

MATHEMATICAL MODELING AND STABILITY ANALYSIS OF
CORRUPTION DYNAMICS



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Acronomys

R_0	Basic reproduction number.
TI	Transparency International.
WB	World Bank.
LAS	Locally Asymptotically Stable.
GAS	Globally Asymptotically Stable.
CFE	Corruption-free Equilibrium.
CEE	Corruption-endemic Equilibrium.
IMF	International Monetary Fund.
ICRG	International Country Risk Guide.
TICPI	Transparency International Corruption Perception Index.
NGM	Next Generation Matrix.
$\mathbb{R}_+^4$	positive coordinate in 4-dimensions.
E_f	Corruption-free equilibrium point.
E_e	Endemic equilibrium point.

Abstract

Several empirical journal have shown that corruption is an impediment to growth, as it mainly constitutes hindrance to investment. It has adverse impacts on the economy and on well-being of a society. In this study, we have investigated corruption dynamics using mathematical model. The corruption dynamics was analyzed in terms of the basic reproduction number, R_0 . Both local and global stability of the proposed model was investigated by linearizing and Lyapunov function approach, respectively. The analysis shows that the system has a locally asymptotically stable corruption-free equilibrium when $R_0 < 1$. The system has locally asymptotically stable endemic equilibrium, if $R_0 > 1$. Forming Anti-Corruption Commission which includes among other things; public awareness on the danger and consequence of corruption, transparency, accountability and very harsh effective penalties and to educating the sustainable and jailed individuals and implementing law enforcement on the Corruption can reduce corruption to a bearable level from its expansion, respectively. Numerical simulations of the study are carried out to verify the analytical results.

Keywords: Anti-Corruption Commission, Corruption, Basic reproduction number, Endemic equilibrium, Stability analysis, Corruption Dynamics.

Chapter 1

Introduction

1.1 Background of the study

Corruption is a pressing global problem and country in the world is not free of its menacing grip. It is a worm within the body of the society and a cancer to economic, social, and political development. In the developing countries, most of the central government institutions are overtly threatened by corruption which is both deliberate and unconscious break down in human morals by public servants. These in turn gravitate to undermine democratic processes in the governance structures and also challenging situation to governance both at centralized and devolved governance (Le Van, C. and M. Maurel., 2006).

Corruption remains one of the major obstacles to economic prosperity in many countries. It is known to distort incentives, impede investment and divert the allocation of productive resources to rent-seeking activities (Shleifer. A and Vishny, 1993) and there are three broad classification of corruption, which are however mutually exclusive.

- **Petty and Grand corruption**

- **Petty corruption:** Practiced on a smaller scale. Defined as the use of public office for private benefit in the course of delivering a public service. The direct victim of this abuse of power is the citizen.
- **Grand corruption:** The most dangerous and covert type of corruption. Instances where policy making, its design and implementation are compromised by corrupt practices. Occurs at financial, political and administrative centers of power.

- **Political and Business corruption**

- **Political corruption:** Occurs predominantly in developing and less developed countries. Usually associated with the electoral process.

Includes:

- * voting irregularities
 - * holding on to power against the will of the people.
 - * false political promises
- **Business corruption:** Often not regarded as a crime, rather as a means to accelerate business processes. Includes bribery, insider trading, money laundering, embezzlement, tax evasion and accounting irregularities.

- **Chaotic and Organized corruption**

- **Chaotic corruption:** A disorganized system where there is no clarity regarding whom to bribe and how much payment should be offered.

There is:

- * no guarantee that further bribes will not have to be paid to other officials;
 - * no reasonable assurance that the favor will be delivered;
 - * no coordination between the recipients of benefits, with the result
- **Organized corruption:** Organized corruption is often perpetrated by crime gangs and syndicates and includes white-collar crime and identity. A well-organized system of corruption in which there is a clear idea:
- * are confident that they will receive the favor in return.
 - * how much should be offered and
 - * of whom to bribe

Different scholars from social sciences, psychology, political sciences, and religious studies have attempted a working definition for corruption from their various disciplines. However all of the working definition are interwoven (Olagunju, 2012). In everyday use, corruption is a term which conveys an element of moral disapproval. The World Bank has defined corruption

as the abuse of public office for private gain (World Bank, 1997). Additionally, Transparency International (TI) defined corruption as the misuse of entrusted power for private benefit (Pope, 2000). In a more legal form, (Ogbu, 2008) defined corruption as an abuse of position or inducement of abuse of position for an undeserved benefit, advantage or relief.

Corruption is rampant in Ethiopia. According to the Global Integrity Report of 2006, corruption is considered a norm of social, economic and political intercourse in Ethiopia. So Ethiopia is also as any other developing country is challenged in its politics, governance and public sector economics by a social vice of corruption.

In an International Monetary Fund (IMF) working paper, Tanzi (1998) summarizes the causes or factors that promote corruption as: ‘those that affect demand (by the public) for corrupt acts and those that affects the supply (by public officials) of act of corruption. Among the factors affecting the demands are: Regulations and Authorizations, certain characteristics of the tax systems, certain spending decisions and Provision of goods and services at below-market prices. Among the factors affecting the supply of acts of corruption are: The bureaucratic tradition, the example set by the leadership, the penalty system, institutional controls, the transparency of rule, laws and processes; and the level of public sector wages. Obtaining robust measure of corruption is a very difficult task due to the illicit and secretive nature of corrupt practices. Though, it is possible to quantify peoples perception indicates have been developed by several institutions and have been used by various researchers. Two widely used of such indicates are the International Country Risk Guide (ICRG) and Transparency International Corruption Perception Index (TICPI). Numerous studies on corruption have been carried out due to increasing public interest and concern over its universal threat to humanity. Although, some studies (Leff, 1964; Huntington, 1968; Friedrich, 1972; Lui, 1986; Beck and Marker, 1986; Lein, 1986) have pointed the desirability of some corruption on the ground that it can be beneficial to the functioning of the economy, most of the study including Myrdal (1991), Rose-Ackerman (1975), Gould and Amaro-Reyes (1983), Baumol (1990), Murphy et al., (1991) and Klitgaard (1991) concluded that corruption undermines the economic, political as well as social development. It is both a major cause and a result of poverty around the world.

There are several sectors where businesses are particularly vulnerable to corruption. Those

of the most widely observed sectors are tax administrations and land administrations. Corruption also occurs when businesses obtain permits and licenses and due to complicated bureaucracy. Public procurement is also seriously hampered by corruption, and different types of irregularities exist, such as non-transparent tender processes and awarding contracts to people with close connection to the government.

The focus of this thesis work is on the dynamic pattern of the evolution of corruption. Nevertheless, the dynamics of corruption and the existence of corruption cycles have been so far theoretically investigated only by a few authors. F. T. Lui(1986) considers the impact of exogenous corruption deterrent on the corruption and views anti-corruption commission as efforts to shift, in an society with multiple equilibrium, from an unfavorable to a favorable equilibrium.

1.2 Statement of the Problems

Corruption is an old phenomenon that has a substantial impact on economic growth (mostly negative) and affects all societies, albeit to different degrees. For example, the organization Transparency International compares corruption internationally and documents a substantial variation across states. Corruption has been actively studied by mathematicians for about forty years: the first mathematical model dedicated to corrupt structures was appeared in (Rose-Ackerman, 1975).

Several authors including Starkermann (1989), Blanchard et al., (2005), Le Van and Maurel (2006), Becker et al., (2008), Brianzoni et al., (2011) and Waykar (2013) Hathroubi (2013), have over the last two and a half decades used mathematical models to evaluate the effects of corruption on national development. Sirajo Abdulrahman,(2014) developed and analyzed a new mathematical model of corruption transmission dynamics to complement and extend the work of Hathroubi (2013).

However; Sirajo Abdulrahman,(2014) made an assumption that, Sustainable and Jailed individual will come to Honest by their willingness. So we develop the model in Sirajo Abdulrahman,(2014) by adding the assumption of Sustainable and Jailed individual will come to Honest by incorporates the standard rate and effort rate against corruption.

This research work, modify a mathematical model of dynamic corruption which describes reducing corruption to bearable level by regarding the initial size of subpopulation in the community. Therefore, this research work attempts to address the following basic questions:

1. What are the basic assumptions to modify a mathematical model which describes the dynamics of corruption?
2. Are the equilibrium points are locally and globally stable?
3. What are the social interpretations of the solution of the mathematical problem?

1.3 Objective of the Study

1.3.1 General Objectives

The general objective of this research work is to understand the dynamics of corruption by using mathematical model.

1.3.2 Specific Objectives

The specific objectives of the study are:

- (a) to develop a mathematical model which describes the dynamics of corruption.
- (b) to study the nature of local and global stability of the equilibrium points.
- (c) to interpret the social implication of the solutions.

1.4 Significance

- To develop awareness about the dynamics of corruption.
- To suggest the possible intervention mechanisms.
- Improve our understanding of the dynamical properties of corruption .

1.5 Expected Outcomes

The expected outcomes of the present study are:

- reduce the expansion of corruption.
- a mathematical model that governs the dynamics of corruption.

1.6 Methodology

We develop a mathematical model which describes the dynamics of corruption using a system of non-linear differential equations. We perform the qualitative analysis of the model. Both local and global stability analysis of the equilibrium points of the model in equations was be done using Jacobian matrix and Lyapunov function respectively. The analytical solutions of the model equations be supplemented with numerical simulations by appropriately choosing the system parameter values using MATLAB.

Chapter 2

Literature Review

Corruption is a problem common to businesses and public enterprises of all kinds (both in public and private organizations). Corruption reduces economic growth through a negative influence on investments in human capital. The rate of change of development per unit corruption is directly proportional to the development (Bardhan P. 1997; Rose-Ackerman S., 1999; Sayaji R. Waykar, 2013).

To quantify people's perception of corruption within a society; Various perception indicators have been developed by several institutions and have been used by various researchers. The two widely used are the International Country Risk Guide (ICRG) and the Transparency International Corruption Perception Index (TICPI).

Numerous studies on corruption have been carried out due to increasing public interest and concern over its universal threat to humanity. Although, some studies (Leff, 1964; Huntington, 1968; Friedrich, 1972; Lui, 1986; Beck and Marker, 1986; Lien, 1986) have pointed out the desirability of some corruption on the ground that it can be beneficial to the functioning of the economy, most of the studies including Myrdal (1968), Rose-Ackerman (1975), Gould and Amaro-Reyes (1983), Baumol (1990), Murphy et al., (1991) and Klitgaard (1991) concluded that corruption undermines the economic, political as well as social development. It is both a major cause and a result of poverty around the world. Rinaldi, Feichtinger, and Wirl (1998) studied the dynamics of corruption at the top, i.e., corruption by politicians. Their framework relates actions (positive actions of the politicians, bribery, consumption, and the unveiling of corruption) to the state variables popularity (as a proxy for power), assets accumulated from bribes, and investigation.

Several authors including Starkermann (1989), Le Van and Maurel (2006), Becker et al., (2008), Brianzoni et al., (2011), Sayaji R. Waykar, (2013) and Abdulrahman, S. (2014) have used mathematical models to evaluate the effects of corruption on national development by modeling corruption as a diseases. Some of the latest model has the problem of recruitment rate from total populations. Furthermore, the anti Corruption commission influence are not considered in most of these studies. The present works aim at studying the dynamics of corruption mathematically and analysis the role of anti corruption commission to minimize corruption from its expansion.

In Hathroubi, S. (2013), an epidemiological corruption threshold based on the approximation of honest population was determined by dividing the total population $N(t)$ into three compartments Susceptible $S(t)$, Corrupt $C(t)$, and Honest $H(t)$ individual without the stability analysis of the equilibrium points. Hathroubi, S. (2013) develop the model based on the assumptions that:

- (i) a corruptible transmits corruption immediately when she/he becomes corrupted;
- (ii) corrupt-corruptible exchange is proportional to the statistical average of contacts;
- (iii) corrupt-honest relationship is proportional to the corrupt population; and
- (iv) the recovered corrupt are immune against corruption.

Abdulrahman, S. (2014) proposed a mathematical model for corruption transmission dynamics with recruitment rate from the total population to complement and extend the work of Hathroubi, S. (2013) by dividing the total population $N(t)$ into four compartments of Susceptible $S(t)$, Corrupt $C(t)$, Jailed $J(t)$ and Honesty $H(t)$ individuals. The total population equation was formulated as:

$$N(t) = S(t) + C(t) + H(t) + J(t). \quad (2.1)$$

with assumption that:

- (i) the TICPI is an efficient way to measure people's perception of corruption of a country;
- (ii) a susceptible individual is one that have never engage in any corrupt practice that will have a negative impact on his/her country's development but he/she is prone to be corrupted;

- (iii) a corrupt individual is one that have engage in a corrupt practice that have negative impact on his/her country's development and capable of influencing a susceptible individual to become corrupt;
- (iv) a jailed individual is one that is corrupt, caught and imprison for a specified period of year(s) during which he/she cannot engage in any corrupt act that will have negative impact on his/her country's development neither influence others to be corrupt;
- (v) an honest individual is one that can never be corrupt no matter the condition he/she find himself/herself;
- (vi) a susceptible, corrupt or jailed individual can willingly become honest.

The governing mathematical model is:

$$\begin{aligned}
 \frac{dS}{dt} &= \Lambda - \frac{\beta C}{N} S - K_1 S, \\
 \frac{dC}{dt} &= \frac{\beta C}{N} S + \psi(1 - \theta)J - K_2 C, \\
 \frac{dJ}{dt} &= \phi C - K_3 J, \\
 \frac{dH}{dt} &= \theta(S + C + \psi J) - \mu H,
 \end{aligned} \tag{2.2}$$

where the model variables are defined as follows: $S(t)$ Susceptible individuals at time t , $C(t)$ Corrupt individuals at time t , $J(t)$ Jailed individuals at time t , $H(t)$ Honest individuals at time t , and $N(t)$ total population at time t .

The model parameters are defined as:

Λ recruitment rate,

μ death removal rate,

τ effort rate against corruption,

$\beta = p(1 - \tau)$ effective corruption contact rate,

p corruption transmission probability per contact,

ϕ rate at which corrupt individuals are caught and imprison,

θ proportion of individuals that leaves S, C or J compartment to H compartment;

with $K_1 = \theta + \mu$, $K_2 = \phi + \theta + \mu$ and $K_3 = \psi + \mu$.

The stability analysis of both corruption free and endemic equilibrium were conducted. In this model, the role of anti-corruption commission was not considered.

According to the findings of Mauro P., (1998), policies to fight corruption could be very beneficial to growth. Most of the existing literature on the long-run economic consequences of corruption (Friedrich C.J., 1972; Andvig J. C., 1991; Chander and Wilde, 1992; Shleifer and Vishny, 1993; Ehrlich I., 1999; B. Erlend., 2001) focuses on rent seeking in the provision of public services. The economic consequences of this phenomenon concern distortions in resources allocation mainly in terms of less private investment, and a reduced rate of human capital formation.

Anti corruption Commission keep an important role in the society to build good governance and conforming transparency in the administration along with preventing prevailing corruption. According to institutional development plan (2011-2020) the role of the anti corruption commission are as follows:

- (a) Educate the public on the dangers of corruption and the benefits of its eradication and to enlist and foster public support in combating corruption;
- (b) Advise and assist any person, on the latters request, on ways in which corruption may be eliminated by such person;
- (c) Advise heads of public or private entities of any changes in practices, procedures or systems compatible with the effective discharge of their duties as in the opinion of the commission necessary to minimize or eliminate the occurrence of corruption;
- (d) Undertake studies and research to identify the trends, causes, types, pervasiveness and seriousness or impacts of corruption;
- (e) Any other functions which in the opinion of the commission may be necessary for the prevention of, and fight against, corruption in the opinion of the commission.

In this study, we follow and modify a mathematical model for dynamic of corruption as a spread in a population with effort rate against corruption by forming anti corruption commission as detailed in the literature survey above.

Chapter 3

Model Formulation and Analysis

3.1 Model Formulation

In this chapter, we formulate a mathematical model of corruption dynamics using a system of non-linear differential equations by extended the work of Sirajo Abdulrahman (2014), that divided the total population $N(t)$ into four compartments Susceptible $S(t)$, Corrupt $C(t)$, Jailed $J(t)$, and Honesty $H(t)$ individuals by adding the influence of anti corruption commission in similar way. The important of adding the role of anti corruption in to Sirajo Abdulrahman (2014) model is to reduce the number of population tends to corrupt from susceptible and jailed compartments. The influences of anti corruption commission are:

- (a) to promote the investigation of serious cases of corruption and fraud;
- (b) to promote awareness among the public of the causes of corruption and its effects on society;
- (c) to educate the public on the dangers of corruption in society; and
- (d) to instruct, advise and assist any officer, agency or institution in the minimization of corruption.

In the model the system of non-linear differential equations describing the dynamics of corruption depending on the category of the type Susceptible $S(t)$, Corrupt $C(t)$, Jailed $J(t)$, and Honest $H(t)$ is derived from the interaction between among all the compartments for

the model equation and the flow chart of model is shown in figure 3.1

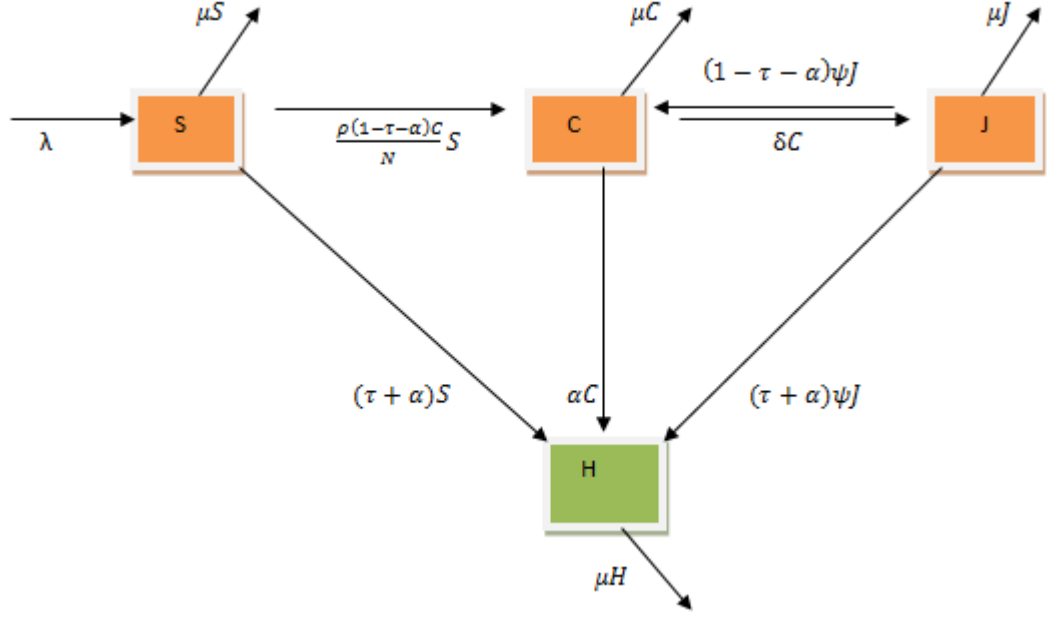


Figure 3.1: Corruption Dynamics Model flow chart.

The proposed model mainly based on the following assumptions:

- (i) a susceptible, corrupt or jailed individual can willingly become to honest,
- (ii) a susceptible individual is one that have never engage in any corrupt practice,
- (iii) a corrupt individual is one that have engage in a corrupts practice and capable of influencing a susceptible individual to become corrupt,
- (iv) a jailed individuals is one that found to be guilty corrupt by law, caught and imprison for a specified period of time, and
- (v) susceptible and jailed individuals become honest through education and giving awareness by anti corruption commission.

Thus, the governing non-linear system of differential equation is given by

$$\begin{aligned}
\frac{dS}{dt} &= \lambda - \frac{\beta C}{N}S - (\mu + \alpha + \tau)S, \\
\frac{dC}{dt} &= \frac{\beta C}{N}S - (\mu + \delta + \alpha)C + \psi(1 - \tau - \alpha)J, \\
\frac{dJ}{dt} &= \delta C - (\mu + \psi)J, \\
\frac{dH}{dt} &= (S + C + \psi J)\alpha + (S + \psi J)\tau - \mu H,
\end{aligned} \tag{3.1}$$

We considered the susceptible $S(t)$ sub-population is generated from constant recruitment of individuals at a rate λ . Individuals in the susceptible, corrupt or jailed class also become honest willingly at the rate α , ($0 < \alpha < 1$). Individual in the susceptible and jailed class become honest at the rate τ , ($0 < \tau < 1$) due to anti-corruption education and awareness to the society on danger of corruption. Susceptible individual is becomes to honest at rate $(\tau + \alpha)$ while acquired infection via horizontal transfer from susceptible individuals in the corrupt class, $C(t)$ at a rate $\beta = p(1 - \tau - \alpha)$ and thus become corrupted. Also, corrupt individuals are caught and imprison at the rate δ .

The losing immunity of jailed individuals is at rate of ψ . Thus, as the jailed individuals spent an average period of $\frac{1}{\psi}$ in prison, then $\psi(\tau + \alpha)$ is the transition rates from J compartment to H compartment while $\psi(1 - \tau - \alpha)$ go back to the corrupted class. The validity of $(1 - \tau - \alpha)$ with reality of $0 \leq (\tau + \alpha) \leq 1$ is as follows:

- If $(1 - \tau - \alpha) = 0$, there is no acquired infection via horizontal transfer from individuals to corrupted class,
- If $(1 - \tau - \alpha) = 1$, there is fully acquired infection via horizontal transfer from individuals to corrupted class,
- If $0 < (1 - \tau - \alpha) < 1$, there is acquired infection via horizontal transfer from individuals to corrupted class.

The variables and parameters of the corruption model defined as follows in table (3.1) and table(3.2), respectively:

Symbol	Description
$S(t)$	The number of susceptible individuals at time t
$C(t)$	The number of corrupt individuals at time t
$J(t)$	The number of jailed individuals at time t
$H(t)$	The number of honest individuals at time t

Table 3.1: State variables of corruption model of equation (3.1).

Symbol	Description
λ	The recruitment rate from sub-population
p	Corruption transmission probability per contact
β	The effective corruption contact rate
τ	The effort rate against corruption due to anti-corruption
δ	The rate at which corrupt individuals are caught and imprison
ψ	The rate of losing immunity
$\frac{1}{\psi}$	The average period jailed individuals spent in prison
μ	The natural death rate.
α	Proportion of individuals that leaves S, C , or J compartment to H compartment willingly

Table 3.2: Parameters of corruption model of equation (3.1).

The total population $N(t)$ can be obtained from the equation:

$$N(t) = S(t) + C(t) + J(t) + H(t) \quad (3.2)$$

Hence, the natural death rate of total population by using equation (3.2) is:

$$\mu N(t) = \mu[S(t) + C(t) + J(t) + H(t)] \quad (3.3)$$

Where, the rate of change of the total population is:

$$\begin{aligned} \frac{dN(t)}{dt} &= \frac{dS(t)}{dt} + \frac{dC(t)}{dt} + \frac{dJ(t)}{dt} + \frac{dH(t)}{dt} \\ &= \lambda - \mu[S(t) + C(t) + J(t) + H(t)] \end{aligned}$$

Hence by substituting equation (3.3) in to the above equation, we obtained

$$\frac{dN(t)}{dt} = \lambda - \mu N(t). \quad (3.4)$$

Thus,

- The total population size N is constant for $\mu N(t) = \lambda$,
- The total population size N is decline for $\mu N(t) > \lambda$,
- The total population size N is growth for $\mu N(t) < \lambda$,

3.2 Basic properties of the model

The basic properties of model represented by the system (3.1) which are useful in the proof of stability are studied. These are the invariant region and positivity of the solutions. The model will be analyzed in the feasible region and since we are working with population all state variables and parameters are assumed to be positive for all $t \geq 0$.

3.2.1 Invariant region

The rate of change of the total population, from equations (3.4) is:

$$\frac{dN}{dt} = \lambda - \mu N, \quad (3.5)$$

If $N(t) \leq \frac{\lambda}{\mu}$, implying $\frac{dN(t)}{dt}$ is bounded above by $\lambda - \mu N(t)$. By separation of variables of differential inequality (3.5), we get

$$\frac{dN}{\lambda - \mu N} \leq dt \quad (3.6)$$

Integrating (3.6) on both side gives, $\int \frac{dN}{\lambda - \mu N} \leq \int dt$

$$\frac{-1}{\mu} \ln(\lambda - \mu N) + c \leq t + c$$

$$\ln(\lambda - \mu N) \geq -\mu(t + c)$$

Therefore,

$$(\lambda - \mu N) \geq Ae^{-\mu t} \quad (3.7)$$

Where A is a constant. Now, applying the initial condition $N(0) = N_0$ in (3.7) we get

$$A = \lambda - \mu N_0 \quad (3.8)$$

substitute (3.8) in to (3.7) gives

$$\lambda - \mu N \geq [\lambda - \mu N_0] e^{-\mu t} \quad (3.9)$$

Making N the subject in (3.9) we have,

$$N \leq \frac{\lambda}{\mu} - \left[\frac{\lambda - \mu N_0}{\mu} \right] e^{-\mu t} \quad (3.10)$$

As $t \rightarrow \infty$ in (3.10), the above total population size N, approaches

$$0 \leq N \leq \frac{\lambda}{\mu} \implies N \rightarrow \frac{\lambda}{\mu} \quad (3.11)$$

Therefore, the feasible solutions set of system (3.1) enters and remain in the region

$$\Omega = \left\{ (S, C, J, H) \in \mathbf{R}_+^4 : N \leq \frac{\lambda}{\mu} \right\}$$

Thus, Ω is positively-invariant (all solution in Ω remain in Ω for all time).

Furthermore, if $N > \frac{\lambda}{\mu}$ then $\frac{dN}{dt} < 0$, hence either the solution enters Ω infinite time or $N(t)$ approaches to $\frac{\lambda}{\mu}$ and the infected variable C and J approaches to Zero. Thus Ω is attracting (all solution in $\mathbf{R}_+^4$ eventually enters in Ω).

3.2.2 Positivity of solutions

Theorem 3.2.1. *Let the initial conditions of system (3.1) be non-negative. Then the solutions $(S(t), C(t), J(t), H(t))$ of the system is non-negative for all $t \geq 0$.*

Proof. Let $Z = \sup \{t > 0 : S > 0, C > 0, J > 0, H > 0\}$. Then

(a). From the first equation of the system of (3.1) we have

$$\frac{dS}{dt} = \lambda - \frac{\beta C}{N} S - (\mu + \tau + \alpha) S = \lambda - \left(\frac{\beta C}{N} + \mu + \tau + \alpha \right) S$$

$$i.e., \quad \frac{dS}{dt} + \left(\frac{\beta C}{N} + \mu + \tau + \alpha \right) S = \lambda$$

The integrating factor (IF) is;

$$IF = e^{\int_0^t \left(\frac{\beta C(r)}{N(r)} + \mu + \tau + \alpha \right) dr} = e^{\int_0^t \frac{\beta C(r)}{N(r)} dr + (\mu + \tau + \alpha)t}$$

We multiply the above equation by integrating factor and we obtain

$$\left(\frac{dS(t)}{dt} + \frac{\beta C}{N} S + (\mu + \tau + \alpha) S \right) e^{\int_0^t \frac{\beta C(r)}{N(r)} dr + (\mu + \tau + \alpha)t} = (\lambda) e^{\int_0^t \frac{\beta C(r)}{N(r)} dr + (\mu + \tau + \alpha)t}$$

After simplifying and integrating the above equation.

$$S(t)e^{\int_0^t \frac{\beta C(r)}{N(r)} dr + (\mu + \tau + \alpha)t} = S(0) + \int_0^t \lambda e^{\int_0^s \frac{\beta C(v)}{N(v)} dv + (\mu + \tau + \alpha)s} ds$$

Since all parameters and variables are non-negative, We simplifying to obtain

$$S(t) \geq S(0)e^{-\int_0^t \frac{\beta C(r)}{N(r)} dr - (\mu + \tau + \alpha)t} + e^{-\int_0^t \frac{\beta C(r)}{N(r)} dr - (\mu + \tau + \alpha)t} \left\{ \int_0^t \lambda e^{\int_0^s \frac{\beta C(v)}{N(v)} dv + (\mu + \tau + \alpha)s} ds \right\}$$

Since the integral of any positive functions must be positive and the exponential functions are always positive for positive coefficients; therefore, $S(t) > 0, \forall t > 0$. $C(t) > 0, \forall t > 0$. $J(t) > 0, \forall t > 0$. $H(t) > 0, \forall t > 0$. Therefore, the solutions $(S(t), C(t), J(t), H(t))$ of the model equation (3.1) are positively invariant for all $t > 0$, if $S(0) \geq 0$, $C(0) \geq 0$, $J(0) \geq 0$, $H(0) \geq 0$. From equation (3.2) we have $N(t) \geq 0$, since $N(t) = S(t) + C(t) + J(t) + H(t) \geq 0$. $\square$

Therefore, the feasible solution set for the corruption dynamics model of equation (3.1), given by $\Omega \in \mathbb{R}_+^4$ is bounded, positively invariant and attractiveness. Hence the model is socially meaningful and mathematically well-posed in the domain Ω .

3.3 Model Analysis

3.3.1 Determination of Basic Reproduction Number

The basic reproductive number R_0 is defined as the expected number of new corrupts from one corrupt individual in a fully susceptible population through the entire duration of corruption prone period. Considering the proposed model, we see that the system equation (3.1) is block recursive, since the first three equations are independent of $H(t)$. From the total population, we have

$$H(t) = N(t) - [S(t) + C(t) + J(t)]$$

allows us as explained in Hethcote (2000) and Benyah (2008) to study the first three equations of system (3.1). Then after rearranging and simplifying on $\frac{dN}{dt}$, we proceed to conclude that the last equation of the system equation (3.1) has a form in equation (3.4). Therefore, we will used the system below in place of the system equation (3.1) in the model to analysis

stability .

$$\begin{aligned}
\frac{dS}{dt} &= \lambda - \frac{\beta C}{N}S - k_1 S, \\
\frac{dC}{dt} &= \frac{\beta C}{N}S - k_2 C + \psi(1 - \tau - \alpha)J, \\
\frac{dJ}{dt} &= \delta C - k_3 J, \\
\frac{dN}{dt} &= \lambda - \mu N,
\end{aligned} \tag{3.12}$$

Where $k_1 = \mu + \tau + \alpha$, $k_2 = \mu + \delta + \alpha$ and $k_3 = \mu + \psi$.

We use the next generation operator approach as described by Diekmann et al. (2000) to define the basic reproduction number R_0 , as the number of secondary infections that one infectious individual would create over the duration of the infectious period, provided that everyone else is susceptible. Reproduction number R_0 is the threshold for many epidemiology models, it determines whether a corruption can invade in the population or not. When $R_0 < 1$ each corrupted individual produces on average less than one new corrupt individual. So we would expect that the corruption to minimize. On the other hand, if $R_0 > 1$, each corrupted individual produces new corrupt individual, so we would expect that the corruption to spread in the population. This means that the threshold quantity for eradicating the corruption is to reduce the value of R_0 to value less than one.

3.3.2 Existence of Corruption Free Equilibrium

Let us outline the steps needed to compute the basic reproduction number. The basic reproduction number cannot be determined from the structure of the mathematical model alone, but depends on the definition of corrupted and uncorrupted compartments.

Let

- $\mathcal{F}_c(x)$ be the rate of appearance of new corrupts in compartments c ,
- $\mathcal{V}_c^+$ is the rate of transfer of individuals into compartment c ,
- $\mathcal{V}_c^-$ is the rate of transfer of individual out of the compartment c by all other means and E_f is the corruption free equilibrium.

It is assumed that each function is continuously differentiable at least twice in each variable. The corruption model consists of non-negative initial conditions together with the following

system of equations:

$$\begin{aligned}\frac{dx_c}{dt} &= f_c(x) = \mathcal{F}_c(x) - \mathcal{V}_c(x), c = 1, 2, 3, \dots, n \\ &= \begin{pmatrix} \frac{\beta C}{N} S - k_2 C + \psi(1 - \tau - \alpha)J \\ \delta C - k_3 J \end{pmatrix}\end{aligned}$$

where $\mathcal{V}_c(x) = \mathcal{V}_c^+(x) - \mathcal{V}_c^-(x)$, $\mathcal{F} = \begin{pmatrix} \frac{\beta C}{N} S \\ 0 \end{pmatrix}$ and $\mathcal{V} = \begin{pmatrix} k_2 C - \psi(1 - \tau - \alpha)J \\ -\delta C + k_3 J \end{pmatrix}$

Thus, the new corrupt terms $\mathcal{F}$ and the transition terms $\mathcal{V}$ are obtained from the corrupted compartment C and J compartment. The corruption-free equilibrium is the state in which the population is free of corruption, so that we have only susceptible and honest individuals. In this state there is no corrupt immigrants. Thus, the model has a corruption-free equilibrium $E_f(S^*, C^*, J^*, N^*) = \left(\frac{\lambda}{k_1}, 0, 0, \frac{\lambda}{\mu}\right)$ obtained by setting the right-hand side of the system equations (3.12) to zero with $C = 0$ and $J = 0$. The existence of the corruption-free equilibrium is established, then the next step is the computation of the $m \times m$ square matrices F and V at the corruption-free E_f equilibrium which are defined by

$$F = \left[\frac{\partial \mathcal{F}_c}{\partial x_j}(E_f) \right] \text{ and } V = \left[\frac{\partial \mathcal{V}_c}{\partial x_j}(E_f) \right] \text{ with } 1 \leq c, j \leq m$$

such that $F = \begin{pmatrix} \frac{\partial \mathcal{F}_{11}}{\partial C}(E_f) & \frac{\partial \mathcal{F}_{12}}{\partial J}(E_f) \\ \frac{\partial \mathcal{F}_{21}}{\partial C}(E_f) & \frac{\partial \mathcal{F}_{22}}{\partial J}(E_f) \end{pmatrix} = \begin{pmatrix} \frac{\beta S^*}{N^*} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\beta \mu}{k_1} & 0 \\ 0 & 0 \end{pmatrix}$ and ,

$$V = \begin{pmatrix} \frac{\partial \mathcal{V}_{11}}{\partial C}(E_f) & \frac{\partial \mathcal{V}_{12}}{\partial J}(E_f) \\ \frac{\partial \mathcal{V}_{21}}{\partial C}(E_f) & \frac{\partial \mathcal{V}_{22}}{\partial J}(E_f) \end{pmatrix} = \begin{pmatrix} k_2 & -\psi(1 - \tau - \alpha) \\ -\delta & k_3 \end{pmatrix}$$

is non-negative by considering the second and the third equations of (3.12) at steady state and from determinant of V , we can deduced that $k_2 k_3 > \psi \delta (1 - \tau - \alpha)$. Then, the square matrix V is invertible.

$$V^{-1} = \frac{1}{(k_2 k_3 - \psi \delta (1 - \tau - \alpha))} \begin{pmatrix} k_3 & \psi(1 - \tau - \alpha) \\ \delta & k_2 \end{pmatrix}$$

is a non-singular $m \times m$ of square matrix. Then the physical meaning of the Next Generation Matrix (NGM) technique described by Diekmann and Heesterbeek (2000) and subsequently analyzed by Van den Driessche and Watmough (2002), is obtaining basic reproduction number $R_0 = \rho(K)$, where $K = FV^{-1}$ of the model (3.12) to computed in order to analyze local and

global stability of the corruption-free equilibrium. Therefore, the next generation matrix is given by

$$\begin{aligned} K = FV^{-1} &= \begin{pmatrix} \frac{\partial \mathcal{F}_{11}}{\partial C}(E_f) & \frac{\partial \mathcal{F}_{12}}{\partial J}(E_f) \\ \frac{\partial \mathcal{F}_{21}}{\partial C}(E_f) & \frac{\partial \mathcal{F}_{22}}{\partial J}(E_f) \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{V}_{11}}{\partial C}(E_f) & \frac{\partial \mathcal{V}_{12}}{\partial J}(E_f) \\ \frac{\partial \mathcal{V}_{21}}{\partial C}(E_f) & \frac{\partial \mathcal{V}_{22}}{\partial J}(E_f) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\beta\mu}{k_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_2 & -\psi(1-\tau-\alpha) \\ -\delta & k_3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{\beta k_3 \mu}{k_1(k_2 k_3 - \psi \delta(1-\tau-\alpha))} & \frac{\beta \mu \psi(1-\tau-\alpha)}{k_1(k_2 k_3 - \psi \delta(1-\tau-\alpha))} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Since, the quantity of R_0 is the spectral radius (maximum eigenvalue) of the matrix. Hence the maximum eigenvalues of FV^{-1} is $\frac{\beta k_3 \mu}{k_1(k_2 k_3 - \psi \delta(1-\tau-\alpha))}$ and becomes the basic reproduction number R_0 of the model.

$$R_0 = \frac{\beta k_3 \mu}{k_1(k_2 k_3 - \psi \delta(1-\tau-\alpha))} \quad (3.13)$$

3.3.3 Local Stability of Corruption Free Equilibrium

Theorem 3.3.1. *The corruption free equilibrium E_f of the system (3.12) is locally asymptotically stable (LAS) when $R_0 < 1$.*

Proof. We used the Jacobian stability approach to prove the local stability of the corruption-free equilibrium state. The Jacobian matrix $\hat{J}$ of system (3.12) is

$$\hat{J} = \begin{pmatrix} -\left(k_1 + \frac{\beta C}{N}\right) & -\frac{\beta S}{N} & 0 & 0 \\ \frac{\beta C}{N} & -\left(k_2 - \frac{\beta S}{N}\right) & \psi(1-\tau-\alpha) & 0 \\ 0 & \delta & -k_3 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}$$

Linearization the above matrix at corruption free equilibrium E_f is,

$$\hat{J}(E_f) = \begin{pmatrix} -k_1 & -\frac{\beta S^*}{N^*} & 0 & 0 \\ 0 & -\left(k_2 - \frac{\beta S^*}{N^*}\right) & \psi(1-\tau-\alpha) & 0 \\ 0 & \delta & -k_3 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix} \quad (3.14)$$

the eigenvalues of equation (3.14) are the solution of the characteristic equation.

$$|\hat{J}(E_f) - \lambda I_4| = 0$$

where I_4 is the identity matrix of order 4. then we have

$$\begin{vmatrix} -k_1 - \lambda & -\frac{\beta S^*}{N^*} & 0 & 0 \\ 0 & -\left(k_2 - \frac{\beta S^*}{N^*}\right) - \lambda & \psi(1 - \tau - \alpha) & 0 \\ 0 & \delta & -k_3 - \lambda & 0 \\ 0 & 0 & 0 & -\mu - \lambda \end{vmatrix} \quad (3.15)$$

The first column of (3.15) is all zero except the first entry, which is $-k_1 - \lambda$. Thus, we have the first eigenvalue; $\lambda = -k_1 < 0$.

Again the fourth column of (3.15) is all zero except the last diagonal entry, which is $-\mu - \lambda$. Thus, we have the second eigenvalue; $\lambda = -\mu < 0$. The remaining eigenvalues are found using the equation:

$$\begin{vmatrix} -\left(k_2 - \frac{\beta S^*}{N^*}\right) - \lambda & \psi(1 - \tau - \alpha) \\ \delta & -k_3 - \lambda \end{vmatrix} = 0 \quad (3.16)$$

By applying the elementary row operations of

$$R_2 \rightarrow \frac{\delta}{\left(k_2 - \frac{\beta S^*}{N^*}\right) + \lambda} R_1 + R_2$$

$$\begin{vmatrix} -\left(k_2 - \frac{\beta S^*}{N^*} + \lambda\right) & \psi(1 - \tau - \alpha) \\ 0 & (-k_3 - \lambda) + \frac{\delta\psi(1 - \tau - \alpha)}{\left(k_2 - \frac{\beta S^*}{N^*}\right) + \lambda} \end{vmatrix} = 0 \quad (3.17)$$

The first column of (3.17) is zero except the first entry, which is $-\left(k_2 - \frac{\beta S^*}{N^*} + \lambda\right)$. Since from the second equation of (3.12) at steady state, we can deduced that $k_2 > \frac{\beta S^*}{N^*}$, thus, we have the third eigenvalue; $\lambda = -\left(k_2 - \frac{\beta S^*}{N^*}\right) < 0$.

The algebraic simplification of $(-k_3 - \lambda) + \frac{\delta\psi(1 - \tau - \alpha)}{\left(k_2 - \frac{\beta S^*}{N^*}\right) + \lambda}$ from (3.17) gives

$$\lambda^2 + \left(\left(k_2 - \frac{\beta S^*}{N^*}\right) + k_3\right) \lambda - k_3 \frac{\beta S^*}{N^*} + k_2 k_3 - \psi\delta(1 - \tau - \alpha) = 0 \quad (3.18)$$

from the second term of equation (3.18) and at steady state of equation (3.12), we can deduced that $k_2 > \frac{\beta S^*}{N^*}$. Hence $\left(\left(k_2 - \frac{\beta S^*}{N^*}\right) + k_3\right) > 0$. Routh-Hurwitz criteria gives a necessary and sufficient condition for the negativity of the roots (eigenvalues) of quadratic equation (3.18). In our case, the condition of Routh-Hurwitz criteria is reduced to the condition

$$\frac{\beta k_3 \mu}{k_1} (1 - R_0) > 0 \quad (3.19)$$

But we know that $\frac{\beta k_3 \mu}{k_1} > 0$. Hence, if we need equation (3.19) to hold, then R_0 should be less than a unity. Therefore, the CFE, E_f is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

In order for corruption minimize to be independent of the initial sizes of the subpopulations of the model when $R_0 < 1$, a global asymptotic stability property must be established for the CFE when $R_0 < 1$. This is explored below. $\square$

3.3.4 Global Stability of Corruption Free Equilibrium

Theorem 3.3.2. *The corruption-free equilibrium, E_f of the system (3.12) globally asymptotically stable (GAS) in Ω if $R_0 \leq 1$.*

Proof. One common approach in studying the global asymptotic stability of the corruption-free is to construct an appropriate Lyapunov function.

Considering the Lyapunov function; $V(S, C, J, N) : \mathbb{R}^4 \rightarrow \mathbb{R}^+$ defined as

$$V = k_3 C + \psi(1 - \tau - \alpha)J \quad (3.20)$$

Its derivatives along the solutions of the model equation with respect to time is:

$$V' = k_3 C'(t) + \psi(1 - \tau - \alpha)J'(t) \quad (3.21)$$

Substituting the respective equations in the system equation (3.12) into (3.21), we have

$$\begin{aligned} V' &= k_3 \left[\frac{\beta C}{N} S - k_2 C + \psi(1 - \tau - \alpha)J \right] + \psi(1 - \tau - \alpha)[\delta C - k_3 J] \\ &= -C \left(k_2 k_3 - \delta \psi(1 - \tau - \alpha) - \left(\frac{k_3 \beta S}{N} \right) \right) \\ &= -C \left(\frac{k_3 \beta S}{N} \right) \left(\frac{(k_2 k_3 - \delta \psi(1 - \tau - \alpha))N}{k_3 \beta S} - 1 \right) \end{aligned}$$

Now, since $\frac{S}{N} \leq \frac{S^*}{N^*}$, we have

$$\begin{aligned} &\leq -C \left(\frac{k_3 \beta S^*}{N^*} \right) \left(\frac{(k_2 k_3 - \delta \psi(1 - \tau - \alpha))N^*}{k_3 \beta S^*} - 1 \right) \\ &= -C \left(\frac{k_3 \beta S^*}{N^*} \right) \left(\frac{1}{R_0} - 1 \right) \end{aligned}$$

Therefore,

$$\begin{aligned} V' &= -C \left(\frac{k_3 \beta S^*}{N^*} \right) \left(\frac{1}{R_0} - 1 \right) \\ &\leq 0, \end{aligned}$$

Thus we have established that $V' \leq 0$ if $R_0 \leq 1$. Therefore, the largest compact invariant set $\{(S, C, J, H) \in R_+^4 : V' = 0\}$ is the singleton E_f . Hence, by the LaSalle invariance principle LaSalle (1976); every solution that start in the region Ω approaches E_f as $t \rightarrow \infty$ is overall globally asymptotically stable for $R_0 \leq 1$ in region R_+^4 and hence, the proof is complete. The above theorem shows that corruption will be minimized regardless of the initial profile of the subpopulation in the community if R_0 can be brought down to a level less than unity. $\square$

3.3.5 Existence of Corruption Endemic equilibrium

At the corruption endemic equilibrium (CEE); E_e is steady state solution where the corruption persists in the population. Thus, at least one of the corrupt classes is greater than zero. In order to find the positive endemic equilibrium of the system (3.12), denoted by $E_e = (\overset{\circ}{S}, \overset{\circ}{C}, \overset{\circ}{J}, \overset{\circ}{N})$ and the equations in the system (3.12) are solved as explained in Gumel (2007) in terms of the associated force of acquired infection at steady state, given by:

$$\overset{\circ}{\lambda} = \frac{\beta \overset{\circ}{C}}{\overset{\circ}{N}} \quad (3.22)$$

Solving the equation of the model (3.12) at the steady state gives:

$$\begin{aligned} \overset{\circ}{S} &= \frac{\lambda}{k_1 + \overset{\circ}{\lambda}} \\ \overset{\circ}{C} &= \frac{\lambda \overset{\circ}{\lambda} k_3}{(k_1 + \overset{\circ}{\lambda})(k_2 k_3 - \delta \psi(1 - \tau - \alpha))} \\ \overset{\circ}{J} &= \frac{\lambda \overset{\circ}{\lambda} \delta}{(k_1 + \overset{\circ}{\lambda})(k_2 k_3 - \delta \psi(1 - \tau - \alpha))} \\ \overset{\circ}{N} &= \frac{\lambda}{\mu} \end{aligned} \quad (3.23)$$

Substitute the second equation of system(3.23) in to (3.22)and its simplifying gives:

$$\overset{\circ}{\lambda} = k_1 \left(\frac{\beta k_3 \lambda}{k_1 (k_2 k_3 - \delta \psi(1 - \tau - \alpha)) N^*} \right)$$

That means.

$$\overset{\circ}{\lambda} = k_1 (R_0 - 1) \quad (3.24)$$

Since all the parameters are assumed non-negative, it follows that $\overset{\circ}{\lambda} > 0$, whenever $R_0 > 1$.

To find all the components of E_e , substitute (3.24) into (3.23). Hence we have:

$$\begin{aligned} \overset{\circ}{S} &= \frac{\lambda}{k_1 R_0} \\ \overset{\circ}{C} &= \frac{\lambda (R_0 - 1) k_3}{(k_2 k_3 - \delta \psi(1 - \tau - \alpha)) R_0} \\ \overset{\circ}{J} &= \frac{\lambda (R_0 - 1) \delta}{(k_2 k_3 - \delta \psi(1 - \tau - \alpha)) R_0} \\ \overset{\circ}{N} &= \frac{\lambda}{\mu} \end{aligned} \quad (3.25)$$

Hence in all the components of E_e , $R_0 > 1$ indicates that the force of acquired infection at steady state $\dot{\lambda}$ is non-negative (which is biologically meaningful) and the system has a unique endemic (positive) equilibrium.

3.3.6 Local Stability of Corruption Endemic Equilibrium

Theorem 3.3.3. *The positive endemic equilibrium E_e of the system (3.12) is locally asymptotically stable (LAS) when $R_0 > 1$.*

Proof. we used the Jacobian stability approach to prove the stability of the corruption-endemic equilibrium state. The Jacobian matrix $\hat{J}$ of system (3.12) is

$$\hat{J} = \begin{pmatrix} -\left(k_1 + \frac{\beta C}{N}\right) & -\frac{\beta S}{N} & 0 & 0 \\ \frac{\beta C}{N} & -\left(k_2 - \frac{\beta S}{N}\right) & \psi(1 - \tau - \alpha) & 0 \\ 0 & \delta & -k_3 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}.$$

Linearization the above matrix at corruption endemic equilibrium E_e is,

$$\hat{J}(E_e) = \begin{pmatrix} -\left(k_1 + \frac{\beta \dot{C}}{\dot{N}}\right) & -\frac{\beta \dot{S}}{\dot{N}} & 0 & 0 \\ \frac{\beta \dot{C}}{\dot{N}} & -\left(k_2 - \frac{\beta \dot{S}}{\dot{N}}\right) & \psi(1 - \tau - \alpha) & 0 \\ 0 & \delta & -k_3 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix}. \quad (3.26)$$

From equation (3.12) at E_e , we have:

$$\begin{aligned} \lambda &= \frac{\beta \dot{C} \dot{S}}{\dot{N}} + k_1 \dot{S} \\ k_2 &= \frac{\beta \dot{S}}{\dot{N}} + \frac{\psi(1 - \tau - \alpha) \dot{J}}{\dot{C}} \\ k_3 &= \frac{\dot{C} \delta}{\dot{J}} \\ \mu &= \frac{\lambda}{\dot{N}} \end{aligned} \quad (3.27)$$

the eigenvalues of equation (3.26) are the solution of the characteristic equation.

$$|\hat{J}(E_e) - \lambda I_4| = 0$$

where I_4 is the identity matrix of order 4. then we have

$$\begin{vmatrix} -\left(k_1 + \frac{\beta\dot{C}}{\dot{N}}\right) - \lambda & -\frac{\beta\dot{S}}{\dot{N}} & 0 & 0 \\ \frac{\beta\dot{C}}{\dot{N}} & -\left(k_2 - \frac{\beta\dot{S}}{\dot{N}}\right) - \lambda & \psi(1 - \tau - \alpha) & 0 \\ 0 & \delta & -k_3 - \lambda & 0 \\ 0 & 0 & 0 & -\mu - \lambda \end{vmatrix} = 0 \quad (3.28)$$

The fourth column of (3.28) is all zero except the last diagonal entry, which is $-\mu - \lambda_1 = 0$. Thus, we have the first eigenvalue; $\lambda_1 = -\mu < 0$. The rest of the eigenvalues are computed from the following matrix.

$$\hat{J}(E_e) = \begin{pmatrix} -\left(k_1 + \frac{\beta\dot{C}}{\dot{N}}\right) & -\frac{\beta\dot{S}}{\dot{N}} & 0 \\ \frac{\beta\dot{C}}{\dot{N}} & -\left(k_2 - \frac{\beta\dot{S}}{\dot{N}}\right) & \psi(1 - \tau - \alpha) \\ 0 & \delta & -k_3 \end{pmatrix} \quad (3.29)$$

By applying elementary row operation $R_2 \rightarrow \frac{\beta\dot{C}}{(\beta\dot{C} + k_1\dot{N})}R_1 + R_2$ and

$$R_3 \rightarrow R_3 + \frac{\delta}{\left(k_2 - \frac{\beta\dot{S}}{\dot{N}}\right) + \frac{\beta^2\dot{S}\dot{C}}{((\beta\dot{C} + k_1\dot{N}))\dot{N}}}R_2$$

Equation (3.29) becomes:

$$\hat{J}(E_e) = \begin{pmatrix} -\left(k_1 + \frac{\beta\dot{C}}{\dot{N}}\right) & -\frac{\beta\dot{S}}{\dot{N}} & 0 \\ 0 & -H & \psi(1 - \tau - \alpha) \\ 0 & 0 & -k_3 + \frac{\delta\psi(1 - \tau - \alpha)}{H} \end{pmatrix} \quad (3.30)$$

Where

$$H = \left(\left(k_2 - \frac{\beta\dot{S}}{\dot{N}} \right) + \frac{\beta^2\dot{S}\dot{C}}{((\beta\dot{C} + k_1\dot{N}))\dot{N}} \right) \quad (3.31)$$

Hence we have eigenvalues;

$$\lambda_1 = -\left(k_1 + \frac{\beta\dot{C}}{\dot{N}} \right) < 0$$

$$\lambda_2 = -\left(\left(k_2 - \frac{\beta\dot{S}}{\dot{N}} \right) + \frac{\beta^2\dot{S}\dot{C}}{((\beta\dot{C} + k_1\dot{N}))\dot{N}} \right) < 0$$

and

$$\lambda_3 = - \left(k_3 - \frac{\delta\psi(1 - \tau - \alpha)}{H} \right) \quad (3.32)$$

Now, for to be λ_3 negative, we must have

$$- \frac{k_3 H + \delta\psi(1 - \tau - \alpha)}{H} < 0$$

Simplify and substitute the value of H, we have

$$\begin{aligned} k_3 \left(\left(k_2 - \frac{\beta \dot{S}}{\dot{N}} \right) + \frac{\beta^2 \dot{S} \dot{C}}{(\beta \dot{C} + k_1 \dot{N}) \dot{N}} \right) - \delta\psi(1 - \tau - \alpha) &> 0 \\ \left(k_2 k_3 - \frac{k_3 \beta \dot{S}}{\dot{N}} \right) + \left(\frac{k_3 \beta^2 \dot{S} \dot{C}}{(\beta \dot{C} + k_1 \dot{N}) \dot{N}} \right) &> \delta\psi(1 - \tau - \alpha) \\ \frac{k_2 k_3 \dot{N} - k_3 \beta \dot{S}}{\dot{N}} + \frac{k_3 \beta^2 \dot{S} \dot{C}}{(\beta \dot{C} + k_1 \dot{N}) \dot{N}} &> \delta\psi(1 - \tau - \alpha) \\ \frac{(k_2 k_3 \dot{N} - k_3 \beta \dot{S}) (\beta \dot{C} + k_1 \dot{N}) + k_3 \beta^2 \dot{S} \dot{C}}{(\beta \dot{C} + k_1 \dot{N}) \dot{N}} &> \delta\psi(1 - \tau - \alpha) \\ \frac{(k_2 k_3 - \delta\psi(1 - \tau - \alpha)) \dot{N}}{k_3 \beta \dot{S}} + \frac{\beta \dot{C}}{(\beta \dot{C} + k_1 \dot{N})} &> 1 \\ \frac{k_1 R_0 (k_2 k_3 - \delta\psi(1 - \tau - \alpha)) \dot{N}}{k_3 \beta \lambda} + \frac{\beta \dot{C}}{(\beta \dot{C} + k_1 \dot{N})} &> 1 \\ \beta \dot{C} \left[\frac{1}{\beta \dot{C}} + \frac{1}{(\beta \dot{C} + k_1 \dot{N})} \right] &> 1 \\ \beta \dot{C} &> \frac{1}{\left[\frac{1}{\beta \dot{C}} + \frac{1}{(\beta \dot{C} + k_1 \dot{N})} \right]} \end{aligned}$$

Hence, we obtained

$$\beta \dot{C} > 0 \quad (3.33)$$

Substitute the second equation of system (3.25) in to system (3.33), we have:

$$\frac{\beta \lambda (R_0 - 1) k_3}{(k_2 k_3 - \delta\psi(1 - \tau - \alpha)) R_0} > 0$$

From the simplification of the above equation we have: $(R_0 - 1) > 0$.

Thus, $\lambda_3 < 0$ if $R_0 > 1$ implying all the eigenvalues have negative real parts. Therefore, the corruption-endemic equilibrium E_e is locally asymptotically stable. $\square$

3.3.7 Global Stability of Corruption Endemic Equilibrium

Theorem 3.3.4. *If $R_0 > 1$, then the corruption-endemic equilibrium, E_e of the system (3.12) is globally asymptotically stable (GAS) in Ω when;*

$$\frac{S}{\dot{S}} = \frac{C}{\dot{C}} \quad (3.34)$$

Proof. Considering the Lyapunov function candidate; $L(S, C, J, N) : \mathbb{R}^4 \rightarrow \mathbb{R}^+$ defined as

$$\begin{aligned} L = & (S - \dot{S} - \dot{S} \ln \left(\frac{S}{\dot{S}} \right)) + (C - \dot{C} - \dot{C} \ln \left(\frac{C}{\dot{C}} \right)) \\ & + \frac{\psi(1 - \tau - \alpha)\dot{J}}{\delta\dot{C}} (J - \dot{J} - \dot{J} \ln \left(\frac{J}{\dot{J}} \right)) + \frac{1}{2\mu} [N - \dot{N}]^2 \end{aligned} \quad (3.35)$$

its derivatives along the solutions of the model equations is:

$$\begin{aligned} L' = & \left(1 - \frac{\dot{S}}{S}\right) S'(t) + \left(1 - \frac{\dot{C}}{C}\right) C'(t) + \frac{\psi(1 - \tau - \alpha)\dot{J}}{\delta\dot{C}} \left(1 - \frac{\dot{J}}{J}\right) J'(t) \\ & + \frac{1}{\mu} [N - \dot{N}] N'(t). \end{aligned} \quad (3.36)$$

Substituting the values of the system (3.12) of model equation into equation (3.36), it becomes:

$$\begin{aligned} L' = & \left(1 - \frac{\dot{S}}{S}\right) \left(\lambda - \frac{\beta C}{N} S - k_1 S\right) + \left(1 - \frac{\dot{C}}{C}\right) \left(\frac{\beta C}{N} S - k_2 C + \psi(1 - \tau - \alpha) J\right) \\ & + \frac{\psi(1 - \tau - \alpha)\dot{J}}{\delta\dot{C}} \left(1 - \frac{\dot{J}}{J}\right) (\delta C - k_3 J) + \frac{1}{\mu} [N - \dot{N}] [\lambda - \mu N] \end{aligned}$$

Substituting from equation (3.27) the values of λ , k_2 and k_3 at E_e into the above equation, it becomes:

$$\begin{aligned} L' = & \left(1 - \frac{\dot{S}}{S}\right) \left(\frac{\beta\dot{C}}{\dot{N}}\dot{S} + k_1\dot{S} - \frac{\beta C}{N} S - k_1 S\right) \\ & + \left(1 - \frac{\dot{C}}{C}\right) \left(\frac{\beta C}{N} S + \psi(1 - \tau - \alpha) J - \frac{\beta C}{\dot{N}}\dot{S} - \frac{\psi(1 - \tau - \alpha)C\dot{J}}{\dot{C}}\right) \\ & , + \frac{\psi(1 - \tau - \alpha)\dot{J}}{\delta\dot{C}} \left(1 - \frac{\dot{J}}{J}\right) \left(\delta C - \frac{\delta\dot{C}J}{\dot{J}}\right) + \frac{1}{\mu} [N - \dot{N}] [\mu\dot{N} - \mu N] \end{aligned}$$

Rearranging the terms and simplifying, we have

$$\begin{aligned}
 L' &= \left[-\frac{k_1(S - \dot{S})^2}{S} - [N - \dot{N}]^2 \right] \\
 &\quad + \frac{\beta \dot{C}}{\dot{N}} \dot{S} \left[\left(1 - \frac{\beta CS}{\frac{\beta}{\dot{N}} \dot{C} \dot{S}} \right) \left(1 - \frac{\dot{S}}{S} \right) + \left(\frac{\beta CS}{\frac{\beta}{\dot{N}} \dot{C} \dot{S}} - \frac{C}{\dot{C}} \right) \left(1 - \frac{\dot{C}}{C} \right) \right] \\
 &\quad + \psi(1 - \tau - \alpha) \dot{J} \left[\left(1 - \frac{\dot{C}}{C} \right) \left(\frac{J}{\dot{J}} - \frac{C}{\dot{C}} \right) + \left(1 - \frac{\dot{J}}{J} \right) \left(\frac{C}{\dot{C}} - \frac{J}{\dot{J}} \right) \right] \\
 &= \left[-\frac{k_1(S - \dot{S})^2}{S} - [N - \dot{N}]^2 \right] + \frac{\beta \dot{C}}{\dot{N}} \dot{S} y + \psi(1 - \tau - \alpha) \dot{J} z
 \end{aligned}$$

where

$$y = \left[\left(1 - \frac{\beta CS}{\frac{\beta}{\dot{N}} \dot{C} \dot{S}} \right) \left(1 - \frac{\dot{S}}{S} \right) + \left(\frac{\beta CS}{\frac{\beta}{\dot{N}} \dot{C} \dot{S}} - \frac{C}{\dot{C}} \right) \left(1 - \frac{\dot{C}}{C} \right) \right]$$

and

$$z = \left[\left(1 - \frac{\dot{C}}{C} \right) \left(\frac{J}{\dot{J}} - \frac{C}{\dot{C}} \right) + \left(1 - \frac{\dot{J}}{J} \right) \left(\frac{C}{\dot{C}} - \frac{J}{\dot{J}} \right) \right]$$

Considering y , with the condition $\frac{S}{\dot{S}} = \frac{C}{\dot{C}}$, we have:

$$y = 2 - \frac{\dot{C}}{C} - \frac{C}{\dot{C}} = 2 - \left(\frac{\dot{C}^2 + C^2}{C\dot{C}} \right) \leq 0.$$

Similarly, for z , we have:

$$z = 2 - \frac{\dot{C}J}{C\dot{J}} - \frac{C\dot{J}}{\dot{C}J} = 2 - \left(\frac{(\dot{C}J)^2 + (C\dot{J})^2}{(C\dot{J})(\dot{C}J)} \right) \leq 0.$$

Thus the derivative of considered lypunov function

$$\begin{aligned}
 L' &= - \left[\frac{k_1(S - \dot{S})^2}{S} + [N - \dot{N}]^2 \right] \\
 &\quad + \frac{\beta \dot{C}}{\dot{N}} \dot{S} y + \psi(1 - \tau - \alpha) \dot{J} z \\
 &< 0
 \end{aligned}$$

Thus, L' is less zero $(S, C, N) \neq (\dot{S}, \dot{C}, \dot{N})$ and $\frac{S}{\dot{S}} = \frac{J}{\dot{J}} = \frac{C}{\dot{C}} = 1$. Thus, $L' = 0$ only if $(S, C, J, N) = (\dot{S}, \dot{C}, \dot{J}, \dot{N})$. Then, By LaSalle's invariance principle (LaSalle, 1976), the largest invariant set of system (3.12) on the set $\{(S, C, J, N) \in \Omega \subset \mathbb{R}_+^4 : L' \leq 0\}$ and the endemic equilibrium E_e is globally asymptotically stable in Ω if $R_0 > 1$. $\square$

Chapter 4

Numerical Simulation and results

In this chapter we present a numerical simulations of the model which is carried out using Matlab ode45. To monitor the dynamics of the full system of equation (3.1) for various values of the parameters and in order to confirm the results on the local and global stability of both the corruption-free and the endemic equilibrium.

4.1 Estimation of Parameter

Parameters values estimation is probably the hardest part of any mathematical modeling. This is because any small under or over estimation of a value can result in a big misleading output. The values of the parameters used in the model are given in Table 4.1, which are directly taken from the work of S. Abdulrahaman, (2014) and the data recorded by the TICPI in the years 2012 and 2013.

No.	Parameter	Baseline value	No.	Parameter	Baseline value
1	λ	30,000	6	α	$(0 < \alpha < 1)$
2	δ	0.0001τ	7	p	0.36,
3	τ	$(0 < \tau < 1)$	8	α	0.000001
4	ψ	0.143	—	—	—
5	μ	$0.013, (0.012 \leq \mu \leq 0.022)$	-	—	—

Table 4.1: Baseline Values for Parameters of equation (3.1).

The recruitment rate, λ in to susceptible subpopulation is assumed to be 30,000. Again the average lifespan of most countries in the world $\left(\frac{1}{\mu}\right)$ is 76.92, a death removal rate

$\mu = 0.013$ (the modal value) is used as hypothetical value for numerical simulations, so that $N = \frac{\lambda}{\mu} = 2,308462$, a transmission probability of corruption ($p = 0.36$) is assumed in the absence of transparency and accountability. The two different values of nations effort rate against corruption due to influence of anti-corruption $\tau = 0.90$ and 0.16 are used as hypothetical values for numerical simulation. The rate at which corrupt individuals are caught and jailed (δ) depends so much on the illicit and secretive nature of corruption as well as its level of prevalence in the population concerned. Thus, it given as $\delta = 0.0001\tau$, where the reduced modification value associated with the illicit and secretive nature of corruption is 0.0001 .

In most countries the average period corrupt individuals spent in prison $\left(\frac{1}{\psi}\right)$ is eight (7) years, so that $\psi = 0.143$ and as it is natural to have a very insignificant number of individuals to be honest in any human set up, a proportion α of the susceptible, corrupt or jailed compartment is assumed to become honest (per year) due to public enlightenment.

The basic reproduction number, R_0 is a measure of the potential for corruption spread in a population, and is one of the foremost and most valuable ideas that mathematical thinking has brought to epidemic theory (Heesterbeek and Dietz, 1996). If $R_0 < 1$, then on average, an corrupted individual produces less than one newly corrupted individual over the course of its corruption period. In this case, the corruption may remove out in the long run. Conversely, if $R_0 > 1$, each corrupted individual produces, on average more than one new corrupt, then corruption will be able to spread in a population. A large value of R_0 may indicate the possibility of a major epidemic.

4.2 Numerical Simulation

The dynamic behavior of corruption model can be studied through numerical simulations by varying the parameters of the model. For this, the list of basic numerical parameters available in Table 4.1

- (i) With $\mu = 0.013$ and $\tau = 0.90$, so that $R_0 = 0.0073 < 1$,

Clearly from figure 1 given below we seen that the solution of system (3.1) converges to the corruption free equilibrium in both the two different initial values of $C(0)$. This confirms the

analytical results of the local as well as the global asymptotic stability of the corruption free equilibrium it describes structures of the system (3.1) with alternative solutions proposed to minimize corruption.

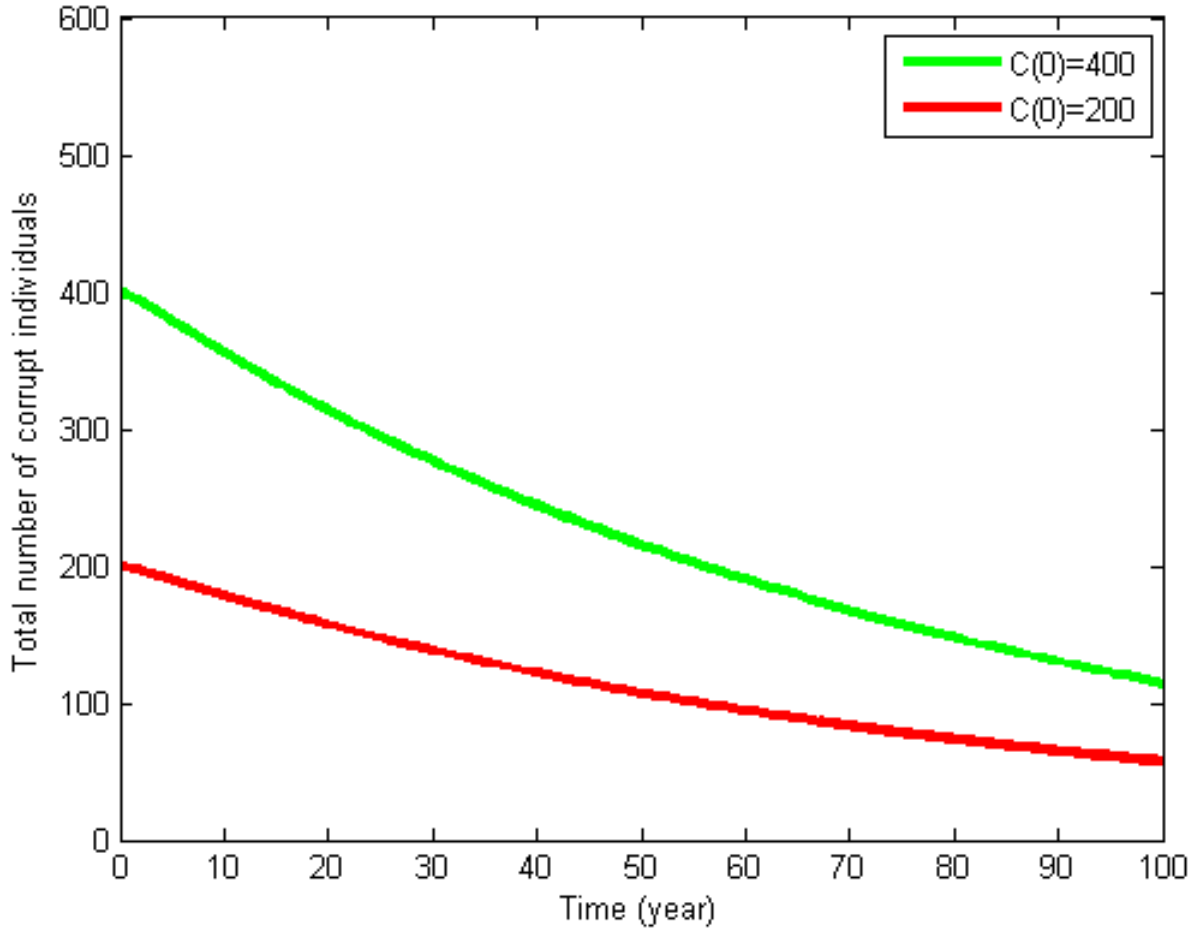


Figure 4.1: Total number of corrupt individuals with different initial population and rate against corruption $\tau = 0.9$.

Total number of corrupt individuals with different initial population of $C(0) = 200$ and 400, with Parameters value used are as in Table 1, $\tau = 0.90$ and $\mu = 0.013$, so that $R_0 = 0.0073$. This confirms the local as well as the global asymptotic stability of the corruption free equilibrium. The highest value of τ (effort rate against corruption), can be reduce corruption to a tolerable level, rather than remove out because of the illicit and secretive nature of corruption.

(ii) With $\mu = 0.013$ and $\tau = 0.16$, so that $R_0 = 1.57 > 1$,

It seen clearly from figure 2 given below the solution of system (3.1) converges to the corruption endemic equilibrium in all the two different initial values of $C(0)$. This confirms the analytical results of the local as well as the global asymptotic stability of the corruption endemic equilibrium. It is observed that the total number of corrupt individuals continued to rise.

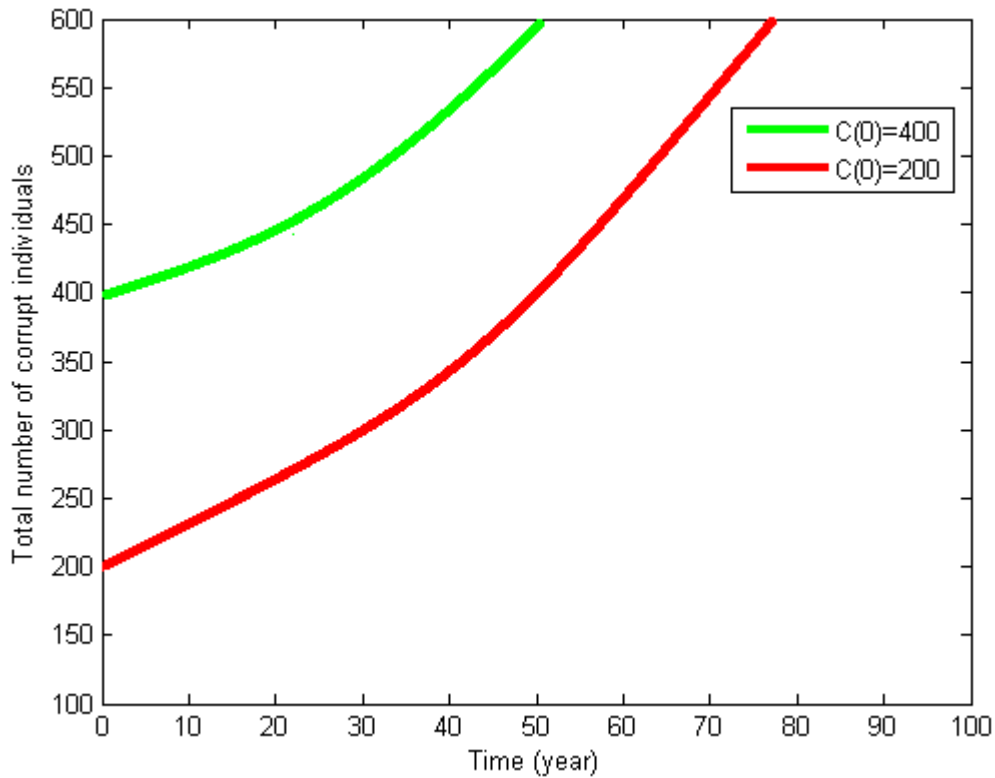


Figure 4.2: Total number of corrupt individuals with different initial population and rate against corruption $\tau = 0.16$.

Total number of corrupt individuals with different initial population of $C(0) = 200$ and 400 . Parameters value used are as in Table 1, with $\tau = 0.16$ and $\mu = 0.013$, so that $R_0 = 1.57$. This confirms the local as well as the global asymptotic stability of the corruption endemic equilibrium and the less value of $\tau = 0.16$ is that the infection will be able to spread in a population. A large value of $R_0 > 1$ may indicate the possibility of a major epidemic.

Chapter 5

Discussion and Conclusion

5.1 Discussion

The transmission of Corruption in a population is a common problem to businesses and reduces economic growth through a negative influence on investments in human capital. Anti-Corruption commission is giving public awareness on the danger, very harsh effective penalties and consequence of corruption to the population. The basic reproduction number R_0 is such an important tool, which allows us to determine the importance of the parameters in the corruption dynamics.

We can study both local and global stability of the model using a Jacobians and Lyapunov function approaches, respectively in both corruption free and endemic equilibrium. In particular, we investigate the stability properties by paying more attention to the basic reproduction number. Furthermore, we have seen that bringing R_0 less than unity is necessary but not sufficient condition for the removing totally corruption from society.

5.2 Conclusion

In this thesis, we have developed and analyzed a mathematical model to study the dynamical nature of corruption based on a system of non-linear differential equation of type susceptible(S), corrupt(C), jailed(J) and honest(H) with constant recruitment rate on the total population. It was showed that there exists a domain Ω in which the model is mathematically and epidemiologically well posed. The next generation matrix was used to derive the basic reproduction number R_0 , which is the average number of new cases that one infected case will generate.

The corruption free equilibrium of the model was proved to be locally asymptotically stable whenever R_0 is less than unity. It is also showed that the corruption free equilibrium is globally asymptotically stable provided that the basic reproduction number is less than some threshold. Finally, we found that the numerical simulations are basic to characterize the behaviors of the solutions of the proposed model equations as in the analytical case.

Thus, anti corruption commission educating the susceptible and jailed individuals and law enforcement on the corrupt individuals as prevention mechanisms and reduction measures of corruption from its expansion. This work can be further extended by using conditions for optimal control of corruption with prevention mechanisms and reduction measures.

Appendix A

Mathematical Preliminaries

A.1 Basic Definitions and Results

In this section, we provide the basic mathematical definitions and results, which are crucial for a better understanding of this thesis work.

Definition A.1.1. Model: is a miniature of something, is a pattern of something to be made.

Definition A.1.2. Mathematical Model: is a representation in mathematical term of the behavior of real devices and objects.

Definition A.1.3. Bounded Function: A function $f(x)$ is said to be bounded on a set S if and only if there exist an $M \in S$ such that

$$|f(x)| \leq M,$$

for any $x \in S$.

Definition A.1.4. Invariant Set: A set M is an invariant set, if for the system

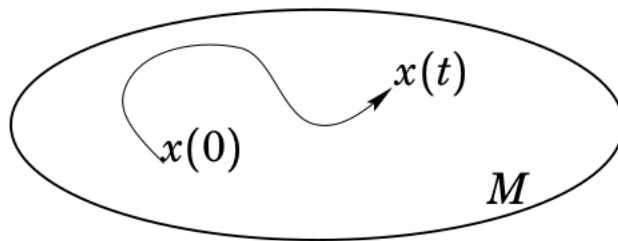
$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M, \forall t \in R$.

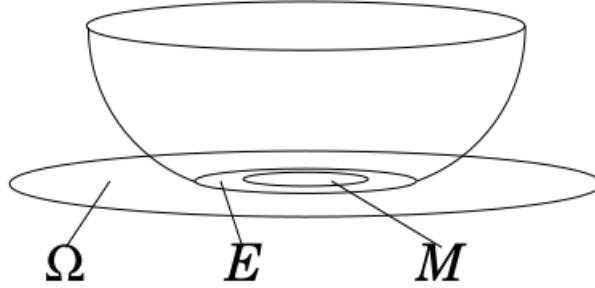
Definition A.1.5. Positively Invariant: A set M is a positively invariant set, if for the system

$$\dot{x} = f(x),$$

$x(0) \in M$ implies that $x(t) \in M, \forall t \geq 0$.



Theorem A.1.1. (LaSalle's Invariant Set:) Let $\Omega \subseteq \mathbb{R}^n$ be compact positively invariant set for $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over Ω such that $V'(x) \leq 0, \forall x \in \Omega$. Let E be the set of all points in Ω where $V'(x) = 0$, and M be the largest invariant set in E . Then every solution starting in Ω approaches M or the trajectory tends to the largest invariant set inside M as $t \rightarrow \infty$.



The prove of the above theorem can be found in (LaSalle, J.P., 1976).

Theorem A.1.2. (Gronwall's Inequality - differential form): Let $I = [t_0, t_1]$. Suppose $p : I \rightarrow \mathbb{R}$ and $q : I \rightarrow \mathbb{R}$ are continuous and suppose $x : I \rightarrow \mathbb{R}$ is continuously differentiable on I and satisfies

$$x'(t) \leq q(t) + p(t)x(t), \quad (\text{A.1})$$

for $t \in I$ and $x(t_0) = x_0$. Then $x(t) \leq x_0 e^{\int_{t_0}^t p(r)dr} + \int_{t_0}^t q(s) e^{\int_s^t p(r)dr} ds$, and thus a solution of the differential inequality is bounded above by the solution of the equality, i.e., the differential equation $x' = q + px$.

The prove of the above theorem can be found in (Gronwall, Thomas H., 1919).

A.2 The Routh-Hurwitz Criteria

Definition A.2.1. A matrix $A = (a_{ij})_{n \times n}$ is said to be Routh-Hurwitz (or asymptotically stable) if and only if

$$\text{Re}(\lambda_i) < 0, \forall i = 1, 2, \dots, n,$$

where λ_i are eigenvalues of the matrix A (Hassan K. Khalil, 2002).

Theorem A.2.1. Given a polynomial $p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n$ where the coefficients a_i are real constants, $i = 1, 2, 3, \dots, n$ define the n Hurwitz matrices using the coefficients of the characteristic polynomial:

$$H_1 = [a_1]$$

$$,H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \cdots \text{ and}$$

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \text{ where } a_j = 0 \text{ for } j > n . \text{ All the roots of polynomial } p(\lambda)$$

are negative or have negative real parts if and only if the determinants of all Hurwitz matrices are positive:

$$\det H_j > 0, j = 1, 2, 3, \dots, n$$

for $n = 1$, then $\det H_1 = a_1 > 0$

for $n = 2$ then $\det H_2 = \det \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} = a_1 a_2 > 0$ or $a_1 > 0$ and $a_2 > 0$ and

for $n = 3$ then $\det H_3 = \det \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} = a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$ polynomials of degree 1, 2 and 3, the Routh Hurwitz Criteria are summarized as follows:

$n = 1$ then $a_1 > 0$

$n = 2$ then $a_1 > 0$ and $a_2 > 0$

$n = 3$ then $a_1 > 0$, $a_2 > 0$ and $a_1 a_2 - a_3 > 0$

A.3 Lyapunov Theory

A powerful method for analyzing the stability of an equilibrium point is based on the use of Lyapunov functions by John Guckenheimer and Philip J Holmes (2013).

Definition A.3.1. (Equilibrium Point) A point x^* is an equilibrium point of $\dot{x} = f(x)$, if and only if $f(x^*) = 0$.

Definition A.3.2. (Locally Stable) An equilibrium point x^* is locally stable if for every $R > 0$ there exists $r > 0$ such that

$$\|x(0) - x^*\| < r \implies \|x(t) - x^*\| < R, \quad t \geq 0$$

Definition A.3.3. (Locally Asymptotically Stable) An equilibrium point is locally asymptotically stable if it is locally stable and

$$\|x(0) - x^*\| < r \implies \lim_{t \rightarrow \infty} x(t) = x^*.$$

Definition A.3.4. (Globally Asymptotically Stable) An equilibrium point is globally asymptotically stable if it is Asymptotically stable for all $x(0) \in \mathbb{R}^n$.

Theorem A.3.1. (Alexandr Mihailovich Lyapunov - Stability Theory): Let $\dot{x} = f(x)$ and $f(x^*) = 0$. If there exists a continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (1) $V(x^*) = 0$
- (2) $V(x) > 0$, for all $x \neq x^*$
- (3) $\dot{V}(x) < 0$, for all $x \neq x^*$ and $\dot{V}(x) = 0$, if $x = x^*$
- (4) $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$,

then x^* is a globally asymptotically stable equilibrium. The function V that fulfills this conditions is called a Lyapunov function for global asymptotic stable. The prove of the above theorem can be found in (Alexandr M. L., 1892).

A.4 Matlab Codes

Matlab codes or ode45 codes for corruption free equilibrium of figure 4.1.

```
function dc=corruption(t,c)

dc=zeros(4,1);

p=0.36;
phi=0.143;
alpha=0.000001;
tau=0.9;
delta=0.0001*tau;
lambda=30000;
mu=0.013;
beta=p*(1-tau-alpha);
dc(1)=lambda- beta*c(2)*c(1)./c(4)-(mu+alpha+tau)*c(1);
dc(2)=beta*c(2)*c(1)./c(4)-(mu+alpha+delta)*c(2)+phi*(1-tau-alpha)*c(3);
```

```

dc(3)=delta*c(2)-(mu+tau+alpha)*c(3);
dc(4)=lambda-mu*c(4);
t=[0 100];
y0=[30000,400,900,2307692];
[T,C1] = ode45(@corruption,t,y0);
plot(T,C1(:,2),'g','Line Width',3)
x label('Time(year)')
y label('Total number of corrupt individuals ')
hold on
t=[0 100];
y0=[30000,200,300, 2307692];
[T,C] = ode45(@corruption,t,y0);
plot(T,C(:,2),'r','Line Width',3)
plot(T,C1(:,2),'g',T,C(:,2),'r','Line Width',3)
legend('C(0)=400','C(0)=200')
axis([0 100 100 600])
x label('Time (year)')
y label('Total number of corrupt individuals ')

```

Matlab codes or ode45 codes for corruption endemic equilibrium of figure 4.2.

```

function dc=corruption(t,c)
dc=zeros(4,1);
p=0.36;
phi=0.143;
alpha=0.000001;
tau=0.16;
delta=0.0001*tau;
lambda=30000;
mu=0.013;
beta=p*(1-tau-alpha);
dc(1)=lambda- beta*c(2)*c(1)./c(4)-(mu+alpha+tau)*c(1);

```

```

dc(2)=beta*c(2)*c(1)./c(4)-(mu+alpha+delta)*c(2)+phi*(1-tau-alpha)*c(3);
dc(3)=delta*c(2)-(mu+tau+alpha)*c(3);
dc(4)=lambda-mu*c(4);
t=[0 100];
y0=[30000,400,40,2307692];
[T,C1] = ode45(@corruption,t,y0);
plot(T,C1(:,2),'g','Line Width',3)
x label('Time(year)')
y label('Total number of corrupt individuals ')
hold on
t=[0 100];
y0=[30000,200,10, 2307692];
[T,C] = ode45(@corruption,t,y0);
plot(T,C(:,2),'r','Line Width',3)
plot(T,C1(:,2),'g',T,C(:,2),'r','Line Width',3)
legend('C(0)=400','C(0)=200')
axis([0 100 100 600])
x label('Time (year)')
y label('Total number of corrupt individuals ')

```

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