A MATHEMATICAL MODEL FOR COFFEE BERRY BORER MANAGEMENT IN COFFEE FARM WITH OPTIMAL CONTROL

Mohammedsultan Abaraya Abawari

A Thesis Submitted to the Department of Applied Mathematics

School of Applied Natural Science

Presented in Partial Fulfillment of the Requirement for the Degree of Master's in Applied Mathematics

Office of Graduate Studies

Adama Science and Technology University

August 4, 2021

Adama, Ethiopia

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Mohammedsultan Abaraya Abawari

Major advisor: Legesse Lemecha (Ph.D)

Co-advisor: Abdisa Shiferaw (Ph.D Candidate)

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Declaration

I hereby declare that this Master Thesis entitled “A mathematical model for Coffee Berry Borer management in Coffee farm with optimal control”is my original work. That is, it has not been submitted for the award of any academic degree, diploma or certificate in any other university. All sources of materials that are used for this thesis have been duly acknowledged through citation.

Name of student	Signature	Date

Recommendation

We, the advisors of this thesis, hereby certify that we have read the revised version of the thesis entitled “A mathematical model for Coffee Berry Borer management in Coffee farm with optimal control”prepared under our guidance by Mohammedsultan Abaraya Abawari submitted in partial fulfillment of the requirements for the degree of Master’s of Science in Applied Mathematics. Therefore, we recommend that the submission of revised version of the thesis to the department following the applicable procedures.

Major Advisor	Signature	Date
Co-advisor	Signature	Date

Approval Page

We, the advisors of the thesis entitled "A mathematical model for Coffee Berry Borer management in Coffee farm with optimal control"and developed by Mohammedsultan Abaraya Abawari, hereby certify that the recommendation and suggestion made by the board of examiners are appropriately incorporated into the final version of the thesis.

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Co-advisor	Signature	Date

We, the undersigned, members of the Board of Examiners of the thesis by Mohammedsultan Abaraya Abawari, have read and evaluated the thesis entitled "A mathematical model for Coffee Berry Borer management in Coffee farm with optimal control"and examined the candidate during open defense. This is, therefore, to certify that the thesis is accepted for partial fulfillment of the requirement of the degree of Master of Science in Applied Mathematics.

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Chairperson	**Signature**	**Date**
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Finally approval and acceptance of the thesis is contingent upon submission of its final copy to the Office of Postgraduate Studies (OPGS) through the Department Graduate committee (DGC) and School Graduate Committee (SGC).

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Abstract

The Coffee berry borer (CBB) is the most destructive pest that affects the productivity of Coffee worldwide. The female CBB drills the coffee berry eat the seed and reproduce inside it. This study investigated the infestation dynamics of CBB and its optimal management. For this, we formulated and analyzed a mathematical model describing the progression of CBB. The proposed model includes; Coffee berry biomass, Coffee berry borer population and awarded farmers'. First, we analyzed the dynamical properties of the governing model including the boundedness of the solution, existence of borer free and endemic equilibrium, local and global stability of equilibrium points. The basic reproduction number $\mathcal{R}_0$ *was calculated using the next-generation matrix and the stability of the equilibrium points were analyzed. From the qualitative analysis, the borer free equilibrium point is both locally and globally stable if* $\mathcal{R}_0 < 1$*, and the endemic equilibrium point is also both locally and globally stable if* $\mathcal{R}_0 > 1$*. Furthermore, sensitivity analysis of the model equation was performed on the key parameters to find out their relative significance and potential impact on the infestation dynamic of CBB. Then we extend the proposed model into an optimal control problem. The optimal control strategy is found by minimizing the number of coffee borer (pest) individual with taking into account the cost of implementation. The existence of optimal controls is proved with the help of Pontryagin's Minimum Principle. Then, numerical simulations of the model equations were carried out using MATLAB. Finally, the numerical simulations show that the agreement with the analytical results.*

Keywords: CBB; Mathematical modelling; Stability; Basic reproduction number; Optimal control, Numerical simulation.

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ACRONYMS

CBB	Coffee Berry Borer
CBD	Coffee Berry Disease
CLR	Coffee Leaf Rust
CWD	Coffee Wilt Disease
CBBFEP	CBB Free Equilibrium Point
EEP	Endemic Equilibrium Point
ECTA	Ethiopia Coffee and Tea Authority
GDP	Gross Domestic Product
ICO	International Coffee Organization
PMP	Pontryagin's Minimum Principle
SEIR	Susceptible, Exposed, Infected, Recovered

Chapter 1

INTRODUCTION

1.1 Background of the study

Coffee is an important crop globally due to its high contribution to National GDP, tax generation, food security, job creation, and inequality reduction role. In both producing and consuming countries, coffee is an important commodity. According to the (International Coffee Organization, 2018) report, more than 120 million people in the world rely on activities relating to coffee production for their livelihoods. And also around 25 million smallholder farmers and workers living in poor socioeconomic conditions (Waller et al., 2007) and 90% of the world's coffee production takes place in developing countries. Namely, Brazil, Vietnam, Colombia, Indonesia, Ethiopia, Honduras, India, Uganda, Mexico, Guatemala while its consumption takes place in industrialized nations.

Figure 1.1: Major coffee producers of the world.
Source:https://coffeelands.crs.org/2018/04/the-time-to-regenerate-coffee-is-now-and-it-starts-with-heatlhy-soil/

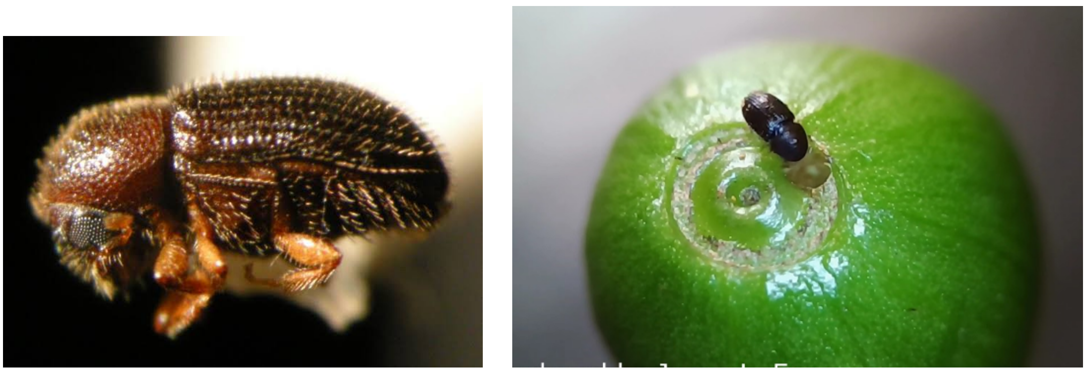
Ethiopia is gifted with enormous biodiversity and diverse agro-ecologies. In particular, the country is a home for the primary origin and diversity of many plants. Among all, Arabica coffee *(Coffea arabica L.,)* is the greatest gift of the country in which it has economical, social, and environmental importance. It is the centre of origin and diversity of Arabica coffee, which is a stimulant beverage crop, and belongs to the family *Rubiaceae* and the genus *Coffea* (Getu, 2011). There are different types of coffee in the world. Among these, the major economic species are coffee Arabica and Coffee Robusta. Arabica accounts for 80% of the world coffee trade, and Robusta most of the remaining 20%. Coffee Liberica and Excelsa together supply less than 1% (Habtamu, 2008).

The Ethiopian economy is highly dependent on agriculture. Among the agricultural production, the coffee sub-sector plays a major role in the economy of the country. It is the biggest source of foreign currency earning and has a major contribution to Gross Domestic Product (GDP)(Habtamu, 2008). As reported by (Arega, 2006), Coffee is not only one of the highly preferred international beverages, but also one of the most important trade commodities in the world next to petroleum. Further, it is used as an input in some food processing industries. For instance, it is used as a flavouring to various pastries, ice-creams, chocolate, and candies.

Coffee production systems in Ethiopia are grouped into four broad categories namely, forest coffee, semi-forest coffee, garden coffee, and coffee plantations. The most important cultivation areas are southwestern and southern Ethiopia. Ethiopia is the only country in the world where coffee grows wild as an understorey shrub or small tree in the Afromontane rain forests. It is believed that forests harbor a large genetic pool of arabica coffee that represents a potential source to develop the crop for the benefit of present and future human generations in the world (Tesfu, 2012). According to Ethiopian Coffee and Tea Authority (ECTA) 2017 report, in Ethiopia, the coffee total area coverage is about 1.2mil ha of which 900,000 ha estimated to be productive. The annual production ranges from 500,000-700,000 tons and average national productivity is seven ql./ha (Tsegaye, 2017). In the same paper, the authors also reported that in Ethiopia, over 25 % of the population is dependent on coffee production. In Ethiopia, coffee is grown in areas of humid (moist) evergreen forests such as Sidamo, Yirgacheffe, Harar, Nekemte, Limu, and Jimma.

Many abiotic and biotic factors are the major constraints of coffee production in Ethiopia. The most important ones are diseases caused by many etioletic agents, mainly the fungi and pests especially insect pests. The insect pests that widely affect coffee production are Antestia bug, Coffee blotch miner, and Coffee berry borer (Fuad, 2010). The crop is prone to several diseases in addition to insect pests that attack fruits, leaves, stems, and roots, and reduce the yield and marketability. As discussed in (Eshetu, 1997; Hindorf & Omondi, 2011), the major coffee diseases in Ethiopia are coffee berry disease (CBD), coffee wilt disease (CWD), and coffee leaf rust (CLR). The most important diseases both in severity and wide distribution are coffee berry disease (CBD) and coffee wilt disease (CWD). Coffee berry disease (CBD) is a disease caused by the fungal pathogen, anthracnose of green and ripe coffee berries. It is one of the three species of Colletotrichum that have been isolated from coffee berries, leaves, and branches: and it is the top major disease of coffee in Ethiopia, which attacks mainly the green berries of coffee. It was first observed in Ethiopia in 1971. Since then it spreads and is found in all coffee-producing areas in which it has been favored by favorable environmental conditions. In the present study, we will focus mainly on coffee berry borer rather than CBD.

As cited in (Vega et al., 1999), Coffee berry borer was first reported in Gabon in 1901, and later in Kenya in 1928. Then spread to almost all the 80 coffee-producing countries. The authors also pointed out that *H.hampei* is a pest of immature and mature coffee berries, causing no damage to the leaves, branches, or stem. It attacks the apex of the coffee berry from about eight weeks after flowering. As shown in Figure (1.2), female beetle bores into the berries through the navel region make tunneling and feed inside content.



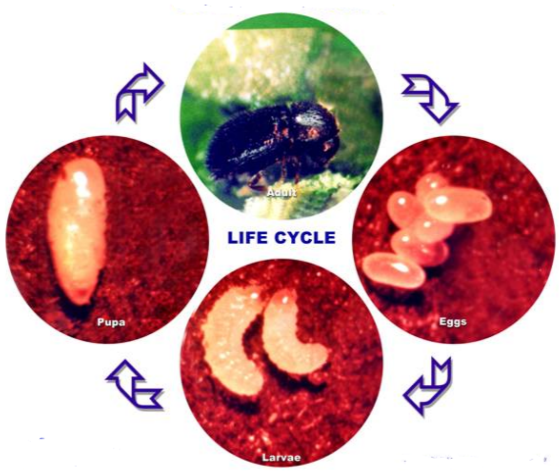
(a) Adult coffee berry borer. **(b)** Coffee borer entering a coffee cherry.

Figure 1.2: Adult Coffee berry borer entering coffee bean.

A small perforation about 1 mm diameter shown in Figure (1.3) is often clearly visible through this may become partly obscured by subsequent growth of the berry or by fungi that attack the borer. During active boring by the adult female, she pushes out the debris, which forms a deposit over the hole. Inside the seed, the CBB female will oviposit and larvae will feed on the same seed to grow and reach the adult stage. Sibling mating inside the berry makes this insect quite difficult to control. The levels of infestation of CBB in coffee-growing areas are estimated at 60% in Mexico, 50-90% in Malaysia, 60% in Colombia, 75% in Jamaica, 80% in Uganda, and 90% in Tanzania (Vega, 2004). It is now present in almost all of the major coffee-producing countries.

It lives the greatest part of its life cycle inside the coffee berry, which involves egg laying followed by the emergence of adult females from the berry. As illustrated in Figure (1.3), the life cycle of the CBB is composed of four stages: eggs, larvae, nymphs, and adults. Mature females are responsible for the dispersal of the population: they emerge from the berries to colonize and lay their eggs in new berries, while males and larvae stages remain inside the berries (Vega et al., 1999). Due to the intensity of the damages, several programs and control methods have been developed by the coffee growers, such as improved cultural practices, chemical, biological control and host plant resistance (Damon, 2000; Aristizábal et al., 2016).

(a) Adult coffee berry borer entering hole.



(b) Life cycle of coffee berry borer.

Figure 1.3: Life cycle of adult coffee berry borer once entered coffee cherry.

1.2 Control strategies of coffee berry borer

For the control of CBB, numerous options have been recommended, of which lies in four categories. These are cultural, chemical, host plant resistance, and biological control strategies. Each control strategy is discussed below in detail.

Cultural Control

Cultural control strategies are usually developed to suit specific pests. The Coffee berry borer easily survives from one cropping season to the next while inside the coffee berries that have either dropped on the ground or dried on the trees. The most effective cultural approach in the control of this pest is to pick up and destroy the berries that are the potential reservoirs for CBB, at the end of the cropping season through deep burying or burning. However, this control method is regarded as tedious and rather very labor intensive especially picking berries off the ground (Le Pelley et al., 1968). Therefore it's advocated that during coffee picking, berry dropping to the ground should be kept to a minimum. Proper drying of coffee beans also helps in reducing the CBB infestation (P. S. Baker, 1999).

Chemical control

Chemical control has limited effectiveness because of the biology and feeding behavior of this pest. Nearly the entire life cycle of CBB takes place inside the coffee cherry. For that reason, insecticides applied to control the CBB can be rarely effective against it. For any insecticides to be effective either wholly or partially, they must be applied before the CBB adults get into the hardened coffee bean (Mugo et al., 2007). In the study, it is also revealed that the insecticide requires to be sprayed four to five months after the crop has flowered since this is the period when the coffee beans hardened and suitable for CBB attack.

Several insecticides have been evaluated and recommended for management of major insect pests of coffee (Le Pelley et al., 1968; Brun et al., 1989), with some reported to reduce populations of biological control agents. Endosulfan, an organo-chlorine, is mostly used in coffee to manage Coffee berry borer in many parts of the world. Its frequent use has led to the development of some resistance against it as reported in New Calendonia (Pacific Ocean) (Brun et al., 1989).

The CBB has an interesting life history that facilitates faster development of resistance enables it to readily develop pesticide resistance. Most CBB are females in a ratio of 10:1 (Brun et al., 1989). The study has also shown that males are flightless. They mate with their sisters because they never leave the cherry in which they are born. This results in genetic inbreeding. When the mutation for endosulfan resistance occurs, this rapidly spread through a population because of this inbreeding. This development of resistance to endosulfan if it becomes widespread may prove devastating to the management of CBB.

Biological control

Biological control through the use of natural enemies (parasitoids, fungal pathogens, and parasitic nematodes) enhanced with effective cultural controls and selective insecticide the application helps to keep this pest in check. These includes parasitoids such as *Proropsnasuta* Watson, *Heterospilus coffeicola* Schmied, *Cephalonomia stephanoderis* Betrem, *Phymastichus coffea*; fungal pathogens, the *Beauveria bassiana* and parasitic nematodes. A number of these biocontrol agents are indigenous to Eastern Africa region with reported parasitism levels ranging from 18 to 59% (Le Pelley et al., 1968). The *Phymastichus coffea* unlike the other parasitoids obtained from East Africa, parasitizes the adult female CBB before it enters the coffee bean and moves from berry to berry (P. S. Baker, 1999). Thus *Phymastichus coffea* is considered to be a potentially useful biological control tool in the management of CBB.

Host plant resistance

The most attractive CBB control option, economically feasible and environmentally welcoming, is growing host plant resistance cultivars (Martin & Shepherd, 2009; Karavina, 2014). These are obtained through either conventional breeding or genetic engineering. Most recent host plant resistance projects have emphasized the need for an interdisciplinary approach to cultivar development. In these studies, it is also presented that despite the past success of these efforts, there are several difficulties in producing conventionally bred coffee genotypes having high degrees of coffee berry borer resistance.

A mathematical model is the application of mathematics to describe real-world problems and investigating important questions that arise from it (Banerjee, 2014). Mathematical models have a great role in science and engineering fields. Developing a mathematical model is the process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into the real world phenomena (Tahir et al., 2019). Mathematical modelling is a means of studying the transmission dynamics of infectious diseases that aim to reveal the progression of the disease process and to provide for controlling and decision making.

Optimal control, also known as dynamic optimization, is used with mathematical modelling to find the best outcome using a predetermined goal or goals (Lenhart & Workman, 2007). Many times, the problem will include tradeoffs between two competing factors and where parameters are transformed into control functions called control variables. Using optimal control, the state variable(s) and control variables(s) are used to maximize or minimize the objective function. Ultimately, optimal control determines the control(s) to optimize a specific set of goals in a dynamic system. Most of the modern mathematical models that have been developed apply the optimal control theory (Karrakchou et al., 2006).

Optimal control theory has been applied to various mathematical models for studying the transmission dynaamics of plant disease for example, see in (Kinene et al., 2015; Eliab et al., 2017; Chowdhury et al., 2019; Haileyesus et al., 2019) and references cited therein. From the above discussion, one can easily judge that several studies have been conducted on the biology and ecology of these insect pests. However, almost very few studies have been conducted from a mathematical perspective. Thus, the present study aims to propose a mathematical model with optimal control to manage the coffee berry borer infestations.

1.3 Statement of the problem

The Coffee berry borer(CBB) or *Hypothenemus hampei* (Ferrari) is the most serious and widespread insect pest of coffee. It affects the productivity of this crop in many countries (Baker et al., 2002).

A wide range of mathematical models have been formulated and analyzed to explain the dynamics of plant disease transmission and assessing their controls. For instance, (Eliab et al., 2017) investigated mathematical model for the vector transmission and control of banana Xanthomonas wilt disease. Further, (Kinene et al., 2015) introduced the optimal control to determine cassava brown streak disease model with chemical and uprooting as control measures. (Haileyesus et al., 2019), have derived mathematical modeling of MSV pathogen interaction with leafhopper vector on maize plant. (Chowdhury et al., 2019), have developed a mathematical model for pest control using bio-pesticides. Moreover, they also incorporate optimal control theory to minimize the cost in pest management due to bio-pesticides.

(Fotso et al., 2019) proposed and analyzed a mathematical model that describes the infestation dynamics of coffee berry borer without farmer's awareness. All the studies from agricultural point of view indicates that managing coffee berry borer requires careful monitoring to keep the pest under control. In this regard, mathematical modeling by including farmer's awareness may play a very important role to manage the threats induced by coffee berry borer and prevent productivity loss. Thus, in this study, we proposed a nonlinear deterministic mathematical model to investigate the dynamics of CBB with optimal controls in the presence of farmers awareness.

1.4 Objective of the study

1.4.1 General objective

The general objective of this study is to propose and analyze a mathematical model for coffee berry borer management with optimal control.

1.4.2 Specific objectives

The specific objective of this study are to:

- formulates a deterministic mathematical model for coffee berry borer;
- perform analysis of the formulated mathematical model;
- extend the developed mathematical model into optimal control analysis;
- investigates the impact of optimal control strategies on the spread of CBB.

1.5 Significant of the study

The study is significant for the following reasons:

- it will help to provide better control strategies on the occurrence of CBB in the coffee farm.
- the finding of the study may help policymakers on optimal pest control mechanisms.
- it also provides useful guidelines to agricultural practitioners on cost-effective elimination strategies of CBB.
- the outcome of the study may contribute to the existing knowledge on the application of the mathematical model. In addition, it may help researchers as a starting point for further studies on CBB and related studies.

1.6 Structure of the thesis

The thesis is organized as follows: Chapter 2 is devoted to mathematical preliminary on model and applications of optimal control methods. Chapter 3 presents the literature review on the plant diseases model and applications of optimal control in epidemiology. Research methodology is presented in Chapter 4. In Chapter 5, we discussed model formulation and analysis in detail. We extended the proposed model into the optimal control problem in Chapter 6. In Chapter 7, we performed the numerical simulations of the model. Finally, the conclusion and recommendation were drawn.

Chapter 2

MATHEMATICAL PRELIMINARY

This chapter is a summary of important mathematical concepts and methodologies that are applied throughout this thesis. The concepts given are mostly standard definitions and results as documented in the literature on mathematical theory. Detail about these technical terms can be found in (Van den Driessche & Watmough, 2002; Lenhart & Workman, 2007; Perko, 2013).

2.1 Linear and non-linear systems equilibria

Given a system of ODEs (ordinary differential equations) as shown below:

$$\dot{y} = g(y, t; \mu), \quad \mu \in \mathcal{U}\ , t \in \mathbb{R}^n\ , and\ \mu \in \mathcal{V} \in \mathbb{R}^m \tag{2.1}$$

where $\mathcal{U}$ and $\mathcal{V}$ are respectively, open sets in $\mathbb{R}^n$ and $\mathcal{R}^m$, and μ is a parameter. Then the right-hand side function $g(y, t; \mu)$ of equation (2.1) is referred to as a vector field. ODEs that do not explicitly depend on time are referred to as autonomous differential equations while those that are explicitly dependent on time are called non-autonomous differential equations. Consider the following general autonomous system

$$\dot{y} = g(y),\ y \in \mathbb{R}^n, \tag{2.2}$$

then the following definitions, theorems and lemmas are stated:

Definition 2.1.1. *The autonomous system (2.2) equilibrium solution is given by $y = \overline{y} \in \mathbb{R}^n$, where $g(\overline{y}) = 0$. The vector or point $\overline{y}$ is referred to as an equilibrium point.*

Theorem 2.1.1. *(Perko, 2013). Suppose D is an open subset of $\mathbb{R}^n$ containing y_0 and assume that $g \in C^1(D)$. Then there exists $c > 0$ such that the initial value problem (IVP)*

$$\dot{y} = g(y)\ , y(0) = y_0 \tag{2.3}$$

has a unique solution $y(t)$ on the interval $[-c, c]$.

Lemma 2.1.1 (See (Perko, 2013))**.** *Let D be an open subset of $\mathbb{R}^n$ and let $g : D \longmapsto \mathbb{R}$. Then, if $g \in C^1(D)$, g is locally Lipsitch on D.*

Definition 2.1.2. *The Jacobian matrix of g at the equilibrium point $\overline{y}$, denoted by $Dg(\overline{y})$, is the matrix of partial derivatives of g evaluated at $\overline{y}$ and can be obtained as*

$$Dg(\overline{y}) = \begin{bmatrix} \frac{\partial g_1(\overline{y})}{\partial y_1} & \dots & \frac{\partial g_m(\overline{y})}{\partial y_m} \\ \vdots & \dots & \vdots \\ \frac{\partial g_m(\overline{y})}{\partial y_1} & \dots & \frac{\partial g_m(\overline{y})}{\partial y_m} \end{bmatrix}$$

Definition 2.1.3. *Suppose $y = \overline{y}$ is an equilibrium solution of (2.2). Then $\overline{y}$ is called hyperbolic if none of the eigenvalues of $Dg(\overline{y})$ has zero real part. An equilibrium point that is not hyperbolic is called non-hyperbolic.*

2.2 Stable and unstable solutions

Here the definitions and theorems that are relevant to analyzing the stability of an autonomous system are stated. Letting $\overline{y}(t)$ be any solution of the general autonomous system (2.2), then $\overline{y}(t)$ is considered stable if solutions starting "close"to ȳ(t) at a given time remain close to $\overline{y}(t)$ for all later times. It is said to be asymptotically-stable if close solutions are attracted to ȳ(t) as $t \longrightarrow \infty$.

Definition 2.2.1. *(Wiggins, 2003). Let $\epsilon > 0$, then a solution $\overline{y}(t)$ is considered to be stable if there exists a $\delta = \delta(\epsilon) > 0$, such that for any solution x(t) of (2.2), $|\overline{y}(t, 0) - x(t, 0)| < \delta$, $|\overline{y}(t) - x(t)| < \epsilon$ for every $t > t_0$, $t_0 \in \mathbb{R}$.*

Definition 2.2.2. *(Wiggins, 2003). The solution $\overline{y}(t)$ is considered to be asymptotically stable if it has the following properties: (i) it is stable and (ii) a constant $c > 0$ exists such that any solution $x(t)$ (2.2) of fulfills conditions $|\overline{y}(t_0) - x(t_0)| < c$ and $\lim_{t\to\infty} |\overline{y}(t) - x(t)| = 0$.*

Definition 2.2.3. *A solution which is not stable is considered to be unstable.*

Theorem 2.2.1. *(Wiggins, 2003). The equilibrium solution of (2.2) is said to be locally asymptotically stable if its Jaccobian matrix $Dg(\overline{y})$ evaluated at equilibrium point $\overline{y}$ has all eigenvalues with negative real part. The equilibrium $\bar{y}$ is said to be unstable if at least one of the eigenvalues has positive real part.*

Definition 2.2.4. *(La Salle, 1976; Sanchez, 2007). A matrix $A \in \mathbb{R}^{n\times n}$ is called Routh-Hurwitz or asymptotically stabe, if and only if $R_e(\lambda_i) < 0,\ \forall i = 1, 2, \cdots, n$, where $\lambda_i' s$ are the eigenvalues of the matrix A.*

Given the polynomial;

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1}a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

where the coefficients a_i are real constants, $i = 1, \cdots, n$.
Define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial such that:

$$H = \begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

then Routh-Hurwitz condition with $a_n > 0$ are described as follows,

$H_1 = a_1 > 0, \quad \det(H_2) = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_3 > 0$ for polynomial degree 2.

$\det(H_3) = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_1a_2a_3 - a_3^2 > 0$ for polynomial degree 3.

$$\vdots$$

$\det(H_n) = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & \\ 1 & a_2 & a_4 & \cdots & \\ 0 & a_1 & a_3 & \cdots & \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0$ for $k = 1, 2, \cdots, n$

2.3 Method for computing next-generation operator method

The basic reproduction number $\mathcal{R}_0$ is defined as the average number of new infections that a typical infected individual gives rise to over the lifetime of the infection when introduced into a wholly susceptible population. (Van den Driessche & Watmough, 2002) developed a technique for calculating the basic reproduction number of disease transmission models. This method is called the next generation operator (NGO) method. The technique is also becoming increasingly popular in establishing the local asymptotic stability of the associated disease-free equilibrium. Here the procedure as given in (Van den Driessche & Watmough, 2000) is briefly outlined below.

Consider a disease transmission model that has non-negative initial conditions and can be expressed in terms of the following autonomous system:

$$\dot{y}_i = f_i(y) = F_i(y) - V_i(y),\ i = 1, \cdots, n. \tag{2.4}$$

where $V_i = V_i^- - V_i^+$ and the functions fulfill the properties $(A1) - (A5)$ stated below.
In terms of disease transmission modeling $F_i(y)$ represent the rate at which new infections occur in compartment i, $V_i^+(y)$ represent the rate at which individuals flow into compartment i by all other means and finally $V_i^-(y)$ represent the rate at which individuals flow out of compartment i (Van den Driessche & Watmough, 2002). The functions $F_i(y)$, $V_i^+(y)$, $V_i^-(y)$ are assumed to be at least twice continuous-differentiable in each variable (Van den Driessche & Watmough, 2002).

First note that $Y_s = y \geq 0 | y_i = 0,\ i = 0, \cdots, m$ define the set of all disease free states (disease free) of the model, where $y = (y_1, \cdots, y_n)^T$, $y_i \geq 0$, account for the number of individuals residing in respective compartments of the model. Properties $(A1) - (A5)$ are:

$(A1)$ If $y_i \geq 0$, then $F_i,\ V_i^-,\ V_i^+ \geq 0$ for all $i = 1, \cdots, m$;

$(A2)$ $F_i = 0$ if $i > m$;

$(A3)$ If $y_i = 0$, then $V_i^- = 0$ That is if $y \in Y_s$ then $V_i^- = 0$ for $i = 1, \cdots, m$;

$(A4)$ If $y \in Y_s$, then $F_i(y) = 0$ and $V_i^+(y) = 0$ for $i = 1, \cdots, m$;

(A5) If $F(y)$ is set to zero, then all eigenvalues of $DF(\overline{y})$ have negative real part.

Definition 2.3.1. *(Berman & Plemmons, 1994). M-Matrix. An $n \times n$ matrix A is an M-Matrix if and only if every off-diagonal entry of A is non-positive, the diagonal entries are all positive and the real parts of the eigenvalues of the matrix A are non-negative.*

Lemma 2.3.1. *(Van den Driessche & Watmough, 2002). If $\overline{y}$ is the DFE of (2.3) and $f_i(y)$ fulfills conditions $(A1) - (A5)$ then the derivatives $DF(\overline{y})$ and $DV(\overline{y})$ are partitioned as*

$$DF(\overline{y}) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \quad DV(\overline{y}) = \begin{bmatrix} V & 0 \\ J_3 & J_4 \end{bmatrix}$$

where$F$ and V are $m \times m$ matrices and can be obtained as

$$F = \left[\frac{\partial F_i(\overline{y})}{\partial y_j}\right] \quad and \quad V = \left[\frac{\partial V_i(\overline{y})}{\partial y_j}\right] with\ 1 \leq i, j \leq m.$$

Note that F is non-negative and V is a non-singular M-Matrix. J_3 and J_4 are matrices associated with the transition terms of the model and all eigenvalues of J_4 have positive real parts. Considering the above discussion and formulations the reproduction number is given as

$$\mathcal{R}_0 = \rho(FV^{-1}),$$

where ρ denotes the spectral radius. Consequently, Theorem (2.3.1) follows:

Theorem 2.3.1. *(Van den Driessche & Watmough, 2002). Consider the system (2.3) representing disease transmission, where $f_i(y)$ fulfills the stated axioms $(A1) - (A5)$. If $\overline{y}$ is a DFE of the model then $\overline{y}$ is locally asymptotically stable whenever $\mathcal{R}_0 = \rho(FV^{-1}) < 1$ and unstable whenever $\mathcal{R}_0 > 1$.*

2.4 Lyapunov functions and Lasalle's invariance principle

Definition 2.4.1. *A function $V : \mathbb{R}^n \longmapsto \mathbb{R}$ is said to be a positive-definite function if:*

- $V(y) > 0\ for\ all\ y \neq 0;$
- $V(y) = 0\ if\ and\ only\ if\ y = 0.$

Definition 2.4.2. *(Wiggins, 2003; Guckenheimer & Holmes, 2013). Consider the following system*

$$\dot{y} = g(y),\ y \in \mathbb{R}^n. \tag{2.5}$$

Let $\overline{y}$ is the equilibrium solution of (2.5) and assume $V : D \in \mathbb{R}^n \longmapsto \mathbb{R}$ is a C^1 function defined on some neighborhood D of $\overline{y}$ such that:

(a) $\mathcal{V}$ is positive definite;

(b) $\dot{V} \leq 0 \ \forall y \in D \backslash \overline{y}$.

Now the general Lyapunov function theorem is given as below:

Theorem 2.4.1. *(Guckenheimer & Holmes, 2013). Consider the system (2.5). Let $S = \{y \in \overline{\mathcal{U}} : \dot{\mathcal{V}}(y) = 0\}$ and suppose M is the largest invariant set of (2.5) in S. Now if $\mathcal{V}$ is a Lyapunov function on $\mathcal{U}$ and $\gamma^{+}(y_0)$ is a bounded orbit of (2.5) which lies in S, then the $\omega-$limit set of $\gamma^{+}(y_0)$ belongs to M (that is $y(t, y_0) \longrightarrow M$ as $t \longrightarrow \infty$).*

Corollary 2.4.2. *If $\mathcal{V}(y) \longrightarrow \infty$ as $|y| \longrightarrow \infty$ and $\dot{\mathcal{V}} \leq 0$ on $\mathbb{R}$,then every solution of (2.5) is bounded and approaches the largest invariant set M of (2.5) in the set where $\dot{\mathcal{V}} = 0$. To be precise if $M = \{0\}$, then the solution $y = 0$ is globally asymptotically stable (GAS).*

2.5 Numerical methods for system of ODEs

Numerical techniques below is summarized from (Lenhart & Workman, 2007). When a system of ODEs involves equations that can not be solved analytically, numerical methods are used to compute its approximate solution. For example, fourth order Runge Kutta method $(RK4)$ is widely used for solving initial value problems (IVP) for ordinary differential equation(ODE) which is given $\dot{x} = \dfrac{dx}{dt} = f(t, x), \ x(t_0) = x_0$ on the interval $t_0 \leq t \leq T$, we divide the interval into N small segments of constant length h, which is called the step size given by $h = \dfrac{T - t_0}{N}$. The numerical methods introduced for a single equation can be easily extended to systems of equations. For example, consider the system of Ordinary differential equation with initial condition.

$$\begin{cases} \dfrac{dx}{dt} = f(t, x, y), \quad x(t_0) = x_0 \\ \dfrac{dy}{dt} = g(t, x, y), \quad y(t_0) = y_0 \end{cases} \tag{2.6}$$

We want to obtain a numerical solution on the interval $t_0 \leq t \leq T$. The first step is discretizing the interval by defining the time steps given by

$$t_n = t_0 + nh, \quad n = 0, 1, 2, \cdots N$$

where N is the number of steps. Again we use the notation x_n and y_n to denote the approximate values of $x(t_n)$ and $y(t_n)$, respectively. We can summarize the numerical techniques as:

Definition 2.5.1. *(Lenhart & Workman, 2007). Given a well-posed initial-value problem (IVP) for the above condition (2.6), the Runge-Kutta method of order four $(RK4)$ constructs a sequence of approximation points $(t, x) \approx (t, x(t))$ and $(t, y) \approx (t, y(t))$ to the exact solution of an ODE by $t_{n+1} = t_n + h$ to get the following values of eight slope:*

$$\begin{cases} k_{11} = f(t_n, x_n, y_n) \\ k_{21} = f(t_n, x_n, y_n) \\ k_{12} = f(t_n + \frac{h}{2}, x_n + \dfrac{h}{2}k_{11}, y_n + \frac{h}{2}k_{21}) \\ k_{22} = f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{11}, y_n + \frac{h}{2}k_{21}) \\ k_{13} = f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{12}, y_n + \frac{h}{2}k_{22}) \\ k_{23} = f(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_{12}, y_n + \frac{h}{2}k_{22}) \\ k_{14} = f(t_n + h, x_n + hk_{13}, y_n + hk_{23}) \\ k_{24} = f(t_n + h, x_n + hk_{13}, y_n + hk_{23}) \end{cases} \tag{2.7}$$

Then we compute the next approximation using weighted averages of the above slopes,

$$\begin{cases} x_{n+1} = x_n + \frac{h}{6}(k_{11} + 2(k_{12} + k_{13}) + k_{14}) \\ y_{n+1} = y_n + \frac{h}{6}(k_{21} + 2(k_{22} + k_{23}) + k_{24}). \end{cases} \tag{2.8}$$

In such a process, the continuous-time model is reduced to an approximate discrete-time model that is amenable to computer simulation.

Chapter 3

LITERATURE REVIEW

Coffee berry borer becomes a serious problem of Coffee in Africa and researchers have focused their attention on understanding how this pest is controlled. Several studies have been conducted from an agricultural point of view to highlight the control of CBB. Also, mathematical modeling on farming awareness to control plant pests and disease is limited. Thus, in this chapter, we review some literature on CBB and mathematical modelling particularly related to pest control.

3.1 Literature review on related mathematical modelling

As cited by (Mesa et al., 2017) coffee berry borer, *Hypothenemus hampei*, is one of the plagues with the highest incidence and affectation in the coffee plantation. Colombia is a region that promotes the growth and infestation of this plague. The infestation is due to several factors such as Colombia's climate, geographic location, and continuous coffee production. Coffee production is one of the most important agricultural activities in the country. They assumed the behaviour of the coffee borer with an epidemiological model type SIR proposed by Kermack and Mckendrick. The model was used to estimate the coefficient that represents the infestation of the plague in terms of the other parameters involved in the dynamic. Besides, different numerical simulations were done in the Matlab environment by changing the values of the parameters of the dynamic system.

To keep the pest population below economic damage, a biological control strategy has been proposed, which consists of introducing into the environment the number of pest parasitoids determined by the State-Dependent Riccati Equation (SDRE) method (Bezerra et al., 2021). They proposed a mathematical model for interaction between the sugarcane borer and its larval parasite (*Cotesia flavipes*). The interaction between these populations is modelled by a host-parasitoid system, for which the study is done considering an influence of seasonal variation on the dynamics of the system. It has been shown that this variation generates chaotic dynamics in the system. As a raw material for ethanol, the production of sugarcane in Brazil is a very important and profitable agribusiness sector. In the last decades, with the prohibition of burning cane fields and the expansion of plantations, there has been a significant increase in the population of pests, the most important of which is a sugarcane borer (*Diatraea saccharalis*). It affects the stalks of plants, causing losses and decreasing the industrial yield in the sugar and alcohol production process.

The mathematical model discussed in (Mandal et al., 2021), applied the Z-type control method to a crop-pest-natural enemy. They assumed the indirect Z-controller in the natural enemy population and investigate the mathematical properties of the model. Furthermore, their analytical results were also numerically validated. Their paper supported that the pest population can be controlled by using an indirect Z-control mechanism in the natural enemy population. Investigations on the crop-pest-natural enemy model also highlight how the Z-control method acts concerning different dynamical regimes of the uncontrolled model.

Fitri et al. (2021), developed and analyzed a mathematical model which describes the interaction between plants and insects as well as the circulation of the pathogen. They evaluated the application of three dynamical measures of controls, namely, green insecticide, mating disruption, and the removal of infected plants, in controlling pest insects. Optimal control measures are sought in such a way they maximize the healthy plant density jointly with the pests' density under the lowest possible control efforts. Their simulation studies show that all strategies succeed in controlling the insects. However, a cost-effectiveness analysis suggests that a strategy with two measures of green insecticide and plant removal is the most cost-effective, followed by one which applies all control measures. The best strategy projects the decrease of potential loss from 65.36% to 6.12%.

A mathematical model developed by the authors in (Yadav & Kumar, 2021), studied an ecological model that is presented of prey, pest, and natural enemy of pest. Different equilibrium points are obtained, their existence and stability of the steady states are investigated. Further, a mathematical model was obtained in the presence of the control variable for designing optimal pest control problems and to study the effect on crop pests. An optimal control technique was one that minimizes the loss due to crop damage and causes minimum harm to the environment. Several chemicals, biological, physical control, and mechanisms are used. They applied Pontryagin's maximum principle to find the existence, characterization, and necessary conditions of the optimal control. Numerical simulations were further verified to support analytical results and to illustrate a different system that can be observed in the model.

In recent times, integrated pest management is gaining more attention among researchers and its application is also increasing in the crop field. The important role of microbial pesticides in integrated pest management is well-known in agriculture, forestry, and public health. (Basir et al., 2017), described the participation of farming communities in Jatropha projects for biodiesel production and protection of plants from mosaic disease, using a mathematical model to forecast the development of renewable energy resources.The study reveals that the spread of mosaic disease can be contained or even eradicated by the awareness campaigns. To attain an effective eradication, awareness campaign should be implemented at sufficiently short time intervals.
Chowdhury et al. (2019), have developed a mathematical model for pest control using biopesticides. Moreover, they also incorporate optimal control theory to minimize the cost in pest management due to bio-pesticides.Bio-pesticides are costly, require a long term process and expensive to impose. But if chemical pesticides are introduced in the farming system along with bio-pesticide, the process will be faster as well as cost effective. Here, they studied the control of pests using integrated approach i.e. using combination of bio-pesticides and chemical pesticides. They identified the parameter for which stability switches may occur. Finally, optimal concentration profile of both pesticides have been determined using optimal control theory to minimize it negative effects and also to make the process cost effective. Numerical simulations justified their main results. For the use of optimal control theory to eradicate the number of parasites in agroecosystems (Silva et al., 2017). The dynamics of vector-borne plant disease dynamics influenced by farming awareness were analyzed in

(Al Basir & Ray, 2020). Here, a mathematical model is formulated to protect crops through awareness campaigns, modelled via saturated terms, and a delayed optimal control problem for bio-pesticides is posed and solved.

Mathematical models describing the life cycle of the insect pest with different types of biological controls were proposed in (Mohammed Abderrehim, 2021). These controls must be positioned in an optimal way to reduce the risk of attacks and the additional costs associated with the intensive use of pesticides. The models used will be of the type systems of hyperbolic partial differential equations of order one with development speeds depending on environmental conditions(temperature, humidity, rainfall, availability of resources, etc.) and non-local transition and mortality rates.

Sharomi and Malik (2017) developed a mathematical model that describes the theoretical bases to evaluate control strategies in terms of the interactions between host, virus, and vector. Vector activity and behaviour, mainly relative to virus transmission, are essential determinants of the rate and extent of epidemic development. The applicability and flexibility of these models are illustrated by reference to specific case studies, including the increasing importance of whitefly transmitted viruses.

As integrated pest management, bio-pesticides give noticeable pest control reliability in case of crops (Bhattacharyya & Bhattacharya, 2006). The use of viruses against insect pests, as pest control agents, is seen in North America and European countries (Franz & Huber, 1979). Correct and relevant knowledge about crops and their pests is very much essential for people engaged in cultivation. The role of electronic media is critical for keeping the farming community updated and by providing them with relevant agricultural information (Kumar et al., 2012). Accessible pesticide information campaigns help farmers to be aware of the serious risks that pesticides have on human health and the environmenta and minimize negative effects (Van Lenteren et al., 2006).

Teklebirhan et al. (2021) investigated a mathematical model in crop pest management, considering plant biomass, pest, and the effect of farming awareness. The pest population is divided into two compartments: susceptible pest and infected pest. They assumed that the growth rate of self-aware people are proportional to the density of healthy pests present in the crop field. Impacts of awareness are modelled via a saturated term. They further assumed that self-aware people will adopt biological control methods, namely integrated pest management. Susceptible pests are detrimental to crops and there may be some time delay in measuring the healthy pests in the crop field. A time delay may also take place while becoming aware of the control strategies or taking necessary steps to control the pest attack. The existence and the stability criteria of the equilibria are obtained in terms of the basic reproduction number and time delays. Stability switches occur through Hopf-bifurcation when time delays cross critical values. Optimal control theory has been applied for the cost-effectiveness of the delayed system. Numerical simulations illustrate the obtained analytical results.

A mathematical model that describes the infestation dynamics of coffee berry by *Hypothenemus hampei* (CBB) was developed and analyzed by (Fotso et al., 2019). This model takes into account control some integrated pest management strategies which are used by coffee growers to eradicate CBB in plantations, represented by two functions depending on time. By using optimal control theory, they shown that an optimal control exists for this problem and Pontryagin's maximum principle was used to characterize an optimal control. Numerical simulations were provided to illustrate their results.

In the above different mathematical pieces of literature are reviewed to understand the control strategies of pests. In all these studies, mathematical modelling for coffee berry borer with optimal control at the presence of farmer awareness is not considered. Thus, this study addressed these gaps.

Chapter 4

RESEARCH METHODOLOGY

In this chapter, we present the methods and materials that are used to attain the general and specific objectives stated in Section 1.4. For this, we briefly discuss: source of data, method of data analysis, the approach of study and mathematical procedures.

4.1 Source of data

In this study to acquire the desired knowledge secondary source of information is used. A secondary source of information is collected by using document analysis both from published and unpublished documents. That is, the source of information that is used in this study are books, journals and related studies from the internet.

4.2 Method of data analysis

The study involved both qualitative and quantitative analysis. Thus, the data is analyzed using MATLAB Software and the results are displayed in the form of tables and figures.

4.3 Description of mathematical procedure

To achieve the objective of this study we employ the analytical and numerical method. Thus, we formulate a mathematical model which describes the infestation of CBB using a system of non-linear ordinary differential equations. Well-posedness of the formulated model will be determined both in the mathematical and epidemiological sense. We then perform the qualitative analysis of the model. Local and global stability analysis of the equilibrium points of the model will be investigated. Finally, we extend the proposed to optimal control problem and present its numerical simulations using Matlab software to predict the goal to minimize the infestation of CBB.

Chapter 5

MODEL FORMULATION AND ANALYSIS

In this chapter, we present a deterministic mathematical model for Coffee berry bore using a system of nonlinear ordinary differential equations. We begin the chapter by discussing model description and formulation, and model analysis in sequence.

5.1 Model description and formulation

In this section, we proposed deterministic mathematical modelling for the dynamics of Coffee berry borer. The Coffee berry borer or Coffee borer beetle (CBB) is an insect found around the world and prevalent in most coffee-producing countries. It is among the most harmful pests to commercial coffee plantations and can attack 50-100% of berries on a farm if no control measures are applied (Le Pelley et al., 1968).

The proposed mathematical model includes Coffee berry biomass, Coffee borer population, and awared farmers. The density of the coffee berry is denoted by B. Due to the finite size of a coffee farm, we assumed logistic growths for the density of coffee berry biomass with a growth rate $r > 0$ and carrying capacity $K > 0$. The density of the coffee borer is represented by Y. The coffee berry borer female attacks immature and mature coffee berries from about eight weeks after flowering up to harvest season. Females bore a hole into the coffee berry and then construct galleries in the seeds (beans) where the eggs are deposited, followed by larval feeding on the coffee seed (Johnson et al., 2020). The main coffee borer management strategies involve different components, including regular monitoring, controlled harvest, and the use of biological control agents. These strategies are useful before the females enter the berries. Once, it entered the bean it is difficult to control the pest (Vega, 2004). Thus, we assumed that infected female bores do not affect the berries. This due to its lack of power to drill into berries. The natural death rate of bores and berries are respectively denoted by d and μ.

The per capita rate at which female bore Y attack coffee berry B is denoted by the term $\gamma B/(c+B)$. Here, γ denote the attack rate of a female borer on berries while c represents the half-saturation constant. Then the founder female remains inside the fruit after oviposition until she dies, taking care of the offspring. There is sibling mating among the adult progeny with a $10:1$ sex ratio favouring females. Therefore, when the new adult females emerge, they are already inseminated and ready to attack another berry, in which they continue the cycle. Male CBB does not infest and remains inside the berry. Most of the life cycle occurs inside the berry.

The variable $A(t)$ measures the information density available among the farmer due to the prevalence of the borer. The level of farmer awareness $A(t)$ is assumed to increase due to media campaigns and agricultural field workers. It is assumed that the rate of farmers' awareness is proportional to the infestation burden and increases with the rate α. As a consequence of awareness farmers surveillances their farm, use biological control or trap to avoid bore infestation. The awareness activity rate β , owing to aware farmers interactions and control activities such as use of chemical, trap and biological, can be introduced in the model via the mass action term by $\beta AY/(\delta+A)$. Here, denotes β farmers' activity rate due to awareness and δ is the constant related to saturation in information-induced intervention. We also assumed farmers memory fade and thus, the level of awareness decrease with a rate of e.

With these descriptions in mind, the dynamics of the governed mathematical model can be written as:

$$\begin{cases} \dfrac{dB}{dt} &= rB\left(1-\dfrac{B}{K}\right) - \dfrac{\gamma BY}{c+B} - \mu B, \\ \dfrac{dY}{dt} &= \dfrac{\sigma\gamma BY}{c+B} - \dfrac{\beta AY}{\delta+A} - dY, \\ \dfrac{dA}{dt} &= \theta + \alpha Y - eA, \end{cases} \tag{5.1}$$

where $B(0) > 0, Y(0) \geq 0, A(0) > 0$. Here σ denotes conversion rate and θ represents rate of awareness due to global source. Compared to the mathematical model introduced in

(Fotso et al., 2019), the present model differ due to farmers awareness and logistic growth model.

5.2 Model analysis

5.2.1 Positivity and Invariant region of the solution

In this subsection, we study a region in which the solution of the governing equation is both epidemiologically and mathematically meaningful. That is, the solution of the proposed model is non-negative and bounded in a certain feasible region. Thus, we prove that all solutions of the system with positive initial data will remain positive for all times $t \geq 0$. Now we have the following theorem.

Theorem 5.2.1. *If the initial condition* $(B(0), Y(0), A(0) \in \mathbb{R}^3_+)$*, then the solution* $(B(t), Y(t), A(t))$ *of the system* (5.1) *are non-negative for all time* $t \geq 0$ *and bounded. Moreover, the feasible compact set is given by*

$$\Omega = \left\{(B, Y, A) \in \mathbb{R}^3_+ : 0 < B \leq K,\ 0 < \sigma B + Y \leq \frac{M(\sigma r + 1)}{\xi},\ 0 \leq A \leq \frac{\xi\theta + \alpha M(\sigma r + 1)}{e\xi}\right\}$$

where $M = max\{B(0), K\}$ *and* $\xi = min\{1, d\}$.

Proof. For all initial condition prescribed in $\mathbb{R}^3_+$,

$$\left.\frac{dB}{dt}\right|_{B=0} \geq 0,\ \left.\frac{dY}{dt}\right|_{Y=0} \geq 0,\ \left.\frac{dA}{dt}\right|_{A=0} = \theta + \alpha Y \geq 0.$$

This implies that all the solution with positive initial data remains nonnegative in $\mathbb{R}^3_+$ for all $t \geq 0$. This means, the region attracts all solutions of the governing system.

Next, we determine the feasible region in which the solution of the model (5.1) is bounded. For this, consider the first equation of system (5.1)

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - \frac{\gamma BY}{c + B} - \mu B \leq rB\left(1 - \frac{B}{K}\right).$$

Integrating by separation of variables, we obtain

$$B(t) \leq \frac{KB(0)}{e^{-rt}(K - B(0)) + B(0)}.$$

From which we deduce that $B(t) \leq K$ and we have $\overline{\lim} B(t) \leq K$.

Let $w = \sigma B + Y$ at any time t and $\xi = min\{1, d\}$, then by adding the first and second equation of system (5.1), we obtain

$$
\begin{aligned}
w' &= \sigma B' + Y', \\
&= \sigma r B\left(1 - \frac{B}{K}\right) - \sigma\frac{\gamma BY}{c+B} - \sigma\mu B + \sigma\frac{\gamma BY}{c+B} - \frac{\beta AY}{\delta + A} - dY, \\
&\leq \sigma r B - dY, \\
&= B(\sigma r + 1) - (B + dY) \leq M(\sigma r + 1) - \xi w.
\end{aligned}
$$

It follows that

$$\frac{dw}{dt} + \xi w \leq M(\sigma r + 1). \tag{5.2}$$

Solving this it gives that $w(t) \leq \dfrac{M(\sigma r + 1)}{\xi}$. If $w(0) \leq \dfrac{M(\sigma r + 1)}{\xi}$, then we have

$$\limsup_{t\to\infty} w(t) \leq \frac{M(\sigma r + 1)}{\xi}.$$

Now, plunging $Y(t) = w - \sigma B(t)$ in the third equation of model system (5.1) and after manipulation we obtain,

$$
\begin{aligned}
A' &\leq \theta + \alpha\frac{M(\sigma r + 1)}{\xi} - eA \\
\implies A' + eA &\leq \frac{\xi\theta + \alpha M(\sigma r + 1)}{\xi}.
\end{aligned}
$$

Again,solving this we get,

$$A(t) \leq \frac{\xi\theta + \alpha M(\sigma r + 1)}{e\xi}.$$

Thus, if $A(0) \leq \dfrac{\xi\theta + \alpha M(\sigma r + 1)}{e\xi}$, then it follows that $\limsup_{t\to\infty} A(t) \leq \dfrac{\xi\theta + \alpha M(\sigma r + 1)}{e\xi}$.
Thus, the invariant region is given by

$$\Omega = \left\{(B, Y, A) \in \mathbb{R}^3_+ : 0 < B \leq K,\ 0 < \sigma B + Y \leq \frac{M(\sigma r + 1)}{\xi},\ 0 \leq A \leq \frac{\xi\theta + \alpha M(\sigma r + 1)}{e\xi}\right\}.$$

Hence, the solutions of the model equations are bounded and nonnegative in the region Ω.

□

5.3 Coffee berry borer free equilibrium point (CBBFEP)

CBB free equilibrium point is steady-state solutions when there is no CBB. That is, the borer class is zero and the entire population is comprise of the coffee berry and the information density available among the farmers. To find the CBB free equilibrium point, we equate the left hand side of model (5.1) equal to zero. Evaluating it gives,

$$\begin{aligned} rB\left(1-\frac{B}{K}\right)-\frac{\gamma BY}{c+B}-\mu B &=0, \\ \frac{\sigma\gamma BY}{c+B}-\frac{\beta AY}{\delta+A}-dY &=0, \\ \theta+\alpha Y-eA &=0, \end{aligned} \tag{5.3}$$

Thus, the borer free equilibrium point becomes

$$E^0=\left(\frac{K(r-\mu)}{r},\ 0,\ \frac{\theta}{e}\right). \tag{5.4}$$

5.3.1 Basic reproduction number

In this subsection, first, we derive an expression for the basic reproduction number $\mathcal{R}_0$ using the method introduced in (Van den Driessche & Watmough, 2002). Then we utilize $\mathcal{R}_0$ to analyze the stability of the equilibrium points. In our context, the basic reproductive number is the average number of new females originated from a single infesting female in the healthy coffee berries populations.

Let $\tilde{x}=(B,Y,A)$ denotes the set of state variables. Then the system (5.1) can be rewritten as $\dfrac{d\tilde{x}}{dt}=\mathcal{F}_i(\tilde{x})-\mathcal{V}_i(\tilde{x})$, where $\mathcal{F}_i$'s is the rate of new recruits (birth of new colonizing females) in compartment i, $\mathcal{V}_i=\mathcal{V}_i^- - \mathcal{V}_i^+$ where $\mathcal{V}_i^+$ is representing the rate of transfer into a compartment i by all other means, and $\mathcal{V}_i^-$ is the rate of transfer out the compartment i.
For this model, $\mathcal{F}$ and $\mathcal{V}$ are given by

$$\mathcal{F}=\begin{bmatrix} 0 \\ \dfrac{\sigma\gamma BY}{c+B} \\ 0 \end{bmatrix};\quad \mathcal{V}=\begin{bmatrix} -rB\left(1-\dfrac{B}{K}\right)+\gamma\dfrac{BY}{c+B}+\mu B \\ \beta\dfrac{AY}{\delta+A}+dY \\ -\theta-\alpha Y+eA. \end{bmatrix}.$$

To obtain the next generation matrix, we compute the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ denoted by $R=\mathcal{J}_{\mathcal{F}}(E^0)$ and $T=\mathcal{J}_{\mathcal{V}}(E^0)$ respectively. Here, we have

$$R=\mathcal{J}_{\mathcal{F}}(E^0)=\begin{bmatrix} 0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\quad T=\mathcal{J}_{\mathcal{V}}(E^0)=\begin{bmatrix} p_1 & p_2 & 0 \\ 0 & p_3 & 0 \\ 0 & -\alpha & e \end{bmatrix}$$

where $p_0 = \dfrac{\sigma\gamma K(r-\mu)}{cr+K(r-\mu)},\ p_1 = r-\mu,\ p_2 = \gamma\dfrac{K(r-\mu)}{cr+K(r-\mu)},\ p_3 = \beta\dfrac{\theta}{e\delta+\theta}+d.$

It can be verified that the matrix $T(E^0)$ is non-singular as its determinant is non-zero. That is

$$det(T(E^0)) = \begin{vmatrix} p_1 & p_2 & 0 \\ 0 & p_3 & 0 \\ 0 & -\alpha & e \end{vmatrix} = ep_1p_3 \neq 0.$$

Since $det(T(E^0)) \neq 0$, then $T(E^0)$ is invertible and the inverse is given by

$$T(E^0)^{-1} = \frac{Adj(T)}{det(T)}.$$

Then after some algebraic computations the inverse matrix is constructed as :

$$\left[T(E^0)\right]^{-1} = \begin{bmatrix} \frac{1}{p_1} & \frac{p_2}{p_1p_3} & 0 \\ 0 & \frac{1}{p_3} & 0 \\ 0 & -\frac{\alpha}{ep_3} & \frac{1}{e} \end{bmatrix}.$$

An expression for $\mathcal{R}_0$ can be obtained from

$$\begin{aligned} \left[R(E^0)\right]\left[T(E^0)\right]^{-1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{p_1} & \frac{p_2}{p_1p_3} & 0 \\ 0 & \frac{1}{p_3} & 0 \\ 0 & -\frac{\alpha}{ep_3} & \frac{1}{e} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{p_0}{p_3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{5.5}$$

Thus, the eigenvalues of the matrix (5.5) are: $\lambda_1 = 0,\ \lambda_2 = \dfrac{p_0}{p_3},\ \lambda_3 = 0.$

Here, $[R(E^0)]\,[T(E^0)]^{-1}$ is the next-generation matrix for the system (5.1). It follows that the spectral radius of the matrix is λ_2 which is the dominant eigenvalue. Therefore, the basic reproduction number is given by

$$\mathcal{R}_0 = \rho(RT^{-1}) = \frac{\sigma\gamma K(r-\mu)(e\delta+\theta)}{(cr+K(r-\mu))\,((\beta+d)\theta+de\delta)}. \tag{5.6}$$

5.3.2 Local stability of coffee berry borer free equilibrium point(CBBFEP)

Theorem 5.3.1. *The CBBFEP is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$*

Proof. The Jacobian matrix of system (5.1) at CBBFEP is given by

$$\mathcal{J}(E^0) = \begin{pmatrix} -r+\mu & -\dfrac{k\gamma(r-\mu)}{rc+k(r-\mu)} & 0 \\ 0 & \dfrac{\sigma\gamma K(r-\mu)}{rc+K(r-\mu)} - \dfrac{\beta\theta}{e\delta+\theta} - d & 0 \\ 0 & \alpha & -e \end{pmatrix} \tag{5.7}$$

From the Jacobian matrix (5.7), we obtained characteristic polynomial:

$$(-r+\mu-\lambda)\left(\frac{\sigma\gamma K(r-\mu)}{rc+K(r-\mu)} - \frac{\beta\theta}{e\delta+\theta} - d - \lambda\right)(-e-\lambda) = 0. \tag{5.8}$$

From Eq. (5.8), we obtain that

$$\lambda = -(r-\mu) < 0,\ \lambda = \frac{\beta\theta + d(e\delta+\theta)}{e\delta+\theta}(\mathcal{R}_0 - 1) < 0,\ \lambda = -e < 0.$$

Here we suppose that the growth rate is greater than natural death rate of coffee berries (i.e., $r-\mu > 0$). Applying Routh-Hurwitz criteria (DeJesus & Kaufman, 1987) on Eq. (5.8) shows that all eigenvalues have negative and so E^0 is local asymptotically stable for $\mathcal{R}_0 < 1$ in Ω.

□

5.3.3 Global stability of coffee berry borer free equilibrium point(CBBFEP)

Theorem 5.3.2. *The borer free equilibrium point E^0 of the model equation (5.1) is globally asymptotically stable if $\mathcal{R}_0 < 1$.*

Proof. To proof this theorem, define the Lyapunov function as

$$U = Y.$$

Then differentiating U with respect to t at E_0, we obtain

$$\begin{aligned} \frac{dU}{dt} &= \left(\frac{\sigma\gamma k(r-\mu)}{rc+k(r-\mu)} + \frac{\beta\theta}{e\delta+\theta} - d\right)Y \\ &= \frac{\beta\theta+d(e\delta+\theta)}{e\delta+\theta}\left(\frac{\sigma\gamma k(r-\mu)(e\delta+\theta)}{(cr+k(r-\mu))(\beta\theta+d(e\delta+\theta))} - 1\right)Y \\ &= \frac{\beta\theta+d(e\delta+\theta)}{e\delta+\theta}(\mathcal{R}_0-1)Y. \end{aligned} \tag{5.9}$$

Thus, if $\mathcal{R}_0 < 1$, then $(dU/dt)(B,\ Y,\ A) \leq 0$ for all $(B,\ Y,\ A) \in \Omega$. Therefore E^0 is globally asymptotically stable for $\mathcal{R}_0 < 1$ in Ω.

□

5.4 Endemic equilibrium point (EEP)

In the presence of CBB, $B(t) > 0;\ Y(t) \geq 0;\ A(t) > 0$, the model has an equilibrium point called endemic equilibrium point denoted by $E^* = (B^*,\ Y^*,\ A^*)$. It is the steady state solution where CBB persist in the population of coffee plants. It can be obtained by equating each equation of the model equal to zero. That is,

$$\begin{aligned} rB^*\left(1-\frac{B^*}{K}\right)-\frac{\gamma B^*Y^*}{c+B^*}-\mu B^* &=0,\\ \frac{\sigma\gamma B^*Y^*}{c+B^*}-\frac{\beta A^*Y^*}{\delta+A^*}-dY^* &=0,\\ \theta+\alpha Y^*-eA^* &=0. \end{aligned} \tag{5.10}$$

Then we obtain B^* and A^* in terms of Y^*

$$\begin{aligned} B^* &= \frac{(c\beta\alpha+dc\alpha)Y^*+c(\beta+d)\theta+cde\delta}{(\sigma\alpha-\beta\alpha-d\alpha)Y^*+(\sigma\gamma-\beta-d)\theta+e(\sigma\gamma-de)\delta},\\ A^* &= \frac{\theta+\alpha Y^*}{e}, \end{aligned} \tag{5.11}$$

and Y^* is the positive root of the equation

$$f(Y^*) = W_4(Y^*)^4+W_3(Y^*)^3+W_2(Y^*)^2+W_1Y^*+W_0=0, \tag{5.12}$$

where

$$
\begin{aligned}
W_4 &= k\gamma(\alpha(\sigma\gamma - d - \beta))^3,\\
W_3 &= (\alpha(\sigma\gamma - d - \beta))^2(k\mu(\alpha + k\gamma(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))) + kc\alpha(\mu - r)(\sigma\gamma - d - \beta))\\
&\quad + 2k\gamma\alpha^2(\sigma\gamma - d - \beta)(\sigma - d - \beta)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))\\
&\quad + r\alpha(\sigma\gamma - d - \beta)(c\alpha(\beta + d))^2,\\
W_2 &= (\alpha(\sigma\gamma - d - \beta))^2[k\mu c\theta(\beta + d) + k\mu cde\delta + kc(\mu - r)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))]\\
&\quad + 2\alpha(\sigma\gamma - d - \beta)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))(k\mu(\alpha + k\gamma(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta)))\\
&\quad + kc\alpha(\mu - r)(\sigma\gamma - d - \beta)) + k\gamma\alpha(\sigma\gamma - d - \beta[(e\delta(\sigma\gamma - d))^2\\
&\quad + \theta(\sigma\gamma - d - \beta)(\theta(\sigma\gamma - d - \beta) + 2e\delta(\sigma\gamma - d))]\\
&\quad + r(c\alpha)^2(\beta + d)[(\sigma\gamma - d - \beta)(ed\delta + \theta(\beta + d))(\beta + d)(e\delta)(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta)],\\
W_1 &= (2\alpha(\sigma\gamma - d - \beta)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta)))(kc\mu\theta(\beta + d) + kcde\mu\delta\\
&\quad + kc(\mu - r)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))) + [(ce\delta(\sigma\gamma - d))^2 + \theta(\sigma\gamma - d - \beta)(\theta(\sigma\gamma - d - \beta)\\
&\quad + 2e\delta(\sigma\gamma - d))][kc\mu\alpha\\
&\quad + k\gamma(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta)) + kc\alpha(\mu - r)(\sigma\gamma - d - \beta)]\\
&\quad + \alpha(\sigma\gamma - d - \beta)((c\theta(\beta + d))^2 + 2drc^2\delta\theta(\beta + d) + r(cde\delta)^2) + (e\delta(\sigma\gamma - d)\\
&\quad + \theta(\sigma\gamma - d - \beta))(2rc^2\alpha(\beta + d)(ed\delta + \theta(\beta + d))),\\
W_0 &= ((e\delta(\sigma\gamma - d))^2 + \theta(\sigma\gamma - d - \beta)(\theta(\sigma\gamma - d - \beta) + 2e\delta(\sigma\gamma - d)))(kc\mu\theta(\beta + d) + kced\mu\delta\\
&\quad + kc(\mu - r)(e\delta(\sigma\gamma - d) + \theta(\sigma\gamma - d - \beta))) + kc\mu\theta(\beta + d) + kced\mu\delta + kc(\mu - r)(e\delta(\sigma\gamma - d)\\
&\quad + \theta(\sigma\gamma - d - \beta)).
\end{aligned}
$$

.

5.4.1 Local stability of endemic equilibrium point

Theorem 5.4.1. *The endemic equilibrium E^* of system (5.1) is locally asymptotically stable in Ω if $\mathcal{R}_0 > 1$, otherwise unstable.*

Proof. First we obtain the Jacobian matrix of system (5.1)

$$
\mathcal{J} = \begin{pmatrix}
r\left(1 - \frac{2B}{K}\right) - c\frac{\gamma Y}{(c+B)^2} - \mu & \frac{-\gamma B}{c+B} & 0\\
c\frac{\sigma\gamma Y}{(c+B)^2} & \frac{\sigma\gamma B}{c+B} - \frac{\beta A}{\delta + A} - d & \frac{-\beta\delta Y}{(\delta + A)^2}\\
0 & \alpha & -e
\end{pmatrix}. \tag{5.13}
$$

Evaluating the Jacobian matrix at the endemic equilibrium $E^* = (B^*,\ Y^*,\ A^*)$, we get

$$\mathcal{J} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

where

$A_{11} = r\left(1 - \dfrac{2B^*}{K}\right) - c\dfrac{\gamma Y^*}{(c+B^*)^2} - \mu, \quad A_{12} = \dfrac{-\gamma B^*}{c+B^*}, \quad A_{13} = 0,$

$A_{21} = c\dfrac{\sigma\gamma Y^*}{(c+B^*)^2}, \ A_{22} = \dfrac{\sigma\gamma B^*}{c+B^*} - \dfrac{\beta A^*}{\delta + A^*} - d, \quad A_{23} = \dfrac{-\beta\delta Y^*}{(\delta+A^*)^2},$

$A_{31} = 0, \quad A_{32} = \alpha, \quad A_{33} = -e.$

From this Jacobian matrix one can easily obtain the characteristic polynomial as

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{5.14}$$

where

$a_2 = -(A_{11} + A_{22} - e),$

$a_1 = A_{22}A_{33} + A_{11}A_{22} + A_{11}A_{33} - A_{23}A_{32} - A_{12}A_{21},$

$a_0 = A_{11}A_{23}A_{32} + A_{12}A_{21}A_{33} - A_{11}A_{22}A_{33}.$

Using the Routh-Hurwitz criterion (DeJesus & Kaufman, 1987), all roots of characteristic polynomial will have negative real roots or imaginary roots with negative real part if and only if $a_2 > 0,\ a_1a_2 > a_0$ and $a_0(a_1a_2 - a_0) > 0$ for $\mathcal{R}_0 > 1$. Hence, the endemic equilibrium E^* is locally asymptotically stable.

□

5.4.2 Global stability of endemic equilibrium point

Theorem 5.4.2. *The endemic equilibrium point of the model equation (5.1) is globally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof. To proof the global stability of the endemic equilibrium point E^* we define the Lyapunov function as

$$L = \epsilon_1\frac{(B-B^*)^2}{2} + \epsilon_2\frac{(Y-Y^*)^2}{2} + \epsilon_3\frac{(A-A^*)^2}{2}, \tag{5.15}$$

where $\epsilon_1 > 0,\ \epsilon_2 > 0$ and $\epsilon_3 > 0$ are to be chosen appropriately such that $\left(\dfrac{dL}{dr}\right)(E^*) = 0$ and $L(B, Y, A) > 0$ for all $(B, Y, A)\backslash E^*$.

A straight forward derivative of L along the solution curve of system (5.1) yields

$$\frac{dL}{dt} = \epsilon_1(B-B^*)\frac{dB}{dt} + \epsilon_2(Y-Y^*)\frac{dY}{dt} + \epsilon_3(A-A^*)\frac{dA}{dt}. \tag{5.16}$$

Now substituting corresponding expressions equations of model (5.1), we obtain

$$\begin{aligned} \frac{dL}{dt} &= \epsilon_1(B-B^*)\left[rB\left(1-\frac{B}{K}\right) - \frac{\gamma BY}{c+B} - \mu B\right] \\ &+ \epsilon_2(Y-Y^*)\left[\frac{\sigma\gamma BY}{c+B} - \frac{\beta AY}{\delta+A} - dY\right] \\ &+ \epsilon_3(A-A^*)\left[\theta+\alpha Y - eA\right]. \end{aligned} \tag{5.17}$$

By rearranging we get,

$$\begin{aligned} \frac{dL}{dt} &= -\epsilon_1(B-B^*)\left[rB\left(-1+\frac{B}{K}\right) + \frac{\gamma BY}{c+B} + \mu B\right] \\ &- \epsilon_2(Y-Y^*)\left[-\frac{\sigma\gamma BY}{c+B} + \frac{\beta AY}{\delta+A} + dY\right] \\ &- \epsilon_3(A-A^*)\left[eA - (\theta+\alpha Y)\right]. \end{aligned} \tag{5.18}$$

Now, we choose

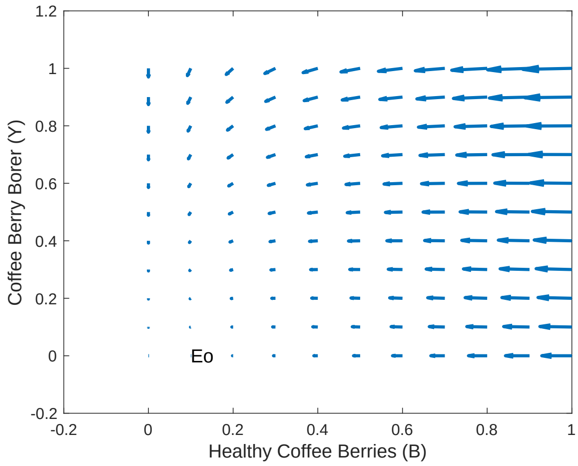
$$\begin{aligned} \epsilon_1 &= \frac{K(c+B)}{\gamma KY + (\mu K - r(K-B))(c+B)}, \\ \epsilon_2 &= \frac{(c+B)(\delta+A)}{\gamma A(c+B) + (d(c+B) - \sigma\gamma B)(\delta+A)}, \\ \epsilon_3 &= \frac{A}{eA - (\theta+\alpha Y)}. \end{aligned}$$

Thus, $(dL/dt)\,(B,Y,A) \leq 0$. Also $dL/dt = 0$, if and only if $B = B^*, Y = Y^*, A = A^*$. Therefore, the largest compact invariant set $\{(B^*,Y^*,A^*) \in \Omega : dL/dt = 0\}$ is the singleton E^*, where E^* is the endemic equilibrium of the system (5.1). Hence, we can conclude that an endemic equilibrium point is globally asymptotically stable by LaSalle's invariant principle (La Salle, 1976) in Ω. $\square$

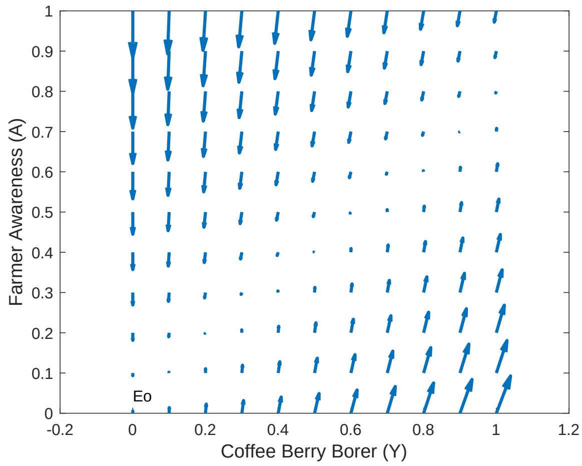
5.5 Phase portrait analysis

The phase portrait of our governing dynamical system (5.1) presents in graphical form in a small region nearby the equilibria. The phase portraits in Figures 5.1 and 5.2 show a typical solution curves and a direction field (with arrows). The CBB free equilibrium $E^0 = \left(\frac{K(r-\mu)}{r}, 0, \frac{\theta}{e}\right)$ is asymptotically stable in Figures 5.1 (a), (b) and (c). That means, in Figure 5.1 (a), (b) and (c) the corresponding trajectories: $(B(t), Y(t))$, $(Y(t), A(t))$ and $(B(t), A(t))$ head into the equilibrium point $\left(\frac{K(r-\mu)}{r}, 0\right)$, $\left(0, \frac{\theta}{e}\right)$ and $\left(\frac{K(r-\mu)}{r}, \frac{\theta}{e}\right)$ as $t \to \infty$ respectively.

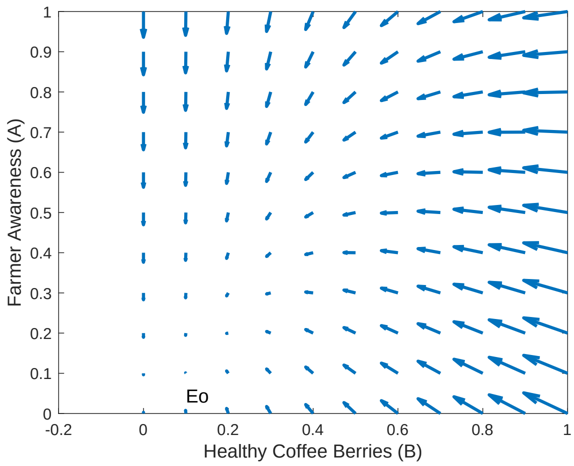
The endemic equilibrium $E^* = (B^*,\ Y^*,\ A^*)$ is asymptotically stable in Figures 5.2 (a), (b) and (c). That is in Figure 5.2 (a), (b) and (c) the corresponding trajectories:$(B(t),\ Y(t))$, $(Y(t),\ A(t))$ and $(B(t),\ A(t))$ head into the equilibrium point $(B^*(t),\ Y^*(t))$, $(Y^*(t), A^*(t))$ and $(B^*(t), A^*(t))$ as $t \to \infty$ respectively. Clearly, the phase portrait analysis agree with the stability analysis presented in the previous section.



(a) Phase portraits for healthy coffee vs CBB.

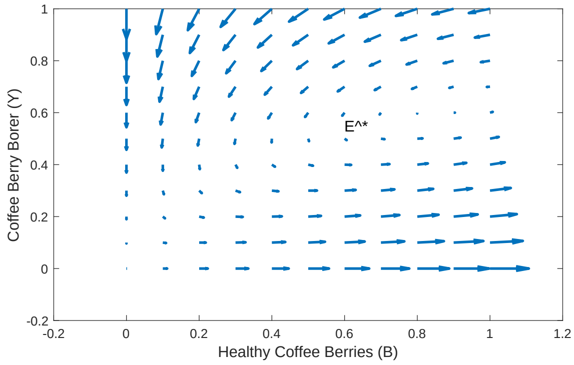


(b) Phase portraits for CBB vs farmer awareness.

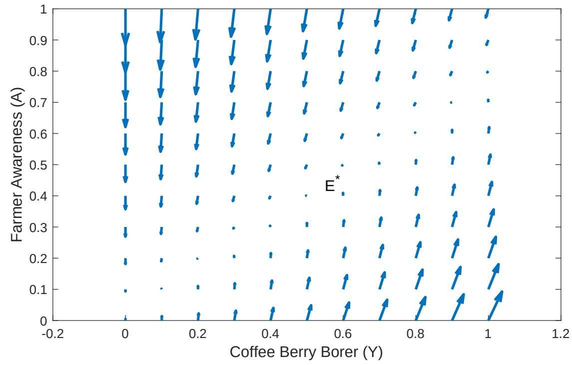


(c) Phase portraits for healthy coffee berries vs farmer awareness.

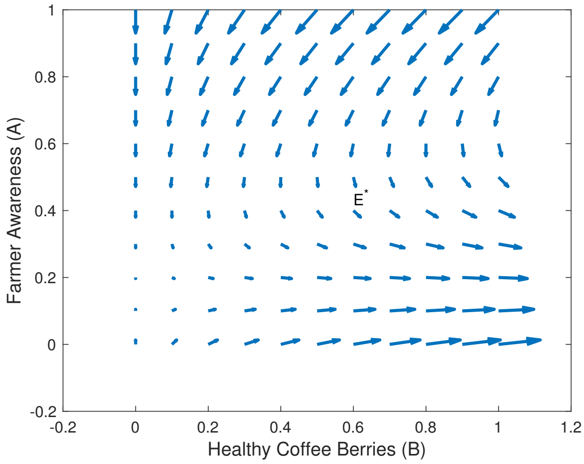
Figure 5.1: Phase portraits for healthy coffee berries, CBB, and farmer awareness population. Parameter values are taken from Table 7.1 except for $r = 0.1,\ k = 1.8,\ d = 0.015$.



(a) Phase portraits for healthy coffee vs CBB.



(b) Phase portraits for CBB vs farmer awareness.



(c) Phase portraits for healthy coffee berries vs farmer awareness.

Figure 5.2: Phase portraits analysis for healthy coffee berries, female CBB, and farmer awareness population. Parameter values are taken from Table 7.1.

5.6 Sensitivity analysis of model parameters

In this section, we carried out sensitivity analysis to identify parameters that can impact the basic reproductive number. Sensitivity analysis tell us how significant each parameter is to borer infestation.

Definition 5.6.1. *(Chitnis et al., 2008) The normalized forward sensitivity index of a variable, g, that depends differentiably on a parameter, p, is defined as*

$$\Lambda_p^g = \frac{\partial g}{\partial p} \times \frac{p}{g}$$

for p represents all the basic parameters.

Notice that Λ_p^g has a maximum value of magnitude 1. Sensitivity index, $\Lambda_p^g = 1$ implies an increase (decrease) of p by $y\%$ increases (decreases) g by $y\%$. On the other hand, $\Lambda_p^g = -1$ indicates an increase (decrease) of p by $y\%$ decreases (increases) g by $y\%$.

Table 5.1: Sensitivity indices and parameters

Model Parameters	**Sensitivity indices of $\mathcal{R}_0$**
r	0.0294424
σ	1
γ	1
K	0.112108
δ	0.547473
e	0.547473
β	-0.583521
μ	-0.0294424
θ	-0.547473
d	-0.416479
c	-0.112108

5.6.1 Interpretation of sensitivity indices

The sensitivity indices of the basic reproductive number concerning main parameters are found in Table 5.1. Those parameters that have positives indices ($\gamma,\ \sigma,\ r,\ K,\ \delta$ and e) show that they have a great impact on expanding the female coffee berry bore in the farm if their values are increasing. Due to the reason that the $\mathcal{R}_0$ increases as their values increases, it means that the average number of new females originated from a single infesting female in the healthy coffee berries populations. And also those parameters in which their sensitivity indices are negative $(c,\ \beta,\ \theta,\ d,\ \mu)$ have an effect of minimizing the burden of the borer in the berry population as their values increase while the others are left constant. And also as their values increase the $\mathcal{R}_0$ decreases, which leads to minimizing the endemicity of the borers in the coffee berry population. The bar diagram of sensitivity indices in Table 5.1 is depicted in Figure 5.3.

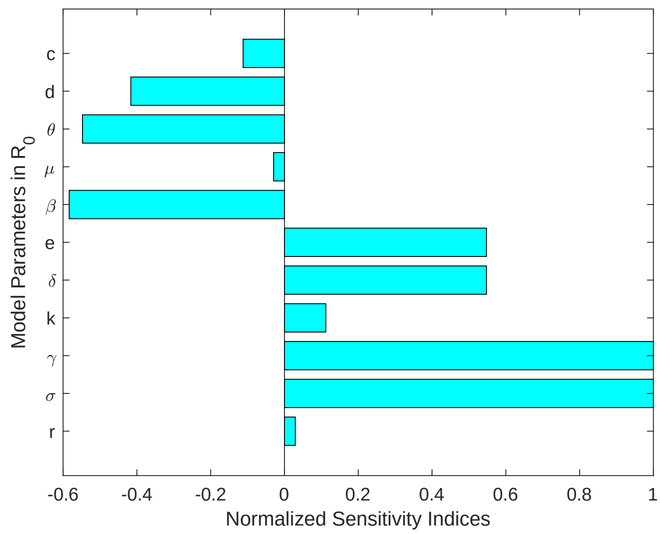


Figure 5.3: Normalized sensitivity indices of $\mathcal{R}_0$ with respect to parameters of the model (5.1). Parameter values are taken from Table 7.1.

Chapter 6

OPTIMAL CONTROL ANALYSIS

In chapter five, we formulated and analyzed a mathematical model for coffee berry borer infestation without providing its control mechanism. In this chapter, the control mechanism is addressed using the theory of optimal control.

6.1 Extension into optimal control model

In this section, we analyse an optimal control describing the infestation of CBB dynamics. The female CBB bores a hole in the coffee bean, and lays eggs. The complete life cycle of the CBB passes within the coffee fruit, except when females emerge to colonize new fruits. The fruit thus protects the CBB from control measures. For instance, insecticides will only be effective if they make contact with female CBBs in their brief trip outside the fruit. This makes difficulty of controlling the pests. Therefore, we follow the integrated pest management method to control the pests to manageable levels by combining several approaches (Marcano et al., 2021). As one approach, we proposed educating (awaring) Coffee farmers on the CBB life cycle and its reservoir for effective management in combination with a trap. Because after harvest, all the remaining coffee berries and fallen must be removed as they are a reservoir where the CBB can live until the next coffee crop is mature enough to infest. The method is environmentally friendly but labour-intensive. The trap also helps to minimize the female bore but it is useful only when the pest is outside the fruit.

The objective is to find the optimal control values $u^* = (u_1^*, u_2^*)$ of controls $u = (u_1, u_2)$ such that the associated state trajectories $(B^*,\ Y^*,\ A^*)$ are solution of system (5.1) in the intervention time interval $[0,\ T]$ with initial condition as given in (5.1) and minimizing the objective functional. The first control u_1 represent the effort made to reduce the colonizing females. It consists mainly of the use of chemicals, traps and the biological control using entomopathogenic fungi such as *Beauveria bassiana*, that is applied to the surface of the Coffee berries and kills the colonizing females of CBB when they drill entry holes into the coffee berry. The second control u_2 represent the effort made to increase the awareness of farmers through media campaign and educating coffee farmer.

The controls are bounded between 0 and 1. When the controls vanish, it means no extra measures are implemented for the reduction of the disease. When the controls take the maximum value 1, it means that the intervention is 100 % perfectly implemented which is not true in reality and thus we assumed $0 \leq u_i \leq u_{max} < 1$ for numerical implementation. The governing mathematical model is expressed by the following system of non-linear ordinary differential equations:

$$\begin{cases} \dfrac{dB}{dt} &= rB\left(1-\dfrac{B}{K}\right) - \dfrac{\gamma(1-u_1)BY}{c+B} - \mu B, \\ \dfrac{dY}{dt} &= \dfrac{(1-u_1)\sigma\gamma BY}{c+B} - \dfrac{\beta AY}{\delta + A} - dY, \\ \dfrac{dA}{dt} &= \theta + u_2 + \alpha Y - eA, \end{cases} \tag{6.1}$$

where $B(0) > 0, Y(0) \geq 0, A(0) > 0$.

Our cost functional considers the number of Coffee berry borer (pest) Y, and the implementation cost of strategies related to the controls $u_i,\ i = 1, 2$. The objective function can be define for the minimization problem as

$$J[u_1, u_2] = \int_0^T \left(CY(t) + \frac{1}{2}\sum_{i=1}^{2} M_i u_i^2\right) dt \rightarrow min \tag{6.2}$$

where constants C and M_i are positive. The weight constants M_1 and M_2 are the measure of relative costs of interventions associated with the controls u_1 and u_2, respectively, and also balances the units of integrand. Larger values of M_1 and M_2 tell us the expensiveness of implementation cost. The terms $\frac{1}{2}M_1u_1$ and $\frac{1}{2}M_2u_2$ describe the cost associated with the traps, biological control and awareness campaign respectively. In the cost function, the term $CY(t)$ describe the cost related to killing colonizing female borers at the end of the season. The quadratic terms in the objective functional is taken because the intervention is nonlinear (Chowdhury et al., 2019; Teklebirhan et al., 2021). Moreover, the functional *J* corresponds

the total cost due to Coffee berry borer infestation and its control strategies.
Further, the integrand function $L(t,\ B,\ Y,\ A,\ u_1,\ u_2) = CY(t) + \frac{1}{2}\sum_{i=1}^{2} M_i u_i^2$ measures the current cost at time t. Lastly, the fixed constant T denotes the terminal treatment time.
The set of admissible controls is defined as follows

$$\Omega = \{u_1(\cdot),\ u_2(\cdot) \in L^{\infty}(0,T) : 0 \le u(t) \le 1,\ \forall t \in [0,t]\} \tag{6.3}$$

Then we consider the optimal control problem of determining $(B^*(\cdot),\ Y^*(\cdot),\ A^*(\cdot))$ associated with an admissible control $(u_1^*,\ u_2^*) \in \Omega$ on the intervention time interval $[0,T]$, subjected to the state control system (6.1) in $\mathbb{R}^3$ with its initial condition and minimizing the cost functional (6.2). More precisely, the optimal control problem can be defined as

$$J[u_1^*,\ u_2^*] = \min_{\Omega} J[u_1(\cdot),\ u_2(\cdot)] \tag{6.4}$$

satisfying (6.1) and its initial condition.

6.2 Existence of the optimal control problem

In this subsection, we prove the existence of such optimal control functions which minimize the cost function in the finite intervention period. The following result guarantees the existence of optimal control functions. A detail and similar analysis on existence of optimal control can be found in (Fleming & Rishel, 2012; Kumar et al., 2020).

Theorem 6.2.1. *There exists an optimal control $(u_1^*,\ u_2^*)$ and a corresponding solution vector $(B^*,\ Y^*,\ A^*)$ to the state initial value problem (6.1) that minimizes the cost functional $J(u)$ of (6.2) over the set of admissible control Ω.*

Proof. All the state variable involved in the model are continuously differentiable. Therefore, we need to verify the following four conditions given in (Fleming & Rishel, 2012).

(i). The set of all solution to system (6.1) with corresponding control functions in (6.3) is non-empty;

(ii). The control set is convex and closed;

(iii). The right hand side of the state system (6.1) is bounded by a linear function in the state and control variable;

(iv). The integrand L of (6.2) is convex on Ω and $L(t,\ B,\ Y,\ A,\ u) \geq g(u)$, where g is continuous and $||u||^{-1}g(u) \longrightarrow +\infty$ as $||u|| \longrightarrow \infty$.

The existence of the solution of the system (6.1) is obtained in using result from (Lukes, 1982),(Theorem 9.2.1), since system (6.1) has bounded coefficients and any solution is bounded on the finite interval time $[0, T]$, so condition (i) is satisfied. To proof (ii), consider

$$\Omega = \{u \in \mathbb{R} : ||u|| \leq 1\}$$

Let $u_1,\ u_2 \in \Omega$ such that $||u_1|| \leq 1$ and $||u_2|| \leq 1$. Then for any $\beta \in [0,\ 1]$,

$$||\beta u_1 + (1-\beta)u_2|| \leq \beta||u_1|| + (1-\beta)||u_2|| \leq 1$$

This implies that the control set Ω is convex and closed.

The state system (6.1) is clearly linear in control variables u_1 and u_2 with coefficients depending on state variables. With this condition (iii) is satisfied. The integrand of the cost functional

$$L(t,\ B,\ Y,\ A,\ u) = CY(t) + \frac{1}{2}\sum_{i=1}^{2} M_i u_i^2 \geq \frac{1}{2}\sum_{i=1}^{2} M_i u_i^2 \tag{6.5}$$

let $\xi = \min\left(\frac{M_1}{2},\ \frac{M_2}{2}\right) > 0$, and define a continuous function $g(u) = \xi||u||^2$. Then from Equation (6.5) we have $L(t,\ B,\ Y,\ A,\ u) \geq g(u)$. Clearly $||u||^{-1}g(u) \longrightarrow +\infty$ as $||u|| \longrightarrow \infty$. Thus, condition (iv) is achieved. Therefore, the existence of an optimal control pair $(X^*,\ u^*)$ satisfying (6.1) and (6.4) is assured by results given in (Fleming & Rishel, 2012). The proof is completed. □

6.3 Characterization of the optimal control solution

In order to derive the necessary conditions for the optimal control pair the Pontryagin's Minimum Principle (Pontryagin, 2018) is used. According to the this principle, for $(u_1^*,\ u_2^*)$ to be optimal with corresponding optimal state $(B^*,\ Y^*,\ A^*)$ which minimizes the objective functional (6.1) for a fixed final time T, then there exists a non-trivial absolutely continuous mapping $\lambda : [0, T] \longrightarrow \mathbb{R}^3,\ \lambda(t) = (\lambda_1(t),\ \lambda_2(t),\ \lambda_3(t))$ called the adjoint vector such that

1. the Hamiltonian function is defined as

$$\mathcal{H} = CY(t) + \frac{1}{2}\sum_{i=1}^{2} M_i u_i^2 + \sum_{i=1}^{3} \lambda_i g_i(t,\ B,\ Y,\ A,\ u_1,\ u_2) \tag{6.6}$$

where g_i stands for the right hands of the constraints (6.1) for $i = 1, 2, 3$.

2. the control system

$$B' = \frac{\partial \mathcal{H}}{\partial \lambda_1},\ Y' = \frac{\partial \mathcal{H}}{\partial \lambda_2},\ A' = \frac{\partial \mathcal{H}}{\partial \lambda_3}: \tag{6.7}$$

3. the adjoint system

$$\lambda_1' = -\frac{\partial \mathcal{H}}{\partial B},\ \lambda_2' = -\frac{\partial \mathcal{H}}{\partial Y},\ \lambda_3' = -\frac{\partial \mathcal{H}}{\partial A}: \tag{6.8}$$

4. The optimality condition

$$\mathcal{H}(B^*(t),\ Y^*(t),\ A^*(t), u^*(t),\ \lambda^*(t)) = \min_{u\in\Omega} \mathcal{H}(B(t),\ Y(t),\ A(t),\ u(t),\ \lambda(t)) \tag{6.9}$$

holds for almost all $t \in [0, T]$

5. Moreover, the transversality condition

$$\lambda_i(T) = 0, \quad i = 1, 2, 3 \text{ also holds true.} \tag{6.10}$$

Theorem 6.3.1. *There exists an optimal solution* $(B^*(t),\ Y^*(t),\ A^*(t))$ *and corresponding optimal control* $u^* = (u_1^*, u_2^*)$ *that minimize* $J(u_1, u_2)$ *over the set of admissible control* Ω. *Moreover, there exist an adjoint function* $\lambda_i^*(t),\ i = 1, 2, 3$ *such that*

$$\begin{cases} \lambda_1' = (\lambda_1 - \sigma\lambda_2)\dfrac{c(1-u_1)\gamma Y}{(c+B)^2} + \lambda_1\left(\dfrac{2rB}{K} - r + \mu\right) \\ \lambda_2' = -C + (\lambda_1 - \sigma\lambda_2)\dfrac{(1-u_1)\gamma B}{(c+B)} + \lambda_2\left(\dfrac{\beta A}{\delta + A} + d\right) - \alpha\lambda_3 \\ \lambda_3' = \lambda_2 \dfrac{\delta\beta Y}{(\delta+A)^2} + e\lambda_3 \end{cases} \tag{6.11}$$

with transiversality conditions $\lambda_i(T) = 0,\ i = 1, 2, 3.$

Furthermore, the following properties holds:

$$\begin{aligned} u_1^* &= \min\left\{\max\left\{0,\ \frac{\gamma BY}{M_1(c+B)}(\sigma\lambda_2 - \lambda_1)\right\}, 1\right\}, \\ u_2^* &= \min\left\{\max\left\{0,\ -\frac{\lambda_3}{M_2}\right\},\ 1\right\}, \end{aligned} \tag{6.12}$$

where $\lambda_3 \le 0$.

Proof. The Hamiltonian function associated with the system is defined as follows:

$$\begin{aligned} \mathcal{H}(t,\ B,\ Y,\ A,\ u_1,\ u_2) &= CY(t) + \frac{1}{2}M_1u_1^2 + \frac{1}{2}M_2u_2^2 \\ &+ \lambda_1\left\{rB(1 - \frac{B}{K}) - \frac{(1-u_1)\gamma BY}{c+B} - \mu B\right\} \\ &+ \lambda_2\left\{\frac{(1-u_1)\sigma\gamma BY}{c+B} - \frac{\beta AY}{\delta + A} - dY\right\} \\ &+ \lambda_3\{\theta + u_2 + \alpha Y - eA\} \end{aligned}$$

where $\lambda_i(t)$, for $i = 1, 2, 3$ are the adjoint functions to be determined suitably. The form of the adjoint equations and transversality conditions are standard results from PMP. The adjoint system can be obtained as follows:

$$\begin{cases} \dfrac{d\lambda_1}{dt} = -\dfrac{\partial H}{\partial B} = (\lambda_1 - \sigma\lambda_2)\dfrac{c(1-u_1)\gamma Y}{(c+B)^2} + \lambda_1\left(\dfrac{2rB}{K} + \mu - r\right) \\ \dfrac{d\lambda_2}{dt} = -\dfrac{\partial H}{\partial Y} = -C + (\lambda_1 - \sigma\lambda_2)\dfrac{(1-u_1)\gamma B}{(c+B)} + \lambda_2\left(\dfrac{\beta A}{\delta + A} + d\right) - \alpha\lambda_3 \\ \dfrac{d\lambda_3}{dt} = -\dfrac{\partial H}{\partial A} = \lambda_2\dfrac{\delta\beta Y}{(\delta + A)^2} + e\lambda_3 \end{cases}$$

The state variables are not assigned at the final time T. Hence, from (Lenhart & Workman, 2007), the transversality conditions (or boundary conditions) becomes $\lambda_i(T) = 0$, for $i = 1, 2, 3$.

By the optimality condition, we have:

$$\frac{\partial \mathcal{H}}{\partial u_i} = 0 \text{ for } i = 1, 2;$$

That is,

(i) $\dfrac{\partial \mathcal{H}}{\partial u_1} = M_1 u_1 + \lambda_1 \dfrac{\gamma BY}{c+B} - \lambda_2 \dfrac{\gamma\sigma BY}{c+B} = 0$, at $u_1(t) = u_1^*(t)$

$$u_1^*(t) = (\sigma\lambda_2 - \lambda_1)\frac{\gamma BY}{M_1(c+B)}$$

(ii) $\dfrac{\partial \mathcal{H}}{\partial u_2} = M_2 u_2 + \lambda_3 = 0$, at $u_2(t) = u_2^*(t)$

$$u_2^*(t) = -\frac{\lambda_3}{M_2}$$

From boundedness on $u_1^*(t)$ and minimality condition, we have:

$$u_1^*(t) = \begin{cases} 0,\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} > 0, \\ (\sigma\lambda_2 - \lambda_1)\dfrac{\gamma BY}{M_1(c+B)},\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} = 0, \\ 1,\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} < 0. \end{cases}$$

Here also from boundedness on $u_2^*(t)$ and minimality condition, we have:

$$u_2^*(t) = \begin{cases} 0,\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} > 0, \\ -\dfrac{\lambda_3}{M_2},\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} = 0, \\ 1,\ if \quad \dfrac{\partial \mathcal{H}}{\partial u_1} < 0. \end{cases}$$

In compact notation:

$$\begin{aligned} u_1^* &= \min\left\{\max\left\{0,\ \frac{\gamma BY}{M_1(c+B)}(\sigma\lambda_2-\lambda_1)\right\},1\right\} \\ u_2^* &= \min\left\{\max\left\{0,\ -\frac{\lambda_3}{M_2}\right\},\ 1\right\} \end{aligned}$$

This complete the proof of the theorem. $\square$

Chapter 7

NUMERICAL SIMULATIONS

7.1 Simulation results and discussion

In this section, first, we illustrate numerically the solution of the governing mathematical model introduced in section (5.1) and the long term behavior of the solution is analyzed. Then, we illustrate numerically the solution of the optimal control problem proposed in Section (6.1) and explain the impact of control strategy on the infestation of CBB. For this purpose, we use the forward-backward sweep method presented in the book of (Lenhart & Workman, 2007). To briefly summarize the procedure, first, we solve the control system with an initial guess for the control variables using forward fourth-order Runge-Kutta scheme and transversality conditions. The adjoint system is then solved applying the backward fourth-order Runge-Kutta scheme using the current solution of state variables. Then, the controls are updated using a convex combination of the previous controls and the values computed in the characterizations process. The updated controls are then utilized to repeat the solution of state and adjoint systems. The iteration continued until a predefined convergence criterion is met.

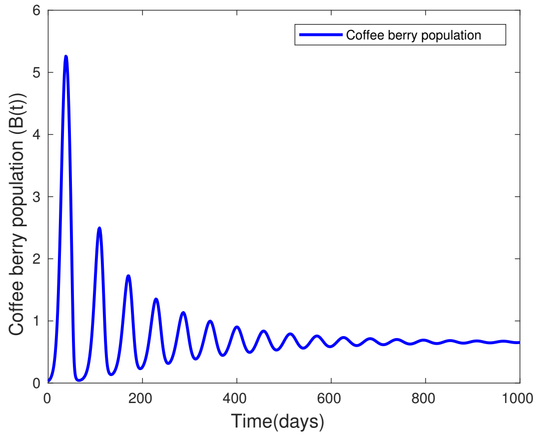
To conduct the study, a set of meaningful values are assigned to the model parameters. These values are either taken from literature or assumed. Using the parameter values given in Table 7.1 and the initial condition: $B(0) = 0.02,\ Y(0) = 0.03$ and $A(0) = 0.01$. For the adjoint system we set terminal conditions $\lambda(T) = 0;\ i = 1, 2, 3$. We considered the numerical value of the controls: $u_i;\ i = 1, 2$, between zero and one as they are not 100% effective. The cost coefficients corresponding to control variables are estimated to be $M_1 = M_2 = 1$ and the relative importance of reducing the associated classes on the infestation of

Table 7.1: The values of parameters used in the simulations.

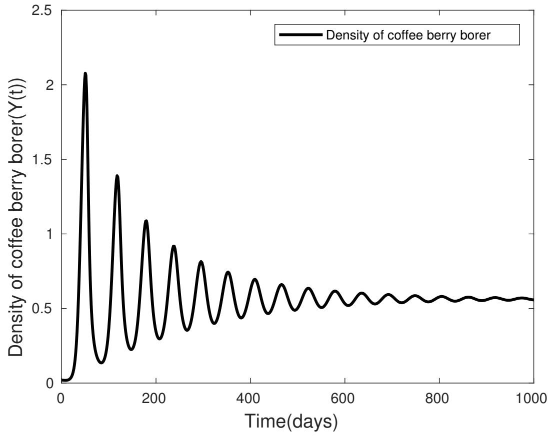
Parameter	Description	Values	Reference
r	Growth rate of coffee biomass	0.25 day^{-1}	Assumed
K	Carrying capacity	10 m^{-2}	Assumed
β	Rate of farmers activity due to awareness	0.28 day^{-1}	Assumed
μ	Natural mortality rate of healthy coffee berries	0.052 day^{-1}	Assumed
σ	Conversion rate from coffee berries to CBB	0.6	(Teklebirhan et al., 2021)
α	Aware farmer growth rate	0.325 day^{-1}	Assumed
γ	Attack rate of female borer on berries	0.535 $pest^{-1}day^{-1}$	Assumed
c	Half saturation rate of healthy coffee berries with CBB	1 day^{-1}	Assumed
d	Natural mortality rate of CBB	1/81 day^{-1}	(Fotso et al., 2019)
e	A rate at which awareness decrease	0.45 day^{-1}	Assumed
δ	Half saturation rate of level awareness with CBB	0.675 day^{-1}	Assumed
θ	Rate awareness from global source	0.02 day^{-1}	Assumed

the pest is $C = 2$. Then the simulation process are performed using software MATLAB $R2018a$ solver.

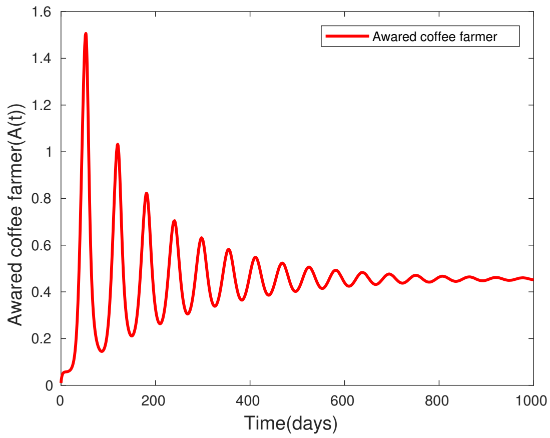
In Figures 7.1 (a), (b), (c) below, the behavioural patterns of the densities of Coffee berry biomass, Coffee berry borer, and aware farmers are presented taking the parameter value from Table 7.1. All the system populations oscillate initially and finally become asymptotically stable and converge to the endemic state value. Here, it is to be noted that all conditions of Theorem 5.4.1 are satisfied and thus the coexistence equilibrium E^* is stable.



(a)



(b)



(c)

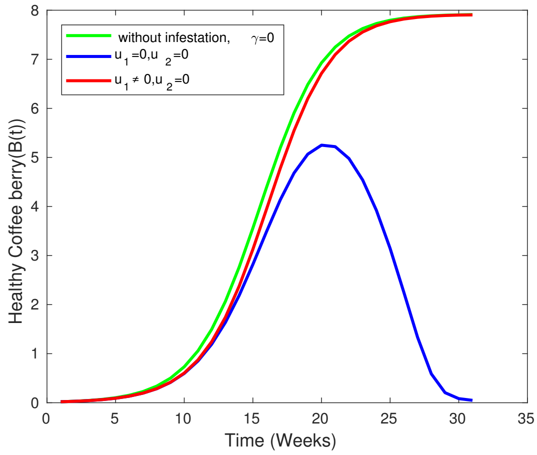
Figure 7.1: Stability behavior of the endemic state for the system (5.1) without controls, i.e., when $u_1 = 0$ and $u_2 = 0$.

In the following, we comparatively analyses the impact of control strategies on the infestation of Coffee berry borer under different scenarios:

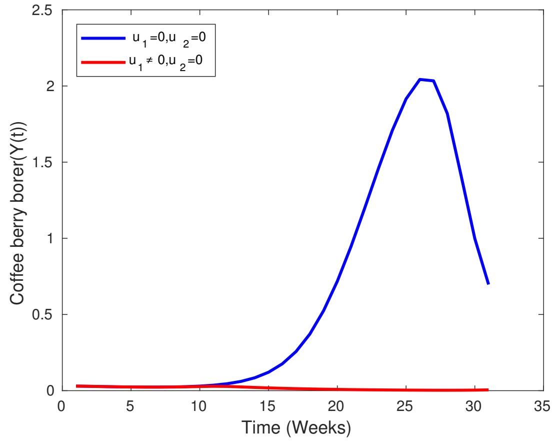
- **Strategy 1**: Applying chemicals, traps and biological control (u_1),
- **Strategy 2**: Using farmers awareness (u_2),
- **Strategy 3**: Using all controls (u_1 and u_2).

1) Strategy with chemicals, traps and biological control

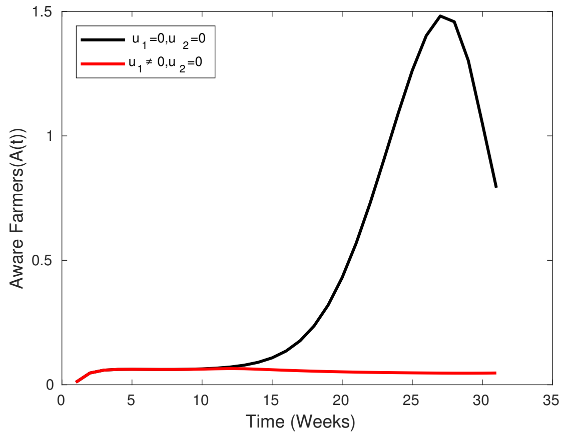
There is a significant difference in the number of healthy coffee with and without control Figure 7.2 (a). More precisely, the number of healthy coffee berries at the end of the period is increased. From Figure 7.2 (b), we observed that integrated pest management application in optimal control mode can make the system stabilize, pest-free. Also, optimal control application time saving and cost-effective. This implies that optimized chemicals, traps, and biological control reduces the burden of CBB infestation. In Figure 7.2 (c), farmers awareness decreased with control and be decreased without control after reaching the peak of the curve. This is because of fading of memory. The control u_1 is at a maximum level (95%) for 13-26 weeks and declines afterwards to zero see (Figure 7.3).



(a) Healthy coffee with and without control



(b) Coffee berry borer with and without control



(c) Farmer aware with and without control

Figure 7.2: The plot shows the effect of controls on healthy coffee berries, coffee berry borer and farmer aware when $u_1 \neq 0, u_2 = 0$.

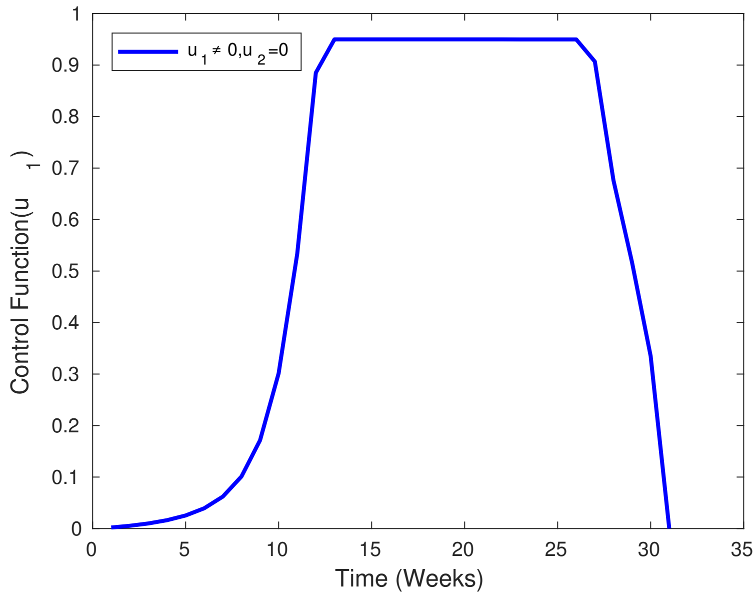
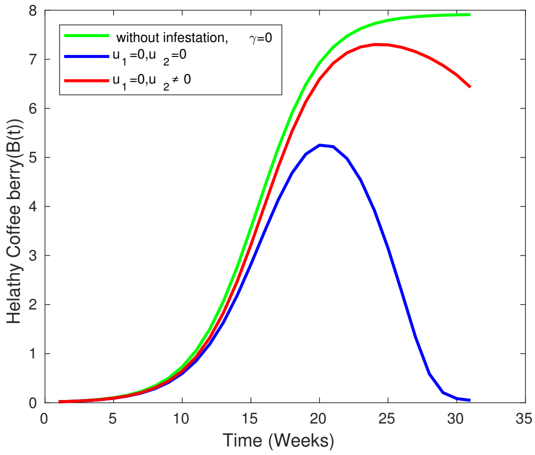


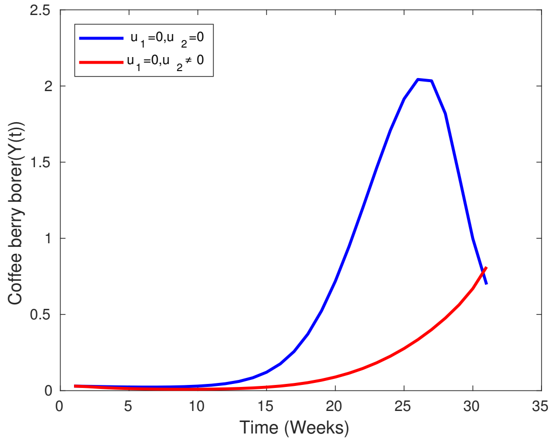
Figure 7.3: Profile of control function (u_1) when $u_2 = 0$.

2) Strategy with farmers awareness

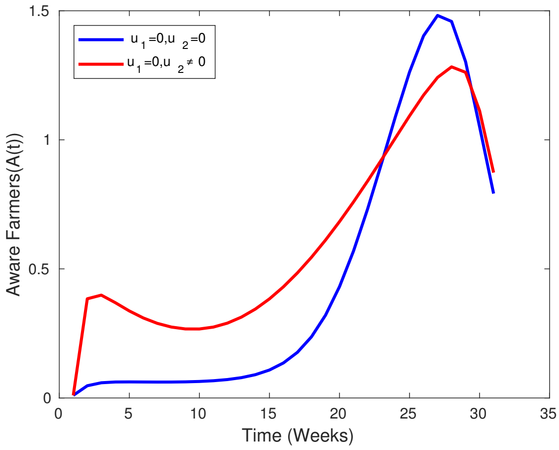
Under this strategies, we simulated the model using farmers awareness u_2 while chemical, traps and biological control u_1 was set to zero. In Figure 7.4 (a), we observed that there is a significantly increasing and after reach maximum peak becomes decreased in the number of healthy Coffee berries with control. In Figure 7.4 (b), initially it is constant and after some period dramatically increased in the number of coffee berry borer with control. In Figure 7.4 (c), there is a significant increase and after reach maximum peak becomes decrease at some period in the number of aware farmers because of fading of memory. Therefore, this strategies is noneffective in controlling the Coffee borer from the coffee farm. The corresponding control profile is shown in Figure 7.5.



(a) Healthy coffee with and without control



(b) Coffee berry borer with and without control



(c) Farmer aware with and without control

Figure 7.4: The plot shows effect of controls on healthy coffee berries, coffee berry borer and farmer aware when $u_1 = 0,\ u_2 \neq 0$.

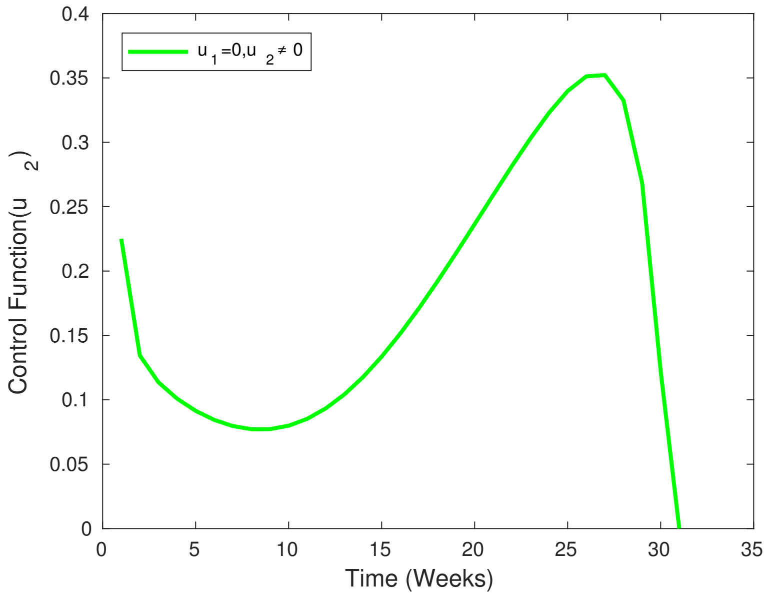
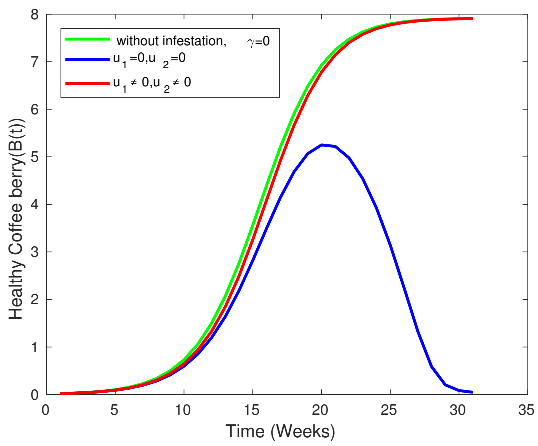


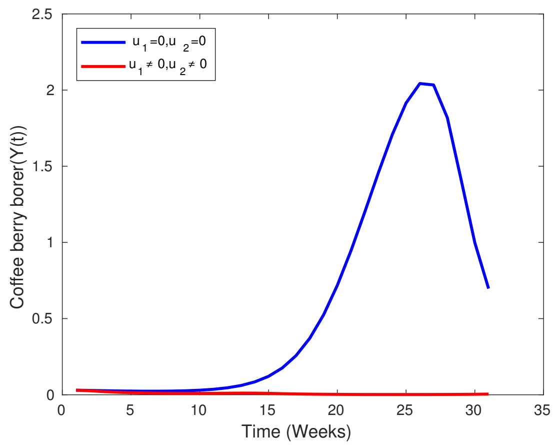
Figure 7.5: Profile of control function (u_2) when $u_1 = 0$.

3) Strategy with the combination of chemical, traps, biological and farmers awareness

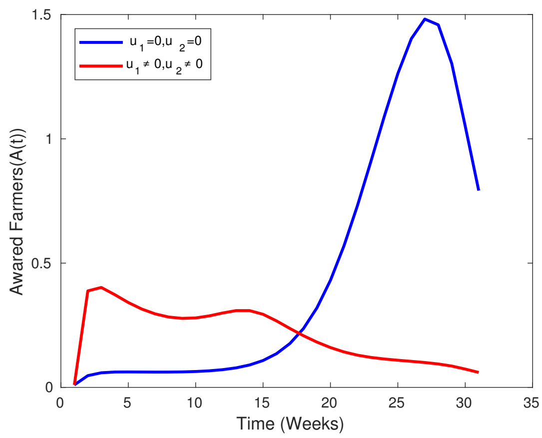
Here we have applied both strategies to optimize the objective function. We simulated the model using both strategies namely, chemical, traps, and biological control u_1 and awareness control u_2 was to optimize the objective function. Clearly, Figure 7.6 (a) shows that as the coffee berry borer infest, the number of coffee berry borer is dramatically dropped to zero in Figure 7.6 (b) and increased coffee population in Figure 7.6 (a) with control. Meanwhile, intervention strategies will witness a decrease in the number of aware farmers in Figure 7.6 (c). Thus, the best choice is to apply both controls together to minimize the impact of CBB on Coffee production.



(a) Healthy coffee with and without control



(b) Coffee berry borer with and without control



(c) Farmer aware with and without control

Figure 7.6: The plot shows effect of controls on healthy coffee berries, coffee berry borer and farmer aware when $u_1 \neq 0,\ u_2 \neq 0$.

The corresponding controls profile is shown in Figure 7.7. The control u_1 is at a maximum level (95%) for 13-26 weeks and declines afterwards to zero. However, u_2 is a little bit increase for a while then decline to zero.

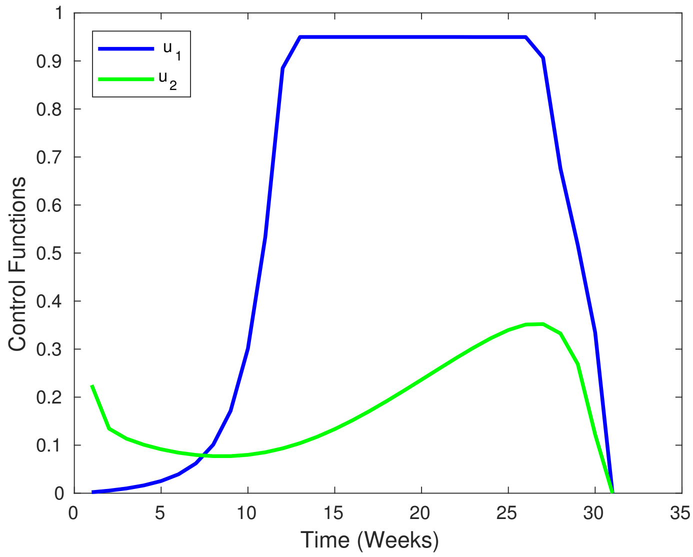


Figure 7.7: Profile of control functions u_1 and u_2.

CONCLUSION AND RECOMMENDATION

Conclusion

In this thesis, a deterministic mathematical model has been formulated to investigate the progression of CBB in Coffee farm. Then the proposed mathematical model is detailed in chapter five and later extended to the optimal control problem.

In chapter five, the proposed mathematical model includes Coffee berry biomass, Coffee berry borer population, and awared farmers. The well-posedness of the model was established both in the mathematical and epidemiological sense by showing that all solution of the model is positive and bounded with initial conditions in a certain meaning full set. The model has two non-negative equilibria, namely the CBB free and endemic equilibrium. Then, we performed the qualitative analysis of the model. Both local and global stability analysis of the equilibrium points of the model has been done using Jacobian matrix and Lyapunov functions respectively. Sensitivity analysis of the model has been studied and determine a parameter that plays a greater role in the infestation of CBB.

In chapter six, the proposed mathematical model is extended to the optimal control problem. Here we have used two controls, chemical, traps and biological (u_1) and farmers awareness (u_2). Then, the Hamiltonian, adjoint variables, the characterization of controls and the optimality of our model were derived and analyzed using Pontryagin's minimum principle.

Finally, different simulation cases were comparatively performed to obtain the best control strategies. The numerical simulation results demonstrate good agreement with our analytical results. The simulation results also clearly shown that the CBB can be minimized by preventing the female borer from entering the berries through chemicals, traps and biological control with optimal implementation cost.

Recommendation

The mathematical models on pest are rather simple, but they give insight into some of the consequences of agriculture policies. Controlling the infestation of pest is now a challenging and important issue to study. So, to predict and identify the cost-effective strategies to control the infestation of CBB and minimize the cost of the control program are the primary goals of agriculture practitioners, policy-makers and researchers. Based on the findings of this study, we recommend to the stakeholders and farmers to manage this pest by applying the combination of control strategies to minimize the occurrence of CBB in the coffee farm.

Future work

Several mathematical questions are still open related to CBB. For instance, the impact of climate variability is not addressed in this work. This will be addressed in the future work.

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Appendix A

The Matlab Code for Numerical Simulation

Listing A.1: Matlab Program for model (6.1) with and without optimal control.

```
%------------------------------------------------
%CBB  Simulation Main program
%---------------------------------------------------
%This function solves the CBB model with control
% x_1 = B-density of coffee berry
% x_2 = Y-Coffee berry borer (pest)
% x_3 = A-Awared farmer
%
   ----------------------------------------------------------------

function y = CBB_runge1
test = -1;
%---------------------------------------------------
%     Model Parameters
%---------------------------------------------------
delta1 = 0.001;
N = 30;
t0=0;
tf=64;
t=linspace(0,tf,N+1);
```

```
h = (tf-t0)/N;
h2 = h/2;

r = 0.25;         % Growth rate of coffee biomass
k = 10;           % Carrying capacity
beta = 0.28;      % Rate of farmers activity due to awareness
mu = 0.052;       % Natural mortality rate of healthy coffee
   berries
sigma = 0.6;      % Conversion rate from coffee berries to CBB
alpha = 0.325;    % Aware farmer growth rate
gamma = 0.535;    % Attack rate of female borer on berries
c = 1;            % Half saturation rate of healthy coffee
   berries with CBB
d = 0.02;         % Natural mortality rate of CBB
e = 0.45;         % A rate at which awareness decrease
delta = 0.675;    % Half saturation rate of level awareness
   with CBB
theta = 0.02;     % Rate awareness from global source

C = 2;
M1 = 1;
M2 = 1;
%-----------------------------------
% Vectors for system state variable
%-----------------------------------
u1=zeros(1,N+1);
u2=zeros(1,N+1);
x1=zeros(1,N+1);
x2=zeros(1,N+1);
x3=zeros(1,N+1);
%-----------------------------------
% Vectors for system restrictions and control
```

```
%-----------------------------------
lambda1 = zeros(1,N+1);
lambda2 = zeros(1,N+1);
lambda3 = zeros(1,N+1);
%-------------------------------------
% Initial conditions of the model
%--------------------------------------
x1(1) = 0.02;
x2(1) = 0.03;
x3(1) = 0.01;
%-------------------------------------
% Iterations of the method
%------------------------------------
while(test < 0)

   oldu1 = u1;
   oldu2 = u2;

   oldx1 = x1;
   oldx2 = x2;
   oldx3 = x3;

   oldlambda1 = lambda1;
   oldlambda2 = lambda2;
   oldlambda3 = lambda3;

       for i = 1:N  % forward in state
   m11 =  r*x1(i)*(1-x1(i)./k)-gamma*x1(i)*x2(i)*(1-u1(i))./(
      c+x1(i))-mu*x1(i);
   m12 =  sigma* gamma*x1(i)*x2(i)*(1-u1(i))./(c+x1(i))-beta*
      x2(i)*x3(i)./(delta+x3(i))-d*x2(i);
   m13 = theta+u2(i)+alpha*x2(i)-e*x3(i);
```

```
%-----------------------------------------
% Second Runge-Kutta parameter in state
%-----------------------------------------
m21 =  r*(x1(i)+h2*m11)*(1-(x1(i)+h2*m11)./k)-gamma*(x1(i)
    +h2*m11)*(x2(i)+h2*m12)*(1-0.5*(u1(i)+u1(i+1)))./(c+(x1
    (i)+h2*m11))-mu*(x1(i)+h2*m11);
m22 =  sigma* gamma*(x1(i)+h2*m11)*(x2(i)+h2*m12)*(1-0.5*(
    u1(i)+u1(i+1)))./(c+(x1(i)+h2*m11))-beta*(x2(i)+h2*m12)
    *(x3(i)+h2*m13)./(delta+(x3(i)+h2*m13))-d*(x2(i)+h2*m12
    );
m23 = theta+0.5*(u2(i)+u2(i+1))+alpha*(x2(i)+h2*m12)-e*(x3
    (i)+h2*m13);
  %-----------------------------------------
  % Third Runge-Kutta parameter in state
  %-----------------------------------------
m31 =  r*(x1(i)+h2*m21)*(1-(x1(i)+h2*m21)./k)-gamma*(x1(i)
    +h2*m21)*(x2(i)+h2*m22)*(1-0.5*(u1(i)+u1(i+1)))./(c+(x1
    (i)+h2*m21))-mu*(x1(i)+h2*m21);
m32 =  sigma* gamma*(x1(i)+h2*m21)*(x2(i)+h2*m22)*(1-0.5*(
    u1(i)+u1(i+1)))./(c+(x1(i)+h2*m21))-beta*(x2(i)+h2*m22)
    *(x3(i)+h2*m23)./(delta+(x3(i)+h2*m23))-d*(x2(i)+h2*m22
    );
m33 = theta+0.5*(u2(i)+u2(i+1))+alpha*(x2(i)+h2*m22)-e*(x3
    (i)+h2*m23);
  %-----------------------------------------
  % Fourth Runge-Kutta parameter in state
  %-----------------------------------------
m41 =  r*(x1(i)+h2*m31)*(1-(x1(i)+h2*m31)./k)-gamma*(x1(i)
    +h2*m31)*(x2(i)+h2*m32)*(1-(u1(i)+u1(i+1)))./(c+(x1(i)+
    h2*m31))-mu*(x1(i)+h2*m31);
m42 =  sigma* gamma*(x1(i)+h2*m31)*(x2(i)+h2*m32)*(1-(u1(i
    )+u1(i+1)))./(c+(x1(i)+h2*m31))-beta*(x2(i)+h2*m32)*(x3
```

```
(i)+h2*m33)./(delta+(x3(i)+h2*m33))-d*(x2(i)+h2*m32);
m43 = theta+(u2(i)+u2(i+1))+alpha*(x2(i)+h2*m32)-e*(x3(i)+
    h2*m33);


    x1(i+1) = x1(i) + (h/6)*(m11 + 2*m21 + 2*m31 + m41);
    x2(i+1) = x2(i) + (h/6)*(m12 + 2*m22 + 2*m32 + m42);
    x3(i+1) = x3(i) + (h/6)*(m13 + 2*m23 + 2*m33 + m43);

  end
%---------------------------------------------------------------
% Backward Runge-Kutta iterations in co-state variables
%---------------------------------------------------------------
for i = 1:N
    j = N + 2 - i;
  %----------------------------------------
  % First Runge-Kutta parameter in co-state
  %----------------------------------------
n11 = (lambda1(j)-sigma*lambda2(j))*c*(1-u1(j))*gamma*x2(j)
   ./(c+x1(j)).^2 +lambda1(j)*(2*r*x1(j)./k-r+mu);
n12 = -C +(lambda1(j)-sigma*lambda2(j))*(1-u1(j))*gamma*x2(j
   )./(c+x1(j))+lambda2(j)*(beta*x3(j)./(delta+x3(j))+d)-
   alpha*lambda3(j);
n13 = lambda2(j)*delta*beta*x2(j)./(delta+x3(j)).^2+e*
   lambda3(j);
  %----------------------------------------
  % Second Runge-Kutta parameter in co-state
  %----------------------------------------
n21 = ((lambda1(j)-h2*n11)-sigma*(lambda2(j)-h2*n12))*c
   *(1-0.5*(u1(j)+u1(j-1)))*gamma*0.5*(x2(j)+x2(j-1))./(c
   +0.5*(x1(j)+x1(j-1))).^2 +(lambda1(j)-h2*n11)*(2*r*0.5*(
   x1(j)+x1(j-1))./k-r+mu);
```

```
n22 = -C +((lambda1(j)-h2*n11)-sigma*(lambda2(j)-h2*n12))
    *(1-0.5*(u1(j)+u1(j-1)))*gamma*0.5*(x2(j)+x2(j-1))./(c
    +0.5*(x1(j)+x1(j-1)))+(lambda2(j)-h2*n12)*(beta*0.5*(x3(j
    )+x3(j-1))./(delta+0.5*(x3(j)+x3(j-1)))+d)-alpha*(lambda3
    (j)-h2*n13);
n23 = (lambda2(j)-h2*n12)*delta*beta*0.5*(x2(j)+x2(j-1))./(
    delta+0.5*(x3(j)+x3(j-1))).^2+e*(lambda3(j)-h2*n13);
     %--------------------------------------
     % Third Runge-Kutta parameter in co-state
     %--------------------------------------
n31 = ((lambda1(j)-h2*n21)-sigma*(lambda2(j)-h2*n22))*c
    *(1-0.5*(u1(j)+u1(j-1)))*gamma*0.5*(x2(j)+x2(j-1))./(c
    +0.5*(x1(j)+x1(j-1))).^2 +(lambda1(j)-h2*n21)*(2*r*0.5*(
    x1(j)+x1(j-1))./k-r+mu);
n32 = -C +((lambda1(j)-h2*n21)-sigma*(lambda2(j)-h2*n22))
    *(1-0.5*(u1(j)+u1(j-1)))*gamma*0.5*(x2(j)+x2(j-1))./(c
    +0.5*(x1(j)+x1(j-1)))+(lambda2(j)-h2*n22)*(beta*0.5*(x3(j
    )+x3(j-1))./(delta+0.5*(x3(j)+x3(j-1)))+d)-alpha*(lambda3
    (j)-h2*n23);
n33 = (lambda2(j)-h2*n22)*delta*beta*0.5*(x2(j)+x2(j-1))./(
    delta+0.5*(x3(j)+x3(j-1))).^2+e*(lambda3(j)-h2*n23);
     %--------------------------------------
     % Fourth Runge-Kutta parameter in co-state
     %--------------------------------------
n41 = ((lambda1(j)-h2*n31)-sigma*(lambda2(j)-h2*n32))*c*(1-(
    u1(j)+u1(j-1)))*gamma*(x2(j)+x2(j-1))./(c+(x1(j)+x1(j-1))
    ).^2 +(lambda1(j)-h2*n31)*(2*r*(x1(j)+x1(j-1))./k-r+mu);
n42 = -C +((lambda1(j)-h2*n31)-sigma*(lambda2(j)-h2*n32))
    *(1-(u1(j)+u1(j-1)))*gamma*(x2(j)+x2(j-1))./(c+(x1(j)+x1(
    j-1)))+(lambda2(j)-h2*n32)*(beta*(x3(j)+x3(j-1))./(delta
    +(x3(j)+x3(j-1)))+d)-alpha*(lambda3(j)-h2*n33);
```

```
n43 = (lambda2(j)-h2*n32)*delta*beta*(x2(j)+x2(j-1))./(delta
   +(x3(j)+x3(j-1))).^2+e*(lambda3(j)-h2*n33);
   %--------------------------------------
   % Runge-Kutta new approximation in co-state
   %--------------------------------------
   lambda1(j-1) = lambda1(j) - h/6*(n11 + 2*n21 + 2*n31 +
      n41);
   lambda2(j-1) = lambda2(j) - h/6*(n12 + 2*n22 + 2*n32 +
      n42);
   lambda3(j-1) = lambda3(j) - h/6*(n13 + 2*n23 + 2*n33 +
      n43);

 end
   u1 = min(max(0,gamma.*x1.*x2.*sigma.*(lambda2-lambda1)./
      M1.*(c+x1)),0.95);
   u1 = 0.5*(u1 + oldu1);
   u2 = min(max(0,-lambda3./M2),0.95);
   u2 = 0*(u2 + oldu2);
   J=sum((C.*x2 + 1/2*M1*u1.^2 +1/2*M2*u2.^2 )*(t(i+1)-t(i))
      );
   %-------------------------------
   % Absolute error for convergence
   %-------------------------------
   temp1 = delta1*sum(abs(u1)) - sum(abs(oldu1 - u1));
   temp2 = delta1*sum(abs(u2)) - sum(abs(oldu2 - u2));
   temp3 = delta1*sum(abs(x1)) - sum(abs(oldx1 - x1));
   temp4 = delta1*sum(abs(x2)) - sum(abs(oldx2 - x2));
   temp5 = delta1*sum(abs(x3)) - sum(abs(oldx3 - x3));

   temp6 = delta1*sum(abs(lambda1)) - sum(abs(oldlambda1 -
      lambda1));
```

```
    temp7 = delta1*sum(abs(lambda2)) - sum(abs(oldlambda2 -
        lambda2));
    temp8 = delta1*sum(abs(lambda3)) - sum(abs(oldlambda3 -
        lambda3));

    test = min(temp1, min(temp2, min(temp3, min(temp4, min(
        temp5, min(temp6, min(temp7, min(temp8))))))));
end

y(1,:) = t;
y(2,:) = u1;
y(3,:) = u2;
y(4,:) = x1;
y(5,:) = x2;
y(6,:) = x3;
y(7,:) = lambda1;
y(8,:) = lambda2;
y(9,:) = lambda3;
y(10,:) = J;

%-----------------------------------------------------------
%  Ploting section
%-----------------------------------------------------------

  box on
 hold on
plot(y(2,:),'b','linewidth',2)
xlabel('Time (Weeks)','fontsize',12)
ylabel('Control Function(u_{1})','fontsize',12)
legend(' u_1\neq0,u_2=0')
hold on
```