk \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{vmatrix} \cdot \begin{vmatrix} r - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & r - \lambda_2 & \cdots & \lambda_1 - \lambda_2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1 - \lambda_2 & \lambda_1 - \lambda_2 & \cdots & r - \lambda_2 \end{vmatrix}^{m-1}.$$

Now taking determinant of the transpose of matrix A and adding suitable multiples of row 1 to all other rows, give $rk(r-\lambda_1)^{n-1}$. Also applying procedure (a) to the second determinant and taking out appropriate constant, give $[r + \lambda_1(n-1) - n\lambda_2]^{m-1}((r-\lambda_1)^{n-1})^{m-1}$. Thus, $|NN^T| = rk(rk - v\lambda_2)^{m-1}(r-\lambda_1)^{m(n-1)}$.

Bose and Connor (1952) have made careful study of combinatorial properties of group divisible designs (GDDs) and classified them into three subtypes depending on the values assumed by the quantities $rk - v\lambda_2$ and $r - \lambda_1$ which are obtained from the above determinant. They classified GDDs as:

(a) singular if $r - \lambda_1 = 0$;

(b) semi-regular if $r-\lambda_1>0$ and $rk-v\lambda_2=0$;

(c) regular if $r-\lambda_1>0$ and $rk-v\lambda_2>0$.

2.3.1 Nonuniform GDDs

Here we consider nonuniform GDDs with $\lambda_1 \neq 0$ and all the groups have different sizes.The general definition for such GDD is given as follows.

Definition 2.3.4. A group divisible design GDD$(n_1,n_2,...,n_m,k;\lambda_1,\lambda_2)$ is a triple $(X,\mathscr{G},\mathscr{B})$, where X is a v-set ($v=n_1+n_2+...+n_m$), $\mathscr{G}$ a partition of X into m subsets (called *groups*) of sizes $n_1,n_2,...,n_m$ respectively and $\mathscr{B}$ is collection of k-subsets of X (called *blocks*) such that

(i) pairs of points with in the same group are called *first associates* of each other and appear together in λ_1 blocks, and

(ii) pairs of points not in the same group are called *second associates* of each other and appear together in λ_2 blocks (Bose and Shimamoto, 1952).

Assuming a GDD$(n_1,n_2,...,n_m,k;\lambda_1,\lambda_2)$ exists and has b blocks, using simple counting argument, the replication number r_i for elements of i^{th} group is given by

$$r_i=\frac{\lambda_1(n_i-1)+\lambda_2\sum_{\substack{j=1\\ j\neq i}}^{m}n_j}{k-1}. \tag{2.3.1}$$

Since $kb=n_ir_i$, we get the following expression for the number of blocks b:

$$b=\frac{\lambda_1\sum_{i=1}^{m}(n_i^2-n_i)+\lambda_2\sum_{i\neq j}n_in_j}{k(k-1)}. \tag{2.3.2}$$

2.3.2 Large set of GDDs

Large sets of disjoint group divisible designs with block size 3 and type 2^n4^1 have been studied by (Schellenberg, 1989) and (Chen et al., 1992) and motivated by their connection

with perfect threshold schemes. Another notation for a group divisible design with block size 3 and type 2^n4^1 is GDD$(2,2,...,2,4,3;0,1)$. They showed that such large sets exist for $n=6$ and $n=3^k$, $k\geq 1$. These large sets have applications in cryptography in the construction of perfect threshold schemes. Later, (Cao et al., 2001) presented new recursive constructions and used them to show such large sets exist for all odd $n\equiv 0 \pmod 3$ and for even $n=24m$, where m is odd and $m\geq 1$. Further, (Cao et al., 2015) gave a generalization of a special large set ${}^*LS(2^n)$ and used it to obtain a similar result for $k=2,4$ and partially for $k=1$.

(Niu et al., 2018) completely solved the existence of large sets of a $(3,\lambda)$-GDDs of type n^m. A $(3,\lambda)$-GDD of type n^m means a GDD$(n,m,3;0,\lambda)$. A GDD$(n,m,3;0,\lambda)$ is said to be simple if it does not contain repeated blocks. Two simple GDDs are called disjoint if the intersection of their block sets is empty. A large set of GDD$(n,m,3;0,\lambda)$ is a collection of $\frac{(m-2)n}{\lambda}$ disjoint GDD$(n,m,3;0,\lambda)$'s. In other words, it is a partition of $\binom{m}{3}n^3$ 3-subsets of type $(1,1,1)$ into $\frac{(m-2)n}{\lambda}$ disjoint GDD$(n,m,3;0,\lambda)$'s.

Example 2.3.2. Let $X=\{1,2,3,a,b,c,x,y,z\}$, $G_1=\{1,2,3\}$, $G_2=\{a,b,c\}$ and $G_3=\{x,y,z\}$. Then the collection $\{\mathscr{B},\mathscr{C},\mathscr{D}\}$ is a large set of GDD$(3,3,3;0,1)$, where
$\mathscr{B}=\{\{1,a,x\},\{1,b,y\},\{1,c,z\},\{2,a,z\},\{2,b,x\},\{2,c,y\},\{3,a,y\},\{3,b,z\},\{3,c,x\}\}$,
$\mathscr{C}=\{\{1,b,x\},\{1,c,y\},\{1,a,z\},\{2,b,z\},\{2,c,x\},\{2,a,y\},\{3,b,y\},\{3,c,z\},\{3,a,x\}\}$ and
$\mathscr{D}=\{\{1,c,x\},\{1,a,y\},\{1,b,z\},\{2,c,z\},\{2,a,x\},\{2,b,y\},\{3,c,y\},\{3,a,z\},\{3,b,x\}\}$.

Theorem 2.3.1. *A large set of GDD$(n,m,3;0,1)$ exists if and only if $n(m-1)\equiv 0 \pmod 2$, $n^2m(m-1)\equiv 0 \pmod 6$, $m\geq 3$ and $(n,m)\neq(1,7)$ (Niu et al., 2018).*

2.4 3-Group divisible designs (3-GDDs)

Recently, (Sarvate and Bezire, 2018) defined 3-GDDs with two groups and block size k by generalizing the definition of t-designs and group divisible designs (GDDs) .

Definition 2.4.1. A 3-GDD$(n,2,k,\lambda_1,\lambda_2)$ is a set X of $2n$ elements partitioned into two parts of size n called groups together with a collection of k-subsets of X called blocks, such that (i) every 3-subset of each group occurs in λ_1 blocks and

(ii) every 3-subset where two elements are from one group and one element from the other group occurs in λ_2 blocks.

Example 2.4.1. A 3-GDD$(4,2,4;0,1)$ with $X=\{1,2,3,4,a,b,c,d\}$, $G_1=\{1,2,3,4\}$, $G_2=\{a,b,c,d\}$. Blocks are written as columns:

1	1	1	1	1	1	2	2	2	2	3	3
2	2	3	3	4	4	3	3	4	4	4	4
a	c	a	b	a	b	a	b	a	b	a	c
b	d	d	c	c	d	c	d	d	c	b	d

(Sarvate and Bezire, 2018) also gave the following fundamental construction.

Theorem 2.4.1. *A 3-GDD$(n,2,4,0,1)$ exists for even n and a 3-GDD$(n,2,4,0,2)$ exists for all positive integers n.*

Later, (Sarvate and Shinde, 2020) generalized Definition 2.4.1 as follows:

Definition 2.4.2. A 3-GDD$(n,m,k,\Lambda_1,\Lambda_2)$ is a set X of mn elements partitioned into $(m>2)$ parts of size n called groups together with a collection of k-subsets of X called blocks, such that
(i) every 3-subset of configuration $(3,0)$, i.e. where all 3 elements are from the same group occur in Λ_1 blocks,
(ii) every 3-subset of configuration $(2,1)$ where two elements are from one group and one element from the other group, or of configuration $(1,1,1)$, i.e. three elements from different groups, occurs in Λ_2 blocks.

If Condition (ii) of Definition 2.4.2 is relaxed in such a way that every 3-subset where each element from different group occurs in Λ_3 blocks, where Λ_3 may not be equal to Λ_2, then such a design will be denoted by a 3-PBIBD$(n,m,k;\Lambda_1,\Lambda_2,\Lambda_3)$.

2.5 f-Factors of graphs

Let G be a graph and let f be a non negative integer-valued function on $V(G)$. An f-factor of G is a spanning subgraph F of G such that $d_F(v) = f(v)$ for all $v \in V(G)$ (West et al., 2001). A k-factor of G is an f-factor with $f(v) = k$ for all $v \in V(G)$. In particular, a 1-factor of G is a spanning subgraph of G whose edge set is a perfect matching and a 2-factor of G is a 2-regular spanning subgraph of G.

Definition 2.5.1. A 1-factorization of a graph G is a partition of the edge set of G into 1-factors.

Definition 2.5.2. A 2-factorization of a graph G is a decomposition of G into 2-factors.

Theorem 2.5.1. *The complete graph K_{2n} has 1-factorization and a complete graph K_{2n+1} has 2-factorization (Liebeck, 1971).*

2.6 Difference sets and difference families

Definition 2.6.1. Let $D = \{b_1, ..., b_k\}$ be a subset of an additive group G. The G-stabilizer of D is the subgroup G_B of G consisting of all elements $g \in G$ such that $D + g = D$. D is full or short according to whether G_B is or is not trivial. The G-orbit of D is the set $Orb_G D$ of all distinct right translates of D, namely, $Orb_G D = \{D + s : s \in S\}$ where S is a complete system of representatives for the right cosets of G_B in G (Abel and Buratti, 2006).

Definition 2.6.2. Suppose $(G, +)$ is a finite group of order v in which the identity element is denoted "0". Let k and λ be positive integers such that $2 \leq k < v$. A (v, k, λ)-difference set in $(G, +)$ is a subset $D \subseteq G$ that satisfies the following properties:

1. $|D| = k$,
2. the multiset $\Delta D = [x - y : x, y \in D, x \neq y]$ contains every element in $G \setminus \{0\}$ exactly λ times (Stinson, 2007).

Definition 2.6.3. Let G be a group of order v. A collection $\{D_1,...,D_t\}$ of k-subsets of G form a (v,k,λ) difference family (or difference system) if every nonidentity element of G occurs λ times in $\partial D_1 \cup ... \cup \partial D_t$. The sets D_i are base blocks. A difference family having at least one short block is partial and $\Delta D = \partial D$ if and only if D is a full block (Abel and Buratti, 2006).

Chapter 3

Research Methodology

3.1 Introduction

This dissertation is devoted to a study of the existence of nonuniform group divisible designs with block size 4 with 2 or 3 groups, and a 4-Group Divisible Design (4-GDD) with block size 5 and 2 groups of equal size.

The known methods to construct block designs are of two main kinds, direct and recursive (Hall, 1998). A direct method helps us to construct a design with particular parameters, usually employing finite fields or congruences, graph decomposition and difference set etc. A recursive method is a way of constructing designs from smaller ones.

In this dissertation, we applied direct methods to construct nonuniform GDDs with block size 4 and a 4-GDD with block size 5. In certain cases we also applied recursive method to construct GDDs. Further, we used combinatorial argument and modular arithmetic to obtain necessary conditions.

3.2 Techniques for constructing dsigns

Several well known techniques such as resolvable designs, 1-factorization and 2-factorization of a complete graph of order n, difference family, relabeling and some times just clever arrangement of points with trial and error for small parameters have been used to construct families of GDDs. We formulated general constructions for a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$ for $\lambda_2 = 1, 2$ by using resolvable and 2-resolvable BIBDs. Further, a general construction for a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ has been formulated by applying 1-factorization and 2-factorization of a complete graph of order n. Also, known designs such as BIBDs with block size 4 were used to construct GDDs along with the general constructions.

3.3 Mathematical procedures

Given a design with certain parameters. First, we obtained the conditions and relationships the parameters must have and decided for which values of parameters, the design may exist. In short, first we obtained necessary conditions for the existence of a design.

Next, for those parameters for which the necessary conditions were satisfied, we proved the designs exist or not. In short, we proved that the necessary conditions were sufficient or not for the existence of the designs.

Chapter 4

Group divisible designs with 2 groups of unequal sizes and block size 4

4.1 Introduction

In this chapter, we study Group Divisible Designs (GDDs) with blocks of size 4 and two groups of different sizes. We present a general construction of GDDs with two groups of different sizes and any block of size k. Further, we present general constructions for such GDDs with block size 4 when λ_2 is 1 and 2. We obtain the spectrum for the existence of such GDDs with block size 4 and two groups and apply these general constructions and known designs to prove that the necessary conditions are sufficient in many cases.

4.2 General construction

Theorem 4.2.1. *If an α-resolvable BIBD(n_1,k,λ) with replication number r and a BIBD$(\frac{r}{\alpha}, k+1,\lambda)$ exist, then a GDD$(n_1,n_2=\frac{r}{\alpha},k+1;\lambda,\alpha)$ exists.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and $n_2=\frac{r}{\alpha}$ respectively. Suppose the blocks of the α-resolvable BIBD(n_1,k,λ) are partitioned into $\frac{r}{\alpha}$ classes namely: π_1, π_2, ..., $\pi_{\frac{r}{\alpha}}$ each of size $\frac{\alpha n_1}{k}$.

First, we construct blocks of the required GDD by taking union of the i-th element of G_2 with each block of the i-th class for $i=1,2,...,\frac{r}{\alpha}$. These blocks provide blocks of size $k+1$, where pairs from G_1 occur λ times and pairs from different groups occur α times. Then, these blocks together with the blocks of a BIBD$(n_2=\frac{r}{\alpha},k+1,\lambda)$ on the points of G_2 provide the blocks for GDD$(n_1,n_2=\frac{r}{\alpha},k+1;\lambda,\alpha)$. □

Example 4.2.1. As a RBIBD$(9,3,3)$ with replication number 12 and a BIBD$(12,4,3)$ exist, we have a GDD$(9,12,4;3,1)$.

4.3 Constructions of GDD$(n_1,n_2,4;\lambda,1)$

Theorem 4.3.1. *Suppose a KTS(n_1) with replication number r and a BIBD$(n_2,4,\frac{n_2}{r})$ exist. Then there exists a GDD$(n_1,n_2,4;\frac{n_2}{r},1)$.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively such that a KTS(n_1) with replication number r and a BIBD$(n_2,4,\frac{n_2}{r})$ exist. Then a KTS(n_1) has r parallel classes namely: $\pi_1,\pi_2,...,\pi_r$ each of size $\frac{n_1}{3}$ and $r \mid n_2$. Let $T_1,T_2,...,T_{\frac{n_2}{r}}$ be partitions of G_2 into parts of size r each.

First we construct the blocks of the required GDD by taking union of the i-th element of T_j with each block of π_i for $i=1,2,...,r$ and $j=1,2,...,\frac{n_2}{r}$. These blocks provide blocks of size 4, where pairs from G_1 occur $\frac{n_2}{r}$ times and pairs from different groups occur only once. Then these blocks together with the blocks of a BIBD$(n_2,4,\frac{n_2}{r})$ on points G_2 provides the blocks for GDD$(n_1,n_2,4;\frac{n_2}{r},1)$. □

Theorem 4.3.2. *Suppose a KTS(n_1) with n_2 blocks and replication number r and a BIBD$(n_2,4,\frac{n_1}{3})$ exist. Then there exists a GDD$(n_1,n_2,4;\frac{n_1}{3},1)$.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Since KTS(n_1) is resolvable with replication number r, it has r parallel classes each of size $\frac{n_1}{3}$ namely: $\pi_1,\pi_2,..,\pi_r$. Since $n_2=\frac{n_1}{3}r$, G_2 can be partitioned into $\frac{n_1}{3}$ parts of size r each, say, $T_1,T_2,...,T_{\frac{n_1}{3}}$.

First we construct the blocks of the required GDD by taking union of the i-th element of T_j with each block of π_i for $i=1,2,...,r$ and $j=1,2,...,\frac{n_1}{3}$. These blocks provide blocks of size 4, where pairs from G_1 occur $\frac{n_1}{3}$ times and pairs from different groups occur only once. Then these blocks together with the blocks of a BIBD$(n_2,4,\frac{n_1}{3})$ on points G_2 provides a GDD$(n_1,n_2,4;\frac{n_1}{3},1)$. □

Example 4.3.1. As a BIBD$(15,3,1)$ is resolvable with repliation number 7 and a BIBD $(21,4,21/7=3)$ exists, we have a GDD$(15,21,4;3,1)$

Example 4.3.2. As a BIBD$(9,3,1)$ is resolvable design having 12 blocks and a BIBD$(12, 4, 9/3 = 3)$ exists, we have a GDD$(9,12,4;3,1)$.

4.4 Necessary conditions

Let r_i be the replication number of elements of the i-th group for $i = 1,2$ and b be the required number of blocks if a GDD$(n_1,n_2,k;\lambda_1,\lambda_2)$ exists. By simple counting argument,

$$r_1 = \frac{(n_1-1)\lambda_1 + n_2\lambda_2}{k-1}. \tag{4.4.1}$$

$$r_2 = \frac{n_1\lambda_2 + (n_2-1)\lambda_1}{k-1}. \tag{4.4.2}$$

$$b = \frac{[n_1(n_1-1)+n_2(n_2-1)]\lambda_1 + \lambda_2(2n_1n_2)}{k(k-1)}. \tag{4.4.3}$$

Theorem 4.4.1. *A necessary condition for the existence of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ is $b \geq max(2r_i - \lambda_1)$.*

Proof. Let (x,y) be a first associate pair from G_i for some $i = 1,2$. Then as both of them come together λ_1 times, there are r_i blocks containing x and $r_i - \lambda_1$ blocks which contain y but do not contain x. So the number of blocks must be at least $2r_i - \lambda_1$ to accommodate x and y r_i times in the design. □

Theorem 4.4.2. *A necessary condition for the existence of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ is $b \geq r_1 + r_2 - \lambda_2$.*

Proof. Let (x,y) be a second associate pair, say $x \in G_1$ and $y \in G_2$. As both of them come together λ_2 times, there are $r_1 + r_2 - \lambda_2$ blocks containing x or y but do not contain both x and y. So the number of blocks must be at least $r_1 + r_2 - \lambda_2$. □

Suppose a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ exists. Then the number of first associate pairs equals $\binom{n_1}{2}\lambda_1 + \binom{n_2}{2}\lambda_1 = \frac{n_1^2+n_2^2-(n_1+n_2)}{2}\lambda_1$ and the number of second associate pairs equals $(n_1n_2)\lambda_2$.

As the block size is 4, there are three types of block configurations: $(4,0)$, $(3,1)$ and $(2,2)$. Let x, y and z be the number of blocks of type $(4,0)$, $(3,1)$ and $(2,2)$ respectively. Note $x+y+z=b$ and they contribute $6x+3y+2z$ first associate pairs in the design. Hence,

$$6x+3y+2z=\frac{n_1^2+n_2^2-(n_1+n_2)}{2}\lambda_1. \tag{4.4.4}$$

Similarly, counting second associate pairs,

$$3y+4z=n_1n_2\lambda_2. \tag{4.4.5}$$

From Equations (4.4.4) and (4.4.5), we have
$2z-6x=n_1n_2\lambda_2-\frac{n_1^2+n_2^2-(n_1+n_2)}{2}\lambda_1\leq 2z\leq 2b$. Using Equation (4.4.3), we obtain $\lambda_2\leq \frac{n_1^2+n_2^2-(n_1+n_2)}{n_1n_2}\lambda_1$. Thus, we have the following theorem.

Theorem 4.4.3. *A necessary condition for the existence of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ is $\lambda_2\leq \frac{n_1^2+n_2^2-(n_1+n_2)}{n_1n_2}\lambda_1$.*

As every block of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ contains at least two first associate pairs, hence $[\binom{n_1}{2}+\binom{n_2}{2}]\lambda_1\geq 2b$. Thus, we have

Theorem 4.4.4. *A necessary condition for the existence of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ is $b\leq \frac{n_1^2+n_2^2-(n_1+n_2)}{4}\lambda_1$.*

Example 4.4.1. A GDD$(7,10,4;1,3)$ does not exist as $\frac{n_1^2+n_2^2-(n_1+n_2)}{4}\lambda_1=33$ which is less than the required number of blocks $b=46$.

Theorem 4.4.5. *A necessary condition for the existence of a GDD$(n_1,n_2,4;\lambda_1,\lambda_2)$ is $r_1\geq\lambda_1$ for $n_1>1$.*

Proof. Let $x\in G_1$, as a first associate pair occurs λ_1 times, $r_1\geq\lambda_1$ for $n_1>1$. □

Example 4.4.2. A GDD$(2,5,4;4,1)$ does not exists as $r_1=3<4=\lambda_1$.

Suppose $\lambda_1 = \lambda$ and $\lambda_2 = 1$. Then from Equations (4.4.1)–(4.4.3), we have

$$r_1 = \frac{(n_1 - 1)\lambda + n_2}{3}. \tag{4.4.6}$$

$$r_2 = \frac{n_1 + (n_2 - 1)\lambda}{3}. \tag{4.4.7}$$

$$b = \frac{[n_1(n_1 - 1) + n_2(n_2 - 1)]\lambda + 2n_1n_2}{12}. \tag{4.4.8}$$

Since r_1 and r_2 must be integers, from equations (4.4.6) and (4.4.7), we have the following.

i. If $\lambda \equiv 0 \pmod 3$, then $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.

ii. If $\lambda \equiv 1 \pmod 3$, $n_1 + n_2 \equiv 1 \pmod 3$.

iii. If $\lambda \equiv 2 \pmod 3$, then there are no integer value for n_1 and n_2 that makes r_1 and r_2 integers.

Since b must be an integer, from equation (4.4.8), we have the following.

i. If $\lambda \equiv 0 \pmod 6$, then $n_1n_2 \equiv 0 \pmod 6$.

ii. If $\lambda \equiv 1 \pmod 6$, then $n_1 + n_2 \equiv 0, 1, 4, 9 \pmod{12}$.

iii. If $\lambda \equiv 2 \pmod 6$, then $(n_1 + n_1)(n_1 + n_2 - 1) - n_1n_2 \equiv 0 \pmod 6$.

iv. If $\lambda \equiv 3 \pmod 6$, then $n_1 + n_2 \equiv 0, 1 \pmod 4$ and $n_1n_2 \equiv 0 \pmod 3$.

v. If $\lambda \equiv 4 \pmod 6$, then $n_1 + n_2 \equiv 0, 1 \pmod 3$ and $n_1n_2 \equiv 0 \pmod 2$.

vi. If $\lambda \equiv 5 \pmod 6$), then $5(n_1 + n_2)(n_1 + n_2 - 1) - 8n_1n_2 \equiv 0 \pmod{12}$.

Combining the cases above, we obtain the following necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda, 1)$.

1. $\lambda \equiv 0 \pmod 6$: $n_1 n_2 \equiv 0 \pmod 6$, $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.

2. $\lambda \equiv 1 \pmod 6$: $n_1 + n_2 \equiv 1,4 \pmod{12}$.

3. $\lambda \equiv 3 \pmod 6$: $n_1 + n_2 \equiv 0,9 \pmod{12}$ and $n_1 n_2 \equiv 0 \pmod 3$.

4. $\lambda \equiv 4 \pmod 6$: $n_1 + n_2 \equiv 1 \pmod 3$ and $n_1 n_2 \equiv 0 \pmod 2$.

4.5 Sufficient conditions for the existence of GDD$(n_1, n_2, 4; \lambda, 1)$ for certain values of n_1, n_2 and λ

Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 1 \pmod 6$, from the necessary conditions, $n_1 + n_2 \equiv 1,4 \pmod{12}$. Then by Theorem 2.1.6, a BIBD$(n_1 + n_2, 4, 1)$ exists. Also when $n_i \geq 4$ for $i = 1,2$, a BIBD$(n_i, 4, 6)$ exists. So the blocks of a BIBD$(n_1 + n_2, 4, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i, 4, 6)$ on points of G_i provide the blocks of a GDD$(n_1, n_2, 4; 6s+1, 1)$, where s is a nonnegative integer. Thus, we have the following:

Theorem 4.5.1. *Necessary conditions are sufficient for the existence of a GDD*$(n_1, n_2, 4; \lambda, 1)$ *when* $\lambda \equiv 1 \pmod 6$ *and* $n_i \geq 4$ *for* $i = 1,2$.

Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 4 \pmod 6$, from the necessary conditions, $n_1 + n_2 \equiv 1 \pmod 3$ and $n_1 n_2 \equiv 0 \pmod 2$. But $n_1 + n_2 \equiv 1 \pmod 3$ implies $n_1 + n_2 \equiv 1,4,7,10 \pmod{12}$. When $n_1 + n_2 \equiv 1,4 \pmod{12}$, by Theorem 2.1.6, a BIBD$(n_1 + n_2, 4, 1)$ exists. As $n_1 + n_2 \equiv 1,4 \pmod{12}$ implies $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$, and $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$, also by Theorem 2.1.6 a BIBD$(n_i, 4, 3)$ for $i = 1,2$ exists. Since a BIBD$(n_i, 4, 6)$ for $n_i \geq 4$ exists, the blocks of a BIBD$(n_1 + n_2, 4, 1)$ on points of $G_1 \cup G_2$ together with the blocks of a BIBD$(n_i, 4, 3)$ and s copies of the blocks of a BIBD$(n_i, 4, 6)$ on points of G_i provide the blocks of a GDD$(n_1, n_2, 4; 6s+4, 1)$, where s is a nonnegative integer. Thus, we have the following:

Theorem 4.5.2. *A GDD*$(n_1, n_2, 4; \lambda, 1)$ *exists for* $\lambda \equiv 4 \pmod 6$ *when* $n_1 \equiv 0 \pmod{12}$ *and* $n_2 \equiv 1, 4 \pmod{12}$*, when* $n_1 \equiv 9 \pmod{12}$ *and* $n_2 \equiv 4 \pmod{12}$*, and when* $n_1 \equiv 8 \pmod{12}$ *and* $n_2 \equiv 5, 8 \pmod{12}$.

Theorem 4.5.3. *A GDD*$(n_1, n_2, 4; \lambda, 1)$ *exists for* $\lambda \equiv 3 \pmod 6$ *when* $n_1 = 24t + 9$ *and* $n_2 = 36t + 12$*, and when* $n_1 = 24t + 15$ *and* $n_2 = 36t + 21$*, where* t *is a nonnegative integer.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 24t + 9$ for a nonnegative integer t. Then a KTS$(n_1, 3, 1)$ with replication number $r = 12t + 4$ exists. As a BIBD$(36t + 12, 4, \frac{36t+12}{r} = 3)$ exists, by Theorem 4.3.1, a GDD$(n_1, n_2, 4; 3, 1)$ exists. Since a BIBD$(n_i, 4, 6)$ for $i = 1, 2$ exists, the blocks of a GDD$(n_1, n_2, 4; 3, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i, 4, 6)$ on points of G_i provide the blocks of a GDD$(n_1, n_2, 4; 3 + 6s, 1)$ for a nonnegative integer s. Similarily, by Theorems 4.3.1 and 2.1.6 we can construct a GDD$(n_1, n_2, 4; 6s + 3, 1)$ for $n_1 = 24t + 15$ and $n_2 = 36t + 21$, where t is a nonnegative integer. □

Theorem 4.5.4. *A GDD*$(n_1, n_2, 4; \lambda, 1)$ *exists for* $\lambda \equiv 4 \pmod 6$ *when* $n_1 = 6t + 3$ *and* $n_2 = 12t + 4$*, where* t *is a nonnegative integer.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 6t + 3$ for a nonnegative integer t. Then a KTS$(n_1, 3, 1)$ with replication number $r = 3t + 1$ exists. As a BIBD$(12t + 4, 4, \frac{12t+4}{r} = 4)$ exists, by Theorem 4.3.1, a GDD$(n_1, n_2, 4; 4, 1)$ exists. Since a BIBD$(n_i, 4, 6)$ for $i = 1, 2$ and $n_1 > 3$ exists, the blocks of a GDD$(n_1, n_2, 4; 4, 1)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i, 4, 6)$ on points of G_i provide the blocks of a GDD$(n_1, n_2, 4; 4 + 6s, 1)$ for a nonnegative integer s. □

Theorem 4.5.5. *A GDD*$(n_1, n_2, 4; \lambda, 1)$ *exists for* $\lambda \equiv 0 \pmod 6$ *when* $n_1 = 6t + 3$ *and* $n_2 = 18t + 6$*, and when* $n_1 = 12t + 9$ *and* $n_2 = 36t + 24$*, where* t *is a nonnegative integer.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 6t + 3$ for a nonnegative integer t. Then a KTS$(n_1, 3, 1)$ with replication number $r = 3t + 1$ exists. As a BIBD$(18t + 6, 4, \frac{18t+6}{r} = 6)$ exists, by Theorem 4.3.1, a GDD$(n_1, n_2, 4; 6, 1)$ exists. Since a BIBD$(n_i, 4, 6)$ for $i = 1, 2$ and $n_1 > 3$ exists, the blocks of a GDD$(n_1, n_2, 4; 6, 1)$ on points

of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i, 4, 6)$ on points of G_i provide the blocks of a GDD$(n_1, n_2, 4; 6+6s, 1)$ for a nonnegative integer s. Similarily, by Theorems 4.3.1 and 2.1.6 we can construct a GDD$(n_1, n_2, 4; 6s+6, 1)$ for $n_1 = 12t+9$ and $n_2 = 36t+24$, where t is a nonnegative integer. □

4.6 Construction of GDD$(n_1, n_2, 4; \lambda, 2)$

Theorem 4.6.1. *Suppose a 2-resolvable BIBD$(n_1, 3, 2)$ with replication number r and a BIBD$(n_2, 4, \frac{4n_2}{r})$ exist. Then there exists a GDD$(n_1, n_2, 4; \frac{4n_2}{r}, 2)$.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively such that a 2-resolvable BIBD$(n_1, 3, 2)$ with replication number r and a BIBD$(n_2, 4, \frac{4n_2}{r})$ exist. Then the blocks of a BIBD$(n_1, 3, 2)$ is partitioned into classes in which each point occurs 2 times. Since each class contains a point of G_1 2 times, the blocks of a BIBD$(n_1, 3, 2)$ are partitioned into $\frac{r}{2}$ classes namely: π_1, π_2,..., $\pi_{\frac{r}{2}}$ each of size $\frac{2n_1}{3}$. Let G_2 be partitioned into parts each of size $\frac{r}{2}$. Then there are $\frac{2n_2}{r}$ partitions of G_2 namely: T_1, T_2, ..., $T_{\frac{2n_2}{r}}$ each of size $\frac{r}{2}$.
First we construct the blocks of the required GDD by taking union of the i-th element of T_j with each block of π_i for $i = 1, 2, ..., \frac{r}{2}$ and $j = 1, 2, ..., \frac{2n_2}{r}$. These blocks provide blocks of size 4, where pairs from G_1 occur $\frac{4n_2}{r}$ times and pairs from different groups occur twice. Then these blocks together with the blocks of a BIBD$(n_2, 4, \frac{4n_2}{r})$ on points of G_2 provides the blocks for GDD$(n_1, n_2, 4; \frac{4n_2}{r}, 2)$.

□

4.7 Necessary conditions for GDD$(n_1, n_2, 4; \lambda, 2)$

For $\lambda_1 = \lambda$ and $\lambda_2 = 2$, from Equations (4.4.1)–(4.4.3), we have the following:

$$r_1 = \frac{(n_1 - 1)\lambda + 2n_2}{3}. \tag{4.7.1}$$

$$r_2 = \frac{2n_1 + (n_2 - 1)\lambda}{3}. \tag{4.7.2}$$

$$b = \frac{[n_1(n_1 - 1) + n_2(n_2 - 1)]\lambda + 4n_1n_2}{12}. \tag{4.7.3}$$

Since r_1 and r_2 must be integers, from equations (4.7.1) and (4.7.2), we have the following.

i. If $\lambda \equiv 0 \pmod 3$, then $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.

ii. If $\lambda \equiv 1 \pmod 3$, then there is no integer value for n_1 and n_2 that makes r_1 and r_2 integers.

iii. If $\lambda \equiv 2 \pmod 3$, then $n_1 + n_2 \equiv 1 \pmod 3$.

Since b must be an integer, from equation (4.7.3), we have the following.

i. If $\lambda \equiv 0 \pmod 6$, then $n_1n_2 \equiv 0 \pmod 3$.

ii. If $\lambda \equiv 1 \pmod 6$, then $(n_1 + n_2)(n_1 + n_2 - 1) + 2n_1n_2 \equiv 0 \pmod{12}$.

iii. If $\lambda \equiv 2 \pmod 6$, then $n_1 + n_2 \equiv 0, 1, 3, 4 \pmod 6$.

iv. If $\lambda \equiv 3 \pmod 6$, then $3(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1n_2 \equiv 0 \pmod{12}$.

v. If $\lambda \equiv 4 \pmod 6$, then $(n_1 + n_2)(n_1 + n_2 - 1) - n_1n_2 \equiv 0 \pmod 3$.

vi. If $\lambda \equiv 5 \pmod 6$, then $5(n_1 + n_2)(n_1 + n_2 - 1) - 6n_1n_2 \equiv 0 \pmod{12}$.

Combining the cases above, we obtain the following necessary conditions for the existence of a GDD$(n_1, n_2, 4; \lambda, 2)$.

1. $\lambda \equiv 0 \pmod 6$: $n_1n_2 \equiv 0 \pmod 3$, $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.

2. $\lambda \equiv 2 \pmod 6$: $n_1 + n_2 \equiv 1, 4 \pmod 6$.

3. $\lambda \equiv 3 \pmod 6$: $3(n_1 + n_2)(n_1 + n_2 - 1) - 2n_1n_2 \equiv 0 \pmod{12}$, $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 0 \pmod 3$.

4. $\lambda \equiv 5 \pmod 6$: $5(n_1+n_2)(n_1+n_2-1)-6n_1n_2 \equiv 0 \pmod{12}$ and $n_1+n_2 \equiv 1,4 \pmod 6$.

4.8 Sufficient conditions for the existence of GDD$(n_1, n_2, 4; \lambda, 2)$ for certain values of n_1, n_2 and λ

Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 2 \pmod 6$, from the necessary conditions, $n_1+n_2 \equiv 1,4 \pmod 6$. But $n_1+n_2 \equiv 1,4 \pmod 6$ implies $n_1+n_2 \equiv 1 \pmod 3$. Then by Theorem 2.1.6, a BIBD$(n_1+n_2,4,2)$ exists. Also when $n_i \geq 4$ for $i=1,2$, a BIBD$(n_i,4,6)$ exists. So the blocks of a BIBD$(n_1+n_2,4,2)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i,4,6)$ on points of G_i provide the blocks of a GDD$(n_1,n_2,4;6s+2,2)$, where s is a nonnegative integer. Thus, we have the following:

Theorem 4.8.1. *Necessary conditions are sufficient for the existence of a GDD*$(n_1, n_2, 4; \lambda, 2)$ *when* $\lambda \equiv 2 \pmod 6$ *and* $n_i \geq 4$ *for* $i = 1,2$.

Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. When $\lambda \equiv 5 \pmod 6$, from the necessary conditions, $5(n_1+n_2)(n_1+n_2-1)-6n_1n_2 \equiv 0 \pmod{12}$ and $n_1+n_2 \equiv 1,4 \pmod 6$. As $n_1+n_2 \equiv 1,4 \pmod 6$ implies $n_1+n_2 \equiv 1 \pmod 3$, a BIBD$(n_1+n_2,4,2)$ exists. As $n_1+n_2 \equiv 1,4 \pmod 6$ implies $n_1+n_2 \equiv 1,4,7,10 \pmod{12}$, for $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, and $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$, a BIBD$(n_i,4,3)$ exists. Since a BIBD$(n_i,4,6)$ for $n_i \geq 4$ exists, the blocks of a BIBD$(n_1+n_2,4,2)$ on points of $G_1 \cup G_2$ together with the blocks of a BIBD$(n_i,4,3)$ and s copies of the blocks of a BIBD$(n_i,4,6)$ on points of G_i provide the blocks of a GDD$(n_1,n_2,4;6s+5,1)$, where s is a nonnegative integer. Thus, we have the following:

Theorem 4.8.2. *A GDD*$(n_1,n_2,4;\lambda,2)$ *exists for* $\lambda \equiv 5 \pmod 6$ *when* $n_1 \equiv 0 \pmod{12}$ *and* $n_2 \equiv 1,4 \pmod{12}$, *when* $n_1 \equiv 9 \pmod{12}$ *and* $n_2 \equiv 1,4 \pmod{12}$, *and when* $n_1 \equiv 8 \pmod{12}$ *and* $n_2 \equiv 5,8 \pmod{12}$.

Theorem 4.8.3. *A GDD$(n_1,n_2,4;\lambda,2)$ exists for $\lambda \equiv 0 \pmod 6$ when $n_1 = 24t+9$ and $n_2 = 36t+12$, and when $n_1 = 24t+3$ and $n_2 = 36t+3$ for $n_1, n_2 \neq 3$, where t is a nonnegative integer.*

Proof. Let G_1 and G_2 be groups of sizes n_1 and n_2 respectively. Let $n_1 = 24t+9$ for a nonnegative integer t. Then a 2-resolvable BIBD$(n_1,3,2)$ with replication number $r = 24t+8$ exists. As a BIBD$(36t+12,4,4\frac{36t+12}{r} = 6)$ exists, by Theorem 4.3.1, a GDD$(n_1,n_2,4;6,2)$ exists. Sinse a BIBD$(n_i,4,6)$ for $i = 1,2$ exists, the blocks of a GDD$(n_1,n_2,4;6,2)$ on points of $G_1 \cup G_2$ together with s copies of the blocks of a BIBD$(n_i,4,6)$ on points of G_i provide the blocks of a GDD$(n_1,n_2,4;6s+6,2)$ for a nonnegative integer s. Similarily, by Theorems 4.3.1 and 2.1.6 we can construct a GDD$(n_1,n_2,4;6s+6,2)$ for $n_1 = 24t+3$ and $n_2 = 36t+3$ where $n_1, n_2 \neq 3$ and t is a nonnegative integer. □

By Theorem 4.2.1, the following families of GDD exist.

Theorem 4.8.4. *For any positive integer t and nonnegative integer s a GDD$(12s+9,(36s+24)t,4;12t,2)$, a GDD$(24t-9,108t-45,4;18t,2)$, a GDD$(12t,36t-3,4;12,2)$, a GDD$(3,6t,4;12t,2)$, a GDD$(3,12t+3,4;24t+6,2)$, a GDD$(3,12t+6,4;24t+12,2)$ and a GDD$(3,12t+10,4;24t+20,2)$ exist.*

Example 4.8.1. A GDD$(3,3,4;3,2)$ with groups $G_1 = \{1,2,3\}$ and $G_2 = \{a,b,c\}$. The collection of blocks is $\{\{1,2,3,a\},\{1,2,3,b\},\{1,2,3,c\},\{1,a,b,c\},\{2,a,b,c\},\{3,a,b,c\}\}$.

For some positive integers n_1, n_2 with $n_1 = n_2$, we can construct recursively a GDD$(n_1,n_2,4;3,2)$.

Lemma 4.8.5. *Let $n = n_1n_2$ and if a GDD$(n_1,n_1,n_1,n_1,4;0,1)$ and a GDD$(n_2,n_2,n_2,n_2,4;0,1)$ exist, then there exist a GDD$(n,n,n,n,4;0,1)$.*

Proof. Let G_1, G_2, G_3 and G_4 be groups of size n each. Partition each group G_i into n_1 parts G_{i1}, G_{i2}, ..., G_{in_1} of size n_2 each for $i = 1,2,3,4$. Consider
$G'_1 = \{G_{11},G_{12},...,G_{1n_1}\}$, $G'_2 = \{G_{21},G_{22},...,G_{2n_1}\}$, $G'_3 = \{G_{31},G_{32},...,G_{3n_1}\}$, and $G'_4 = \{G_{41},G_{42},...,G_{4n_1}\}$. Then by the hypothesis, a GDD$(n_1,n_1,n_1,n_1,4;0,1)$ exists with groups

G'_1, G'_2, G'_3 and G'_4. This GDD provides n_1^2 blocks of the form $\{G_{11}, G_{21}, G_{31}, G_{41}\}$, $\{G_{11}, G_{22}, G_{32}, G_{42}\}$, etc.

From each block of GDD$(n_1, n_1, n_1, n_1, 4; 0, 1)$, construct a GDD$(n_2, n_2, n_2, n_2, 4; 0, 1)$ where the groups are elements of the block. As a GDD$(n_2, n_2, n_2, n_2, 4; 0, 1)$ exists, there are $n_1^2(n_2^2) = (n_1 n_2)^2 = n^2$ blocks of type $(1, 1, 1, 1)$ which are the blocks of a GDD$(n, n, n, n, 4; 0, 1)$. □

Kreher and Stinson (1997), investigated the existence of group divisible designs with block size 4, $\lambda_1 = 0$ and $\lambda_2 = 1$, having up to 30 points. They listed such GDDs with groups of different types that exist or do not exist. The following GDDs are from the list that exist: GDD$(3, 3, 3, 3, 4; 0, 1)$, GDD$(4, 4, 4, 4, 4; 0, 1)$, GDD$(5, 5, 5, 5, 4; 0, 1)$ and GDD$(7, 7, 7, 7, 4; 0, 1)$. So by Lemma 4.8.5, we can construct GDD$(9, 9, 9, 9, 4; 0, 1)$ GDD$(12, 12, 12, 12, 4; 0, 1)$, GDD$(15, 15, 15, 15, 4; 0, 1)$ GDD$(16, 16, 16, 16, 4; 0, 1)$, GDD $(20, 20, 20, 20, 4; 0, 1)$, GDD$(21, 21, 21, 21, 4; 0, 1)$,GDD$(25, 25, 25, 25, 4; 0, 1)$, GDD$(28, 28, 28, 28, 4; 0, 1)$, GDD$(35, 35, 35, 35, 4; 0, 1)$ and GDD$(49, 49, 49, 49, 4; 0, 1)$.

Theorem 4.8.6. *For $n \equiv 0, 3 \pmod{12}$, if a GDD$(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}, \frac{n}{3}, 4; 0, 1)$ exists,then there exists a GDD$(n, n, 4; 3, 2)$.*

Proof. Let $n = 12t$ for $t > 0$. Let G_1 and G_2 be groups of size $12t$ each. Partition each group G_i into three subsets G_{i1}, G_{i2}, G_{i3} of size $4t$ each for $i = 1, 2$. Consider $G'_1 = \{G_{11}, G_{12}, G_{13}\}$ and $G''_2 = \{G_{21}, G_{22}, G_{23}\}$. By Example **??**, a GDD$(3, 3, 4; 3, 2)$ with groups G'_1 and G''_2 exists. The blocks are $\{\{G_{11}, G_{12}, G_{13}, G_{21}\}, \{G_{11}, G_{12}, G_{13}, G_{22}\}, \{G_{11}, G_{12}, G_{13}, G_{23}\}, \{G_{11}, G_{21}, G_{22}, G_{23}\}, \{G_{12}, G_{21}, G_{22}, G_{23}\}, \{G_{13}, G_{21}, G_{22}, G_{23}\}\}$. From each block of GDD $(3, 3, 4; 3, 2)$, construct a GDD$(4t, 4t, 4t, 4t, 4; 0, 1)$ where the groups are elements of the block. By hypothesis as GDD$(4t, 4t, 4t, 4t, 4; 0, 1)$ exists, there are $6(16t^2) = 96t^2$ blocks in which first associate pairs occur 3 times and second associate pairs occur twice. Thus the above blocks constructed so far together with the blocks of a BIBD$(4t, 4, 3)$ on each G_{ij} for $i = 1, 2$ and $j = 1, 2, 3$ provide a GDD$(12t, 12t, 4; 3, 2)$. Similarly, we can construct a GDD$(12t + 3, 12t + 3, 4; 3, 2)$ for $t > 0$ if a GDD$(4t + 1, 4t + 1, 4t + 1, 4t + 1, 4; 0, 1)$ exists. □

Example 4.8.2. GDD$(12,12,4;3,2)$, GDD$(15,15,4;3,2)$, GDD$(36,36,4;3,2)$, GDD$(48,48,4;3,2)$, GDD$(60,60,4;3,2)$ and GDD$(63,63,4;3,2)$ exist.

4.9 Summary

We presented an important general construction for GDDs with two groups of sizes n_1 and n_2 with block size k. Further, we obtained general constructions for such GDDs with block size 4 and $\lambda_2 = 1,2$. First, we obtained necessary conditions for the existence of a GDD$(n_1,n_2,4;\lambda,1)$ and proved that they were sufficient for

- $\lambda \equiv 0 \pmod 6$ when $n_1 = 6t+3$ and $n_2 = 18t+6$, and when $n_1 = 12t+9$ and $n_2 = 36t+24$, where t is a nonnegative integer.
- $\lambda \equiv 1 \pmod 6$ when $n_i \geq 4$ for $i = 1,2$.
- $\lambda \equiv 3 \pmod 6$ when $n_1 = 24t+6$ and $n_2 = 36t+12$, and when $n_1 = 24t+15$ and $n_2 = 36t+21$, where t is a nonnegative integer.
- $\lambda \equiv 4 \pmod 6$ when $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, when $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$, when $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$, and when $n_1 \equiv 3 \pmod 6$ and $n_2 \equiv 4 \pmod{12}$.

Next, we obtained necessary conditions for the existence of a GDD$(n_1,n_2,4;\lambda,2)$ and proved that they were sufficient for

- $\lambda \equiv 0 \pmod 6$ when $n_1 = 24t+9$ and $n_2 = 36t+12$ and when $n_1 = 24t+3$ and $n_2 = 36t+3$ for $n_1,n_2 \neq 3$, where t is a nonnegative integer.
- $\lambda \equiv 2 \pmod 6$ when $n_i \geq 4$ for $i = 1,2$.
- $\lambda \equiv 5 \pmod 6$ when $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, when $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$, and when $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$.

Chapter 5

Group divisible designs with 3 groups of sizes 1, n, n and block size 4

5.1 Introduction

In this chapter, we study the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ with three groups G_1, G_2, G_3 of sizes 1, n, and n respectively.

In general, when the number of groups is less than the block size the work is more involved and possibly making them harder to construct. It is well known that GDDs are used as a building block for BIBDs, but the converse is also true, for example, an easy observation is the following result.

Theorem 5.1.1. *If a BIBD$(n_1+n_2+...+n_m,k,\lambda_2)$ and a BIBD(n_i,k,λ_1) exist for $i=1,2,...m$, then a GDD$(n_1,n_2,...,n_m,k,\lambda_1+\lambda_2,\lambda_2)$ exists.*

Proof. Let G_i be a group of size n_i for $i=1,2,...,m$. Then the blocks of a BIBD$(n_1+n_2+...+n_m,k,\lambda_2)$ on $G_1\cup G_2\cup...\cup G_m$ along with the blocks of a BIBD(n_i,k,λ_1) on each G_i if they exist provide a GDD$(n_1,n_2,...,n_m,k;\lambda_1+\lambda_2,\lambda_2)$. □

Corollary 5.1.2. *If a BIBD$(mn+1,4,\lambda_2)$ and a BIBD$(n,4,\lambda)$ exist, then a GDD$(1,n,n,...,n,4,\lambda_1=\lambda_2+\lambda,\lambda_2)$ exists.*

The converse of the above corollary is not true, for example, we show in Section 5.5 that a GDD$(1,n,n,4;\lambda_2+\lambda,\lambda_2)$ exist for $n=2$ or $n=3$ but clearly a BIBD$(2,4,\lambda)$ or BIBD$(3,4,\lambda)$ does not exist. One can find such examples for larger values of n by using the construction given in the next section. For example, the construction gives a GDD$(1,7,7,4;9,6)$ but a BIBD$(7,4,3)$ does not exist.

Another observation gives,

Theorem 5.1.3. *A GDD$(n_1, n_2, ..., n_m, k, \lambda_1, 0)$ exists if and only if a BIBD(n_i, k, λ_1) exist for $i = 1, 2, ...m$.*

Corollary 5.1.4. *A GDD$(1, n, n, ..., n, k, \lambda_1 = \lambda, 0)$ exists if and only if a BIBD(n, k, λ) exists.*

One may notice that it is much easier to construct GDDs with $\lambda_1 \geq \lambda_2$, especially when a BIBD(n, k, λ) exists. In this chapter, first we present an important construction technique which produces GDDs when $\lambda_1 < \lambda_2$ as well as $\lambda_1 \geq \lambda_2$. Next we obtain necessary conditions for the existence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ and prove that these necessary conditions are sufficient for specific values of n, specific values of λ_1 and for $n \equiv 0, 1, 4, 5, 8, 9 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.

5.2 New construction of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$

A K_n on G_i means the vertices of the complete graph K_n are labeled with the elements of G_i for $i = 2, 3$. Let n be even. Then the complete graph K_n on G_2 (respectively on G_3) has a 1-factorization, say $\{E_1, E_2, ..., E_{n-1}\}$ (respectively $\{F_1, F_2, ..., F_{n-1}\}$). For $x = 1, 2, ..., n-1$, if $E_x = \{e_1, e_2, ..., e_{\frac{n}{2}}\}$ and $F_x = \{f_1, f_2, ..., f_{\frac{n}{2}}\}$, then we can form blocks $e_l \cup f_m$ of size 4, for $1 \leq l, m \leq \frac{n}{2}$.

On the other hand when n is odd, a K_n on G_2 (respectively on G_3) has a 2-factorization, say $\{E_1, E_2, ..., E_{\frac{n-1}{2}}\}$ (respectively $\{F_1, F_2, ..., F_{\frac{n-1}{2}}\}$). For $x = 1, 2, ..., \frac{n-1}{2}$, if $E_x = \{e_1, e_2, ..., e_n\}$ and $F_x = \{f_1, f_2, ..., f_n\}$, then we can form blocks $e_l \cup f_m$ of size 4, for $1 \leq l, m \leq n$. Now we define

$\mathscr{B}_4 = \{e_l \cup f_m : e_l \in E_x$ and $f_m \in F_x$ for $x = 1, 2, ..., n-1$ and $1 \leq l, m \leq \frac{n}{2}\}$ if n is even, and

$\mathscr{B}_4 = \{e_l \cup f_m : e_l \in E_x$ and $f_m \in F_x$ for $x = 1, 2, ..., \frac{n-1}{2}$ and $1 \leq l, m \leq n\}$ if n is odd.

Theorem 5.2.1. *Suppose a BIBD$(2n, 3, \lambda)$ and a BIBD$(n, 3, \mu)$ exist.*

(a) Suppose n be even and there are nonnegative integers i, j, u and v such that

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + v(n-1). \tag{5.2.1}$$

Then there exists a GDD$(1,n,n,4; i\lambda + j\mu + un\mu + vn/2, \frac{i(2n-1)\lambda+j\mu(n-1)}{2})$.

(b) Suppose n be odd and there are nonnegative integers i, j, u and v such that

$$\frac{i(2n-1)\lambda + j\mu(n-1)}{2} = i\lambda + u\mu(n-1) + 2v(n-1). \tag{5.2.2}$$

Then there exists a GDD$(1,n,n,4; i\lambda + j\mu + un\mu + vn, \frac{i(2n-1)\lambda+j\mu(n-1)}{2})$.

Proof. Let $G_1 = \{x\}$, $G_2 = \{a_1, a_2, ..., a_n\}$ and $G_3 = \{b_1, b_2, ..., b_n\}$. Then consider the following sets:

- $\mathscr{B}_1 = \{G_1 \cup B : B \text{ is a block of BIBD}(2n, 3, \lambda) \text{ on } G_2 \cup G_3\}$;
- $\mathscr{B}_2 = \{G_1 \cup B : B \text{ is a block of BIBD}(n, 3, \mu) \text{ on } G_2 \text{ and } G_3\}$;
- $\mathscr{B}_3 = \{\{a\} \cup B : a \in G_i \text{ and } B \text{ is a block of BIBD}(n, 3, \mu) \text{ on } G_j \text{ for } i, j = 2, 3 \text{ and } i \neq j\}$.

In $\mathscr{B}_1$, every element from $G_2 \cup G_3$ comes with the point of G_1 $\frac{\lambda(2n-1)}{2}$ times and every pair of elements from $G_2 \cup G_3$ comes λ times. In $\mathscr{B}_2$, every element from G_2 and G_3 comes with the point of G_1 $\frac{\mu(n-1)}{2}$ times and first associate pair from G_2 and G_3 comes μ times. In $\mathscr{B}_3$, first associate pair from G_2 and G_3 comes μn times and second associate pair from G_2 and G_3 comes $\mu(n-1)$ times. In $\mathscr{B}_4$, first and second associate pairs from G_2 and G_3 occur n and $2(n-1)$ times respectively if n is odd while first and second associate pairs from G_2 and G_3 occur $\frac{n}{2}$ and $n-1$ times respectively if n is even.

Suppose we have i copies of $\mathscr{B}_1$, j copies of $\mathscr{B}_2$, u copies of $\mathscr{B}_3$ and v copies of $\mathscr{B}_4$. Then the following matrix displays the replication number of each pair (a_1, x), (a_1, a_2) and (a_1, b_1) in $i\mathscr{B}_1$, $j\mathscr{B}_2$, $u\mathscr{B}_3$ and $v\mathscr{B}_4$, where i, j, u and v are any nonnegative integers.

For n even,

$$
\begin{array}{c}
 \\
i\mathscr{B}_1 \\
j\mathscr{B}_2 \\
u\mathscr{B}_3 \\
v\mathscr{B}_4
\end{array}
\begin{array}{c}
\begin{array}{ccc}
(a_1,x) & (a_1,a_2) & (a_1,b_1)
\end{array} \\
\left(\begin{array}{ccc}
\frac{i(2n-1)\lambda}{2} & i\lambda & i\lambda \\
\frac{j(n-1)\mu}{2} & j\mu & 0 \\
0 & un\mu & u\mu(n-1) \\
0 & vn/2 & v(n-1)
\end{array}\right)
\end{array}.
$$

So we have a GDD$(1,n,n,4;i\lambda+j\mu+un\mu+vn/2,\frac{i(2n-1)\lambda+j\mu(n-1)}{2})$ when

$$\frac{i(2n-1)\lambda+j\mu(n-1)}{2} = i\lambda+u\mu(n-1)+v(n-1). \tag{5.2.3}$$

For n odd,

$$
\begin{array}{c}
 \\
i\mathscr{B}_1 \\
j\mathscr{B}_2 \\
u\mathscr{B}_3 \\
v\mathscr{B}_4
\end{array}
\begin{array}{c}
\begin{array}{ccc}
(a_1,x) & (a_1,a_2) & (a_1,b_1)
\end{array} \\
\left(\begin{array}{ccc}
\frac{i(2n-1)\lambda}{2} & i\lambda & i\lambda \\
\frac{j(n-1)\mu}{2} & j\mu & 0 \\
0 & un\mu & u\mu(n-1) \\
0 & vn & 2v(n-1)
\end{array}\right)
\end{array}.
$$

So we have a GDD$(1,n,n,4;i\lambda+j\mu+un\mu+vn,\frac{i(2n-1)\lambda+j\mu(n-1)}{2})$ when

$$\frac{i(2n-1)\lambda+j\mu(n-1)}{2} = i\lambda+u\mu(n-1)+2v(n-1). \tag{5.2.4}$$

□

Theorem 5.2.2. *(a) If a BIBD$(2n,b,r,3,\lambda)$ exists for odd n and if $r-\lambda=2(n-1)t$, then a GDD$(1,n,n,4;\lambda+nt,r)$ exists.*

(b) If a BIBD$(2n,b,r,3,\lambda)$ exists for even n and if $r-\lambda=(n-1)t$, then a GDD$(1,n,n,4;\lambda+\frac{nt}{2},r)$ exists.

Proof. Let $\mathscr{B}=\{G_1\cup B : B \text{ is a block of BIBD}(2n,b,r,3,\lambda)\}$. Then t copies of $\mathscr{B}_4$ along with $\mathscr{B}$ give the required GDD. □

Example 5.2.1. As a BIBD$(6,40,20,3,8)$ exists, $r-\lambda=12=4\times3$. We get a GDD$(1,3,3,4;8+3\times3=17,20=(8+3\times4))$ using 3 copies of $\mathscr{B}_4$.

Essentially, $\mathscr{B}_4$ is a GDD on two groups of size n, where the indices depend on n odd or even.

For n even, $\mathscr{B}_4$= GDD$(n,n,4;\lambda_1=\frac{n}{2},\lambda_2=n-1)$ and for n odd, $\mathscr{B}_4$= GDD$(n,n,4;\lambda_1=n,\lambda_2=2(n-1))$. Now the replication number r for a BIBD$(2n,3,\lambda)$ is $\frac{\lambda(2n-1)}{2}$. If we wish $r-\lambda$ to be a multiple of $(n-1)$, say $s(n-1)$ when n is even (respectively $2s(n-1)$ when n is odd), then $\lambda=\frac{2s(n-1)}{2n-3}$ (respectively $\lambda=\frac{4s(n-1)}{2n-3}$). For $s=2n-3$, $\lambda=2(n-1)$ for n even (respectively $\lambda=4(n-1)$ for n odd).

Example 5.2.2. For $n=4$, we have GDD$(1,4,4,4;16,21)$ by using blocks of a BIBD$(8,b,21,3,6)$ and $s=5$ copies of $\mathscr{B}_4$ =GDD$(4,4,4;2,3)$.

Example 5.2.3. For $n=5$, we have GDD$(1,5,5,4;51,72)$ by using blocks of a BIBD$(10,b,72,3,16)$ and $s=7$ copies of $\mathscr{B}_4$ =GDD$(5,5,4;5,8)$.

In general :

Theorem 5.2.3. *For n even, using a BIBD*$(2n,3,2(n-1))$ *and* $2n-3$ *copies of a GDD*$(n,n,4;\frac{n}{2},n-1)$, *a GDD*$(1,n,n,4;\frac{2n^2+n-4}{2},(n-1)(2n-1))$ *and for n odd, using a BIBD*$(2n,3,4(n-1))$ *and* $2n-3$ *copies of a GDD*$(n,n,4;n,2(n-1))$, *a GDD*$(1,n,n,4;2n^2+n-4,2(n-1)(2n-1))$ *exists.*

In the next section we obtain some necessary conditions for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$. Towards this aim, assuming a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ exists, we count the number of blocks, r_i, containing a given element x of G_i for $i=1,2,3$, and the required number of blocks, say b, for the GDD.

5.3 Necessary conditions

Suppose a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ exists with groups G_1, G_2, G_3 of size 1, n, n respectively. Let r_i be the replication number of each element of G_i for $i=1,2,3$. As the size of G_2 is equal to the size of G_3, $r_2=r_3$. Then by counting argument, $r_1=\frac{2n\lambda_2}{3}$ and $r_2=r_3=$

$\frac{\lambda_1(n-1)+\lambda_2(n+1)}{3}$. Let b be the required number of blocks for a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ if it exists. Since $4\times b = r_1\times 1 + r_2\times(n+n)$, we have $b = \frac{\lambda_1(n^2-n)+\lambda_2(n^2+2n)}{6}$.

As r_1 and r_2 must be integers, we have the following.

i. If $n \equiv 0 \pmod 3$, then $\lambda_1 \equiv \lambda_2 \pmod 3$.

ii. If $n \equiv 1 \pmod 3$, then $\lambda_2 \equiv 0 \pmod 3$.

iii. If $n \equiv 2 \pmod 3$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

Since b must be an integer, we have the following.

i. If $n \equiv 0,4 \pmod 6$, then no restriction on λ_1 and λ_2.

ii. If $n \equiv 1,3 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 2$.

iii. If $n \equiv 2 \pmod 6$, then $\lambda_1+\lambda_2 \equiv 0 \pmod 3$.

iv. If $n \equiv 5 \pmod 6$, then $2\lambda_1+5\lambda_2 \equiv 0 \pmod 6$.

Hence some basic necessary conditions for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ are

1. if $n \equiv 0 \pmod 6$, then $\lambda_1 \equiv \lambda_2 \pmod 3$.

2. if $n \equiv 1 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 6$.

3. if $n \equiv 2 \pmod 6$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

4. if $n \equiv 3 \pmod 6$, then $\lambda_1 \equiv \lambda_2 \pmod 3$, $\lambda_2 \equiv 0 \pmod 2$.

5. if $n \equiv 4 \pmod 6$, then $\lambda_2 \equiv 0 \pmod 3$.

6. if $n \equiv 5 \pmod 6$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$.

Above necessary conditions are summarized in Table 5.3.1, where "None" means the design does not exist for any n. λ_1 is given in modulo 3 and λ_2 is given in modulo 6.

A side application of the table is that for $n \equiv 3 \pmod 6$ instead of constructing three families: GDD$(1,6m+3,6m+3,4;3t,6s)$, GDD$(1,6m+3,6m+3,4;3t+1,6s+4)$ and

Table 5.3.1: The necessary conditions for GDD$(1,n,n,4;\lambda_1,\lambda_2)$

$\lambda_1\backslash\lambda_2$	0	1	2	3	4	5
0	all n	None	None	n even	None	None
1	$n\equiv 1 \pmod 3$	$n\equiv 0 \pmod 6$	None	$n\equiv 4 \pmod 6$	$n\equiv 0 \pmod 3$	None
2	$n\equiv 1 \pmod 3$	None	$n\equiv 0 \pmod 3$	$n\equiv 4 \pmod 6$	None	$n\equiv 0 \pmod 6$

GDD$(1,6m+3,6m+3,4;3t+2,6s+2)$, one needs to construct just one family GDD$(1,6m+3,6m+3,4;3t,6s)$. Then the family GDD$(1,6m+3,6m+3,4;3t+2,6s+2)$ can be obtained by taking the blocks of a GDD$(1,6m+3,6m+3,4;3t,6s)$ and the blocks of a BIBD$(12m+7,4,2)$. Similarly, a GDD$(1,6m+3,6m+3,4;3t+1,6s+4)$ can be obtained by taking the blocks of a GDD$(1,6m+3,6m+3,4;3(t-1),6s)$ and the blocks of a BIBD$(12m+7,4,4)$ where m is any nonnegative integer.

As a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ has 3 groups and blocks of size 4, each block contains at least one associate pair. Then $b \leq [\binom{n}{2}+\binom{n}{2}]\lambda_1 = n(n-1)\lambda_1$. Now substituting the value of b, we have the following theorem:

Theorem 5.3.1. *A necessary condition for the existence of a GDD*$(1,n,n,4;\lambda_1,\lambda_2)$ *is*

$$\lambda_2 \leq \frac{5(n-1)}{n+2}\lambda_1.$$

Corollary 5.3.2. *For the existence of a GDD*$(1,n,n,4,\lambda_1,\lambda_2)$, $\lambda_2 \leq 5\lambda_1$.

The blocks of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$, if exists, have $(n^2+2n)\lambda_2$ second associate pairs. There can be at most r_1 blocks of type $(1,1,2)$ which account for $5r_1$ second associate pairs, we have $b-r_1 \geq \frac{(n^2+2n)\lambda_2-5r_1}{4}$ as all other blocks can have at the most 4 second associate pairs. Thus, we have the following:

Theorem 5.3.3. *A necessary condition for the existence of a GDD*$(1,n,n,4;\lambda_1,\lambda_2)$ *is* $b \geq \frac{(n^2+2n)\lambda_2-r_1}{4}$.

Corollary 5.3.4. *A necessary condition for the existence of a GDD*$(1,n,n,4;\lambda_1,\lambda_2)$ *is* $\lambda_2 \leq \frac{2(n-1)}{n}\lambda_1 < 2\lambda_1$.

Proof. Substituting the values of b and r_1 in $b \geq \frac{(n^2+2n)\lambda_2 - r_1}{4}$, we have $\lambda_2 \leq \frac{2(n-1)}{n}\lambda_1 < 2\lambda_1$. □

As a consequence of the above corollary, we have

Corollary 5.3.5. *For any nonnegative integer t and any positive integers x and s, where* $x \leq s$ *following GDDs do not exist.*

1. *A GDD*$(1,6t+4,6t+4,4;3x+1,6s+3)$.
2. *A GDD*$(1,3t,3t,4;3x+1,6s+4)$.
3. *A GDD*$(1,6t,6t,4;3x+2,6s+5)$.
4. *A GDD*$(1,n,n,4;3x,6s)$.

5.4 Existence of families of GDD$(1,n,n,4;\lambda_1,\lambda_2)$

Unless otherwise stated in this section we are assuming $\lambda_1 \geq \lambda_2$.

Remark 1. A GDD$(1,n,n,4;0,\lambda_2)$ does not exist as the number of groups is less than the block size.

As a consequence of Theorem 5.1.1 we have following theorem where unless otherwise stated λ, s and t are nonnegative integers and $n > 1$.

Theorem 5.4.1. *If a BIBD*$(2n+1,4,\lambda_2)$ *and a BIBD*$(n,4,\lambda)$ *exist, then a GDD*$(1,n,n,4;\lambda_1 = \lambda_2+\lambda,\lambda_2)$ *exists. In particular, we have*

1. *A GDD*$(1,n,n,4;6t,6s)$ *exists for all* $n \geq 4$, *where* $t \geq s$.
2. *A GDD*$(1,n,n,4;6s+3t,6s)$ *exists for* $n \equiv 0,1 \pmod 4$.
3. *A GDD*$(1,n,n,4;6t+3s,3s)$ *exists for* $n \equiv 0 \pmod 2$.

4. *A* $GDD(1,n,n,4;6t+2s,2s)$ *exists when* $n \equiv 0 \pmod 3$.

5. *A* $GDD(1,n,n,4;3t+s,s)$ *exists for* $n \equiv 0 \pmod{12}$.

6. *A* $GDD(1,n,n,4;3t+2s,2s)$ *exists for* $n \equiv 9 \pmod{12}$.

7. *A* $GDD(1,n,n,4;6t+\lambda,\lambda)$ *exists for* $n \equiv 0 \pmod 6$.

8. *A* $GDD(1,n,n,4;3t,3s)$ *exists for* $n \equiv 0 \pmod 4$, *where* $t \geq s$.

9. *A* $GDD(1,n,n,4;2t+6s,6s)$ *exists for* $n \equiv 1 \pmod 6$.

10. *A* $GDD(1,n,n,4;2t+3s,3s)$ *exists for* $n \equiv 4 \pmod 6$.

11. *A* $GDD(1,n,n,4;6s+\lambda,6s)$ *exists for* $n \equiv 1 \pmod{12}$.

12. *A* $GDD(1,n,n,4;3s+\lambda,3s)$ *exists for* $n \equiv 4 \pmod{12}$.

Case 1: $\lambda_2 \equiv 0 \pmod 6$

From Theorem 5.4.1(1) we have

Corollary 5.4.2. *Necessary conditions are sufficient for the existence of a* $GDD(1, n,n,4; \lambda_1,\lambda_2)$ *for* $\lambda_1 \equiv 0 \pmod 6$ *and* $\lambda_2 \equiv 0 \pmod 6$.

From Theorem 5.4.1(2) we have

Corollary 5.4.3. *Necessary conditions are sufficient for the existence of a* $GDD(1, n,n,4; \lambda_1,\lambda_2)$ *for* $n \equiv 5 \pmod{12}$.

In the above family $\lambda_1 - \lambda_2 \equiv 3 \pmod 6$.

From the necessary conditions, when $\lambda_2 \equiv 3 \pmod 6$ and $\lambda_1 \equiv 0 \pmod 3$, n has to be even. From Theorem 5.4.1 (3), a $GDD(1,n,n,4;6t+3,6s+3)$ exists for any even n and any nonnegative integers s and t, where $t \geq s$. From Theorem 5.4.1(8), a $GDD(1,n,n,4;6t,6s+3)$ exists when $n \equiv 0 \pmod 4$ for any nonnegative integers s and t, where $t > s$. Hence we have

Corollary 5.4.4. *Necessary conditions are sufficient for the existence of a GDD*$(1,\ n,n,4;\ \lambda_1,\lambda_2)$ *for* $n \equiv 0 \pmod 4$, $\lambda_1 \equiv 0 \pmod 3$ *and* $\lambda_2 \equiv 0 \pmod 3$.

When $\lambda_1 \equiv 2 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$, from the necessary conditions, we have $n \equiv 1 \pmod 3$. For $n \equiv 1 \pmod 3$, a BIBD$(n,4,2)$ and a BIBD$(2n+1,4,6)$ exist for $n \geq 4$. From Theorem 5.4.1(9) and (10), we have

Lemma 5.4.5. *A GDD*$(1,n,n,4;6t+2,6s)$ *exists when* $n \equiv 1 \pmod 3$ *for any nonnegative integers s and t, where* $t \geq s$.

For $n \equiv 1,4 \pmod{12}$, a BIBD$(n,4,5)$ and a BIBD$(2n+1,4,6)$ exist. From Theorem 5.4.1(11) and (12), we have

Lemma 5.4.6. *A GDD*$(1,n,n,4;6t+5,6s)$ *exists when* $n \equiv 1,4 \pmod{12}$ *for any nonnegative integers s and t, where* $t \geq s$.

For $\lambda_2 \equiv 0 \pmod 6$, and $\lambda_1 \equiv 1,2 \pmod 3$ from the necessary conditions, we have $n \equiv 1 \pmod 3$. From Theorem 5.4.1 (9) and (10), we have

Lemma 5.4.7. *A GDD*$(1,n,n,4;6t+4,6s)$ *exists when* $n \equiv 1 \pmod 3$ *for any nonnegative integers s and t, where* $t \geq s$.

From Theorem 5.4.1 (11) and (12), we have the following:

Lemma 5.4.8. *A GDD*$(1,n,n,4;6t+1,6s)$ *exists for* $n \equiv 1,4 \pmod{12}$, *where s and t are nonnegative integers such that* $t \geq s$.

For $n \equiv 9 \pmod{12}$, a BIBD$(n,4,3)$ and BIBD$(2n+1,4,6)$ exist. So we have

Lemma 5.4.9. *A GDD*$(1,n,n,4;\lambda_1 = 3t,\lambda_2 = 6s)$ *exists when* $n \equiv 9 \pmod{12}$ *for any nonnegative integers s and t, where* $\lambda_1 \geq \lambda_2$.

Hence, we have :

Corollary 5.4.10. *Necessary conditions are sufficient for the existence of a GDD*$(1,n,n,4;\ \lambda_1,\lambda_2)$ *for* $\lambda_1 \geq \lambda_2$ *for* $n \equiv 1,4 \pmod{12}$, $\lambda_1 \equiv 1 \pmod 3$ *and* $\lambda_2 \equiv 0 \pmod 6$.

Case 2: $\lambda_2 \equiv 1 \pmod 6$

In this case from the necessary conditions, we have $\lambda_1 \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 6$. From Theorem 5.4.1 (7):

Lemma 5.4.11. *A GDD$(1,n,n,4;6t+1,6s+1)$ exists when $n \equiv 0 \pmod 6$ for any nonnegative integers s and t, where $t \geq s$.*

From Theorem 5.4.1 (5):

Lemma 5.4.12. *A GDD$(1,n,n,4;6t+4,6s+1)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integers s and t, where $t \geq s$.*

Corollary 5.4.13. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for $n \equiv 0 \pmod{12}$, and $\lambda_2 \equiv 1 \pmod 6$.*

Case 3: $\lambda_2 \equiv 2 \pmod 6$

In this case, $\lambda_1 \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 3$. From Theorem 5.4.1(4) we have the following lemma.

Lemma 5.4.14. *A GDD$(1,n,n,4;6t+2,6s+2)$ exists when $n \equiv 0 \pmod 3$ for any nonnegative integers s and t, where $t \geq s$.*

From Theorem 5.4.1(5) and (6), we have

Lemma 5.4.15. *A GDD$(1,n,n,4;6t+5,6s+2)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integer s and t, where $t \geq s$.*

Hence we have:

Corollary 5.4.16. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for $n \equiv 0,9 \pmod{12}$, and $\lambda_2 \equiv 2 \pmod 6$.*

Case 4: $\lambda_2 \equiv 3 \pmod 6$

In this case from the necessary conditions, when $\lambda_1 \equiv 1,2 \pmod 3$, $n \equiv 4 \pmod 6$. For $n \equiv 4 \pmod 6$, a BIBD$(n,4,2)$, a BIBD$(n,4,4)$ and a BIBD$(2n+1,4,3)$ exist. From Theorem 5.4.1(10), we have the following lemma:

Lemma 5.4.17. *When $n \equiv 4 \pmod 6$, a GDD$(1,n,n,4;6t+1,6s+3)$ exists for any nonnegative integers s and t, where $t > s$ and a GDD$(1,n,n,4;6t+5,6s+3)$ exists where $t \geq s$.*

For $n \equiv 4 \pmod{12}$, a BIBD$(n,4,1)$ and a BIBD$(2n+1,4,3)$ exist. Hence we have a GDD$(1,n,n,4;3,3)$, a GDD$(1,n,n,4;4,3)$ and a GDD$(1,n,n,4;5,3)$. From Theorem 5.4.1 (12), we have

Corollary 5.4.18. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1, 6s+3)$ when $n \equiv 4 \pmod{12}$ for any nonnegative integers s and λ_1, where $\lambda_1 \geq 6s+3$.*

When $\lambda_1 \equiv 0 \pmod 3$, $n \equiv 0 \pmod 2$, from Theorem 5.4.1 (8), we have

Corollary 5.4.19. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1, 6s+3)$ when $n \equiv 0 \pmod 4$ for any nonnegative integers s and λ_1, where $\lambda_1 \geq 6s+3$.*

Case 5: $\lambda_2 \equiv 4 \pmod 6$

In this case from the necessary conditions, we have $\lambda_1 \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$. For $n \equiv 0 \pmod{12}$, a BIBD$(2n+1,4,1)$, and a BIBD$(n,4,3)$ exist. Hence, a GDD$(1,n,n,4;3x+1,4)$ exists for $x > 1$. Hence

Corollary 5.4.20. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1 = 3t+1, \lambda_2 = 6s+4)$ for $n \equiv 0 \pmod{12}$.*

Similarly from Theorem 5.4.1 (6), we have a GDD$(1,n,n,4;\lambda_1 = 3x+4, \lambda_2 = 4)$ for a nonnegative integer x, hence

Corollary 5.4.21. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1 = 3t+1, \lambda_2 = 6s+4)$ for $n \equiv 9 \pmod{12}$ where $\lambda_1 > \lambda_2$.*

For $n \equiv 0 \pmod 3$, a BIBD$(2n+1,4,2)$, a BIBD$(2n+1,4,4)$ and a BIBD$(n,4,6)$ exist for $n \geq 4$. From Theorem 5.4.1(4), we have

Lemma 5.4.22. *A GDD$(1,n,n,4;6t+4,6s+4)$ exists when $n \equiv 0 \pmod 3$ for any nonnegative integers s and t, where $t \geq s$.*

Case 6: $\lambda_2 \equiv 5 \pmod 6$

Here, from the necessary conditions, we have $\lambda_1 \equiv 2 \pmod 3$ and $n \equiv 0 \pmod 6$. But for $n \equiv 0 \pmod{12}$, a BIBD$(2n+1,4,5)$ and a BIBD$(n,4,3)$ exist. From Theorem 5.4.1(5) and (6), we have

Lemma 5.4.23. *A GDD$(1,n,n,4;6t+2,6s+5)$ exists when $n \equiv 0 \pmod{12}$ for any nonnegative integers s and t, where $t > s$.*

From Theorem 5.4.1(7), we have

Lemma 5.4.24. *A GDD$(1,n,n,4;6t+5,6s+5)$ exist when $n \equiv 0 \pmod 6$ for any nonnegative integers s and t, where $t \geq s$.*

Hence we have

Corollary 5.4.25. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;$ $\lambda_1 = 3t+2, \lambda_2 = 6s+5)$ for $n \equiv 0 \pmod{12}$ where $\lambda_1 > \lambda_2$.*

We have summarized main results from this section in Table 5.4.1: From Table 5.4.1, we have

Theorem 5.4.26. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;\lambda_1,$ $\lambda_2)$ for $n \equiv 0,1,4,5,8,9 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.*

5.5 Specific GDDs

In this section we study the existence of GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for specific values of the parameters.

Table 5.4.1: For the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$, $\lambda_1 \geq \lambda_2$

$n \equiv$		The existence is not known for
0 (mod 12)	Corollary 5.4.2, Corrolary 5.4.4, Corollary 5.4.13,Corollary 5.4.16, Corollary 5.4.19, Corollary 5.4.20, Corollary 5.4.25, Lemma 5.4.11, Lemma 5.4.12, Lemma 5.4.14, Lemma 5.4.15, Lemma 5.4.22, Lemma 5.4.23	
1 (mod 12)	Corollary 5.4.2, Corollary 5.4.10, Lemma 5.4.5, Lemma 5.4.6, Lemma 5.4.7, Lemma 5.4.8, Theorem 5.4.1 (11)	
2 (mod 12)	Corollary 5.4.2	GDD$(1,n,n,4;3t,6s+3)$, GDD$(1,n,n,4;6t+3,6s)$
3 (mod 12)	Corollary 5.4.2, Lemma 5.4.14, Lemma 5.4.22	GDD$(1,n,n,4;6t+3,6s)$, GDD$(1,n,n,4;6t+1,6s+4)$ GDD$(1,n,n,4;6t+5,6s+2)$
4 (mod 12)	Corollary 5.4.2, Corrolary 5.4.4, Corollary 5.4.13,Corollary 5.4.10, Corollary 5.4.18, Corollary 5.4.19, Lemma 5.4.5, Lemma 5.4.6, Lemma 5.4.7, Lemma 5.4.8, Lemma 5.4.17	
5 (mod 12)	Corollary 5.4.2, Corollary 5.4.3	
6 (mod 12)	Corollary 5.4.2, Lemma 5.4.11, Lemma 5.4.14, Lemma 5.4.24	GDD$(1,n,n,4;6t+3,6s)$, GDD$(1,n,n,4;3t,6s+3)$, GDD$(1,n,n,4;6t+4,6s+1)$, GDD$(1,n,n,4;6t+5,6s+2)$, GDD$(1,n,n,4;6t+1,6s+4)$, GDD$(1,n,n,4;6t+2,6s+5)$
7 (mod 12)	Corollary 5.4.2, Lemma 5.4.5, Lemma 5.4.7	GDD$(1,n,n,4;6t+1,6s)$, GDD$(1,n,n,4;6t+3,6s)$, GDD$(1,n,n,4;6t+5,6s)$
8 (mod 12)	Corollary 5.4.2, Corollary 5.4.4, Corollary 5.4.19	
9 (mod 12)	Corollary 5.4.2, Corollary 5.4.16, Corollary 5.4.21, Lemma 5.4.9, Lemma 5.4.14, Lemma 5.4.22	
10 (mod 12)	Corollary 5.4.2, Lemma 5.4.5, Lemma 5.4.7, Lemma 5.4.17	GDD$(1,n,n,4;6t+3,6s)$, GDD$(1,n,n,4;6t+1,6s)$, GDD$(1,n,n,4;6t+5,6s)$, GDD$(1,n,n,4;6t+2,6s+3)$
11 (mod 12)	Corollary 5.4.2	GDD$(1,n,n,4;6t+3,6s)$

5.5.1 $\lambda_1 = 1$

Theorem 5.5.1. *Necessary conditions are sufficient for the existence of a GDD*$(1,n,n,4;1,\lambda_2)$. *Specifically, a GDD*$(1,n,n,4;1,\lambda_2)$ *exists when* $\lambda_2 = 1$ *and* $n \equiv 0 \pmod 6$ *and when* $\lambda_2 = 0$ *and* $n \equiv 1,4 \pmod{12}$.

Proof. For $\lambda_1 = 1$, by Corollary 5.3.4, $\lambda_2 < 2$, hence λ_2 can only be 0 or 1. A GDD$(1,n,n,4;1,1)$ exists for $n \equiv 0 \pmod 6$ as a BIBD$(2n+1,4,1)$ on $G_1 \cup G_2 \cup G_3$ exists. A GDD$(1,n,n,4;1,0)$ exists for $n \equiv 1,4 \pmod{12}$ as a BIBD$(n,4,1)$ on G_i for $i = 2,3$, where G_1, G_2, G_3 are groups of size 1, n, n respectively. □

5.5.2 $\lambda_1 = 2$

Theorem 5.5.2. *A GDD*$(1,n,n,4;2,\lambda_2)$ *exists for* $\lambda_2 \leq 2$, *specifically when* $\lambda_2 = 2$ *and* $n \equiv 0 \pmod 3$ *and when* $\lambda_2 = 0$ *and* $n \equiv 1,4 \pmod 6$.

Proof. For $\lambda_1 = 2$, by Corollary 5.3.4, $\lambda_2 < 4$, hence λ_2 can be 0, 1, 2 and 3. A GDD$(1,n,n,4;2,0)$ exists for $n \equiv 1 \pmod 3$ as a BIBD$(n,4,2)$ on G_i for $i = 2,3$, where G_1, G_2, and G_3 are groups of size 1, n, and n respectively. A GDD$(1,n,n,4;2,1)$ does not exists for any n from the necessary conditions. A GDD$(1,n,n,4;2,2)$ exists for $n \equiv 0 \pmod 3$ as a BIBD$(2n+1,4,2)$ on $G_1 \cup G_2 \cup G_3$ exists. □

As a GDD$(1,n,n,4;2,3)$ exists for $n = 4$ (see Example 5.5.1) and for $n = 10$ (see Example 5.6.6), we have

Theorem 5.5.3. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;2,\lambda_2)$ except possibly for $n \equiv 4 \pmod 6$, $n \neq 4$, 10 and $\lambda_2 = 3$.*

5.5.3 $\lambda_1 = 3$

Theorem 5.5.4. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n,4;3,\lambda_2)$. Specifically, a GDD$(1,n,n,4;3,\lambda_2)$ exists when $\lambda_2 = 3$ and $n \equiv 0 \pmod 2$ and when $\lambda_2 = 0$ and $n \equiv 0,1 \pmod 4$.*

Proof. For $\lambda_1 = 3$, by Corollary 5.3.4, $\lambda_2 < 6$. Hence λ_2 can be 0 and 3. A GDD$(1,n,n,4;3,0)$ exists for $n \equiv 0,1 \pmod 4$ as a BIBD$(n,4,3)$ on G_i for $i = 2,3$ exists. But a GDD$(1,n,n,4;3,0)$ does not exist for $n \equiv 2,3 \pmod 4$ by Corollary 5.1.4. A GDD$(1,n,n,4;3,3)$ exists for $n \equiv 0 \pmod 2$ as a BIBD$(2n+1,4,3)$ on $G_1 \cup G_2 \cup G_3$ exists. □

5.5.4 $n = 2$

When $n = 2$, both λ_1 and λ_2 are 0 modulo 3. In a GDD$(1,2,2,4;\lambda_1,\lambda_2)$, there are no blocks of type $(0,4)$ and $(1,3)$. Hence, a GDD$(1,2,2,4;\lambda_1,0)$ does not exist. Let b_1 and b_2 be the number of blocks of type $(1,1,2)$ and $(2,2)$ for a GDD$(1,2,2,4;\lambda_1,\lambda_2)$ if it exists. Then $b_1 + 2b_2 = 2\lambda_1$ and $5b_1 + 4b_2 = 8\lambda_2$. Hence

$$3b_1 = 8\lambda_2 - 4\lambda_1.$$

As $b_1 = r_1 = \frac{4}{3}\lambda_2$, we have

Lemma 5.5.5. *A necessary condition for the existence of a GDD*$(1,2,2,4;\lambda_1,\lambda_2)$ *is* $\lambda_1 = \lambda_2$.

As a GDD$(1,n,n,k;\lambda,\lambda)$ is a BIBD$(2n+1,k,\lambda)$ and as a BIBD$(5,4,3)$ exists by Theorem 2.1.6, we have

Theorem 5.5.6. *Necessary conditions are sufficient for the existence of a GDD*$(1,2,2,4;\lambda_1,\lambda_2)$.

5.5.5 $n = 3$

Let $G_1 = \{x\}$, $G_2 = \{a,b,c\}$ and $G_3 = \{1,2,3\}$. In a GDD$(1,3,3,4;\lambda_1,\lambda_2)$, there is no block of type $(0,4)$. Let b_1, b_2 and b_3 be the number of blocks of type $(1,1,2)$, $(1,3)$ and $(2,2)$ respectively. Then $b_1 + 3b_2 + 2b_3 = 6\lambda_1$ and $5b_1 + 3b_2 + 4b_3 = 15\lambda_2$. As $6\lambda_1 = b_1 + 3b_2 + 2b_3 \leq 5b_1 + 3b_2 + 4b_3 = 15\lambda_2$, we have $\lambda_1 \leq \frac{5}{2}\lambda_2$. Also, by Corollary 5.3.4, $\lambda_2 \leq \frac{4}{3}\lambda_1$. Hence,

Lemma 5.5.7. *A necessary condition for the existence of a GDD*$(1,3,3,4;\lambda_1,\lambda_2)$ *is* $\frac{3}{4}\lambda_2 \leq \lambda_1 \leq \frac{5}{2}\lambda_2$.

In other words, $\frac{2}{5}\lambda_1 \leq \lambda_2 \leq \frac{4}{3}\lambda_1$.

Remark 2. A GDD$(1,3,3,4;\lambda_1,0)$ does not exist as there are no blocks of type $(0,4)$.

Case 1.

$\lambda_1 < \lambda_2$.

For $n = 3$, $\lambda_1 \equiv \lambda_2 \pmod 3$. Let $\lambda_2 - \lambda_1 = 3s$ for some positive integer s.

Let $\lambda_1 = 3t + i$, for $i = 0,1,2$. Then from $\lambda_2 \leq \frac{4\lambda_1}{3}$, $\lambda_2 \leq 4t + i$.

Therefore the difference $\lambda_2 - \lambda_1 = 3s$ is less than or equal to t. Using $\lambda_2 = 3s + \lambda_1$ and $\lambda_2 \leq \frac{4\lambda_1}{3}$, the smallest λ_1 will be $9s$ and λ_2 will be $12s$. Hence the smallest GDD where $\lambda_2 - \lambda_1 = 3s$ is a GDD$(1,3,3,4;9s,12s)$. For $s = 1$, we construct a GDD$(1,3,3,4;9,12)$ as follows.

A relabeling construction

Let $X = \{0,1,2,3,4,5,6\}$, $G_1 = \{0\}$, $G_2 = \{1,2,3\}$ and $G_3 = \{4,5,6\}$.
Then $\mathscr{B} = \{\{2,4,5,6\},\{3,5,6,0\},\{4,6,0,1\},\{5,0,1,2\},\{6,1,2,3\},\{0,2,3,4\},\ \{1,3,4,5\}\}$ is a collection of blocks of a BIBD$(7,4,2)$ on $G_1 \cup G_2 \cup G_3 = X$.
We relabel elements of these blocks using different permutations (say α) on X to get following six isomorphic BIBDs.

- $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha(2) = 3$, $\alpha(3) = 2$, $\alpha(4) = 4$, $\alpha(5) = 6$ and $\alpha(6) = 5$. Then, when we relabel the points of X using α, the blocks of $\mathscr{B}$ become
  $\mathscr{B}_1 = \{\{3,4,6,5\},\{2,6,5,0\},\{4,5,0,1\},\{6,0,1,3\},\{5,1,3,2\},\{0,3,2,4\},\{1,2,4,6\}\}$.

- $\alpha(0) = 0$, $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 3$, $\alpha(4) = 6$, $\alpha(5) = 5$ and $\alpha(6) = 4$. Then the blocks of $\mathscr{B}$ become
  $\mathscr{B}_2 = \{\{1,6,5,4\},\{3,5,4,0\},\{6,4,0,2\},\{5,0,2,1\},\{4,2,1,3\},\{0,1,3,6\},\{2,3,6,5\}\}$.

- $\alpha(0) = 2$, $\alpha(1) = 1$, $\alpha(2) = 0$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 6$. Then the blocks of $\mathscr{B}$ become
  $\mathscr{B}_3 = \{\{0,4,5,6\},\{3,5,6,2\},\{4,6,2,1\},\{5,2,1,0\},\{6,1,0,3\},\{2,0,3,4\},\{1,3,4,5\}\}$.

- $\alpha(0) = 2$, $\alpha(1) = 1$, $\alpha(2) = 0$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 6$. Then the blocks of $\mathscr{B}$ become
  $\mathscr{B}_4 = \{\{0,4,5,6\},\{3,5,6,2\},\{4,6,2,1\},\{5,2,1,0\},\{6,1,0,3\},\{2,0,3,4\},\{1,3,4,5\}\}$.

- $\alpha(0) = 6$, $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 0$. Then the blocks of $\mathscr{B}$ become
  $\mathscr{B}_5 = \{\{2,4,5,0\},\{3,5,0,6\},\{4,0,6,1\},\{5,6,1,2\},\{0,1,2,3\},\{6,2,3,4\},\{1,3,4,5\}\}$.

- $\alpha(0) = 6$, $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 3$, $\alpha(4) = 4$, $\alpha(5) = 5$ and $\alpha(6) = 0$. Then the blocks of $\mathscr{B}$ become
  $\mathscr{B}_6 = \{\{2,4,5,0\},\{3,5,0,6\},\{4,0,6,1\},\{5,6,1,2\},\{0,1,2,3\},\{6,2,3,4\},\{1,3,4,5\}\}$.

So $\mathscr{B} \cup \mathscr{B}_1 \cup ... \cup \mathscr{B}_6$ gives a BIBD$(7,4,14)$ which contains 49 blocks. Removing the ten blocks containing all three points $1,2,3$ and all three points $4,5,6$, we have a GDD$(1,3,3,4;$ $14-5=9, 14-2=12)$.

In general, as a BIBD$(7,4,14t)$ exists for any positive integer t, we have the following result.

Lemma 5.5.8. *A GDD$(1,3,3,4;14t-5s,14t-2s)$ for $s=0,1,\cdots,t$ exists for any positive integer t where $s=0$ gives a BIBD.*

For example, when $t=1$, we have a GDD$(1,3,3,4;14-5=9,14-2=12)$, and hence by using BIBD(7,4,2) repeatedly, we have GDD$(1,3,3,4;9+2m,12+2m)$, specifically we are interested in GDD$(1,3,3,4;13,16)$, GDD$(1,3,3,4;15,18)$, GDD$(1,3,3,4;17,20)$, GDD$(1,3,3,4;19,22)$, GDD$(1,3,3,4;21,24)$.

Lemma 5.5.9. *Necessary conditions are sufficient for the existence of a GDD$(1,3,3,4;\lambda_1,\lambda_2)$ for $\lambda_1 \leq \lambda_2$.*

Proof. A GDD$(1,3,3,4;9,12)$ and a BIBD$(7,4,2)$ exist. The smallest GDD$(1,3,3,4;\lambda_1,\lambda_2)$ when the difference $\lambda_2-\lambda_1=3s$ is GDD$(1,3,3,4;9s,12s)$. s copies of GDD$(1,3,3,4;9,12)$ and m copies of BIBD$(7,4,2)$ together give all required GDD$(1,3,3,4;9s+2m,12s+2m)$ where $\lambda_2-\lambda_1=3s$. Recall that for $n=3$, λ_2 is always even. When $\lambda_1=\lambda_2$, necessary conditions are the same as the conditions for the existence of a BIBD$(7,4,\lambda_1)$. □

Case 2

$\lambda_1 > \lambda_2$.

Since a BIBD$(6,3,2)$ and a BIBD$(3,3,1)$ exist, we have the following from Theorem 5.2:

$$
\begin{array}{c} \\ i\mathscr{B}_1 \\ j\mathscr{B}_2 \\ u\mathscr{B}_3 \\ v\mathscr{B}_4 \end{array}
\begin{array}{c}
\begin{array}{ccc} (1,x) & (1,2) & (1,a) \end{array} \\
\left(\begin{array}{ccc} 5i & 2i & 2i \\ j & j & 0 \\ 0 & 3u & 2u \\ 0 & 3v & 4v \end{array} \right)
\end{array}
$$

So when $5i+j = 2i+2u+4v$, then we have a

$$GDD(1,3,3,4;2i+j+3u+3v,5i+j). \tag{5.5.1}$$

Let $\lambda_1 - \lambda_2 = 3s$ for some nonnegative integer s. As $\lambda_1 \leq \frac{5\lambda_2}{2}$, $3s \leq 1.5\lambda_2$. Hence when the difference $\lambda_1 - \lambda_2 = 3s$, smallest value of λ_2 is $2s$ and corresponding smallest parameter GDD will be GDD$(1,3,3,4;5s,2s)$. For $s = 1$, the required GDD is GDD$(1,3,3,4;5,2)$ which can be constructed from 5.5.1 by letting $i = 0$, $j = 2$, $u = 1$, $v = 0$.

Lemma 5.5.10. *Necessary conditions are sufficient for the existence of a GDD*$(1,3,3,4;\lambda_1,\lambda_2)$ *for* $\lambda_1 \geq \lambda_2$.

Proof. A GDD$(1,3,3,4;5,2)$ and a BIBD$(7,4,2)$ exist. The smallest GDD$(1,3,3,4;\lambda_1,\lambda_2)$ when the difference $\lambda_1 - \lambda_2 = 3s$ is GDD$(1,3,3,4;5s,2s)$. Note that s copies of GDD$(1,3,3,4;5,2)$ and m copies of BIBD$(7,4,2)$ together give all required GDD$(1,3,3,4;5s+2m,2s+2m)$ where the $\lambda_1 - \lambda_2 = 3s$. Recall that for $n = 3$, λ_2 is always even.
When $\lambda_1 = \lambda_2$, necessary conditions are the same as the conditions for the existence of a BIBD$(7,4,\lambda_1)$. □

Lemma 5.5.9 and Lemma 5.5.10 together complete the case for $n = 3$ and we have

Theorem 5.5.11. *Necessary conditions are sufficient for the existence of a GDD*$(1,3,3,4;\lambda_1,\lambda_2)$.

5.5.6 $n = 4$

Example 5.5.1. A GDD$(1,4,4,4;2,3)$ with $G_1 = \{x\}$, $G_2 = \{a,b,c,d\}$ and $G_3 = \{1,2,3,4\}$. The blocks are given below in columns.

x	x	x	x	x	x	x	x	a	c	a	c	a	b	a	b
a	d	b	c	b	a	a	b	b	d	b	d	c	d	c	d
1	1	2	2	c	d	d	c	1	1	3	3	1	1	2	2
4	4	3	3	1	2	3	4	2	2	4	4	3	3	4	4

For $n = 4$, from Theorem 7.3.6, $\lambda_2 \leq \frac{3}{2}\lambda_1$. Hence, we have the following corollary:

Corollary 5.5.12. *A necessary condition for the existence of a GDD*$(1,4,4,4;\lambda_1,\lambda_2)$ *is* $\lambda_1 \geq \frac{2}{3}\lambda_2$.

From Table 5.3.1, we need to construct two families: GDD$(1,4,4,4;\lambda_1,6s)$ and GDD$(1,4,4,4;\lambda_1,6s+3)$ where s and λ_1 are nonnegative integers. For the first family, by Corollary 5.5.12, $\lambda_1 \geq 4s$. Using $2s$ copies of the GDD$(1,4,4,4;2,3)$ and λ copies of BIBD$(4,4,1)$, we have a GDD$(1,4,4,4;4s+\lambda,6s)$, for any λ.

To construct a GDD$(1,4,4,4;\lambda_1,6s+3)$, we observe that by Corollary 5.5.12, $\lambda_1 \geq 4s+2$. Hence using $2s+1$ copies of the GDD$(1,4,4,4;2,3)$ and λ copies of a BIBD$(4,4,1)$, we have GDD$(1,4,4,4;4s+2+\lambda,6s+3)$, for any nonnegative integer λ. Hence we have:

Theorem 5.5.13. *Necessary conditions are sufficient for the existence of a GDD*$(1,4,4,4;\lambda_1,\lambda_2)$.

5.6 Difference families constructions

The aim of this section is to construct some examples of GDDs with a difference family. In the process we make some comments to show sufficiency in certain cases.

5.6.1 $n = 5$

Recall for $n = 5$, from the necessary conditions, $\lambda_2 \equiv 0 \pmod 6$. So let $\lambda_2 = 6t$ where t is a nonnegative integer. Now from the necessary condition, we have $\lambda_1 < \lambda_2 < 2\lambda_1$. Let $\lambda_2 - \lambda_1$ =$3s$. Then using $\lambda_2 < 2\lambda_1$, $3s < \lambda_1$. If smallest possible value of λ_1 is $3s+3$, then $\lambda_2 = 6s+3$ not 0 mod 6. Hence the smallest possible value of λ_1 has to be $3s+6$ and $\lambda_2 = 6s+6$. For example, when $s = 1$ the smallest possible GDD will be GDD$(1,5,5,4;9,12)$.

We present a difference family construction for GDD$(1,5,5,4;9,12)$.

Example 5.6.1. Let the groups be $G_1 = \{\infty\}$, $G_2 = \{1,3,5,7,9\}$ and $G_3 = \{0,2,4,6,8\}$. Difference family is $\{\{\infty,0,1,3\},\{\infty,0,1,4\},\{\infty,0,3,4\},\{\infty,0,3,6\},\{0,5,1,2\},\{0,5,1,3\},$ $\{0,5,1,4\},\{0,5,2,3\},\{0,5,2,4\},\{0,5,3,4\}\}$.

Hence, we also have GDD$(1,5,5,4;6m+9t,6m+12t)$ for nonnegative integers m and t.

Theorem 5.6.1. *The necessary conditions for the existence of a GDD*$(1,5,5,4;6t,6s)$ *are sufficient for* $t \geq s$ *and the necessary conditions for the existence of a GDD*$(1,5,5,4;\ 6t+3,6s)$ *are sufficient for* $t \geq s-1$

On the other hand, let $\lambda_1 - \lambda_2$ =$3s$, as $\lambda_2 \equiv 0 \pmod 6$, a GDD$(1,5,5,4;6t,6t)$ and a BIBD$(5,4,3)$ exist, hence a GDD$(1,5,5,4;6t+3s,6t)$ exists.

Theorem 5.6.2. *The necessary conditions for the existence of a GDD*$(1,5,5,4;\lambda_1,\lambda_2)$ *are sufficient for* $\lambda_1 \geq \lambda_2$.

5.6.2 $n = 6$

Let $G_1 = \{\infty\}$, $G_2 = \{1,3,5,7,9,11\}$ and $G_3 = \{0,2,4,6,8,10\}$ be groups.

Example 5.6.2. The following multiset $\{\{\infty,0,2,4\},\{0,1,4,5\},\{0,2,7,8\},\{0,2,4,6\},\{0,$ $3,6,9\},\{0,\ 3,6,9\}\}$ is a difference family for a GDD$(1,6,6,4;6,3)$. Note that $\{0,3,6,9\}$ is a short difference set and gives only three blocks. These bocks cover difference 3 and 6 pairs only once. Hence, we also have a GDD$(1,6,6,4;m+6t,m+3t)$ for any nonnegative integers m and t.

Example 5.6.3. The following multiset $\{\{\infty,0,1,5\},\{\infty,0,2,3\},\{\infty,0,3,4\},\{0,1,2,5\},\{0,1,4,5\},\ \{0,1,7,10\},\{0,2,7,8\},\{0,3,5,10\},\{0,3,6,9\},\{0,3,6,9\}\}$ is a difference family for a GDD$(1,6,6,4;\ 6,9)$. Hence, we also have a GDD$(1,6,6,4;m+6t,m+9t)$ for any nonnegative integers m and t.

Example 5.6.4. A GDD$(1,6,6,4;4,1)$ can be constructed by difference family: $\{\{\infty,0,4,8\},\{0,1,4,6\},\ \{0,2,4,6\}\}$.

Example 5.6.5. The difference family for a GDD$(1,6,6,4;5,2)$ is $\{\{\infty,0,4,8\},\{\infty,0,4,8\},\{0,1,6,7\},\ \{0,2,4,5\},\{0,2,4,10\},\{0,3,6,9\}$ where G_1=$\{\infty\}$, G_2=$\{1,3,5,7,9,11\}$ and G_3=$\{0,2,4,6,8,10\}$.

So we have a GDD$(1,6,6,4;3t+1,3s+1)$ for $t>s+1$, GDD$(1,6,6,4;3t+2,3s+2)$ for $t>s+1$, GDD$(1,6,6,4;6t+1,6s+1)$, GDD$(1,6,6,4;6t+1,6s+1)$ and GDD$(1,6,6,4;6t+3,6s+3)$ for $t\geq s$.

5.6.3 $n=10$

Example 5.6.6. $G_1=\{\infty\}$, $G_2=\{1,3,5,...,19\}$ and $G_3=\{0,2,4,...,18\}$.The difference family $\{\{\infty,0,3,6\},\{0,1,3,7\},\{0,1,8,9\},\{0,5,7,16\},\{0,5,10,15\},\ \{0,5,10,15\}\}$ provides a GDD$(1,10,10,4;2,3)$.

5.7 Summary

We used interesting construction techniques to construct specific examples for GDDs and obtained an important general construction for GDDs with three groups of sizes $1,n,n$ with block size 4. We obtained necessary conditions for the existence of these GDDs and proved that they were sufficient for specific values of n, specific values of λ_1 and for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for $n\equiv 0,1,4,5,8,9\pmod{12}$ when $\lambda_1\geq\lambda_2$. The work leads to several open problems including questions on the existence of unknown families of GDDs.

Chapter 6

Group divisible designs with 3 groups of different sizes with block size 4

6.1 Introduction

In this chapter, we study two problems of group divisible designs (GDDs) with three groups and blocks of size four. The first problem involves group sizes 1, n and $n+2$ respectively. We show that if equal number of blocks of block type configurations is required, a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ where $t>0$ does not exist. Then we focus our study on the case where $t=2$ with no block configuration restriction. We obtain the necessary conditions for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ and prove that such necessary conditions are sufficient for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ when $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$ except n even, $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 3 \pmod 6$, and for $n \equiv 2 \pmod 6$, $\lambda_1 \equiv 1,4 \pmod 6$ and $\lambda_2 \equiv 1,5 \pmod 6$. The second problem involves group sizes 3, n, and $n+1$ respectively. We obtain necessary conditions for the existence of a GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$ and provide constructions for certain families of such GDDs.

Note that a GDD$(n_1,n_2,n_3,k;\lambda,\lambda)$ is just a BIBD$(n_1+n_2+n_3,k,\lambda)$. Also, a GDD$(n_1,n_2,n_3,4;\lambda_1,\lambda_2)$ exists when $\lambda_2 \leq \lambda_1$ and $\lambda_1 \equiv \lambda_2 \pmod 6$, if a BIBD$(n_1+n_2+n_3,4,\lambda_2)$ exists as a BIBD$(v,4,6)$ exists for all integers $v \geq 4$. More generally,

Theorem 6.1.1. *If a BIBD$(n_1+n_2+n_3,4,\lambda_2)$ and a BIBD$(n_i,4,\lambda_1)$ exist for $i=1,2,3$, then a GDD$(n_1,n_2,n_3,4;\lambda_1+\lambda_2,\lambda_2)$ exists.*

Corollary 6.1.2. *A GDD$(n_1,n_2,n_3,4;\lambda_1=\lambda_2+6s,\lambda_2)$ exists if a BIBD $(n_1+n_2+n_3,4,\lambda_2)$ exists and $n_i \geq 4$ for $i=1,2,3$.*

Corollary 6.1.3. *For all values of* n_1, n_2, n_3 *where* $n_1 + n_2 + n_3 \equiv 1,4 \pmod{12}$ *and a* $BIBD(n_i, 4, \lambda)$ *exists for* $i = 1,2,3$*, a* $GDD(n_1, n_2, n_3, 4; \lambda_1 = \lambda + \lambda_2, \lambda_2)$ *exists for all* λ_2*'s.*

6.2 Non-existence result of GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ with configuration restriction

As the first group is of size 1, these GDDs if exist, must have blocks of type (1,3) or $(1,1,2)$.

Suppose a GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ exists where $t \geq 1$. Then the number of first associate pairs equals $\binom{n}{2}\lambda_1 + \binom{n+t}{2}\lambda_1 = (n^2 + (t-1)n + \frac{t(t-1)}{2})\lambda_1$ and the number of second associate pairs equals $(n + (n+t) + n(n+t))\lambda_2$. As the block size is 4, there are four types of blocks for a GDD$(1, n, n+t, 4, \lambda_1, \lambda_2)$: blocks of configuration $(0,4)$, $(1,1,2)$, $(1,3)$ and $(2,2)$ respectively. Let b_1, b_2, b_3 and b_4 be the number of blocks of type $(0,4)$, $(1,1,2)$, $(1,3)$ and $(2,2)$ respectively. Then there are $6b_1 + b_2 + 3b_3 + 2b_4$ first associate pairs in the design. Hence,

$$6b_1 + b_2 + 3b_3 + 2b_4 = (n^2 + (t-1)n + \frac{t(t-1)}{2})\lambda_1. \tag{6.2.1}$$

In addition, there are $5b_2 + 3b_3 + 4b_4$ second associate pairs in the design. Hence,

$$5b_2 + 3b_3 + 4b_4 = \lambda_2(n^2 + (t+2)n + t). \tag{6.2.2}$$

Since the point of G_1 can be paired with any one element of G_2 and G_3, there are $\lambda_2(2n + t)$ second associate pairs containing the point of G_1 in the design. Let r_1 be the number of blocks containing the point of G_1. Then there are $3r_1$ second associate pairs containing the point of G_1 in the design. Hence, $r_1 = \frac{\lambda_2(2n+t)}{3}$.

6.2.1 Equal number of blocks of type $(1,1,2)$ and $(2,2)$

Now, we establish the non-existence results for GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ $(n \geq 2)$ with equal number of blocks having configuration $(1,1,2)$ and $(2,2)$ only.

Suppose a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks of configuration $(1,1,2)$ and $(2,2)$ exists and has b blocks, so $b_2 = b_4 = \frac{b}{2}$. Then from equation (6.2.2), we have $b = \frac{2\lambda_2(n^2+(t+2)n+t)}{9}$.

Since the point of G_1 must occur in $\frac{b}{2}$ blocks of configuration $(1,1,2)$ but does not occur in any blocks of configuration $(2,2)$, $r_1 = \frac{b}{2}$. Then $\frac{\lambda_2(2n+t)}{3} = \frac{\lambda_2(n^2+(t+2)n+t)}{9}$. This implies $n^2+(t-4)n-2t = 0$ which has integer solution only when $n = 3$ and $t = 3$. For this case, $b = 6\lambda_2$, and counting the number of first associate pairs, $b = 12\lambda_1$. Hence $\lambda_2 = 2\lambda_1$. Now the number of first associate pairs from the third group is $15\lambda_1$, as each block has at most one first associate pair from a group, $15\lambda_1$ must be less than or equal to b. Hence for nonzero λ_1, a GDD$(1,3,6,4;\lambda_1,\lambda_2)$ does not exist. Hence, we have the following theorem.

Theorem 6.2.1. *A GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks having configuration $(1,1,2)$ and $(2,2)$ respectively does not exist.*

6.2.2 Equal number of blocks of type $(1,1,2)$ and $(0,4)$

Here we establish non-existence results for GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ $(n \geq 2)$ with equal number of blocks having configuration $(1,1,2)$ and $(0,4)$ only.

Consider a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks of configuration $(1,1,2)$ and $(0,4)$ exists and has b blocks. Then from equation (6.2.2), we have $b = \frac{2\lambda_2(n^2+(t+2)n+t)}{5}$.

Since the point of G_1 must occur in $\frac{b}{2}$ blocks of configuration $(1,1,2)$ but does not occur in any block of configuration $(0,4)$, we must have $r_1 = \frac{b}{2}$. Then $\frac{\lambda_2(2n+t)}{3} = r_1 = \frac{b}{2} = \frac{\lambda_2(n^2+(t+2)n+t)}{5}$. This implies $3n^2+(3t-4)n-2t = 0$, but this does not have an integer solution. Hence, we have the following theorem.

Theorem 6.2.2. *A GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks having configuration $(1,1,2)$ and $(0,4)$ respectively does not exist.*

6.2.3 Equal number of blocks of type $(0,4)$, $(1,1,2)$ and $(2,2)$

First, we establish non-existence results for GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ $(n \geq 2)$ with equal number of blocks having configuration $(0,4)$, $(1,1,2)$ and $(2,2)$ only.

Suppose a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ with equal number of blocks having configuration $(0,4)$, $(2,2)$ and $(1,1,2)$ exists and has b blocks. Then from equation (6.2.2), we have $b=\frac{3\lambda_2(n^2+4n+2)}{9}$. This implies b is a multiple of 3 and satisfies the requirement for having equal number of blocks of configuration $(0,4)$, $(1,1,2)$ and $(2,2)$ respectively.

Since the point of G_1 occurs only in blocks of configuration $(1,1,2)$, $r_1=\frac{b}{3}$. Then $\frac{\lambda_2(2n+2)}{3}=\frac{\lambda_2(n^2+4n+2)}{9}$. This implies $n^2-2n-4=0$, which has no integer solution. Thus, we have the following theorem,

Theorem 6.2.3. *A GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ with equal number of blocks of configuration $(0,4)$, $(2,2)$ and $(1,1,2)$ respectively does not exist.*

Now we reproduce similar result for GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ $(n\geq 2)$ with equal number of blocks having configuration $(0,4)$, $(1,1,2)$ and $(2,2)$ only.

Suppose a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks having configuration $(0,4)$, $(2,2)$ and $(1,1,2)$ exists and has b blocks.Then from equation (6.2.2), we have $b=\frac{3\lambda_2(n^2+(t+2)n+t)}{9}$.

Since the point of G_1 occurs only in blocks of configuration $(1,1,2)$, $r_1=\frac{b}{3}$. Then $\frac{\lambda_2(2n+t)}{3}=\frac{\lambda_2(n^2+(t+2)n+t)}{9}$. This implies $n^2+(t-4)n-2t=0$ which has an integer solution only when $n=3$ and $t=3$. If the design GDD$(1,3,6,4;\lambda_1,\lambda_2)$ exists, it has $b=9\lambda_2$ blocks and $r_1=3\lambda_2$. Now counting the number of first associate pairs in these blocks , if the design exists, we get $18\lambda_1 = 27\lambda_2$. Hence $b=6\lambda_1$. As per the condition, $2\lambda_1$ first associate pairs of the group of size three (second group) are in (the bocks of type $(2,2)$ and hence remaining $\lambda_1=3t$ first associate pairs of second group must come in the blocks of type $(1,1,2)$. It means that $6t$ second associate pairs of first and second group elements have occurred in these λ_1 blocks already and hence it is impossible to construct remaining $(1,1,2)$ blocks. Thus, we have the following theorem,

Theorem 6.2.4. *A GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ with equal number of blocks of configuration $(0,4)$, $(2,2)$ and $(1,1,2)$ respectively does not exist.*

6.3 General construction of GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ without block configuration restriction

A GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ exists when $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$ if a BIBD$(2n+3,4,\lambda_2)$ exists as a BIBD$(n,4,6)$ exists for $n \geq 4$. More generally, we have the following theorem:

Theorem 6.3.1. *If a BIBD$(2n+3,4,\lambda_2)$, a BIBD$(n,4,\lambda)$ and a BIBD$(n+2,4,\lambda)$ exist, then a GDD$(1,n,n+2,4;\lambda_1=\lambda+\lambda_2,\lambda_2)$ exists.*

Proof. Let G_1, G_2 and G_3 be groups with size 1, n and $n+2$ respectively and let a BIBD$(2n+3,4,\lambda_2)$, a BIBD$(n,4,\lambda)$ and a BIBD$(n+2,4,\lambda)$ exist. Then the blocks of a BIBD$(2n+3,4,\lambda_2)$ on $G_1 \cup G_2 \cup G_3$ along with the blocks of a BIBD$(n,4,\lambda)$ on G_2 and the blocks of a BIBD$(n+2,4,\lambda)$ on G_3 provide the blocks of a GDD$(1,n,n+2,4;\lambda_1=\lambda+\lambda_2,\lambda_2)$. □

For example, a GDD$(1,8,10,4,12,6)$ exists.

Corollary 6.3.2. *A GDD$(1,n,n+2,4;\lambda_1=6s+\lambda_2,\lambda_2)$ exists for $n \equiv 5 \pmod 6$ and any non-negative integer s.*

Proof. Let G_1, G_2 and G_3 be groups with size 1, n and $n+2$ respectively. For $n \equiv 5 \pmod 6$, a BIBD$(n,4,6)$ and a BIBD$(n+2,4,6)$ exist. As $n \equiv 5 \pmod 6$ implies $2n+3 \equiv 1 \pmod{12}$, a BIBD$(2n+3,4,1)$ exists. Then taking the blocks of λ_2 copies of a BIBD$(2n+3,4,1)$ on $G_1 \cup G_2 \cup G_3$ along with the blocks of s copies of a BIBD$(n,4,\lambda)$ on G_2 and the blocks of s copies of a BIBD(n + 2, 4, λ) on G_3 provide the required GDD. □

Corollary 6.3.3. *A GDD$(1,n,n+2,4;\lambda_1=6s+\lambda_2,\lambda_2)$ exists for $n \equiv 2 \pmod 6$, $\lambda_2 \equiv 0 \pmod 2$, $n \neq 2$ and any non-negative integer s.*

Proof. Let G_1, G_2 and G_3 be groups with size 1, n and $n+2$ respectively and let $\lambda_2 = 2t$ for some positive integer t. For $n \equiv 2 \pmod 6$, a BIBD$(n,4,6)$ and a BIBD$(n+2,4,6)$ exist. As $n \equiv 2 \pmod 6$ implies $2n+3 \equiv 1 \pmod 3$, a BIBD$(2n+3,4,2)$ exists. Then the

blocks of t copies of a BIBD$(2n+3,4,2)$ on $G_1 \cup G_2 \cup G_3$ along with the blocks of s copies of a BIBD$(n,4,6)$ on G_2 and the blocks of s copies a BIBD$(n+2,4,6)$ on G_3 provide the required GDD. □

Corollary 6.3.4. *A GDD*$(1,n,n+2,4;\lambda_1 = 6s+\lambda_2,\lambda_2)$ *exists for* $n \equiv 3 \pmod 6$, $\lambda_2 \equiv 0 \pmod 3$, $n \neq 3$ *and any non-negative integer s.*

Proof. Let G_1, G_2 and G_3 be groups with size 1, n and $n+2$ respectively and let $\lambda_2 = 3t$ for some positive integer t. For $n \equiv 3 \pmod 6$, a BIBD$(n,4,6)$ and a BIBD$(n+2,4,6)$ exist. As $n \equiv 3 \pmod 6$ implies $2n+3 \equiv 1 \pmod 4$, a BIBD$(2n+3,4,3)$ exists. Then the blocks of t copies of a BIBD$(2n+3,4,2)$ on $G_1 \cup G_2 \cup G_3$ along with the blocks of s copies of a BIBD$(n,4,6)$ on G_2 and the blocks of s copies a BIBD$(n+2,4,6)$ on G_3 provide the required GDD. □

Theorem 6.3.5. *If a GDD*$(n_1,n_2,k;\lambda_1,\lambda_2)$ *and a BIBD*$(n_1+n_2,k-1,\lambda)$ *with* $r = \lambda_2+\lambda$ *exist, then a GDD*$(1,n_1,n_2,k;\lambda_1+\lambda,\lambda_2+\lambda)$ *exists.*

Proof. Let G_1, G_2 and G_3 be groups with size 1, n_1 and n_2 respectively and let a GDD$(n_1,n_2,k;\lambda_1,\lambda_2)$ and a BIBD$(n_1+n_2,k-1,\lambda)$ with $r = \lambda_2+\lambda$ exist on $G_1 \cup G_2$. Then taking the union of each block of a BIBD$(n_1+n_2,k-1,\lambda)$ with the point of G_1 gives blocks of size k in which each first associate pair occurs λ times, each second associate pair containing the point of G_1 occurs $\lambda_2+\lambda$ times and each second associate pair containing elements from G_2 and G_3 occurs λ times. So the above blocks together with the blocks of GDD$(n_1,n_2,k;\lambda_1,\lambda_2)$ contain $\lambda_1+\lambda$ first associate pairs and $\lambda_2+\lambda$ second associate pairs. Hence, a GDD$(1,n_1,n_2,k;\lambda_1+\lambda,\lambda_2+\lambda)$ can be obtained by taking the union of blocks of a GDD$(n_1,n_2,k;\lambda_1,\lambda_2)$ on $G_2 \cup G_3$ along with the blocks obtained by adjoining each block of BIBD$(n_1+n_2,k-1,\lambda)$ with the point of G_1. □

Example 6.3.1. A GDD$(1,2,4,4;8,5)$ with groups G_1, G_2, G_3 of size 1, 2, 4 respectively can be constructed by taking union of the blocks of a GDD$(2,4,4;6,3)$ on $G_2 \cup G_3$ along with the blocks obtained by adjoining the point of G_1 with each block of BIBD$(6,3,2)$ on $G_2 \cup G_3$. The blocks of GDD$(2,4,4;6,3)$ can be obtained by taking the union of G_2 with every 2-subset of G_3 along with 5 copies of G_3.

But we can not use Theorem 6.3.5 to construct a GDD$(1,4,6,4;6,9)$ as the BIBD$(10, 3,2)$ with replication number 9 exists and a GDD$(4,6,4;4,7)$ does not exist.

In the next sections, we study GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ without configuration restrictions.

6.4 The necessary conditions for GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$

For $n=1$, a GDD$(1,1,3,4;\lambda_1,\lambda_2)$ exists only when $\lambda_1=\lambda_2$ and it is a BIBD$(5,4,\lambda_1)$ on $G_1\cup G_2\cup G_3$. As BIBD$(5,4,3)$ exists, we have the following.

Lemma 6.4.1. *A GDD$(1,1,3,4;\lambda_1,\lambda_2)$ exists only when $\lambda_1=\lambda_2$ and its blocks are λ_1 copies of BIBD$(5,4,3)$ on $G_1\cup G_2\cup G_3$.*

Proof. BIBD$(5,4,3)$ on $V=G_1\cup G_2\cup G_3$ can be obtained by taking all the 4-subsets of V. Hence, taking λ_1 copies of all the 4-subsets of V provide the required GDD. □

Example 6.4.1. A GDD$(1,1,3,4;3,3)$ with $G_1=\{x\},G_2=\{a\}$ and $G_3=\{1,2,3\}$. Here $b=5$, $r_1=r_2=r_3=4$. The blocks of the GDD are given below in columns.

x	x	x	x	a
a	a	a	1	1
1	2	3	2	2
2	3	1	3	3

As a result of the above lemma, we can assume that $n>1$ throughout this section and Section 6.5.

In this section we will find necessary conditions for the existence of a GDD$(1,n,\ n+2,4;\lambda_1,\lambda_2)$. Assuming a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ exists, by counting argument, the replication numbers r_i for the elements of i^{th} group are :

$$r_1=\frac{(2n+2)\lambda_2}{3}, \tag{6.4.1}$$

$$r_2 = \frac{(n-1)\lambda_1 + (n+3)\lambda_2}{3} \tag{6.4.2}$$

and

$$r_3 = \frac{(n+1)\lambda_1 + (n+1)\lambda_2}{3}. \tag{6.4.3}$$

Since $4b = r_1 + nr_2 + (n+2)r_3$, we have

$$b = \frac{(n^2+n+1)\lambda_1 + (n^2+4n+2)\lambda_2}{6}. \tag{6.4.4}$$

As r_1, r_2 and r_3 must be integers, from equations (6.4.1) – (6.4.3), we have the following.

i. If $n \equiv 0,1 \pmod 3$, then $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

ii. If $n \equiv 2 \pmod 3$, then $\lambda_1 \equiv \lambda_2 \pmod 3$.

Since b must be an integer, from equation (6.4.4), we have the following.

i. If $n \equiv 0,2 \pmod 6$, then $\lambda_1 + 2\lambda_2 \equiv 0 \pmod 6$.

ii. If $n \equiv 1 \pmod 6$, then $3\lambda_1 + \lambda_2 \equiv 0 \pmod 6$.

iii. If $n \equiv 3,5 \pmod 6$, then $\lambda_1 + 5\lambda_2 \equiv 0 \pmod 6$.

iv. If $n \equiv 4 \pmod 6$, then $3\lambda_1 + 4\lambda_2 \equiv 0 \pmod 6$.

Combining the cases above, we obtain the following necessary conditions for the existence of a GDD$(1, n, n+2, 4; \lambda_1, \lambda_2)$.

1. $n \equiv 0 \pmod 6$: $\lambda_1 \equiv 0 \pmod 6$, $\lambda_2 \equiv 0 \pmod 3$.

2. $n \equiv 1 \pmod 6$: $3\lambda_1 + \lambda_2 \equiv 0 \pmod 6$, $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$.

3. $n \equiv 2 \pmod 6$: $\lambda_1 + 2\lambda_2 \equiv 0 \pmod 6$.

4. $n \equiv 3 \pmod 6$: $\lambda_1 + 5\lambda_2 \equiv 0 \pmod 6$, $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$.

5. $n \equiv 4 \pmod 6$: $3\lambda_1 + 4\lambda_2 \equiv 0 \pmod 6$, $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$.

6. $n \equiv 5 \pmod 6$: $\lambda_1 + 5\lambda_2 \equiv 0 \pmod 6$.

Table 6.4.1 summarizes certain restrictions on n where "None" means the design does not exist for any n. Here λ_1 and λ_2 are given in modulo 6.

Table 6.4.1: The necessary conditions for GDD$(1, n, n+2, 4; \lambda_1, \lambda_2)$

$\lambda_1 \backslash \lambda_2$	0	1	2	3	4	5
0	all n	None	None	n even	None	None
1	None	$n \equiv 5 \pmod 6$	None	None	None	None
2	None	None	$n \equiv 2,5 \pmod 6$	None	None	$n \equiv 2 \pmod 6$
3	None	None	None	n odd	None	None
4	None	$n \equiv 2 \pmod 6$	None	None	$n \equiv 2,5 \pmod 6$	None
5	None	None	None	None	None	$n \equiv 5 \pmod 6$

Namyalo et al. (2017) obtained the following necessary condition for GDD$(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$.

Lemma 6.4.2. *A necessary condition for the existence of a GDD$(n_1, n_2, n_3, 4; \lambda_1, \lambda_2)$ is $b \geq max(2r_i - \lambda_1)$, $i = 1, 2, 3$. Note that when $n_1 = 1$, $b \geq max(2r_i - \lambda_1)$, $i = 2, 3$.*

An example that does not satisfy the above necessary condition is given below.

Example 6.4.2. A GDD$(1, 2, 4, 4; 6, 9)$ does not exist as $b = 28$, $r_1 = 18, r_2 = 17$, $r_3 = 15$ and $28 = b < 2r_1 - \lambda_1 = 30$.

The following is an example of a GDD that exists and satisfy the above necessary conditions.

Example 6.4.3. A GDD$(1, 2, 4, 4; 6, 3)$ with $G_1 = \{x\}, G_2 = \{a, b\}$ and $G_3 = \{1, 2, 3, 4\}$. Here $b = 14$, $r_1 = 6$, $r_2 = 7$ and $r_3 = 9$. The blocks of the GDD are given below in columns.

x	x	x	x	x	x	a	a	a	a	1	1	1	1
a	a	a	b	1	1	b	b	b	b	2	2	2	2
b	b	3	3	2	2	1	2	3	4	3	3	3	3
1	2	4	4	4	3	4	3	1	2	4	4	4	4

Consider a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ with groups G_1,G_2,G_3 of cardinality $1,n,n+2$ respectively. Observe that every block contains at least two first associate pairs except for the blocks that contain the element of G_1. Then the r_1 blocks that contain the element of G_1 contain at least 1 first associate pair. So we must have at least $r_1+2(b-r_1)=2b-r_1$ first associate pairs. This implies $[\binom{n}{2}+\binom{n+1}{2}]\lambda_1 \geq 2b-r_1$. Hence, we have

Lemma 6.4.3. *A necessary condition for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ is* $\lambda_1(n^2+n+1) \geq 2b-r_1$.

The following example does not satisfy the above necessary condition.

Example 6.4.4. A GDD$(1,2,4,4;2,5)$ does not exist as $r_1=10, r_2=9, r_3=7$ and $b=14$. But $14=(n^2+n+1)\lambda_1 \not\geq 2b-r_1=26$. Also a GDD$(1,8,10,4;2,5)$ does not exist as $r_1=30, r_2=23, r_3=21$ and $b=106$. But $146=(n^2+n+1)\lambda_1 \not\geq 2b-r_1=182$.

Though general results are hard to obtain, we have an interesting bound for how far away λ_2 can go from λ_1.

Theorem 6.4.4. *A necessary condition for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ is* $\lambda_2 < 2\lambda_1$.

Proof. Substituting the values of b and r_1 in $\lambda_1(n^2+n+1) \geq 2b-r_1$, we have $\lambda_2 \leq \frac{2(n^2+n+1)\lambda_1}{n^2+2n}$ $< 2\lambda_1$ for $n \geq 2$. □

6.5 Existence results for GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$

In this section we will study the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$.

6.5.1 GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ when $\lambda_1 \leq 3$

We are considering $\lambda_1 \leq 3$ as in these cases GDDs are just BIBDs. We start with $\lambda_1 = 3$.

GDD$(1,n,n+2,4;3,\lambda_2)$

For $\lambda_1 = 3$, $\lambda_2 < 6$ from Theorem 6.4.4. A GDD$(1,n,n+2,4;3,3)$ is just a BIBD$(2n+3,4,3)$ and a BIBD$(2n+3,4,3)$ exists when $n \equiv 1 \pmod 2$. Also, from Table 6.4.1, a GDD$(1,n,n+2,4;3,\lambda_2)$ for $\lambda_2 = 0,1,2,4,5$ does not exist. Hence, we have the following result.

Lemma 6.5.1. *A GDD$(1,n,n+2,4;3,\lambda_2)$ does not exist except when $\lambda_2 = 3$ and in this case it exists only for $n \equiv 1 \pmod 2$ and it is a BIBD$(2n+3,4,3)$.*

GDD$(1,n,n+2,4;1,\lambda_2)$

For $\lambda_1 = 1$, $\lambda_2 \leq 1$ from Theorem 6.4.4. Also, from Table 6.4.1, GDD$(1,n,n+2,4;1,\lambda_2)$ does not exist for $\lambda_2 = 0$. So a GDD$(1,n,n+2,4;1,1)$ is just a BIBD$(2n+3,4,1)$ which exists when $n \equiv 5 \pmod 6$. Thus we have

Lemma 6.5.2. *A GDD$(1,n,n+2,4,1,\lambda_2)$ does not exist for any λ_2 except when $\lambda_2 = 1$ and in this case it exists only for $n \equiv 5 \pmod 6$ and it is just a BIBD$(2n+3,4,1)$.*

GDD$(1,n,n+2,4;2,\lambda_2)$

When $\lambda_1 = 2$, from Theorem 6.4.4 we have $\lambda_2 \leq 3$. Also, from Table 6.4.1, one can easily see that a GDD$(1,n,n+2,4;2,\lambda_2)$ does not exist for $\lambda_2 = 0,1$ or 3. A GDD$(1,n,n+2,4;2,2)$ is just a BIBD$(2n+3,4,2)$ which exists when $n \equiv 2 \pmod 3$. Therefore, we have

Lemma 6.5.3. *A GDD$(1,n,n+2,4,2,\lambda_2)$ does not exist for any λ_2 except when $\lambda_2 = 2$ and in this case it exists only for $n \equiv 2 \pmod 3$ and it is just a BIBD$(2n+3,4,2)$.*

In summary, a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ for $\lambda_1 \leq 3$ exists only when $\lambda_1 = \lambda_2$ and in that case it exists if and only if a BIBD$(2n+3,4,\lambda_1)$ exists.

6.5.2 GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ when $\lambda_1 \equiv \lambda_2 \pmod 6$.

In this subsection, we study a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ for the cases on the main diagonal of Table 6.4.1. Here we consider when $\lambda_1 \geq \lambda_2$ and $\lambda_1 < \lambda_2$.

$\lambda_1 \geq \lambda_2$

$\lambda_1 \equiv 0 \pmod 6$ **and** $\lambda_2 \equiv 0 \pmod 6$.

For any value of n, a GDD$(1,n,n+2,4;6s,6t)$ is possible, but it does not exist in some situations. From Theorem 6.4.4, a GDD$(1,n,n+2,4;6s,6t)$ does not exist for some $t \geq 2s$. For instance, a GDD$(1,n,n+2,4;6,12)$ does exist for any $n \geq 2$ by Lemma 6.4.3.

Lemma 6.5.4. *For all values of $n \geq 4$, a GDD$(1,n,n+2,4;6s,6t)$ exists for some positive integers s and t with $t \leq s$.*

Proof. Let G_1, G_2, G_3 be groups with size $1, n, n+2$ respectively. The blocks of GDD$(1,n,n+2,4;6s,6t)$ can be obtained by taking union of the collection of blocks of BIBD$(2n+3,4,6t)$ on $G_1 \cup G_2 \cup G_3$, 2-$(n,4,6(s-t))$ on G_2 and BIBD$(n+2,4,6(s-t))$ on G_3. □

$\lambda_1 \equiv 1 \pmod 6$ **and** $\lambda_2 \equiv 1 \pmod 6$.

From Table 6.4.1, for this case $n \equiv 5 \pmod 6$. Then by Corollary 6.3.2, the required GDD exists.

$\lambda_1 \equiv 2 \pmod 6$ **and** $\lambda_2 \equiv 2 \pmod 6$.

From the necessary conditions (i.e., from Table 6.4.1) for this case, $n \equiv 2,5 \pmod 6$. Hence, by Corollary 6.3.2 and Corollary 6.3.3, we have the required GDD.

$\lambda_1 \equiv 3 \pmod 6$ **and** $\lambda_2 \equiv 3 \pmod 6$.

Lemma 6.5.5. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n+2,4;\lambda_1 \equiv 3 \pmod 6, \lambda_2 \equiv 3 \pmod 6)$ for $\lambda_2 \leq \lambda_1$.*

Proof. From Table 6.4.1, for this case n is odd, say $n = 2t+1$ for some positive integer t. As $4t+5 \equiv 1 \pmod 4$, a BIBD$(4t+5,4,\lambda_2)$ exists. Then the blocks of BIBD$(4t+5,4,\lambda_2)$ on $G_1 \cup G_2 \cup G_3$ along with the blocks of $\frac{\lambda_1-\lambda_2}{6}$ copies of BIBD$(2t+1,4,6)$ on G_2 and BIBD$(2t+3,4;6)$ on G_3 provide the blocks of a GDD$(1,2t+1,2t+3,4;\lambda_1,\lambda_2)$. That is, a GDD$(1,2t+1,2t+3,4;6r+3,6s+3)$ when $r \geq s$. □

$\lambda_1 \equiv 4 \pmod 6$ **and** $\lambda_2 \equiv 4 \pmod 6$**.**

From Table 6.4.1, for this case $n \equiv 2,5 \pmod 6$ and it is identical to the case when $\lambda_1 \equiv 2 \pmod 6$ and $\lambda_2 \equiv 2 \pmod 6$. Then a GDD$(1,n,n+2,4;\lambda_1 \equiv 4 \pmod 6, \lambda_2 \equiv 4 \pmod 6)$ can be obtained by taking 2 copies of the blocks of a GDD$(1,n,n+2,4;\lambda_1 \equiv 2 \pmod 6, \lambda_2 \equiv 2 \pmod 6)$. Hence, we proved the existence of the required GDD.

$\lambda_1 \equiv 5 \pmod 6$ **and** $\lambda_2 \equiv 5 \pmod 6$**.**

From Table 6.4.1, for this case $n \equiv 5 \pmod 6$. Then by Corollary 6.3.2 we have the required GDD.

In summary, the above subcases together prove the following theorem.

Theorem 6.5.6. *Necessary conditions are sufficient for the existence of a GDD*$(1,n,n+2,4;\lambda_1,\lambda_2)$ *when* $\lambda_1 \geq \lambda_2$ *and* $\lambda_1 \equiv \lambda_2 \pmod 6$.

$\lambda_1 < \lambda_2$

Let $\lambda_1 \equiv 1 \pmod 6$, say $n = 6t+5$ for some integer t, then from the necessary condition (see Table 6.4.1) $\lambda_2 \equiv 1 \pmod 6$. But $\lambda_2 < 2\lambda_1$ by Theorem 6.4.4. As $\lambda_1 < \lambda_2$, the only choices for λ_2 are from $\{6s+7, 6s+13, ..., 12s+1\}$. So when $\lambda_1 = 1$, GDD$(1,n,n+2,4;1,\lambda_2)$ does not exist. As $n \equiv 5 \pmod 6$ when $\lambda_1 \equiv 1 \pmod 6$, the smallest design to construct will be a GDD$(1,5,7,4;7,13)$. But this design does not exist by Lemma 6.4.3. For $\lambda_1 = 13$, $\lambda_2 < 26$ implies that $\lambda_2 = 19$ or 25. But GDD$(1,5,7,4;13,25)$ does not exist. Also for $\lambda_1 = 19$,

$\lambda_2 < 38$ implies that $\lambda_2 = 25, 31$ or 37 and GDD$(1,5,7,4;19,37)$ does not exist.

For $n \equiv 5 \pmod 6$, when $\lambda_1 = 1$ and $\lambda_2 = 6s+1$ for some positive integer s, $\lambda_1(n^2+n+1) \ngeq 2b - r_1$. Hence, we have the following:

Lemma 6.5.7. *A GDD$(1,n,n+2,4;1,6s+1)$ for $n \equiv 5 \pmod 6$ and a GDD$(1,5,7,4;\lambda_1 = 6s+1, \lambda_2 = 12s+1)$ does not exist for any positive integer s.*

Since we are considering for the cases on the main diagonal entries of Table 6.4.1, if we let $\lambda_1 \equiv 2 \pmod 6$, say $\lambda_1 = 6s+2$ for some integer s, then $\lambda_2 \equiv 2 \pmod 6$. But for the existence of GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$, $\lambda_2 < 2\lambda_1$ by Theorem 6.4.4. As $\lambda_1 < \lambda_2$, the only choice for λ_2 is from $\{6s+8, 6s+14, ..., 12s+2\}$. So when $\lambda_1 = 2$, GDD$(1,n,n+2,4;2,\lambda_2)$ does not exist. As $n \equiv 2,5 \pmod 6$ when $\lambda_1 \equiv 2 \pmod 6$, the smallest design to construct is a GDD$(1,2,4,4;8,14)$. But this design does not exist. For $\lambda_1 = 14$, $\lambda_2 < 28$ implies that $\lambda_2 = 20$ or 26 and the designs GDD$(1,2,4,4;14,20)$, GDD$(1,2,4,4;14,26)$ and a GDD$(1,5,7,4;14,26)$ does not exist.

For $\lambda_1 = 20$, λ_2 can be 26, 32, or 38 but a GDD$(1,2,4,4;20,26)$, a GDD$(1,2,4,4;20,32)$, a GDD$(1,2,4,4;20,38)$ and a GDD$(1,5,7,4;20,38)$ do not exist. Moreover, for $n \equiv 2,5 \pmod 6$, $\lambda_1 = 2$ and $\lambda_2 = 6s+2$ for some positive integer s, $\lambda_1(n^2+n+1) \ngeq 2b - r_1$. So we have the following:

Lemma 6.5.8. *A GDD$(1,n,n+2,4;2,6s+2)$ for $n \equiv 2,5 \pmod 6$ and a GDD$(1,2,4,4;6s+2,12s+2)$ does not exist for any positive integer s and a GDD$(1,5,7,4;6s+2,12s+2)$ does not exist for any positive integer $s > 1$.*

By the same argument, it can be easily shown that a GDD$(1,n,n+2,4;3,6s+3)$, a GDD$(1,n,n+2,4;4,6s+4)$ and a GDD$(1,n,n+2,4;5,6s+5)$ for $n \equiv 1,3,5 \pmod 6$, $n \equiv 2,5 \pmod 6$ and $n \equiv 5 \pmod 6$ respectively do not exist for some positive integer s. Moreover, a GDD$(1,2,4,4;6s+9,12s+15)$ and a GDD$(1,2,4,4;6s+4,12s+4)$ does not exist for a non negative integer s and for a positive integer t as $\lambda_1(n^2+n+1) \ngeq 2b - r_1$.

6.5.3 GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ with the cases on the off-diagonal entries of Table 6.4.1.

$\lambda_1 \equiv 0 \pmod 6$ **and** $\lambda_2 \equiv 3 \pmod 6$**.**

For $n \geq 4$, BIBD$(n,4,6)$ and BIBD$(n+2,4,6)$ exist and hence a GDD$(1,n,n+2,4;6t,0)$ exists for some positive integer t. But a GDD$(1,n,n+2,4;0,\lambda_2)$ does not exist for any n as the size of the block is greater than the number of groups. Hence, a GDD$(1,n,n+2,4;0,3)$ does not exist for $n \equiv 0,2,4 \pmod 6$. Although, it is difficult to obtain general existence result for a GDD$(1,n,n+2,4;6,3)$ when $n \equiv 0,2,4 \pmod 6$, we may construct such GDD for small values of n. For instance, a GDD$(1,2,4,4;6,3)$ and a GDD$(1,2,4,4;8,5)$ exist (see Examples 6.4.3, 6.3.1).

$\lambda_1 \equiv 2 \pmod 6$ **and** $\lambda_2 \equiv 5 \pmod 6$

Consider a GDD$(1,n,n+2,4;2,5)$ for $n \equiv 2 \pmod 6$. Let $n = 6t+2$ for some nonnegative integer t. Then $r_1 = 20t+10, r_2 = 14t+9, r_3 = 14t+7$ and $b = 42t^2+50t+14$. Since $\lambda_1(n^2+n+1) = 72t^2+60t+14$ and $2b-r_1 = 84t^2+80t+18$, by Lemma 6.4.3, it does not exist. Hence, we have

Lemma 6.5.9. *A GDD$(1,n,n+2,4,2s,5s)$ does not exist for $n \equiv 2 \pmod 6$ and any positive integer s.*

$\lambda_1 \equiv 4 \pmod 6$ **and** $\lambda_2 \equiv 1 \pmod 6$

Suppose a GDD$(1,2,4,4;4,1)$ with groups G_1, G_2, G_3 of cardinality $1,2,4$ respectively exists. Then $r_1 = 2, r_2 = 3, r_3 = 5$ and $b = 7$. As every pair of elements of G_i for $i = 2,3$ occurs in exactly 4 blocks, each element of G_2 occurs 4 times in the design. But this is a contradiction as $r_2 = 3$. Thus, a GDD$(1,2,4,4;4,1)$ does not exist.

As a summary of this section, we have the following theorem.

Theorem 6.5.10. *Necessary conditions are sufficient for the existence of a GDD$(1,n,n+$*

$2,4;\lambda_1,\lambda_2)$ *when* $\lambda_1 \geq \lambda_2$ *and* $\lambda_1 \equiv \lambda_2 \pmod 6$ *except for* n *even,* $\lambda_1 \equiv 0 \pmod 6$ *and* $\lambda_2 \equiv 3 \pmod 6$*, and for* $n \equiv 2 \pmod 6, \lambda_1 \equiv 2,4 \pmod 6$ *and* $\lambda_2 \equiv 1,5 \pmod 6$.

6.6 GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$

Recall that (Henson et al., 2007) and (Zhu and Ge, 2010), proved the necessary conditions are sufficient for the existence of a GDD$(n,3,4;\lambda_1,\lambda_2)$ with Configuration $(1,1,2)$, i.e. each block in a design contains an element from the two of the three groups, respectively, and the two remaining elements in the block are from the third group. Note that three groups have the same size n in this problem. Namyalo et al. (2017), established some general results for GDD$(n_1,n,n+1,4;\lambda_1,\ \lambda_2)$ where $n_1 = 1$ or $n_1 = 2$. In the next sections we study GDD$(n_1,n,n+1,4;\lambda_1,\ \lambda_2)$ for $n_1 = 3$ and $n \geq 3$. If $n_1 = 3, n_2 = n$ and $n_3 = n+1$, a GDD$(3,n,n+1,4;\lambda,\lambda)$ exists whenever a BIBD$(2n+4,4,\lambda)$ exists. We have the following result from Corollary 6.1.3.

Corollary 6.6.1. *A GDD*$(3,n,n+1,4;\lambda,\lambda)$ *exists if* $n \equiv 0 \pmod 6$. *Also, a GDD*$(3,n,n+1,4;\lambda,\lambda)$ *exists if* $n \equiv 2,4 \pmod 6$ *and* $\lambda \equiv 0,3 \pmod 6$*, and a GDD*$(3,n,n+1,4;\lambda,\lambda)$ *exists if* $n \equiv 1,5 \pmod 6$ *and* $\lambda \equiv 0 \pmod 6$*, and a GDD*$(3,n,n+1,4;\lambda,\lambda)$ *exists if* $n \equiv 3 \pmod 6$ *and* $\lambda \equiv 0,2,4 \pmod 6$.

6.7 Necessary conditions for GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$

Suppose a GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$ exists and has b blocks. Notice that he number of the first associate pairs equals $\binom{3}{2}+\binom{n}{2}+\binom{n+1}{2} = n^2+3$, and the number of the second associate pairs equals $3(2n+1)+n(n+1) = n^2+7n+3$. Since each block of size 4 has $\binom{4}{2} = 6$ pairs, the number of blocks $b = \frac{\lambda_1(n^2+3)+\lambda_2(n^2+7n+3)}{6}$. Since b must be integers, we have the following.

i. $n \equiv 0 \pmod 6$: $\lambda_1+\lambda_2 \equiv 0 \pmod 2$, i.e., $\lambda_1 \equiv \lambda_2 \pmod 2$.

ii. $n \equiv 1 \pmod 6$: $4\lambda_1+5\lambda_2 \equiv 0 \pmod 6$.

iii. $n \equiv 2 \pmod 6$: $\lambda_1 + 3\lambda_2 \equiv 0 \pmod 6$.

iv. $n \equiv 3 \pmod 6$: $\lambda_2 \equiv 0 \pmod 2$, i.e., λ_2 is even.

v. $n \equiv 4 \pmod 6$: $\lambda_1 + 5\lambda_2 \equiv 0 \pmod 6$.

vi. $n \equiv 5 \pmod 6$: $4\lambda_1 + 3\lambda_2 \equiv 0 \pmod 6$.

The replication number for an arbitrary point in group G_1 is $r_1 = \frac{2\lambda_1+\lambda_2(2n+1)}{3}$. Similarly, $r_2 = \frac{\lambda_1(n-1)+\lambda_2(n+4)}{3}$ and $r_3 = \frac{\lambda_1 n+\lambda_2(n+3)}{3}$. Since r_1, r_2 and r_3 must be integers, we have the following.

i. $n \equiv 0 \pmod 3$: From r_1, we have $2\lambda_1 + \lambda_2 \equiv 0 \pmod 3$. From r_2, we have $\lambda_2 - \lambda_1 \equiv 0 \pmod 3$. Since there is no condition on r_3, $\lambda_2 - \lambda_1 \equiv 0 \pmod 3$ for this case.

ii. $n \equiv 1 \pmod 3$: From r_1, we have $\lambda_1 \equiv 0 \pmod 3$. From r_2, we have $\lambda_2 \equiv 0 \pmod 3$. From r_3, we have $\lambda_1 + \lambda_2 \equiv 0 \pmod 3$. Hence, $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

iii. $n \equiv 2 \pmod 3$: From both r_1 and r_3, we have $\lambda_1 + \lambda_2 \equiv 0 \pmod 3$. From r_2, we have $\lambda_1 \equiv 0 \pmod 3$. Hence, $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$.

Combining the cases above, we have the following necessary conditions for the existence of a GDD$(3, n, n+1, 4; \lambda_1, \lambda_2)$.

1. $n \equiv 0 \pmod 6$: $\lambda_2 - \lambda_1 \equiv 0 \pmod 6$.

2. $n \equiv 1 \pmod 6$: $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$.

3. $n \equiv 2 \pmod 6$: $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$, and λ_1 and λ_2 have the same parity.

4. $n \equiv 3 \pmod 6$: $\lambda_2 - \lambda_1 \equiv 0 \pmod 3$ and λ_2 is even.

5. $n \equiv 4 \pmod 6$: $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$, and λ_1 and λ_2 have the same parity.

6. $n \equiv 5 \pmod 6$: $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 6$.

Notice that λ_1 should be less than or equal to the minimum of $\{r_1, r_2, r_3\}$ in general. Since $r_1 = \min\{r_1, r_2, r_3\}$ for GDD$(3, n, n+1, 4; \lambda_1, \lambda_2)$, $\lambda_1 \leq r_1 = \frac{2\lambda_1+\lambda_2(2n+1)}{3}$, i.e., $\lambda_1 \leq \lambda_2(2n+1)$. Also, since every block has at least one first associate pair, we have $b \leq \lambda_1(\binom{3}{2} + \binom{n}{2} + \binom{n+1}{2})$, i.e. $\frac{\lambda_1(n^2+3)+\lambda_2(n^2+7n+3)}{6} \leq \lambda_1(n^2+3)$. Simplify this and combine the above, we have $\frac{\lambda_2(n^2+7n+3)}{5(n^2+3)} \leq \lambda_1 \leq \lambda_2(2n+1)$.

From the necessary conditions, if λ_1 is even and λ_2 is odd, a GDD$(3, n, n+1, 4; \lambda_1, \lambda_2)$ does not exist. We have the following.

Corollary 6.7.1. *A GDD$(3, n, n+1, 4; 2s, 2t+1)$ does not exist.*

6.8 GDD$(3, 3, 4, 4; \lambda_1, \lambda_2)$

If $n = n_1 = 3$, then $\lambda_1 < r_1$, i.e., a GDD$(3, 3, 4, 4; \lambda_1, \lambda_2)$ does not exist if $\lambda_1 = r_1$. Using the bounds for λ_1 obtained in Section 6.7, we have $\frac{11\lambda_2}{20} \leq \lambda_1 < 7\lambda_2$. Also, $r_1 = r_2 = \frac{2\lambda_1+7\lambda_2}{3}$ implies that all elements of the same group occur together, and as block size is greater than 3, we have second associate pairs from G_1 and G_2.

The upper bound on λ_1 can be improved further. Suppose $G_1 = \{a, b, c\}$, $G_2 = \{x, y, z\}$, and a GDD$(3, 3, 4, 4; \lambda_1, \lambda_2)$ exists. Since $r_1 > \lambda_1$, let $r_1 = \lambda_1 + s$ where $s = r_1 - \lambda_1 = \frac{2\lambda_1+7\lambda_2}{3} - \lambda_1 = \frac{7\lambda_2-\lambda_1}{3}$. Notice that there are at most $\lambda_1 - s$ blocks of type $\{a, b, c, *\}$ and $\{x, y, z, *\}$ in the design, respectively, where $*$ is an element from a different group. This implies that the second associate pairs of elements from G_1 and G_2 appear at most $2 \times 3 \times (\lambda_1 - s)$ times in these blocks. Since G_3 has four elements, there are at most $4\lambda_2$ blocks of the type $\{a, b, c, *\}$ where $*$ is an element in G_3, and there are at most $4\lambda_2$ blocks of the type $\{x, y, z, *\}$ where $*$ is an element in G_3. In the remaining of the blocks of the type $\{a, b, c, *\}$ or $\{x, y, z, *\}$ in the design, $*$ must be an element from G_2 or G_1, respectively. That is, the second associate pairs of elements from G_1 and G_2 appear at most $6(\lambda_1 - s) - 2 \times 3 \times 4\lambda_2$ times in those remaining blocks. Since the second associate pairs of elements from G_1 and G_2 appears $9\lambda_2$ times in the design, $6(\lambda_1 - s) - 24\lambda_2 \leq 9\lambda_2$, i.e., $2(\lambda_1 - s) \leq 11\lambda_2$. Replace s with $\frac{7\lambda_2-\lambda_1}{3}$ and simplify, we have $\lambda_1 \leq \frac{47\lambda_2}{8}$.

Note that from the necessary conditions, a GDD$(3, 3, 4, 4; \lambda_1, 2t+1)$ does not exist.

6.8.1 GDD$(3,3,4,4;\lambda_1,2)$

If $n = 3$ and $\lambda_2 = 2$, from the necessary conditions, we have $\lambda_1 \equiv 2 \pmod 3$ and $\frac{11}{10} \leq \lambda_1 \leq \frac{47}{4}$, i.e. $\lambda_1 = 2,5,8$ or 11. A GDD$(3,3,4,4;2,2)$ is just a BIBD$(10,4,2)$ which exists. The blocks of a GDD$(3,3,4,4;\lambda_1,2)$ where $\lambda_1 = 5,8,11$, respectively, are given below. Let $G_1 = \{1,2,3\}$, $G_2 = \{a,b,c\}$ and $G_3 = \{w,x,y,z\}$.

Example 6.8.1. A GDD$(3,3,4,4;5,2)$:

1	1	1	a	a	a	1	1	1	1	2	2	a	a	a	a	b	b	x	x	x
2	2	2	b	b	b	2	2	3	3	3	3	b	b	c	c	c	c	y	y	y
3	3	3	c	c	c	x	z	x	y	x	y	x	z	x	y	x	y	z	z	z
a	b	c	1	2	3	y	w	z	w	w	z	y	w	z	w	w	z	w	w	w

Example 6.8.2. A GDD$(3,3,4,4;8,2)$:

1	1	1	1	1	1	1	1	a	a	a	a	a	a
2	2	2	2	2	2	2	2	b	b	b	b	b	b
3	3	3	3	3	3	3	3	c	c	c	c	c	c
x	y	z	w	x	y	z	w	1	2	3	1	2	3

a	a	a	a	b	b	x	x	x	x	x	x	x	
b	b	c	c	c	c	y	y	y	y	y	y	y	
x	z	x	y	x	y	z	z	z	z	z	z	z	
y	w	z	w	w	z	w	w	w	w	w	w	w	

Example 6.8.3. A GDD$(3,3,4,4;11,2)$:

1	1	1	1	1	1	1	1	a	a	a	a	a	a	a	a
2	2	2	2	2	2	2	2	b	b	b	b	b	b	b	b
3	3	3	3	3	3	3	3	c	c	c	c	c	c	c	c
x	y	z	w	x	y	z	w	x	y	z	w	x	y	z	w

1	1	1	a	a	a	x	x	x	x	x	x	x	x	x	x
2	2	2	b	b	b	y	y	y	y	y	y	y	y	y	y
3	3	3	c	c	c	z	z	z	z	z	z	z	z	z	z
a	b	c	1	2	3	w	w	w	w	w	w	w	w	w	w

Lemma 6.8.1. *Necessary conditions are sufficient for the existence of a GDD*$(3,3,4,4;\lambda_1,2)$.

6.8.2 GDD$(3,3,4,4;\lambda_1,4)$

If $n=3$ and $\lambda_2=4$, from the necessary conditions, we have $\lambda_1 \equiv 1 \pmod 3$ and $\frac{11}{5} \leq \lambda_1 \leq \frac{47}{2}$, i.e. $\lambda_1 = 3t+1$ where $1 \leq t \leq 7$.

A GDD$(3,3,4,4;4,4)$ is just a BIBD$(10,4,4)$ which exists. A GDD $(3,3,4,4;7,4)$ can be obtained by combining the blocks of a BIBD$(10,4,2)$ and the blocks of a GDD$(3,3,4,4;5,2)$. A GDD$(3,3,4,4;10,4)$ can be obtained by putting together two copies of the blocks of a GDD$(3,3,4,4;5,2)$. A GDD$(3,3,4,4;13,4)$ can be obtained by combining the blocks of a GDD $(3,3,4,4;5,2)$ and the blocks of a GDD$(3,3,4,4;8,2)$. A GDD $(3,3,4,4;\ 16,4)$ can be obtained by putting together two copies of the blocks of a GDD$(3,3,4,4;8,2)$. A GDD$(3,3,4,4;19,4)$ can be obtained by combining the blocks of a GDD $(3,3,4,4;8,2)$ and the blocks of a GDD $(3,3,4,4;11,2)$. A GDD $(3,3,4,4;\ 22,4)$ can be obtained by putting together two copies of the blocks of a GDD $(3,3,4,4;11,2)$.

Lemma 6.8.2. *Necessary conditions are sufficient for the existence of a GDD*$(3,3,4,4;\lambda_1,4)$.

6.8.3 GDD$(3,3,4,4;\lambda_1,6)$

If $n=3$ and $\lambda_2=6$, from the necessary conditions, we have $\lambda_1 \equiv 0 \pmod 3$ and $\frac{33}{10} \leq \lambda_1 \leq \frac{141}{4}$, i.e. $\lambda_1 = 3t$ where $2 \leq t \leq 11$.

Let a D(λ_1,λ_2) represent a GDD$(3,3,4,4;\lambda_1,\lambda_2)$. A D$(6,6)$ can be obtained by combining three copies of the blocks of a D$(2,2)$ (it is also a BIBD$(10,4,6)$ which exists). A D$(9,6)$ can be obtained by combining two copies of the blocks of a D$(2,2)$ and the blocks of a D$(5,2)$. A D$(12,6)$ can be obtained by combining two copies of the blocks of a D$(5,2)$ and the blocks of a D$(2,2)$. A D$(15,6)$ can be obtained by combining three copies of the blocks of a D$(5,2)$. A D$(18,6)$ can be obtained by combining two copies of the blocks of a D$(5,2)$ and the blocks of a D$(8,2)$. A D$(21,6)$ can be obtained by combining two copies of the blocks of a D$(5,2)$ and the blocks of a D$(11,2)$. A D$(24,6)$ can be obtained by combining three copies of the blocks of a D$(8,2)$. A D$(27,6)$ can be obtained by combining two copies of the blocks of a D$(11,2)$ and the blocks of a D$(5,2)$. A D$(30,6)$ can be obtained by combining two copies of the blocks of a D$(11,2)$ and the blocks of a D$(8,2)$. A D$(33,6)$ can be obtained by combining three copies of the blocks of a D$(11,2)$. We have the following.

Lemma 6.8.3. *Necessary conditions are sufficient for the existence of a GDD*$(3,3,4,4;\lambda_1,6)$.

6.9 GDD$(3,4,5,4;\lambda_1,\lambda_2)$

If $n=4$, from the necessary conditions, we have $\lambda_1 \equiv 0 \pmod 3$, $\lambda_2 \equiv 0 \pmod 3$, and λ_1 and λ_2 have the same parity. In addition, $\frac{47\lambda_2}{95} \leq \lambda_1 \leq 9\lambda_2$.

6.9.1 GDD$(3,4,5,4;3(3+t),3(1+t))$

If $\lambda_1 = 3(3+t)$ and $\lambda_2 = 3(1+t)$ for $t \geq 0$, the necessary conditions are satisfied. We first construct the 52 blocks of a GDD$(3,4,5,4;9,3)$ (where $t=0$) as follows. Suppose $G_1=\{x,y,z\}$, $G_2=\{a,b,c,d\}$ and $G_3=\{1,2,3,4,5\}$:

The 28 blocks on the top consist of $\{a,b,c,i\}$, $\{a,b,d,i\}$, $\{a,c,d,i\}$, $\{b,c,d,i\}$ where $i \in G_1$, 3 copies of $\{a,b,c,d\}$, $\{x,y,z,i\}$ where $i \in G_3$, $\{1,2,x,y\}$, $\{3,4,x,y\}$, $\{1,3,x,z\}$, $\{2,4,x,z\}$, $\{1,4,y,z\}$, $\{2,3,y,z\}$, and 2 copies of $\{x,y,z,5\}$.

A BIBD$(5,3,6)$ on G_3 is 3- resolvable in four 3- resolvable classes. The first 20 blocks on the bottom are formed by combining the triples of the first class with a, of the second with b, of the third with c and the fourth with d. The last four blocks are the blocks of a BIBD$(5,4,3)$ on G_3 except for $\{1,2,3,4\}$.

A BIBD$(12,4,3)$ exists (it is also a GDD$(3,4,5,4;3,3)$). Combining the blocks of t copies of a BIBD$(12,4,3)$ and the blocks of a GDD$(3,4,5,4;9,3)$, we have a GDD$(3,4,5,4;9+3t,3+3t)$ for all $t \geq 0$.

6.10 GDD$(3,6,7,4;\lambda_1,\lambda_2)$

If $n=6$, from the necessary conditions, we have $\lambda_2-\lambda_1 \equiv 0 \pmod 6$ and $\frac{27\lambda_2}{65} \leq \lambda_1 \leq 13\lambda_2$.

6.10.1 GDD$(3,6,7,4;\lambda_1,1)$ and GDD$(3,6t,6t+1,4;12t+1,1)$

If $n=6$ and $\lambda_2=1$, from the necessary conditions, $\lambda_1=7$ and 13.

Let $G_1=\{a,b,c\}$, $G_2=\{x,y,z,w,u,v\}$ and $G_3=\{1,2,3,4,5,6,7\}$. The 59 blocks of a GDD$(3,6,7,4;7,1)$ are as follows: $G_1 \cup \{i\}$ where $i \in G_2$, $\{ab12, ac34, bc56, a756, b734, c712\}$, the blocks from a BIBD$(13,4,1)$ on $G_2 \cup G_3$, the blocks of a BIBD$(6,4,6)$ on G_2, and the blocks in $\{7135,7135,7136,7145,7146,7234,7245,7246,7236,7256,1245, 1246,1235,1236,1436,1456,2345,2356,3456\}$.

Let $G_1=\{a,b,c\}$, $G_2=\{x_1,x_2,\cdots,x_{6t}\}$ and $G_3=\{y_1,y_2,\cdots,y_{6t+1}\}$. The blocks of a GDD$(3,6t,6t+1,4;12t+1,1)$ are as follows: $G_1 \cup \{i\}$ where $i \in G_2$, $G_1 \cup \{j\}$ where $j \in G_3$, the blocks of a BIBD$(v,4,1)$ where $v \in G_2 \cup G_3$, the blocks of a BIBD$(v,4,12t)$ where $v \in G_2$, and the blocks of a BIBD$(v,4,12t)$ where $v \in G_3$. Specifically, if $t=1$, we have a GDD$(3,6,7,4;13,1)$. We have the following lemma.

Lemma 6.10.1. *The necessary conditions of a GDD$(3,6,7,4;\lambda_1,1)$ are sufficient.*

6.11 Summary

In this chapter we proposed and studied two new problems of group divisible designs with three groups and block size 4.The first problem involved group sizes 1, n and $n+2$, respectively. We showed that if equal number of blocks of block type configurations was required, a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ where $t>0$ did not exist. We then focused our study on the case where $t=2$ where there was no block configuration restriction. We provided the necessary conditions for the existence of a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$. Also, we proved that the necessary conditions were sufficient for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ when $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$ except for n even, $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 3 \pmod 6$, and for $n \equiv 2 \pmod 6$, $\lambda_1 \equiv 2,4 \pmod 6$ and $\lambda_2 \equiv 1,5 \pmod 6$.

The second problem involved group sizes 3, n and $n+1$, respectively. Group size of the first group being 3 made the problem much more challenging. We obtained necessary conditions for the existence of a GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$, and we provided constructions for certain families of the problem and showed that the necessary conditions were sufficient for these families.

Chapter 7

4-GDDs with block size 5

7.1 Introduction

In this chapter, we present a 4-Group Divisible Designs (4-GDDs) with blocks of size 5. First, we give general definitions for t-GDD when every t subset from different groups occurs the same number of times as well as every t subset from different groups does not occur the same number of times. Then we study the existence of a 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ and obtain necessary conditions for its existence and prove that these necessary conditions are sufficient for $n = 3,4,5,6,7,9,10$.

Sarvate and Shinde (2020) generalized Definition 2.4.1 for any number of groups and any block size k (see Definition 2.4.2), but it can be stated in general for any t-GDDs instead of 3-GDDs, and the most straight forward generalization is:

Definition 7.1.1. A t-GDD$(n,m,k;\Lambda_1,\Lambda_2)$ is a triple $(X,\mathscr{G},\mathscr{B})$ where X is a set of mn elements, $\mathscr{G} = \{G_1,G_2,...,G_m\}$ is a partition of X into m parts called groups of size n, and $\mathscr{B}$ is a collection of k-subsets called blocks of X such that

(i) every t-subset of configuration $(t,0)$, i.e., where all t elements are from the same group occurs in Λ_1 blocks,

(ii) any other t-subset of X where elements are not from the same group occurs in Λ_2 blocks.

If we relax Condition (ii), the above definition can be restated for $m = 2$ and $t > 3$ as a t-PBIBD (t-partially balanced incomplete block design):

Definition 7.1.2. (i) For odd t, a t-PBIBD$(n,2,k;\Lambda_1,\Lambda_2,...,\Lambda_{\lceil \frac{t}{2} \rceil})$ is a triple $(X,\mathscr{G},\mathscr{B})$ where X is a set of $2n$ elements, $\mathscr{G}$ is a partition of X into 2 parts called groups of size n each and $\mathscr{B}$

is a collection of k-subsets of X called blocks such that every t-subset of configuration $(t,0)$, $(t-1,1)$, ..., $(\frac{t+1}{2},\frac{t-1}{2})$ occurs in Λ_1, Λ_2, ..., $\Lambda_{\lceil \frac{t}{2} \rceil}$ blocks respectively.
(ii) For even t, a t-PBIBD$(n,2,k;\Lambda_1,\Lambda_2,...,\Lambda_{\lceil \frac{t+1}{2} \rceil})$ is a triple $(X,\mathscr{G},\mathscr{B})$ where X is a set of $2n$ elements, $\mathscr{G}$ is a partition of Xinto 2 parts called groups of size n each and $\mathscr{B}$ is a collection of k-subsets of X called blocks such that every t-subset of configuration $(t,0)$, $(t-1,1)$, ..., $(\frac{t}{2},\frac{t}{2})$ occurs in Λ_1, Λ_2, ..., $\Lambda_{\lceil \frac{t+1}{2} \rceil}$ blocks respectively.

In the next section we will study 4-GDDs with $k=5$ and $m=2$.

7.2 4-GDDs with 2 groups and block Size 5

Let X be a set of mn elements where m and n are positive integers. For any positive integers t and k such that $t \leq k$ and $t \leq n$, consider all possible k-subsets of X. Then a t-subsets of X can be joined with any one of $(k-t)$-subsets from the remaining $(mn-t)$ elements and is contained in $\binom{mn-t}{k-t}$ k-subsets. It is well known that the set of all k-subsets of a v-set X is a t-$(v,k,\binom{v-t}{k-t})$ design for every t, $1 \leq t \leq k$. Let $v=mn$ and we partition X into m groups $G_1,G_2,...,G_m$ of size n each. By removing all k-subsets from the t-$(v,k,\binom{v-t}{k-t})$ design, we get a t-GDD$(n,m,k;\Lambda_1=\binom{v-t}{k-t}-(m-1)\binom{n-t}{k-t},\Lambda_2=\binom{v-t}{k-t})$. Thus, we have

Theorem 7.2.1. *A t-GDD$(n,m,k;\Lambda_1=\binom{v-t}{k-t}-(m-1)\binom{n-t}{k-t},\Lambda_2=\binom{v-t}{k-t})$ exists for $t \leq k$ and $t \leq n$.*

For $t=k-1$, $\Lambda_1=\binom{mn-k+1}{k-k+1}-(m-1)\binom{n-k+1}{k-k+1}=n+(m-2)(k-1)$ and $\Lambda_2=mn-k+1$. Hence we have

Corollary 7.2.2. *A $(k-1)$-GDD$(n,m,k;n+(m-2)(k-1),mn-k+1)$ exists for $k-1 \leq n$. Specifically for $m=2$, a $(k-1)$-GDD$(n,2,k;n,2n-k+1)$ exists.*

For ease of reference, Definition 7.1.1 can be restated as

Definition 7.2.1. A 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ is a triple $(X,\mathscr{G},\mathscr{B})$ where X is a set of $2n$ elements, $\mathscr{G}$ is a partition of X into 2 parts of size n each called groups and $\mathscr{B}$ is a collection of 5-subsets of X called blocks such that

(i) every 4-subset of configuration $(4,0)$ occurs in Λ_1 blocks,

(ii) every 4-subset of configuration $(3,1)$ or of configuration $(2,2)$ occurs in Λ_2 blocks.

In addition, we can relax condition (ii) of Definition 7.2.1 as done earlier and define a 4-PBIBD$(n,2,5;\Lambda_1,\Lambda_2,\Lambda_3)$. A trivial family of a 4-PBIBD$(n,2,5;\Lambda_1=0,\Lambda_2=n-1,\Lambda_3=2(n-2))$ is obtained by constructing block of size five by taking union of every triple (3-subset) of each group with every pair (2-subset) of the other group for $n \geq 3$. The first trivial example is needed for Theorem 7.2.4 where $\Lambda_2 = \Lambda_3$.

Example 7.2.1. A 4-GDD$(3,2,5;0,2)$ exists.

Example 7.2.2. A 4-GDD$(4,2,5;4,4)$ = 4-$(8,5,4)$ exists with $X = \{1,2,3,4,a,b,c,d\}$, $G_1 = \{1,2,3,4\}$, $G_2 = \{a,b,c,d\}$.

In a $(k-1)$-GDD$(n,2,k;n,2n-k+1)$, every $(k-1)$-subset of configuration $(k-2,1)$ occurs in $(n-k+2)$ blocks of type $(k-1,1)$ and in $(n-1)$ blocks of type $(k-2,2)$. If we delete all the blocks of type $(k-1,1)$, then $\Lambda_1 = 0$ and $(k-1)$-subsets of configuration $(k-2,1)$ occur $(n-1)$ times in the design. But other $(k-1)$-subsets such as of configuration $(k-3,2)$, $(k-4,3)$ etc. occur $(2n-k+1)$ times as they do not occur in the blocks of type $(k-1,1)$. Hence, we have the following theorem.

Theorem 7.2.3. *A $(k-1)$-PBIBD$(n,2,k;0,n-1,2n-k+1)$ exists. In particular, a $(n+1)$-GDD$(n,2,n+2;0,n-1)$ exists.*

Example 7.2.3. Removing 8 blocks of type $(4,1)$ from Example 7.2.2, we get a 4-PBIBD$(4,2,5;0,3,4)$.

7.2.1 4-GDDs with 2 groups and block size 5 for $\Lambda_1 = 0$

A 4-GDD$(n,2,5;0,\Lambda_2)$ contains only blocks of type $(3,2)$ if it exists. The following theorem gives a necessary and sufficient condition for the existence of a 4-GDD$(n,2,5;0,\Lambda_2)$.

Theorem 7.2.4. *Necessary conditions, $n = 3$ and Λ_2 is even, are sufficient for the existence of 4-GDD$(n,2,5;0,\Lambda_2)$.*

Proof. First we prove that the necessary conditions for the existence of a 4-GDD$(n,2,5;0,\Lambda_2)$ are $n=3$ and Λ_2 is even. Assume that a 4-GDD$(n,2,5;0,\Lambda_2)$ exists. Then the design contains only one type of blocks of configuration $(3,2)$ where 3 elements are from one group and 2 from the other group. As every block has 3 quadruples of type $(2,2)$ and 2 quadruples of type $(3,1)$, we have $b=\frac{n^2(n-1)(n-2)}{6}\Lambda_2=\frac{n^2(n-1)^2}{12}\Lambda_2$. But this equation holds only for $n=3$.

Further, as $n=3$, there are $3\Lambda_2$ quadruples containing a given group of size 3 in the blocks of the design if it exists and each block has only two 2 quadruples of type $(3,1)$. Therefore, Λ_2 is even.

A 4-GDD$(3,2,5;0,2)$ is given in Example 7.2.1. Hence, taking t copies of the blocks of a 4-GDD$(3,2,5;0,2)$ provide the required 4-GDD for some positive integer t. □

The necessary and sufficient conditions for the existence of a 4-GDD$(n,2,k;\Lambda_1,0)$ are the same as the conditions for a 4-(n,k,Λ_1). In the next subsection, we obtain the spectrum for the existence of a 4-GDD$(n,2,5;\Lambda_1,0)$.

7.2.2 4-GDDs with 2 groups and block size 5 for $\Lambda_2=0$

A 4-GDD$(n,2,5,\Lambda_1,0)$ is the collection of blocks of a 4-$(n,5,\Lambda_1)$ on n elements of G_i for $i=1,2$. Hence, necessary condition for the existence of a 4-GDD$(n,2,5;\Lambda_1,0)$ is same as the conditions for the existence of a 4-$(n,5,\Lambda_1)$, viz.,

$$\lambda_0=\frac{n(n-1)(n-2)(n-3)}{120}\Lambda_1 \tag{7.2.1}$$

$$\lambda_1=\frac{(n-1)(n-2)(n-3)}{24}\Lambda_1 \tag{7.2.2}$$

$$\lambda_2=\frac{(n-2)(n-3)}{6}\Lambda_1 \tag{7.2.3}$$

$$\lambda_3 = \frac{n-3}{2}\Lambda_1. \tag{7.2.4}$$

As λ_s is an integer for $s = 0,1,2,3$, from the divisibility requirements, we get the spectrum for the existence of a 4-$(n,5,\Lambda_1)$ as follows.

Case 1: Λ_1 odd.

As Λ_1 is odd, from λ_3, n is odd. Then $n \equiv 1,3,5 \pmod 6$. When $n \equiv 1 \pmod 6$, from λ_2, $\Lambda_1 \equiv 0 \pmod 3$ but for $n \equiv 3,5 \pmod 6$ there is no restrictin on Λ_1. From λ_1, when $n \equiv 1 \pmod 6$, $\Lambda_1 \equiv 0 \pmod 3$ and when $n \equiv 3,5 \pmod 6$ no restriction on Λ_1.
Let $n = 6t+1$. By taking cases of $t \equiv 0,1,2,3,4 \pmod 5$, from λ_0, we obtain $n \equiv 1,7,13,25 \pmod{30}$, $\Lambda_1 \equiv 0 \pmod 3$ when $t \equiv 0,1,2,4 \pmod 5$ and $n \equiv 19 \pmod{30}$, $\Lambda_1 \equiv 15 \pmod{30}$ when $t \equiv 3 \pmod 5$.
Let $n = 6t+3$. By taking cases of $t \equiv 0,1,2,3,4 \pmod 5$, from λ_0, we obtain $n \equiv 3,15,21,27 \pmod{30}$ and no restriction on Λ_1. But when $t \equiv 1 \pmod 5$, $n \equiv 9 \pmod{30}$ and $\Lambda_1 \equiv 5 \pmod{10}$.
Similarily, for $n = 6t+5$, $n \equiv 5,11,17,23 \pmod{30}$ and no restriction on Λ_1 when $t \equiv 0,1,2,3 \pmod 5$. But $n \equiv 29 \pmod{30}$ and $\Lambda_1 \equiv 5 \pmod{10}$ when $t \equiv 4 \pmod 5$. Hence we have the following:

Lemma 7.2.5. *Necessary conditions for the existence of a* 4-$(n,5,\Lambda_1)$ *when* Λ_1 *odd:*

i. $n \equiv 1,7,13,25 \pmod{30}$, $\Lambda_1 \equiv 0 \pmod 3$*;*

ii. $n \equiv 3,5,11,15,17,21,23,27 \pmod{30}$*, no restriction on* Λ_1*;*

iii. $n \equiv 9,29 \pmod{30}$, $\Lambda_1 \equiv 5 \pmod{10}$*;*

iv. $n \equiv 19 \pmod{30}$ *and* $\Lambda_1 \equiv 15 \pmod{30}$.

Case 2: Λ_1 even.

As Λ_1 is even, from λ_3, no restriction on n. From λ_2, when $\Lambda_1 \equiv 0 \pmod 6$, $n \equiv 1,2,3 \pmod 4$. From λ_1, for $n \equiv 1,2,3 \pmod 4$, $\Lambda_1 \equiv 0 \pmod 6$.

Let $n = 4t+1$. From λ_0, we obtain $n \equiv 1,5,13,17 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod 6$ when $t \equiv 0,1,3,4 \pmod 5$ but $n \equiv 9 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod{30}$ when $t \equiv 2 \pmod 5$. For $n = 4t+2$, from λ_0, we obtain $n \equiv 2,6,10,18 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod 6$ when $t \equiv 0,1,2,4 \pmod 5$ but $n \equiv 14 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod{30}$ when $t \equiv 3 \pmod 5$. For $n = 4t+3$, $n \equiv 3,7,11,15 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod 6$ when $t \equiv 0,1,2,3 \pmod 5$ but $n \equiv 19 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod{30}$ when $t \equiv 4 \pmod 5$.

For $\Lambda_1 \not\equiv 0 \pmod 6$, from λ_2, $n \equiv 0,2 \pmod 3$. Consider $n \equiv 0 \pmod 3$. When $n \equiv 6 \pmod{12}$, from λ_1, there is no restriction on Λ_1 but when $n \equiv 0 \pmod{12}$, $\Lambda_1 \equiv 0 \pmod 4$. Let $n = 12t$. From λ_0, we obtain $n \equiv 0,12,36,48 \pmod{60}$, $\Lambda_1 \equiv 0 \pmod 4$ when $t \equiv 0,1,3,4 \pmod 5$ but $n \equiv 24 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{20}$ when $t \equiv 2 \pmod 5$.

Let $n = 12t+6$. From λ_0, we obtain $n \equiv 6,18,30,42 \pmod{60}$ and no restriction on Λ_1 when $t \equiv 0,1,2,3 \pmod 5$ but $n \equiv 54 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{10}$ when $t \equiv 4 \pmod 5$.

For $n \equiv 3 \pmod 6$, from λ_1, no restriction on Λ_1. Let $n = 6t+3$. By taking cases of $t \equiv 0,1,2,3,4 \pmod 5$, from λ_0, we obtain $n \equiv 3,15,21,27 \pmod{30}$ and no restriction on Λ_1 but $n \equiv 9 \pmod{30}$ and $\Lambda_1 \equiv 0 \pmod{10}$ when $t \equiv 1 \pmod 5$.

Consider $n \equiv 2 \pmod 3$. Then $n \equiv 2,5,8,11 \pmod{12}$. From λ_1, when $n \equiv 8 \pmod{12}$, $\Lambda_1 \equiv 0 \pmod 4$ and when $n \equiv 2,5,11 \pmod{12}$ no restriction on Λ_1. For $n \equiv 12t+2$, from λ_0, we have $n \equiv 2,26,38,50 \pmod{60}$ and no restriction on Λ_1 when $t \equiv 0,2,3,4 \pmod 5$ but $n \equiv 14 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{10}$ when $t \equiv 1 \pmod 5$. For $n \equiv 12t+5$, from λ_0, we have $n \equiv 5,17,41,53 \pmod{60}$ and no restriction on Λ_1 when $t \equiv 0,1,3,4 \pmod 5$ but $n \equiv 29 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{10}$ when $t \equiv 2 \pmod 5$. For $n \equiv 12t+11$, from λ_0, we have $n \equiv 11,23,35,47 \pmod{60}$ and no restriction on Λ_1 when $t \equiv 0,1,2,3 \pmod 5$ but $n \equiv 59 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{10}$ when $t \equiv 4 \pmod 5$.

For $n \equiv 12t+8$, from λ_0, we have $n \equiv 8,20,32,56 \pmod{60}$, $\Lambda_1 \equiv 0 \pmod 4$ when $t \equiv$

$0,1,2,4 \pmod 5$ but $n \equiv 44 \pmod{60}$ and $\Lambda_1 \equiv 0 \pmod{20}$ when $t \equiv 3 \pmod 5$.
For $n \equiv 0 \pmod 4$, from λ_2, $\Lambda_1 \equiv 0 \pmod 6$. But from λ_1, when $n \equiv 0 \pmod 4$, $\Lambda_1 \equiv 0 \pmod{12}$. Let $n = 4t$. From λ_0, we obtain $n \equiv 0,8,12,16 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod{12}$ when $t \equiv 0,2,3,4 \pmod 5$ but $n \equiv 4 \pmod{20}$ and $\Lambda_1 \equiv 0 \pmod{60}$ when $t \equiv 1 \pmod 5$. Hence we have,

Lemma 7.2.6. *Necessary conditions for the existence of a* 4-$(n,5,\Lambda_1)$ *when* Λ_1 *even:*

i. $n \equiv 0,8,12,20,32,36,48,56 \pmod{60}$, $\Lambda_1 \equiv 0 \pmod 4$*;*

ii. $n \equiv 1,2,3,5,6,7,10,11,13,15,17,18 \pmod{20}$, $\Lambda_1 \equiv 0 \pmod 6$*;*

iii. $n \equiv 2,3,5,6,11,15,17,18,21,23,26,27,30,33,35,38,41,42,45,47,50,51,53,57 \pmod{60}$*, no restriction on* Λ_1*;*

iv. $n \equiv 19,34,49 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{30}$*;*

v. $n \equiv 9,14,29,39,54,59 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{10}$*;*

vi. $n \equiv 24,44 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{20}$*;*

vii. $n \equiv 4 \pmod{20}$ *and* $\Lambda_1 \equiv 0 \pmod{60}$*;*

viii. $n \equiv 0,8,12,16 \pmod{20}$ *and* $\Lambda_1 \equiv 0 \pmod{12}$.

From Lemma 7.2.5 and Lemma 7.2.6, we have the following theorem.

Theorem 7.2.7. *Necessary conditions for the existence of a* 4-$(n,5,\Lambda_1)$ *are*

1. $n \equiv 0,2,6,8,12,18,20,26 \pmod{30}$ *and* $\Lambda_1 \equiv 0 \pmod 4$*;*
2. $n \equiv 1,7,13,25 \pmod{30}$ *and* $\Lambda_1 \equiv 0 \pmod 3$*;*
3. $n \equiv 2,6,18,26,30,38,42,50 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod 2$*;*
4. $n \equiv 3,5,11,15,17,21,23,27 \pmod{30}$ *and any* Λ_1*;*
5. $n \equiv 4 \pmod{20}$ *and* $\Lambda_1 \equiv 0 \pmod{60}$*;*

6. $n \equiv 9,29 \pmod{30}$ *and* $\Lambda_1 \equiv 0 \pmod{5}$*;*

7. $n \equiv 10,16,22,28 \pmod{30}$ *and* $\Lambda_1 \equiv 0 \pmod{12}$*;*

8. $n \equiv 10,22,46,58 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{6}$*;*

9. $n \equiv 14,54 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{10}$*;*

10. $n \equiv 19,49 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{15}$*;*

11. $n \equiv 24,44 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{20}$*;*

12. $n \equiv 34 \pmod{60}$ *and* $\Lambda_1 \equiv 0 \pmod{30}$.

If each $\lambda_s = \frac{\Lambda_1 \binom{n-s}{t-s}}{\binom{k-s}{t-s}}$ for $0 \leq s \leq t$ is an integer, then an order quadruple of positive integers $(\Lambda_1;t,k,n)$ is admissible and denoted by a t-(n,k,Λ_1). An admissible t-(n,k,Λ_1) is realizable if a t-(n,k,Λ_1) design exists. Admissible but not realizable parameter quadruples for $t = 4$ and $n \leq 30$ are 4-$(15,5,1)$, 4-$(12,6,2)$, 4-$(12,6,6)$ and 4-$(18,6,1)$. Many simple 4-designs with block size 5 exist and can be used to construct 4-GDDs, for example, 4-$(11,5,\Lambda_1)$, $\Lambda_1 \in \{1,2,3\}$; 4-$(12,5,4)$; 4-$(13,5,3)$; 4-$(15,5,\Lambda_1)$, $\Lambda_1 \in \{3,4,5\}$; 4-$(17,5,\Lambda_1)$, $\Lambda_1 \in \{3,4,...,6\}$; 4-$(18,5,\Lambda_1)$, $\Lambda_1 \in \{2,4,6\}$; 4-$(20,5,\Lambda_1)$, $\Lambda_1 \in \{4,8\}$; 4-$(21,5,\Lambda_1)$, $\Lambda_1 \in \{2,3,...,8\}$; 4-$(22,5,6)$; 4-$(23,5,\Lambda_1)$, $\Lambda_1 \in \{1,2,...,9\}$ (Laue and Khosrovshahi, 2007, pp 84-87).

In the next section, we obtain some necessary conditions for the existence of a 4-GDD$(n, 2,5;\Lambda_1,\Lambda_2)$ assuming it exists. We count the number of blocks, λ_1, containing a first associate pair, λ_2, the number of blocks containing a second associate pair, r_1, the number of blocks containing a triple from the same group, r_2, the number of blocks containing a triple of configuration $(2,1)$, the number of blocks containing a given element x (called the replication number r for x) and the required number of blocks, say b, for a 4-GDD.

7.3 Necessary Conditions

Suppose a 4-GDD$(n,2,k;\Lambda_1,\Lambda_2)$ exists. For a first associate pair $\{x_1,x_2\}$, 3 types of quadruples can be formed:

(i) $\{x_1,x_2,x_i,x_j\}$ where all are from the same group.

(ii) $\{x_1,x_2,x_i,y_i\}$ where y_i is from the other group.

(iii) $\{x_1,x_2,y_i,y_j\}$ where y_i and y_j are from the other group.

For each first associate pair $\{x_1,x_2\}$, there are $\binom{n-2}{2}$ quadruples of type $\{x_1,x_2,x_i,\ x_j\}$ that appear in Λ_1 blocks. Also the pair $\{x_1,x_2\}$ can form $(n-2)n$ quadruples of type $\{x_1,x_2,x_i,y_i\}$ and $\binom{n}{2}$ quadruples of type $\{x_1,x_2,y_i,y_j\}$ which appear in Λ_2 blocks. So, there are $\binom{n-2}{2}\Lambda_1+(n-2)n\Lambda_2+\binom{n}{2}\Lambda_2$ quadruples containing both x_1 and x_2 in the design. On the other hand, if a block contains $\{x_1,x_2\}$ then it has exactly $\binom{k-2}{2}$ quadruples containing both x_1 and x_2. As there are λ_1 blocks containing the first associate pair $\{x_1,x_2\}$, there are $\lambda_1\binom{k-2}{2}$ quadruples containing both x_1 and x_2 in the design. Thus, we have

$$\lambda_1=\frac{(n-2)(n-3)\Lambda_1+n(3n-5)\Lambda_2}{(k-2)(k-3)}. \tag{7.3.1}$$

Similar counting for a pair of second associates $\{x,y\}$ leads to:

$$\lambda_2=\frac{2(n-1)(2n-3)}{(k-2)(k-3)}\Lambda_2. \tag{7.3.2}$$

From Equations (7.3.1) and (7.3.2), every 4-GDD$(n,2,k;\Lambda_1,\Lambda_2)$ is also a GDD$(n,2,k;\lambda_1,\lambda_2)$.

Let r_1 be the number of blocks containing triples, say $\{x_1,x_2,x_3\}$, from the same group. For each triple $\{x_1,x_2,x_3\}$, where x_1,x_2,x_3 are from the same group; the following two types of quadruples can be formed:

(i) $\{x_1,x_2,x_3,x_i\}$ where x_1,x_2,x_3 and x_i are from the same group.

(ii) $\{x_1, x_2, x_3, y_i\}$ where y_i is from the other group.

The triple $\{x_1, x_2, x_3\}$ can form $(n-3)$ quadruples with an element x_i belonging to the same group. Such quadruples appear in Λ_1 blocks. Also the triple $\{x_1, x_2, x_3\}$ can form n quadruples with an element y_i belonging to the other group. Such quadruples appear Λ_2 times in a 4-GDD$(n, 2, k; \Lambda_1, \Lambda_2)$. On the other hand, if a block contains $\{x_1, x_2, x_3\}$ then it has exactly $(k-3)$ quadruples containing x_1, x_2 and x_3. Thus, we have

$$r_1 = \frac{(n-3)\Lambda_1 + n\Lambda_2}{k-3}. \tag{7.3.3}$$

Let r_2 be the number of blocks containing triples, say $\{x_1, x_2, y\}$, where x_1 and x_2 are from the same group and y is from the other group. From similar counting argument, we have

$$r_2 = \frac{2n-3}{k-3}\Lambda_2. \tag{7.3.4}$$

Thus, every 4-GDD$(n, 2, k; \Lambda_1, \Lambda_2)$ is also a 3-GDD$(n, 2, k; r_1, r_2)$.

With out loss of generality, let $x \in G_1$ and let r be the replication number for x. There are $\binom{n-1}{3}$ 4-subsets of type $\{x, x_i, x_j, x_k\}$ where all the elements are from the same group G_1. There are $\binom{n-1}{2}n$ 4-subsets of type $\{x, x_i, x_j, y_i\}$ where $x_i, x_j \in G_1$, $y_i \in G_2$ and $\binom{n}{3}$ 4-subsets of type $\{x, , y_i, y_j, y_k\}$ where $y_i, y_j, y_k \in G_2$. Also there are $(n-1)\binom{n}{2}$ 4-subsets of type $\{x, x_i, y_i, y_j\}$ where $x_i \in G_1$ and $y_i, y_j \in G_2$. Hence there are $\binom{n-1}{3}\Lambda_1 + \binom{n-1}{2}n\Lambda_2 + (n-1)\binom{n}{2}\Lambda_2 + \binom{n}{3}\Lambda_2$ quadruples containing x in the design. If a block contains x then it has $\binom{k-1}{3}$ quadruples containing x. So, there are $r\binom{k-1}{3}$ quadruples containing x in the design. Thus, we have

$$r = \frac{(n-1)(n-2)(n-3)\Lambda_1 + n(n-1)(7n-11)\Lambda_2}{(k-1)(k-2)(k-3)}. \tag{7.3.5}$$

Let b be the required number of blocks in a 4-GDD, if it exists. Using $2nr = bk$, we get

$$b = \frac{2n(n-1)(n-2)(n-3)\Lambda_1 + 2n^2(n-1)(7n-11)\Lambda_2}{k(k-1)(k-2)(k-3)}. \tag{7.3.6}$$

Specifically, for $k = 5$, we get the following parameters for a 4-GDD$(n, 2, 5; \Lambda_1, \Lambda_2)$ if it exists.

$$\lambda_1 = \frac{(n-2)(n-3)\Lambda_1 + n(3n-5)\Lambda_2}{6} \tag{7.3.7}$$

$$\lambda_2 = \frac{(n-1)(2n-3)}{3}\Lambda_2 \tag{7.3.8}$$

$$r_1 = \frac{(n-3)\Lambda_1 + n\Lambda_2}{2} \tag{7.3.9}$$

$$r_2 = \frac{2n-3}{2}\Lambda_2 \tag{7.3.10}$$

$$r = \frac{(n-1)(n-2)(n-3)\Lambda_1 + n(n-1)(7n-11)\Lambda_2}{24} \tag{7.3.11}$$

$$b = \frac{n(n-1)(n-2)(n-3)\Lambda_1 + n^2(n-1)(7n-11)\Lambda_2}{60}. \tag{7.3.12}$$

From Equations (7.3.9) and (7.3.10), if n is even both Λ_1 and Λ_2 are even and when n is odd Λ_2 is even but there is no restriction for Λ_1. In addition, from Theorem 7.2.4, we see that $n = 3$ is the smallest case for which $\Lambda_1 = 0$ and for all $n \neq 3$, Λ_1 can not be zero.

From Equations (7.3.7)–(7.3.11), we have the following lemma for n even.

Lemma 7.3.1. *For the existence of a* 4*-GDD*$(n, 2, 5; \Lambda_1, \Lambda_2)$,

(*i*) *if* $n \equiv 0 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{2}$;

(*ii*) *if* $n \equiv 2 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{12}$;

(*iii*) *if* $n \equiv 4 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;

(*iv*) *if* $n \equiv 6 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{2}, \Lambda_2 \equiv 0 \pmod{4}$;

(*v*) *if* $n \equiv 8 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{6}$;

(*vi*) *if* $n \equiv 10 \pmod{12}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{4}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$.

From Equations (7.3.7)–(7.3.11), we have the following lemma for n odd.

Lemma 7.3.2. *For the existence of a* 4-$GDD(n,2,5;\Lambda_1,\Lambda_2)$,

(*i*) *if* $n \equiv 1,7 \pmod{12}$ *then* $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;

(*ii*) *if* $n \equiv 3,9 \pmod{12}$ *then* $\Lambda_2 \equiv 0 \pmod{2}$;

(*iii*) *if* $n \equiv 5,11 \pmod{12}$ *then* $\Lambda_2 \equiv 0 \pmod{6}$.

Note that the conditions given in Lemmas 7.3.1 and 7.3.2 are needed for λ_1, λ_2, r_1, r_2 and r to be integers but may not satisfy the divisibility requirements for b.
So from Equation (7.3.12), we have the following lemma.

Lemma 7.3.3. *For the existence of a* 4-$GDD(n,2,5;\Lambda_1,\Lambda_2)$,

(*i*) *if* $n \equiv 0,1,3 \pmod{5}$, *then no restrictions on* Λ_1 *and* Λ_2 *except* $\Lambda_1 \neq 0$ *for* $n \neq 3$;

(*ii*) *if* $n \equiv 2 \pmod{5}$, *then* $\Lambda_2 \equiv 0 \pmod{5}$;

(*iii*) *if* $n \equiv 4 \pmod{5}$, *then* $\Lambda_1 \equiv \Lambda_2 \pmod{5}$.

Now combining Lemma 7.3.1 and Lemma 7.3.3, we get the following theorem for even n.

Theorem 7.3.4. *Necessary conditions for the existence of a* 4-$GDD(n,2,5;\Lambda_1,\Lambda_2)$ *when n is even:*

1. *if* $n \equiv 0, 36, 48 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{2}$;
2. *if* $n \equiv 2 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{60}$;
3. *if* $n \equiv 4 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}, \Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{15}$;
4. *if* $n \equiv 6, 18, 30 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{4}$;
5. *if* $n \equiv 8, 20, 56 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{6}$;
6. *if* $n \equiv 10, 46, 58 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{4}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;
7. *if* $n \equiv 12 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{10}$;
8. *if* $n \equiv 14 \pmod{60}$, $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{12}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$;
9. *if* $n \equiv 16, 28, 40 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;
10. *if* $n \equiv 22 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{20}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;
11. *if* $n \equiv 24 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$;
12. *if* $n \equiv 26, 38, 50 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{12}$;
13. *if* $n \equiv 32 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{30}$;
14. *if* $n \equiv 34 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{4}$, $\Lambda_1 \equiv \Lambda_2 \pmod{15}$;
15. *if* $n \equiv 42 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{20}$;
16. *if* $n \equiv 44 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{6}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$;
17. *if* $n \equiv 52 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{4}$, $\Lambda_2 \equiv 0 \pmod{10}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$;
18. *if* $n \equiv 54 \pmod{60}$, *then* $\Lambda_1 \equiv 0 \pmod{2}$, $\Lambda_2 \equiv 0 \pmod{4}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$.

Also combining Lemma 7.3.2 and Lemma 7.3.3, we get the following theorem for odd n.

Theorem 7.3.5. *Necessary conditions for the existence of a* 4-*GDD*$(n,2,5;\Lambda_1,\Lambda_2)$ *when n is odd:*

1. *if* $n \equiv 1,13,25,31,43,55 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$*;*
2. *if* $n \equiv 3,15,21,33,45,51 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{2}$*;*
3. *if* $n \equiv 5,11,23,35,41,53 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{6}$*;*
4. *if* $n \equiv 7,37 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{10}$, $\Lambda_1 \equiv \Lambda_2 \pmod{3}$*;*
5. *if* $n \equiv 9,39 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$*;*
6. *if* $n \equiv 17,47 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{30}$*;*
7. *if* $n \equiv 19,49 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{2}$, $\Lambda_1 \equiv \Lambda_2 \pmod{15}$*;*
8. *if* $n \equiv 27,57 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{10}$*;*
9. *if* $n \equiv 29,59 \pmod{60}$, *then* $\Lambda_2 \equiv 0 \pmod{6}$, $\Lambda_1 \equiv \Lambda_2 \pmod{5}$.

Suppose a 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ exists and let b_1, b_2 and b_3 be the number of blocks of type $(5,0)$, $(4,1)$ and $(3,2)$ respectively. As each block of type $(3,2)$ contains 3 quadruples of configuration $(2,2)$, there are $3b_3$ quadruples of configuration $(2,2)$ in the design. On the other hand, there are $\binom{n}{2}^2\Lambda_2$ quadruples of configuration $(2,2)$ in the design. Hence, $3b_3 = \binom{n}{2}^2\Lambda_2$ and we have

$$b_3 = \frac{n^2(n-1)^2}{12}\Lambda_2.$$

There are $2n\binom{n}{3}\Lambda_2$ quadruples of type $(3,1)$ in the design. On the other hand, as each block of type $(4,1)$ has 4 quadruples of configuration $(3,1)$ and each block of type $(3,2)$ has 2 quadruples of configuration $(3,1)$, there are $4b_2+2b_3$ quadruples of configuration $(3,1)$. Hence, $4b_2+2b_3 = 2n\binom{n}{3}\Lambda_2$ and we have

$$b_2 = \frac{n^2(n-1)(n-3)}{24}\Lambda_2.$$

In addition, there are $2\binom{n}{4}\Lambda_1$ quadruples of configuration $(4,0)$ in the design. Since each block of type $(5,0)$ contains 5 quadruples of configuration $(4,0)$ and each block of type $(4,1)$ contains 1 quadruple of configuration $(4,0)$, there are $5b_1+b_2$ quadruples of configuration $(4,0)$. Then $5b_1+b_2=2\binom{n}{4}\Lambda_1$. Hence, we have

$$b_1=\frac{2n(n-1)(n-2)(n-3)\Lambda_1-n^2(n-1)(n-3)\Lambda_2}{120}.$$

As $b_1\geq 0$, $2n(n-1)(n-2)(n-3)\Lambda_1\geq n^2(n-1)(n-3)\Lambda_2$. Simplifying, we have the following theorem.

Theorem 7.3.6. *Let $n\geq 4$. For the existence of a 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$,*

$$\Lambda_1\geq\frac{n}{2n-4}\Lambda_2.$$

7.3.1 $n=3$

From the necessary conditions for the existence of a 4-GDD$(3,2,5;\Lambda_1,\Lambda_2)$, the smallest $\Lambda_2=2$ as $\Lambda_2\equiv 0 \pmod 2$ but as $n=3$, any Λ_1 is allowed as vacuously the balance of Λ_1 will hold. The blocks of a 4-GDD$(3,2,5;\Lambda_1,2)$ where $G_1=\{1,2,3\}$ and $G_2=\{a,b,c\}$ are $\{\{1,2,3,a,b\},\{1,2,3,a,c\},\{1,2,3,b,c\},\{1,2,a,b,c\},\{1,3,a,b,c\},\{2,3,a,b,c\}$. Using t copies of theses blocks we can construct any 4-GDD$(3,2,5;\Lambda_1,2t)$. Hence, we have

Lemma 7.3.7. *A 4-GDD$(3,2,5;\Lambda_1,\Lambda_2)$ exists.*

7.3.2 $n=4$

Lemma 7.3.8. *A 4-GDD$(4,2,5;\Lambda_1,\Lambda_2)$ for $\Lambda_1\neq\Lambda_2$ does not exist.*

Proof. From Theorem 7.3.4, the necessary conditions for the existence of a 4-GDD$(4,2,5;\Lambda_1,\Lambda_2)$ are $\Lambda_1\equiv 0 \pmod 4$, $\Lambda_2\equiv 0 \pmod 2$ and $\Lambda_1\equiv\Lambda_2 \pmod{15}$. Suppose a 4-GDD$(4,2,5;\Lambda_1,\Lambda_2)$ exists and $\Lambda_1=4t$ for some positive integer t. Then there are

$8t$ blocks of type $(4,1)$, where 4 elements are from one group and 1 element is from the other group. These blocks of type $(4,1)$ cover $32t$ quadruples of configuration $(3,1)$. As there are $32\Lambda_2$ quadruples of configuration $(3,1)$ in the design, the remaining $32(\Lambda_2 - t)$ quadruples of configuration $(3,1)$ are covered in the blocks of type $(3,2)$. In addition, blocks of type $(3,2)$ cover $36\Lambda_2$ quadruples of configuration $(2,2)$. Hence $\frac{32(\Lambda_2-t)}{2} = \frac{36\Lambda_2}{3}$. This implies $\Lambda_2 = 4t$ which is equal to Λ_1. Thus, a 4-GDD$(4,2,5;\Lambda_1,\Lambda_2)$ does not exist when $\Lambda_1 \neq \Lambda_2$. On the other hand, a 4-GDD$(4,2,5;\Lambda_1,\Lambda_2)$ exists when $\Lambda_1 = \Lambda_2 = 4t$ as it can be obtained by taking t copies of the blocks of a 4-GDD$(4,2,5;4,4)$. □

7.3.3 $n = 5$

From the necessary conditions for the existence of a 4-GDD$(5,2,5;\Lambda_1,\Lambda_2)$, the smallest $\Lambda_2 = 6$.Then by Theorem 7.3.6, $\Lambda_1 \geq 5s$. But by Corollary 7.2.2, a 4-GDD$(5,2,5;5,6)$ exists. As a 4-$(5,5,1)$ exists, by taking t copies of the blocks of 4-$(5,5,1)$ on each group G_1 and G_2 of size 5 along with s copies of the blocks of a 4-$(5,2,5;5,6)$ on $G_1 \cup G_2$, we get a 4-GDD$(5,2,5;5s+t,6s)$. Hence, we have the following:

Lemma 7.3.9. *A* 4-*GDD*$(5,2,5;\Lambda_1,\Lambda_2)$ *exists.*

7.3.4 $n = 6$

Example 7.3.1. As per (Laue and Khosrovshahi, 2007, pp 81) a 5-$(13,6,4)$ exists , and as per (Alltop, 1975) and (Denniston, 1983) the derived design of a 5-$(13,6,4)$ yield a 4-$(12,5,4)$ design. Thus, considering any partition of twelve elements into two sets of size 6 as groups we hav a 4-GDD$(6,2,5;4,4)$.

From the necessary conditions for the existence of a 4-GDD$(6,2,5;\Lambda_1,\Lambda_2)$, $\Lambda_2 = 8s$ or $\Lambda_2 = 8s+4$ for nonnegative integer s. When $\Lambda_2 = 8s$, by Theorem 7.3.6, $\Lambda_1 \geq 6s$. By Corollary 7.2.2, a 4-GDD$(6,2,5;6,8)$ exists. Let G_1 and G_2 be groups of size 6 each. As a 4-$(6,5,2)$ exists, a 4-GDD$(6,2,5;6s+2t,8s)$ can be constructed by taking s copies of the blocks of a 4-GDD$(6,2,5;6,8)$ on $G_1 \cup G_2$ along with t copies of the blocks of a 4-$(6,5,2)$

on G_1 and G_2, where t is a nonnegative integer. When $\Lambda_2 = 8s+4$, by Theorem 7.3.6, $\Lambda_1 \geq 6s+3$. Since Λ_1 is even, $\Lambda_1 = 6s+4$. As a 4-GDD$(6,2,5;4,4)$ exists (see Example 7.3.1), a 4-GDD$(6,2,5;6s+2t+4s',8s+4s')$ can be constructed by taking the blocks of a 4-GDD$(6,2,5;6s+2t,8s)$ along with s' copies of the blocks of a 4-GDD$(6,2,5;4,4)$, where s' is a nonnegative integer. Thus, we have the following:

Lemma 7.3.10. *A 4-GDD$(6,2,5;\Lambda_1,\Lambda_2)$ exists.*

7.3.5 $n = 7$

From the necessary conditions for the existence of a 4-GDD$(7,2,5;\Lambda_1,\Lambda_2)$, $\Lambda_2 = 10s$. Then by Theorem 7.3.6, $\Lambda_1 \geq 7s$. A 4-GDD$(7,2,5;7,10)$ exists by Corollary 7.2.2. As a 4-$(7,5,3)$ exists, a 4-GDD$(7,2,5;7s+3t,10s)$ can be obtained by taking t copies of the blocks of a 4-$(7,5,3)$ on each groups G_1 and G_2 along with s copies of the blocks of a 4-GDD$(7,2,5;7,10)$ on $G_1 \cup G_2$ for any non negative integer s and t. Since $\Lambda_1 \equiv \Lambda_2 \pmod 3$, we have the following theorem.

Lemma 7.3.11. *A 4-GDD$(7,2,5;\Lambda_1,\Lambda_2)$ exists.*

7.3.6 $n = 8$

From the necessary conditions for the existence of a 4-GDD$(8,2,5;\Lambda_1,\Lambda_2)$, $\Lambda_1 = 4s$, $\Lambda_2 = 6s'$. Then by Theorem 7.3.6, $\Lambda_1 \geq 4s'$. The smallest such 4-GDD is 4-GDD$(8,2,5;4,6)$. If it exists, then a 4-GDD$(8,2,5;4s',6s')$ exists, and as a 4-$(8,5,4)$ exists, a 4-GDD$(8,2,5;4s,6s')$ exists for all $s \geq s'$. Unfortunately we are not able to construct 4-GDD$(8,2,5;4,6)$ yet. As a 4-GDD$(8,2,5;8,12)$ exists from Corollary 7.2.2, a 4-GDD$(8,2,5;8s'+4s,12s')$ exists for all s. If a 4-$(16,5,6)$ exists, then a 4-GDD$(8,2,5;6s'+4s,6s')$ exists for all s,s', but then a 4-GDD$(8,2,5;6+4,6)$ will exist, a contradiction, so a 4-$(16,5,6)$ does not exist. Which is actually coming from the necessary conditions for the existence of 4-$(v,5,\lambda)$ theorem.

7.3.7 $n = 9$

The necessary conditions for the existence of a 4-GDD$(9,2,5;\Lambda_1,\Lambda_2)$ are $\Lambda_2 \equiv 0 \pmod 2$ and $\Lambda_1 - \Lambda_2 \equiv 0 \pmod 5$. Let $\Lambda_2 = 14s + x$, where $x = 2s'$ for nonnegative integers s' and s. Then by Theorem 7.3.6, $\Lambda_1 \geq 9s + \frac{9}{14}x$.

When $x = 0$, $\Lambda_1 = 9s$. By Corollary 7.2.2, a 4-GDD$(9,2,5;9,14)$ exists. As a 4-$(9,5,5)$ exists, a 4-GDD$(9,2,5;9s+5t,14s)$ can be constructed by taking s copies of the blocks of a 4-GDD$(9,2,5;9,14)$ along with t copies of the blocks of a 4-$(9,5,5)$ on G_1 and G_2 for a nonnegative integer t.

When $x \geq 2$, as $\Lambda_1 - \Lambda_2 \equiv 0 \pmod 5$, $\Lambda_1 = 9s + 2s'$ for a positive integer s'. Since a 4-GDD$(9,2,5,2,2)$ exists as a 4-$(18,5,2)$, a 4-GDD$(9,2,5;9s+2s',14s+2s')$ can be constructed by taking s copies of the blocks of a 4-GDD$(9,2,5;9,14)$ along with s' copies of the blocks of a 4-GDD$(9,2,5;2,2)$. Thus, we have the following:

Lemma 7.3.12. *A 4-GDD$(9,2,5;\Lambda_1,\Lambda_2)$ exists.*

7.3.8 $n = 10$

The necessary conditions for the existence of a 4-GDD$(10,2,5;\Lambda_1,\Lambda_2)$ are $\Lambda_1 \equiv 0 \pmod 2$, $\Lambda_2 \equiv 0 \pmod 4$ and $\Lambda_1 - \Lambda_2 \equiv 0 \pmod 3$.

Let $\Lambda_2 = 16s + 4s'$, where s and $0 \leq s' \leq 3$ are nonnegative integers . Then by Theorem 7.3.6, $\Lambda_1 \geq 10s + \frac{5}{2}s'$.

The lower bound for Λ_1 is $10s + \frac{5}{2}s'$. Using $\Lambda_1 - \Lambda_2 \equiv 0 \pmod 3$ and $\Lambda_1 \equiv \pmod 2$, one can check for each value of s', that the smallest allowed Λ_1 is $10s + 4s'$. By Corollary 7.2.2, a 4-GDD$(10,2,5;10,16)$ exists. As a 4-$(20,5,4)$ exists, a 4-GDD$(10,2,5;10s+4s'+6t,16s+4s')$ can be constructed by taking s copies of the blocks of a 4-GDD$(10,2,5;10,16)$ along with s' copies of the blocks of a 4-GDD$(10,2,5;4,4)$ on $G_1 \cup G_2$ and t copies of the blocks of a 4-$(10,5,6)$ on G_1 and G_2. Thus, we have the following:

Lemma 7.3.13. *A* 4-*GDD*$(10,2,5;\Lambda_1,\Lambda_2)$ *exists.*

In summary, the above lemmas together prove the following theorem.

Theorem 7.3.14. *Necessary conditions are sufficient for the existence of a* 4-*GDD*$(n,2,5;\Lambda_1,\Lambda_2)$ *when* $n = 3,4,5,6,7,9,10$.

7.4 Summary

We gave general definitions for t-GDD when every t-subset from different groups occurs the same number of times as well as every t-subset from different groups does not occurs the same number of times. After a short survey of t-GDDs, the actual spectrum for 4-$(n,5,\Lambda_1)$ and for 4-GDD$(n,2,5,\Lambda_1,\Lambda_2)$ were obtained as a ready source for researchers interested in the constructions of such designs though the necessary conditions for the existence of t-designs are well known. Further, we proved the necessary conditions were sufficient for the existence of a 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ when group sizes are 3, 4, 5, 6, 7, 9 and 10.

Chapter 8

Conclusions and future work

8.1 Conclusions

We proposed and studied nonuniform GDDs with block size 4 with 2 or 3 groups and a 4-GDD with block size 5 and 2 groups of equal size. In chapter 4, we studied nonuniform GDD with block size 4 and 2 groups of different sizes. We gave a general construction of GDD$(n_1,n_2,k,\lambda_1,\lambda_2)$ by using α-resolvable BIBDs. Then we obtained necessary conditions for the existence of a GDD$(n_1,n_2,4;\lambda,1)$ by counting argument and proved that the necessary conditions were sufficient for

i. $\lambda \equiv 0 \pmod 6$ when $n_1 = 6t+3$ and $n_2 = 18t+6$, and when $n_1 = 12t+9$ and $n_2 = 36t+24$, where t is a nonnegative integer.

ii. $\lambda \equiv 1 \pmod 6$ when $n_i \geq 4$ for $i = 1,2$.

iii. $\lambda \equiv 3 \pmod 6$ when $n_1 = 24t+6$ and $n_2 = 36t+12$, and when $n_1 = 24t+15$ and $n_2 = 36t+21$, where t is a nonnegative integer.

iv. $\lambda \equiv 4 \pmod 6$ when $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, when $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$, when $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$, and when $n_1 \equiv 3 \pmod 6$ and $n_2 \equiv 4 \pmod{12}$.

We also obtained necessary conditions for the existence of a GDD$(n_1,n_2,4;\lambda,2)$ and proved that they were sufficient for

i. $\lambda \equiv 0 \pmod 6$ when $n_1 = 24t+9$ and $n_2 = 36t+12$ and when $n_1 = 24t+3$ and $n_2 = 36t+3$ for $n_1,n_2 \neq 3$, where t is a nonnegative integer.

ii. $\lambda \equiv 2 \pmod 6$ when $n_i \geq 4$ for $i = 1,2$.

iii. $\lambda \equiv 5 \pmod 6$ when $n_1 \equiv 0 \pmod{12}$ and $n_2 \equiv 1,4 \pmod{12}$, when $n_1 \equiv 9 \pmod{12}$ and $n_2 \equiv 4 \pmod{12}$, and when $n_1 \equiv 8 \pmod{12}$ and $n_2 \equiv 5,8 \pmod{12}$.

In chapter 5, we concentrated on nonuniform GDDs with block size 4 and three groups of sizes 1, n, n respectively. We employed 1-factorization and 2-factorization of complete graphs of order n to obtain an important general construction for such GDD. We obtained necessary conditions for the existence of these GDDs and proved that they were sufficient for specific values of n, specific values of λ_1 and for the existence of a GDD$(1,n,n,4;\lambda_1,\lambda_2)$ for $n \equiv 0,1,4,5,8,9 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.

In chapter 6, we focused on two new problems of group divisible designs with three groups and block size 4.The first problem dealt with group sizes 1, n and $n+2$, respectively. We proved that if equal number of blocks of block type configurations was required, a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ where $t > 0$ did not exist. We then concentrated our study on the case where $t = 2$ where there was no block configuration restriction. We obtained the necessary conditions for the existence of a GDD$(1,n,n+t,4;\lambda_1,\lambda_2)$ and proved that the necessary conditions were sufficient for the existence of a GDD$(1,n,n+2,4;\lambda_1,\lambda_2)$ when $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$ except for n even, $\lambda_1 \equiv 0 \pmod 6$ and $\lambda_2 \equiv 3 \pmod 6$, and for $n \equiv 2 \pmod 6, \lambda_1 \equiv 2,4 \pmod 6$ and $\lambda_2 \equiv 1,5 \pmod 6$. The second problem dealt with group sizes 3, n and $n+1$, respectively. We obtained necessary conditions for the existence of a GDD$(3,n,n+1,4;\lambda_1,\lambda_2)$, and we provided constructions for certain families of the problem and showed that the necessary conditions were sufficient for these families.

In chapter 7 we studied a t-group divisible design (t-GDD), a new approach defined by combining the definition of a t-design and a group divisible design. After a short survey of t-GDD, we concentrated on the existence of a 4-group divisible designs (4-GDDs) with two groups of equal size and blocks of size 5. We obtained necessary conditions for the existence of a 4-GDD$(n,2,5;\Lambda_1,\Lambda_2)$ and showed that the necessary conditions were sufficient for $n = 3,4,5,6,7,9,10$.

8.2 Future work

Further work can be carried out on the existence of nonuniform GDDs and t-GDDs. We state a few challenging open problems on the existence of such GDDs.

Problem 1. Find the necessary and sufficient conditions for the existence of a GDD$(n_1, n_2, 4; \lambda_1, \lambda_2)$ when $\lambda_2 = 3, 4, 5$.

Problem 2. Show the existence or nonexistence of a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ for $n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}$ when $\lambda_1 \geq \lambda_2$.

Problem 3. Prove that a GDD$(1, n, n, 4; \lambda_1, \lambda_2)$ does or does not exist for $\lambda_1 < \lambda_2$.

Problem 4. Does a GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ with equal number of blocks of configurations $(0,4)$ and $(1,3)$; $(0,4)$ and $(2,2)$; $(1,3)$ and $(2,2)$; $(1,3)$ and $(1,1,2)$ respectively exist for a nonnegative integer t?

Problem 5. Does a GDD$(1, n, n+t, 4; \lambda_1, \lambda_2)$ with equal number of blocks of configurations $(0,4)$, $(1,3)$ and $(1,1,2)$; $(0,4)$, $(1,3)$ and $(2,2)$; $(1,3)$, $(2,2)$ and $(1,1,2)$ respectively exist for a nonnegative integer t?

Problem 6. Prove the existence or nonexistence of a GDD$(1, n, n+2, 4; \lambda_1, \lambda_2)$ for n even when $\lambda_1 \equiv 0 \pmod{6}$ and $\lambda_2 \equiv 3 \pmod{6}$, and for $n \equiv 2 \pmod{6}$ when $\lambda_1 \equiv 2, 4 \pmod{6}$ and $\lambda_2 \equiv 1, 5 \pmod{6}$.

Problem 7. Prove the necessary conditions for the existence of a GDD$(1, n, n+2, 4; \lambda_1, \lambda_2)$ are sufficient when $\lambda_1 < \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod{6}$.

Problem 8. Prove the necessary conditions are sufficient for the existence of a 4-GDD$(n, 2, 5; \Lambda_1, \Lambda_2)$ except for $n = 3, 4, 5, 6, 7, 9, 10$.

Bibliography

Abel, R. J. R. and Buratti, M. (2006). 16 difference families. *Handbook of Combinatorial Designs*, page 392.

Abel, R. J. R., Ge, G., and Yin, J. (2006). Resolvable and near-resolvable designs. In *Handbook of Combinatorial Designs*, pages 150–158. Chapman and Hall/CRC.

Ajoodani-Namini, S. (1996). Extending large sets oft-designs. *journal of combinatorial theory, Series A*, 76(1):139–144.

Alltop, W. O. (1975). Extending t-designs. *Journal of Combinatorial Theory, Series A*, 18(2):177–186.

Atkinson, A., Donev, A., and Tobias, R. (2007). *Optimum experimental designs, with SAS*, volume 34. Oxford University Press.

Bailey, R. A. (2004). *Association schemes: Designed experiments, algebra and combinatorics*, volume 84. Cambridge University Press.

Beth, T., Jungnickel, D., and Lenz, H. (1999). *Design Theory: Volume 1*. Cambridge University Press.

Bose, R. and Connor, W. (1952). Combinatorial properties of group divisible incomplete block designs. *The Annals of Mathematical Statistics*, pages 367–383.

Bose, R. C. (1939). On the construction of balanced incomplete block designs. *Annals of Eugenics*, 9(4):353–399.

Bose, R. C. and Nair, K. R. (1939). Partially balanced incomplete block designs. *Sankhyā: The Indian Journal of Statistics*, pages 337–372.

Bose, R. C. and Shimamoto, T. (1952). Classification and analysis of partially balanced incomplete block designs with two associate classes. *Journal of the American Statistical Association*, 47(258):151–184.

Brouwer, A. E., Schrijver, A., and Hanani, H. (1977). Group divisible designs with block-size four. *Discrete Mathematics*, 20:1–10.

Bruck, R. H. and Ryser, H. J. (1949). The nonexistence of certain finite projective planes. *Canadian Journal of Mathematics*, 1(1):88–93.

Cao, H., Lei, J., and Zhu, L. (2001). Large sets of disjoint group-divisible designs with block size three and type 2n41. *Journal of Combinatorial Designs*, 9(4):285–296.

Cao, H., Lei, J., and Zhu, L. (2015). Constructions of large sets of disjoint group-divisible designs ls (2n41) using a generalization of* ls (2n). *Discrete Mathematics*, 338(8):1449–1459.

Chaiyasena, A., Hurd, S. P., Punnim, N., and Sarvate, D. G. (2012). Group divisible designs with two association classes.

Chen, D., Lindner, C. C., and Stinson, D. R. (1992). Further results on large sets of disjoint group-divisible designs. *Discrete mathematics*, 110(1-3):35–42.

Chowla, S. and Ryser, H. J. (1950). Combinatorial problems. *Canadian Journal of Mathematics*, 2:93–99.

Clatworthy, W. H. (1973). *Tables of two-associate-class partially balanced designs*, volume 63. US] National Bureau of Standards.

Colbourn, C. J., Dinitz, J. H., and Stinson, D. R. (1999). *Applications of combinatorial designs to communications, cryptography, and networking*. Citeseer.

Colbourn, C. J., Hoffman, D. G., and Rees, R. (1992). A new class of group divisible designs with block size three. *Journal of Combinatorial Theory, Series A*, 59(1):73–89.

Das, A., Dean, A. M., and Gupta, S. (1998). On optimality of some partial diallel cross designs. *Sankhyā: The Indian Journal of Statistics, Series B*, pages 511–524.

Denniston, R. (1983). A small 4-design. In *North-Holland Mathematics Studies*, volume 78, pages 291–294. Elsevier.

Du, D., Hwang, F. K., and Hwang, F. (2000). *Combinatorial group testing and its applications*, volume 12. World Scientific.

Fisher, R. A. (1940). 174: An examination of the different possible solutions of a problem in incomplete blocks. *Annals of Eugenics*.

Fisher, R. A. and Yates, F. (1934). The 6$\times$ 6 latin squares. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 30, pages 492–507. Cambridge University Press.

Fu, H.-L. and Rodger, C. A. (1998). Group divisible designs with two associate classes: n= 2 orm= 2. *Journal of Combinatorial Theory, Series A*, 83(1):94–117.

Fu, H.-L., Rodger, C. A., Sarvate, D. G., et al. (2000). The existence of group divisible designs with first and second associates, having block size 3. *Ars Combinatoria*, 54:33–50.

Gao, F. and Ge, G. (2011). A complete generalization of clatworthy group divisible designs. *SIAM Journal on Discrete Mathematics*, 25(4):1547–1561.

Ge, G., Li, S., and Wei, H. (2016). A new construction of group divisible designs with nonuniform group type. *Journal of Combinatorial Designs*, 24(8):369–382.

Ge, G. and Ling, A. C. (2004). Group divisible designs with block size four and group type gum1 for small g. *Discrete mathematics*, 285(1-3):97–120.

Ghosh, D. and Divecha, J. (1997). Two associate class partially balanced incomplete block designs and partial diallel crosses. *Biometrika*, 84(1):245–248.

Hall, M. (1998). *Combinatorial theory*, volume 71. John Wiley & Sons.

Hanani, H. (1961). The existence and construction of balanced incomplete block designs. *The Annals of Mathematical Statistics*, 32(2):361–386.

Hanani, H. (1963). On some tactical configurations. *Canadian Journal of Mathematics*, 15:702–722.

Hanani, H. (1975). Balanced incomplete block designs and related designs. *Discrete Mathematics*, 11(3):255–369.

Harwit, M. and Sloane, N. (1979). Hadamard transform optics academic. *New York*.

Henson, D. and Sarvate, D. G. (2006). A family of group divisible designs of block size four and three groups with lambda˜ 1= 2 and lambda˜ 2= 1 using mols. *ARS COMBINATORIA-WATERLOO THEN WINNIPEG-*, 78:123.

Henson, D., Sarvate, D. G., and Hurd, S. P. (2007). Group divisible designs with three groups and block size four. *Discrete mathematics*, 307(14):1693–1706.

Hurd, S. P., Mishra, N., and Sarvate, D. G. (2007). A new construction for group divisible designs with block size five and few groups. *ARS COMBINATORIA-WATERLOO THEN WINNIPEG-*, 84:243.

Hurd, S. P., Mishra, N., and Sarvate, D. G. (2009). Group divisible designs with two groups and block size five with fixed block configuration. *J. Combin. Math. Comput*, 70:15–31.

Hurd, S. P. and Sarvate, D. G. (2004). Odd and even group divisible designs with two groups and block size four. *Discrete mathematics*, 284(1-3):189–196.

Hurd, S. P. and Sarvate, D. G. (2008). Group divisible designs with block size four and two groups. *Discrete mathematics*, 308(13):2663–2673.

Jia-Xi, L. (1984). On large sets of disjoint steiner triple systems, v. *Journal of Combinatorial Theory, Series A*, 37(2):164–188.

Keranen, M. S. and Laffin, M. R. (2012). Fixed block configuration group divisible designs with block size six. *Discrete Mathematics*, 312(4):745–756.

Kirkman, T. P. (1850). Query vi,. lady's and gentleman's diary. *48*.

Kreher, D. and Stinson, D. (1997). Small group-divisible designs with block size four. *Journal of statistical planning and inference*, 58(1):111–118.

Lam, C. W., Thiel, L., and Swiercz, S. (1989). The non-existence of finite projective planes of order 10. *Canadian Journal of Mathematics*, 41(6):1117–1123.

Lapchinda, W., Punnim, N., and Pabhapote, N. (2014). Gdds with two associate classes and with three groups of sizes 1, n and n. *Australas. J Comb.*, 58:292–303.

Laue, R. and Khosrovshahi, G. (2007). *t-designs, t¿=3*, pages 79–101.

Liebeck, P. (1971). Graph theory. by frank harary. pp. ix, 274. 1969.(addison-wesley.). *The Mathematical Gazette*, 55(393):338–338.

Lu, J.-X. (1983). On large sets of disjoint steiner triple systems iii. *J. Comb. Theory, Ser. A*, 34(2):156–182.

MacWilliams, F. J. and Sloane, N. J. A. (1977). *The theory of error correcting codes*, volume 16. Elsevier.

Mullin, R. C. and Gronau, H.-D. O. (2006). Pbds and gdds: the basics. In *Handbook of Combinatorial Designs*, pages 257–261. Chapman and Hall/CRC.

Mwesigwa, R., Sarvate, D. G., and Zhang, L. (2016). Group divisible designs of four groups and block size five with configuration (1; 1; 1; 2). *Journal of Algebra Combinatorics Discrete Structures and Applications*, 3(3):187–194.

Namyalo, K., Sarvate, D. G., and Zhang, L. (2017). Gdd (n1, n, n+ 1, 4; λ1, λ2): n1= 1 or 2.

Ndungo, I. and Sarvate, D. G. (2016). Gdd (n, 2, 4; $\lambda 1$, $\lambda 2$) with equal number of even and odd blocks. *Discrete Mathematics*, 339(4):1344–1354.

Niu, X., Cao, H., and Javadi, R. (2018). The spectrum for large sets of-gdds of type. *Journal of Combinatorial Designs*, 26(12):616–639.

Pabhapote, N. (2012). Group divisible designs with two associate classes and with two unequal groups. *International Journal of Pure and Applied Mathematics*, 81(1):191–198.

Pabhapote, N. and Punnim, N. (2011). Group divisible designs with two associate classes and $\lambda 2$= 1. *Int. J. Math. Math. Sci.*, 2011:148580–1.

Rodger, C. and Rogers, J. (2010). Generalizing clatworthy group divisible designs. *Journal of statistical planning and inference*, 140(9):2442–2447.

Rodger, C. and Rogers, J. (2012). Generalizing clatworthy group divisible designs ii. *JCMCC-Journal of Combinatorial Mathematicsand Combinatorial Computing*, 80:299.

Sakda, A. and Uiyyasathian, C. (2017). Group divisible designs gdd (n, n, n, 1; $\lambda 1$, $\lambda 2$). *Australas. J Comb.*, 69:18–28.

Sarvate, D., Mishra, N., and Namyalo, K. (2018). Group divisible designs with block size five from clatworthy's table. *Communications in Statistics-Theory and Methods*, 47(9):2085–2097.

Sarvate, D. G. and Bezire, W. (2018). 3-gdds with block size 4. *Bull. ICA*, 82:56–71.

Sarvate, D. G. and Cowden, W. (2018). A simple construction of 3-gdds with block size 4 using sqs(v). *Bull. ICA*, 84:40–44.

Sarvate, D. G. and Shinde, P. N. (2020). A short note on 3-gdds with three groups and blocksize 4. *J. Combin. Math. Combin. Comput.*

Schellenberg, P. (1989). Threshold schemes from combinatorial designs. *J. Combinatorial Mathematics and Combinatorial Computing*, 5:143–160.

Schützenberger, M.-P. (1947). A non-existence theorem for an infinite family of symmetrical block designs. *Annals of Eugenics*, 14(1):286–287.

See, K., Song, S. Y., and Stufken, J. (1997). On a class of partially balanced incomplete block designs with applications in survey sampling. *Communications in Statistics-Theory and Methods*, 26(1):1–13.

Stinson, D. (2007). *Combinatorial designs: constructions and analysis*. Springer Science & Business Media.

Street, A. P. and Street, D. J. (1986). *Combinatorics of experimental design*. Oxford University Press, Inc.

Street, D. J. and Street, A. P. (2006). Partially balanced incomplete block designs. In *Handbook of Combinatorial Designs*, pages 588–591. Chapman and Hall/CRC.

Teirlinck, L. (1984). On large sets of disjoint quadruple systems. *Ars Combin*, 17:173–176.

Vakil, F. and Parnes, M. (1994). On the structure of a class of sets useful in nonadaptive group testing. *Journal of statistical planning and inference*, 39(1):57–69.

van Trung, T. (2019). Recursive constructions for s-resolvable t-designs. *Designs, Codes and Cryptography*, 87(12):2835–2845.

Wang, J. and Yin, L. (2010). A class of group divisible designs with block size three and index λ. *Australas. J Comb.*, 46:51–66.

West, D. B. et al. (2001). *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River.

Yates, F. (1936). Incomplete randomized blocks. *Annals of eugenics*, 7(2):121–140.

Yener, B., Ofek, Y., and Yung, M. (1997). Combinatorial design of congestion-free networks. *IEEE/ACM transactions on networking*, 5(6):989–1000.

Zhu, M. and Ge, G. (2010). Mixed group divisible designs with three groups and block size 4. *Discrete mathematics*, 310(17-18):2323–2326.