THE PHOTON STATISTICS OF THE SUPERPOSED LIGHT PRODUCED BY V-LASER AND PARAMETRIC OSCILLATOR.

by
GETAHUN EDOSA DANU

A Thesis Submitted to

the School of Graduate Studies

Adama Science and Technology University

In Partial Fulfillment of the Requirement for the

Degree of Masters of Science in Physics(Quantum Optics)

Office of Graduate Studies

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Approved by the Examination Committee

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DECLARATION

I here by declare that this thesis is my original work and has not been presented for a degree in any other universities and all sources of material used for this thesis have been duly acknowledged.

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This M.sc thesis has been submitted for examination with my approval as thesis advisor supervisor.

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Signature: —————

date of submission:————————————

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ABSTRACT

In this thesis, We obtained c-number Langevin equations using the pertinent master equation. Making use of the solutions of the resulting c-number Langevin equations, we calculated the anti-normally ordered characteristic function defined in schrõdinger picture. With the help of this characteristic function, we determined the Q-function for a two-mode parametric oscillator as well as for two-mode V-type three-level laser light. Then the Q-function were used to calculate the photon statistics for cavity modes. Our result showed that the mean photon number of mode a and mode b for the two-mode parametric oscillator light are the same and as well as the mean photon number differences vanishes, the mean photon number for the two-mode V-type three-level laser light are different except $\eta = 0$.

On other hand, using the aforementioned Q-functions, we determined the Q-function for the superposition of the signal-signal and idler-idler modes of parametric oscillator and V-type three-level laser light beams. Finally, using the resulting signal-signal mode Q-function and idler-idler mode Q-function, we obtained the mean photon number sum and difference and the variance of photon number. The results showed that the mean and variance of the photon number for the superposed light was the sum of the mean and variance of the photon number for the constituent lights.

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INTRODUCTION

1.1 Back Ground of the Study

Quantum optics deals mainly with the quantum properties of the light generated by various quantum optical systems such as lasers and with the effect of light on the dynamics of atoms. Quantum optics is the branch of optics dealing with light as a stream of photons, each possessing a quantum of energy proportional to the frequency of light. It is also a field of quantum physics that deals specifically with the interaction of photon with matter.

A light has played a special role in our attempts to understand nature both classically and quantum mechanically. Classically, light field consists of waves with well defined amplitude and phase when we treat the field quantum mechanically. Recent development technology greatly improved our ability to control individual quantum system. This brings quantum technology beyond academic research to control the level of concrete industrial programs. Lasers and masers are the most obvious application of quantum optics. It was developed due to the developments of maser(emitted coherent microwaves) in 1935 and laser(emitted coherent light) in 1960. Laser is an acronym for Light Amplification by Stimulated Emission of Radiation. It is the powerful source of light having extraordinary properties which are not found in the normal light sources like tungsten lumps, mercury lamps, etc. The unique property of laser is that its light waves travel very long distances with a very little divergence. Maser is an acronym for Microwaves Amplifications by Stimulated Emission of Radiation. It emitted coherent microwaves.

The quantum properties of light are largely determined by the quantum state of the light mode and the well known state are: the coherent state, the chaotic state, the number state and the squeezed state. Squeezing is one of the non-classical feature of light that has been attracting attention and studied by many authors [**1-29**]. It is generated as a result of quantum optical process such as subharmonic generation, second harmonic generation and three-level lasing under certain conditions.

A light mode is said to be in a squeezed state if the noise in one quadrature is below the coherent level at the expense of enhanced fluctuation in the conjugate quadrature with the product of the uncertainties in the two quadrature satisfying the uncertainty relation. Squeezing has potential application in area of spectroscopy, interferometer, precision, low noise communication and weak-signal detection. Squeezing like photon ant-bunching and subpoissonial. Photons are nonclassical feature of light which are not well described in classical electrodynamics. The squeezing state of light has been observed by quantum optical system such as parametric oscillator. In a parametric oscillator, nonlinear crystal is pumped by laser-light through a single port-mirror. The dynamics of the system can be analyzed quantum mechanically by master equation, the Stochastic differential equation, the Fokker-plank equation or quantum Langevin equation.

The photon statistics and quantum noise of a two-level laser have been investigated by several authors[**10-13**]. It has been found that a two-level laser generates chaotic light when operating below threshold and coherent light when operating well above threshold. A coherently prepared two-mode V laser has also been studied by some authors[**13-15**]. This type of laser is defined as a quantum optical system in which three-level atoms in V configuration and initially prepared in a coherent superposition of the two upper levels are injected at a certain rate in to a cavity coupled to vacuum reservoir via a single port-mirror.

The optical parametric oscillator converts an in put laser wave called pump with frequency ω_p in to out put of lower frequencies ω_s and ω_i in which $\omega_p = \omega_s + \omega_i$ in non degenerate parametric oscillator. In the case of degenerate parametric oscillator, the out put frequency is one half of the pump frequency, therefore($\omega_p = 2\omega_s = 2\omega_i$) . The down conversion process occurs in the single photon each pump photon. That is, annihilation inside the cavity gives rise to a pair of photons in signal and idler intra cavity modes. This leads correlation between the intensities of modes and therefore there is squeezing in the substraction of intensities which motivates the same twin beam.

1.2 Statement of the Problem

In previous works, different authors have studied about the quantum dynamics of the V-Laser and parametric oscillator alone using different methods. In this thesis, we study the statistical properties of the parametric oscillator, the V-type three-level laser and the superposed light produced by V-laser and parametric oscillator coupled to vacuum reservoir.

1.3 Objectives of the Study

1.3.1 General Objectives of the Study

The general objective of this research work is to study **the photon statistics of the superposed light produced by V laser and parametric oscillator coupled to vacuum reservoir.**

1.3.2 Specific Objectives of the Study

The specific objective of this studies are;

- ▶ to determine the c-number langevin equation and their solution.
- ▶ to determine the Q function.
- ▶ to calculate the mean and variance of photon number.
- ▶ to calculate the mean and variance of photon number sum and differences.

2

REVIEW OF LITERATURE

The failure of classical mechanics to account for many experimental results such as the stability of atoms and matter, blackbody radiation, etc. led physicists to the realization of new tool to deal with these problems and thus quantum mechanics came to the vicinity of life at the beginning of the last century. Indeed, quantum mechanics is one of the crowning achievements of modern physics where the quantized electromagnetic field is supported by the experimental observations of nonclassical states of the radiation field, such as; squeezed states, sub-Poissonian photon statistics and photon antibunching. Nowadays, quantum optics, the union of quantum field theory and physical optics, is undergoing a time of revolutionary change. The subject has evolved from early studies on the coherence properties of radiation to the laser in the modern areas of study involving, such as, the role of squeezed states of the radiation field and atomic coherence in quenching quantum noise in interferometry and optical amplifiers. Furthermore, quantum optics provides a powerful new probe for addressing fundamental issues of quantum mechanics such as complementarity, hidden variables, and other aspects central to the foundations of quantum physics.

For the description of a quantum system, the concept of state is used, which is the same, a wave function, a state vector or a density matrix containing the information about the possible results of measurement on the system. Quantum optics has statistical origin and therefore the state of a quantum system contains all information necessary to completely determine its statistics (the probabilistic nature of a quantum system). Among the wide variety of possible radiation-field states, there are some fundamental types that play a special role in quantum optics. Strictly speaking, there are four types of these states which are widely used, namely, chaotic, coherent, number and squeezed states. Following the development of the quantum theory of radiation and with the advent of the laser, the coherent states of the field, that mostly describe a classical electromagnetic field, were widely studied. Indeed, these states are more appropriate basis for many optical fields. However, number states are purely nonclassical states and they are a useful representation of high-energy photons. However, there are experimental difficulties which have prevented the generation of photon number states with more than a small number of photons. Despite this the number states of the electromagnetic field have been used as a basis for several problems in quantum optics including some laser theories. The squeezed states (i.e. the states of light with reduced fluctuations in one quadrature below the level associated with the vacuum state) are very important owing to their potential applications,such as: in the optical communication systems, interferometric techniques, and in an optical waveguide tap.

Properties of Light

What is light? This question has been debated for many centuries. The sun radiates light, electric lights brighten our darkness, and many other uses of light impact our lives daily. The answer, in short, is light is a special kind of electromagnetic energy. The speed of light, although quite fast, is not infinite. The speed of light in a vacuum is expressed as $c = 2.99 \times 10^8 m/s$. Light travels in a vacuum at a constant speed, and this speed is considered a universal constant. It is important to note that speed changes for light traveling through nonvacuum media such as air (0.03 percent slower) or glass (30.0 percent slower). For most purposes, we may represent light in terms of its magnitude and direction. In a vacuum, light will travel in a straight line at fixed speed, carrying energy from one place to another. Two key properties of light interacting with a medium are:

1. It can be deflected upon passing from one medium to another (refraction).
2. It can be bounced off a surface (reflection).

Concept of a photon

The particle-like nature of light is modeled with photons. A photon has no mass and no charge. It is a carrier of electromagnetic energy and interacts with other discrete particles (e.g., electrons, atoms, and molecules). A beam of light is modeled as a stream of photons, each carrying a well-defined energy that is dependent upon the wavelength of the light. The energy of a given photon can be calculated by:

Photon energy (E) = $\frac{hc}{\lambda}$

where E is in joules

h = Planckfs constant = 6.625×10^{-34}J.s

c = Speed of light = $2.998 \times 10^{8} m/s$

λ= Wavelength of the light in meters

Superposition

For many kinds of waves, including electromagnetic, two or more waves can traverse the same space at the same time independently of one another. This means that the electric field at any point in space is simply the vector sum of the electric fields that the individual waves alone produce at the point. This is the superposition principle. Both the electric and magnetic fields of an electromagnetic wave satisfy the superposition principle. Thus, given multiple waves, the field at any given point can be calculated by summing each of the individual wave vectors.When two or more waves are superimposed, the resulting physical effect is called interference.

Quantum optics deals mainly with quantum properties of the light generated by various quantum optical system, such as; Laser and maser with their effect on the dynamics of atoms. Laser is an acronym for Light Amplification by Stimulated Emission of Radiation. Maser is an acronym for Microwaves Amplifications by Stimulated Emission of Radiation. It emitted coherent microwaves. The unique property of laser is that its light waves travel very long distances with a very little divergence.

The three-level atomic systems are: V-type, λ-type and cascade-type. A three-level cascade laser is a quantum optical system in which three level atoms in a cascade configuration. In this system the top, intermediate and bottom levels of a three level cascade atom can be denoted by $\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$, respectively. In λ-type there are also three-states:$\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$.However, in this type, $\mid a\rangle$ is at the highest energy level, while $\mid b\rangle$, $\mid c\rangle$ are at lower energy levels, respectively.

A v laser consists of a cavity into which three-level atoms in v configuration are injected at a constant rate r_a and removed from the cavity after a certain time τ . The V-type atoms have three states:$\mid a\rangle$, $\mid b\rangle$ and $\mid c\rangle$. The atoms are initially prepared in a coherent supper position of the upper levels $\mid a\rangle$ and $\mid b\rangle$. The laser cavity is arranged such that it would be at resonance with the transitions between $\mid a\rangle \longleftrightarrow \mid c\rangle$ and $\mid b\rangle \longleftrightarrow \mid c\rangle$, with direct transitions between levels $\mid a\rangle$ and $\mid b\rangle$ to be dipole forbidden. The transitions from the upper levels $\mid a\rangle$ and $\mid b\rangle$ to the lower level $\mid c\rangle$ produce two photons of different frequencies. The energy levels of $\mid a\rangle$ and

$\mid b\rangle$ higher than that of $\mid c\rangle$.

The optical parametric oscillator converts an in put laser wave called pump photon with frequency in to out put of lower frequencies in non degenerate parametric oscillator. If the two photons have the same frequency, the parametric oscillator is called degenerate otherwise non-degenerate.

T.Yirgashewa[8] has studied squeezing and statistical properties of the light produced by non degenerate parametric oscillator with the cavity modes driven by coherent and coupled to a two-mode squeezed vacuum reservoir.He has found that the two-mode squeezed vacuum is increase the degree of squeezing of the radiation and the two-mode coherent light and the two-mode squeezed vacuum reservoir increase the mean and variance of the photon number.

S.Girma[9] has studied that the two-mode V laser dynamics with coherent superposition of atomic states. He has found that both quadrature operators have the same variances and the light generated by two-mode V laser is not in squeezed state. For the atom initially prepared in a coherent superposition of the upper states such that $\epsilon = 0$ and $\theta = \pi$, the superposition of the two-modes turns and out to be in a coherent state.

Y.Siyum[16] has studied that the two-mode coherent and subharmonic light beams coupled to vacuum reservoir. He has found that the mean of the photon number for the superposition of two-mode coherent and subharmonic light beams is the sum of the individual mean photon number and the variance of photon number for modes a is the same as that of mode b. He have seen that the mean of the photon number differences vanishes.

Eyob has considered a non degenerate three-level laser in which the atoms in cascade configuration prepared initially in the coherent superposition of the top and the bottom levels are injected in to cavity and the cavity mode coupled to squeezed vacuum reservoir via single port-mirror. Applying the pertinent the master equation, he analyzed the effect of the injected squeezed light on the quadrature squeezing,entanglement and the normalized intensity difference fluctuations.He has shown that the injected squeezed light increases the degree of squeezing and entanglement for certain initial condition as well as the mean photon numbers of the cavity modes[3]

Lu and Zhu have considered a non degenerate three-level laser with the atom initially prepared in coherent superposition of the top and the the bottom levels. They have estimated the maximum of 50% interactivity two-mode squeezing [4].

Tewodros and Fesseha have studied the squeezing and statistical properties of light produced by a coherently driven degenerate three-level laser with a parametric amplifier. They have considered the case in which the atoms injected in to the cavity are prepared in a coherent superposition of the top and the bottom levels coupled by the pump mode emerging from the parametric amplifier. Their results shows that the presence of the parametric amplifier increases the squeezing and the mean photon number significantly. Moreover, they have found that the maximum interactivity squeezing is 93 percent in the presence of the coupling and when the superposition has no contribution $(\eta = 0)$. Next to this, they have shown that the effect of coupling the top and the bottom level is to decrease the mean and variance of the photon number[5]

S.tesfa[6] has studied the squeezing properties of the cavity modes produced by non degenerate three-level laser applying the solution of stochastic differential equations. He has found that the two-mode cavity radiation exhibits squeezing if the atoms are initially prepared with more atoms in the bottom level than in the upper level, and the degree of squeezing increases with the linear gain coefficient.

Friew[7], He has studied that squeezing and statistical properties of the light generated by a non degenerate three-level laser whose cavity mode contains two parametric amplifier coupld to ordinary vacuum reservoir. He has found that parametric amplifier increase the degree of squeezing.

3

MATERIALS AND METHODOLOGY

3.1 MATERIALS

The materials that we used in this thesis were:

⧫ Different articles,

⧫ Difference reference books on quantum optics and M.sc thesis,

⧫ PhD.dissertations.

In addition to these, we used software such as:

► WinEdit for writing the thesis

► MATHLAB for plotting different graphs.

3.2 METHODOLOGY

In this thesis, we seek to study the statistical properties of a two-mode light by non degenerate parametric oscillator, V-type three-level laser and the superposition of the two systems whose cavity modes are coupled to ordinary vacuum reservoir. This thesis contains three parts.

In the first part, we wish to investigate the statistical properties of two-mode light produced by a non-degenerate parametric oscillator whose cavity is couple to a two-mode ordinary vacuum reservoir. In order to carry out analysis, we find the c-number Langevin equations for the cavity light produced by a parametric oscillator systems using the pertinent master equation. Applying the solutions of the resulting c-number Langevin equation along with the properties of noise forces, we obtain the anti-normally ordered characteristics function, with the aid of which we determine the Q-function. The resulting Q-function is then used to calculate the mean and variances of the photon number. In addition to that, we have determine the mean and variance of the photon number sum and differences

In the second part, we wish to investigate the statistical properties of two-mode light produced by a non-degenerate V-type three-level laser light whose cavity is coupled to a two-mode ordinary vacuum reservoir. In order to carry out analysis, we find the c-number Langevin equations for the cavity light produced by a V-type three-level laser systems using the pertinent master equation. Applying the solutions of the resulting c-number Langevin equation along with the properties of noise forces, we obtain the anti-normally ordered characteristics function, with the aid of which we determine the Q-function. The resulting Q-function is then used to calculate the mean and variances of the photon number. In addition to that, we see to determine the mean and variance of the photon number sum and differences.

In the third part, we investigate the statistical properties of a pair of superposed two-mode parametric amplifier and V-type three-level laser light beams. In order to perform our analysis, we first drive Q-functions for a pair of superposed parametric oscillator and V-type three-level laser light beams. The resulting of Q-function is then used to calculate the mean and variances of photon number. In addition, we have determined the mean photon number sum and differences and variances of a photon number for the superposed two-mode cavity light.

3.2.1 Master Equation for Parametric Oscillator

In this section we wish to drive the equation of the reduced density operator for the cavity modes coupled to a two-mode vacuum reservoir.

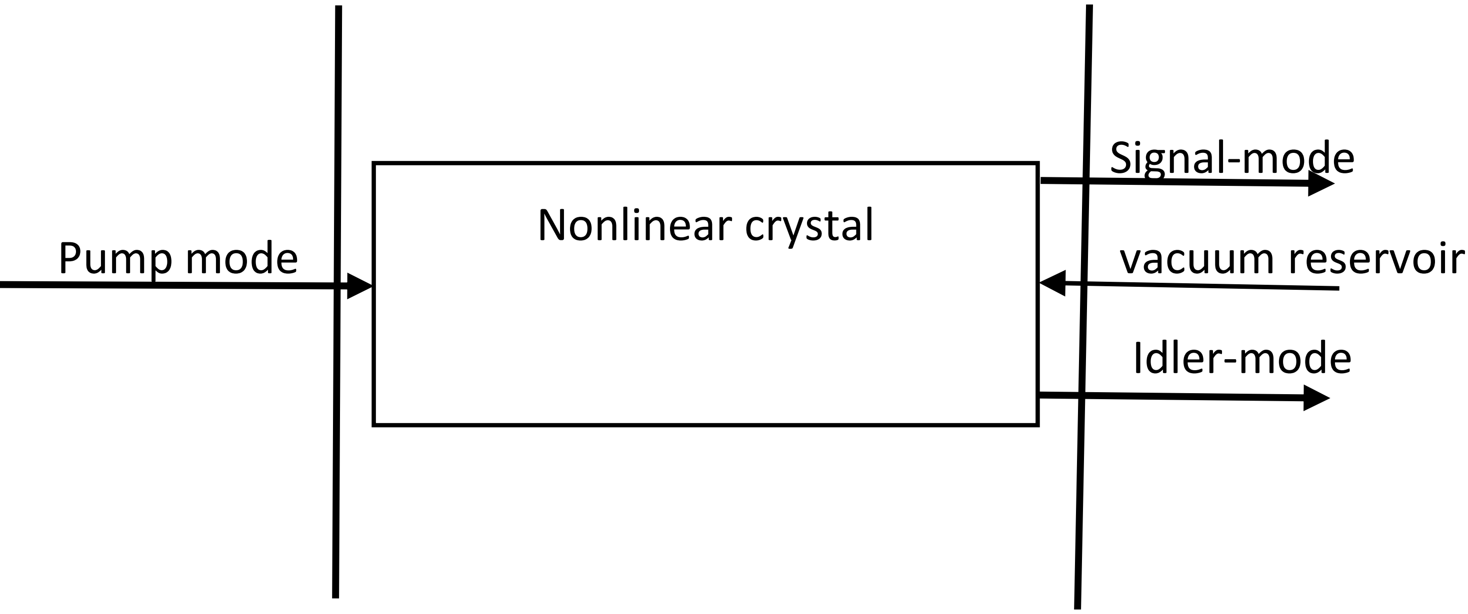

Fig.3.1 Schematic representation of a non degenerate parametric oscillator system in a cavity

The interaction of the system with the reservoir can be described by the Hamiltonian

$$\hat{H} = \hat{H}_S + \hat{H}_{SR}, \tag{3.1}$$

where $\hat{H}_S$ is the hamiltonian for the interaction in the cavity and $\hat{H}_{SR}$ is the hamiltonian for the interaction between the cavity modes and reservoir,suppose $\hat{\rho}_{SR}(t)$ is the total density operator for the cavity modes coupled to vacuum reservoir. The time evolution of this density operator is

$$\frac{d}{dt}\hat{\rho}_{SR}(t) = -i[\hat{H}_S(t) + \hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{3.2}$$

We note that

$$\hat{\rho}(t) = TrR(\hat{\rho}_{SR}(t)). \tag{3.3}$$

is the density operator for the cavity mode only. In which TrR indicates trace over the reservoir variables only. Hence taking in to account (3.2), we see that the time evolution of the reduced density operator takes the form

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}_S, \hat{\rho}(t)] - iTrR[\hat{H}_{SR}(t), \hat{\rho}_{SR}(t)]. \tag{3.4}$$

Furthermore, a formal solution of Eq.(3.2)can be written as

$$\hat{\rho}_{SR}(t) = \hat{\rho}_{SR}(0) - i\int_0^t [\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}_{SR}(t')]dt'. \tag{3.5}$$

To proceed further, we need to introduce a certain approximation scheme. To this end assuming here is a week interaction between the system and reservoir, we can write approximately valid relation $\hat{\rho}_{SR}(t') = \hat{\rho}(t')R$, where $\hat{R}$ is the density operator for the reservoir assumed to be constant in time. Using this approximation Eq.(3.5)

$$\hat{\rho}_{SR}(t) = \hat{\rho}(0)\hat{R} - i\int_0^t [\hat{H}_S(t') + \hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}]dt'. \tag{3.6}$$

Inserting this result in to Eq.(3.4), we get

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& -[\hat{H}_S, \hat{\rho}(t)] - i[\langle H_{SR}\rangle R, \hat{\rho}(0)] \\ &- \int_0^t [\langle H_{SR}(t)\rangle R, [H_S(t'), \rho(t)]]dt' \\ &- \int_0^t TrR[\hat{H}_{SR}(t), [\hat{H}_{SR}(t'), \hat{\rho}(t')\hat{R}]]dt'.\end{aligned} \tag{3.7}$$

It should be noted that $\hat{\rho}(0)$ represents the radiation initially in the cavity and the density that describes the chaotic light in the cavity. Further more ,the interaction of the two mode cavity radiation with a two mode reservoir can be described in the interaction picture by the Hamiltonian

$$\hat{H}_{SR} = i\sum \lambda_1(\hat{a}^+\hat{a}_1 e^{i(\omega_a-\omega_1)t} - \hat{a}\hat{a}_1^+ e^{-i(\omega_a-\omega_1)t}) + i\sum \lambda_2(\hat{b}^+\hat{b}_2 e^{i(\omega_b-\omega_2)t} - \hat{b}\hat{b}_2^+ e^{-i(\omega_b-\omega_2)t}) \tag{3.8}$$

where $\omega_0 = \frac{\omega_a+\omega_b}{2}$, with ω_a and ω_b stands for the frequencies and $(\hat{a}, \hat{b})$ are the annihilation operators for the cavity modes. In addition (a_1, b_2) are the annihilation operators for the reservoir modes having frequencies ω_1 and ω_2 respectively. λ_1 and λ_2 are coupling constants between the cavity modes and reservoir modes. Two mode vacuum reservoir operators satisfy the following properties;

$$\langle\hat{a}_1\rangle R = \langle\hat{b}_2\rangle R = 0. \tag{3.9}$$

$$\langle\hat{a}_1^+\hat{a}_2\rangle R = \langle\hat{b}_1^+\hat{b}_2\rangle R = 0. \tag{3.10}$$

$$\langle\hat{a}_1\hat{a}_2^+\rangle R = \langle\hat{b}_1\hat{b}_2^+\rangle R = \delta_{12}. \tag{3.11}$$

$$\langle\hat{a}_1\hat{a}_2\rangle R = \langle\hat{a}_1^+\hat{a}_2^+\rangle R = 0. \tag{3.12}$$

$$\langle \hat{b}_1 \hat{b}_2 \rangle R = \langle \hat{b}_1^+ \hat{b}_2^+ \rangle R = 0. \tag{3.13}$$

$$\langle \hat{a}_1^+ \hat{b}_2 \rangle R = \langle \hat{a}_1^+ \hat{b}_2 \rangle R = 0. \tag{3.14}$$

$$\langle \hat{a}_1 \hat{b}_2^+ \rangle R = \langle \hat{a}_1^+ \hat{b}_2 \rangle R = 0. \tag{3.15}$$

In view of Eq.(3.9), one easily shows that

$$\langle H_{SR} \rangle R = 0 \tag{3.16}$$

Taking in to account Eq.(3.16), along with $H_{SR}(t)H_{SR}(t')$ = $H_{SR}(t')H_{SR}(t)$, Eq.(3.7), reduces to

$$\begin{aligned} \frac{d}{dt}\hat{\rho}(t) = &- i[H_S(t), \hat{\rho}(t)] - \int_0^t \langle H_{SR}(t)H_{SR}(t')\rangle_R \hat{\rho}(t')\partial t' \\ &- \int_0^t \hat{\rho}(t')\langle H_{SR}(t')H_{SR}(t)\rangle_R \partial t' + 2\int_0^t \langle H_{SR}(t')\hat{\rho}(t')H_{SR}(t)\rangle_R \hat{\rho}(t')\partial t'. \end{aligned} \tag{3.17}$$

Applying the Hamiltonian , Eq.(3.8) and Eqs.(3.9)-(3.15), we readily obtain

$$-TrR(H_{SR}(t)H_{SR}(t'))\hat{R} = [q_1 \hat{a}\hat{a}^+ + q_2 \hat{b}\hat{b}^+], \tag{3.18}$$

where

$$q_1 = -\sum_1 \lambda_1^2 e^{-i(\omega_a - \omega_1)(t-t')}. \tag{3.19}$$

$$q_2 = -\sum_2 \lambda_1^2 e^{-i(\omega_b - \omega_2)(t-t')}. \tag{3.20}$$

Assuming the frequencies of the reservoir modes to be closely spaced, the summation can be changed to the integral over frequency. Then applying Markov's approximation to the resulting integral we find the following result;

$$-\sum_1 \lambda_1^2 e^{-i(\omega_a - \omega_1)(t-t')} = k_j \delta(t-t'), \tag{3.21}$$

in which $k_j = 2\pi g(\omega_j)\lambda^2(\omega_j)(j = a, b)$ is defined to be the damping constant for cavity mode j with $g(\omega_j)$ being the density operator for the reservoir modes which assumed to be at resonance with the cavity mode j. In our analysis,we assume that $k_a = k_b = k$ for convenience. Therefore, on account of these results, Eq.(3.19) and Eq.(3.20) take the form

$$q_1 = q_2 = -k\delta(t-t'). \tag{3.22}$$

Substituting Eq.(3.22) in to Eq.(3.18) yields

$$-iTrR(\hat{H}_{SR}(t)\hat{H}_{SR}(t'))\hat{R} = -k\delta(\hat{a}\hat{a}^{+} + \hat{b}\hat{b}^{+})(t-t'). \tag{3.23}$$

It follows that

$$-\int_{0}^{t} TrR(\hat{H}_{SR}(t)\hat{H}_{SR}(t'))\hat{R}\hat{\rho}(t)\partial t' = -\frac{k}{2}(\hat{a}\hat{a}^{+}\hat{\rho} + \hat{b}\hat{b}^{+}\hat{\rho}). \tag{3.24}$$

In similar way, one can easily verify that

$$-\int_{0}^{t} \hat{\rho}(t)TrR(\hat{R}\hat{H}_{SR}(t')\hat{H}_{SR}(t))\partial t' = -\frac{k}{2}(\hat{\rho}\hat{a}\hat{a}^{+} + \hat{\rho}\hat{b}\hat{b}^{+}). \tag{3.25}$$

Furthermore, applying Eq.(3.9), we can get

$$TrR(\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)) = -[q_1\hat{a}^{+}\hat{\rho}\hat{a} + q_2\hat{b}^{+}\hat{\rho}\hat{b}]. \tag{3.26}$$

With the aid of Eq.(3.22), we find

$$TrR(\hat{H}_{SR}(t')\hat{\rho}(t')\hat{R}\hat{H}_{SR}(t)) = \frac{k}{2}(\hat{a}\hat{\rho}\hat{a}^{+} + \hat{b}\hat{\rho}\hat{b}^{+}). \tag{3.27}$$

In similar way, one can get

$$TrR(\hat{H}_{SR}(t')\hat{R}\hat{\rho}(t')\hat{H}_{SR}(t)) = \frac{k}{2}(\hat{a}\hat{\rho}\hat{a}^{+} + \hat{b}\hat{\rho}\hat{b}^{+}). \tag{3.28}$$

Up on substituting Eqs.(3.24)-(3.28) into (3.17), we can obtain the master equation for the cavity modes coupled to a two-mode vacuum reservoir to be

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}_S(t),\hat{\rho}(t)] + \frac{k}{2}(2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}) + \frac{k}{2}(2\hat{b}\hat{\rho}\hat{b}^{+} - \hat{b}^{+}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{+}\hat{b}). \tag{3.29}$$

Eq.(3.29) represents the equation of evolution of the reduced density operator for cavity modes coupled to ordinary vacuum reservoir.

3.2.2 The Down Conversion Process

We seek to derive the master equation for the non-degenerate parametric oscillator coupled to vacuum reservoir via a single port-mirror.

In a non-degenerate parametric oscillator a pump photon of frequency ω is down converted in a highly correlated signal and idler photons of frequencies ω_a and ω_b such that $\omega = \omega_a + \omega_b$. The process of the parametric interaction can be described by the Hamiltonian

$$\hat{H} = i\lambda(\hat{a}\hat{b}\hat{c}^{+} - \hat{a}^{+}\hat{b}^{+}\hat{c}). \tag{3.30}$$

Where $\hat{a}(\hat{b})$ is the annihilation operator for the signal(idler) modes, $\hat{c}$ is the annihilation operator for the pump mode and λ is the coupling constant with the pump mode treated classically, the Hamiltonian can be written as

$$\hat{H} = i\varepsilon(\hat{a}\hat{b} - \hat{a}^{+}\hat{b}^{+}). \tag{3.31}$$

Where ε , considered to be real and constant, is proportional to the amplitude of the pump mode. The time evolution of the density operator corresponding to this Hamiltonian is

$$\frac{d}{dt}\hat{\rho}(t) = \varepsilon(\hat{a}\hat{b}\hat{\rho} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}). \tag{3.32}$$

Finally , with the aid of Eqs.(3.29) and Eqs.(3.32) , one can write

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) =& \varepsilon(\hat{a}\hat{b}\hat{\rho} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho} - \hat{\rho}\hat{a}\hat{b} + \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}) \\ &+ \frac{k}{2}(2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}) + \frac{k}{2}(2\hat{b}\hat{\rho}\hat{b}^{+} - \hat{b}^{+}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{+}\hat{b}).\end{aligned} \tag{3.33}$$

Therefore, Eq.(3.33) represents the master equation of a non-degenerate parametric oscillator coupled to vacuum reservoir. In which the cavity damping constant κ is assumed to be the same for both the signal and idler modes.

3.2.3 The C-Number Langevin Equation

The dynamics of a cavity mode coupled to the reservoir can be also described using the c-number Langevin equation. In this section, we seek to obtain the c-number Langevin of equation. The time evolution of the expectation value of an operator $\hat{A}$ in the schrodinger picture can be expressed as [1]

$$\frac{d}{dt}\langle A\rangle = Tr(\frac{d}{dt}\hat{\rho}\hat{A}). \tag{3.34}$$

Now taking in to of Eq.(3.33) along with Eq.(3.34) , one can write

$$\begin{aligned}\frac{d}{dt}\langle a\rangle =& \varepsilon Tr(\hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{a}^{\dagger}\hat{b}^{\dagger}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a} + \hat{\rho}\hat{a}^{\dagger}\hat{b}^{\dagger}\hat{a}) \\ &+ \frac{k}{2}Tr(2\hat{a}\hat{\rho}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^{\dagger}\hat{a}\hat{a}) + \frac{k}{2}Tr(2\hat{b}\hat{\rho}\hat{b}^{\dagger}\hat{a} - \hat{b}^{\dagger}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^{\dagger}\hat{b}\hat{a}).\end{aligned} \tag{3.35}$$

Applying the cyclic property of the trace operation together with the commutation relations

$$[\hat{a}, \hat{a}^{\dagger}] = 1. \tag{3.36}$$

$$[\hat{a}, \hat{b}] = [\hat{a}^{\dagger}, \hat{b}^{\dagger}] = [\hat{a}, \hat{b}^{\dagger}] = [\hat{a}^{\dagger}, \hat{b}] = 0 \tag{3.37}$$

and

$$[\hat{a}^{\dagger}, \hat{a}^2] = -2\hat{a}. \tag{3.38}$$

Then, we readily find

$$\frac{d}{dt}\langle a(t)\rangle = -\frac{k}{2}\langle a(t)\rangle - \varepsilon\langle b\hat{\dagger}(t)\rangle \tag{3.39}$$

It can also be shown in similar fashion that

$$\frac{d}{dt}\langle b(t)\rangle = -\frac{k}{2}\langle b(t)\rangle - \varepsilon\langle a^{\dagger}(t)\rangle. \tag{3.40}$$

$$\frac{d}{dt}\langle a(t)b(t)\rangle = -\varepsilon\langle a^{\dagger}(t)a(t)\rangle - \varepsilon\langle \hat{b}^{\dagger}(t)b(t)\rangle - k\langle a(t)b(t)\rangle - \varepsilon. \tag{3.41}$$

$$\frac{\partial}{\partial t}\langle a(t)a(t)\rangle = -2\varepsilon\langle a(t)b^{\dagger}(t)\rangle - k\langle a^2(t)\rangle. \tag{3.42}$$

$$\frac{d}{dt}\langle a^{\dagger}(t)a(t)\rangle = -k\langle a^{\dagger}(t)a(t)\rangle - \varepsilon(\langle a(t)b(t)\rangle + \langle a^{\dagger}(t)b^{\dagger}(t)\rangle). \tag{3.43}$$

The c-number equations corresponding to Eqs.(3.39)-(3.43) become

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\frac{k}{2}\langle \alpha(t)\rangle - \varepsilon\langle \beta^*(t)\rangle. \tag{3.44}$$

$$\frac{d}{dt}\langle \beta(t)\rangle = -\frac{k}{2}\langle \beta(t)\rangle - \varepsilon\langle \alpha^*(t)\rangle. \tag{3.45}$$

$$\frac{d}{dt}\langle \alpha(t)\beta(t)\rangle = -\varepsilon\langle \alpha^*(t)\alpha(t)\rangle - \varepsilon\langle \beta^*(t)\beta(t)\rangle - k\langle \alpha(t)\beta(t)\rangle - \varepsilon. \tag{3.46}$$

$$\frac{d}{dt}\langle a(t)a(t)\rangle = -2\varepsilon\langle \alpha(t)\beta^*(t)\rangle - k\langle \alpha^2(t)\rangle. \tag{3.47}$$

$$\frac{d}{dt}\langle \alpha^*(t)\alpha(t)\rangle = -k\langle \alpha^*(t)\alpha(t)\rangle - \varepsilon(\langle \alpha(t)\beta(t)\rangle + \langle \alpha^*(t)\beta^*(t)\rangle). \tag{3.48}$$

On the account of Eqs.(3.44) and (3.45), one can write

$$\frac{d}{dt}\langle \alpha(t)\rangle = -\frac{k}{2}\langle \alpha(t)\rangle - \varepsilon\langle \beta^*(t)\rangle + f_{\alpha}(t) \tag{3.49}$$

$$\frac{d}{dt}\langle \beta(t)\rangle = -\frac{k}{2}\langle \beta(t)\rangle - \varepsilon\langle \alpha^*(t)\rangle + f_{\beta}(t) \tag{3.50}$$

where f_α and f_β are noise forces. We observe that Eq.(3.44) and the expectation value of Eq.(3.45) as well as Eq.(3.45) and the expectation value of Eq.(3.50) will have the same form if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.51}$$

Using Eqs.(3.49) and (3.50) together with the relation

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = \langle\frac{d}{dt}\alpha(t)\beta(t)\rangle + \langle\alpha(t)\frac{d}{dt}\langle\beta(t)\rangle. \tag{3.52}$$

We find

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -k\langle\alpha(t)\beta(t)\rangle - \varepsilon\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon\langle\beta^*(t)\beta(t)\rangle + \langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle. \tag{3.53}$$

Comparison of Eqs.(3.46) and (3.53) indicates that

$$\langle\alpha(t)f_\beta(t)\rangle + \langle\beta(t)f_\alpha(t)\rangle = -\varepsilon. \tag{3.54}$$

The formal solution of Eqs.(3.49) and (3.50) can be as

$$\alpha(t) = \alpha(0)e^{-\frac{k}{2}t} + \int_0^t e^{-\frac{k}{2}(t-t')}[f_\alpha(t') - \varepsilon\beta^*(t')]dt'. \tag{3.55}$$

$$\beta(t) = \beta(0)e^{-\frac{k}{2}t} + \int_0^t e^{-\frac{k}{2}(t-t')}[f_\beta(t') - \varepsilon\alpha^*(t')]dt'. \tag{3.56}$$

On the basis of Eqs.(3.55) and (3.56), one can write

$$\langle\alpha(t)f_\beta(t)\rangle = \langle\alpha(0)f_\beta(t)\rangle e^{-\frac{k}{2}t} + \int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\beta(t)f_\alpha(t')\rangle - \varepsilon\langle\beta^*(t')f_\beta(t)\rangle]dt'. \tag{3.57}$$

$$\langle\beta(t)f_\alpha(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle e^{-\frac{k}{2}t} + \int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\alpha(t)f_\beta(t')\rangle - \varepsilon\langle\alpha^*(t')f_\alpha(t)\rangle]dt'. \tag{3.58}$$

Assuming a noise forces $f_\alpha(t)$ and $f_\beta(t)$ at time t should not affect the system variables at earlier times, we note that

$$\langle\alpha(0)f_\beta(t)\rangle = \langle\beta(0)f_\alpha(t)\rangle = 0. \tag{3.59}$$

$$\langle\alpha^*(t')f_\alpha(t)\rangle = \langle\alpha^*(t')\rangle\langle f_\alpha(t)\rangle = 0. \tag{3.60}$$

$$\langle\beta^*(t')f_\beta(t)\rangle = \langle\beta^*(t')\rangle\langle f_\beta(t)\rangle = 0. \tag{3.61}$$

Thus on account of Eqs.(3.51),(3.59),(3.60) and (3.61),Eqs.(3.57), reduces to

$$\langle\alpha(t)f_\beta(t)\rangle = \int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\beta(t)f_\alpha(t')\rangle]\partial t'. \tag{3.62}$$

It can also be verified in a similar fashion that,Eq.(3.58) reduces to

$$\langle \beta(t) f_\alpha(t) \rangle = \int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\alpha(t) f_\beta(t') \rangle]\partial t'. \tag{3.63}$$

Therefore, in view of Eqs.(3.54),(3.62) and (3.63) assuming that

$$\langle f_\alpha(t) f_\beta(t') \rangle = \langle f_\beta(t) f_\alpha(t') \rangle. \tag{3.64}$$

we have

$$\int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\beta(t) f_\alpha(t') \rangle]dt' = \int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\alpha(t) f_\beta(t') \rangle]dt' = -\frac{\varepsilon}{2}. \tag{3.65}$$

Therefore, on account of [1]

$$\int_0^t e^{-\frac{k}{2}(t-t')}\langle f(t) g(t') \rangle dt' = D. \tag{3.66}$$

we assert that

$$\langle f(t) g(t') \rangle = 2D\delta(t-t'), \tag{3.67}$$

where κ is a constant and D is some function of the time.We note that

$$\langle f_\alpha(t) f_\beta(t') \rangle = \langle f_\beta(t) f_\alpha(t') \rangle = -\varepsilon\delta(t-t'). \tag{3.68}$$

Furthermore, using Eq.(3.49) along with the relation

$$\frac{d}{dt}\langle \alpha(t)\alpha(t) \rangle = \langle \frac{d}{dt}\alpha(t)\alpha(t) \rangle + \langle \alpha(t)\frac{d}{dt}\alpha(t) \rangle. \tag{3.69}$$

We easily see that

$$\frac{d}{dt}\langle \alpha(t)\alpha(t) \rangle = -\kappa\langle \alpha^2(t) \rangle = 2\varepsilon\langle \alpha(t)\beta^*(t) \rangle + 2\langle \alpha(t) f_\alpha(t) \rangle \tag{3.70}$$

Comparison of Eqs.(3.47) and (3.70),shows that

$$\langle \alpha(t) f_\alpha(t) \rangle = 0. \tag{3.71}$$

Applying Eq.(3.55) together with (3.71), one can write

$$\int_0^t e^{-\frac{k}{2}(t-t')}[\langle f_\alpha(t) f_\alpha(t') \rangle]dt' = 0. \tag{3.72}$$

From which follows that

$$\langle f_\alpha(t) f_\alpha(t') \rangle = 0. \tag{3.73}$$

It can be shown in similar way that

$$\langle f_\alpha(t) f_\alpha(t') \rangle = \langle f_\beta(t) f_\beta(t') \rangle = 0. \tag{3.74}$$

Moreover employing Eq.(3.49) and its complex conjugate along with

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = \langle\frac{d}{dt}\alpha^*(t)\alpha(t)\rangle + \langle\alpha^*(t)\frac{d}{\partial t}\alpha(t)\rangle, \tag{3.75}$$

we find

$$\frac{d}{dt}\langle\alpha^*(t)\alpha(t)\rangle = -k\langle\alpha^*(t)\alpha(t)\rangle - \varepsilon\langle\beta(t)\alpha(t)\rangle - \varepsilon\langle\alpha^*(t)\beta^*(t)\rangle + \langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle. \tag{3.76}$$

Comparison of Eqs.(3.48) and (3.76), gives as

$$\langle\alpha(t)f_\alpha^*(t)\rangle + \langle\alpha^*(t)f_\alpha(t)\rangle = 0. \tag{3.77}$$

With the aid of Eq.(3.55) and its complex conjugate, one can get

$$\langle\alpha(t)f_\alpha^*(t)\rangle = \int_0^t e^{-\frac{k}{2}(t-t')}\langle f_\alpha^*(t)f_\alpha(t)\rangle\partial t'. \tag{3.78}$$

$$\langle\alpha^*(t)f_\alpha(t)\rangle = \int_0^t e^{-\frac{k}{2}(t-t')}\langle f_\alpha(t)f_\alpha^*(t)\rangle dt'. \tag{3.79}$$

In view of Eqs.(3.77),(3.78),(3.79) and the assumption that

$$\langle f_\alpha^*(t)f_\alpha(t)\rangle = \langle f_\alpha(t)f_\alpha^*(t)\rangle. \tag{3.80}$$

we obtain

$$\int_0^t e^{-\frac{k}{2}(t-t')}\langle f_\alpha^*(t)f_\alpha(t)\rangle dt' = \int_0^t e^{-\frac{k}{2}(t-t')}\langle f_\alpha(t)f_\alpha^*(t)\rangle dt'. \tag{3.81}$$

From which one can get

$$\langle f_\alpha(t)f_\alpha^*(t)\rangle = \langle f_\alpha(t)f_\alpha^*(t)\rangle = 0. \tag{3.82}$$

It can also be established in a similar procedure that

$$\langle f_\beta(t)f_\beta^*(t)\rangle = \langle f_\beta(t)f_\beta^*(t)\rangle = 0. \tag{3.83}$$

Now Eqs.(3.51),(3.68),(3.73),(3.74),(3.82),and (3.83) describe the correlation properties of the noise forces f_α and f_β associated with the normal ordering.

We next proceed to obtain the solutions of coupled differential equations Eqs.(3.49) and (3.50). Introducing new variable defined by

$$z_\pm = \alpha(t) \pm \beta^*(t) \tag{3.84}$$

and applying Eqs.(3.49) as well as (3.50) along with their complex conjugate, we obtain

$$\frac{d}{dt}z_\pm = -\frac{1}{2}\lambda_\pm z_\pm + f_\alpha(t) \pm f_\beta^*(t), \tag{3.85}$$

where

$$\lambda_{\pm} = k \pm 2\varepsilon. \tag{3.86}$$

According to Eq.(3.85) together with Eq.(3.86), the equation of evolution for $z_{-}(t)$ has no solution for $\kappa < 2\varepsilon$. We then identity $\kappa = 2\varepsilon$ as the threshold condition. Thus, for $2\varepsilon < \kappa$, the solution of Eq.(3.85) can be written as

$$z_{\pm} = z_{\pm}(0)e^{-\frac{1}{2}\lambda_{\pm}t} + \int_0^t e^{-\frac{1}{2}\lambda_{\pm}(t-t')}[f_{\alpha}(t') + f_{\beta}^*(t')]\partial t'. \tag{3.87}$$

Now with the aid of Eqs.(3.84) and (3.87), we obtain

$$\alpha(t) = A_{\dagger}\alpha(0) + A_{-}\beta^*(0) + F_{+}(t) + F_{-}(t). \tag{3.88}$$

$$\beta(t) = A_{\dagger}\beta(0) + A_{-}\alpha^*(0) + F_{+}^*(t) + F_{-}^*(t). \tag{3.89}$$

where

$$E_{\pm}(t) = \frac{1}{2}(e^{-}\frac{\lambda_{\dagger}t}{2} \pm e^{-}\frac{\lambda_{-}t}{2}). \tag{3.90}$$

$$F_{\pm}(t) = \frac{1}{2}\int_0^t e^{-\frac{\lambda_{\pm}}{2}(t-t')}[f_{\alpha}(t') \pm f_{\beta}^*(t')]dt'. \tag{3.91}$$

Employing the resulting solution of the c-number Langevin equation along with the correlation relations of the noise forces,we will study the Q-function, the mean and the variances the photon number.

3.2.4 The Q-function for a Non-Degenerate Parametric oscillator

We next seek to obtain the **Q**-function for the the signal-idler modes,assumed to be initially in a vacuum state. The **Q**-function for a two-mode light is expressible as

$$Q(\alpha, \beta, t) = \frac{1}{\pi^4}\int d^2z d^2\chi \phi_a(z, \chi, t) exp(z^*\alpha - z\alpha^* + \chi^*\beta - \chi\beta^*), \tag{3.92}$$

where the anti-normally ordered characteristics function is defined in the Heisenberg picture by

$$\phi_a(z, \chi, t) = Tr(\rho(0) exp(z\hat{a}^{\dagger}(t) - z^*\hat{a}(t) + \chi b^{\dagger}(t) - \chi^* b(t))). \tag{3.93}$$

This can be put employing the identity [1]

$$e^A e^B = e^B e^A e^{[A,B]} \tag{3.94}$$

in the form

$$\phi_a(z, \chi, t) = exp[-z^*z - \chi^*\chi] Tr[\rho(0) exp(z\hat{a}^{\dagger}(t) - z^*\hat{a}(t) + \chi b^{\dagger}(t) - \chi^* b(t))]. \tag{3.95}$$

The characteristic function can be expressed in terms of c-number variables associated with the normal ordering in the form

$$\phi_a(z,\chi,t) == exp[-z^*z - \chi^*\chi]\langle exp(z\hat{\alpha}^* - z^*\alpha + \chi\beta^* - \chi^*\beta)\rangle. \tag{3.96}$$

Eqs.(3.88) and (3.89) can be written as

$$\alpha(t) = E_\dagger\alpha(0) + E_-\beta^*(0) + \gamma_\alpha(t). \tag{3.97}$$

$$\beta(t) = E_\dagger\beta(0) + E_-\alpha^*(0) + \gamma_\beta(t). \tag{3.98}$$

Where $A_\pm$ is defined by Eq.(3.90).

$$\gamma_\alpha(t) = \frac{1}{2}[\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}(f_\alpha(t') + f^*_\beta(t'))dt' + \int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}(f_\alpha(t') - f^*_\beta(t'))dt'] \tag{3.99}$$

and

$$\gamma_\beta(t) = \frac{1}{2}[\int_0^t e^{-\frac{1}{2}\lambda_+(t-t')}(f^*_\alpha(t') + f_\beta(t'))dt' - \int_0^t e^{-\frac{1}{2}\lambda_-(t-t')}(f^*_\alpha(t') - f_\beta(t'))\partial t']. \tag{3.100}$$

Now in view of Eqs.(3.97) and (3.98), one can get

$$\begin{aligned}\phi_a(z,\chi,t) =& exp[-z^*z - \chi^*\chi]\langle exp(zA_+\alpha^*(0) + zA_-\beta(0) + z\gamma_\alpha - z^*A_+\alpha(0) \\ &- z^*A_-\beta^*(0) - z^*\gamma_\alpha + \chi A_+\beta^*(0) + \chi A_-\alpha(0) \\ &+ \chi\gamma^*_\beta - \chi^*A_+\beta(0) - \chi^*A_-\alpha^*(0) - \chi^*\gamma_\beta)\rangle\end{aligned} \tag{3.101}$$

$$\begin{aligned}\phi_a(z,\chi,t) =& exp[-z^*z - \chi^*\chi]\langle exp[(\chi A_- - z^*A_+)\alpha(0) + (zA_+ - \chi^*A_-)\alpha^*(0) \\ &+ (zA_- - \chi^*A_+)\beta(0) + (\chi A_+ - z^*A_-)\beta^*(0)]\rangle \\ &\times \langle exp[z\gamma^*_\alpha - z^*\gamma_\alpha + \chi\gamma^*_\beta - \chi^*\gamma_\beta]\rangle.\end{aligned} \tag{3.102}$$

Considering the cavity radiation to be initially in the two-mode vacuum state, we obtain

$$\langle exp[(\chi A_- - z^*A_+)\alpha(0) + (zA_+ - \chi^*A_-)\alpha^*(0) + (zA_- - \chi^*A_+)\beta(0) + (\chi A_+ - z^*A_-)\beta^*(0)]\rangle = 1 \tag{3.103}$$

On account of Eqs.(3.102) and (3.103), we have

$$\phi_a(z,\chi,t) = exp[-z^*z - \chi^*\chi]X\langle exp[z\gamma^*_\alpha - z^*\gamma_\alpha + \chi\gamma^*_\beta - \chi^*\gamma_\beta]\rangle. \tag{3.104}$$

We recall that Gaussian variables with vanishing mean satisfy the relation

$$\langle exp(z\alpha^* - z^*\alpha)\rangle = exp[\frac{1}{2}\langle(z\alpha^* - z^*\alpha)^2\rangle]. \tag{3.105}$$

We next seek to show that $\gamma_\alpha(t)$ and $\gamma_\beta(t)$ are Gaussian variables. One can rewrite Eq.(3.99) as

$$\gamma_\alpha = \gamma_+ + \gamma_-, \tag{3.106}$$

in which

$$\gamma_+ = \frac{1}{2}\int_0^t e^{\frac{1}{2}\lambda_+(t-t')}(f_\alpha(t') + f_\beta^*(t'))dt' \tag{3.107}$$

and

$$\gamma_- = \frac{1}{2}\int_0^t e^{\frac{1}{2}\lambda_-(t-t')}(f_\alpha(t') - f_\beta^*(t'))dt'. \tag{3.108}$$

Using Eqs.(3.107) and (3.108), the time evolution for the expectation values γ_+ and γ_- can be expressed as

$$\frac{d}{dt}\langle\gamma_+\rangle = -\frac{\lambda_+}{4}\langle\gamma_+\rangle \tag{3.109}$$

and

$$\frac{d}{dt}\langle\gamma_-\rangle = -\frac{\lambda_-}{4}\langle\gamma_-\rangle. \tag{3.110}$$

In view of Eqs.(3.109) and (3.110), indicates that γ_+ and γ_- are Gaussian variables. Thus in view of Eq.(3.106), we see that γ_α is also Gaussian variables. Similarly we rewrite Eq.(3.100) as

$$\gamma_\beta = \delta_+ - \delta_-, \tag{3.111}$$

where

$$\delta_+ = \frac{1}{2}\int_0^t e^{-}\frac{1}{2}\lambda_+(t-t')(f_\alpha^*(t') + f_\beta(t'))dt' \tag{3.112}$$

and

$$\delta_- = \frac{1}{2}\int_0^t e^{\frac{1}{2}\lambda_+(t-t')}(f_\alpha^*(t') - f_\beta(t'))dt'. \tag{3.113}$$

Now the time evolution for the expectation values of δ_+ and δ_- can be expressed as

$$\frac{d}{dt}\langle\delta_+\rangle = -\frac{\lambda_-}{4}\langle\delta_+\rangle \tag{3.114}$$

and

$$\frac{d}{dt}\langle\delta_-\rangle = -\frac{\lambda_-}{4}\langle\delta_-\rangle. \tag{3.115}$$

In view of Eqs.(3.111),(3.114), and (3.115) ,we see that γ_β is Gaussian variables. In addition, using Eqs.(3.99) and (3.110), one can easily get

$$\langle\gamma_\alpha(t)\rangle = \langle\gamma_\beta(t)\rangle = 0. \tag{3.116}$$

Therefore, we have that γ_α and γ_β are Gaussian variables with vanishing mean. Then the characteristics function can be put in the form

$$\phi_a(z,\chi,t) = exp[-z^*z - \chi^*\chi]exp[\frac{1}{2}\langle(z\gamma_\alpha^* - z^*\gamma_\alpha + \chi\gamma_\beta^* - \chi^*\gamma_\beta)^2\rangle] \tag{3.117}$$

which is equivalent to

$$\phi_a(z,\chi,t) = exp[-z^*z-\chi^*\chi]exp[\frac{z^2}{2}\langle\gamma_\alpha^{*2}\rangle+\frac{z^{*2}}{2}\langle\gamma_\alpha^2\rangle+\frac{\chi^2}{2}\langle\gamma_\beta^{*2}\rangle+\frac{\chi^{*2}}{2}\langle\gamma_\beta^2\rangle-zz^*\langle\gamma_\alpha\gamma_\alpha^*\rangle$$
$$+z\chi\langle\gamma_\alpha^*\gamma_\beta^*\rangle-z^*\chi\langle\gamma_\alpha\gamma_\beta^*\rangle-z\chi^*\langle\gamma_\alpha^*\gamma_\beta\rangle+z^*\chi^*\langle\gamma_\alpha\gamma_\beta\rangle-\chi\chi^*\langle\gamma_\beta\gamma_\beta^*\rangle]. \quad (3.118)$$

On account of Eqs.(3.109) and (3.110), one can obtains

$$\langle\gamma_\alpha^{*2}\rangle=\langle\gamma_\alpha^2\rangle=\langle\gamma_\beta^2\rangle=\langle\gamma_\beta^{*2}\rangle=0. \quad (3.119)$$

$$\langle\gamma_\alpha^*\gamma_\beta\rangle=\langle\gamma_\alpha\gamma_\beta^*\rangle=0. \quad (3.120)$$

$$\langle\gamma_\alpha\gamma_\alpha^*\rangle=\langle\gamma_\beta\gamma_\beta^*\rangle=-\frac{\eta}{2}(F-H). \quad (3.121)$$

$$\langle\gamma_\alpha\gamma_\beta\rangle=\langle\gamma_\alpha^*\gamma_\beta^*\rangle=-\frac{\eta}{2}(F+H), \quad (3.122)$$

where

$$F=\frac{1}{\lambda_+}(1-e^{-\lambda_+t}). \quad (3.123)$$

$$H=\frac{1}{\lambda_-}(1-e^{-\lambda_-t}). \quad (3.124)$$

Thus in view of Eqs.(3.119),(3.120),(3.121),and (3.122)expression of Eq.(3.118), goes over in to

$$\phi_a(z,\chi,t)=exp[-(1-\frac{\varepsilon}{2})(F-H)(z^*z+\chi^*\chi)-\frac{\varepsilon}{2}(F+H)(\chi z+\chi^*z^*)] \quad (3.125)$$

Finally , we can also put the characteristics function in the form

$$\phi_a(z,\chi,t)=exp[-a(z^*z+\chi^*\chi)-b(\chi z+\chi^*z^*)] \quad (3.126)$$

where

$$a=1-\frac{\varepsilon}{2}(F-H)=1-\frac{\varepsilon}{2\lambda_+}(1-e^{-\lambda_+t})+\frac{\epsilon}{2\lambda_-}(1-e^{-\lambda_-t}). \quad (3.127)$$

$$b=\frac{\varepsilon}{2}(F+H)=\frac{\epsilon}{2\lambda_+}(1-e^{-\lambda_+t})+\frac{\varepsilon}{2\lambda_-}(1-e^{-\lambda_-t}). \quad (3.128)$$

Now up on introducing Eqs.(3.116) in to Eqs.(3.92), we readily obtain

$$Q(\alpha,\beta,t)=\frac{1}{\pi^2}\int_0^t\frac{\partial^2z\partial^2\chi}{\pi^2}exp[-a(z^*z+\chi^*\chi)-b(\chi z+\chi^*z^*)+\alpha z^*-\alpha^*z+\beta\chi^*-\beta^*\chi]. \quad (3.129)$$

Furthermore, using the relation [1]

$$\int \frac{\partial^2 z}{\pi} exp[-az*z+bz+cz*+Az^2+Bz*^2] = [\frac{1}{a^2-4AB}]^{\frac{1}{2}} exp(\frac{abc+Ac^2+Bb^2}{a^2-4AB}), a>0, \tag{3.130}$$

and performing the integration employing Eq.(3.130),the Q-function for the signal-idler modes turns out to be

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}(u^2-v^2)exp[-u(\alpha^*\alpha+\beta^*\beta)-v(\alpha\beta+\alpha^*\beta^*)], \tag{3.131}$$

where

$$u = \frac{a}{a^2-b^2}. \tag{3.132}$$

$$v = \frac{b}{a^2-b^2}. \tag{3.133}$$

Eq.(3.131) represent the Q-function for the signal-idler mode of the parametric oscillator system under consideration.

The Q-function for a signal mode and idler mode

Next, we integrate Eq.(3.131) over β, we can get the Q-function for the signal-mode

$$Q(\alpha,t)=\frac{1}{\pi^2}(u^2-v^2)[\int\partial^2\beta exp(-u\beta^*\beta-v\alpha\beta-v\alpha^*\beta^*)]\times exp(-u\alpha^*\alpha) \tag{3.134}$$

which is equivalent to

$$Q(\alpha,t)=\frac{(u^2-v^2)}{\pi u}exp[-(\frac{u^2-v^2}{u})\alpha^*\alpha]. \tag{3.135}$$

Using the same procedure the Q-function for the idler mode can be written as

$$Q(\beta,t)=\frac{(u^2-v^2)}{\pi v}exp[-(\frac{u^2-v^2}{v})\beta^*\beta]. \tag{3.136}$$

3.2.5 Master Equation for V-Type Three-Level Laser Light

A two-mode v laser consists of a cavity into which three-level atoms in v configuration are injected at a constant rate r_a and removed from the cavity after a certain time τ . We denote the two upper closely lying levels of the atom are by $\mid a\rangle$ and $\mid b\rangle$ and a single lower level by $\mid c\rangle$. The atoms are initially prepared in a coherent supper position of the upper levels $\mid a\rangle$ and $\mid b\rangle$. The laser cavity is arranged such that it would be at resonance with the transitions between $\mid a\rangle \longleftrightarrow \mid c\rangle$ and $\mid b\rangle \longleftrightarrow \mid c\rangle$, with direct transitions between levels $\mid a\rangle$ and $\mid b\rangle$ to be dipole forbidden. The transitions from the upper levels $\mid a\rangle$ and $\mid b\rangle$ to the lower level $\mid c\rangle$ produce two photons of different frequencies.

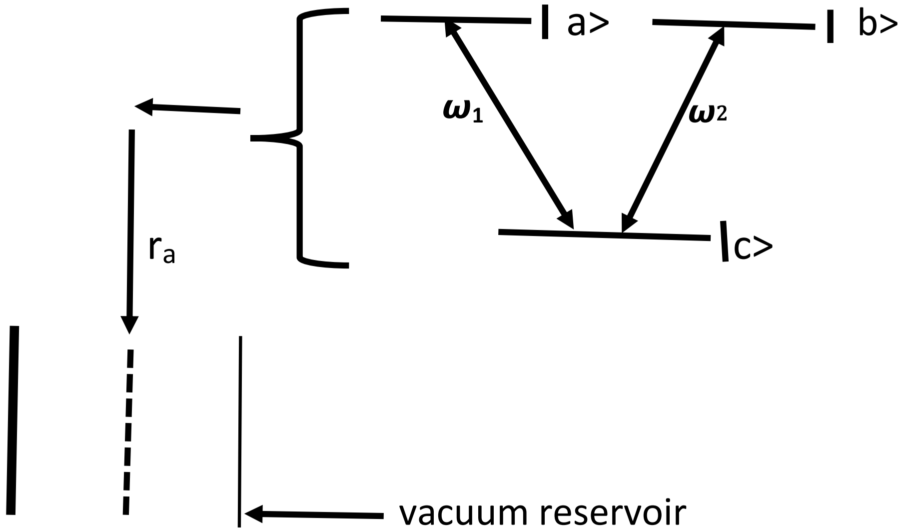

Fig.3.2 Schematic representation of a two-mode three-level atom in V configuration

The interaction of a single three-level atom with a two-mode radiation can be described in the interaction picture by the Hamiltonian is

$$\hat{H}_A = ig[\hat{a}^\dagger \mid c\rangle\langle a \mid -\hat{a} \mid a\rangle\langle c \mid +\hat{b}^\dagger \mid c\rangle\langle b \mid -\hat{b} \mid b\rangle\langle c \mid], \tag{3.137}$$

where g is the atom-radiation coupling constant ,$\hat{a}(\hat{b})$ are the annihilation operators for the cavity modes. We take the initial state of a three-level atom to be

$$\mid \Psi_A(0)\rangle = C_a(0) \mid a\rangle + C_b(0) \mid b\rangle \tag{3.138}$$

The density operator corresponding to this state for a single atom

$$\hat{\rho}_A(0) = \mid \psi_A(0)\rangle\langle\psi_A(0) \mid . \tag{3.139}$$

can be written as

$$\begin{aligned}\hat{\rho}_A(0) =& \hat{\rho}_{aa}^{(0)} \mid a\rangle\langle a \mid +\hat{\rho}_{ab}^{(0)} \mid a\rangle\langle b \mid +\hat{\rho}_{ba}^{(0)} \mid b\rangle\langle a \mid \\ &+ \hat{\rho}_{bb}^{(0)} \mid b\rangle\langle b \mid,\end{aligned} \tag{3.140}$$

where

$$\hat{\rho}_{aa}^{(0)} = \mid c_a(0)|^2, \hat{\rho}_{bb}^{(0)} = \mid c_b(0) \mid^2, \hat{\rho}_{ab}^{(0)} = c_a(0)c_b^*(0), and\hat{\rho}_{ba}^{(0)} = c_b(0)c_a^*(0). \tag{3.141}$$

in which $\hat{\rho}_{aa}^{(0)}$ and $\hat{\rho}_{bb}^{(0)}$ are the probability for the atom to be initially in the upper levels $\mid a\rangle$ and $\mid b\rangle$ respectively and $\hat{\rho}_{ab}^{(0)}$ and $\hat{\rho}_{ba}^{(0)}$ represent the atomic coherence at initial time.

Suppose $\hat{\rho}_{AR}(t, t_j)$ is the density operator for a single atom plus the cavity modes at time t,with the atom injected at time t_j such that extend from $t - \tau \leq t_j \leq t$. Then the density operator for all atoms in the cavity modes at time t can be written as [1]

$$\hat{\rho}_{AR}(t) = r_a \sum_j \hat{\rho}_{AR}(t, t_j)\Delta t, \tag{3.142}$$

where r_a is the rate of atoms injecting in to the cavity and $r_a\Delta t$ represents the number of atoms injected in to the cavity at time t_j . Then converting the summation in to integration in the limit $\Delta t \longrightarrow 0$, we have

$$\hat{\rho}_{AR}(t) = r_a \int_{t-\tau}^{t} \hat{\rho}_{AR}(t, t')dt'. \tag{3.143}$$

Upon differentiating with respect to t with the help of the Leibnitz rule

$$\frac{d}{dx}\int_p^q f(x, x')dx' = [f(x, q)\frac{dq}{dx} - f(x, p)\frac{dp}{dx}] + \frac{d}{dx}\int_p^q f(x, x')dx'$$

there follows

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a(\hat{\rho}_{AR}(t, t) - \hat{\rho}_{AR}(t, t-\tau)) + r_a \int_{t-\tau}^{t} \frac{d}{dt}\hat{\rho}_{AR}(t, t')dt'. \tag{3.144}$$

We observe that $\hat{\rho}_{AR}(t, t)$ is density operator for the cavity modes with an atom being injected in to the cavity at time t and $\hat{\rho}_{AR}(t, t-\tau))$ represents the density operator for an atom and the cavity mode at time t with the atom being removed from the cavity at this time. Since the cavity modes and the atom at time t are independents; the cavity operator $\hat{\rho}_{AR}(t, t)$ can be written as

$$\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(t, t)\hat{\rho}(t), \tag{3.145}$$

in which $\hat{\rho}(t)$ is the density operator for the cavity modes alone. The density operator $\hat{\rho}_{AR}(t, t-\tau)$ can also be expressed as

$$\hat{\rho}_{AR}(t, t-\tau)) = \hat{\rho}_A(t, t-\tau))\hat{\rho}(t) \tag{3.146}$$

Then by inserting Eq.(3.145) and Eq.(3.146) into Eq.(3.144), we obtain

$$\frac{d}{dt}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau)] + r_a \int_{t-\tau}^{t} \frac{\partial}{\partial t}\hat{\rho}_{AR}(t, t')dt', \tag{3.147}$$

where $\hat{\rho}_{AR}(t, t) = \hat{\rho}_A(0)$. In the absence of damping of the cavity modes by the vacuum reservoir, the density operator $\hat{\rho}_{AR}(t, t')$ evolves in time is

$$\frac{d}{dt}\hat{\rho}_{AR}(t, t') = -i[\hat{H}, \hat{\rho}_{AR}(t, t')]. \tag{3.148}$$

Now , on substituting Eq.(3.148) into Eq.(3.147) and using Eq.(3.143), we write as

$$\frac{\partial}{\partial t}\hat{\rho}_{AR}(t) = r_a[\hat{\rho}_A(0) - \hat{\rho}_A(t-\tau)]\hat{\rho}(t) - i[\hat{H}, \hat{\rho}_{AR}(t)]. \tag{3.149}$$

Since we are mainly interested in the density operator for the cavity modes alone,we trace Eq.(3.149) over the atomic variables. And taking in to account of Eq.(3.29), the damping of the cavity modes by a vacuum reservoir, we can obtain the equation of evolution of the reduced density operator for cavity modes coupled to a two-mode vacuum reservoir to be

$$\frac{d}{dt}\hat{\rho} = -iTr[\hat{H}, \hat{\rho}_{AR}(t)] + \frac{k}{2}(2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}) + \frac{k}{2}(2\hat{b}\hat{\rho}\hat{b}^{+} - \hat{b}^{+}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{+}\hat{b}), \tag{3.150}$$

where κ is the cavity damping constant assumed to be $\kappa_a = \kappa_b = \kappa$ for convenience (the same for both modes).

On substituting Eq.(3.137) into Eq.(3.150) and applying the cyclic property of trace operation, we can write in form

$$\begin{aligned}\frac{d}{dt}\hat{\rho} =& g[\hat{a}^{\dagger}\hat{\rho}_{ac} - \hat{\rho}_{ac}\hat{a}^{\dagger} + \hat{\rho}_{ca}\hat{a} - \hat{a}\hat{\rho}_{ca} + \hat{b}^{\dagger}\hat{\rho}_{bc} - \hat{\rho}_{bc}\hat{b}^{\dagger} + \hat{\rho}_{cb}\hat{b} - \hat{b}\hat{\rho}_{cb}] \\ &+ \frac{k}{2}(2\hat{a}\hat{\rho}\hat{a}^{+} - \hat{a}^{+}\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^{+}\hat{a}) + \frac{k}{2}(2\hat{b}\hat{\rho}\hat{b}^{+} - \hat{b}^{+}\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^{+}\hat{b}),\end{aligned} \tag{3.151}$$

in which the matrix element $\hat{\rho}_{\alpha\beta}$ are defined by

$$\hat{\rho}_{\alpha\beta} = \langle\alpha|\hat{\rho}_{AR}(t)|\beta\rangle \tag{3.152}$$

with

$$\alpha, \beta = a, b, c.$$

On the other way, taking Eq.(3.149), multiplying on the left side by $\langle\alpha|$ and on right hand side by $|\beta\rangle$, we obtain

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta} = r_a(\langle\alpha \mid \hat{\rho}_A(0)|\beta\rangle - \langle\alpha \mid \hat{\rho}_A(t-\tau) \mid \beta\rangle)\hat{\rho}(t) - i\langle\alpha \mid [\hat{H}, \hat{\rho}_{AR}(t)] \mid \beta\rangle - \gamma\hat{\rho}_{\alpha\beta}, \tag{3.153}$$

where the last term is included to account the damping (decay) of the atoms due to spontaneous emission. Here γ considered to be the same for all three-levels, the atomic decay rate. We assume that the atoms are removed from the cavity after they have decayed to the level other than the bottom level $|c\rangle$. so that

$$\langle\alpha|\hat{\rho}_A(t-\tau) \mid \beta\rangle = 0 \tag{3.154}$$

Hence Eq.(3.153) reduces to

$$\frac{d}{dt}\hat{\rho}_{\alpha\beta} = r_a\langle\alpha \mid \hat{\rho}(0) \mid \beta\rangle\hat{\rho}(t) - i\langle\alpha \mid \hat{H}\hat{\rho}_{AR}(t) \mid \beta\rangle + i\langle\alpha \mid \hat{\rho}_{AR}(t)\hat{H} \mid \beta\rangle - \gamma\hat{\rho}_{\alpha\beta} \tag{3.155}$$

so that applying Eqs.(3.137) and (3.137) in to Eq.(3.155), one can write as

$$\begin{aligned}\frac{d}{dt}\hat{\rho}_{\alpha\beta} =& r_a[\rho^{(0)}_{aa}\delta_{\alpha a}\delta_{a\beta} + \rho^{(0)}_{ab}\delta_{\alpha a}\delta_{b\beta} + \rho^{(0)}_{ba}\delta_{\alpha b}\delta_{a\beta} + \rho^{(0)}_{bb}\delta_{\alpha b}\delta_{b\beta}]\hat{\rho}(t)\\ &+ g[\hat{a}^{\dagger}\hat{\rho}_{a\beta}\delta_{\alpha c} - \hat{a}\hat{\rho}_{c\beta}\delta_{\alpha a} + \hat{b}^{\dagger}\hat{\rho}_{b\beta}\delta_{\alpha c} - \hat{b}\hat{\rho}_{c\beta}\delta_{\alpha b} - \hat{\rho}_{\alpha c}\hat{a}^{\dagger}\delta_{a\beta}\\ &+ \hat{\rho}_{\alpha a}\hat{a}\delta_{c\beta} - \hat{\rho}_{\alpha c}\hat{b}^{\dagger}\delta_{b\beta} + \hat{\rho}_{\alpha b}\hat{b}\delta_{c\beta}] - \gamma\hat{\rho}_{\alpha\beta}. \end{aligned} \tag{3.156}$$

with the aid of Eq.(3.156), we obtain

$$\frac{d}{dt}\hat{\rho}_{aa} = r_a\rho^{(0)}_{aa}\hat{\rho} - g(\hat{a}\hat{\rho}_{ca} + \hat{\rho}_{ac}\hat{a}^{\dagger}) - \gamma\hat{\rho}_{aa}, \tag{3.157}$$

$$\frac{d}{dt}\hat{\rho}_{bb} = r_a\rho^{(0)}_{bb}\hat{\rho} - g(\hat{b}\hat{\rho}_{cb} + \hat{\rho}_{bc}\hat{b}^{\dagger}) - \gamma\hat{\rho}_{bb}, \tag{3.158}$$

$$\frac{d}{dt}\hat{\rho}_{cc} = g(\hat{a}^{\dagger}\hat{\rho}_{ac} + \hat{\rho}_{ca}\hat{a} + \hat{b}^{\dagger}\hat{\rho}_{bc} + \hat{\rho}_{cb}\hat{b}) - \gamma\hat{\rho}_{cc}, \tag{3.159}$$

$$\frac{d}{dt}\hat{\rho}_{ab} = r_a\rho^{(0)}_{ab}\hat{\rho} - g(\hat{a}\hat{\rho}_{cb} + \hat{\rho}_{ac}\hat{b}^{\dagger}) - \gamma\hat{\rho}_{ab}, \tag{3.160}$$

$$\frac{\partial}{\partial t}\hat{\rho}_{ac} = g(\hat{\rho}_{aa}\hat{a} + \hat{a}\hat{\rho}_{cc} + \hat{\rho}_{ab}\hat{b}) - \gamma\hat{\rho}_{cc}, \tag{3.161}$$

$$\frac{d}{dt}\hat{\rho}_{bc} = g(\hat{\rho}_{ba}\hat{a} + \hat{b}\hat{\rho}_{cc} + \hat{\rho}_{bb}\hat{b}) - \gamma\hat{\rho}_{cc}. \tag{3.162}$$

Now we confine our selves to linear analysis of the system under consideration and this can be achieved by dropping the g terms in Eqs.(3.157)-(3.162). Dropping the g terms Eqs.(3.157), (3.158), (3.159), (3.160) and applying the adiabatic approximation scheme, we get

$$\hat{\rho}_{aa} = \frac{r_a\rho^{(0)}_{aa}\hat{\rho}}{\gamma}, \tag{3.163}$$

$$\hat{\rho}_{bb} = \frac{r_a\rho^{(0)}_{bb}\hat{\rho}}{\gamma}, \tag{3.164}$$

$$\hat{\rho}_{cc} = 0, \tag{3.165}$$

$$\hat{\rho}_{ab} = \frac{r_a\rho^{(0)}_{ab}\hat{\rho}}{\gamma}. \tag{3.166}$$

Substituting the above results in Eq.(3.160) and (3.161), we get

$$\frac{\partial}{\partial t}\hat{\rho}_{ac} = -\frac{gr_a}{\gamma}(\rho^{(0)}_{aa}\hat{\rho}\hat{a} + \rho^{(0)}_{ab}\hat{\rho}\hat{b}) - \gamma\hat{\rho}_{ac}, \tag{3.167}$$

$$\frac{\partial}{\partial t}\hat{\rho}_{bc} = -\frac{gr_a}{\gamma}(\rho_{bb}^{(0)}\hat{\rho}\hat{b} + \rho_{ab}^{(0)}\hat{\rho}\hat{b}) - \gamma\hat{\rho}_{bc}. \tag{3.168}$$

Applying once more the large time approximation scheme, we find

$$\hat{\rho}_{ac} = \frac{gr_a}{\gamma^2}(\rho_{aa}^{(0)}\hat{\rho}\hat{a} + \rho_{ab}^{(0)}\hat{\rho}\hat{b}), \tag{3.169}$$

$$\hat{\rho}_{bc} = \frac{gr_a}{\gamma^2}(\rho_{bb}^{(0)}\hat{\rho}\hat{a} + \rho_{ba}^{(0)}\hat{\rho}\hat{b}). \tag{3.170}$$

Finally on account the result in Eqs.(3.169) and (3.170) and their complex conjugate in to Eq.(3.151), the equation evolution of the density operator for the cavity modes takes the form

$$\begin{aligned}\frac{\partial}{\partial t}\hat{\rho} =& \frac{1}{2}A\rho_{aa}^{(0)}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger - \hat{a}\hat{a}^\dagger\hat{\rho}) + \frac{1}{2}A\rho_{ab}^{(0)}(2\hat{a}^\dagger\hat{\rho}\hat{b} - \hat{\rho}\hat{b}\hat{a}^\dagger - \hat{b}\hat{a}^\dagger\hat{\rho})\\ &+ \frac{1}{2}A\rho_{ba}^{(0)}(2\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{b}^\dagger) + \frac{1}{2}A\rho_{bb}^{(0)}(2\hat{b}^\dagger\hat{\rho}\hat{b} - \hat{b}\hat{b}^\dagger\hat{\rho} - \hat{\rho}\hat{b}\hat{b}^\dagger)\\ &+ \frac{1}{2}k(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}),\end{aligned} \tag{3.171}$$

in which we have

$$A = \frac{2g^2 r_a}{\gamma^2}. \tag{3.172}$$

Eq.(3.172) is the linear gain coefficient. The quantum properties of the light generated by the two-mode V laser are determined by the master equation (3.171)

3.2.6 C-number Langevin Equation

We here proceed, first to derive the equation of evolution for the expectation value of cavity mode operators in the normal ordering. We obtain c-number Langevin equations, associated with the normal ordering, for the dynamical variable of the cavity modes.

The master equation (3.171) allows us to derive the equations of evolution for the expectation values of cavity mode operators. Generally, the equations of evolutions for the expectation value of an operator $\hat{A}$ in the schrödinger picture is expressible as

$$\frac{d}{dt}\langle\hat{A}\rangle = T_r(\frac{d\rho}{dt}\hat{A}). \tag{3.173}$$

On account of Eq.(3.173) and (3.171), the expectation values of the operator $\hat{a}$ can be written as

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle =& T_r[\frac{1}{2}\rho_{aa}^{(o)}(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a}) + \frac{1}{2}\rho_{ab}^{(o)}(2\hat{a}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{\rho}\hat{b}\hat{a}^\dagger\hat{a})]\\ &+T_r[\frac{1}{2}\rho_{ba}^{(o)}(2\hat{b}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}) + \frac{1}{2}\rho_{bb}^{(o)}(2\hat{b}^\dagger\hat{\rho}\hat{b}\hat{a} - \hat{b}\hat{b}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}\hat{b}^\dagger\hat{a})]\end{aligned}$$

$$+T_r[\frac{1}{2}k(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a})].$$

Applying the cyclic property of the trace operation and the commutation relations, $[a_i, a_j^\dagger] = \delta_{ij}, [a_i, a_j] = 0, [a_i^2, a_j^\dagger] = 2a_i\delta_{ij}, [a_i^2, a_j^{\dagger 2}] = (2 + 4a_i^\dagger a_i)\delta_{ij}$ and $[\hat{a}, \hat{a}^\dagger] = 1$, with $\hat{a}_i$ being a and b, we get

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = -\frac{1}{2}\mu\langle\hat{a}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{b}(t)\rangle, \tag{3.174}$$

where

$$\mu = k - A\rho_{aa}^{(o)}. \tag{3.175}$$

Following the same procedure, it can be verified that

$$\frac{d}{dt}\langle\hat{b}(t)\rangle = -\frac{1}{2}\nu\langle\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{a}(t)\rangle, \tag{3.176}$$

where

$$\nu = k - A\rho_{bb}^{(o)}. \tag{3.177}$$

Furthermore, in view of (3.173) and (3.171), there follows

$$\frac{d}{dt}\langle\hat{a}^2(t)\rangle = -\mu\langle\hat{a}^2(t)\rangle + A\rho_{ab}^{(o)}\langle\hat{a}(t)\hat{b}(t)\rangle. \tag{3.178}$$

It can also be established in a similar way that

$$\frac{d}{dt}\langle\hat{b}^2(t)\rangle = -\nu\langle\hat{b}^2(t)\rangle + A\rho_{ba}^{(o)}\langle\hat{b}(t)\hat{a}(t)\rangle. \tag{3.179}$$

Moreover, taking in to account (3.173) and (3.171), we find

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}(t)\rangle = -\frac{1}{2}(\mu+\nu)\langle\hat{a}(t)\hat{b}(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{b}^2(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{a}^2(t)\rangle. \tag{3.180}$$

We can verify in a similar procedure that

$$\frac{d}{dt}\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle = -\frac{1}{2}(\mu+\nu)\langle\hat{a}(t)\hat{b}^\dagger(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)}[\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle + \langle\hat{b}^\dagger(t)\hat{b}(t)\rangle] + A\rho_{ab}^{(o)}, \tag{3.181}$$

$$\frac{d}{dt}\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle = -\mu\langle\hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{b}\hat{a}^\dagger\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\hat{b}^\dagger(t)\hat{a}\rangle + A\rho_{aa}^{(o)}, \tag{3.182}$$

$$\frac{d}{dt}\langle\hat{b}^\dagger\hat{b}\rangle = -\nu\langle\hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\hat{a}\hat{b}^\dagger\rangle + A\rho_{bb}^{(o)}. \tag{3.183}$$

We note that the c-number Langevin equations corresponding to Eqs.(3.174), (3.176), (3.178), (3.179), (3.180), (3.181), (3.182) and (3.183) which are in normal order are

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu\langle\alpha(t)\rangle + \frac{1}{2}A\rho_{ab}^{(o)\langle\beta(t)\rangle}, \tag{3.184}$$

$$\frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}\nu\langle\beta(t)\rangle + \frac{1}{2}A\rho_{ba}^{(o)\langle\alpha(t)\rangle}, \tag{3.185}$$

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + A\rho_{ab}^{(o)}\langle\alpha\beta\rangle, \tag{3.186}$$

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -\nu\langle\beta^2\rangle + A\rho_{ba}^{(o)}\langle\alpha\beta\rangle, \tag{3.187}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu+\nu)\langle\alpha\beta\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\beta^2\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha^2\rangle, \tag{3.188}$$

$$\frac{d}{dt}\langle\alpha\beta^*(t)\rangle = -\frac{1}{2}(\mu+\nu)\langle\alpha\beta^*\rangle + A\rho_{ab}^{(o)}[\langle\alpha^*\alpha\rangle + \langle\beta^*\beta\rangle] + A\rho_{ab}^{(o)}, \tag{3.189}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu\langle\alpha^*\alpha\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\beta\alpha^*\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\beta^*\alpha\rangle + A\rho_{aa}^{(o)}, \tag{3.190}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\nu\langle\beta^*\beta\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha^*\beta\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha\beta^*\rangle + A\rho_{bb}^{(o)}. \tag{3.191}$$

On the basis of Eqs.(3.184) and (3.185), we can write as

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu\alpha(t) + \frac{1}{2}A\rho_{ab}^{(o)}\beta(t) + f_\alpha(t), \tag{3.192}$$

$$\frac{d}{dt}\beta(t) = -\frac{1}{2}\nu\beta(t) + \frac{1}{2}A\rho_{ba}^{(o)}\alpha(t) + f_\beta(t), \tag{3.193}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are noise forces whose properties remain to be determined. Moreover, we see that Eqs. (3.192) and the expectation value of Eq.(3.192) as well as Eq.(3.193) and the expectation value of Eq.(3.193) will have the same forces if

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0. \tag{3.194}$$

We note that $\frac{d}{dt}\langle\alpha^2(t)\rangle = 2\langle\alpha(t)\frac{d}{dt}\langle\alpha(t)\rangle$. so that on substituting (3.192) in to this equation, we obtain

$$\frac{d}{dt}\langle\alpha^2(t)\rangle = -\mu\langle\alpha^2(t)\rangle + A\rho_{ab}^{(o)}\langle\alpha\beta\rangle + 2\langle\alpha f_\alpha(t)\rangle. \tag{3.195}$$

In a similar fashion, we can verify that

$$\frac{d}{dt}\langle\beta^2(t)\rangle = -\nu\langle\beta^2(2)\rangle + A\rho_{ba}^{(o)}\langle\alpha\beta\rangle + 2\langle\beta f_\beta(t)\rangle. \tag{3.196}$$

We also note that

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu\langle\alpha^*\alpha\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\frac{d}{dt}\alpha^*(t)\alpha(t)\rangle + \langle\alpha^*(t)\frac{d}{dt}\alpha(t)\rangle$$

. So that on account of Eq.(3.192) and its complex conjugate, we get

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu\langle\alpha^*\alpha\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha^*\beta\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha\beta^*\rangle + \langle f_\alpha^*\beta\rangle + \langle f_\alpha\alpha^*(t)\rangle. \tag{3.197}$$

In similar way, we can get

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\nu\langle\beta^*\beta\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\alpha^*\beta\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\beta^*\alpha\rangle + \langle f_\beta^*\beta(t)\rangle + \langle f_\beta\beta^*(t)\rangle. \tag{3.198}$$

In addition, we also note that

$$\frac{d}{dt}\langle\alpha\beta\rangle = \langle\frac{d}{dt}\alpha(t)\beta(t)\rangle + \langle\alpha(t)\frac{d}{dt}\beta(t)\rangle$$

.

It then follows from Eq.(3.192) and (3.193), we see that

$$\frac{d}{dt}\langle\alpha(t)\beta(t)\rangle = -\frac{1}{2}(\mu+\nu)\langle\alpha\beta\rangle + \frac{1}{2}A\rho_{ab}^{(o)}\langle\beta^2\rangle + \frac{1}{2}A\rho_{ba}^{(o)}\langle\alpha^2\rangle + \langle f_\alpha\beta(t)\rangle + \langle f_\beta\alpha(t)\rangle. \tag{3.199}$$

We can verify in similar fashion that

$$\frac{d}{dt}\langle\alpha\beta^*(t)\rangle = -\frac{1}{2}(\mu+\nu)\langle\alpha\beta^*\rangle + A\rho_{ab}^{(o)}[\langle\alpha^*\alpha\rangle + \langle\beta^*\beta\rangle] + \langle f_\alpha\beta^*\rangle + \langle f_\beta^*\alpha(t)\rangle. \tag{3.200}$$

Comparison of Eqs.(3.186) and (3.195) as well as Eqs.(3.187) and (3.193) will have the same forms provided that

$$\langle\alpha(t)f_\alpha(t)\rangle = 0 \tag{3.201}$$

and

$$\langle\beta(t)f_\beta(t)\rangle = 0. \tag{3.202}$$

Moreover, Eqs.(3.182) and (3.197), (3.183) and (3.198), (3.180) and (3.199), (3.181) and (3.200) will have the same forms if

$$\langle f_\alpha^*(t)\alpha(t)\rangle + \langle f_\alpha(t)\alpha^*(t)\rangle = A\rho_{aa}^{(o)}, \tag{3.203}$$

$$\langle f_\alpha^*(t)\beta(t)\rangle + \langle f_\beta(t)\beta^*(t)\rangle = A\rho_{bb}^{(o)}, \tag{3.204}$$

$$\langle f_\alpha(t)\beta(t)\rangle + \langle f_\beta(t)\alpha(t)\rangle = 0, \tag{3.205}$$

$$\langle f_\alpha(t)\beta^*(t)\rangle + \langle f_\beta^*(t)\alpha(t)\rangle = A\rho_{ab}^{(o)}. \tag{3.206}$$

The formal solutions of Eqs.(3.184) and (3.185) can be written as

$$\alpha(t) = \alpha(0)e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}[\frac{1}{2}A\rho_{ab}^{(0)}\beta(t') + f_\alpha(t')]dt'. \tag{3.207}$$

and

$$\beta(t) = \alpha(t)e^{-\frac{1}{2}\eta t} + \int_0^t e^{-\frac{1}{2}\nu(t-t')}[\frac{1}{2}A\rho_{ba}^{(0)}\alpha(t') + f_\beta(t')]dt'. \tag{3.208}$$

Then in view of Eq.(3.207), we see that

$$\langle\alpha(t)f_\alpha(t)\rangle = \langle\alpha(t)f_\alpha(t)\rangle e^{-\frac{1}{2}\mu t} + \int_0^t e^{-\frac{1}{2}\mu(t-t')}[\langle\beta(t')f_\alpha(t)\rangle + \langle f_\alpha(t')f_\alpha(t)\rangle]dt'. \tag{3.209}$$

Assuming that the noise force f at time t does not affect the cavity mode variables at earlier time t' and taking in to account (3.160), we have

$$\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\alpha(t')f_\alpha(t)\rangle dt' = 0.$$

It then follows from this result that

$$\langle f_\alpha(t')f_\alpha(t)\rangle = 0. \tag{3.210}$$

In similar procedure, we can get

$$\langle f_\beta(t)f_\beta(t')\rangle = 0. \tag{3.211}$$

Moreover employing (3.207) along with (3.203), we can find

$$\langle\alpha(0)f_\alpha^*\rangle e^{-\frac{1}{2}\mu t} + \langle f_\alpha\alpha^*(0)\rangle\int_0^t e^{-\frac{1}{2}\mu(t)} + \frac{1}{2}A\rho_{ab}^{(0)}\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\alpha^*(t)\beta(t')\rangle$$
$$+\frac{1}{2}A\rho_{ba}^{(0)}\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\alpha(t)\beta^*(t')\rangle + \int_0^t e^{-\frac{1}{2}\mu(t-t')}[\langle f_\alpha^*(t)f_\alpha(t')\rangle + \langle f_\alpha(t)f_\alpha^*(t')\rangle]dt' = A\rho_{aa}^{(0)}.$$

So that on recalling the noise force f at time t does not affect the cavity mode variables at earlier times, the above expression reduces to

$$\int_0^t e^{-\frac{1}{2}\mu(t-t')}[\langle f_\alpha^*(t)f_\alpha(t')\rangle + \langle f_\alpha(t)f_\alpha^*(t')\rangle]dt' = A\rho_{aa}^{(0)}. \tag{3.212}$$

We can write on the basis of this result

$$\langle f_\alpha^*(t)f_\alpha(t')\rangle = langle f_\alpha(t)f_\alpha^*(t')\rangle]dt' = A\rho_{aa}^{(0)}\delta(t-t'). \tag{3.213}$$

Following a similar fashion, we can easily verify that

$$\langle f_\beta^*(t)f_\beta(t')\rangle = langle f_\beta^*(t')f_\beta(t)\rangle = A\rho_{bb}^{(0)}\delta(t-t'), \tag{3.214}$$

$$\langle f_\alpha(t) f_\beta(t')\rangle = 0, \tag{3.215}$$

$$\langle f_\beta(t) f_\alpha(t')\rangle = 0. \tag{3.216}$$

Generally, application of Eqs.(3.205), (3.206) and (3.207) yields

$$\langle f_\alpha(t)\beta(0)\rangle e^{-\frac{1}{2}\nu t} + \frac{1}{2}A\rho_{ab}^{(0)}\int_0^t e^{-\frac{1}{2}\eta(t-t')}\langle f_\alpha(t)\alpha^*(t')\rangle dt' + \int_0^t e^{-\frac{1}{2}\nu(t-t')}\langle f_\alpha(t) f_\beta^*(t)\rangle dt'$$
$$+\langle f_\beta^*(t)\alpha(0)\rangle e^{-\frac{1}{2}\mu t} + \frac{1}{2}A\rho_{ab}^{(0)}\int_0^t e^{-\frac{1}{2}\mu(t-t')}\langle f_\beta^*(t)\beta(t')\rangle dt' + \int_0^t e^{-\frac{1}{2}\mu t}\langle f_\beta^*(t) f_\alpha(t')\rangle dt' = A\rho_{ab}^{(0)}. \tag{3.217}$$

In view of the fact that the noise force at time t does not affect the cavity mode variables at earlier times, there follows

$$\int_0^t e^{\frac{1}{2}\nu(t-t')}\langle f_\alpha(t) f_\beta^*(t')\rangle dt' + \int_0^t e^{\frac{1}{2}\nu(t-t')}\langle f_\beta^*(t) f_\alpha(t')\rangle dt' = A\rho_{ab}^{(0)}.$$

and on the basis of this result, one can write

$$\langle f_\alpha(t) f_\beta^*(t')\rangle dt' = \langle f_\beta^*(t) f_\alpha(t')\rangle = A\rho_{ab}^{(0)}\delta(t-t'). \tag{3.218}$$

We note that equations (3.210)-(3.216) and (3.218) describes the correlation properties of the noise forces f(t) associated with normal ordering.

We next proceed to obtain the solutions of coupled differential equations Eqs.(3.192) and (3.193) simultaneously.

To solve such coupled differential equations, we can write a single matrix equation

$$\frac{d}{dt}y(t) = My(t)x(t), \tag{3.219}$$

where

$$Q(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}, \tag{3.220}$$

$$M(t) = \begin{pmatrix} -\frac{\mu}{2} & \frac{A}{2}\rho_{ab}^{(o)} \\ \frac{A}{2}\rho_{ba}^{(o)} & -\frac{\eta}{2} \end{pmatrix}, \tag{3.221}$$

$$S(t) = \begin{pmatrix} f_\alpha(t) \\ f_\beta(t) \end{pmatrix}. \tag{3.222}$$

In order to solve (3.92), we need the eigen vectors and the eigen values of M such that

$$MV_i = \lambda_i V_i \qquad\qquad (i = 1, 2). \tag{3.223}$$

Let the egein vectors be

$$Vi = \begin{pmatrix} x_i \\ y_i \end{pmatrix}. \tag{3.224}$$

with the normalization condition

$$x_i^2 + y_i^2 = 1. \tag{3.225}$$

It can be shown using(3.94) and (3.96) that

$$\begin{vmatrix} -\frac{\mu}{2} - \lambda & \frac{A}{2}\rho_{ab}^{0} \\ \frac{A}{2}\rho_{ba}^{(o)} & -\frac{\nu}{2} - \lambda \end{vmatrix} = 0. \tag{3.226}$$

Then one readily obtains the eigen values to be

$$\lambda_1 = \frac{-(\mu+\nu) + \sqrt{(\mu-\nu)^2 + 4A^2 \mid \rho_{ab}^{(o)} \mid^2}}{4}. \tag{3.227}$$

$$\lambda_2 = \frac{-(\mu+\nu) - \sqrt{(\mu-\nu)^2 + 4A^2 \mid \rho_{ab}^{(o)} \mid^2}}{4}. \tag{3.228}$$

So that in view of (3.52) and (3.54) along with the fact that

$$\mid \rho_{ab}^{(o)} \mid^2 = \rho_{aa}^{(o)} \rho_{bb}^{(o)} \tag{3.229}$$

and

$$\rho_{aa}^{(o)} + \rho_{bb}^{(o)} = 1. \tag{3.230}$$

Eqs. (100) and (101) reduce to

$$\lambda_1 = -\frac{k-A}{2}. \tag{3.231}$$

$$\lambda_2 = -\frac{k}{2}. \tag{3.232}$$

with the aid of (3.96),(3.97) and (3.98), the corresponding eigen vectors are found to be

$$V_1 = \frac{1}{\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}}} \begin{pmatrix} \rho_{ab}^{(o)} \\ \rho_{bb}^{(o)} \end{pmatrix}. \tag{3.233}$$

$$V_2 = \frac{1}{\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}}} \begin{pmatrix} -\rho_{ab}^{(o)} \\ \rho_{aa}^{(o)} \end{pmatrix}. \tag{3.234}$$

We now construct a matrix V consisting of the eigen vectors of the matrix M as column matrices

$$V = (V_1 \qquad V_2).$$

That is

$$V = \begin{pmatrix} \frac{\rho_{ab}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{bb}^{(o)2}}} & \frac{-\rho_{ab}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{aa}^{(o)2}}} \\ \frac{\rho_{bb}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{bb}^{(o)2}}} & \frac{\rho_{aa}^{(o)}}{\sqrt{\rho_{ab}^{(o)2}+\rho_{aa}^{(o)2}}} \end{pmatrix}. \tag{3.235}$$

The inverse of V turns out to be

$$V^{-1} = \frac{1}{\rho_{ab}^{(o)}} \begin{pmatrix} \rho_{aa}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}} & \rho_{ab}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{bb}^{(o)2}} \\ -\rho_{bb}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}} & \rho_{ab}^{(o)}\sqrt{\rho_{ab}^{(o)2} + \rho_{aa}^{(o)2}} \end{pmatrix}. \tag{3.236}$$

Since $VV^{-1} = V^{-1}V = I$, we can rewrite Eq.(3.92) as

$$\frac{d}{dt}Q(t) = VV^{-1}MVV^{-1}Q(t) + S(t)$$

or

$$\frac{d(V^{-1}Q(t))}{dt} = DV^{-1}Q(t) + V^{-1}S(t), \tag{3.237}$$

where

$$D = V^{-1}MV = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{3.238}$$

The formal solution of Eq.(3.110) is

$$V^{-1}Q(t) = e^{Dt}V^{-1}Q(0) + \int_o^t e^{D(t-t^{'})}V^{-1}S(t^{'})dt^{'}. \tag{3.239}$$

Now multiplying EQ.(3.85) on the left by V it follows that

$$Q(t) = Ve^{Dt}V^{-1}Q(0) + \int_o^t Ve^{D(t-t^{'})}V^{-1}S(t^{'})dt^{'}. \tag{3.240}$$

In view of the fact that D is diagonal, e^{Dt} is simply equal to

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \tag{3.241}$$

and

$$e^{D(t-t^{'})} = \begin{pmatrix} e^{\lambda_1 t-t^{'}} & 0 \\ 0 & e^{\lambda_2 t-t^{'}} \end{pmatrix}. \tag{3.242}$$

Now we see that once V and V^{-1} are known, $\alpha(t)$ and $\beta(t)$ can be derived from Eq.(3.240) by straight forward matrix multiplication. Thus from Eqs.(3.235),(3.236),(3.241) and (3.242), one can find

$$Ve^{Dt}V^{-1}Y(0) = \begin{pmatrix} [\rho_{aa}^{(o)}e^{\lambda_1 t} + \rho_{bb}^{(o)}e^{\lambda_2 t}]\alpha(0) + \rho_{ab}^{(o)}[e^{\lambda_1 t} - e^{\lambda_2 t}]\beta(0) \\ \rho_{ab}^{(o)}[e^{\lambda_1 t} - e^{\lambda_2 t}]\alpha(0) + [\rho_{aa}^{(o)}e^{\lambda_1 t} + \rho_{bb}^{(o)}e^{\lambda_2 t}]\beta(0) \end{pmatrix} \tag{3.243}$$

and

$$\int_0^t Ve^{D(t-t')}V^{-1}X(t')\partial t' = \begin{pmatrix} \int_0^t[\rho_{aa}^{(o)}e^{\lambda_1(t-t')} + \rho_{bb}^{(o)}e^{\lambda_2(t-t')}]f_\alpha(t')dt' + \rho_{ab}^{(o)}\int_0^t[e^{\lambda_1 t} - e^{\lambda_2 t}]f_\beta(t')dt' \\ \rho_{ab}^{(o)}\int_0^t[e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}]f_\alpha(t')dt' + \int_0^t[\rho_{aa}^{(o)}e^{\lambda_1(t-t')} + \rho_{bb}^{(o)}e^{\lambda_2(t-t')}]f_\beta(t')dt' \end{pmatrix}. \tag{3.244}$$

Using Eqs.(3.240),(3.243) and (3.244), we can get

$$\alpha(t) = A_1(t)\alpha(0) + B_1\beta(0) + F_1(t) \tag{3.245}$$

and

$$\alpha(t) = A_2(t)\alpha(0) + B_2\beta(0) + F_2(t). \tag{3.246}$$

where

$$A_1(t) = \rho_{aa}^{(o)}e^{\lambda_1 t} + \rho_{bb}^{(o)}e^{\lambda_2 t}, \tag{3.247}$$

$$B_1(t) = \rho_{ab}^{(o)}(e^{\lambda_1 t} - e^{\lambda_2 t}), \tag{3.248}$$

$$A_2(t) = \rho_{ba}^{(o)}(e^{\lambda_1 t} - e^{\lambda_2 t}), \tag{3.249}$$

$$B_2(t) = \rho_{bb}^{(o)}e^{\lambda_1 t} + \rho_{aa}^{(o)}e^{\lambda_2 t}, \tag{3.250}$$

$$F_1(t) = \int_0^t [\rho_{aa}^{(o)}e^{\lambda_1(t-t')} + \rho_{bb}^{(o)}e^{\lambda_2(t-t')}]f_\alpha(t')dt' + \rho_{ab}^{(o)}\int_0^t [e^{\lambda_1 t} - e^{\lambda_2 t}]f_\beta(t')dt', \tag{3.251}$$

$$F_2(t) = \rho_{ab}^{(o)}\int_0^t [e^{\lambda_1(t-t')} - e^{\lambda_2(t-t')}]f_\alpha(t')dt' + \int_0^t [\rho_{aa}^{(o)}e^{\lambda_1(t-t')} + \rho_{bb}^{(o)}e^{\lambda_2(t-t')}]f_\beta(t')dt'. \tag{3.252}$$

These solutions are used to determine the Q-function, the mean and variances of photon number.

3.2.7 The Q-function for two-mode V-Type Three-Level Laser Light

Here we wish to obtain the Q function for the cavity modes produced by the system under consideration. The Q function of a two-mode light can be expressed as

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z\partial^2\eta\phi(z,\eta,t)e^{[z*\alpha(t) - z\alpha*(t) + \eta\beta*(t) - \eta*\beta(t)]} \tag{3.253}$$

From Eq.(3.253), with $\alpha(t)$ and $\beta(t)$ replaced by α and β, we have

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2z\partial^2\eta\phi(z,\eta,t)e^{[z*\alpha - z\alpha* + \eta\beta* - \eta*\beta]}, \tag{3.254}$$

where

$$\phi(z,\eta,t) = Tr[\hat{\rho}e^{-z\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-\eta\hat{b}}e^{\eta\hat{a}^\dagger}]. \tag{3.255}$$

is the anti-normally ordered characteristic function defined in the Heisenberg picture. Employing the Baker-Hausdorff identity, we can rewrite Eq.(3.256) in normal ordering as

$$\phi(z,\eta,t) = e^{-z^*z - \eta^*\eta}\langle e^{z\hat{a}^\dagger(t) - z^*\hat{a}(t)}e^{\eta\hat{a}^\dagger(t) - \eta^*\hat{a}(t)}\rangle \tag{3.256}$$

so that the corresponding c-number equation is

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta}\langle e^{z\alpha^*(t)-z^*\alpha(t)}e^{\eta\beta^*(t)-\eta^*\beta(t)}\rangle. \tag{3.257}$$

Now taking in to account Eqs.(3.246) and (3.247) along with their complex conjugate,Eq.(3.257) can be put in the form

$$\phi(z,\eta,t) = e^{-z^*z-\eta^*\eta}\langle e^{z\alpha^*(t)-z^*\alpha(t)}e^{\eta\beta^*(t)-\eta^*\beta(t)}\rangle, \tag{3.258}$$

where

$$\alpha(t) = A_1(t)\alpha(0) + B_1(t)\beta(0) + F_1(t) \tag{3.259}$$

and

$$\beta(t) = A_2(t)\alpha(0) + B_2(t)\beta(0) + F_2(t). \tag{3.260}$$

With the aid of Eqs.(3.232),(3.233),(3.234),(3.248),(3.250),(3.249),(3.2451),(3.252) and (3.253), we have

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}(k - A\rho^0_{aa})\langle\alpha(t)\rangle + \frac{1}{2}A\rho^0_{ab}\langle\beta(t)\rangle \tag{3.261}$$

and

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}(k - A\rho^0_{bb})\langle\beta(t)\rangle + \frac{1}{2}A\rho^0_{ba}\langle\alpha(t)\rangle. \tag{3.262}$$

We see that Eqs.(3.262) and (3.263) are linear differential equations for $\alpha(t)$ and $\beta(t)$. On the other hand, taking in Eqs.(3.252),(3.253),(3.214) and the assumption that the cavity modes are initially in vacuum state, we have

$$\langle\alpha(t)\rangle = \langle\beta(t)\rangle \tag{3.263}$$

Thus we observe $\alpha(t)$ and $\beta(t)$ are Gaussian variable with vanishing mean.On account of Eq.(3.259) can be expressed as

$$\phi(z,\eta,t) = e^{(-z^*z-\eta^*\eta)}exp[\frac{1}{2}\langle(z\alpha^*(t) - z^*\alpha(t)\eta\beta^*(t) - \eta^*\beta(t))^2\rangle] \tag{3.264}$$

which is equivalent to

$$\begin{aligned}\phi(z,\eta,t) =& exp(-z^*z - \eta^*\eta)exp[\frac{z^2}{2}\langle F_1^{*2}\rangle + \frac{z^2}{2}\langle F_1^2\rangle + \frac{z^2}{2}\eta^2\langle F_2^{*2}\rangle + \frac{z^2}{2}\eta^{*2}\langle F_2^{*2}\rangle - z^*z\langle F_1^*F_1\rangle \\ &+ z\eta\langle F_1^*F_2^*\rangle - z^*\eta\langle F_1^*F_2\rangle - z\eta^*\langle F_2^*F_1\rangle + z^*\eta^*\langle F_1F_1\rangle - \eta^*\eta\langle F_2^*F_2\rangle]\end{aligned} \tag{3.265}$$

Now in view of Eq.(3.260), we have

$$\langle\alpha^2(t)\rangle = \langle(A_1(t)\alpha(0) + B_1(t)\beta(0))^2\rangle + 2\langle(A_1(t)\alpha(0)F_1)^2\rangle + 2\langle(B_1(t)\beta(0)F_1)^2\rangle + \langle F_1^2\rangle \tag{3.266}$$

With the aid of Eqs.(3.252),(3.211) and (3.212), along with the assumption that the cavity modes are initially in a vacuum state and the fact that a noise force at a given instant time does not affect the cavity mode variables at earlier time, we obtain

$$\langle \alpha^2(t) \rangle = 0. \tag{3.267}$$

Using a similar procedure, we can get

$$\langle F_1^2(t) \rangle = \langle F_1^{*2}(t) \rangle = \langle F_2^2(t) \rangle = \langle F_2^{*2}(t) \rangle = 0, \tag{3.268}$$

$$\langle F_1 F_2(t) \rangle = \langle F_1^* F_2^*(t) \rangle = 0, \tag{3.269}$$

$$\langle f_\alpha^* f_\beta(t') \rangle = \langle F_1^* F_2 \rangle, \tag{3.270}$$

$$\langle f_\alpha^* f_\alpha(t') \rangle = \langle F_1^* F_1 \rangle, \tag{3.271}$$

$$\langle f_\beta^* f_\beta(t') \rangle = \langle F_2^* F_2 \rangle. \tag{3.272}$$

Now on account of Eqs.(3.268),(3.269),(3.270),(3.271) and (3.272), then the characteristic function of Eq(3.266), can be put in the form

$$\phi(z, \eta, t) = exp[-Uz^* z - Tz^* \eta - T' z\eta^* - V\eta^* \eta]. \tag{3.273}$$

From Eq.(3.273) with T' replaced by T^*, we have

$$\phi(z, \eta, t) = exp[-Uz^* z - Tz^* \eta - T^* z\eta^* - V\eta^* \eta], \tag{3.274}$$

where

$$U = 1 + \langle F_1^*(t) F_1(t) \rangle, \tag{3.275}$$

$$T = \langle F_1^*(t) F_2(t) \rangle, \tag{3.276}$$

$$T' = \langle F_1(t) F_2^*(t) \rangle, \tag{3.277}$$

$$V = 1 + \langle F_2^*(t) F_2(t) \rangle. \tag{3.278}$$

Furthermore, from Eq.(3.252) and its complex conjugate, then making use of Eqs.(3.214),(3.215),(3.219) and carrying out the integration, one easily find

$$\begin{aligned}\langle F_1^*(t)F_1(t)\rangle =&\frac{A}{2\lambda_1}[\rho_{aa}^{(0)3}+\rho_{bb}^0|\rho_{ab}^0|^2+2\rho_{aa}^0|\rho_{ab}^0|^2](e^{2\lambda_1 t}-1)\\ &\frac{A}{2\lambda_2}[\rho_{aa}^0\rho_{bb}^{(0)2}-\rho_{bb}^0|\rho_{ab}^{(0)}|^2](e^{2\lambda_2 t}-1)\\ &\frac{A}{\lambda_1+\lambda_2}[\rho_{aa}^{(0)2}\rho_{bb}^0-\rho_{aa}^0|\rho_{ab}^{(0)}|^2](e^{2\lambda_1 t}-1).\end{aligned} \tag{3.279}$$

This equation after employing Eqs.(3.230) and (3.231), takes the form

$$\langle F_1^*(t)F_1(t)\rangle = \frac{A\rho_{aa}^0}{2\lambda_1}(e^{2\lambda_1 t}-1). \tag{3.280}$$

Now for the sake of convenience, we introduce a new parameter defined by

$$\rho_{aa}^0-\rho_{bb}^0=\eta, \tag{3.281}$$

$$\rho_{aa}^0=\frac{1+\eta}{2}, \tag{3.282}$$

$$\rho_{bb}^0=\frac{1-\eta}{2} \tag{3.283}$$

and

$$|\rho_{ab}^0|=\frac{1}{2}(1-\eta)^{\frac{1}{2}}. \tag{3.284}$$

Applying Eq.(3.283) along with Eqs.(3.232) and (3.233), we rewrite Eq.(3.281) as

$$\langle F_1^*(t)F_1(t)\rangle = \frac{A(1+\eta)}{2(k-A)}[1-e^{-(k-A)t}]. \tag{3.285}$$

It can be established in a similar fashion that

$$\langle F_1^*(t)F_2(t)\rangle = \frac{A(1-\eta^2)^{\frac{1}{2}}e^{-i\theta}}{2(k-A)}[1-e^{-(k-A)t}], \tag{3.286}$$

$$\langle F_2^*(t)F_2(t)\rangle = \frac{A(1-\eta)}{2(k-A)}[1-e^{-(k-A)t}]. \tag{3.287}$$

where we have used Eqs.(3.284) and (3.285) and setting

$$\rho_{ab}^0=|\rho_{ab}^0|e^{i\theta} \tag{3.288}$$

with the aid of Eqs.(3.252),(3.253),(3.214),(3.216),(3.217) and (3.219), we can write Eqs.(3.275),(3.276) and (3.277) as

$$U=1+\frac{A(1+\eta)}{2(k-A)}[1-e^{-(k-A)t}], \tag{3.289}$$

$$T = \frac{A(1-\eta^2)^{\frac{1}{2}}e^{-i\theta}}{2(k-A)}[1-e^{-(k-A)t}], \tag{3.290}$$

$$V = 1 + \frac{A(1-\eta)}{2(k-A)}[1-e^{-(k-A)t}]. \tag{3.291}$$

Now using Eqs.(3.274) and (3.254), we have

$$Q(\alpha,\beta,t) = \frac{1}{\pi^4}\int d^2zd^2\eta exp[-Uz^*z-\alpha^*z+\alpha z^*]\times\int d^2\eta exp[-V\eta^*\eta+(\beta^*-z^*T)\eta-(\beta+T^*z)\eta^*]. \tag{3.292}$$

so that carrying out the integration with the help of Eq.(3.130), the Q function for the signal-idler modes turns out to be

$$Q(\alpha,\beta,t) = \frac{GM-H^*H}{\pi^2}exp[-G\alpha^*\alpha - H\alpha^*\beta - H^*\alpha\beta^* - M\beta^*\beta], \tag{3.293}$$

where

$$M = \frac{U}{VU-T^*T}, \tag{3.294}$$

$$H = \frac{T}{VU-T^*T}, \tag{3.295}$$

$$G = \frac{V}{VU-T^*T}. \tag{3.296}$$

Eq.(3.293) represent the Q-function for the signal-idler modes of the v-type three-level laser light.

The Q-function for a signal mode and idler mode

Next, we integrate Eq.(3.293) over β, we can get the Q-function for the signal-mode.

$$Q(\alpha,t) = \frac{GM-H^*H}{\pi^2}[\int d^2\beta exp(-M\beta^*\beta - H\alpha^*\beta - H^*\alpha\beta^*) - G\alpha^*\alpha]. \tag{3.297}$$

Eq.(3.294) can be rewritten as

$$Q(\alpha,t) = \frac{GM-H^*H}{\pi M}exp[(-\frac{GM-H^*H}{M})\alpha^*\alpha] \tag{3.298}$$

one can show that

$$GM - H^*H = \frac{1}{VU-T^*T} \tag{3.299}$$

and

$$\frac{GM-H^*H}{M} = \frac{1}{U}. \tag{3.300}$$

Therefore, substituting Eq.(3.297)in to Eq.(3.293), one readily get

$$Q(\alpha,t) = \frac{1}{\pi U} exp[(-\frac{\alpha^*\alpha}{U})]. \tag{3.301}$$

Using the same procedure the Q-function for the idler-mode can be written as

$$Q(\beta,t) = \frac{1}{\pi V} exp[(-\frac{\beta^*\beta}{V})]. \tag{3.302}$$

3.2.8 The Q function for superposed two-mode Parametric oscillator and V-type laser light.

In this section, we are interested to obtain the Q-function for the superposed two-mode parametric oscillator and V-type laser light beams.

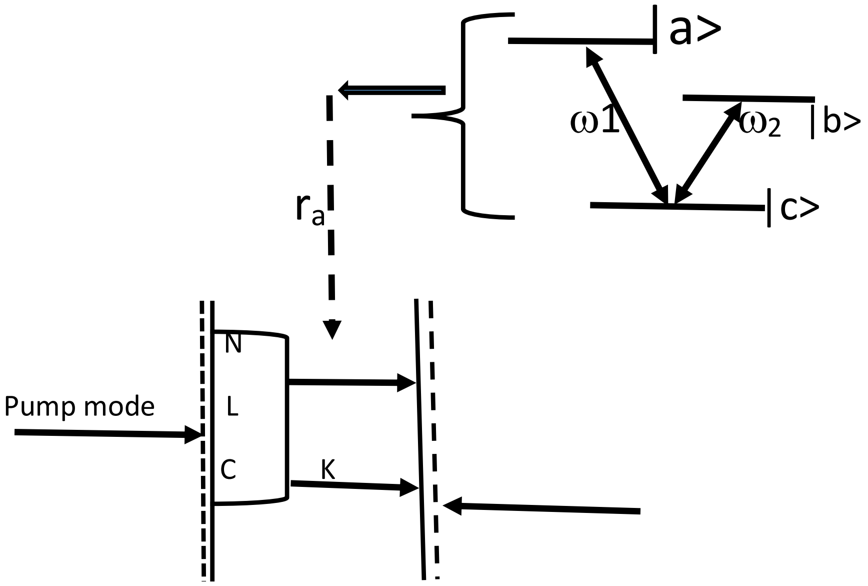

Fig.3.3, Schematic representation of superposed light produced by v-laser and parametric oscillator.

The Q-function for the superposed signal-signal and idler-idler mode light beams.

The Q-function for the superposed signal-signal modes are expressed as [1]

$$\begin{aligned} Q(\alpha^*,\alpha,t) =& \frac{1}{\pi}\int d^2\beta d^2\lambda Q_p(\lambda^*,\alpha-\beta)\times Q_v(\beta^*,\alpha-\lambda) \\ & exp[-\alpha^*\alpha-\beta^*\beta-\lambda^*\lambda+\alpha^*\beta+\alpha\beta^*+\alpha^*\lambda+\alpha\lambda^*-\beta^*\lambda-\beta\lambda^*] \end{aligned} \tag{3.303}$$

With the aid of Eqs.(3.135) and (3.298), one can write

$$Q_p(\lambda^*,\alpha-\beta) = (\frac{u^2-v^2}{\pi u})exp[-(\frac{u^2-v^2}{u})\alpha\lambda^* + (\frac{u^2-v^2}{u})\beta\lambda^*] \tag{3.304}$$

and

$$Q_v(\beta^*, \alpha - \lambda) = (\frac{GM - H^*H}{\pi M})exp[-(\frac{GM - H^*H}{M})\alpha\beta^* + (\frac{GM - H^*H}{M})\beta^*\lambda]. \tag{3.305}$$

where $Q_p(\lambda^*, \alpha - \beta)$ and $Q_v(\beta^*, \alpha - \lambda)$ represents the Q-function for the parametric oscillator and V-type three-level laser light respectively. With the aid of Eqs.(3.304) and (3.305), one can put Eq.(3.303) in the form

$$\begin{aligned} Q(\alpha^*, \alpha, t) =& \frac{(u^2 - v^2)(GM - H^*H)}{\pi^3 uM} \int d^2\beta d^2\lambda exp[-\alpha^*\alpha - \beta^*\beta - \lambda^*\lambda + \frac{(u^2 - v^2)}{u}\beta\lambda^* - (\frac{u^2 - v^2}{u})\alpha\lambda^* \\ &+ (\frac{GM - H^*H}{M})\beta^*\lambda - (\frac{GM - H^*H}{M})\alpha\beta^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\lambda + \alpha\lambda^* - \beta^*\lambda - \beta\lambda^*] \end{aligned} \tag{3.306}$$

In order to have a mathematically manageable analysis, we take

$$A = \frac{u^2 - v^2}{u} \tag{3.307}$$

$$B = \frac{GM - H^*H}{M} \tag{3.308}$$

With the aid of Eqs.(3.307) and (3.308), one can put Eq.(3.306) in the form

$$\begin{aligned} Q(\alpha^*, \alpha, t) =& \frac{AB}{\pi^3} \int d^2\beta d^2\lambda exp[-\alpha^*\alpha - \beta^*\beta - \lambda^*\lambda + A\beta\lambda^* - A\alpha\lambda^* + B\beta^*\lambda \\ &- B\alpha\beta^* + \alpha^*\beta + \alpha\beta^* + \alpha^*\lambda + \alpha\lambda^* - \beta^*\lambda - \beta\lambda^*] \end{aligned} \tag{3.309}$$

Eq.(3.309) can be rewritten as

$$\begin{aligned} Q(\alpha^*, \alpha, t) =& \frac{AB}{\pi^2}[\int d^2\beta exp(-\alpha^*\alpha - \beta^*\beta + \alpha^*\beta + \alpha\beta^* - B\alpha\beta^*) \\ &\times \int \frac{d^2\lambda}{\pi} exp(-\lambda^*\lambda + (B\beta^* + \alpha^* - \beta^*)\lambda + (AB - A\alpha + \alpha - \beta)\lambda^*)]. \end{aligned} \tag{3.310}$$

Integrating with respect to the variable λ using the relation given in Eq.(3.130), we get

$$Q(\alpha^*, \alpha, t) = \frac{AB}{\pi} e^{-A\alpha^*\alpha} \times \int \frac{d^2\beta}{\pi} exp[-(A + B - AB)\beta^*\beta + A\alpha^*\beta + (A\alpha - AB\alpha)\beta^*] \tag{3.311}$$

Finally, integrating over the variable β, we have

$$Q(\alpha^*, \alpha, t) = \frac{AB}{\pi(A + B - AB)} e^{-(\frac{AB}{A+B-AB})\alpha^*\alpha}. \tag{3.312}$$

Eq.(3.312) represent the Q-function for the superposed parametric oscillator and V-type three-level laser of signal-signal modes.

Using the same procedure, the Q-function for the superposed parametric oscillator and V-type three-level laser of idler-idler modes can be found

$$Q(\beta^*, \beta, t) = \frac{CD}{\pi(C + D - CD)} e^{-(\frac{CD}{C+D-CD})\beta^*\beta}, \tag{3.313}$$

where

$$C = \frac{u^2 - v^2}{v}, \tag{3.314}$$

$$D = \frac{GM - H^*H}{G}. \tag{3.315}$$

4

RESULT AND DISCUSSION

4.1 Photon Statistics for Parametric Oscillator

Here we seek to study, employing the Q-function, the photon statistics of the signal and idler modes.It would be help full to classify the photon statistics of light modes based on the relation between the variance and the mean of the photon number. Hence the photon statistics of a light mode for which $(\triangle n)^2 = \bar{n}$ is referred to as poissonian and the photon statistics of a light mode for which $(\triangle n)^2 > \bar{n}$ is called super-poissonian. Other wise the photon statistics is said to be sub-poissonian.[1]

4.1.1 The Mean of the Photon Number Sum and Difference

Now we proceed to determine the mean and the variance of the photon number sum and difference for the signal-idler modes by applying the Q-function. The photon number operators for the two-modes are defined

$$\hat{n_a} = \hat{a}^\dagger \hat{a}, \tag{4.1}$$

$$\hat{n_b} = \hat{b}^\dagger \hat{b}. \tag{4.2}$$

The photon annihilation and creation operators obey the commutation rule

$$[\hat{a}^\dagger, \hat{a}] = 1 \tag{4.3}$$

On account of Eq.(3.135), the mean photon number of the signal mode in terms of the Q-function is expressible as

$$\bar{n}_a = \int d^2\alpha Q(\alpha^*, \alpha, t)\hat{n}_a(\alpha) \tag{4.4}$$

where

$$n_a(\alpha) = \alpha^*\alpha - 1. \tag{4.5}$$

is the c-number variable corresponding to $\bar{n}_a(\hat{a}^\dagger\hat{a},)$ in the anti normal order. In view Eqs.(4.4) and (4.5), the mean photon number of the signal-mode is given by

$$\bar{n}_a = (\frac{u^2 - v^2}{\pi u})\int d^2\alpha exp[-(\frac{u^2 - v^2}{u})\alpha^*\alpha]\alpha^*\alpha - 1. \tag{4.6}$$

This equation can be rewritten as

$$\bar{n}_a = (\frac{u^2 - v^2}{u})\frac{dd}{dxdy}[\int \frac{d^2\alpha}{\pi}exp(-(\frac{u^2 - v^2}{u})\alpha^*\alpha + x\alpha^* + y\alpha)] - 1. \tag{4.7}$$

Up on carrying out the integration with the help of Eq.(3.130), we obtain

$$\bar{n}_a = \frac{dd}{dxdy}exp[\frac{uxy}{u^2 - v^2}]|_{x=y=0} - 1 \tag{4.8}$$

$$\bar{n}_a = \frac{u}{u^2 - v^2} - 1. \tag{4.9}$$

Hence, with the aid of Eqs.(3.132) and (3.133), one readily obtains

$$\bar{n}_a = a - 1. \tag{4.10}$$

so that on account of Eq.(3.127), there follows

$$\bar{n}_a = \frac{\varepsilon}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\varepsilon}{2\lambda_+}(1 - e^{-\lambda_+ t}). \tag{4.11}$$

We observe that at steady state the mean photon number reduces to

$$\bar{n}_a = \frac{\varepsilon}{2\lambda_-} - \frac{\varepsilon}{2\lambda_+}. \tag{4.12}$$

and in view of Eq.(3.86), we see that

$$\bar{n}_a = \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.13}$$

It can also be establish in a similar fashion that the mean photon number of the idler-mode is

$$\bar{n}_b = \frac{\varepsilon}{2\lambda_-}(1 - e^{-\lambda_- t}) - \frac{\varepsilon}{2\lambda_+}(1 - e^{-\lambda_+ t}). \tag{4.14}$$

Thus at steady state, introducing the value for $\lambda_\pm$, we see that

$$\bar{n}_b = \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.15}$$

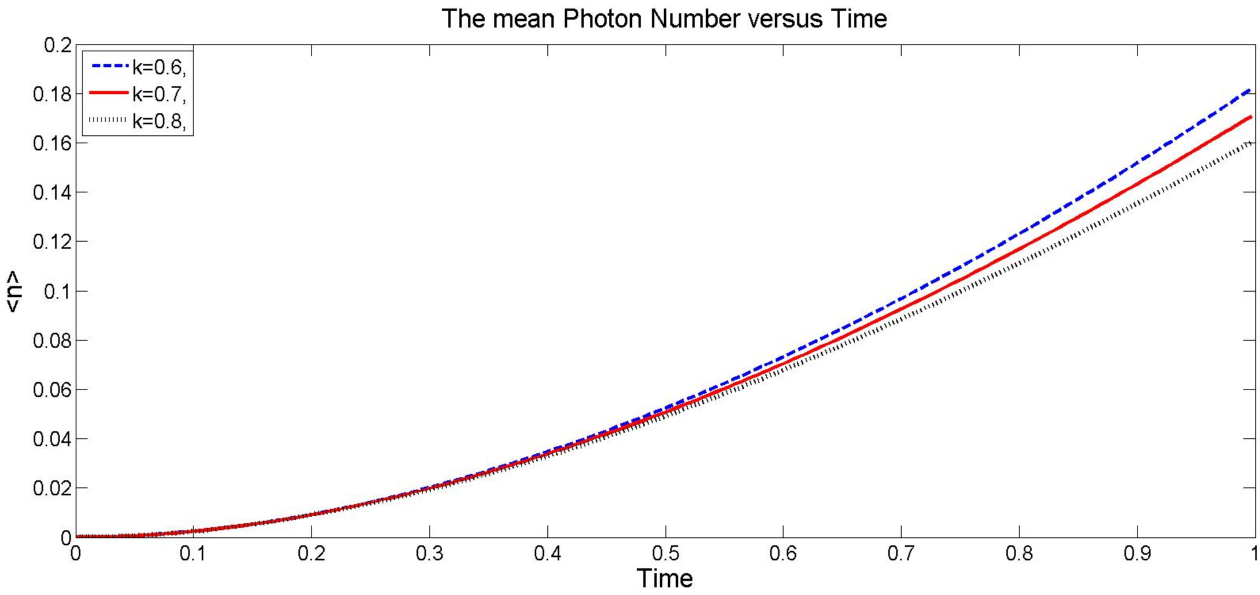

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Fig. 4.1: Plots of the mean photon number[Eq.(4.11) or (4.14)] versus time for $\epsilon = 0.2$, and different values of damping constant ,for κ = 0.8(Dotted curve), κ = 0.6(dash- curve) and κ = 0.7(Solid curve).

The plots in the Fig.4.1. indicates that the mean photon number for the cavity signal and idler modes are equal and increases with the time and decrease with the damping constant (κ)

We define the operators representing the photon number sum and differences of mode a and mode b by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b. \tag{4.16}$$

Up on taking the expectation value of Eq.(4.16), and applying Eqs.(4.13) and (4.15), the mean of photon number sum and differences can be written in the form

$$\bar{n}_{\pm} = \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2} \pm \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.17}$$

Then the mean of the photon number sum can be put at steady state in the form

$$\bar{n}_{+} = \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2} + \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.18}$$

In view of Eq.(4.18), we have

$$\bar{n}_{+} = \frac{4\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.19}$$

On the other hand, the mean of the photon number difference is given by

$$\bar{n}_{-} = \bar{n}_a - \bar{n}_b. \tag{4.20}$$

$$\bar{n}_a = \bar{n}_b \tag{4.21}$$

Then the mean of the photon number difference turns out to be

$$\bar{n}_- = 0. \tag{4.22}$$

4.1.2 The Variances of the Photon Number Sum and Differences

Next,we seek to obtain the variances of the photon number sum and difference. The variance of the photon number can be expressed as

$$(\triangle n_\pm)^2 = \langle n_\pm^2\rangle - \langle n_\pm\rangle^2 \tag{4.23}$$

In view of Eqs.(4.16) and (4.23), the variances of the photon number sum and difference can be put in the form

$$(\triangle n_\pm)^2 = \langle \bar{n}_a^2\rangle - \langle \bar{n}_a\rangle^2 + \langle \bar{n}_b^2\rangle - \langle \bar{n}_b\rangle^2 \pm 2[\langle \bar{n}_a\bar{n}_b\rangle - \langle \bar{n}_a\rangle\langle \bar{n}_b\rangle] \tag{4.24}$$

This can also be rewritten in the form

$$(\triangle n_\pm)^2 = (\triangle n_a)^2 + (\triangle n_b)^2 \pm 2[\langle \bar{n}_a\bar{n}_b\rangle - \bar{n}_a\bar{n}_b], \tag{4.25}$$

where

$$(\triangle n_a)^2 = \langle \bar{n}_a^2\rangle - \langle \bar{n}_a\rangle^2, \tag{4.26}$$

and

$$(\triangle n_b)^2 = \langle \bar{n}_b^2\rangle - \langle \bar{n}_b\rangle^2. \tag{4.27}$$

are the photon number variance for the separate mode. On the basis of Eq.(4.3), the photon number operator for mode a in the anti normal-order is obtained as

$$\hat{n}_a = \hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1, \tag{4.28}$$

which implies

$$\langle \hat{n}_a^2\rangle = \langle \hat{a}^2\hat{a}^{\dagger 2}\rangle - 3\langle \hat{a}^\dagger\hat{a}\rangle + 1. \tag{4.29}$$

On account of Eq.(4.3), the above expression can be rewritten as

$$\langle \hat{n}_a^2\rangle = \langle \hat{a}^2\hat{a}^{\dagger 2}\rangle - 3\langle \hat{a}^\dagger\hat{a}\rangle - 2 \tag{4.30}$$

Or

$$\langle \hat{n}_a^2\rangle = \langle \hat{a}^2\hat{a}^{\dagger 2}\rangle - 3\bar{n}_a - 2. \tag{4.31}$$

Substituting of Eq.(4.26)in to Eq.(4.31),we can get

$$(\triangle n_a)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2 \tag{4.32}$$

Using the Q-function of Eq.(3.135), we readily find

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u^2 - v^2}{u} \int \frac{d^2\alpha}{\pi} exp[-(\frac{u^2 - v^2}{u})\alpha^* \alpha]\alpha^{*2}\alpha^2 \tag{4.33}$$

which can be rewritten as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{u^2 - v^2}{u} \frac{dd}{dxdy}[\int \frac{d^2\alpha}{\pi} exp(-(\frac{u^2 - v^2}{u})\alpha^* \alpha + x\alpha^{*2} + y\alpha^2)]|_{y=x=0}, \tag{4.34}$$

so that performing the integration using Eq.(3.130), one readily obtains

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = (\frac{u^2 - v^2}{u}) \frac{dd}{dxdy}[\frac{1}{(\frac{u^2-v^2}{u}) - 4xy}]|_{x=y=0}. \tag{4.35}$$

Thus carrying out the differentiation and applying the condition $y = x = 0$, one can get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2a^2. \tag{4.36}$$

On account of Eq.(4.36) along with Eq.(4.10), we put into Eq.(4.32), we have

$$(\triangle \bar{n}_a)^2 = a^2 - a, \tag{4.37}$$

which can be rewritten as

$$(\triangle \bar{n}_a)^2 = \bar{n}_a^2 + \bar{n}_a. \tag{4.38}$$

Following the same procedure, one can readily establish that the variance of the photon number of the idler-mode is exactly the same as those of the signal-mode. From which follows

$$(\triangle \bar{n}_b)^2 = \bar{n}_b^2 + \bar{n}_b. \tag{4.39}$$

Eqs.(4.38) and (4.39), indicates that the signal-idler modes are in a chaotic state. Now on account of Eq.(3.126), we can find

$$(\triangle \bar{n}_a)^2 = \frac{\epsilon^2}{4\lambda_+^2}(1-e^{-\lambda_+ t})^2 + \frac{\epsilon^2}{4\lambda_-^2}(1-e^{-\lambda_- t})^2 - \frac{\varepsilon}{2\lambda_+}(1-e^{-\lambda_+ t}) + \frac{\varepsilon}{2\lambda_-}(1-e^{-\lambda_- t}) - \frac{\epsilon^2}{2\lambda_+\lambda_-}(1-e^{-\lambda_+ t})(1-e^{-\lambda_- t}). \tag{4.40}$$

Thus at steady state, it reduces to

$$(\triangle \bar{n}_a)^2 = (\frac{2\varepsilon^2}{k^2-4\varepsilon^2})^2 + (\frac{2\varepsilon^2}{k^2-4\varepsilon^2}.) \tag{4.41}$$

Similarly , at steady state, we also have

$$(\triangle \bar{n}_b)^2 = (\frac{2\varepsilon^2}{k^2-4\varepsilon^2}.)^2 + (\frac{2\varepsilon^2}{k^2-4\varepsilon^2}.) \tag{4.42}$$

Furthermore,on account of Eq.(3.131), one finds

$$\langle n_a n_b\rangle = (u^2 - v^2)\int \frac{d^2\alpha d^2\beta}{\pi^2} exp[-u(\alpha^*\alpha + \beta^*\beta) - v(\alpha\beta + \alpha^*\beta^*)]n_a(\alpha)n_b(\beta) \tag{4.43}$$

where

$$n_a(\alpha) = \alpha^*\alpha - 1, \tag{4.44}$$

$$n_b(\beta) = \beta^*\beta - 1. \tag{4.45}$$

are the c-number variables corresponding to the operator $\hat{n}_a$ and $\hat{n}_b$ in the anti normal order. Therefore, Eq.(4.43) can be rewritten as

$$\begin{aligned}\langle n_a n_b\rangle =&(u^2 - v^2)[\frac{1}{u^2}\frac{dd}{dxdy}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp(-xu\alpha^*\alpha - yu\beta^*\beta - v(\alpha\beta + \alpha^*\beta^*)) \\ &+ \frac{1}{u}\frac{d}{dx}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp(-xu\alpha^*\alpha - yu\beta^*\beta - v(\alpha\beta + \alpha^*\beta^*)) \\ &+ \frac{1}{u}\frac{d}{dy}\int \frac{d^2\alpha d^2\beta}{\pi^2} exp(-xu\alpha^*\alpha - yu\beta^*\beta - v(\alpha\beta + \alpha^*\beta^*))]|_{x=y=1} + 1,\end{aligned} \tag{4.46}$$

so that performing the integration using Eq.(3.130), we obtain

$$\langle n_a n_b\rangle = (u^2 - v^2)[\frac{1}{u^2}\frac{dd}{dxdy}(\frac{1}{xyu^2 - v^2}) + \frac{1}{u}\frac{d}{dx}(\frac{1}{xyu^2 - v^2}) + \frac{1}{u}\frac{d}{dy}(\frac{1}{xyu^2 - v^2})]|_{x=y=1} + 1. \tag{4.47}$$

Thus carrying out the differentiation and applying the condition a=b=1, one readily obtains

$$\langle n_a n_b\rangle = \frac{u^2 - v^2}{(u^2 - v^2)^2} - \frac{2u}{u^2 - v^2} + 1. \tag{4.48}$$

Introducing Eqs.(3.131) and (3.132) in to (4.48), we readily obtains

$$\langle n_a n_b\rangle = a^2 + b^2 - 2a + 1 = \bar{n}_a^2 + b^2. \tag{4.49}$$

Hence, the result of the mean and variance of the photon number for the signal-mode are the the same as those of the idler-mode. Therefore, the variance of the photon number sum and difference defined by Eq.(4.25), can be rewritten in the form

$$(\triangle n_\pm)^2 = 2(\triangle n_a)^2 \pm (\langle n_a n_b\rangle - \bar{n}^2). \tag{4.50}$$

In view of Eq.(4.37) and (4.49), we readily obtain

$$(\triangle n_\pm)^2 = 2[a^2 - a \pm b^2]. \tag{4.51}$$

Finally, applying Eqs.(4.49) and (4.50), the variance of the photon number sum is obtained as

$$(\triangle n_+)^2 = \frac{\varepsilon^2}{\lambda_+^2}(1 - e^{\lambda_+ t})^2 + \frac{\varepsilon^2}{\lambda_-^2}(1 - e^{\lambda_- t})^2 + \frac{\varepsilon}{\lambda_-}(1 - e^{\lambda_- t}) - \frac{\varepsilon}{\lambda_+}(1 - e^{\lambda_+ t}). \tag{4.52}$$

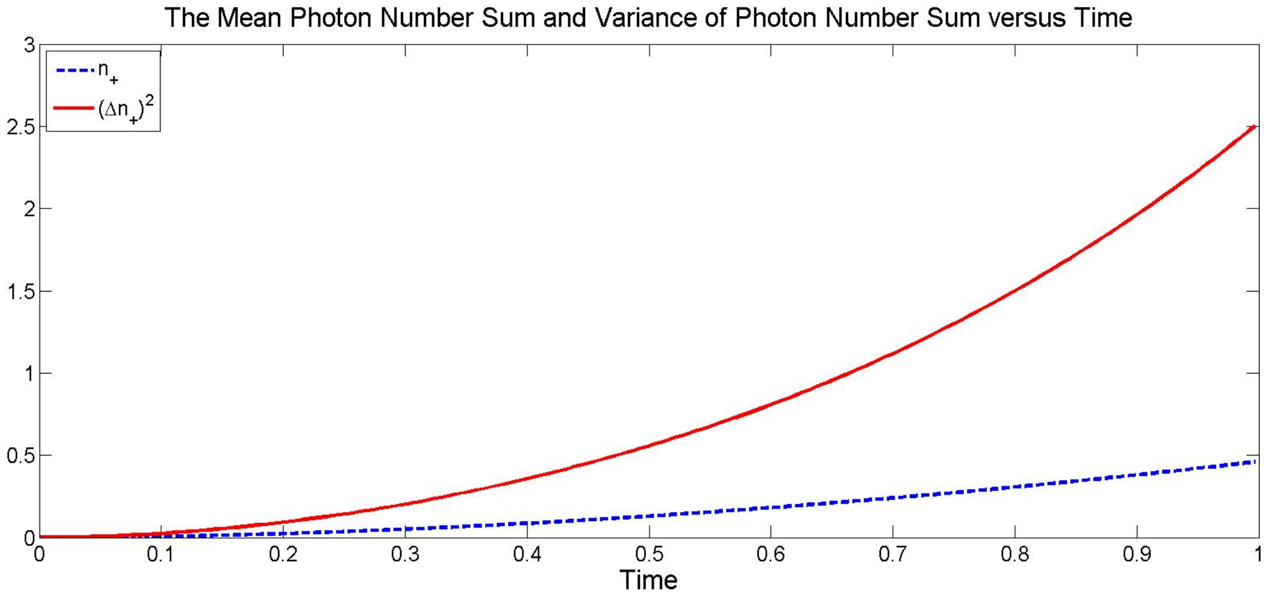

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Fig. 4.2: Plots of the variance of photon number[Eq.(4.53) and (4.19)] versus time for $\kappa = 0.2, \epsilon = 0.5$, and the mean photon number sum and variance of photon number sum ,for $\bar{n}_+$(dash curve) and $(\Delta n_+)^2$ (Solid curve).

The sum of the photon number variance takes at steady state take the form

$$(\triangle n_+)^2 = \frac{2\varepsilon^2(3k^2 - 4\varepsilon^2)}{(k^2 - 4\varepsilon^2)^2}. \tag{4.53}$$

From Fig.4.2 we observe that the variance of the photon number sum is greater than the mean of the photon number sum. Furthermore, we have also observed that the photon number statistics is super-Poissonian.

The variances of the photon number difference can be found

$$(\triangle n_-)^2 = \frac{\varepsilon^2}{\lambda_-^2}(1 - e^{\lambda_- t})^2 - \frac{\varepsilon^2}{\lambda_+^2}(1 - e^{\lambda_+ t})^2 - \frac{2\varepsilon^2}{\lambda_+\lambda_-}(1 - e^{\lambda_+ t})(1 - e^{\lambda_- t}). \tag{4.54}$$

Thus the variance of the photon number difference at steady state takes the form

$$(\triangle n_-)^2 = \frac{2\varepsilon^2}{k^2 - 4\varepsilon^2}. \tag{4.55}$$

The plots in the Fig.4.3, indicates that the variance of photon number difference for the cavity mode increases with time and pump mode (ϵ).

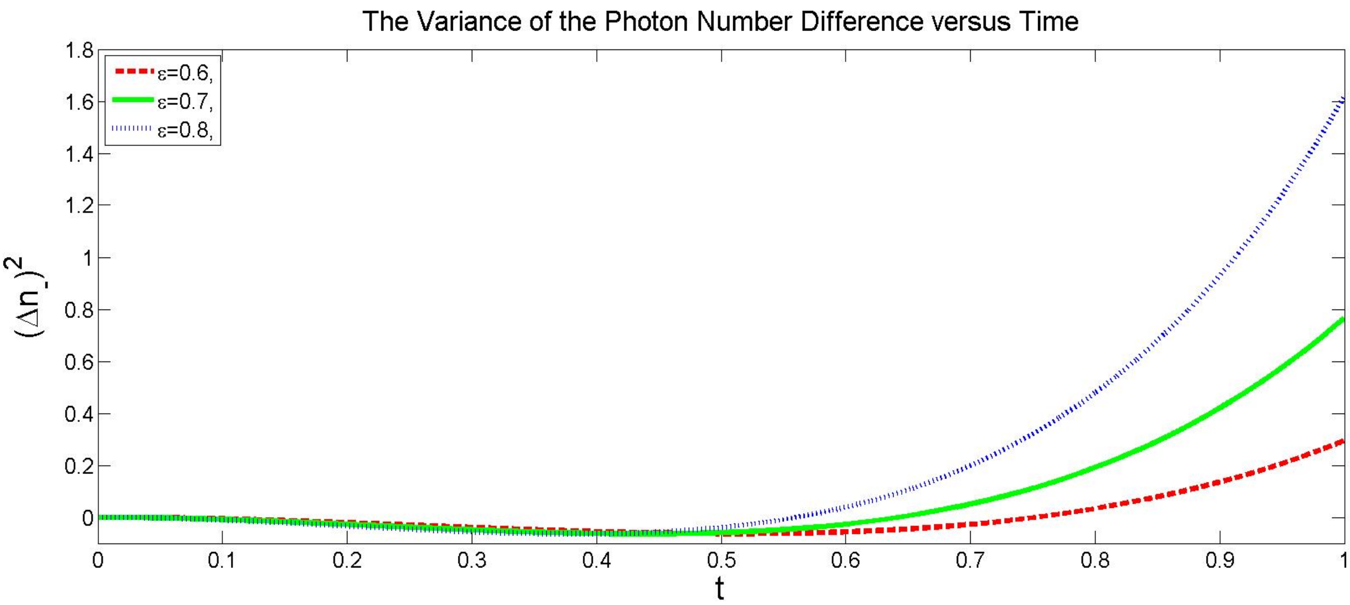

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Fig. 4.3: Plots of the variance of photon number[Eq.(4.54)] versus time for $\kappa = 0.8$, and different values of amplitude of pump mode ,for ϵ = 0.8(Dotted curve), ϵ = 0.6(dash curve) and ϵ = 0.7(Solid curve).

4.2 Photon Statistics For V-Type Laser Light

Here, we seek to study, employing the Q-function, the statistical properties of the signal-mode and idler-modes.

4.2.1 The Mean of the Photon Number Sum and Difference.

Now we proceed to determine the mean and the variance of the photon number sum and difference for signal-idler modes by applying the Q-function.The photon annihilation and creation operators obey the commutation relation

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \tag{4.56}$$

The photon number operators for the two-modes are defined by

$$\hat{n}_a = \hat{a}^{\dagger}\hat{a} \tag{4.57}$$

and

$$\hat{n}_b = \hat{b}^{\dagger}\hat{b}, \tag{4.58}$$

where $\bar{n}_a$ and $\bar{n}_b$ are the photon number operator for mode a and mode b . The mean photon number for a can be written in terms of the Q-function as

$$\bar{n}_a = \int d^2\alpha d^2\beta Q(\alpha,\beta,t)(\alpha^*\alpha - 1). \tag{4.59}$$

On account of Eq.(3.288), we see that

$$\bar{n}_a = (FG - H^*H)\int \frac{d^2\alpha d^2\beta}{\pi^2} exp[-G\alpha^*\alpha + H^*\alpha^*\beta + H\alpha\beta^* - M\beta^*\beta](\alpha^*\alpha - 1). \tag{4.60}$$

This equation can be rewritten as

$$\bar{n}_a = \frac{(FG - H^*H)}{\pi^2}\frac{dd}{dxdy}\int d^2\alpha d^2\beta exp[-G\alpha^*\alpha + x\alpha + y\alpha^* + H^*\alpha^*\beta + H\alpha\beta^* - M\beta^*\beta]|_{x=y=0} - 1. \tag{4.61}$$

Up on carrying out the integration with the help of Eq.(3.130), we obtain

$$\bar{n}_a = (FG - H^*H)\frac{dd}{dxdy}exp[\frac{Gxy}{(MG - H^*H)}]|_{x=y=0} - 1. \tag{4.62}$$

Performing the differentiation and putting the condition a=b=0, readily obtains

$$\bar{n}_a = \frac{M}{(FG - H^*H)} - 1. \tag{4.63}$$

Introducing Eq.(3.289),(3.290) and (3.291) in to Eq.(4.62), we obtain

$$\bar{n}_a = U - 1, \tag{4.64}$$

so that on account of Eq.(3.284), there follows

$$\bar{n}_a = \frac{A(1+\eta)}{2(k-A)}[1 - e^{-(k-A)t}]. \tag{4.65}$$

The mean photon number at steady state reduces to

$$\bar{n}_a = \frac{A(1+\eta)}{2(k-A)}. \tag{4.66}$$

It can also establish in a similar fashion that the mean photon number of the idler-mode is found to be

$$\bar{n}_b = \frac{A(1-\eta)}{2(k-A)}[1 - e^{-(k-A)t}]. \tag{4.67}$$

Thus at steady state, the mean photon number of the idler-mode reduces to

$$\bar{n}_b = \frac{A(1-\eta)}{2(k-A)}. \tag{4.68}$$

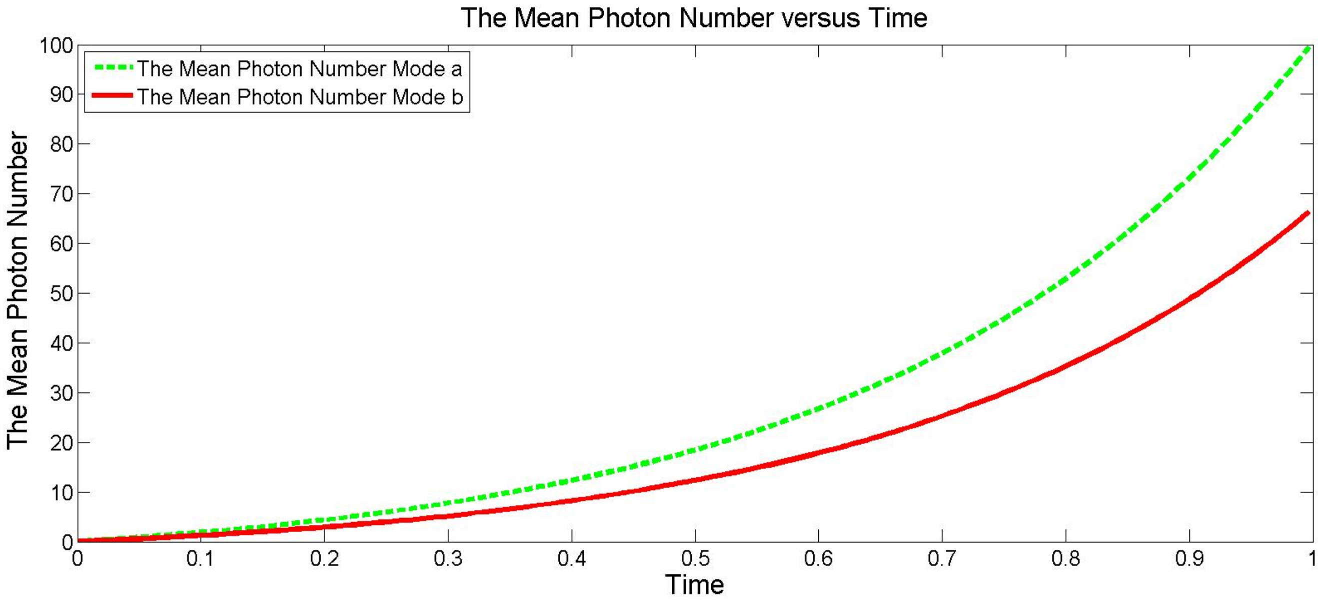

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Fig. 4.4: Plots of the mean photon number[Eq.(4.66) and (4.63)] versus Time for $\kappa = 0.2, \eta = 0.5$, the value of linear gain coefficient for A=3 and mode a (dash curve) and mode b(solid curve) .

The plots in the fig.4.4, indicates that the mean photon number for the cavity mode a increases with initial preparation of three-level atom (η) and time,the mode b increases with time and decreases with initial preparation of three-level atom (η). This implies that η, increases, the mean photon number of mode a increases and the mean photon number of mode b decreases.

From Eq.(4.66) and Eq.(4.68),It is possible to see that for $\eta = 0$, $\bar{n}_a = \bar{n}_b$. Moreover as η, increases, the mean photon number of mode a increases and the mean photon number of mode b decreases.

We define the operators representing the photon number sum and difference of mode a and mode b by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b. \tag{4.69}$$

Up on taking the expectation value of Eq.(4.69) applying Eqs.(4.66) and (4.68), the mean of the photon number sum and difference can be written in the form

$$\bar{n}_{\pm} = \frac{A(1+\eta)}{2(k-A)} \pm \frac{A(1-\eta)}{2(k-A)}. \tag{4.70}$$

Then the mean of the photon number sum can be put at steady state in the form

$$\bar{n}_{+} = \frac{A(1+\eta)}{2(k-A)} + \frac{A(1-\eta)}{2(k-A)}. \tag{4.71}$$

In view of Eq.(4.70), we have

$$\bar{n}_+ = \frac{A}{k - A}. \tag{4.72}$$

On the other hand, the mean the photon number of difference is given by

$$\bar{n}_- = \bar{n}_a - \bar{n}_b \tag{4.73}$$

Then the mean of the photon number difference can be put at steady state in the form

$$\bar{n}_- = \frac{A(1+\eta)}{2(k-A)} - \frac{A(1-\eta)}{2(k-A)} \tag{4.74}$$

In view of Eq.(4.74), we have

$$\bar{n}_- = \frac{A\eta}{k - A} \tag{4.75}$$

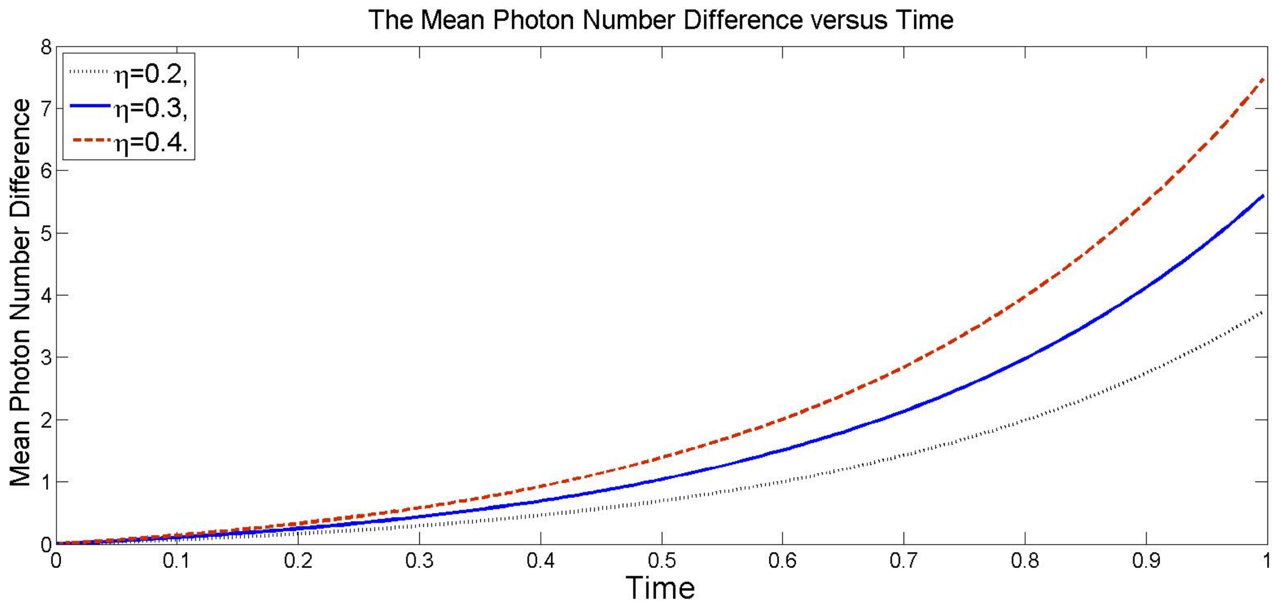

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Fig. 4.5: Plots of the mean photon number[Eq.(4.75)] versus Time for $\kappa = 0.2$,the value of linear gain coefficient for, A=5 and different values of initial preparation of three-level atom,for $\eta = 0.4$(dash curve), $\eta = 0.2$(dot curve) and $\eta = 0.3$(solid line) .

The plots in the fig.4.5, indicates that the mean photon number difference for the cavity mode increases with time and pump mode (η)

From Eq.(4.75), the mean photon number difference increases as η increases and zero at $\eta = 0$.

4.2.2 The Variance of the Photon Number Sum and Difference for a V-Type Laser Light

Hence, we seek to obtain the variances of the photon number sum and difference. The variances of the photon number sum and differences are given by

$$\triangle n_{\pm}^2 = \langle n_{\pm}^2 \rangle - \langle n_{\pm} \rangle^2. \tag{4.76}$$

In view of Eqs.(4.69) and (4.76), the variances of the photon number sum and difference can be put in the form

$$(\triangle n_{\pm})^2 = \langle n_a^2 \rangle - \langle n_a \rangle^2 + \langle n_b^2 \rangle - \langle n_b \rangle^2 \pm 2[\langle n_a n_b \rangle - n_a n_b]. \tag{4.77}$$

This can be also rewritten in the form

$$(\triangle n_{\pm})^2 = (\triangle n_a)^2 + (\triangle n_b)^2 \pm 2[\langle n_a n_b \rangle - n_a n_b] = (\triangle n_a)^2 + (\triangle n_b)^2 \pm 2n_{ab}, \tag{4.78}$$

where

$$(\triangle n_a)^2 = \langle n_a^2 \rangle - \langle n_a \rangle^2, \tag{4.79}$$

$$(\triangle n_b)^2 = \langle n_b^2 \rangle - \langle n_b \rangle^2. \tag{4.80}$$

are the photon number variances for the separate mode. On the basis of (4.56), the photon number for mode a in the anti normal-order is obtained as

$$\hat{n}_a = \hat{a}^\dagger \hat{a} = \hat{a}\hat{a}^\dagger - 1, \tag{4.81}$$

which implies

$$\langle n_a^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}^\dagger \hat{a} \rangle + 1. \tag{4.82}$$

On account of Eq.(4.56), the above expression can be rewritten as

$$\langle n_a^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - n_a^2 - 3n_a - 2. \tag{4.83}$$

The first term on the write side of Eq.(4.83), expressible in terms of Q-function as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int d^2\alpha d^2\beta Q(\alpha, \beta, t) \alpha^{*2} \alpha^2. \tag{4.84}$$

On account of Eq.(3.288), we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{MG - H^* H}{\pi^2} \int dl^2\alpha d^2\beta exp[-G\alpha^*\alpha + H^*\alpha^*\beta + H\alpha\beta^* - M\beta^*\beta] \alpha^{*2} \alpha^2. \tag{4.85}$$

This equation can be rewritten as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{(FG - H^*H)}{\pi^2} \frac{dd}{dxdy} \int d^2\alpha d^2\beta exp[-G\alpha^*\alpha + H^*\alpha^*\beta + H\alpha\beta^* - M\beta^*\beta + x\alpha^2 + y\alpha^{*2}]|_{x=y=0}, \tag{4.86}$$

so that performing the integration using Eq.(3.130), one readily obtains

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = (MG - H^*H) \frac{dd}{dxdy} [((FG - H^*H)^2 - 4M^2xy)^{\frac{-1}{2}}]. \tag{4.87}$$

Thus carrying out the differentiation and applying the condition y=x=0, one readily get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2(\frac{M}{MG - H^*H})^2. \tag{4.88}$$

With the aid of Eq.(4.63), one can write as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2(\bar{n}_a + 1)^2. \tag{4.89}$$

Therefore, of Eq.(4.90) into Eq.(4.84), we have

$$\Delta n_a^2 = \bar{n}_a^2 + \bar{n}_a. \tag{4.90}$$

Following the same procedure, one can readily establish that the variance of the photon number of the idler-mode is put in the form

$$\triangle n_b^2 = \bar{n}_b^2 + \bar{n}_b \tag{4.91}$$

and

$$\bar{n}_{ab} = \bar{n}_a \bar{n}_a + \bar{n}_a + \bar{n}_b + 1 - \frac{1}{GM - H^*H}. \tag{4.92}$$

With the aid of Eqs.(3.284),(3.285),(3.286),(3.289),(3.290) and (3.291), we can write

$$\Delta n_a^2 = \frac{A^2(1+\eta)^2}{4(k-A)^2} + \frac{A(1+\eta)}{2(K-A)}, \tag{4.93}$$

$$\Delta n_b^2 = \frac{A^2(1-\eta)^2}{4(k-A)^2} + \frac{A(1-\eta)}{2(K-A)}, \tag{4.94}$$

$$n_{ab} = \frac{A^2(1+\eta)^2 + 4A\eta(k-A) - A^2(1-\eta^2)}{4(k-A)^2}. \tag{4.95}$$

Now substituting Eqs.(4.94),(4.95) and (4.96) into Eq.(4.78), we get the variance of the photon number sum and difference to be

$$\triangle n_\pm^2 = \frac{A^2(1+\eta)^2}{4(k-A)^2} + \frac{A(1+\eta)}{2(K-A)} + \frac{A^2(1-\eta)^2}{4(k-A)^2} + \frac{A(1-\eta)}{2(K-A)} \pm 2\frac{A^2(1+\eta)^2 + 4A\eta(k-A) - A^2(1-\eta^2)}{4(k-A)^2}, \tag{4.96}$$

which is rewritten as

$$\begin{aligned}\Delta n_{\pm}^{2} =& \frac{A^2(1+\eta)^2 + 2A(1+\eta)(k-A)}{4(k-A)^2} + \frac{A^2(1-\eta)^2 + 2A(1-\eta)(k-A)}{4(k-A)^2} \\ &\pm 2\frac{4(k-A)^2 + A^2(1-\eta^2) + 2A(k-A)(1+\eta) + 2A(k-A)(1-\eta)}{4(k-A)^2}.\end{aligned} \tag{4.97}$$

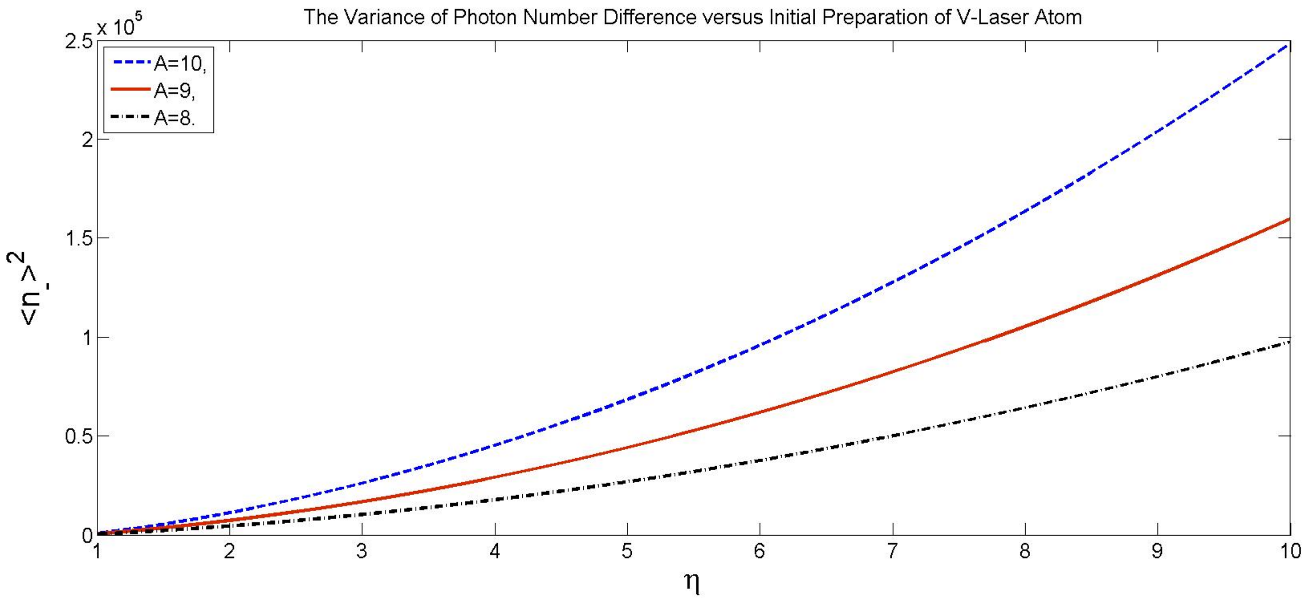

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Fig. 4.6: Plots of the variance of photon number[Eq.(4.97)] versus η for $\kappa = 0.8$, and different values of linear gain coefficient, for A= 10(dash curve), A = 9(dash-dot curve) and A = 8(Solid curve).

The plots in the Fig.4.6, indicates that the variance of the photon number difference for the cavity mode increases with linear gain coefficient(A) and initial preparation of three-level atom (η)

4.3 Photon Statistics for the Superposed Light Beams

Here we seek to study, employing the Q-function, the statistical properties of the superposed light beams.

4.3.1 The Mean Photon Number Sum and Difference

We now proceed to calculate the mean photon number sum and differences of the superposed parametric oscillator and V-type three-level laser light beams. To this end, the mean photon number for signal modes for the superposition of the parametric oscillator and V-type three-level laser light can be written in terms of the Q-function as [1]

$$\bar{n}_a = \int d^2\alpha Q(\alpha^*, \alpha, t)(\alpha^*\alpha - 1) \tag{4.98}$$

Taking in to account Eq.(3.312), the mean photon number can be put in the form

$$\bar{n}_a = \frac{AB}{\pi(A+B-AB)} \int d^2\alpha exp[-(\frac{AB}{A+B-AB})\alpha^*\alpha]\alpha^*\alpha - 1. \tag{4.99}$$

One can write the above equation in the form

$$\bar{n}_a = \frac{AB}{(A+B-AB)} \frac{dd}{dxdy} \int \frac{d^2\alpha}{\pi} exp[-(\frac{AB}{A+B-AB})\alpha^*\alpha + x\alpha + y\alpha^*]|_{x=y=0} - 1. \tag{4.100}$$

Upon carrying out the integration with the help of Eq.(3.130), we obtain

$$\bar{n}_a = \frac{dd}{dxdy}[e^{(\frac{A+B-AB}{AB})xy}]|_{x=y=0} - 1, \tag{4.101}$$

so that performing the differentiation and applying the condition x=y=0, we find

$$\bar{n}_a = (\frac{A+B-AB}{AB}) - 1 = \frac{1}{B} + \frac{1}{A} - 1 - 1 = \frac{M}{GM - H^*H} + \frac{u}{u^2 - v^2} - 2 = U + a - 2. \tag{4.102}$$

With the aid of Eqs.(3.127), (3.312), (3.289), (3.294),(3.307) and (3.308), we can write

$$\bar{n}_a = \frac{A(1+\eta)}{2(k-A)} + \frac{2\epsilon^2}{k^2 - 4\epsilon^2}. \tag{4.103}$$

It can also establish in a similar fashion that the mean photon number of the superposition of the parametric oscillator and V-type three-level light of idler-idler modes is found to be

$$\bar{n}_b = \frac{A(1-\eta)}{2(k-A)} + \frac{2\epsilon^2}{k^2 - 4\epsilon^2}. \tag{4.104}$$

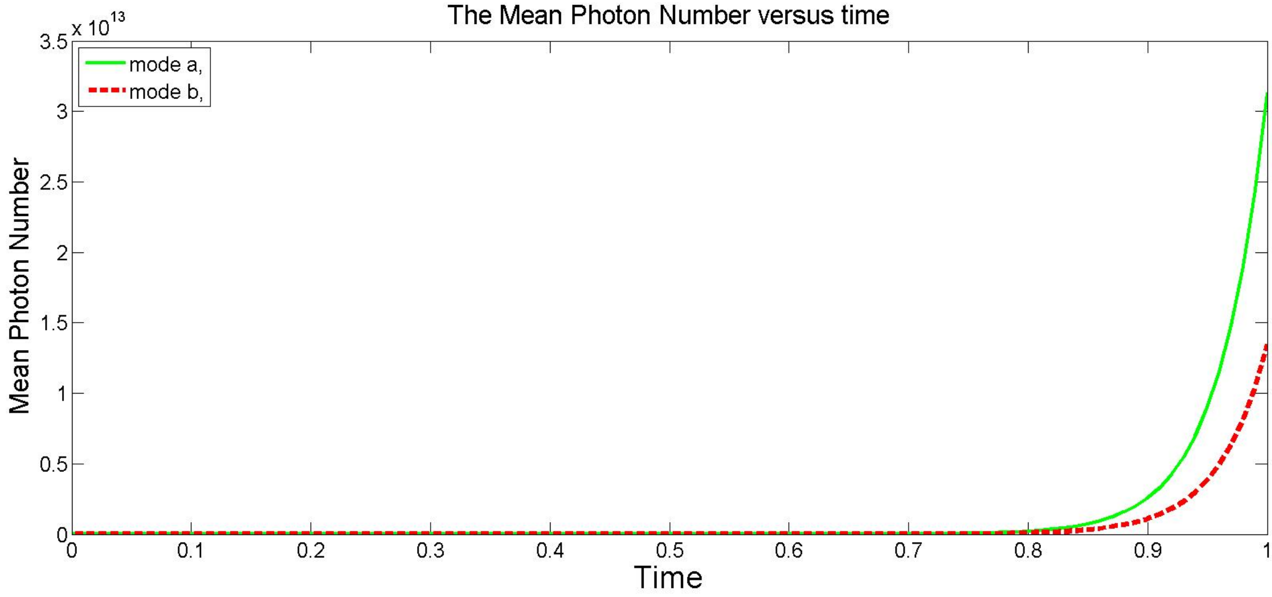

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Fig. 4.7: Plots of the superposed mean photon number[Eq.(4.103) and (4.104)] versus time for $\kappa = 0.8, \eta = 0.6, A = 3, \epsilon = 0.6$ and the superposed mean photon number, for $\bar{n}_b$(dash curve) and $(\bar{n}_a)$ (Solid curve).

The plots in the fig.4.7, indicates that the superposed mean photon number for the cavity mode increases with time and initial preparation of three-level atom and the superposed mean photon number for mode a is greater than mode b

We are also interested in the mean of the photon number sum and difference of the two-mode light beams. We define the operators representing the photon number sum and difference of the mode a and mode b by

$$\hat{n}_{\pm} = \hat{n}_a \pm \hat{n}_b \tag{4.105}$$

Upon taking the expectation values of Eq.(4.105) and applying Eqs.(4.103) and (4.104), the mean photon number sum and difference can be written in the form

$$\bar{n}_{\pm} = \frac{A(1+\eta)}{2(k-A)} + \frac{2\epsilon^2}{k^2-4\epsilon^2} \pm \frac{A(1-\eta)}{2(k-A)} + \frac{2\epsilon^2}{k^2-4\epsilon^2} \tag{4.106}$$

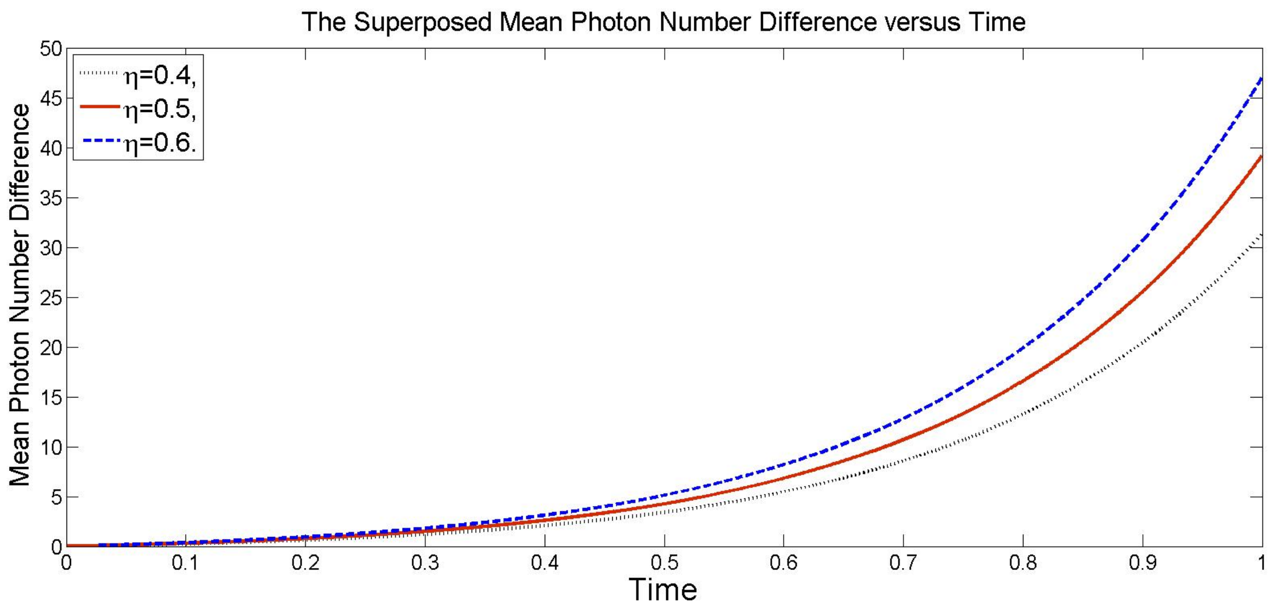

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Fig. 4.8: Plots the superposed mean photon number difference[Eq.(4.106)] versus time for $\kappa = 0.8, \epsilon = 0.8$,A=5, and different values of pump mode for η= 0.6(dash curve), η = 0.5(solid curve) and η = 0.4(dot curve).

The plots in the fig.4.8, indicates that the superposed mean photon number difference for the cavity mode increases with time and pump mode (ϵ).

4.3.2 The Variance of the Photon Number

We next proceed to obtain the variance of the photon number for the superposed two-mode parametric oscillator and V-type three-level laser light beams. The variance of the photon number is defined by

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle n \rangle^2. \tag{4.107}$$

Furthermore, the variance of the photon number for signal-sinal mode is given by

$$(\Delta n_a)^2 = \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \bar{n}_a^2. \tag{4.108}$$

Using Eq.(4.56), one can check that

$$\langle (\hat{a}^\dagger \hat{a})^2 \rangle = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - 3\langle \hat{a}^\dagger \hat{a} \rangle - 2, \tag{4.109}$$

so that expression (4.108) can be written as

$$(\Delta n_a)^2 = \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle - \bar{n}_a^2 - 3\bar{n}_a - 2. \tag{4.110}$$

Thus employing the Q-function, we have

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \int \partial^2 \alpha Q(\alpha^*, \alpha, t) \alpha^{*2} \alpha^2. \tag{4.111}$$

This can be put in the form

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{AB}{A+B-AB} \frac{d^4}{dx^2 dy^2} [\frac{\int d^2\alpha}{\pi} e^{-(\frac{AB}{A+B-AB})\alpha^*\alpha + x\alpha + y\alpha^*}]|_{x=y=0}. \tag{4.112}$$

On carrying out the integration using Eq.(3.130), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = \frac{d^4}{dx^2 dy^2} [(\frac{ABxy}{A+B-AB})]|_{x=y=0}, \tag{4.113}$$

so that carrying out the differentiation and applying the condition x=y=0, one can finds

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2[\frac{A+B-AB}{AB}]^2. \tag{4.114}$$

With the aid of Eqs.(3.307) and (3.308), Eq.(4.114) can be rewritten as

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2[\frac{M}{MG - H^*H} + \frac{u}{u^2 - v^2}]^2 \tag{4.115}$$

On account of Eqs.(3.131) and (3.294), we get

$$\langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle = 2U^2 + 2a^2 + 2Ua + 4U + 4a + 2. \tag{4.116}$$

Now with the aid of Eqs.(4.102) and (4.116), Eq.(4.110) can be put in the form

$$(\Delta n_a)^2 = U^2 + a^2 + 2Ua + U + a. \tag{4.117}$$

Finally, applying Eqs.(3.131) and (3.294), the variance of the signal-signal modes photon number is expressed as

$$(\Delta n_a)^2 = \frac{A(1+\eta)}{2(k-\eta)}[\frac{A(1+\eta)}{2(k-\eta)} + \frac{4\epsilon^2}{k^2-4\epsilon^2} + 1] + \frac{2\epsilon^2}{k^2-4\epsilon^2}[\frac{2\epsilon^2}{k^2-4\epsilon^2} + 1]. \tag{4.118}$$

Following the same procedure, the variance of the photon number idler-idler modes is written as

$$(\Delta n_b)^2 = \frac{A(1-\eta)}{2(k-\eta)}[\frac{A(1-\eta)}{2(k-\eta)} + \frac{4\epsilon^2}{k^2-4\epsilon^2} + 1] + \frac{2\epsilon^2}{k^2-4\epsilon^2}[\frac{2\epsilon^2}{k^2-4\epsilon^2} + 1]. \tag{4.119}$$

We observe that the variance of the photon number is greater than the mean photon number. Thus the photon number statistics is super-poissonian

5

CONCLUSION

In this thesis, we have studied the statistical properties of the light produced by: a two-mode parametric oscillator, a V-type three-level laser and the superposition of the parametric oscillator and the V-type three-level laser. We first obtained the master equation for the system under consideration. We obtained c-number Langevin equations using the pertinent master equation. Applying the c-number Langavin equations, we have calculated the anti normally ordered characteristic function. With the help of this characteristic function, we determined the Q functions for the two-mode parametric oscillator as well as V-type three-level laser light. Then the Q function were used to calculate the mean, the variance and the mean and variances of photon number sum and difference. Our result showed that the mean photon number of mode a and mode b for the two-mode parametric oscillator light are the same therefore, the mean photon number difference was vanished. Moreover, the mean and variance of the photon number is increases as the time increases and decreases as damping constant κ increases. The mean photon number for the two-mode V-type three-level laser light are different except $\eta = 0$. From the result of the mean photon number difference, we have seen that the mean photon number difference increases as both initial preparation of three-level atom (η) and time increases and in addition to from the result of the variance of photon number difference, we have also seen that the variance of photon number difference increases as both linear gain coefficient(A) and initial preparation of three-level atom (η) increases. Furthermore, we have also observed that the photon number statistics is super-Poissonian.

Finally, using the aforementioned Q functions, we have determined the Q function for the superposition of two-mode parametric oscillator and V-type three-level laser light signal-signal and idler-idler modes. Using the resulting Q-function, we have calculated the mean photon number sum and difference and the variance of the photon number. Our result showed that the mean and variance of the photon number for the superposed light was the sum of the mean and variance of the photon number for the constituent lights.

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